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A Simple Theory of Structural Transformation

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Abstract

The paper presents a theory of the industrial transformation amongst sectors along the balanced growth path equilibrium, using endogenous growth theory. Allowing only a slight upward trend in the productivity of the human capital sector, combined with ascending degrees of human capital shares of sectoral output, in say, agriculture, manufacturing and services, output gradually shifts relatively over time from agriculture to manufacturing and to services. Abstracting from international trade theory, sectors intensive in the factor that is becoming relatively more plentiful find their relative outputs expanding. Adding more sectors of greater human capital intensity causes labor time to decrease within each sector, as shown for agriculture, and in general for any number of sectors. The number of sectors is also allowed to depend endogenously on the human capital productivity level.

JEL Classification: E25, F11, J24, O14

Keywords: Human Capital Intensity, Sectoral Allocation, Labor Shares, Secular Endogenous Growth

PRELIMINARY DRAFT

1 Introduction

The gradual industrial transformation as countries develop, of relative output and input shares, remains a topic defying easy explanation. T.W.Schultz (1964) describes how human capital accumulation enables the movement from traditional to modern agriculture, similar to Cochrane (1993), who suggested that by raising education levels, farmers could transform their agricultural methods to modern ones using advanced farming equipment. Yair Mundlak (2000,2005) focuses on going from agriculture to manufacturing, while D.G. Johnson (2002) emphasizes the rural to urban migration. Lucas (2002) can be thought of as extending Schultz (1964) by using human capital to explain the industrial revolution from agriculture to manufacturing, as well as in Lucas (2004) to explain the rural to urban migration. Explanations without human capital of historical growth rate changes, using a two sector model, is found in Hansen and Prescott (2002).

This paper follows the Lucas (1988) human capital approach in order to offer a simple yet complete theory of the structural industry transformation over time, using the balanced growth equilibrium. It uses standard homothetic production and utility functions, with only one simple assumption that has commonality with the Solow exogenous growth theory used in many structural transformation theories. The only parameter that changes over time is the productivity of the human capital sector, with a very slight exogenous upward trend, similar to the exogenous upward trend of the goods sector productivity in the neoclassical growth model. This explains the changing relative shares of output. By including additional sectors, with more human capital intensity, labor shifts across sectors are also explained.

The slight upward trend in human capital productivity, with King and Rebelo (1990) CRS production functions for each sector, output shifts gradually over time to the sectors that are more human capital intensive, thus explaining the relative output shift. This first key part includes relative prices of the sectors moving opposite of relative output changes, an application of the Rybzynsky theorem, if you will, in that more productive human capital cause more human capital accumulation and sectors intensive in that factor see a relative price decline and a relative output expansion.

Second, by adding one more sector, with greater human capital intensity than the other sectors that the model arbitrarily starts with, labor shifts broadly across sectors, towards the more human capital intensive sectors. It is well known that labor and capital shares stay relatively unchanged in models with a set number of sectors, a seeming problem. But one of the key descriptive feature of structural transformation is that economies start with only agriculture, then agriculture and manufacturing, say. Then a third sector services. Then a fourth sector, technology, and so on. Thereby to explain theoretically how relative labor shares change, more human capital intensive sectors are added, and this can go on indefinitely as in the actual economy.[1]

2 Endogenous Growth Sectoral Model

The simplest statement of the theory is to start with only two sectors. Let there be a representative agent and initially two sectoral goods, with no aggregate good per se. The goods are agriculture output y_{At}, and manufacturing output y_{Mt}, with real prices of p_{At} and p_{Mt}. The consumer current period utility u_t is a simple log form, with parameters $\alpha > 0$, $\alpha_A > 0$, and $\alpha_M > 0$, where

$$u_t = \alpha \ln x_t + \alpha_A \ln y_{At} + \alpha_M \ln y_{Mt}.$$

The consumer buys these goods for a total cost of $p_{At}y_{At} + p_{Mt}y_{Mt}$, and invests i_t in physical capital (k_t) accumulation, with a depreciation rate of δ_k, and with

$$i_t = k_{t+1} - k_t (1 - \delta_k).$$

And the consumer also invests i_{Ht} in Lucas (1988) human capital (h_t) accumulation, where i_{Ht} is produced using a production function linear in human

[1] The Wall Street Journal now has interesting sectoral breakdowns of the entire US economy, with agriculture not even included any longer as a sector. It is an interactive that shows the size and composition by firm of every on of its dozen or so sectoral classification of the economy.

capital. With a depreciation rate of δ_h, with $A_H > 0$, with l_{Ht} denoting the time spent in producing human capital investment, and so with

$$A_{Ht} l_{Ht} h_t = i_{Ht} = h_{t+1} - h_t \left(1 - \delta_h\right). \tag{1}$$

Consumer income is from time spent working at the wage rate w_t, per unit of human capital, and from renting physical capital at the rate r_t, per unit of physical capital. The consumer's time is divided between time spent working in the three sectors of output production, and in human capital investment production. With a time endowment of 1, and x_t for leisure, this makes total working time for wages equal to $1 - l_{Ht} - x_t$, wages earned equal to $w_t \left(1 - l_{Ht} - x_t\right) h_t$ and the time allocation as given by

$$1 = l_{At} + l_{Mt} + l_{Ht} + x_t.$$

Capital is being rented by the consumer to each sector, with shares of capital being denoted by s_{At} and s_{Mt}, and with these adding to one:

$$1 = s_{At} + s_{Mt}.$$

With r_t the real interest rate, rental income from the two sectors in total is $r_t k_t$.

Recursively, the consumer's problem is given as the maximization of utility subject to income and human capital accumulation constraints:

$$\begin{aligned} V\left(k_t, h_t\right) &= \underset{y_{At}, y_{Mt}, l_{Ht}, k_{t+1}, h_{t+1}, x_t}{Max} \left\{ \left(\alpha_A \ln y_{At} + \alpha_M \ln y_{Mt} + \alpha \ln x_t\right) + \beta V\left(k_{t+1}, h_{t+1}\right) \right. \\ & +\lambda_t \left[w_t \left(1 - l_{Ht} - x_t\right) h_t + r_t k_t - k_{t+1} + k_t \left(1 - \delta_k\right) - p_{At} y_{At} - p_{Mt} y_{Mt}\right] \\ & \left. +\nu_t \left[h_t \left(1 + A_H l_{Ht} - \delta_h\right) - h_{t+1}\right] \right\}. \end{aligned}$$

Eliminating the constraints, the problem is

$$\begin{aligned} & V\left(k_t, h_t\right) \\ = & \underset{l_{Ht}, x_t, y_{At}, y_{Mt}}{Max} \left\{ \left(\alpha_A \ln y_{At} + \alpha_M \ln y_{Mt} + \alpha \ln x_t\right) + \right. \\ & \left. \beta V\left(\left[w_t \left(1 - l_{Ht} - x_t\right) h_t + k_t \left(1 + r_t - \delta_k\right) - p_{At} y_{At} - p_{Mt} y_{Mt}\right], h_t \left(1 + A_H l_{Ht} - \delta_h\right)\right) \right\} \end{aligned}$$

The standard first order equilibrium conditions are

$$
\begin{aligned}
l_{Ht} &: -\beta \frac{\partial V\left(k_{t+1}, h_{t+1}\right)}{\partial k_{t+1}} w_t h_t + \beta \frac{\partial V\left(k_{t+1}, h_{t+1}\right)}{\partial h_{t+1}} A_H h_t = 0, : \\
x_t &: \frac{\alpha}{x_t} - \beta \frac{\partial V\left(k_{t+1}, h_{t+1}\right)}{\partial k_{t+1}} w_t h_t = 0, \\
y_{At} &: \frac{\alpha_A}{y_{At}} - p_{At} \beta \frac{\partial V\left(k_{t+1}, h_{t+1}\right)}{\partial k_{t+1}} = 0, \\
y_{Mt} &: \frac{\alpha_M}{y_{Mt}} - p_{Mt} \beta \frac{\partial V\left(k_{t+1}, h_{t+1}\right)}{\partial k_{t+1}} = 0;
\end{aligned}
$$

while the envelope conditions,

$$
\begin{aligned}
h_t &: \frac{\partial V\left(k_t, h_t\right)}{\partial h_t} = \beta \frac{\partial V\left(k_{t+1}, h_{t+1}\right)}{\partial k_{t+1}} w_t \left(1 - l_{Ht} - x_t\right) + \beta \frac{\partial V\left(k_{t+1}, h_{t+1}\right)}{\partial h_{t+1}} \left(1 + A_H l_{Ht} - \delta_h\right), \\
k_t &: \frac{\partial V\left(k_t, h_t\right)}{\partial k_t} = \beta \frac{\partial V\left(k_{t+1}, h_{t+1}\right)}{\partial k_{t+1}} \left(1 + r_t - \delta_k\right),
\end{aligned}
$$

yield the intertemporal growth conditions along the balanced growth path (BGP) equilibrium, with g_t denoting the BGP growth rate:

$$
\begin{aligned}
1 + g_t &= \beta \left[1 + A_H \left(1 - x_t\right) - \delta_h\right], && (2) \\
1 + g_t &= \beta \left(1 + r_t - \delta_k\right). && (3)
\end{aligned}
$$

These show how the return to human and physical capital are equal on the BGP equilibrium. A third intertemporal growth condition and the BGP comes from the human capital investment function, quickly yielding an expression for consumer time in this sector, l_{Ht}, in terms of the growth rate. From equation (1), on the BGP,

$$
l_{Ht} = \frac{g + \delta_h}{A_H}.
$$

Note that combined with the growth equation in (2), and assuming that $\frac{1}{1+\rho} \equiv \beta$, the solution for leisure in terms of the BGP growth rate $g:$

$$
\begin{aligned}
l_{Ht} &= \frac{g + \delta_h}{A_H} = \frac{\frac{1 + A_H(1 - x_t) - \delta_h}{1+\rho} - 1 + \delta_h}{A_H}, \\
x_t &= 1 - \frac{\left[\left(1 + g\right)\left(1 + \rho\right) - 1 + \delta_h\right]}{A_H}.
\end{aligned}
$$

This leaves the labor sum l_t in the agriculture and manufacturing sector to be simply

$$l_t \equiv l_{At} + l_{Mt} = \frac{\rho\,(1+g)}{A_H}. \tag{4}$$

Meanwhile the intratemporal marginal rate of substitution between goods and leisure shows how leisure is related to the value of each sector's:

$$x_t = \frac{\alpha p_{At} y_{At}}{\alpha_A w_t h_t} = \frac{\alpha p_{Mt} y_{Mt}}{\alpha_M w_t h_t}. \tag{5}$$

2.1 Sectoral Goods Producers

The representative firm in each sector produces output with Cobb-Douglas production functions in the amount of human capital and physical capital being allocated to each sector. With $l_{At}h_t$ the amount of human capital allocated to agriculture production, $s_{At}k_t$ the amount of physical capital allocated to agriculture production, with a_{At} a positive productivity parameter, and with γ_A the share of human capital income in total agriculture revenue, the production technology in agriculture is

$$y_{At} = a_{At}\left(l_{At}h_t\right)^{\gamma_A}\left(s_{At}k_t\right)^{1-\gamma_A}.$$

The profit maximization problem is

$$\underset{l_{At}, s_{At}}{Max}\ \Pi_{At} = p_{At}a_{At}\left(l_{At}h_t\right)^{\gamma_A}\left(s_{At}k_t\right)^{1-\gamma_A} - w_t l_{At} h_t - r_t s_{At} k_t.$$

