On $P(\phi)_2$ Interactions At Positive Temperature

Florian Robl
School of Mathematics
Cardiff University

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Abstract

The Schwinger functions of the thermal $P(\phi)_2$ model on the real line and the vacuum $P(\phi)_2$ model on the circle are equal up to interpretation of their time and space coordinates. This is called Nelson symmetry. In the present work this correspondence is exploited to construct and prove results for the thermal $P(\phi)_2$ model. The results are existence of the thermal Wightman functions, the relativistic KMS condition, verification of the Wightman axioms and spatial exponential decay.

A Hölder inequality for general KMS states is proven, employing non-commutative L_p-spaces. This inequality is key in the proof of the existence of the thermal Wightman functions.

For the vacuum $P(\phi)_2$ model on the circle a version of the Glimm-Jaffe ϕ-bound is proven. Furthermore the Källén-Lehmann representation for general vacuum two-point functions are proven and general facts about the damping factor are established. The consequences for the damping factor in the thermal case are briefly discussed.

Contents

Chapter 1

Preliminaries

1.1 Introduction

The present work can topically be decomposed into two blocks. The construction of the thermal $P(\phi)_2$ model on the one hand, and the investigation of properties of the same model, especially particle aspects, on the other hand. The original motivation was the latter, but certain subtleties in the construction of the thermal Wightman functions made the former necessary.

In order to convey the motivation behind this investigation, the following questions should be answered - at the very least from the personal point of view of the author: Why constructive quantum field theory? Why thermal quantum field theory? Why the $P(\phi)_2$ model? Why is it important to contemplate the particle content of a thermal theory?

Constructive quantum field theory is concerned with the axiomatisation of quantum field theory and with the mathematically rigorous construction of interesting models. Interesting models are in general those, which arise in Lagrangian quantum field theory in theoretical physics. This is justified by the in general very good and in some cases amazing agreement of these models with high energy collider experiments. Yet, as most physicists will agree, the prediction of mentioned measurements has gotten along very well without the heavy mathematical machinery, that constructive theory seems to necessitate. But what have we learnt, if we do not break a verified theory down into its properties and foundations, compare them with other theories and in doing so gain insight into its fundamental properties. Put differently, from the author's point of view the answer to the first question is the following. Only through axiomatisation and mathematical rigour can physics claim to give a view of the world, a Weltbild. This discussion is of course a philosophical one and will not be pursued any further.

A second argument comes from the realisation, that the advent of quantum theory has made it difficult to gain intuitions for microscopic physical processes. Pursuers of the constructive approach feel, that the danger of going wrong, when not employing mathematical rigour, is too great. And the difficulties which arise treating thermal quantum fields perturbatively might serve as an example.

The interest in thermal quantum field theory stems mainly from the discovery of the Quark Gluon Plasma - a strongly interacting matter state in which neither effects of special relativity nor thermal properties can be neglected in its scientific description. The Quark Gluon Plasma is believed to have appeared shortly after the Big Bang as well as in some heavy neutron stars and it has been reproduced in several heavy ion collision experiments. Furthermore the ease with which path integral techniques have been applied to many body systems/quantum statistical mechanics certainly instigates further study towards thermal and special relativistic theories from the constructive point of view.[1]

At first the choice of a scalar, polynomially interacting equilibrium theory in two space-time dimensions seems a poor choice when the physical system one is interested in, is the Quark Gluon Plasma in a collider experiment. However, the mathematical difficulties arising already in such a comparatively simple model, warrant the lower dimensionality and simpler field structure. It is in the tradition of constructive quantum field theory to start with a simple model and to proceed stepwise to more involved models. As the $P(\phi)_2$ model without temperature, i.e. the vacuum $P(\phi)_2$ model, is the best understood theory of the desired complexity, the choice is natural. Since the same model has also been constructed in three space-time dimensions it is furthermore safe to assume, that many results presented in this thesis can be proven in that case as well. Another striking argument for the $P(\phi)_2$ model is the availability of *Nelson symmetry*. In general the correlation functions in quantum field theory have analytic continuations in the time variable, called Schwinger functions. Nelson symmetry states, that the Schwinger functions of the thermal theory on $\mathbb{R}^2$ and the Schwinger functions for the vacuum model, where the space-time manifold is a cylinder, are the same, albeit with exchanged meanings of space and time coordinate. For the vacuum theory on the cylinder the slices, which are considered to be spatial, are circles. Therefore this model is also often referred to as the vacuum $P(\phi)_2$ model on the circle.

With regards to the particle interpretation the first observation to be made is, that

[1] It should be mentioned here, that such connections have been investigated already some time ago in [32,33].

viewing particles as irreducible representations of the Poincaré group according to Wigner does not work in the thermal case [55]. This is not surprising as a thermal equilibrium state distinguishes a rest frame. Yet one does not want to abandon the particle interpretation as crutches to the intuition. The only attempt to give a precise mathematical meaning to the notion of a particle so far comes from Bros & Buchholz [15]. In their work they derive Källén-Lehmann and Jost-Lehmann-Dyson type representations for thermal fields, that is the 2-point functions are given as an integral over free correlators with differently weighed masses. They argue, that stable particles are to be identified with Dirac delta contributions to the integral, while short lived thermal excitations belong to the absolutely continuous part of the involved weights. The ultimate goal of this work is to confirm or disconfirm this claim. Unfortunately the question could not be resolved in the given time frame but partial results are presented. Further discussion is given in Chapter 5.

The present work builds directly on the three papers [21–23]. The first two works contain a construction of the thermal $P(\phi)_2$ model in the Haag-Kastler framework, i.e. a corresponding algebra of observables, interacting dynamics and KMS-state are provided. In the third one the Wightman two-point function has been constructed and the relativistic KMS condition has been proven for it.

Doubtlessly the main new result in this thesis is the existence of the general n-point functions for the thermal $P(\phi)_2$ model as tempered distributions. Another result is a Hölder inequality for KMS states, which forms one of the two main pillars of the proof of the existence theorem, the other being Nelson symmetry. The Hölder inequality has its roots in non-commutative L_p spaces and modular theory. Further results are proofs of the relativistic KMS condition as well as spatial exponential decay of the thermal correlation functions - also derived with the help of Nelson symmetry - and the Källén-Lehmann representation for the vacuum theory on the circle. Since in this case some information about the common spectrum of the Hamiltonian and the momentum operator is available, some basic statements can readily be made about the weights.

The remainder of this chapter contains an introduction to several approaches to quantum field theory and a treatment of the free fields corresponding to the models of interest. Chapter 2 is devoted to the statement and proof of the Hölder inequality for KMS states. Chapter 3 treats the construction of the two models and the proofs of the existence of the Wightman functions as well as the relativistic KMS condition. The proofs of the Wightman axioms, the exponential decay of correlation for the thermal field as well as the Källén-Lehmann representation for the vacuum model are contained in Chapter 4. Chapter 5 presents a short conclusion and a conjecture on the mass gap for the thermal

$P(\phi)_2$ model.

This thesis is a result of close collaboration between the supervisor C. Jäkel the author. The following lists which of the *original* proofs in this work are results of the supervisor's, the student's and common work. The proofs of Lemma 2.3.1 and Lemma 3.2.2 are due to C. Jäkel. The proofs of Lemma 2.3.2, Theorem 2.1.1, Proposition 3.3.1, Theorem 3.4.1, Lemma 3.4.2, Proposition 3.4.3, Lemma 3.4.4, Theorem 3.4.6, Lemma 4.1.1 and Lemma 4.1.2 are due to C. Jäkel and the author. The proofs of Lemma 3.3.5, Proposition 4.1.4, Lemma 4.2.1, Lemma 4.2.2, Theorem 4.2.4, Theorems 4.3.1 and 4.3.5 and Proposition 4.3.6 are due to the author.

1.2 Three Approaches to Quantum Field Theory

It is the purpose of this section to give an introduction to the different approaches to quantum field theory employed in this work. From a mathematical physics point of view the respective axiom sets are at the heart of the approaches. These are

- the Wightman axioms in the Hamiltonian approach [60, 65],
- the Osterwalder-Schrader and Nelson axioms in the Euclidean approach [29, 64],
- and the Haag-Kastler axioms in Algebraic Quantum Field Theory (AQFT) [34].

It is the goal of Chapter 3 to construct the thermal $P(\phi)_2$ model in the sense of Wightman. For technical reasons, however, it is not possible to dispense with either of the other approaches. While the Wightman axioms best capture what one wants in a fully fledged quantum field theory useful to physicists, the Euclidean approach is best suited for the technical construction of such. The basic idea behind it is to solve an elliptic equation instead of the hyperbolic *Klein-Gordon Equation* and obtain the desired solutions by analytic continuation. Since the corresponding elliptic equation is in general easier to solve, this is a technical simplification. AQFT is based on the idea to shift the main focus of the description away from computational devices (Lagrangian QFT) to the basic physical concepts of *observables* and *states*. This naturally leads to an operator algebraic formulation, which in turn makes powerful technical tools available. The exploitation of these tools is not even finished yet, as the Hölder inequality presented in Chapter 2 shows. Since a thorough exhibition of the three approaches is beyond the scope of this thesis, the reader is referred to the standard textbooks [29, 34, 60, 64, 65].

It should be mentioned, that originally the axiom sets were intended to capture field theories, which transform covariantly under Lorentz transformations, i.e. vacuum theories on Minkowski space. Presently the situation is different - none of the considered models is Lorentz invariant. Therefore the axiom sets have to be modified in the appropriate manner. They will still be called by their respective original names, as the spirit and mathematical tools are in essence unchanged. Tightly associated with these axiom sets are reconstruction theorems, which show under what circumstances the different axiom sets are equivalent - both for different formulations inside their respective approaches and across the different approaches. As they are too great in number, the analogs of these reconstruction theorems will not be given here, except for those, which are relevant for the present work.

1.2.1 Some Notation

Since the technical part starts here, some comments on notation seem in order. S_β denotes a circle of circumference β. The coordinates $(\alpha, x) \in S_\beta \times \mathbb{R}$ of a point in the cylinder will refer to either one of the charts $[-\beta/2, \beta/2) \times \mathbb{R}$ or $[0, \beta) \times \mathbb{R}$ and addition as well as subtraction of the angular coordinates are always understood modulo β.

Definition 1.2.1. *Let X be either $\mathbb{R}$, S_β or any finite Cartesian product thereof. The* Schwartz functions $\mathcal{S}(X)$ *($\mathcal{S}_\mathbb{R}(X)$) are defined as the (real-valued) functions f on X fulfilling*

$$\sup_{x\in X} |(1+|x|)^\gamma \, \partial_x^\gamma f(x)| \leq C_\gamma \,, \tag{1.1}$$

where γ is a multi-index and C_γ is a constant. The space of tempered distributions *$\mathcal{S}'(X)$ is defined as the (topological) dual space of $\mathcal{S}(X)$. $\mathcal{S}'_\mathbb{R}(X)$ is defined as the real-linear subspace of real valued tempered distributions in $\mathcal{S}'(X)$. $C^\infty(X)$ ($C_0^\infty(X)$) is defined as the space of infinitely often continuously differentiable functions (of compact support) on X.*

Remark 1.2.2. *(i) $\mathcal{S}(X)$ is a Fréchet space.*

(ii) For $X = S_\beta$ the Schwartz functions coincide with $C^\infty(S_\beta)$.