Assume that manufacturing is more human capital intensive than agriculture, so that

$$\gamma_A < \gamma_M,$$

where the production function in manufacturing is

$$y_{Mt} = a_{Mt}\left(l_{Mt}h_t\right)^{\gamma_M}\left(s_{Mt}k_t\right)^{1-\gamma_M},$$

and the firm problem similarly is

$$\underset{l_{Mt}, s_{Mt}}{Max}\ \Pi_{Mt} = p_{Mt}a_{Mt}\left(l_{Mt}h_t\right)^{\gamma_M}\left(s_{Mt}k_t\right)^{1-\gamma_M} - w_t l_{Mt} h_t - r_t s_{Mt} k_t.$$

The equilibrium conditions give the marginal products of labor and capital as

$$r_t = p_{At} a_{At} (1-\gamma_A) (l_{At} h_t)^{\gamma_A} (s_{At} k_t)^{-\gamma_A}, \quad (6)$$
$$w_t = p_{At} a_{At} \gamma_A (l_{At} h_t)^{\gamma_A - 1} (s_{At} k_t)^{1-\gamma_A}; \quad (7)$$

$$r_t = p_{Mt} a_{Mt} (1-\gamma_M) (l_{Mt} h_t)^{\gamma_M} (s_{Mt} k_t)^{-\gamma_M},$$
$$w_t = p_{Mt} a_{Mt} \gamma_M (l_{Mt} h_t)^{\gamma_M - 1} (s_{Mt} k_t)^{1-\gamma_M}.$$

2.2 Sectoral Allocations along the Balanced Growth Path

The equilibrium finds that the shares of capital in each sector are constant for any growth rate g, and that the shares of labor are constant for a given growth rate g, but change as g changes. The constant capital shares result because of the assumption of using only human capital in the production of human capital. This represents the simplest, and analytically solvable, way to show the structural transformation theory, with changes in the labor shares causing relative output levels to also change. More generally, with physical capital also in the human capital production function, the shares of capital would depend on the growth rate g.

Proposition 1 *The sectoral shares of capital in each sector are constant.*

Proof. By production's Cobb-Douglas nature,

$$p_{At} y_{At} = \frac{r s_{At} k_t}{(1-\gamma_A)}, \quad (8)$$
$$p_{Mt} y_{Mt} = \frac{r s_{Mt} k_t}{(1-\gamma_M)}.$$

From the consumer side of the equilibrium, we know that

$$\frac{\alpha_A}{p_{At} y_{At}} = \frac{\alpha_M}{p_{Mt} y_{Mt}},$$

which combined with the firm conditions and the consumer's sum of capital shares equaling one, gives a solution for the capital shares in terms of preference and technology parameters.

$$
\begin{aligned}
\frac{\alpha_A}{\frac{r(1-s_{Mt})k_t}{(1-\gamma_A)}} &= \frac{\alpha_M}{\frac{r s_{Mt} k_t}{(1-\gamma_M)}}, \\
s_{Mt}\frac{\alpha_A}{\alpha_M} &= (1-s_{Mt})\frac{(1-\gamma_M)}{(1-\gamma_A)}, \\
s_{Mt} &= \frac{\alpha_M(1-\gamma_M)}{\alpha_A(1-\gamma_A)+\alpha_M(1-\gamma_M)}; \qquad (9) \\
s_{At} &= \frac{\alpha_A(1-\gamma_A)}{\alpha_A(1-\gamma_A)+\alpha_M(1-\gamma_M)}. \qquad (10)
\end{aligned}
$$

■

Second, it can be shown that the labor shares in each sector depend only upon the BPG growth rate g and the utility and technology parameters.

Proposition 2 *The sectoral shares of labor in each sector are simple rising functions of the balanced growth path growth rate, as given by*

$$
\begin{aligned}
l_A &= \frac{\gamma_A\alpha_A}{(\gamma_A\alpha_A+\gamma_M\alpha_M)}\frac{\rho(1+g)}{A_H}, \qquad (11) \\
l_M &= \frac{\gamma_M\alpha_M}{(\gamma_A\alpha_A+\gamma_M\alpha_M)}\frac{\rho(1+g)}{A_H}. \qquad (12)
\end{aligned}
$$

Proof. From the firm's first order conditions, it true that

$$
\begin{aligned}
l_A &= \frac{rk}{wh}\frac{\gamma_A}{1-\gamma_A}s_A, \\
l_M &= \frac{rk}{wh}\frac{\gamma_M}{1-\gamma_M}s_M.
\end{aligned}
$$

Substituting in the solutions for the capital shares from Proposition 1,

$$
\begin{aligned}
l_A &= \frac{rk}{wh}\frac{\gamma_A\alpha_A}{\alpha_A(1-\gamma_A)+\alpha_M(1-\gamma_M)}, \qquad (13) \\
l_M &= \frac{rk}{wh}\frac{\gamma_M\alpha_M}{\alpha_A(1-\gamma_A)+\alpha_M(1-\gamma_M)}. \qquad (14)
\end{aligned}
$$

The ratio of total rental income to wage income can be solved from equation (4) giving the sum of sectoral labor allocation as a function of the growth rate:

$$\begin{aligned}
\frac{\rho\left(1+g\right)}{A_H} &= l = l_A + l_M \\
&= \frac{rk}{wh}\frac{\gamma_A\alpha_A + \gamma_M\alpha_M}{\alpha_A\left(1-\gamma_A\right)+\alpha_M\left(1-\gamma_M\right)}; \\
\frac{rk}{wh} &= \frac{\left[\alpha_A\left(1-\gamma_A\right)+\alpha_M\left(1-\gamma_M\right)\right]\rho\left(1+g\right)}{\left(\gamma_A\alpha_A+\gamma_M\alpha_M\right)A_H}. \qquad (15)
\end{aligned}$$

Substituting the solution for $\frac{rk}{wh}$ back into equations (13)-(14), proves the proposition. ■

3 Sectoral Effects of Growth on Output and Labor

Having shown how the share of capital amongst sectors is fixed while the labor share rises with the growth rate, consider next how relative output levels and labor shares depend on the growth rate. Then the growth rate will be solved, and changes in parameters determining the growth rate can be seen to affect the sectoral output ratios and labor share ratios.

Proposition 3 *A rise in the human capital productivity factor A_H causes output levels to shift relatively towards the more human capital intensive good.*

Proof. With output levels in each sector given by the production functions,

$$\begin{aligned}
y_{At} &= a_{At}\left(l_{At}h_t\right)^{\gamma_A}\left(s_{At}k_t\right)^{1-\gamma_A}, \\
y_M &= a_{Mt}\left(l_{Mt}h_t\right)^{\gamma_M}\left(s_{Mt}k_t\right)^{1-\gamma_M},
\end{aligned}$$

the output ratio can be expressed in terms of the capital ratio state variable $\frac{k}{h}$ and the growth rate g, by substituting in the capital and labor shares from equations (9), (10), (11), and (12):

$$\begin{aligned}
\frac{y_{At}}{y_M} &= \frac{a_{At}\left(l_{At}h_t\right)^{\gamma_A}\left(s_{At}k_t\right)^{1-\gamma_A}}{a_{Mt}\left(l_{Mt}h_t\right)^{\gamma_M}\left(s_{Mt}k_t\right)^{1-\gamma_M}} = \frac{a_{At}\left(\frac{\left(l_{At}h_t\right)}{\left(s_{At}k_t\right)}\right)^{\gamma_A}}{a_{Mt}\left(\frac{\left(l_{Mt}h_t\right)}{\left(s_{Mt}k_t\right)}\right)^{\gamma_M}}\frac{s_{At}}{s_{Mt}} \\
&= \left(\frac{k}{h}\right)^{\left(\gamma_M-\gamma_A\right)}\frac{a_{At}\left(s_{At}\right)^{1-\gamma_A}}{a_{Mt}\left(s_{Mt}\right)^{1-\gamma_M}}\frac{\left(l_A\right)^{\gamma_A}}{\left(l_M\right)^{\gamma_M}} \qquad (16)
\end{aligned}$$

To solve for the capital ratio $\frac{k}{h}$, normalize p_A to one, and use the marginal product of labor condition (6), plus equations (10), (11), and (3) to get

$$\begin{aligned}
r_t &= a_{At}\left(1-\gamma_A\right)\left(\frac{l_{At}h_t}{s_{At}k_t}\right)^{\gamma_A}, \\
\frac{k_t}{h_t} &= \frac{l_{At}}{s_{At}}\left(\frac{a_{At}\left(1-\gamma_A\right)}{r_t}\right)^{\frac{1}{\gamma_A}} \\
&= \frac{\frac{rk}{wh}\gamma_A}{\left(1-\gamma_A\right)}\left(\frac{a_{At}\left(1-\gamma_A\right)}{r_t}\right)^{\frac{1}{\gamma_A}} \\
&= \frac{k}{wh}r^{\left(1-\frac{1}{\gamma_A}\right)}\gamma_A\left(a_{At}\right)^{\frac{1}{\gamma_A}}\left(1-\gamma_A\right)^{\frac{1-\gamma_A}{\gamma_A}}. \qquad (17)
\end{aligned}$$

The solution for $\frac{k}{hw}$ in terms of g from equation (15), and the growth rate equation (3) is

$$\frac{k}{wh} = \frac{\left[\alpha_A\left(1-\gamma_A\right)+\alpha_M\left(1-\gamma_M\right)\right]\rho\left(1+g\right)}{\left[\left(1+g\right)\left(1+\rho\right)+\delta_k-1\right]\left(\gamma_A\alpha_A+\gamma_M\alpha_M\right)A_H}. \qquad (18)$$

Substituting this back into the expression solution for $\frac{k_t}{h_t}$ in equation (17),

$$\frac{k_t}{h_t} = \frac{\rho\left(1+g\right)\left[\alpha_A\left(1-\gamma_A\right)+\alpha_M\left(1-\gamma_M\right)\right]\gamma_A\left(a_{At}\right)^{\frac{1}{\gamma_A}}\left(1-\gamma_A\right)^{\frac{1-\gamma_A}{\gamma_A}}}{\left[\left(1+g_t\right)\left(1+\rho\right)-1+\delta_k\right]^{\left(\frac{1}{\gamma_A}\right)}A_H\left(\gamma_A\alpha_A+\gamma_M\alpha_M\right)} \qquad (19)$$

To solve the growth rate g, use a second equation that solves for $\frac{k}{hw}$ in terms of g as given by the equations involving leisure, on the consumer and firm side in equations (5) and (8), and the growth rate g in equation (2):

$$\begin{aligned}
x_t &= \frac{\alpha p_{At}y_{At}}{\alpha_A w_t h_t} = \frac{\alpha r s_{At}}{\alpha_A\left(1-\gamma_A\right)}\frac{k}{wh}; \\
x_t &= 1-\frac{\left[\left(1+g_t\right)\left(1+\rho\right)-1+\delta_h\right]}{A_H}.
\end{aligned}$$

The second solution for $\frac{k}{wh}$ then follows as

$$\frac{k}{wh} = \frac{\left(1-\frac{\left[\left(1+g_t\right)\left(1+\rho\right)-1+\delta_h\right]}{A_H}\right)}{\frac{\alpha\left[\left(1+g_t\right)\left(1+\rho\right)-1+\delta_k\right]}{\left[\alpha_A\left(1-\gamma_A\right)+\alpha_M\left(1-\gamma_M\right)\right]}}. \qquad (20)$$

Combining equations (18) and (20) gives the solution for the growth rate in terms of only exogenous parameters:

$$1+g = \frac{1+A_H-\delta_h}{1+\rho\left(1+\frac{\alpha}{\left(\gamma_A\alpha_A+\gamma_M\alpha_M\right)}\right)}. \qquad (21)$$

Now given the solution for g in equation (21), and the expression for $\frac{k_t}{h_t}$ in equation (19), the relative output relation in equation (16) can be solved as

$$\frac{y_{At}}{y_{Mt}} = \frac{\frac{(a_{At})^{\frac{\gamma_M}{\gamma_A}}}{a_{Mt}} \frac{(1-\gamma_A)^{\frac{\gamma_M(1-\gamma_A)}{\gamma_A}}}{(1-\gamma_M)^{1-\gamma_M}} \frac{\alpha_A}{\alpha_M} \left(\frac{\gamma_A}{\gamma_M}\right)^{\gamma_M}}{\left[\left(\frac{(1+A_H-\delta_h)(1+\rho)}{1+\rho\left(1+\frac{\alpha}{(\gamma_A\alpha_A+\gamma_M\alpha_M)}\right)}\right) - 1 + \delta_k\right]^{\left(\frac{\gamma_M-\gamma_A}{\gamma_A}\right)}}$$

Clearly, as A_H increases, with $\gamma_M > \gamma_A$, $\frac{\partial\left(\frac{y_{At}}{y_M}\right)}{\partial A_H} < 0$. ■

The result on relative output carries through inversely to relative prices.

Corollary 4 *As A_H increases, the relative price of the human capital intensive sector falls.*

Proof. $\frac{p_{Mt}}{p_{At}} = \frac{y_{At}\alpha_M}{y_M\alpha_A}$. $\frac{\partial\left(\frac{p_{Mt}}{p_{At}}\right)}{\partial A_H} = \frac{\alpha_M}{\alpha_A}\frac{\partial\left(\frac{y_{At}}{y_M}\right)}{\partial A_H} < 0$. ■

Similarly it can be shown that as human capital productivity and the growth rate increase, the sectoral labor shares individually decrease while relative sectoral labor remains constant.

Proposition 5 *As A_H increases, the growth rate rises, the sectoral labor shares fall, while the ratio of labor in manufacturing relative to agriculture remains constant.*

Proof. From equation (21), $\frac{\partial g}{\partial A_H} = \frac{1}{1+\rho\left(1+\frac{\alpha}{(\gamma_A\alpha_A+\gamma_M\alpha_M)}\right)} > 0$. The labor shares from equations (11), (12) and equation (21) are

$$l_A = \frac{\gamma_A\alpha_A}{(\gamma_A\alpha_A+\gamma_M\alpha_M)} \frac{\rho(1+A_H-\delta_h)}{A_H\left[1+\rho\left(1+\frac{\alpha}{(\gamma_A\alpha_A+\gamma_M\alpha_M)}\right)\right]},$$

$$l_M = \frac{\gamma_M\alpha_M}{(\gamma_A\alpha_A+\gamma_M\alpha_M)} \frac{\rho(1+A_H-\delta_h)}{A_H\left[1+\rho\left(1+\frac{\alpha}{(\gamma_A\alpha_A+\gamma_M\alpha_M)}\right)\right]}.$$

The derivatives are $\frac{\partial l_A}{\partial A_H} = \frac{\gamma_A\alpha_A\rho}{(\gamma_A\alpha_A+\gamma_M\alpha_M)\left[1+\rho\left(1+\frac{\alpha}{(\gamma_A\alpha_A+\gamma_M\alpha_M)}\right)\right]}\frac{-(1-\delta_h)}{(A_H)^2} < 0$, $\frac{\partial l_M}{\partial A_H} = \frac{\gamma_M\alpha_M\rho}{(\gamma_A\alpha_A+\gamma_M\alpha_M)\left[1+\rho\left(1+\frac{\alpha}{(\gamma_A\alpha_A+\gamma_M\alpha_M)}\right)\right]}\frac{-(1-\delta_h)}{(A_H)^2} < 0$; and $\frac{l_A}{l_M} = \frac{\gamma_A\alpha_A}{\gamma_M\alpha_M}$. ■

And finally, note that the capital ratio $\frac{k_t}{h_t}$ falls as A_H and g rise given standard ranges of values for parameters:

Corollary 6 *The physical capital to human capital ratio falls as A_H increases, for small enough leisure preference α.*

Proof. By equation (18), $\frac{k_t}{h_t} = \frac{(1+g)\left(\frac{\rho[\alpha_A(1-\gamma_A)+\alpha_M(1-\gamma_M)]\gamma_A(a_{At})^{\frac{1}{\gamma_A}}(1-\gamma_A)^{\frac{1-\gamma_A}{\gamma_A}}}{(\gamma_A\alpha_A+\gamma_M\alpha_M)}\right)}{[(1+g_t)(1+\rho)-1+\delta_k]^{\left(\frac{1}{\gamma_A}\right)}A_H} \equiv \frac{(1+g)(Z)}{[(1+g_t)(1+\rho)-1+\delta_k]^{\left(\frac{1}{\gamma_A}\right)}A_H}$, $\frac{\partial\left(\frac{k_t}{h_t}\right)}{\partial A_H} = Z\frac{\partial\left(\frac{1+g}{[(1+g_t)(1+\rho)-1+\delta_k]^{\left(\frac{1}{\gamma_A}\right)}A_H}\right)}{\partial A_H}$

$= Z\frac{\partial\left(\frac{\frac{1+A_H-\delta_h}{1+\rho\left(1+\frac{\alpha}{(\gamma_A\alpha_A+\gamma_M\alpha_M)}\right)}}{\left[\left(\frac{(1+A_H-\delta_h)(1+\rho)}{1+\rho\left(1+\frac{\alpha}{(\gamma_A\alpha_A+\gamma_M\alpha_M)}\right)}\right)-1+\delta_k\right]^{\left(\frac{1}{\gamma_A}\right)}A_H}\right)}{\partial A_H} = \left[\left(\frac{(1+A_H-\delta_h)(1+\rho)}{1+\rho\left(1+\frac{\alpha}{(\gamma_A\alpha_A+\gamma_M\alpha_M)}\right)}\right)-1+\delta_k\right]^{\left(\frac{1}{\gamma_A}\right)} X$

$\left[(A_H)-[1+A_H-\delta_h]\left\{1+\frac{A_H\left(\frac{1}{\gamma_A}\right)\left(\frac{(1+\rho)}{1+\rho\left(1+\frac{\alpha}{(\gamma_A\alpha_A+\gamma_M\alpha_M)}\right)}\right)}{\left[\left(\frac{(1+A_H-\delta_h)(1+\rho)}{1+\rho\left(1+\frac{\alpha}{(\gamma_A\alpha_A+\gamma_M\alpha_M)}\right)}\right)-1+\delta_k\right]}\right\}\right]$. Evaluated at $\alpha=\varepsilon$, for small ε, $\frac{\partial\left(\frac{k_t}{h_t}\right)}{\partial A_H} \simeq (A_H-\delta_h+\delta_k)^{\left(\frac{1}{\gamma_A}\right)}\left[(A_H)-[1+A_H-\delta_h]\left\{1+\frac{A_H\left(\frac{1}{\gamma_A}\right)}{[A_H-\delta_h+\delta_k]}\right\}\right]$ < 0, given a large enough A_H so that $A_H-\delta_h+\delta_k>0$. ■

In contrast to some exogenous growth theories, when the BGP growth rate rises as a result of A_H rising, the input ratio of the wage rate to the interest rate falls.

Proposition 7 *A rise in A_H causes $\frac{w}{r}$ to fall.*

Proof. By equations (15), (18), and (21),

$$\frac{w}{r} = \frac{k}{h}\frac{(\gamma_A\alpha_A+\gamma_M\alpha_M)A_H}{[\alpha_A(1-\gamma_A)+\alpha_M(1-\gamma_M)]\rho(1+g)};$$

$$= \frac{\frac{\left(\frac{\rho[\alpha_A(1-\gamma_A)+\alpha_M(1-\gamma_M)]\gamma_A(a_{At})^{\frac{1}{\gamma_A}}(1-\gamma_A)^{\frac{1-\gamma_A}{\gamma_A}}}{(\gamma_A\alpha_A+\gamma_M\alpha_M)}\right)(\gamma_A\alpha_A+\gamma_M\alpha_M)}{[\alpha_A(1-\gamma_A)+\alpha_M(1-\gamma_M)]\rho}}{\left[\left(\frac{1+A_H-\delta_h}{1+\rho\left(1+\frac{\alpha}{(\gamma_A\alpha_A+\gamma_M\alpha_M)}\right)}\right)(1+\rho)-1+\delta_k\right]^{\left(\frac{1}{\gamma_A}\right)}}.$$

$\frac{\partial\left(\frac{w}{r}\right)}{\partial A_H} < 0$ for $A_H > \delta_h$. ■

4 Adding an Additional Sector

The shift in output towards the more human capital intensive sector is established by Proposition 3, when the human capital sector productivity rises. And Proposition 5 similarly establishes that the labor time shares fall in both sectors when human capital becomes more productive. But this does not establish a relative movement in the labor time towards more human capital intensive sectors. In fact, equations (13) and (14) show that $\frac{l_A}{l_M}$ is constant at $\frac{\gamma_A \alpha_A}{\gamma_M \alpha_M}$.

However consider that as economies develop new sectors are constantly being created. First there is agriculture, then manufacturing, then inclusion of services, and now inclusion of technology. Where one sector begins and the other ends is a priori extremely hard to determine. For example the Wall Street Journal interactive online (www.smartmoney.com/sectormaps/) shows a firm size based decomposition of all of the major sectors of the US economy, listing these as 10 sectors: Basic materials, consumer cyclical, consumer non-cyclical, energy, financial, healthcare, industrial, technology, telecommunications, utilities. It would be a heroic effort to force these 10 sectors into the three or four standard sectors used in the structural transformation literature.

Consider the conceptual proposition that as the economy expands, the extent of the market grows, the division of labor increases, and the new sectors that come into existence tend to be more human capital intensive that sectors they are replacing or that they are adding onto. This is a refinement of Adam Smith's notion of labor specialization that 1) new goods are created as a result and that 2) it is the more human capital intensive sectors that arise out of this process over long periods of time. Put differently in Sherwin Rosen's (1974) hedonic characteristics, which hedonic characteristics arise over time within any one product. Again the proposition here is that these are the more human capital based features, such as new cars with non-internal fuel combustion propagation engines.