In the context of this work two important space-times are $\mathbb{R}^2$ and[2] $\mathbb{R} \times S_\beta$. For a space-time X and $f_j \in \mathcal{S}(X)$, $j \in \{1, \ldots, n\}$, the *Wightman distributions* $W^{(n)}$ are given

[2] $\mathbb{R} \times S_\beta$ is called the Einstein cylinder in physics.

as continuous functionals[3]

$$W^{(n)}(f_1, \ldots, f_n) = (\Omega, \phi(f_1) \cdots \phi(f_n)\, \Omega)\,. \tag{1.2}$$

The *Nuclear Theorem* [65] guarantees, that $W^{(n)}$ can also be viewed as a tempered distribution of one argument taking values in the n-fold tensor product of X, that is $W^{(n)} \in \mathcal{S}'(X^n)$. $W^{(n)}$ will be decorated with a sub-index β for the thermal theory and a sub-index C for the vacuum theory. All the occurring Wightman distributions will be translation invariant. Therefore [60, p.66] there exists a distribution $\mathfrak{W}^{(n-1)}$, such that

$$W^{(n)}(f) = \int_X \mathfrak{W}^{(n-1)}(f_{(x)})\; \mathrm{d}x\,, \tag{1.3}$$

where $f \in \mathcal{S}(X^n)$ and $f_{(x)}(\xi_1, \ldots, \xi_{n-1}) = f(x, x-\xi_1, x-\xi_1-\xi_2, \ldots, x-\xi_1-\ldots\xi_{n-1})$. Heuristically this can be written as

$$W^{(n)}(x_1, \ldots, x_n) = \mathfrak{W}^{(n-1)}(x_1 - x_2, \ldots, x_{n-1} - x_n)\,. \tag{1.4}$$

In general Wightman distributions $\mathfrak{W}^{(n-1)}$ are boundary values of analytic functions on various domains. These analytic functions will typically be denoted by $\mathcal{W}^{(n-1)}$. In rare cases the notation $\mathcal{W}^{(n)}(t_1, x_1, \ldots, t_n, x_n)$ for purely real t and x will be used. On such occasions, the distributional character of $\mathcal{W}^{(n)}$ will be explicitly mentioned. For $\mathfrak{W}^{(n-1)}$ and $\mathcal{W}^{(n-1)}$ the same sub-script conventions will be used as for $W^{(n)}$.

Furthermore the following conventions and notations will be used throughout:

(i) The forward and backward light-cones are defined by

$$V^+ := \{(t,x) \in \mathbb{R}^2 \mid t > |x|\}\,, \tag{1.5}$$

and by $V^- := -V^+$, respectively.

(ii) The following intersection of light-cones will play an important role,

$$V_\beta := V^+ \cap (\beta e - V^+)\,, \tag{1.6}$$

where e is the unit vector in time direction.

(iii) Units are chosen such, that all physical constants are one, i.e. $k_B = \hbar = c = 1$. The

[3]For the definition of the involved objects cf. Subsection 1.2.2 on the Wightman axioms.

temperature parameter $\beta = \frac{1}{k_B T} = \frac{1}{T} > 0$ as well as the mass parameter $m > 0$ will be fixed but arbitrary positive constants throughout.

(iv) Defining $D_x = -i\partial_x$ the Laplace operator on $S_\beta \times \mathbb{R}$ and the d'Alembert operator on $\mathbb{R}^2$ are defined as

$$-\triangle := D_\alpha^2 + D_x^2 \quad \text{and} \quad \square := -D_t^2 + D_x^2 \,. \tag{1.7}$$

(v) As approximations of the Dirac delta on S_β and $\mathbb{R}$, respectively,

$$\delta_k(\alpha) := \beta^{-1} \sum_{|n| \le k} \mathrm{e}^{i\nu_n \alpha} \quad \text{and} \quad \delta_\kappa(x) := \kappa\chi(\kappa x) \,, \tag{1.8}$$

will be used, where $k \in \mathbb{N}$, $\kappa \in \mathbb{R}^+$, $\nu_n = 2\pi n/\beta$, $n \in \mathbb{N}$, and χ is a function[4] in $C_0^\infty(\mathbb{R})$ with $\int \chi(x)\, \mathrm{d}x = 1$.

(vi) Translations on $X = \mathbb{R}^2$ or $X = S_\beta \times \mathbb{R}$ are denoted by $\mathfrak{t}$ and are written

$$\mathfrak{t}_y \colon x \mapsto x + y \,. \tag{1.9}$$

On $S_\beta \times \mathbb{R}$ two reflections are defined as $\mathfrak{r} \colon (\alpha, x) \mapsto (-\alpha, x)$ and $\mathfrak{r}' \colon (\alpha, x) \mapsto (\alpha, -x)$. The pull-backs

$$(\mathfrak{t}_*^y f)(x) := f\left(\mathfrak{t}_y^{-1}(x)\right) = f(x - y)$$

acting on the test functions $f \in \mathcal{S}(X)$, induce actions on the tempered distributions $q \in \mathcal{S}(X)'$:

$$(t_y q)(f) := \langle q, (\mathfrak{t}_*^y)^{-1} f\rangle, \quad (rq)(f) := \langle q, \mathfrak{r}_* f\rangle, \quad \text{and} \quad (r'q)(f) := \langle q, \mathfrak{r}'_* f\rangle.$$

For $f \in \mathcal{S}(X^n)$ define

$$(\mathfrak{t}_*^y f)(x_1, \ldots, x_n) := f(\mathfrak{t}_y^{-1} x_1, \ldots, \mathfrak{t}_y^{-1} x_n) \,. \tag{1.11}$$

1.2.2 Hamiltonian and Algebraic Approach

As they are closely related, the Hamiltonian and the algebraic approach will be presented in parallel. Usually the Wightman and the Haag-Kastler axioms demand a representa-

[4]Independently of the choice of χ the limit of δ_κ in $\mathcal{S}'(\mathbb{R})$ is the Dirac delta.

tion of the Poincaré group acting on the respective objects. But neither of the models considered in this work, has Lorentz invariant correlation functions. Therefore this requirement will be omitted where necessary and the Wightman reconstruction theorem will be re-proven for the present case.

Let X be a space-time - in this case either $\mathbb{R}^2$ or $\mathbb{R} \times S_\beta$. The Wightman axioms below are formulated in terms of the correlation functions. The equivalent formulation in terms of field operators is recovered by the Reconstruction Theorem 1.2.4.

Axiom W1 (Wightman Distributions). *There exists a sequence of tempered distributions* $W^{(n)} \in \mathcal{S}'(X^n)$, $n \in \mathbb{N}$.

Axiom W2 (Hermiticity). *For* $f \in \mathcal{S}(X^k)$ *and* $g \in \mathcal{S}(X^{n-k})$ *there holds*

$$\overline{W^{(n)}(f \otimes g)} = W^{(n)}(\overline{g} \otimes \overline{f})\,. \tag{1.12}$$

Axiom W3 (Locality). *For* $f_j \in \mathcal{S}(X)$, $j \in \{1, \ldots, n\}$, *such that supp* f_j *and supp* f_{j+1} *are space like with respect to each other there holds*

$$W^{(n)}(f_1 \otimes \ldots \otimes f_n) = W^{(n)}(f_1 \otimes \ldots \otimes f_{j-1} \otimes f_{j+1} \otimes f_j \otimes f_{j+2} \otimes \ldots \otimes f_n)\,. \tag{1.13}$$

Axiom W4 (Positive Definiteness). *For all sequences* $(f_j)_{j \in J}$, *where* $f_j \in \mathcal{S}(X^j)$ *and where* J *is a finite index set, there holds*

$$\sum_{j,k \in J} W^{(j+k)}(\overline{f}_j \otimes f_k) \geq 0\,. \tag{1.14}$$

Axiom W5 (Translation Invariance). *For all* $n \in \mathbb{N}$ *and any translation* $\mathfrak{t}_y$ *in* X *there holds*

$$W^{(n)}(f) = t_y W^{(n)}(f)\,. \tag{1.15}$$

Remark 1.2.3. *If spontaneous symmetry braking takes place, it is possible, that translation invariance W5 in spatial directions is broken. However, in the models in this work this is not the case. W5 might also fail on general curved space-times.*

In space-times which have infinite spatial slices it is reasonable to require the following axiom.

Axiom W6 (Space-like clustering). *For all* $f \in \mathcal{S}(X^k)$, $g \in \mathcal{S}(X^{n-k})$, $0 < k < n$, *and a space like vector* y *there holds*

$$\lim_{|s|\to\infty} W^{(n)}(f \otimes \mathfrak{t}_*^{sy} g) = W^{(k)}(f)\, W^{(n-k)}(g)\,. \tag{1.16}$$

The next theorem is the Wightman reconstruction theorem without Lorentz covariance. Naturally the statement and proof thereof are very similar to the vacuum case presented in [65, Theorem 3-7].

Theorem 1.2.4. *Assuming Axioms W1 to W4 there exists*

(i) a separable Hilbert space $\mathcal{H}$,

(ii) a unit vector $\Omega \in \mathcal{H}$,

(iii) and a hermitian operator $\phi(f)$ *for every* $f \in \mathcal{S}(X)$,

such that

$$W^{(n)}(f_1 \otimes \ldots \otimes f_n) = (\Omega,\, \phi(f_1)\ldots\phi(f_n)\,\Omega), \tag{1.17}$$

for $f_j \in \mathcal{S}(X)$, $j \in \{1, \ldots, n\}$. *Furthermore, if* $f, g \in \mathcal{S}(X)$ *have space-like separated supports, then for every* Φ *and* Ψ *in the domain of both* $\phi(f)$ *and* $\phi(g)$, *there holds*

$$(\phi(f)\,\Phi,\, \phi(g)\,\Psi) = (\phi(\overline{g})\,\Phi,\, \phi(\overline{f})\,\Psi)\,. \tag{1.18}$$

Assuming Axioms W1 to W5 there exists a strongly continuous group of unitary operators $U(y)$ *implementing the space-time translations:*

$$\phi(\mathfrak{t}_*^y f) = U(y)\phi(f)U(y)^*\,. \tag{1.19}$$

Assuming Axioms W1 to W6 the vector $\Omega \in \mathcal{H}$ *is the unique vector satisfying* (1.17).

Remark 1.2.5. *As is shown in [6] and [7], one can always find self-adjoint extensions of the hermitian field operators, although it might be necessary to enlarge the Hilbert space.*

Proof. Consider the Borchers-Uhlmann algebra, which is the tensor algebra over the space of test functions:

$$\underline{\mathcal{S}} \doteq \bigoplus_{k=0}^{\infty} \mathcal{S}_n\,, \tag{1.20}$$

with $\mathcal{S}_0 = \mathbb{C}$ and $\mathcal{S}_n = \mathcal{S}(X)^n$. The elements of $\underline{\mathcal{S}}$ are linear combinations of finite sequences

$$\underline{f} \doteq \{f_0, \ldots f_N, 0, \ldots\} \ , \qquad N < \infty \ , \tag{1.21}$$

with $f_k = f_k^{(1)} \otimes \ldots \otimes f_k^{(k)}$ and $f_k^{(j)} \in \mathcal{S}(X)$. The product in $\underline{\mathcal{S}}$ is the tensor product

$$(\underline{f} \otimes \underline{g})_n(x_1, \ldots, x_n) := \sum_{k=0}^{n} f_k(x_1, ..., x_k) g_{n-k}(x_{k+1}, \ldots, x_n) \ . \tag{1.22}$$

For $g \in \mathcal{S}(X)$ define $\phi(g)\colon \underline{\mathcal{S}} \to \underline{\mathcal{S}}$ by

$$\phi(g)\{f_0, f_1, \ldots, f_n\} \doteq \{0,\, g f_0,\, g \otimes f_1, \ldots, g \otimes f_n\} \ . \tag{1.23}$$

Now define a sesquilinear form $\langle \underline{f}, \underline{g} \rangle$ for two terminating sequences $\underline{f} = \{f_k\}_{k \in \mathbb{N}_0}$ and $\underline{g} = \{g_j\}_{j \in \mathbb{N}_0}$ in $\underline{\mathcal{S}}$ by

$$\langle \underline{f}, \underline{g} \rangle = \sum_{j,k=0}^{\infty} W^{(j+k)} \left(\overline{f}_j \otimes g_k \right) \ . \tag{1.24}$$

The sesquilinear form $\langle \cdot\,, \cdot \rangle$ fulfils

(i) $\langle \underline{f}, \underline{g} \rangle = \overline{\langle \underline{g}, \underline{f} \rangle}$ (by Axiom W2);

(ii) $\|\underline{f}\| \doteq \langle \underline{f}, \underline{f} \rangle^{1/2} \geq 0$ (by Axiom W4).

Let $\mathcal{N}$ be the kernel of $\| \cdot \|$. The quotient space $\underline{\mathcal{S}}/\mathcal{N}$ is a pre-Hilbert space with inner product $\langle \cdot\,, \cdot \rangle$. Its completion is denoted by $\mathcal{H}$. As $\underline{\mathcal{S}}$ has a countable dense subspace, $\mathcal{H}$ is separable. $\mathcal{H}$ being a space of equivalence classes of Cauchy sequences with values in $\underline{\mathcal{S}}/\mathcal{N}$, Ω is identified with the equivalence class of the constant sequence $\{a, a, \ldots\}$ where $a = \{1, 0, \ldots\} \in \underline{\mathcal{S}}$.