Given this rationale, now add one more sector, call it services, whereby the human capital intensity is greater than agriculture and manufacturing. Let the representative agent choose amongst the goods, y_{At}, y_{Mt}., and services output y_{St}, with real prices of p_{At}, p_{Mt} and p_{St}. The consumer current period extended utility u_t is again a simple log form, with parameters $\alpha > 0$, $\alpha_A > 0$, $\alpha_M > 0$ and $\alpha_S > 0$, where

$$u_t = \alpha \ln x_t + \alpha_A \ln y_{At} + \alpha_M \ln y_{Mt} + \alpha_S \ln y_{St}.$$

With the same investment i_t in physical capital accumulation,

$$i_t = k_{t+1} - k_t \left(1 - \delta_k\right),$$

and the same human capital investment function i_{Ht},whereby

$$A_{Ht} l_{Ht} h_t = i_{Ht} = h_{t+1} - h_t \left(1 - \delta_h\right), \tag{22}$$

the allocation of time constraint now includes time spent in the services sector l_{St} :

$$1 = l_{At} + l_{Mt} + l_{St} + l_{Ht} + x_t,$$

while the allocation of physical capital shares now also includes that of services s_{St} :

$$1 = s_{At} + s_{Mt} + s_{St}.$$

The production function in services is given by

$$y_{St} = a_{St} \left(l_{St} h_t\right)^{\gamma_S} \left(s_{St} k_t\right)^{1-\gamma_S},$$

where

$$\gamma_A < \gamma_M < \gamma_S.$$

The recursive consumer's problem is

$$\begin{aligned} & V\left(k_t, h_t\right) \\ = \; & \underset{y_{At}, y_{Mt}, y_{St}, l_{Ht}, x_t}{Max} \left\{ \left(\alpha_A \ln y_{At} + \alpha_M \ln y_{Mt} + \alpha_S \ln y_{St} + \alpha \ln x_t\right) \right. \\ & \left. + \beta V \left(\begin{array}{c} \left[w_t \left(1 - l_{Ht} - x_t\right) h_t + k_t \left(1 + r_t - \delta_k\right) - p_{At} y_{At} - p_{Mt} y_{Mt} - p_{St} y_{St}\right], \\ h_t \left(1 + A_H l_{Ht} - \delta_h\right) \end{array} \right) \right\}, \end{aligned}$$

with the same intertemporal conditions as in the two sector economy, and now with the intratemporal conditions including the additional sector:

$$\frac{\alpha}{x_t w_t h_t} = \frac{\alpha_A}{p_{At} y_{At}} = \frac{\alpha_M}{p_{Mt} y_{Mt}} = \frac{\alpha_S}{p_{St} y_{St}}.$$

Proposition 8 *The addition of the new service sector makes each the share of capital and the share of labor in the other two existing sectors smaller.*

Proof. From the firm side, the sectoral shares of capital are now found in equilibrium to be

$$\begin{aligned} s_{At} &= \frac{\alpha_A (1-\gamma_A)}{\alpha_A (1-\gamma_A) + \alpha_M (1-\gamma_M) + \alpha_S (1-\gamma_S)}, \\ s_{Mt} &= \frac{\alpha_M (1-\gamma_M)}{\alpha_A (1-\gamma_A) + \alpha_M (1-\gamma_M) + \alpha_S (1-\gamma_S)}, \\ s_{St} &= \frac{\alpha_S (1-\gamma_S)}{\alpha_A (1-\gamma_A) + \alpha_M (1-\gamma_M) + \alpha_S (1-\gamma_S)}. \end{aligned} \tag{23}$$

Using s'_{At} to indicate the two-sector only economy, clearly $\frac{s_{At}}{s'_{At}} = \frac{\alpha_A(1-\gamma_A)+\alpha_M(1-\gamma_M)}{\alpha_A(1-\gamma_A)+\alpha_M(1-\gamma_M)+\alpha_S(1-\gamma_S)} < 1$, so that $s_{At} < s'_{At}$. Similarly, $s_{Mt} < s'_{Mt}$. The labor shares are found to be

$$\begin{aligned} l_A &= \frac{\gamma_A \alpha_A}{(\gamma_A \alpha_A + \gamma_M \alpha_M + \gamma_S \alpha_S)} \frac{\rho (1 + A_H - \delta_h)}{A_H \left[1 + \rho \left(1 + \frac{\alpha}{(\gamma_A \alpha_A + \gamma_M \alpha_M + \gamma_S \alpha_S)}\right)\right]}, \\ l_M &= \frac{\gamma_M \alpha_M}{(\gamma_A \alpha_A + \gamma_M \alpha_M + \gamma_S \alpha_S)} \frac{\rho (1 + A_H - \delta_h)}{A_H \left[1 + \rho \left(1 + \frac{\alpha}{(\gamma_A \alpha_A + \gamma_M \alpha_M + \gamma_S \alpha_S)}\right)\right]}, \\ l_S &= \frac{\gamma_S \alpha_S}{(\gamma_A \alpha_A + \gamma_M \alpha_M + \gamma_S \alpha_S)} \frac{\rho (1 + A_H - \delta_h)}{A_H \left[1 + \rho \left(1 + \frac{\alpha}{(\gamma_A \alpha_A + \gamma_M \alpha_M + \gamma_S \alpha_S)}\right)\right]}. \end{aligned} \tag{24}$$

$$\begin{aligned}
l_A &= \frac{\gamma_A \alpha_A}{(\gamma_A \alpha_A + \gamma_M \alpha_M + \gamma_S \alpha_S)} \frac{\rho(1 + A_H - \delta_h)}{A_H \left[1 + \rho\left(1 + \frac{\alpha}{(\gamma_A \alpha_A + \gamma_M \alpha_M + \gamma_S \alpha_S)}\right)\right]} \\
&< \frac{\gamma_A \alpha_A}{(\gamma_A \alpha_A + \gamma_M \alpha_M)} \frac{\rho(1 + A_H - \delta_h)}{A_H \left[1 + \rho\left(1 + \frac{\alpha}{(\gamma_A \alpha_A + \gamma_M \alpha_M)}\right)\right]} \equiv l'_A, \\
l_M &= \frac{\gamma_M \alpha_M}{(\gamma_A \alpha_A + \gamma_M \alpha_M + \gamma_S \alpha_S)} \frac{\rho(1 + A_H - \delta_h)}{A_H \left[1 + \rho\left(1 + \frac{\alpha}{(\gamma_A \alpha_A + \gamma_M \alpha_M + \gamma_S \alpha_S)}\right)\right]} \\
&< \frac{\gamma_M \alpha_M}{(\gamma_A \alpha_A + \gamma_M \alpha_M)} \frac{\rho(1 + A_H - \delta_h)}{A_H \left[1 + \rho\left(1 + \frac{\alpha}{(\gamma_A \alpha_A + \gamma_M \alpha_M)}\right)\right]} \equiv l'_M.
\end{aligned}$$

$$1 < \frac{[(\gamma_A \alpha_A + \gamma_M \alpha_M + \gamma_S \alpha_S) + \rho((\gamma_A \alpha_A + \gamma_M \alpha_M + \gamma_S \alpha_S) + \alpha)]}{[(\gamma_A \alpha_A + \gamma_M \alpha_M) + \rho((\gamma_A \alpha_A + \gamma_M \alpha_M) + \alpha)]}.$$

and so $l_{At} < l'_{At}$, and $l_{Mt} < l'_{Mt}$. ■

Therefore even though the ratio of labor in agriculture and manufacturing stay the same, labor is moving from both sectors into the new services sector. This is going to happen regardless of the human capital intensity of the services sector. But what is dependent on services being more human capital intensive is that relative output of the services sector will rise over time if the human capital investment sectoral productivity gradually rises over time. This is a corollary from Proposition 3.

Corollary 9 *An increase in human capital productivity A_H causes output to rise in services relative to both agriculture and manufacturing, and for manufacturing output again to rise relative to agriculture.*

Proof. With output levels in each sector given by the production functions,

$$\begin{aligned}
y_{At} &= a_{At} (l_{At} h_t)^{\gamma_A} (s_{At} k_t)^{1-\gamma_A}, \\
y_{Mt} &= a_{Mt} (l_{Mt} h_t)^{\gamma_M} (s_{Mt} k_t)^{1-\gamma_M}, \\
y_{St} &= a_{St} (l_{St} h_t)^{\gamma_S} (s_{St} k_t)^{1-\gamma_S},
\end{aligned}$$

then

$$\begin{aligned}
\frac{y_{At}}{y_{Mt}} &= \left(\frac{k}{h}\right)^{(\gamma_M-\gamma_A)} \frac{a_{At}\,(s_{At})^{1-\gamma_A}}{a_{Mt}\,(s_{Mt})^{1-\gamma_M}} \frac{(l_A)^{\gamma_A}}{(l_M)^{\gamma_M}}, \\
\frac{y_{Mt}}{y_{St}} &= \left(\frac{k}{h}\right)^{(\gamma_S-\gamma_M)} \frac{a_{Mt}\,(s_{Mt})^{1-\gamma_M}}{a_{St}\,(s_{St})^{1-\gamma_S}} \frac{(l_M)^{\gamma_M}}{(l_S)^{\gamma_S}}, \\
\frac{y_{At}}{y_{St}} &= \left(\frac{k}{h}\right)^{(\gamma_S-\gamma_A)} \frac{a_{At}\,(s_{At})^{1-\gamma_A}}{a_{St}\,(s_{St})^{1-\gamma_S}} \frac{(l_A)^{\gamma_A}}{(l_S)^{\gamma_S}},
\end{aligned}$$

where the capital ratio $\frac{k}{h}$, with p_A normalized to one, can be expressed by

$$\frac{k_t}{h_t} = \frac{\rho\,(1+g)\,[\alpha_A\,(1-\gamma_A)+\alpha_M\,(1-\gamma_M)+\alpha_S\,(1-\gamma_S)]\,\gamma_A\,(a_{At})^{\frac{1}{\gamma_A}}\,(1-\gamma_A)^{\frac{1-\gamma_A}{\gamma_A}}}{[(1+g_t)\,(1+\rho)-1+\delta_k]^{\left(\frac{1}{\gamma_A}\right)}\,A_H\,(\gamma_A\alpha_A+\gamma_M\alpha_M+\gamma_S\alpha_S)}, \tag{25}$$

and the growth rate g is given by

$$1+g = \frac{1+A_H-\delta_h}{1+\rho\left(1+\frac{\alpha}{(\gamma_A\alpha_A+\gamma_M\alpha_M+\gamma_S\alpha_S)}\right)}. \tag{26}$$

Substituting in for $\frac{k_t}{h_t}$ and g,

$$\begin{aligned}
\frac{y_{At}}{y_{Mt}} &= \frac{\frac{(a_{At})^{\frac{\gamma_M}{\gamma_A}}}{a_{Mt}} \frac{(1-\gamma_A)^{\frac{\gamma_M(1-\gamma_A)}{\gamma_A}}}{(1-\gamma_M)^{1-\gamma_M}} \frac{\alpha_A}{\alpha_M}\left(\frac{\gamma_A}{\gamma_M}\right)^{\gamma_M}}{\left[\left(\frac{(1+A_H-\delta_h)(1+\rho)}{1+\rho\left(1+\frac{\alpha}{(\gamma_A\alpha_A+\gamma_M\alpha_M+\gamma_S\alpha_S)}\right)}\right)-1+\delta_k\right]^{\left(\frac{\gamma_M-\gamma_A}{\gamma_A}\right)}}, \\
\frac{y_{Mt}}{y_{St}} &= \frac{\frac{(a_{Mt})^{\frac{\gamma_S}{\gamma_M}}}{a_{St}} \frac{(1-\gamma_M)^{\frac{\gamma_S(1-\gamma_M)}{\gamma_M}}}{(1-\gamma_S)^{1-\gamma_S}} \frac{\alpha_M}{\alpha_S}\left(\frac{\gamma_M}{\gamma_S}\right)^{\gamma_S}}{\left[\left(\frac{(1+A_H-\delta_h)(1+\rho)}{1+\rho\left(1+\frac{\alpha}{(\gamma_A\alpha_A+\gamma_M\alpha_M+\gamma_S\alpha_S)}\right)}\right)-1+\delta_k\right]^{\left(\frac{\gamma_S-\gamma_M}{\gamma_M}\right)}}, \\
\frac{y_{At}}{y_{St}} &= \frac{\frac{(a_{At})^{\frac{\gamma_S}{\gamma_A}}}{a_{St}} \frac{(1-\gamma_A)^{\frac{\gamma_S(1-\gamma_A)}{\gamma_A}}}{(1-\gamma_S)^{1-\gamma_S}} \frac{\alpha_A}{\alpha_S}\left(\frac{\gamma_A}{\gamma_S}\right)^{\gamma_S}}{\left[\left(\frac{(1+A_H-\delta_h)(1+\rho)}{1+\rho\left(1+\frac{\alpha}{(\gamma_A\alpha_A+\gamma_M\alpha_M+\gamma_S\alpha_S)}\right)}\right)-1+\delta_k\right]^{\left(\frac{\gamma_S-\gamma_A}{\gamma_A}\right)}}.
\end{aligned}$$

With $\gamma_S > \gamma_M > \gamma_A$, $\frac{\partial\left(\frac{y_{At}}{y_{Mt}}\right)}{\partial A_H} = -\frac{\left[\frac{(a_{At})^{\frac{\gamma_M}{\gamma_A}}}{a_{Mt}} \frac{(1-\gamma_A)^{\frac{\gamma_M(1-\gamma_A)}{\gamma_A}}}{(1-\gamma_M)^{1-\gamma_M}} \frac{\alpha_A}{\alpha_M}\left(\frac{\gamma_A}{\gamma_M}\right)^{\gamma_M}\right]\left(\frac{\gamma_M-\gamma_A}{\gamma_A}\right)}{\left[1+\rho\left(1+\frac{\alpha}{(\gamma_A\alpha_A+\gamma_M\alpha_M+\gamma_S\alpha_S)}\right)\right]\left[\left(\frac{(1+A_H-\delta_h)(1+\rho)}{1+\rho\left(1+\frac{\alpha}{(\gamma_A\alpha_A+\gamma_M\alpha_M+\gamma_S\alpha_S)}\right)}\right)-1+\delta_k\right]^{\frac{\gamma_M}{\gamma_A}}}$

0, $\frac{\partial\left(\frac{y_{At}}{y_{St}}\right)}{\partial A_H} < 0$, and $\frac{\partial\left(\frac{y_{Mt}}{y_{St}}\right)}{\partial A_H} < 0$. ■

The relative quantity change in the three sector economy, from a change in the human capital productivity A_H, is smaller in absolute value, or less negative, as compared to that in the two sector economy. While the calculus gets involved in proving this, take an example, one used more extensively below, with

$$\begin{aligned} \alpha_A &= \alpha_M = \alpha_S = 1, \\ \gamma_A &= \frac{1}{3}, \ \gamma_M = \frac{1}{2}, \ \gamma_S = \frac{3}{5}, \end{aligned}$$

and $\rho = 0.03$, $A_H = 0.045$, $\delta_k = 0.03$, and $\delta_h = 0.015$. Then for the 2 sector economy, with just agriculture and manufacturing, $\left.\frac{\partial\left(\frac{y_{At}}{y_{Mt}}\right)}{\partial A_H}\right|_{2-good} = -\frac{Z}{0.01005}$, with

$$Z \equiv \left[\frac{(a_{At})^{\frac{\gamma_M}{\gamma_A}}}{a_{Mt}} \frac{(1-\gamma_A)^{\frac{\gamma_M(1-\gamma_A)}{\gamma_A}}}{(1-\gamma_M)^{1-\gamma_M}} \frac{\alpha_A}{\alpha_M}\left(\frac{\gamma_A}{\gamma_M}\right)^{\gamma_M}\right]\left(\frac{\gamma_M-\gamma_A}{\gamma_A}\right),$$

while in the 3 sector economy, also including the more human capital intensive services, $\left.\frac{\partial\left(\frac{y_{At}}{y_{Mt}}\right)}{\partial A_H}\right|_{3-good} = -\frac{Z}{0.01166}$. Since $\left|-\frac{Z}{0.01166}\right| < \left|\frac{Z}{0.01005}\right|$, the 3 sector economy has a smaller relative output change.

5 Three Sector Model with Upward Trend in Human Capital Productivity

Consider assuming an exogenous trend upwards in the human capital productivity factor A_H, so that now it is specified as time varying, denoted by A_{Ht}. And let this productivty trend upwards over a 250 year period, say from 1750 to 2000. This is similar to the time from Malthus's zero growth world to the modern world after a continuous gradual industrial revolution.

In this example, let tastes be similar between the different goods and leisure, in that

$$\alpha = \alpha_A = \alpha_M = \alpha_S = 1,$$

and let the sectoral productivities be constant over time at 1, so that

$$a_{At} = a_{Mt} = a_{St} = 1.$$

Further, consider again a simple specification of the human capital intensities whereby

$$\gamma_A = \frac{1}{3},\ \gamma_M = \frac{1}{2},\ \gamma_S = \frac{3}{5}.$$

This gives equal sectoral value shares of aggregate output at $\frac{1}{3}$:

$$\begin{aligned} \frac{p_A y_A}{y} &= \frac{\alpha_A}{\alpha_A + \alpha_M + \alpha_S} = \frac{1}{3}, \\ \frac{p_M y_M}{y} &= \frac{\alpha_M}{\alpha_A + \alpha_M + \alpha_S} = \frac{1}{3}, \\ \frac{p_S y_S}{y} &= \frac{\alpha_S}{\alpha_A + \alpha_M + \alpha_S} = \frac{1}{3}. \end{aligned}$$

Target a Malthusian zero growth rate in 1750 at the beginning of the industrial revolution, and between 2 to 3% growth by 2000. Then at time 0,

$$\begin{aligned} 1 + g_0 &= \frac{1 + A_{H0} - \delta_h}{1 + \rho\left(1 + \frac{\alpha}{(\gamma_A\alpha_A + \gamma_M\alpha_M + \gamma_S\alpha_S)}\right)} \\ &\Longrightarrow \\ A_{H0} &= \rho\left(1 + \frac{\alpha}{(\gamma_A\alpha_A + \gamma_M\alpha_M + \gamma_S\alpha_S)}\right) + \delta_h. \end{aligned}$$

Let $\rho = 0.03$, $.\delta_h = 0.015$, $\delta_k = 0.03$ and this implies that

$$A_{H0} = 0.015 + 0.03\left(1 + \frac{1}{\frac{1}{3} + \frac{1}{2} + \frac{3}{5}}\right) = 0.06593,$$

while $r = \rho + \delta_k = 0.06$. Also then total sectoral labor time is $\frac{\rho(1+g)}{A_H} = \frac{0.03}{0.06593} = 0.455$, while leisure is $x = 1 - \frac{(1+g)(1+\rho)+\delta_h-1}{A_{H0}} = 1 - \frac{(1.03)+0.015-1}{0.06593} = 0.3175$, and human capital investment time is $l_{H0} = \frac{g+\delta_h}{A_H} = \frac{0.015}{0.06593} = 0.2275$. And total time is $0.455 + 0.3175 + 0.2275 = 1.0$.

Now assume that

$$A_{Ht+1} = A_{Ht}\left(1+\mu\right),$$

where

$$\mu = 0.0015.$$

Then the growth rate over time increases, so that

$$\frac{g_{t+1}}{g_t} = \frac{\frac{1+A_{Ht}(1+\mu)-\delta_h}{1+\rho\left(1+\frac{\alpha}{(\gamma_A\alpha_A+\gamma_M\alpha_M+\gamma_S\alpha_S)}\right)}-1}{\frac{1+A_{Ht}-\delta_h}{1+\rho\left(1+\frac{\alpha}{(\gamma_A\alpha_A+\gamma_M\alpha_M+\gamma_S\alpha_S)}\right)}-1},$$

and the growth rate at any time t is given by

$$g_t = \frac{1+A_{H0}\left(1+\mu\right)^t-\delta_h}{1+\rho\left(1+\frac{\alpha}{(\gamma_A\alpha_A+\gamma_M\alpha_M+\gamma_S\alpha_S)}\right)}-1. \tag{27}$$

At time $t=0$,

$$g_0 = \frac{1+A_H-\delta_h}{1+\rho\left(1+\frac{1}{\frac{1}{3}+\frac{1}{2}+\frac{3}{5}}\right)}-1 = \frac{1+0.06593-0.015}{1+0.03\left(1+\frac{1}{\frac{1}{3}+\frac{1}{2}+\frac{3}{5}}\right)}-1=0,$$

while at time $t=1$,

$$\begin{aligned} A_{H1}\left(1+\mu\right) &= 0.06593\left(1.0015\right) = 0.066029 \\ g_1 &= \frac{1+0.06593\left(1.00152\right)-0.015}{1+0.03\left(1+\frac{1}{\frac{1}{3}+\frac{1}{2}+\frac{3}{5}}\right)}-1 = 0.000095. \end{aligned}$$

After 250 years,

$$g_{250} = \frac{1+0.06593\left(1.0015\right)^{250}-0.015}{1+0.03\left(1+\frac{1}{\frac{1}{3}+\frac{1}{2}+\frac{3}{5}}\right)}-1 = 0.02852.$$

So the growth rate reaches 2.85% in the year 2000 for the world. The change over time is graphed in Figure 1.

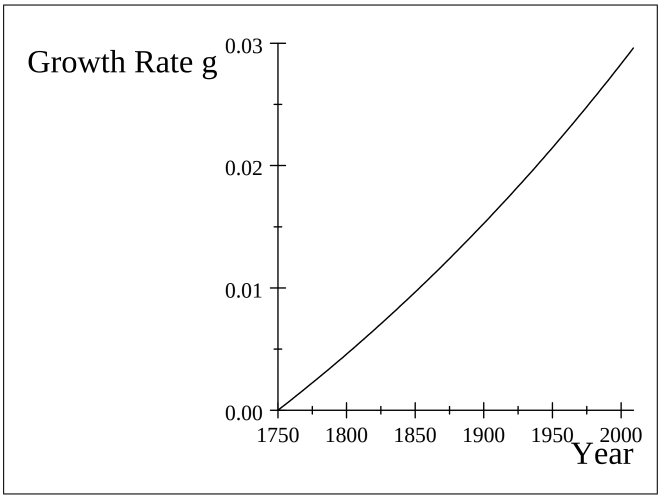

Figure 1: Example: Change in Balanced Growth Path Growth Rate over 250 Years.