By definition, $\phi(g)$ is symmetric. The kernel $\mathcal{N}$ of $\langle \cdot\,, \cdot \rangle$ is invariant under the application of $\phi(g)$: if $\|\underline{f}\| = 0$, then

$$\|\phi(g)\underline{f}\| = \langle \underline{f},\, \phi(\overline{g})\phi(g)\underline{f} \rangle \leq \|\underline{f}\| \; \|\phi(\overline{g})\phi(g)\underline{f}\| = 0 \ , \tag{1.25}$$

as a consequence of the Cauchy-Schwarz inequality. Thus the action of $\phi(g)$, $g \in \mathcal{S}(X)$, on $\underline{\mathcal{S}}/\mathcal{N}$ is well-defined and $\phi(g)$ can be lifted to a densely defined hermitian operator

$\phi(g)$ on $\mathcal{H}$. Equ. (1.17) is a consequence of Equ. (1.24) and the identification

$$|\underline{f}\rangle = \sum_{k=0}^{N} \phi(f_k^{(1)}) \cdots \phi(f_k^{(k)})\, \Omega\,. \tag{1.26}$$

(1.18) is a direct consequence of Axiom W3.

For a translation y define $U(y)\colon \underline{\mathcal{S}} \to \underline{\mathcal{S}}$ by

$$U(y)\underline{f} \doteq \{f_0, \mathfrak{t}_*^y f_1, \mathfrak{t}_*^y f_2, \ldots\}\ . \tag{1.27}$$

Thus, using the definition (1.23),

$$U(y)\phi(f)U(y)^* = \phi(\mathfrak{t}_*^y f)\ . \tag{1.28}$$

$\langle\cdot\,,\cdot\rangle$ is invariant under space-time translations by Axiom W5:

$$\|\underline{f}\| = 0 \quad \Rightarrow \quad \|U(y)\underline{f}\| = 0\ . \tag{1.29}$$

Thus the actions of $U(y)$, on $\underline{\mathcal{S}}/\mathcal{N}$ are well-defined. Because of (1.29) the space-time translations (1.27) extend to unitary operators $U(y)$ on $\mathcal{H}$ by the B.L.T. Theorem [59, Theorem I.7]. Strong continuity of $U(y)$ follows from the strong continuity of the map $y \mapsto U(y)$ on $\underline{\mathcal{S}}$ (see (1.27)).

Axiom W6 implies that Ω is the unique (up to a phase) translation invariant unit vector in $\mathcal{H}$: suppose there was another translation invariant unit vector $\Omega' \in \mathcal{H}$. Let (Ω'_n) be a sequence of vectors in $\underline{\mathcal{S}}/\mathcal{N}$ approximating Ω' in $\mathcal{H}$. Then, for y a translation in space-like direction,

$$\langle\Omega'_n, \Omega'_n\rangle = \lim_{\lambda\to\infty} \langle\Omega'_n, U(\lambda y)\Omega'_n\rangle = \langle\Omega'_n, \Omega\rangle\, \langle\Omega, \Omega'_n\rangle\,. \tag{1.30}$$

The second equality follows from Axiom W6. In the limit $n \to \infty$ the l.h.s. in Equ. (1.30) equals $\|\Omega'\|^2 = 1$, while the r.h.s. is equal to $|\langle\Omega', \Omega\rangle|^2$, which is less then 1, unless Ω' and Ω are equal, up to a phase.

It remains to show that axiom W6 implies uniqueness up to unitary equivalence. For this purpose assume that there exists a second set of objects $(\widehat{\mathcal{H}}, \widehat{\Omega}, \widehat{U}(\cdot), \widehat{\phi})$, which satisfies

(1.17)-(1.19). Then define $V : \underline{\mathcal{S}} \to \widehat{\mathcal{H}}$ by linear extension of the map

$$\underline{f} \mapsto |\widehat{\underline{f}}\rangle \equiv \sum_{k=0}^{N} \widehat{\phi}(f_k^{(1)}) \cdots \widehat{\phi}(f_k^{(k)})\, \widehat{\Omega}\,. \tag{1.31}$$

As the expectation values remain unchanged, i.e.

$$\widehat{\langle \underline{f} \,|\, \underline{g} \rangle} = \langle \underline{f} | \underline{g} \rangle\,, \qquad \underline{f}, \underline{g} \in \underline{\mathcal{S}}/\mathcal{N}\,, \tag{1.32}$$

the equivalence class $\widehat{|\underline{f}\rangle}$ remains unchanged, if $\underline{f}$ is changed by an element of $\mathcal{N}$. Hence V lifts to a map $V : \underline{\mathcal{S}}/\mathcal{N} \to \widehat{\mathcal{H}}$. Moreover, Equ. (1.32) implies that V is an isometry and therefore continuous. Thus we can extend V to a unitary operator from $\mathcal{H}$ to $\widehat{\mathcal{H}}$. One easily verifies that $V\Omega = \widehat{\Omega}$,

$$\widehat{\phi}(g) = V\phi(g)V^{-1}$$

and

$$\widehat{U}(a) = VU(a)V^{-1}\,. \qquad \square$$

Remark 1.2.6. *(i) The reconstruction theorem implies that in the translation invariant case the vectors*

$$|f_1, \ldots, f_k\rangle \doteq \phi(f_k) \ldots \phi(f_1)\Omega \in \mathcal{H}\,. \tag{1.33}$$

transform under translations as

$$U(y)|f_1, \ldots, f_k\rangle = |\mathfrak{t}_*^y f_1, \ldots, \mathfrak{t}_*^y f_k\rangle\,. \tag{1.34}$$

Note that at positive temperature the existence of state vectors, which obey such a simple transformation law, is not obvious from the outset.

(ii) Translation invariant Wightman distributions $W^{(n)}$ can be viewed as distributions in $n-1$ coordinates, cf. (1.3). *The resulting distributions are denoted by $\mathfrak{W}^{(n-1)}$.*

The following are the axioms of AQFT. They encode, what can reasonably be assumed about the observables, without any further specification of the physical system one is interested in. For more details the reader is referred to [34].

Axiom HK.1. *For every open, bounded $\mathcal{O} \subset X$ there exists a von Neumann algebra*[5] $\mathcal{A}(\mathcal{O})$.

Axiom HK.2. *For two bounded, open regions $\mathcal{O}_1 \subset X$ and $\mathcal{O}_2 \subset X$ there holds*[6]

$$\mathcal{A}(\mathcal{O}_1) \vee \mathcal{A}(\mathcal{O}_2) = \mathcal{A}(\mathcal{O}_1 \cup \mathcal{O}_2)\,. \tag{1.35}$$

Axiom HK.3 (Locality). *If two bounded, open $\mathcal{O}_1 \subset X$ and $\mathcal{O}_2 \subset X$ are space-like with respect to each other, there holds*

$$\forall A \in \mathcal{A}(\mathcal{O})_1,\ \forall B \in \mathcal{A}(\mathcal{O})_2 : \qquad [A, B] = 0\,. \tag{1.36}$$

Axiom HK.4 (Isotony). *For two bounded, open regions $\mathcal{O}_1 \subset \mathcal{O}_2 \subset X$ there holds*

$$\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)\,. \tag{1.37}$$

The map $\mathcal{O} \to \mathcal{A}(\mathcal{O})$ is a *net*. The C^*-inductive limit[7] of

$$\bigcup_{\mathcal{O}\subset X} \mathcal{A}(\mathcal{O})$$

is called the *algebra of local observables* - or observable algebra for short - and it is denoted by $\mathcal{A}$.

Axiom HK.5. *The space and time translations in X are implemented by a group of automorphisms α on $\mathcal{A}$,*

$$\forall y \in X : \qquad \alpha_y\big(\mathcal{A}(\mathcal{O})\big) = \mathcal{A}(\mathfrak{t}_y\mathcal{O})\,. \tag{1.38}$$

For $X = \mathbb{R}^2$ the following additional Axiom is to be expected[8].

Axiom HK.6. *If $X = \mathbb{R}^2$, the net $\mathcal{O} \to \mathcal{A}(\mathcal{O})$ is Lorentz covariant, i.e. for every bounded, open $\mathcal{O} \in \mathbb{R}^2$ there is an action $\alpha_\Lambda \in Aut(\mathcal{A})$ of $O(1,1)$ on $\mathcal{A}(\mathcal{O})$ such that*

$$\forall \Lambda \in O(1,1) : \qquad \alpha_\Lambda(\mathcal{A}(\mathcal{O})) = \mathcal{A}(\Lambda\mathcal{O})\,. \tag{1.39}$$

---

[5] The references for basic notions of operator algebras are [9, 44].

[6] $\mathcal{A}_1 \vee \mathcal{A}_2$ denotes the von Neumann algebra generated by the algebras $\mathcal{A}_1$ and $\mathcal{A}_2$.

[7] For this construction the reader is referred to [45, Proposition 11.4.1].

[8] The cylinder $\mathbb{R} \times S_\beta$ is not invariant under Lorentz transformations. Hence no such axiom will hold there.

Notice that, if the field ϕ from the Reconstruction Theorem is self-adjoint - in the present work this will always be the case - a Haag-Kastler net is simply obtained by passing to the *Weyl algebra*

$$\mathcal{A}(\mathcal{O}) := \{\mathrm{e}^{i\phi(f)} \mid \mathrm{supp}\, f \subset \mathcal{O}\}''. \tag{1.40}$$

Axioms HK.1 to HK.5 formalise the notion of observables of a theory. Taking measurements of these observables results in numbers. In the algebraic approach these numbers are given as states evaluated on elements of the observable algebra. Hence the Wightman functions in the Hamiltonian approach and the states in AQFT are analogous concepts. This claim is supported by the next theorem [9, Theorem 2.3.16], which puts the Wightman and the algebraic approach in line.

Theorem 1.2.7 (Gelfand-Naimark-Segal (GNS) Construction). *Let ω be a state on a C^*-algebra $\mathcal{A}$. Then there exists a Hilbert space $\mathcal{H}_\omega$, a vector $\Omega_\omega \in \mathcal{H}_\omega$ and a representation π_ω of $\mathcal{A}$ on $\mathcal{B}(\mathcal{H})$, such that*

(i) $\forall A \in \mathcal{A}: \quad \omega(A) = (\Omega_\omega, \pi_\omega(A)\,\Omega_\omega)$,

(ii) Ω_ω *is cyclic for* $\pi_\omega(A)$.

$(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ *is unique up to unitary equivalence.*

The next task is to characterise “interesting” states. The first of which are positive energy states [5] characterised by the *spectrum condition.*

Definition 1.2.8 (Spectrum Condition). *(i) Let ω be a state on a Haag-Kastler net $\mathcal{A}$. Let the space and time translations be unitarily implemented on the GNS Hilbert space, i.e. there are d-parameter groups of unitaries $U(y)$ such that for all $A \in \mathcal{A}$ and all $y \in X$*

$$\pi_\omega(\alpha_y A) = U(y)\pi_\omega(A)U(y)^*. \tag{1.41}$$

(d is the dimension of X.) ω is called a positive energy state*, if the generators of $U(y)$ have common spectrum in the forward light cone V^+.*

(ii) A quantum field theory, satisfying Axioms W1 to W5, fulfils the spectrum condition*, if the unitaries $U(y)$ in equation* (1.19) *have generators with common spectrum in the forward light cone V^+.*

The Spectrum Condition can be translated into analyticity properties of the Wightman distributions or of the functions

$$y \mapsto \omega\big(A\,\alpha_y(B)\big), \tag{1.42}$$

where $A, B \in \mathcal{A}$. Wightman distributions $W^{(n)}$, which fulfil the spectrum condition, are boundary values of analytic functions on $X - iV^+$ and their Fourier transforms $\widetilde{W}^{(n)}$ have support in the forward light cone [60, Theorem IX.32].

Requiring invariance under Lorentz transformations in addition to the Spectrum Condition defines vacuum states. But since none of the states considered in this work will be Lorentz invariant, these will be omitted here.

The next class of states are thermal equilibrium states, which are characterised by the KMS condition.

Definition 1.2.9 (KMS Condition). *(i) Let ω be a state over a von Neumann algebra or a C^*-algebra $\mathcal{A}$ and let $(\tau_t)_{t\in\mathbb{R}}$ be a strongly continuous automorphism group of $\mathcal{A}$. ω is called a* KMS state *with inverse temperature $\beta > 0$, if there exists for every $A \in \mathcal{A}$ and $B \in \mathcal{A}$ an analytic function $F_{A,B}$, which is analytic in the strip $\mathbb{R} + i\,(0, -\beta)$, continuous on the closure thereof, and which fulfils*

$$F_{A,B}(t) = \omega\big(\tau_t(A)\,B\big) \qquad \text{and} \qquad F_{A,B}(t - i\beta) = \omega\big(B\,\tau_t(A)\big), \tag{1.43}$$

for all $t \in \mathbb{R}$.