5.1 Trends in Relative Output

During this period output gradually realigns towards a higher relative quantity of the human capital intensive sectors. Consider a graph of the 3 sector economy over the 250 years, in terms of the balanced growth path equilibrium ratio of agriculture to manufacturing output. The ratio initially is $\frac{y_{A0}}{y_{M0t}} = 1.5714$ at time 0, using the following expressions.

$$\frac{y_{At}}{y_{Mt}} = \frac{\frac{(a_{At})^{\frac{\gamma_M}{\gamma_A}}}{a_{Mt}} \frac{(1-\gamma_A)^{\frac{\gamma_M(1-\gamma_A)}{\gamma_A}}}{(1-\gamma_M)^{1-\gamma_M}} \frac{\alpha_A}{\alpha_M} \left(\frac{\gamma_A}{\gamma_M}\right)^{\gamma_M}}{\left[\left(\frac{\left(1+A_{H0}(1.0015)^t-\delta_h\right)(1+\rho)}{1+\rho\left(1+\frac{\alpha}{(\gamma_A\alpha_A+\gamma_M\alpha_M+\gamma_S\alpha_S)}\right)}\right)-1+\delta_k\right]^{\left(\frac{\gamma_M-\gamma_A}{\gamma_A}\right)}},$$

$$\frac{y_{A,t}}{y_{M,t}} = \frac{\left(\frac{\left(1-\frac{1}{3}\right)^{\frac{0.5\left(1-\frac{1}{3}\right)}{\frac{1}{3}}}}{(1-0.5)^{1-0.5}} \left(\frac{\frac{1}{3}}{0.5}\right)^{0.5}\right) \left(\frac{0.5-\frac{1}{3}}{\frac{1}{3}}\right)}{\left(\left(\frac{\left(1+0.06593(1.0015)^t-0.015\right)(1+0.03)}{1+0.03\left(1+\frac{1}{\frac{1}{3}+\frac{1}{2}+\frac{3}{5}}\right)}\right)-1+0.03\right)^{\left(\frac{0.5-\frac{1}{3}}{\frac{1}{3}}\right)}}.$$

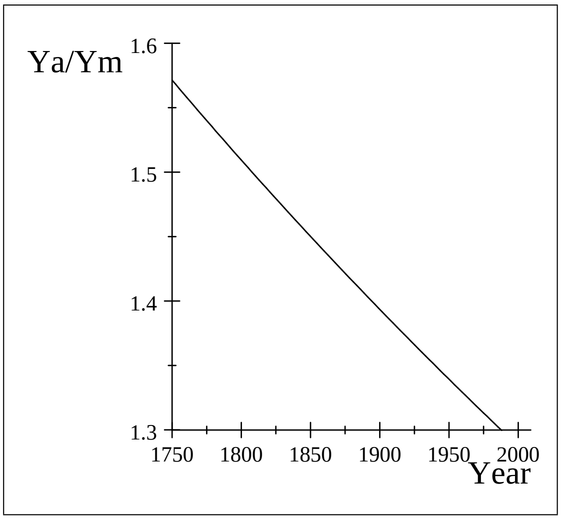

Figure 2: Example: Structural Transformation of the Ratio of Agriculture to Manufacturing from 1750 to 2000.

After 250 years, the balanced growth path equilibrium agriculture to manufacturing ratio falls continuously from 1.5714 to 1.2875, given only 3 sectors this entire time. This can be graphed, as in Figure 2, using Ya/Ym to denote $\frac{y_{At}}{y_{Mt}}$:

5.2 Trend in BGP Human Capital Time

The human capital time is tied to the growth rate in that

$$l_{Ht} = \frac{g_t + \delta_h}{A_{Ht}}.$$

Consider how time in human capital changes over the 250 year period given the calibration of the example economy. The human capital time can be rewritten with the trend in A_{Ht} included, as

$$l_{Ht} = \frac{g_t + \delta_h}{A_{H0}\left(1+\mu\right)^t},$$

with the growth rate given by

$$g_t = \frac{1 + A_{H0}\left(1+\mu\right)^t - \delta_h}{1 + \rho\left(1 + \frac{\alpha}{(\gamma_A \alpha_A + \gamma_M \alpha_M + \gamma_S \alpha_S)}\right)} - 1.$$

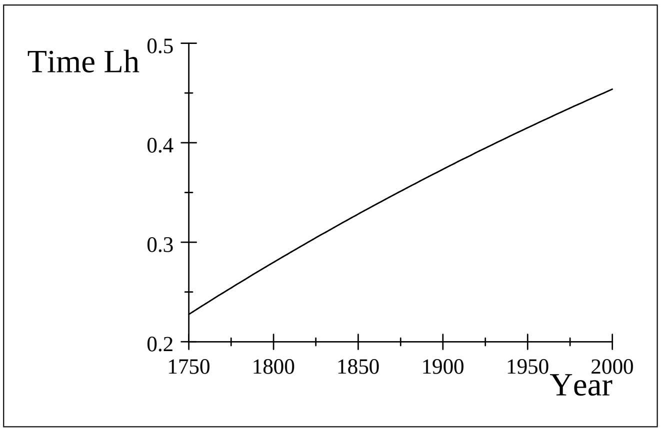

Figure 3: Example Trend Upwards in Human Capital Investment Time: 1750 to 2000.

Then the trend human capital time is solved as

$$l_{Ht}=\frac{\frac{1+A_{H0}(1+\mu)^{t}-\delta_h}{1+\rho\left(1+\frac{\alpha}{(\gamma_A\alpha_A+\gamma_M\alpha_M+\gamma_S\alpha_S)}\right)}-1+\delta_h}{A_{H0}\left(1+\mu\right)^{t}}.$$

With the example calibration this becomes

$$l_{Ht}=\frac{\left(\frac{1+0.06593(1.0015)^{t}-0.015}{1+0.03\left(1+\frac{1}{\frac{1}{3}+\frac{1}{2}+\frac{3}{5}}\right)}\right)-1+0.015}{0.06593\left(1.0015\right)^{250}}.$$

When the growth rate is zero in 1750 during Malthusian times, then $l_{H0}=0.2275$, or a bit more than one-fifth. This time in such a model would be interpreted to include all Beckerian (1975) time in terms of the household child-raising time, and wife household time, and any other forms of early human capital time.

The balanced growth path equilibrium human capital time also rises continuously as the human capital investment sector productivity A_{Ht} trends up. Figure 3, with Lh denoting l_{Ht}, shows this trend upwards, with the years ranging from 1750 to 2000, and $l_{H,250}=0.45378$.The high level of human capital investment time in the year 2000 reflects the steadily rising level of formal education, from no schooling, to primary level average education, to high school average levels, and now to tertiary college and even graduate education as standards. In addition, in such a model, time in research and development must also be interpreted as entering such a time allocation. However, note that by adding physical capital into the human capital sector, the time in human capital investment would not rise quite as high, but the model would then prove less analytically tractible, requiring numerical simulation.

Similarly the input price ratio $\frac{w_t}{r_t}$ can be examined over time, to see that it steadily falls. Using equations (6), (7), (23), (24), and (25), and the example parameters, the input price ratio can be expressed as

$$\begin{aligned}
\frac{w_t}{r_t} &= \frac{\gamma_A}{(1-\gamma_A)}\frac{s_{At}k_t}{l_{At}h_t} \\
&= \frac{k_t}{h_t}\frac{\gamma_A}{(1-\gamma_A)}\frac{\left(\frac{\alpha_A(1-\gamma_A)}{\alpha_A(1-\gamma_A)+\alpha_M(1-\gamma_M)+\alpha_S(1-\gamma_S)}\right)}{\left(\frac{\gamma_A\alpha_A}{(\gamma_A\alpha_A+\gamma_M\alpha_M+\gamma_S\alpha_S)}\frac{\rho(1+A_H-\delta_h)}{A_H\left[1+\rho\left(1+\frac{\alpha}{(\gamma_A\alpha_A+\gamma_M\alpha_M+\gamma_S\alpha_S)}\right)\right]}\right)} \\
&= \frac{\gamma_A\left(a_{At}\right)^{\frac{1}{\gamma_A}}\left(1-\gamma_A\right)^{\frac{1-\gamma_A}{\gamma_A}}}{\left[\left(\frac{1+A_{H0}(1+\mu)^t-\delta_h}{1+\rho\left(1+\frac{\alpha}{(\gamma_A\alpha_A+\gamma_M\alpha_M+\gamma_S\alpha_S)}\right)}\right)(1+\rho)-1+\delta_k\right]^{\left(\frac{1}{\gamma_A}\right)}} \\
&= \frac{\frac{1}{3}\left(\frac{2}{3}\right)^2}{\left[\left(\frac{1+0.06593(1.0015)^{x-1750}-0.015}{1+0.03\left(1+\frac{1}{\frac{1}{3}+\frac{1}{2}+\frac{3}{5}}\right)}\right)(1.03)-1+0.03\right]^{(3)}}
\end{aligned}$$

Figure 4 illustrates the ratio as the human capital productivity steadily rises.And the sectoral physical capital to human capital ratios follow the input price ratio, moving downwards in tandem with $\frac{w}{r}$ since

$$\frac{w_t}{r_t} = \frac{\gamma_A}{(1-\gamma_A)}\frac{s_{At}k_t}{l_{At}h_t} = \frac{\gamma_M}{(1-\gamma_M)}\frac{s_{Mt}k_t}{l_{Mt}h_t} = \frac{\gamma_S}{(1-\gamma_S)}\frac{s_{St}k_t}{l_{St}h_t}.$$

Figure 5 illustrates the three sectoral capital ratios over time.

Despite the fact that the wage to real interest rate are falling, the effective wage to real interest rate is rising. To see this, consider that with each time

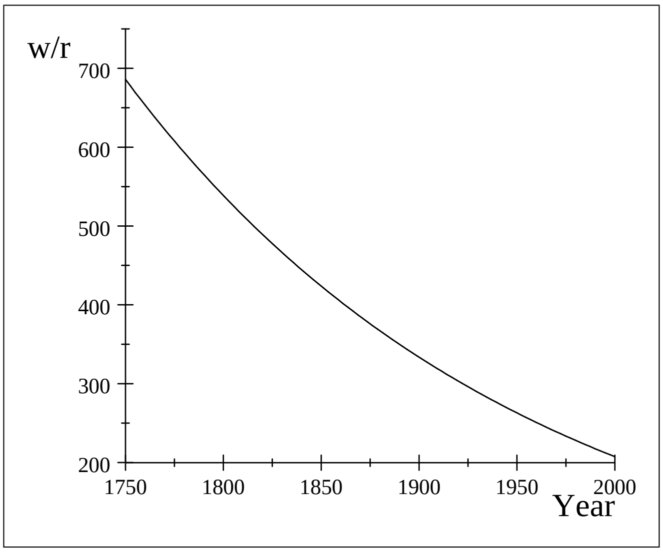

Figure 4: Fall in Input Price Ratio $\frac{w}{r}$ from 1750 to 2000

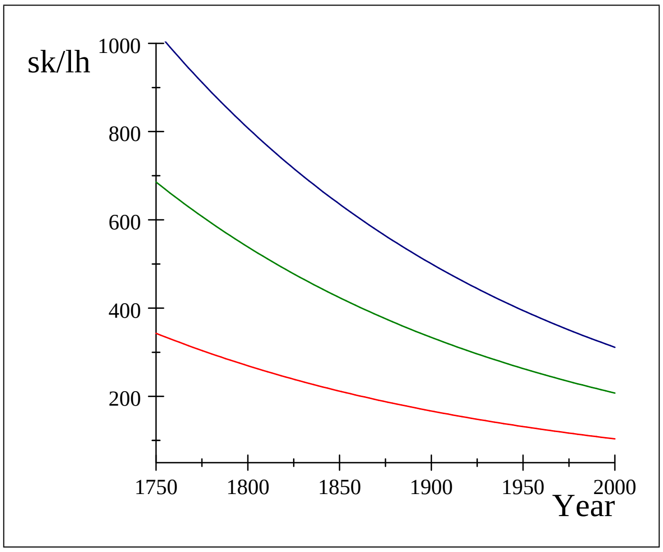

Figure 5: Fall in Sectoral Physical Capital to Human Capital Ratio from 1750 to 2000; Red: Agriculture; Green: Manufacturing; Blue; Services

t being at the balanced growth path equilibrium, and with $h_0 = 1$, that

$$
\begin{aligned}
\frac{w_t h_t}{r_t} &= \frac{w_t}{r_t} h_0 \left(1 + g_0\right)\left(1 + g_1\right) ... \left(1 + g_t\right) \\
\ln\left(\frac{w_t h_t}{r_t}\right) &\simeq \ln\left(\frac{w_t}{r_t} h_0\right) + g_0 + g_1 + ... + g_t \\
\ln\left(\frac{w_t h_t}{r_t}\right) &\simeq \ln\left(\frac{w_t}{r_t}\right) + \sum_{j=0}^{t} g_j; \\
\frac{w_t h_t}{r_t} &\simeq e^{\left(\ln\left(\frac{w_t}{r_t}\right) + \sum_{j=0}^{t} g_j\right)}.
\end{aligned}
$$