(ii) Wightman distributions $W^{(n)}$ satisfying Axioms W1 to W5 satisfy the KMS condition, if for every $(f_1, \ldots, f_n) \in \mathcal{S}(X)^n$ there exists an analytic function $F_{(f_1,\ldots,f_n)}$ on the strip $\mathbb{R} + i\,(0, -\beta)$ such that

$$F_{(f_1,\ldots,f_n)}(t) = (\Omega,\, \phi(f_1)\cdots\phi(f_k)\,U((t,0))^*\,\phi(f_{k+1})\cdots\phi(f_n)\,\Omega)\ , \tag{1.44}$$

and

$$F_{(f_1,\ldots,f_n)}(t - i\beta) = (\Omega,\, \phi(f_{k+1})\cdots\phi(f_n)\,U((t,0))\,\phi(f_1)\cdots\phi(f_k)\,\Omega)\ . \tag{1.45}$$

Remark 1.2.10. *(i) In thermal systems the generator of time translations - generally called* Liouvillean *L - has all of $\mathbb{R}$ as spectrum*[9] *[67]. Therefore the KMS Condition*

[9] The given reference actually treats the simpler case of fixed particle number. There is no reason why

is defined in terms of analyticity properties of correlation functions and not spectral properties of operators.

(ii) For the connection to Gibbs states and how to directly derive the KMS condition, using notions like passivity *and* stability*, the reader is referred to [34, V.1 and V.3].*

If a thermal state is defined on a *local* algebra satisfying axioms HK.1 to HK.5, it is expected to obey the stronger *Relativistic KMS Condition* [14].

Definition 1.2.11 (Relativistic KMS Condition). *(i) Let ω be a state over a von Neumann algebra or a C^*-algebra $\mathcal{A}$, let $(\alpha_{(t,x)})_{(t,x)\in\mathbb{R}^2}$ be a strongly continuous automorphism group of $\mathcal{A}$ and let $\beta > 0$. ω satisfies the* Relativistic KMS Condition*, if it is a β-KMS state with respect to $\alpha_{(\cdot,x)}$, for all $x \in \mathbb{R}$, and if $F_{A,B}$, defined by*

$$F_{A,B}(t,x) := \omega(\alpha_{t,x}(A)\,B)\,,$$

has an analytic continuation to $\mathbb{R}^2 - iV_\beta$.

(ii) Wightman distributions $W^{(n)}$ satisfying Axioms W1 to W5 satisfy the Relativistic KMS Condition*, if they satisfy the KMS condition, and if there exist analytic functions $F^{(n)}$ on a domain*

$$\lambda_1 \mathcal{T}_\beta \times \cdots \lambda_{n-1}\mathcal{T}_\beta\,, \tag{1.46}$$

where $0 < \lambda_j < 1$, $j \in \{1,\ldots,n-1\}$, $\sum_j \lambda_j = 1$, where $\mathcal{T}_\beta = X - iV_\beta$ and such that the associated Wightman distributions in relative coordinates, $\mathfrak{W}^{(n-1)}$, are the boundary values of $F^{(n)}$ in the sense of distributions.

Definition 1.2.12 (Commutant). *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$. The* commutant *$\mathcal{M}'$ is defined as*

$$\mathcal{M}' := \{A \in \mathcal{B}(\mathcal{H}) \mid \forall B \in \mathcal{M} :\ [A,B] = 0\}\,. \tag{1.47}$$

In thermal quantum field theory it is the case that the commutant of the algebra of observables is not trivial [34, Section V.1.4], i.e. it is bigger than $\mathbb{C}\,\mathbb{1}$. This fact has many consequences. In order to exhibit these, one passes[10] on to the weak closure of $\pi_\omega(\mathcal{A})$,

the Liouvillean should have simpler spectral properties for theories with non-fixed particle number.

[10]The reason why this is done is simply, that in the theory of von Neumann algebras more technical tools are available than for the C^* counterpart.

denoted by $\mathcal{M}$, where ω is the KMS state. By *von Neumann's Theorem* the resulting von Neumann algebra can be written as the double commutant of $\pi_\omega(\mathcal{A})$; $\mathcal{M} = \pi_\omega(\mathcal{A})''$. According to [9, Chapter 5.3] the state ω can be lifted to $\mathcal{M}$. Furthermore Ω_ω is not only cyclic but also separating for $\mathcal{M}$ [9, Corollary 5.3.9]. A cyclic and separating vector for a von Neumann algebra is the starting point of *Tomita-Takesaki Modular Theory*. Chapter 2 contains a short introduction into relative modular theory.

1.2.3 Euclidean Approach

Historically, the vacuum $P(\phi)_2$ theory was investigated before its thermal version. Since, in the vacuum case, the Hamiltonian - for this subsection denoted by H - is a positive operator, the semi-group $\{\mathrm{e}^{-tH} \mid t > 0\}$ is well-behaved and therefore amenable to direct construction. The desired real-time correlation functions can then be recovered as analytic continuations of the Schwinger Functions. That is, it is possible to construct the vacuum vector Ω and the field ϕ, such that the Schwinger functions are

$$S(t_1, x_1, \ldots, t_n, x_n) = \big(\Omega,\, \phi(x_1)\, \mathrm{e}^{-(t_2-t_1)H}\, \phi(x_2) \cdots \phi(x_{n-1})\, \mathrm{e}^{-(t_n-t_{n-1})H}\, \phi(x_n)\, \Omega\big)\,, \quad (1.48)$$

for $t_j - t_{j-1} > 0$. For the precise definitions of these objects in the vacuum case the reader is once more referred to [64]. In the thermal case, however, the Liouvillean is not bounded from below - as pointed out before (cf. [67]). That the Euclidean approach works despite this fact can be understood by viewing the field as weak operator valued solution of the Klein-Gordon Equation[11],

$$(\Box + \kappa^2)\phi(x) = -\, {:}P'(\phi(x)){:}\,, \quad (1.49)$$

where $\Box$ is the Klein-Gordon operator in two dimensions, x is a space-time point in $\mathbb{R}^2$, ${:}P'(\phi){:}$ is a Wick-ordered interaction polynomial in the field ϕ and κ is a mass parameter. This equation alone does not fix the solutions. The choice between these different fields is made by imposing analyticity properties on the resulting correlation functions (i.e. the Spectrum and the KMS Condition). The elliptic equation, which corresponds to (1.49) (i.e. (1.49), where t is replaced by it), is

$$(-\triangle + \kappa^2)\phi(x) = -\, {:}P'(\phi(x)){:}\,, \quad (1.50)$$

[11] Of course the field should not be evaluated at a point x. Instead a test function should be used. But, for this introduction, heuristic considerations will suffice.

$\triangle$ being the Laplace operator in two dimensions. In order to achieve the mentioned analyticity properties, equation (1.50) must be solved in different geometries. For the vacuum case the correct geometry is $\mathbb{R}^2$. In the thermal case the KMS condition demands analyticity in a time strip and equation (1.43). These can be realised by solving (1.50) on the cylinder $S_\beta \times \mathbb{R}$. Accordingly β will become the inverse temperature parameter from the KMS condition.

In Euclidean geometry there is of course no distinction between time or space coordinates. Therefore reconstructing the same solutions of (1.50) on $S_\beta \times \mathbb{R}$, and instead of choosing $\mathbb{R}$ in $S_\beta \times \mathbb{R}$ as spatial slices, interpreting the angular coordinate on S_β as the position, results in the vacuum $P(\phi)_2$ model on the circle S_β. Hence the Schwinger functions of the thermal model on $\mathbb{R}$ and the vacuum model on S_β are the same, but with interchanged interpretations of the coordinates. This is Nelson symmetry.

Although there is a construction of the vacuum $P(\phi)_2$ model on $\mathbb{R}$, which uses solely the Hamiltonian approach [24–27][12], the Euclidean construction [29, 64] is certainly simpler. Furthermore certain techniques, which give information about the spectrum of the Hamiltonian, are restricted to the Euclidean Approach. In the thermal case the situation seems to be slightly more complicated. There is no direct Hamiltonian construction of the interacting model as of yet. Furthermore, as pointed out before, the spectrum of the Liouvillean is unbounded in both directions - even in the free case.

For the vacuum case there are several different Euclidean axiom sets and reconstruction theorems. Most notably Nelson's Axioms and the Osterwalder-Schrader Axioms and the corresponding reconstruction theorems. Furthermore there is a version [47] of the Euclidean Approach, which incorporates the technical power of Nelson's formulation in terms of stochastic processes, the technical simplification the Osterwalder-Schrader reconstruction represents and the technical simplification, which comes from the use of bounded observables as in AQFT. Only this version of the Euclidean Approach has been generalised to the thermal case ([49] is a self-contained, complete review.). In what follows this thermal reconstruction theorem will be briefly recalled. Before the statement of the theorem a few notions have to be introduced.

Let $\omega = (\Omega, \cdot\, \Omega)$ be a KMS state on a C^*-algebra $\mathcal{M}$ with respect to the automorphism group $(\alpha_t)_{t\in\mathbb{R}}$. For $A_1, \ldots, A_n \in \mathcal{M}$ define the functions

$$\Gamma_{A_1,\ldots,A_n}(t_1, \ldots, t_n) = (\Omega,\, \alpha_{t_1}(A_1) \cdots \alpha_{t_n}(A_n)\, \Omega). \tag{1.51}$$

[12] A self-contained construction can be found in the thesis [71] (German language).

According to [49, Theorem 2.1 & Equation (3.1)][13] $\Gamma_{A_1,\dots,A_n}$ has an analytic continuation to

$$\{(t_1 + is_1, \dots, t_n + is_n) \in \mathbb{C}^n \mid -\beta/2 \leq s_1 \leq \dots \leq s_n \leq \beta/2\}\,.$$

Definition 1.2.13 (Stochastically Positive KMS Systems [49]).
A stochastically positive KMS system $(\mathcal{M}, \mathfrak{A}, (\alpha_t)_{t\in\mathbb{R}}, \omega)$ *at inverse temperature* $\beta > 0$ *consists of*

(i) a C^**-algebra* $\mathcal{M}$*,*

(ii) a one-parameter group of automorphisms $(\alpha_t)_{t\in\mathbb{R}}$ *of* $\mathcal{M}$*,*

(iii) an Abelian sub-C^**-algebra*[14] $\mathfrak{A}$ *of* $\mathcal{M}$*, such that the* C^**-algebra generated by*

$$\bigcup_{t\in\mathbb{R}} \alpha_t(\mathfrak{A})$$

is equal to $\mathcal{M}$*,*

(iv) a faithful state ω *on* $\mathcal{M}$ *satisfying the KMS condition at inverse temperature* β*, relative to* $(\alpha_t)_{t\in\mathbb{R}}$*;*

such that

(v) for all $A_1 \dots A_n \in \mathfrak{A}_+$ *there holds* stochastic positivity*, i.e. for* $0 < s_j < \beta/2$*,* $j \in \{1, \dots, n\}$*,*

$$\Gamma_{A_1,\dots,A_n}(is_1, \dots, is_n) \geq 0\,. \tag{1.52}$$

Let $(X_t)_{t\in T}$ be a *stochastic process* indexed by the set T with values in the compact Hausdorff space K. The underlying *probability space* is denoted by (Q, Σ, μ) and the expectation of a function $F \in L^1(Q, \Sigma, \mu)$ is denoted by

$$\langle F \rangle := \int_Q F \;\mathrm{d}\mu\,. \tag{1.53}$$

In what follows the notations, definitions and results of [49, Sections 4 and 5] are used freely.

[13] The multiple-time analyticity theorem is due to Araki.
[14] This algebra is to be thought of as the algebra of time-zero observables.

Theorem 1.2.14 (Klein & Landau [49]). *Let* $(\mathcal{M}, \mathfrak{A}, (\alpha_t)_{t\in\mathbb{R}}, \omega)$ *be a stochastically positive KMS system at inverse temperature* β*. Then there exists a stochastic process* $(X_t)_{t\in\mathbb{R}}$ *with values in the spectrum of* $\mathfrak{A}$*, which is periodic with period* β*, stationary, symmetric, weakly stochastically continuous, faithful, and Osterwalder-Schrader positive on* S_β*, such that*

$$\langle A_1(X_{t_1}) \cdots A_n(X_{t_n}) \rangle = \Gamma_{A_1,\dots,A_n}(it_1, \dots, it_n) \,. \tag{1.54}$$

Conversely, let $(X_t)_{t\in\mathbb{R}}$ *be a stochastic process with values in a compact Hausdorff space* K*, periodic with period* β*, stationary, symmetric, weakly stochastically continuous, faithful and Osterwalder-Schrader positive on* S_β*. Then there exists a stochastically positive KMS system* $(\mathcal{M}, \mathfrak{A}, (\alpha_t)_{t\in\mathbb{R}}, \omega)$ *at inverse temperature* β *such that* $\mathfrak{A} \cong C(K)$ *and* (1.54) *holds. Such a KMS system is unique up to isomorphism.*

For the current purpose the second part of the theorem is important. The proof of the theorem is not given here as the construction of the stochastic process and the reconstruction of the real time correlations will be done explicitly in Chapter 3.

1.3 Free Fields

Since the free field is the starting point for the construction of the interacting field, the necessary essentials about it will be treated here.

1.3.1 The Free Euclidean Field

The first step for constructing the free Euclidean field is to procure a measure on the chosen *path space*. The correlation functions are then defined as the moments of the measure.

In this work the (generalised) path space Q is defined as $Q := \mathcal{S}'_{\mathbb{R}}(S_\beta \times \mathbb{R})$. This is of course not a space of one-parameter paths in some manifold. Here the paths are indexed by the infinite dimensional set of Schwartz functions. The choice of Q is different compared to [49]. This is done because in the present formulation the construction of the quantum field as an operator valued distribution - something with which [49] is not concerned - is especially simple.

In the free case the existence of an appropriate measure on Q is guaranteed by *Minlos'* *theorem*[15]. Before stating it, a few things should be introduced. The evaluation map $\phi(f)$, $f \in \mathcal{S}_{\mathbb{R}}(S_\beta \times \mathbb{R})$,

$$\phi(f) \colon Q \to \mathbb{R}, \quad q \mapsto \langle q, f \rangle \,, \tag{1.55}$$

is defined in terms of the duality bracket $\langle \,.\,, \,.\, \rangle$. In the present context ϕ is called the *Euclidean quantum field*. The Borel σ-algebra Σ on Q is the minimal σ-algebra containing all open sets in the $\sigma(\mathcal{S}', \mathcal{S})$-topology.

Remark 1.3.1. *Every function $\widetilde{F} : \mathbb{R} \to \mathbb{C}$ gives rise to a function F on (Q, Σ, μ).*

$$F : Q \to \mathbb{C}, \quad q \mapsto F(\phi(f))(q) := \widetilde{F}(\langle q, f \rangle),$$

for every $f \in \mathcal{S}_{\mathbb{R}}(S_\beta \times \mathbb{R})$. Expressions like $\mathrm{e}^{i\phi(f)}$ in (1.57) below are to be understood in this way.

The translations $t_{(\alpha,x)}$ and reflections r and r' of distributions $q \in Q$ (defined in subsection 1.2.1) lift to translations $U(\alpha, x)$ and reflections R and R' of functions F on Q via

$$\left(U(\alpha, x)F\right)(q) = F(t_{\alpha,x}^{-1} q)\,, \qquad (RF)(q) = F(rq) \qquad \text{and} \qquad (R'F)(q) = F(r'q)\,. \tag{1.56}$$

Theorem 1.3.2 (Minlos [8, 20, 29, 53, 64]). *Let $\mathcal{S}$ be a linear topological space and let E be a functional on $\mathcal{S}$ such that*

(i) $E(0) = 1$,

(ii) $\mathcal{S} \ni f \mapsto E(f) \in \mathbb{C}$ *is continuous,*

(iii) $\sum_{i,j=1}^{n} z_i \bar{z}_j E(f_i - f_j) \geq 0$ *for all* $f_i, f_j \in \mathcal{S}$, $z_i, z_j \in \mathbb{C}$, $i, j \in \{1, \ldots, n\}$.

Then there exists a probability measure μ on $\mathcal{S}'$ such that

$$E(f) = \int_{\mathcal{S}'} \mathrm{e}^{i\phi(f)} \, \mathrm{d}\mu \,, \tag{1.57}$$

for every $f \in \mathcal{S}$.

[15] Minlos' theorem can be viewed as a generalisation of Bochner's theorem for infinite dimensional spaces.

Let C be the *covariance*

$$C(f_1, f_2) := \big(f_1, (D_\alpha^2 + D_x^2 + m^2)^{-1} f_2\big)\,, \quad f_1, f_2 \in \mathcal{S}(S_\beta \times \mathbb{R})\,, \tag{1.58}$$

where $m > 0$ is the mass parameter and $D_\alpha = -i\partial_\alpha$, $D_x = -i\partial_x$. The scalar product $(\,.\,,\,.\,)$ in (1.58) is the one in $L^2(S_\beta \times \mathbb{R})$. The *generating functional*

$$E(f) = \mathrm{e}^{-C(f,f)/2}, \quad f \in \mathcal{S}_\mathbb{R}(S_\beta \times \mathbb{R})\,, \tag{1.59}$$

satisfies the conditions of Minlos' theorem. The resulting measure $\mathrm{d}\phi_C$ on Q is called the *Gaussian measure* with mean zero and covariance C.

The moments of $\mathrm{d}\phi_C$ can be computed explicitly, cf. [64, Theorem II.16] and [64, Theorem III.1].

Theorem 1.3.3. *For $n \in \mathbb{N}$ and for a Gaussian measure $\mathrm{d}\phi_C$ with mean zero and covariance C there holds*

$$\int \mathrm{d}\phi_C\, \phi(f_1) \cdots \phi(f_{2n-1}) = 0\,, \tag{1.60}$$

and

$$\int \mathrm{d}\phi_C\, \phi(f_1) \cdots \phi(f_{2n}) = \sum_{pairs} C(f_{i_1}, f_{j_1}) \cdots C(f_{i_n}, f_{j_n})\,, \tag{1.61}$$

where the sum[16] *is over all $(2n)!/2^n n!$ ways of writing $\{1, \dots, 2n\}$ as n distinct (unordered) pairs $(i_1, j_1), \dots, (i_n, j_n)$.*

The *Euclidean two-point function* ($n = 1$) is therefore just the covariance C. Observe the following properties of the covariance [22, p. 131]:

(i) for $h_1, h_2 \in \mathcal{S}_\mathbb{R}(\mathbb{R})$ and $0 \le \alpha_1, \alpha_2 \le \beta$,

$$\begin{aligned}\lim_{k,k' \to \infty} & C\big(\delta_k(.-\alpha_1) \otimes h_1, \delta_{k'}(.-\alpha_2) \otimes h_2\big) \\ &= \Big(h_1, \frac{\mathrm{e}^{-|\alpha_1-\alpha_2|\epsilon} + \mathrm{e}^{-(\beta-|\alpha_1-\alpha_2|)\epsilon}}{2\epsilon(1-\mathrm{e}^{-\beta\epsilon})} h_2\Big)_{L^2(\mathbb{R},\,\mathrm{d}x)}\,,\end{aligned} \tag{1.62}$$

with $\epsilon := (D_x^2 + m^2)^{\frac{1}{2}}$;

[16] [64, Proposition II.1] is also helpful.

(ii) for $g_1, g_2 \in \mathcal{S}_{\mathbb{R}}(S_\beta)$ and $x_1, x_2 \in \mathbb{R}$,

$$\lim_{\kappa,\kappa'\to\infty} C\big(g_1 \otimes \delta_\kappa(. - x_1), g_2 \otimes \delta_{\kappa'}(. - x_2)\big) = \Big(g_1, \frac{\mathrm{e}^{-|x_1-x_2|\nu}}{2\nu} g_2\Big)_{L^2(S_\beta,\,\mathrm{d}\alpha)}, \tag{1.63}$$

with $\nu := (D_\alpha^2 + m^2)^{\frac{1}{2}}$.

Definition 1.3.4 (Free Schwinger Functions). *The* free Schwinger functions $S^{(n)}_{\mathit{free}} \in \mathcal{S}'\left((S_\beta \times \mathbb{R})^n\right)$ *are defined by*

$$S^{(n)}_{\mathit{free}}(f_1 \otimes \cdots \otimes f_n) := \int_Q \mathrm{d}\phi_C\, \phi(f_1) \cdots \phi(f_n)\,. \tag{1.64}$$

In (1.62) and (1.63) it is possible to take the limits of approximations of Dirac Delta functions $\delta(\cdot - y_j)$ also in the remaining test functions g_j and h_j as long as $y_i \neq y_j$, $i \neq j$, resulting in the free *non-coincident Schwinger functions*, denoted by the same symbol $S^{(n)}_{\text{free}}$, such that[17]

$$S^{(n)}_{\text{free}}(x_1, \ldots, x_n) = S^{(n)}_{\text{free}}(\delta^{(2)}(\cdot - x_1), \ldots, \delta^{(2)}(\cdot - x_n))\,, \tag{1.65}$$

where $x_j \in S_\beta \times \mathbb{R}$, $x_i \neq x_j$ for $i \neq j$ and where $\delta^{(2)}$ is the two-dimensional Dirac Delta on $S_\beta \times \mathbb{R}$.

1.3.2 The Free, Scalar Vacuum Field on the Circle S_β

A pedagogic introduction to the free, scalar bosonic field can be found in [60, Section X.7]. Here the free fields are introduced in a slightly different manner, so as to make the identification of Euclidean and Fock fields easy.

The Sobolev space $H^{-1/2}(S_\beta)$ over S_β is equipped with the inner product

$$(g_1, g_2) \mapsto \big(g_1, (2\nu)^{-1} g_2\big)_{L^2(S_\beta,\,\mathrm{d}\alpha)}\,, \qquad \nu = \big(D_\alpha^2 + m^2\big)^{\frac{1}{2}}\,. \tag{1.66}$$

Define $\mathcal{H}^{(0)}_C$ as the bosonic Fock space over $H^{-1/2}(S_\beta)$,

$$\mathcal{H}^{(0)}_C := \bigoplus_{n=0}^{\infty} H^{-1/2}(S_\beta)^{\otimes_s n}, \tag{1.67}$$

[17] No Confusion will arise from this abuse of notation, as it will always be clear whether the entries of $S^{(n)}_{\text{free}}$ are functions or points.

where $\otimes_s$ denotes the symmetric tensor product and where $H^{-1/2}(S_\beta)^{\otimes_s 0} := \mathbb{C}$. The vacuum vector for the free field is denoted by $\Omega_C^{(0)} := (1, 0, 0, \dots) \in \mathcal{H}_C^{(0)}$. On each level of $\mathcal{H}_C^{(0)}$ let $a^*(g)$ act as

$$a^*(g)\,(g_{n,1} \otimes_s \cdots \otimes_s g_{n,n}) := \sqrt{n+1}\, g \otimes_s g_{n,1} \otimes_s \cdots \otimes_s g_{n,n}\,, \qquad g \in H^{-1/2}(S_\beta). \quad (1.68)$$

Denote the closure of $a^*(g)$ by the same symbol and denote the adjoint of $a^*(g)$ by $a(g)$, such that $a(g)^* = a^{*(g)}$. These are used to define the *time-zero field operator*. Denote the subspace of real-valued elements of the Sobolev space by $H_{\mathbb{R}}^{-1/2}(S_\beta) \subset H^{-1/2}(S_\beta)$. For *real-valued* $g \in H_{\mathbb{R}}^{-1/2}(S_\beta)$ define

$$\phi_C^{(0)}(g) := \frac{1}{\sqrt{2}}\big(a(\nu^{-1/2}g) + a^*(\nu^{-1/2}g)\big)^- . \quad (1.69)$$

$\phi_C^{(0)}(g)$ is unbounded and self-adjoint [60, Theorem X.41 (a)]. The free Hamiltonian is defined as

$$H_C^{(0)} = \mathrm{d}\Gamma(\nu), \quad (1.70)$$

where $\mathrm{d}\Gamma$ is the *second quantisation*. On each level $H^{-1/2}(S_\beta)^{\otimes_s n}$ of $\mathcal{H}_C^{(0)}$ the action of $\mathrm{d}\Gamma(A)$ for an operator A on $H^{-1/2}(S_\beta)$ is given by

$$A \otimes 1\!\!1 \otimes \cdots \otimes 1\!\!1 + 1\!\!1 \otimes A \otimes \cdots \otimes 1\!\!1 + 1\!\!1 \otimes \cdots \otimes 1\!\!1 \otimes A\,. \quad (1.71)$$

The free Wightman distributions $W_{C,\mathrm{free}}^{(n)}$ are defined by

$$W_{C,\mathrm{free}}^{(n)}(t_1, g_1, \dots, t_n, g_n) = \Big(\Omega_C^{(0)},\, \phi_C^{(0)}(g_1)\mathrm{e}^{-i(t_1-t_2)H_C^{(0)}} \cdots \phi_C^{(0)}(g_{n-1})\mathrm{e}^{-i(t_{n-1}-t_n)H_C^{(0)}}\phi_C^{(0)}(g_n)\,\Omega_C^{(0)}\Big)\,. \quad (1.72)$$

A simple calculation establishes the following proposition.