Now substitute in the solution for g_t from equation (27), and use the example calibration,

$$
\frac{w_t h_t}{r_t} \simeq e^{\left(\ln\left(\frac{w_t}{r_t}\right) + \sum_{j=0}^{t}\left[\frac{1 + A_{H0}(1+\mu)^j - \delta_h}{1+\rho\left(1 + \frac{\alpha}{(\gamma_A \alpha_A + \gamma_M \alpha_M + \gamma_S \alpha_S)}\right)} - 1\right]\right)}. \tag{28}
$$

Substituting in the parameter values, the ratio $\frac{w_t h_t}{r_t}$ can be expressed as

$$
e^{\left(\ln\left(\frac{\frac{1}{3}\left(\frac{2}{3}\right)^2}{\Big[\Big[\left(\frac{1+0.06593(1.0015)^{x-1750}-0.015}{1+0.03\left(1+\frac{1}{\frac{1}{3}+\frac{1}{2}+\frac{3}{5}}\right)}\right)(1.03)-1+0.03\Big]^{(3)}}\right) + \sum_{j=0}^{t}\left(\frac{1+0.06593(1.0015)^{j}-0.015}{1+0.03\left(1+\frac{1}{\frac{1}{3}+\frac{1}{2}+\frac{3}{5}}\right)} - 1\right)\right)}
$$

This trend in the input ratio $\frac{w_t}{r_t}$ as factored by the level of human capital h_t can be graphed as a result, using the above log approximation that $\ln(1 + x) \simeq x$ for small x. Figure 6 graphs equation (28) as parameterized: It is noteworthy that initially the effective wage to interest rate does fall but then rises steadily after 1800, despite the continuous fall in $\frac{w_t}{r_t}$.

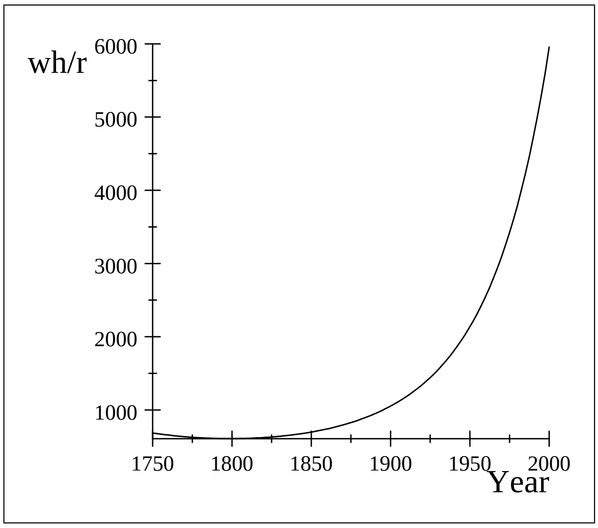

Figure 6: Effective Wage to Interest Rate Ratio from 1750 to 2000 in Example Economy

6 Extension to any n sectors

For any number of sectors denoted now by the index j, with $j = 1, ..., n$, the value of the aggregate output would be defined as y_t, where

$$y_t = \sum_{j=1}^{n} p_{jt} a_{jt} \left(l_{jt} h_t\right)^{\gamma_j} \left(s_{jt} k_t\right)^{1-\gamma_j},$$

and with $\gamma_1 < \gamma_2 < ... < \gamma_n$. Similarly utility would now be given as

$$u_t = \alpha \ln x_t + \sum_{j=1}^{n} \alpha_j \ln y_{jt}. \tag{29}$$

The previous section's corollary carries through to the n-sector economy.

Corollary 10 *An increase in human capital productivity A_H causes output to rise in more human capital intensive sectors relative to less human capital intensive sectors, for all n sectors.*

Proof. Relative output levels between any two sectors, say sector q and sector z, are given by

$$\frac{y_{qt}}{y_{zt}} = \left(\frac{k}{h}\right)^{\left(\gamma_z - \gamma_q\right)} \frac{a_{qt} \left(s_{qt}\right)^{1-\gamma_q}}{a_{zt} \left(s_{zt}\right)^{1-\gamma_S}} \frac{\left(l_q\right)^{\gamma_q}}{\left(l_z\right)^{\gamma_z}},$$

where the capital ratio $\frac{k}{h}$, with p_1 normalized to one, can be expressed by

$$\frac{k_t}{h_t} = \frac{\rho\left(1+g\right)\left[\sum_{j=1}^{n}\alpha_j\left(1-\gamma_j\right)\right]\gamma_1\left(a_{1t}\right)^{\frac{1}{\gamma_1}}\left(1-\gamma_1\right)^{\frac{1-\gamma_1}{\gamma_1}}}{\left[\left(1+g_t\right)\left(1+\rho\right)-1+\delta_k\right]^{\left(\frac{1}{\gamma_1}\right)}A_H\left(\sum_{j=1}^{n}\alpha_j\gamma_j\right)},$$

and the growth rate g is given by

$$1+g = \frac{1+A_H-\delta_h}{1+\rho\left(1+\frac{\alpha}{\left(\sum_{j=1}^{n}\alpha_j\gamma_j\right)}\right)}.$$

Substituting in for $\frac{k_t}{h_t}$ and g,

$$\frac{y_{qt}}{y_{zt}} = \frac{\frac{\left(a_{qt}\right)^{\frac{\gamma_z}{\gamma_q}}}{a_{zt}}\frac{\left(1-\gamma_q\right)^{\frac{\gamma_z\left(1-\gamma_q\right)}{\gamma_q}}}{\left(1-\gamma_z\right)^{1-\gamma_z}}\frac{\alpha_q}{\alpha_z}\left(\frac{\gamma_q}{\gamma_z}\right)^{\gamma_z}}{\left[\left(\frac{\left(1+A_H-\delta_h\right)\left(1+\rho\right)}{1+\rho\left(1+\frac{\alpha}{\left(\sum_{j=1}^{n}\alpha_j\gamma_j\right)}\right)}\right)-1+\delta_k\right]^{\left(\frac{\gamma_z-\gamma_q}{\gamma_q}\right)}}.$$

With $\gamma_z > \gamma_q$, $\frac{\partial\left(\frac{y_{qt}}{y_{zt}}\right)}{\partial A_H} < 0$. ■

Similarly, adding an $n+1$ sector to the n-sector economy, causes the labor time allocations in each of the other n sectors to decrease.

The model can be changed to any number of sectors. Reducing it down to an agriculture, manufacturing model would end up seeing a much greater fraction of time devoted to agriculture than in modern times.

Thus this theory explains the large shift in labor from agriculture to other sectors through the continuing development of technology that opens up new sectors, and transfers labor into those sectors. And with these sectors being more human capital intensive than existing sectors, a slight historical trend upwards in human capital productivity A_H would predict the relative shift of output towards the more human capital intensive, "new" sectors.

The analysis started with just the two sectors. Then the "structural transformation" is shown for the three sectors, and then to any number n sectors. And the story could go on. For instance, it may be that it is the human capital accumulation that allows such new sectors to come about, in some endogenous sense. The creation of new goods/sectors, in this simple model, nor in any other standard models, is not taken up here but would be the next most interesting extension of this simple theory.

However an algorithm method of showing the change for example in sectoral labor shares over time as sectors are added is possible using the following assumption for the labor share in the any n sector. Let γ_n be defined as

$$\gamma_n = \frac{n}{n+2}.$$

Then for the 3 sector economy, the human capital intensity of agriculture would be $\frac{1}{3}$, that of manufacturing, the second sector, would be $\frac{1}{2}$, and the third sector, services, would be $\frac{3}{5}$, as specified in the example 3 sector economy of the last section. Further assuming as in the 3 sector economy that there are equal preferences across sectors, at

$$\alpha = \alpha_1 = \alpha_2 = \alpha_3 = ... = \alpha_n = 1.$$

Then the solution for the labor share in agriculture, where it is designated as sector 1, for a given year t and corresponding growth rate (given the trend in A_{Ht}) would be

$$l_{1t} = \frac{\gamma_1 \alpha_1}{\sum_{j=1}^{n} \gamma_j \alpha_j} \frac{\rho\left(1 + A_{Ht} - \delta_h\right)}{A_{Ht}\left[1 + \rho\left(1 + \frac{\alpha}{\sum_{j=1}^{n} \gamma_j \alpha_j}\right)\right]}.$$

The following proposition results.

Proposition 11 *Assuming that* $\gamma_n = \frac{n}{n+2}$, *and that* $\alpha = \alpha_n = 1$ *for all* n, *as the number of sectors* n *goes to infinity, the share in labor at any given time* t *goes to zero.*

Proof. $l_{1t} = \frac{\gamma_1 \alpha_1}{\sum_{j=1}^{n} \gamma_j \alpha_j} \frac{\rho\left(1 + A_{H0}(1.0015)^t - \delta_h\right)}{A_H\left[1 + \rho\left(1 + \frac{\alpha}{\sum_{j=1}^{n} \gamma_j \alpha_j}\right)\right]}$,

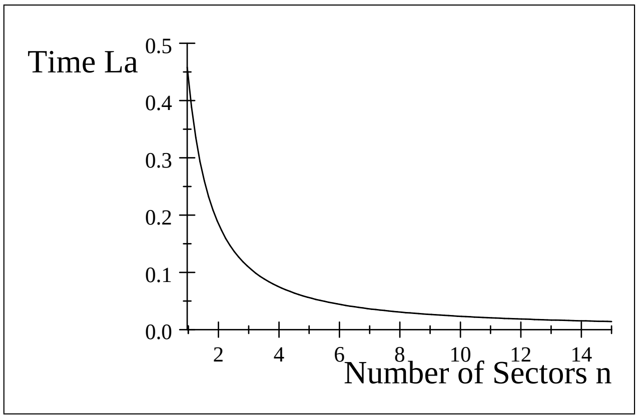

Figure 7: Example Change in Labor Time in Agriculture as Number of Sectors Increases

$$= \frac{\frac{1}{3}}{\sum_{j=1}^{n} \frac{j}{j+2}} \frac{\rho\left(1+A_{H0}(1.0015)^{t}-\delta_{h}\right)}{A_{H}\left[1+\rho\left(1+\left(\frac{1}{\sum_{j=1}^{n} \frac{j}{j+2}}\right)\right)\right]}.$$

$\lim_{n\to\infty}(l_{1t}) = 0.$ ■

A gradual labor share decrease in agriculture over time would be a natural result of adding increasingly human capital sectors to the economy. Figure 7, with La denoting agriculture time l_{At}, illustrates the decrease in time in agriculture as the number of sectors rises from 1 to 15 using the same example parameters as in previous sections, at the year 2000 :

At first, with one sector, all goods production labor is spent in agriculture. As more human capital intensive sectors are added, the labor time in agriculture exponentially falls.