Proposition 1.3.5. *(i) The free two-point function*

$$W_{C,\mathit{free}}^{(2)} \colon \big(\mathbb{R} \times \mathcal{S}_{\mathbb{R}}(S_\beta)\big)^2 \to \mathbb{C} \quad (1.73)$$

for the free scalar field on the circle is given by

$$W^{(2)}_{C,free}(t_1, g_1, t_2, g_2) := \left(g_1, \frac{\mathrm{e}^{-i(t_1-t_2)\nu}}{2\nu}\, g_2 \right) . \tag{1.74}$$

(ii) For $n \in \mathbb{N}$ the Wightman distributions

$$W^{(n)}_{C,free} \colon \big(\mathbb{R} \times \mathcal{S}_{\mathbb{R}}(S_\beta)\big)^n \mapsto \mathbb{C} \tag{1.75}$$

for the free scalar field on the circle are given by $W^{(2n-1)}_{C,free} := 0$ and by

$$W^{(2n)}_{C,free}(t_1, g_1, \ldots, t_{2n}, g_{2n}) := \sum_{pairs} \prod_{i_l, j_k} W^{(2)}_{C,free}(t_{i_l}, g_{i_l}, t_{j_k}, g_{j_k}) , \tag{1.76}$$

where the sum is over all $(2n)!/2^n n!$ ways of writing $\{1, \ldots, 2n\}$ as n distinct (unordered) pairs $(i_1, j_1), \ldots, (i_n, j_n)$.

Note that, using (1.63), there holds

$$\begin{aligned} \Big(\Omega^{(0)}_C,\ \phi^{(0)}_C(g_1)\, \mathrm{e}^{-tH^{(0)}_C}\, \phi^{(0)}_C(g_2)\, \Omega^{(0)}_C\Big) &= \left(g_1, \frac{\mathrm{e}^{-t\nu}}{2\nu} g_2\right) \\ &= \int \mathrm{d}\phi^{(0)}_C\ \phi(g_1 \otimes \delta)\, \phi(g_2 \otimes \delta(\cdot - t)) , \end{aligned} \tag{1.77}$$

as long as $t > 0$ and $g_1, g_2 \in H^{-1/2}_{\mathbb{R}}(S_\beta)$. Evidently the analytic continuations in the time variables of $W^{(n)}_{C,\mathrm{free}}$ coincide with the Schwinger functions $S^{(n)}_{\mathrm{free}}$ for $g_j \in \mathcal{S}_{\mathbb{R}}(S_\beta)$ and $t_j - t_{j+1} > 0$, $j \in \{1, \ldots, n\}$,

$$W^{(n)}_{C,\mathrm{free}}(-it_1, g_1, \ldots, -it_n, g_n) = S^{(n)}_{\mathrm{free}}(g_1 \otimes \delta(\cdot - t_1), \ldots, g_n \otimes \delta(\cdot - t_n)) . \tag{1.78}$$

The free scalar field on $S_\beta \times \mathbb{R}$ is a weak operator-valued solution to the Klein-Gordon equation. This follows either from [60, Theorem X.42] or by simply noting that the two-point function $W^{(2)}_{C,\mathrm{free}}$ is a weak solution of the Klein-Gordon equation and using Definition 1.3.5.

Instead of the fields one can regard the *Weyl-operators* $\mathrm{e}^{i\phi_C(g)}$ as the principal objects. This is more in the spirit of AQFT. The advantage of the Weyl-operators is their boundedness, their drawback is the non-continuity (in operator norm) in the test function g.

1.3.3 The Free, Scalar Thermal Field on $\mathbb{R}$

In contrast to the free vacuum field the free, thermal field is constructed using *two* copies of Fock space. In physics literature, this is known as "doubling of degrees of freedom".

Some terminology has to be fixed before the thermal field can be defined. Let $\mathfrak{h}$ be a complex vector space. Then the *conjugate vector space* $\overline{\mathfrak{h}}$ is defined as the real vector space $\mathfrak{h}$ equipped with the complex structure $-i$. If h is in $\mathfrak{h}$, the corresponding element in $\overline{\mathfrak{h}}$ is denoted by $\overline{h}$. If a is a linear operator on $\mathfrak{h}$, then $\overline{a}$ denotes the operator on $\overline{\mathfrak{h}}$, which acts as $\overline{a}\overline{h} := \overline{ah}$.

The Sobolev space $H^{-1/2}(\mathbb{R})$ is equipped with the inner product

$$(h_1, h_2) \mapsto \big(h_1, (2\epsilon)^{-1} h_2\big), \qquad \epsilon = (D_x^2 + m^2)^{\frac{1}{2}} . \tag{1.79}$$

The Araki-Woods Hilbert space $\mathcal{H}_{AW}$ is defined as the Fock space over $H^{-1/2}(\mathbb{R}) \oplus \overline{H^{-1/2}(\mathbb{R})}$,

$$\mathcal{H}_{AW} := \bigoplus_{n=0}^{\infty} \Big(H^{-1/2}(\mathbb{R}) \oplus \overline{H^{-1/2}(\mathbb{R})}\Big)^{\otimes_s n} , \tag{1.80}$$

with the same conventions for the symmetric tensor product as in the preceding subsection. Define $\Omega_{AW} := (1, 0, 0, \ldots) \in \mathcal{H}_{AW}$ and set $\rho := (e^{\beta\epsilon} - 1)^{-1}$. The next task is to define the annihilation and creation operators for the thermal case. On each level of $\mathcal{H}_{AW}$ let $a^*(h_1, \overline{h}_2)$ act as

$$\begin{aligned} a^*(h_1, \overline{h}_2) \ &\big((h_{n,1} \oplus \overline{h}_{n,1}) \otimes_s \cdots \otimes_s (h_{n,n} \oplus \overline{h}_{n,n})\big) \\ &:= \sqrt{n+1}\, (h_1 \oplus \overline{h}_2) \otimes_s (h_{n,1} \oplus \overline{h}_{n,1}) \otimes_s \cdots \otimes_s (h_{n,n} \oplus \overline{h}_{n,n}) . \end{aligned} \tag{1.81}$$

Denote the closure of $a^*(h_1, \overline{h}_2)$ by the same symbol. Denote the adjoint of $a^*(h_1, \overline{h}_2)$ by $a(h_1, \overline{h}_2)$, such that $a(h_1, \overline{h}_2)^* = a^*(h_1, \overline{h}_2)$. Furthermore let the Fock field and the *Weyl* operators be defined by

$$\phi_F(h_1, \overline{h}_2) := (a^*(\epsilon^{-1/2} h_1, \epsilon^{-1/2} \overline{h}_2) + a(\epsilon^{-1/2} h_1, \epsilon^{-1/2} \overline{h}_2))/\sqrt{2} \tag{1.82}$$

and by $W_F(h_1, \overline{h}_2) = \mathrm{e}^{i\phi_F(h_1, \overline{h}_2)}$, respectively. The *Araki-Woods* representation of the Weyl relations is given by

$$W_{AW}(h) := W_F((1+\rho)^{1/2} h, \overline{\rho}^{1/2} \overline{h}) . \tag{1.83}$$

It is possible to define an Araki-Woods field operator

$$\phi_{AW}(h) := \phi_F\left((1+\rho)^{1/2}h, \overline{\rho}^{1/2}\overline{h}\right) = -i\,\frac{\mathrm{d}}{\mathrm{d}s}W_{AW}(sh)\bigg|_{s=0}\,. \tag{1.84}$$

$\phi_{AW}(h)$ is the *free thermal time zero field* for $h \in H_{\mathbb{R}}^{-1/2}$. $\phi_{AW}(h)$ is unbounded and self-adjoint, the proof of which is similar to the one of [60, Theorem X.41 (a)]. The free Liouvillean is defined as

$$L_{AW} := \mathrm{d}\Gamma(\epsilon \oplus -\overline{\epsilon})\,. \tag{1.85}$$

($\mathrm{d}\Gamma$ as in (1.71).) The free thermal Wightman distributions are defined by

$$\begin{aligned} W^{(n)}_{\beta,\mathrm{free}}(t_1,h_1,\ldots,t_n,h_n) := \big(\Omega_{AW},\ \phi_{AW}(h_1)\mathrm{e}^{-i(t_1-t_2)L_{AW}}\cdots \\ \cdots\phi_{AW}(h_{n-1})\mathrm{e}^{-i(t_{n-1}-t_n)L_{AW}}\phi_{AW}(h_n)\,\Omega_{AW}\big)\,. \end{aligned} \tag{1.86}$$

Proposition 1.3.6. *(i) The free two-point function*

$$W^{(2)}_{\beta,free}\colon \left(\mathbb{R}\times\mathcal{S}_{\mathbb{R}}(\mathbb{R})\right)^2 \to \mathbb{C} \tag{1.87}$$

for the free scalar field on the circle is given by

$$W^{(2)}_{\beta,free}(t_1,h_1,t_2,h_2) := \left(h_1, \frac{\mathrm{e}^{-i(t_1-t_2)\epsilon} + \mathrm{e}^{-(\beta-i(t_1-t_2))\epsilon}}{2\epsilon(1-\mathrm{e}^{-\beta\epsilon})}\,h_2\right)\,. \tag{1.88}$$

(ii) For $n \in \mathbb{N}$ the Wightman distributions

$$W^{(n)}_{\beta,free}\colon \left(\mathbb{R}\times\mathcal{S}_{\mathbb{R}}(\mathbb{R})\right)^n \mapsto \mathbb{C} \tag{1.89}$$

for the free scalar field on the circle are given by $W^{(2n-1)}_{\beta,free} := 0$ and by

$$W^{(2n)}_{\beta,free}(t_1,h_1,\ldots,t_{2n},h_{2n}) := \sum_{pairs}\prod_{i_l,j_k} W^{(2)}_{\beta,free}(t_{i_l},h_{i_l},t_{j_k},h_{j_k})\,, \tag{1.90}$$

where the sum is over all $(2n)!/2^n n!$ ways of writing $\{1,\ldots,2n\}$ as n distinct (unordered) pairs $(i_1,j_1),\ldots,(i_n,j_n)$.

Now for $h_1, h_2 \in H_{\mathbb{R}}^{-1/2}(\mathbb{R})$ and $0 < t < \beta$ there holds

$$\begin{aligned}\big(\Omega_{AW},\, \phi_{AW}(h_1)\, \mathrm{e}^{-tL_{AW}}\, \phi_{AW}(h_2)\, \Omega_{AW}\big) &= \left(h_1, \frac{\mathrm{e}^{-t\epsilon} + \mathrm{e}^{-(\beta-t)\,\epsilon}}{2\epsilon(1-\mathrm{e}^{-\beta\epsilon})}\, h_2\right) \\ &= \int \mathrm{d}\phi_C\, \phi(\delta \otimes h_1)\, \phi(\delta(\cdot - t) \otimes h_2) \end{aligned} \tag{1.91}$$

by (1.62). Again the free Schwinger functions $S^{(n)}_{\text{free}}$ are the analytic continuations of the free Wightman functions $W^{(n)}_{\beta,\text{free}}$,

$$W^{(n)}_{\beta,\text{free}}(-it_1, h_1, \ldots, -it_n, h_n) = S^{(n)}_{\text{free}}(\delta(\cdot - t_1) \otimes h_1, \ldots, \delta(\cdot - t_n) \otimes h_n) \tag{1.92}$$

for $t_j \in (0, \beta)$, $t_j - t_{j+1} > 0$ and $h_j \in \mathcal{S}_{\mathbb{R}}(\mathbb{R})$, $j \in \{1, \ldots, n\}$.

As the two-point function $W^{(2)}_{\beta,\text{free}}$ is a weak solution to the Klein-Gordon equation, so is the free, scalar, thermal field.

It is also possible to define two operators $a^*_{AW}(h)$ and $a_{AW}(h)$ by

$$\phi_{AW}(h) = \frac{1}{\sqrt{2}}\,(a^*_{AW}(h) + a_{AW}(h)) \tag{1.93}$$

and by

$$\pi_{AW} = \frac{i}{\sqrt{2}}\,(a^*_{AW}(h) - a_{AW}(h))\ . \tag{1.94}$$

This results in

$$a_{AW}(h) = a\left((1+\rho)^{1/2}h, 0\right) + a^*\left(0, \overline{\rho}^{1/2}\overline{h}\right) \tag{1.95}$$

and

$$a^*_{AW}(h) = a^*\left((1+\rho)^{1/2}h, 0\right) + a\left(0, \overline{\rho}^{1/2}\overline{h}\right)\ . \tag{1.96}$$

However, these cannot be considered as true ladder operators as $a_{AW}(h)\Omega_{AW} \neq 0$.

Chapter 2

The Hölder Inequality for KMS States

2.1 The Inequality

The starting point of this chapter is a KMS state ω_β on a C^*-algebra $\mathcal{A}$. The KMS-condition implies that ω_β is invariant under τ [10, Proposition 5.3.3] and therefore the latter can be unitarily implemented in the GNS representation $(\pi, \mathcal{H}, \Omega)$ associated to the pair $(\mathcal{A}, \omega_\beta)$. Weak continuity of τ ensures the existence of a generator L, the Liouvillean, such that $\pi(\tau_t(A))\Omega = \mathrm{e}^{-itL}\pi(A)\Omega$ and $L\Omega = 0$.

As the vector Ω is cyclic and separating for the von Neumann algebra $\mathcal{M} \doteq \pi(\mathcal{A})''$, the algebraic operations on $\mathcal{M}$ define maps on the dense set $\mathcal{M}\Omega \subset \mathcal{H}$. Tomita's idea to study the $*$-operation on $\mathcal{M}$ turned out to be especially fruitful. It leads to an anti-linear operator $S_\circ$,

$$S_\circ \colon A\Omega \mapsto A^*\Omega \, , \qquad A \in \mathcal{M} \, ,$$

which is closable, and thus allows a polar decomposition for the closure $S = J\Delta^{1/2}$. The anti-linear involution J is called the *modular conjugation* and the positive albeit in general unbounded operator Δ is called the *modular operator*. The modular conjugation J satisfies $J^* = J$ and $J^2 = \mathbb{1}$, and induces a $*$-anti-isomorphism $j \colon A \mapsto JA^*J$ between the algebra $\mathcal{M}$ and its commutant $\mathcal{M}'$ (*Tomita's theorem*). For an introduction to modular theory the reader is referred to [9, Section 2.5] or [34, Section 5.2].

More generally, an arbitrary normal faithful state over a von Neumann algebra $\mathcal{M}$ is a $(\sigma, -1)$-KMS state with respect to the modular automorphisms σ given by $A \mapsto$

$\Delta^{is} A \Delta^{-is}$, $A \in \mathcal{M}$, $s \in \mathbb{R}$, at temperature[1] $\beta = -1$ (see, e.g., [10]). To be precise, the strong continuity assumption, which is part of Definition 1.2.9, holds on the restricted C^*-dynamical system [62, Proposition 1.18] associated to the W^*-dynamical system $(\mathcal{M}, \sigma)$. Uniqueness of the modular automorphism ensures that $\Delta^{1/2} = \mathrm{e}^{-\beta L/2}$.

The *standard positive cone* $\mathcal{P}^\sharp \subset \mathcal{H}$ is defined as

$$\mathcal{P}^\sharp \doteq \overline{\{JAJA\Omega : A \in \mathcal{M}\}} = \overline{\{\Delta^{1/4} A\Omega : A \in \mathcal{M}^+\}} \, ,$$

where the bar denotes norm closure [2]. Consequently, a KMS state on a C^*-dynamical system $(\mathcal{A}, \tau)$ gives rise to a von Neumann algebra in *standard form*, namely a quadruple $(\mathcal{H}, \mathcal{M}, J, \mathcal{P}^\sharp)$, where $\mathcal{H}$ is a Hilbert space, $\mathcal{M}$ is a von Neumann algebra, J is an anti-unitary involution on $\mathcal{H}$ and $\mathcal{P}^\sharp$ is a self-dual cone in $\mathcal{H}$ such that:

(i) $J\mathcal{M}J = \mathcal{M}'$;

(ii) $JAJ = A^*$ for A in the centre of $\mathcal{M}$;

(iii) $J\Psi = \Psi$ for $\Psi \in \mathcal{P}^\sharp$;

(iv) $AJA\mathcal{P}^\sharp \subset \mathcal{P}^\sharp$ for $A \in \mathcal{M}$.

The vector state induced by Ω extends the KMS state ω_β from $\mathcal{A}$ to $\mathcal{M}$, and we denote this state by the same symbol. Now set, for even $p \in \mathbb{N}$ and a *positive* operator A

$$|\!|\!| A |\!|\!|_p := \Big((\mathrm{e}^{-\beta L/2p} \, A \, \mathrm{e}^{-\beta L/2p})^{p/2} \, \Omega, \, (\mathrm{e}^{-\beta L/2p} \, A \, \mathrm{e}^{-\beta L/2p})^{p/2} \, \Omega \Big)^{1/p} \, , \tag{2.1}$$

if the r.h.s. exists. The rest of this chapter is devoted to the proof of the following theorem [42]. It will be used in Subsection 3.4.3 to prove the existence of the Wightman distributions for the thermal $P(\phi)_2$ model.

Theorem 2.1.1 (Hölder inequality). *Consider a (τ, β)-KMS state ω_β over a C^*-dynamical system $(\mathcal{A}, \tau)$. Let $(z_1, \ldots, z_n) \in \mathbb{C}^n$ and $1 \leq m \leq n$ be such, that $0 \leq \Re z_j$, $\sum_{j=1}^m \Re z_j \leq 1/2$ and $\sum_{j=m+1}^n \Re z_j \leq 1/2$. Furthermore let p_j be the smallest, positive, even integer such that*

$$\frac{1}{p_j} \leq \min\{\Re z_{j+1}, \Re z_j\} \, , \tag{2.2}$$

[1] Any KMS state with parameter β can be transformed into one with parameter -1 by rescaling time [10, p. 77]. Since it is customary to work with temperature $\beta = -1$ in mathematics literature, this convention will be adopted for this chapter.

with $z_{n+1} = z_n$ *and* $z_0 = z_1$. *Then*

$$\left|\left(\mathrm{e}^{-z_{m+1}L}\, A_{m+1} \cdots \mathrm{e}^{-z_n}\, A_n\, \Omega,\; \mathrm{e}^{-z_m L}\, A_m \cdots \mathrm{e}^{-z_1}\, A_1\, \Omega\right)\right| \leq |||\, A_0 |||_{p_0} \cdots |||\, A_n |||_{p_n} \tag{2.3}$$

for all $A_1, \ldots, A_n \in \mathcal{M}^+$.

Remark 2.1.2. *(i) Although the multi-boundary Poisson kernels [62, Lemma 4.4.8] for the domain* $I^{(n)}_{\beta/2}$ *(defined in* (2.5) *below) can be computed explicitly (the computation can be traced back to Widder [70]), it seems unlikely that the Hölder inequality (2.3) can be derived using only methods of complex analysis (unless* $n = 2$*).*

(ii) Let $\mathcal{M}_0$ *denote a weakly dense sub-algebra of entire analytic[2] elements in* $\mathcal{M}$*. It follows that, for* $p \in \mathbb{N}$ *and* $A \in \mathcal{M}_0^+$*,*

$$|||\, A |||_p = \omega_\beta\left(\tau_{i\beta/2p}(A) \cdots \tau_{i(2p-1)\beta/2p}(A)\tau_{i\beta}(A)\right)^{1/p} . \tag{2.4}$$

Thus Theorem 2.1.1 is a generalisation of the Hölder inequality for Gibbs states, as stated, for example, in [51, 52].

(iii) Fröhlich conjectured an inequality similar to (2.3) *in [19].*

Two more aspects of Theorem 2.1.1 are notable. Firstly, it estimates a non-commutative expression in terms of essentially commutative ones, which can be evaluated using spectral theory, and secondly, the bounds are uniform in $\Im z_j$, $j \in \{1, \ldots, n\}$. The proof of Theorem 2.1.1 relies on the theory of non-commutative L^p-spaces, but the appeal of the theorem may well be that knowledge of non-commutative integration theory is not required in order *to apply* the inequality.

In quantum statistical mechanics the uniformity in imaginary time is useful for establishing the existence of real time Greens functions from the Schwinger functions. A direct application of (2.3) is given in the next chapter. Additionally, in a forthcoming work by M. Rouleux and C. Jäkel, the Hölder inequality is used to show that the Wightman distributions of the $P(\phi)_2$ model on the de Sitter space satisfy a micro-local spectrum condition.

[2] An element $A \in \mathcal{M}$ is called analytic for τ_t if there exists a strip $I_\lambda = \{z \in \mathbb{C} : |\Im z| < \lambda\}$ in $\mathbb{C}$, and a function $f\colon I_\lambda \mapsto \mathcal{M}$, such that (i) $f(t) = \tau_t(A)$ for $t \in \mathbb{R}$, and (ii) $z \mapsto \phi(f(z))$ is analytic for all $\phi \in \mathcal{M}_*$ [9, Definition 2.5.20].

2.2 Non-commutative L_p-spaces

Normal states over von Neumann algebras provide a non-commutative extension of classical integration theory, i.e. commutative L^p-spaces, and one recovers the latter in case the algebra is Abelian [54]. Among the many approaches to non-commutative L_p-spaces [16, 35, 39, 50, 57, 63, 68], Araki's and Masuda's approach [4] is best suited for the present purpose. The next two subsections contain short introductions to relative modular operators for weights and non-commutative L_p spaces, respectively. A complete treatment of relative modular operators can be found in [66]. Relative modular operators for finite dimensional Hilbert spaces are laid out in Appendix A.

2.2.1 Relative Modular Operators

Let $\mathcal{M}$ be a σ-finite von Neumann algebra. Furthermore let ϕ be a normal semi-finite weight on $\mathcal{M}$, i.e. $\phi\colon \mathcal{M}_+ \to [0,\infty]$ such that [66, Definition 1.1]

(i) for A and B in $\mathcal{M}_+$ and $\lambda \geq 0$ there holds $\phi(A+B) = \phi(A) + \phi(B)$ and $\phi(\lambda A) = \lambda\phi(A)$, where the convention $0(+\infty) = 0$ is used;

(ii) $\{A \in \mathcal{M}_+ \mid \phi(A) < \infty\}$ generates $\mathcal{M}$;

(iii) and for every bounded increasing net (A_j) in $\mathcal{M}_+$ there holds $\phi(\sup A_j) = \sup \phi(A_j)$.

The semi-cyclic representation[3] [66] makes it possible to define an anti-linear operator $S_{\phi,\Omega}$ by

$$S_{\phi,\Omega} A\Omega := \xi_\phi(A^*) \,, \qquad A \in \mathcal{N}_\phi^* \,,$$

where $\mathcal{N}_\phi \doteq \{A \in \mathcal{M} : \phi(A^*A) < \infty\}$, and $\xi_\phi(A)$ is the semi-cyclic representation of $A \in \mathcal{N}_\phi$ in

$$\mathcal{H}_\phi := \overline{\mathcal{N}_\phi / \ker \phi} \,.$$

$S_{\phi,\Omega}$ is closable and the closure $\overline{S_{\phi,\Omega}}$ has a polar decomposition $\overline{S_{\phi,\Omega}} \doteq J_{\phi,\Omega}\Delta_{\phi,\Omega}^{1/2}$. It is noteworthy that

$$\Delta_{\phi,\Omega} = S_{\phi,\Omega}^* \overline{S_{\phi,\Omega}} \,,$$

is a positive, in general unbounded, operator on the *original* Hilbert space $\mathcal{H}$. If ϕ is a vector state associated to $\xi \in \mathcal{H}$ such that $\phi(x) = (\xi, x\xi)$, then $\xi_\phi(A) = A\xi$ and we denote

[3] The semi-cyclic representation is a generalisation of the GNS representation to weights.