One simple way in which the number of sectors can be endogenized, while relaxing the assumption that n must take on an integer value, is to let n be a function of the level of human capital productivity A_{Ht}. With $A_{Ht} = A_{H0}(1+\mu)^{t}$, so that human capital productivity exogenously trends upwards over time, then $n = n(A_{Ht})$ makes n trend upwards also as a simple function of time, and labor time allocated to agriculture endogenous fall over time as in Figure 4.

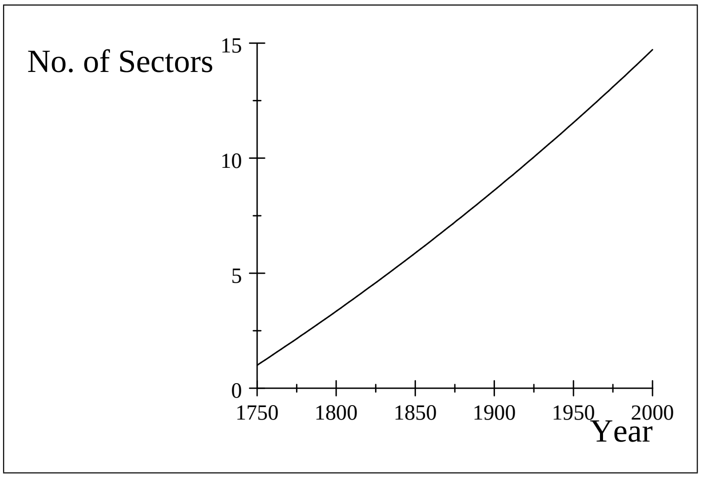

Figure 8: Numbers of Sectors n as an Endogenous Function of the Human Capital Productivity

In particular, specify n such that

$$n\left(A_{Ht}\right) = z_1 A_{Ht} - z_2,$$

where $z_1 = 1992.6$, $z_2 = \frac{1750}{60}$, and A_{Ht} is specified as in the example above, whereby $A_{Ht} = 0.06593\left(1.0015\right)^t$. This means that n is given by the following function of time t :

$$n\left(A_{Ht}\right) = \frac{1992.6\left(0.06593\right)\left(1.0015\right)^t}{60} - \frac{1750}{60},.$$

Figure 8 shows that over the 250 years from 1750 to 2000 the number of sectors rises from 1 to almost 15, as in Figure 6. Consequently, with this endogenous formulation of n, the labor time in agriculture would similarly decline over the same time interval as the number of sectors rises, as in Figure 6.

7 Extension with Intermediate Goods

It is straightforward to show that the thrust of the theory applies if alternatively intermediate goods are postulated. Assume instead that there are n sectors which ease use an intermediate good that is produced as a separate output. Let the jth sectoral output production function, with $\gamma_{lj} > 0$, $\gamma_{kj} > 0$, and $\gamma_{lj} + \gamma_{kj} < 1$, and with d_{jt} denoting the intermediate good output that is an input into sector j's production, be given as

$$y_{jt} = a_{jt} \left(l_{jt} h_t\right)^{\gamma_{lj}} \left(s_{jt} k_t\right)^{\gamma_{kj}} \left(d_{jt}\right)^{1-\gamma_{lj}-\gamma_{kj}},$$

where the intermediation good production for each j sector is given as

$$d_{jt} = a_{djt} \left(l_{djt} h_t\right)^{\gamma_{dlj}} \left(s_{djt} k_t\right)^{\gamma_{dkj}},$$

where l_{djt} and s_{djt} are the shares of human capital and physical capital devoted to the jth intermediate good, and $\gamma_{dlj} > 0$, $\gamma_{dkj} > 0$, and $\gamma_{dlj} + \gamma_{dkj} = 1$. Then the value of aggregate output y_t, with p_{jt} again denoting the price of the jth good, is given by

$$\begin{aligned} y_t &= \sum_{j=1}^{n} p_{jt} a_{jt} \left(l_{jt} h_t\right)^{\gamma_{lj}} \left(s_{jt} k_t\right)^{\gamma_{kj}} \left[a_{djt} \left(l_{djt} h_t\right)^{\gamma_{dlj}} \left(s_{djt} k_t\right)^{\gamma_{dkj}}\right]^{1-\gamma_{lj}-\gamma_{kj}}, \\ &= \sum_{j=1}^{n} p_{jt} a_{jt} \left(a_{djt}\right)^{1-\gamma_{lj}-\gamma_{kj}} \left(l_{jt} h_t\right)^{\gamma_{lj}} \left(s_{jt} k_t\right)^{\gamma_{kj}} \left(l_{djt} h_t\right)^{\frac{\gamma_{dlj}}{1-\gamma_{lj}-\gamma_{kj}}} \left(s_{djt} k_t\right)^{\frac{\gamma_{dkj}}{1-\gamma_{lj}-\gamma_{kj}}} \end{aligned}$$

Assume in addition that the output good production function differs only in its intermediation good input, in that the labor intensity and capital intensity in producing the jth good is the same across all n sectors, equal to γ_l and γ_k. This means that

$$\begin{aligned} \gamma_l &= \gamma_{l1} = \gamma_{l2} = ... = \gamma_{l\,n}, \\ \gamma_k &= \gamma_{k1} = \gamma_{k2} = ... = \gamma_{kn}, \end{aligned}$$

and the aggregate output can be written as

$$y_t = \sum_{j=1}^{n} p_{jt} a_{jt} \left(a_{djt}\right)^{1-\gamma_l-\gamma_k} \left(l_{jt} h_t\right)^{\gamma_l} \left(l_{djt} h_t\right)^{\frac{\gamma_{dlj}}{1-\gamma_{lj}-\gamma_{kj}}} \left(s_{jt} k_t\right)^{\gamma_k} \left(s_{djt} k_t\right)^{\frac{\gamma_{dkj}}{1-\gamma_{lj}-\gamma_{kj}}}.$$

Now assume the model has the same utility function as in equation (29) and the same human capital investment function as in equation (22), along with physical capital accumulation as before by the consumer. Then an increase in human capital productivity A_H would again cause output to shift towards the relatively human capital intensive sector, as determined by the human capital intensity of the intermediate good.

8 Discussion

The theory can be thought of with any number of sectors, or with intermediate goods. When there is only agriculture, everyone works in agriculture, but also in human capital if that sector still is in the model. Then agriculture is the aggregate output good, and the main capital is the value of the land (see Mundlak, 2005). TW Schultz (1964) added a second goods sector, with it still being a part of agriculture, but now termed modern agriculture versus traditional agriculture. His explanation was that with a zero return to human capital, it was not accumulated and the modern sector did not emerge. But once the investment became worthwhile in human capital, so as to accumulate the knowledge to introduce the modern technology of physical capital machines, then the modern agriculture sector could emerge. And so as human capital became more productive, more of agriculture would shift from the traditional to the modern type of agriculture.

This is exactly consistent with the theory of this model, once another agriculture sector is added, with the modern sector having a higher Cobb-Douglas parameter for human capital share of output than the traditional agriculture sector, just as manufacturing is more human capital intensive in the model above than the agriculture sector.

Mundlak (2000, 2005) and many others added manufacturing as the second sector, in addition to agriculture, taking a more in-time view that can be viewed as an alternative but also as an update of TW Schultz's (1964) approach. Rogerson (2008) focuses on two sectors, excluding agriculture, and also uses a time allocation approach, albeit one in which tax rates play a key function. Ngai and Pissarides (2007) considers Baumol's (1967, Baumol et al. 1985) work on a two sector model and find that a balanced growth rate is still feasible within such a structure unlike Baumol's conjectures, but consistent with the balanced growth path approach of this paper's simple theory.

But the model can accommodate any n number of sectors, in an alternative approach to Dixit-Stiglitz of having some finite number of differentiated goods. With perfect competition in the model here, the n sector version would be more akin perhaps to Rosen's (1974) hedonic price view of equilibria with differentiated goods. Here a different quality of a good makes it a slightly different good. But in the model, that view is still consistent. This gets to the empirical issues of measuring prices in the three sector structural transformation literature. With n goods sectors, the sectors become closer in nature, but still would have some ranking based on human capital intensity. To illustrate further, simply let $n = 4$ instead of 3 as above, with the fourth goods sector called Technology.

Clearly the microsofts, facebooks, and googles could be in this category, even though now they would be traditionally lumped into the service sector. Or would they be lumped into the manufacturing sector since microsoft is so big? Of course there are bureaucratic statistics agency answers, with certain categorizations, but there is some unavoidable arbitrariness of these categorizations. With only $n = 4$ sectors, these three named companies would probably be considered technology. This paper's ranking of this sector would only be that it has the highest human capital share of output than the other three sectors.

Now then the labor in the other three goods sectors falls compared to the model with only three sectors. So by adding new sectors that are more human capital intensive, the labor naturally moves from the less human capital intensive sectors towards the new sector. This can explain the dramatic reduction in the labor in any one sector such as agriculture: the development of new more human capital sectors as the economy evolves. Thus the model when extended to more sectors becomes consistent with D. Gale Johnson (1982) analysis of rural to urban labor movements and the growth of labor in the cities. And so in this way the labor theory becomes consistent with evidence, even though within any given fixed number of sectors the relative labor use remains fixed. The model then explains changes in the relative labor amounts between sectors only as the sectors are further subdivided or otherwise added to. And of course this subdivision between sectors is the basis of Smith's (1776) theory of the division of labor being limited by the extent of the markets, and so a completely natural extension.

The modeling approach to n goods sectors could be further extended in much more difficult ways, in particular by adding how new sectors come into existence. This would be based on a theory that as human capital productivity rises, and the price of human capital intensive sectors falls, that such sectors would come into existence via Coase theorem logic on the creation of new markets (combined here with Boldrin and Levine's (2008) fixed cost view of competitive markets). As human capital productivity increases, and sectors intensive in human capital have lower prices, then the fixed cost of starting new more human capital intensive sectors is finally overcome by the profit of the new more human capital sector and it comes into being. At this point there would be $n + 1$ goods sectors, with each good in the utility function just as in Rosen's (1974) hedonic equilibrium each quality difference enters the utility function. Or perhaps more innovative theories may help make this step such as in Boldrin and Levine (2009).

Of course this would be a very significant extension that is beyond the scope of a single paper on the subject. But the point is to argue that this model is consistent with encompassing any number of goods. And the consequence of that logic is important: in taking such models to the data, the arbitrariness of the number of sectors is intricately involved in any, and all, categorizations into sectors. Therefore despite evidence that the price of services may or many not be falling relative to manufacturing as measured in categorizations of the data is not based strictly on the human capital intensity of each sector, and so does not represent a contradiction of the theory of this paper.

Still the paper is not vacuous since its theory can be contradicted by evidence. It assumes that more human capital intensive sectors tend to be added as the economy evolves. This can be contradicted. And the main implications of the paper are widely accepted in international trade theory and macroeconomics: that economies shift towards sectors in which the relative price is reduced because of factor augmentation, as in the Rybczynski (1955) theorem. And it is also agreed that agriculture output falls relative to manufacturing which falls relative to services (which falls relative to technology) as economies develop. So the paper does explain the main output trends, but will admittedly not try in one paper to do another categorization that might show that relative prices move opposite of relative output levels. That part of the theory of the paper is not confirmed here. Yet clearly the theory appears consistent with the evolution of industry, the gradual rise in the growth rate, and the rise in human capital time l_{Ht} as A_{Ht} trends up and education levels continuously rise.

This paper then adds only a very simple theory that is consistent with the development of this literature within a strand that goes back through a long respected tradition. It does not resolve the issues, but it shows another cohesive, and probably simplest, way to explain them potentially.

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