$\Delta_{\phi,\Omega}$ by $\Delta_{\xi,\Omega}$ and $J_{\phi,\Omega}$ by $J_{\xi,\Omega}$. In order to keep the notation simple, $\mathrm{e}^{-\beta L/2}$ will from now on be written as $\Delta^{1/2} \equiv \Delta^{1/2}_{\Omega,\Omega}$.

A key role in the proof of Theorem 2.1.1 will be played by the following estimate: define, for any $\alpha > 0$, a set

$$I_\alpha^{(n)} \doteq \{(z_1, \ldots, z_n) \in \mathbb{C}^n : \sum_{j=1}^n \Re z_j \leq \alpha,\ 0 \leq \Re z_j\} . \tag{2.5}$$

Let $z \in I^{(n)} \equiv I_1^{(n)}$ and $z'_m, z''_m \in \mathbb{C}$ be such that $\Re z'_m, \Re z''_m > 0$, $z'_m + z''_m = z_m$ and

$$\Re z_1 + \ldots \Re z_{m-1} + \Re z''_m \leq 1/2 , \tag{2.6}$$

$$\Re z_n + \ldots \Re z_{m+1} + \Re z'_m \leq 1/2 . \tag{2.7}$$

Under these conditions, Araki [4, Lemma A] has shown[4] that for $\phi_1, \ldots, \phi_n \in \mathcal{M}_*^+$ and $X_0, \ldots, X_n \in \mathcal{M}$

$$\begin{aligned} &\Big|\Big(\Delta^{\bar{z}'_m}_{\phi_m,\Omega} X^*_m\ \Delta^{\bar{z}_{m+1}}_{\phi_{m+1},\Omega} \ldots \Delta^{\bar{z}_n}_{\phi_n,\Omega} X^*_n \Omega, \Delta^{z''_m}_{\phi_m,\Omega} X_{m-1} \Delta^{z_{m-1}}_{\phi_{m-1},\Omega} \ldots \Delta^{z_1}_{\phi_1,\Omega} X_0 \Omega\Big)\Big| \\ &\qquad \leq \Big(\prod_{j=0}^n \|X_j\|\Big) \underbrace{(\Omega, \mathbb{1}\Omega)^{z_0}}_{=1} \Big(\prod_{j=1}^n \phi_j(\mathbb{1})^{\Re z_j}\Big) , \end{aligned} \tag{2.8}$$

with $z_0 = 1 - \sum_{j=1}^n \Re z_j$. This inequality is the direct generalisation of the finite-dimensional case (A.10).

2.2.2 Positive Cones and L_p-Spaces for von Neumann Algebras

Consider a general (σ-finite) von Neumann algebra $\mathcal{M}$ in standard form with cyclic and separating vector Ω. For $2 \leq p \leq \infty$, Araki and Masuda define [4, Equ. (1.3), p. 340]

$$L_p(\mathcal{M}, \Omega) \doteq \Big\{\zeta \in \bigcap_{\xi \in \mathcal{H}} D\big(\Delta_{\xi,\Omega}^{\frac{1}{2}-\frac{1}{p}}\big) : \|\zeta\|_p < \infty\Big\},$$

where

$$\|\zeta\|_p = \sup_{\|\xi\|=1} \|\Delta_{\xi,\Omega}^{\frac{1}{2}-\frac{1}{p}} \zeta\| .$$

[4]Note that, in contrast to [4], the inner product here is linear in the second entry.

For $1 \leq p < 2$, $L_p(\mathcal{M}, \Omega)$ is defined as the completion of $\mathcal{H}$ with respect to the norm

$$\|\zeta\|_p = \inf\{\|\Delta_{\xi,\Omega}^{\frac{1}{2}-\frac{1}{p}}\zeta\| : \|\xi\| = 1, s_{\mathcal{M}}(\xi) \geq s_{\mathcal{M}}(\zeta)\}.$$

Here $s_{\mathcal{M}}(\xi)$ denotes the smallest projection in $\mathcal{M}$, which leaves ξ invariant. The cones [4, Equ. (1.13)]

$$\mathcal{P}^\alpha \doteq \overline{\{\Delta^\alpha A\Omega : A \in \mathcal{M}^+\}}\ , \qquad 0 \leq \alpha \leq 1/2\ ,$$

can be used to define the positive part of $L_p(\mathcal{M}, \Omega)$ [4, Equ. (1.14), p. 341]:

$$L_p^+(\mathcal{M}, \Omega) \doteq L_p(\mathcal{M}, \Omega) \cap \mathcal{P}_\Omega^{1/(2p)}\ , \qquad 2 \leq p \leq \infty\ . \tag{2.9}$$

Note that these are not operator spaces. The connection to the operator algebra $\mathcal{M}$ is made through auxiliary spaces $\mathcal{L}_p(\mathcal{M}, \Omega)$, which consist of formal expressions $A = u\Delta_{\phi,\Omega}^{1/p}$ with $\phi \in \mathcal{M}_*^+$ and u a partial isometry satisfying $u^*u = s(\phi)$ (the support projection of ϕ). Furthermore, denote the set of formal products

$$X_0\Delta_{\phi_1,\Omega}^{z_1}X_1 \cdots \Delta_{\phi_n,\Omega}^{z_n}X_n, \tag{2.10}$$

by $\mathcal{L}_p^*(\mathcal{M}, \Omega)$. Here is $X_j \in \mathcal{M}$, $\phi_j \in \mathcal{M}_*^+$, $j \in \{1, \dots, n\}$ and $\vec{z} = (z_1, \dots, z_n) \in I_{1-(1/p)}^{(n)}$. On the subset $\mathcal{L}_{p,0}^*(\mathcal{M}, \Omega) \subset \mathcal{L}_p^*(\mathcal{M}, \Omega)$, characterised by the condition $\sum_{j=1}^n \Re z_j = 1 - (1/p)$, it is possible to implement the star operation. The adjoint of a generic element (2.10) in $\mathcal{L}_{p,0}^*(\mathcal{M}, \Omega)$ is defined to be

$$X_n^*\Delta_{\phi_n,\Omega}^{\overline{z_n}} \cdots X_1^*\Delta_{\phi_1,\Omega}^{\overline{z_1}}X_0^*. \tag{2.11}$$

As shown in [4], the spaces $L_p(\mathcal{M}, \Omega)$ and $\mathcal{L}_p(\mathcal{M}, \Omega)$ are isomorphic. For $p \geq 2$ the isomorphism is given by $u\Delta_{\phi,\Omega}^{1/p} \mapsto u\Delta_{\phi,\Omega}^{1/p}\Omega$. Thus a multiplication in the $L_p(\mathcal{M}, \Omega)$ spaces can be defined, using the product of operators to connect the formal expressions $BC \in \mathcal{L}_{r,0}^*(\mathcal{M}, \Omega)$ for $B \in \mathcal{L}_{p,0}^*(\mathcal{M}, \Omega)$, $C \in \mathcal{L}_{q,0}^*(\mathcal{M}, \Omega)$ and $r^{-1} = p^{-1} + q^{-1} - 1$.

Araki's inequality (2.8) now entails a Hölder inequality: let $\zeta_1 \in L_p(\mathcal{M}, \Omega)$ and $\zeta_2 \in L_{p'}(\mathcal{M}, \Omega)$ for $p^{-1} + p'^{-1} = r^{-1}$, then

$$\|\zeta_1\zeta_2\|_r \leq \|\zeta_1\|_p\, \|\zeta_2\|_{p'}. \tag{2.12}$$

Thus the product $\zeta_1\zeta_2$ is in $L_r(\mathcal{M}, \Omega)$ and, as the case $p^{-1} + p'^{-1} = 1$ suggests, the topological dual $L_p(\mathcal{M}, \Omega)^*$ of $L_p(\mathcal{M}, \Omega)$ is $L_{p'}(\mathcal{M}, \Omega)$. For $A \in \mathcal{L}_p(\mathcal{M}, \Omega)$ and $B \in$

$\mathcal{L}_p(\mathcal{M},\Omega)^*$, the corresponding duality bracket is given by

$$\langle A, B\rangle = (A\Omega, B\Omega)\ , \tag{2.13}$$

if Ω is in the domain of A and B. According to [4, Notation 2.3 (4)] A and B in $\mathcal{L}_p^*(\mathcal{M},\Omega)$ are said to be equivalent, if (i) $1 \le p \le 2$ and $A\Omega = B\Omega$; (ii) if $2 \le p \le \infty$ and

$$\langle C, A\rangle = \langle C, B\rangle \tag{2.14}$$

for all C in $\mathcal{L}_p(\mathcal{M},\Omega)$.

Another important property is, that for $1 \le p \le \infty$, $x \in \mathcal{M}$ and $\zeta \in L_p(\mathcal{M},\Omega)$, the following inequality holds:

$$\|x\zeta\|_p \le \|x\|\,\|\zeta\|_p. \tag{2.15}$$

It is evident from the definition of the L_p-spaces, that $\mathcal{H}$ and $L_2(\mathcal{M},\Omega)$ are equal. It is proven in [4] that $\mathcal{M} \cong L_\infty(\mathcal{M},\Omega)$ as well as $\mathcal{M}_* \cong L_1(\mathcal{M},\Omega)$.

2.3 Proof of the Inequality

The following two lemmas are necessary for the subsequent proof of Theorem 2.1.1.

Lemma 2.3.1. *Let $A_1,\dots,A_n \in \mathcal{M}^+$. Then there exist unique $\phi_j \in \mathcal{M}_*^+$ such that for $0 \le p_j^{-1} \le 1/2$*

$$\Delta_{\phi_j,\Omega}^{1/p_j}\Omega = \Delta^{1/2p_j}A_j\Omega\,, \qquad j \in \{1,\dots,n\}\,, \tag{2.16}$$

and $\phi_j(\mathbb{1})^{1/p_j} = \|\Delta^{1/2p_j}A_j\Omega\|_{p_j}$. If also $\sum_{j=1}^n 1/p_j = 1/2$ holds, then

$$\Delta_{\phi_n,\Omega}^{1/p_n}\cdots\Delta_{\phi_1,\Omega}^{1/p_1}\Omega = \Delta^{1/2p_n}A_n\Delta^{1/2p_n}\cdots\Delta^{1/2p_1}A_1\Omega\ \in \mathcal{H}. \tag{2.17}$$

Proof. Let $A_1,\dots,A_n \in \mathcal{M}^+$ and $0 \le p_j^{-1} \le 1/2$, $j \in \{1,\dots,n\}$. Then, by definition $\zeta_j := \Delta^{1/2p_j}A_j\Omega \in \mathcal{P}_\Omega^{1/2p_j}$. An application of inequality 2.8 yields

$$\begin{aligned}
\|\zeta_j\|_{p_j}^2 &= \sup_{\|\xi\|=1}\|\Delta_{\xi,\Omega}^{(1/2)-(1/p_j)}\zeta_j\|^2 && (2.18)\\
&= \sup_{\|\xi\|=1}\left(\Delta_{\xi,\Omega}^{(1/2)-(1/p_j)}\Delta^{1/2p_j}A_j\Omega, \Delta_{\xi,\Omega}^{(1/2)-(1/p_j)}\Delta^{1/2p_j}A_j\Omega\right) && (2.19)\\
&\le \sup_{\|\xi\|=1}(\xi,\mathbb{1}\xi)^{1-(2/p_j)}\omega(\mathbb{1})^{2/p_j}\|A_j\|^2 = \|A_j\|^2 < \infty\ , && (2.20)
\end{aligned}$$

which establishes, that $\zeta_j \in L_{p_j}(\mathcal{M}, \Omega)$. Thus, according to (2.9), $\zeta_j \in L^+_{p_j}(\mathcal{M}, \Omega)$. By [4, Theorem 3 (4), p. 342] there exists a unique $\phi_j \in \mathcal{M}^+_*$ such that $\zeta_j = \Delta^{1/p_j}_{\phi_j,\Omega}\Omega$ and $\phi_j(\mathbb{1})^{1/p_j} = \|\zeta_j\|_{p_j} = \|\Delta^{1/2p_j} A_j \Omega\|_{p_j}$.

Thus, by definition [4, Notation 2.3 (4)], $\Delta^{1/2p_j} A_j \Delta^{1/2p_j} \equiv \Delta^{1/p_j}_{\phi_j,\Omega}$ as elements in $\mathcal{L}^*_{p'_j,0}(\mathcal{M}, \Omega)$, where $p_j^{-1} + p_j'^{-1} = 1$. Even though $\Delta^{1/p_j}_{\phi_j,\Omega}$ and $\Delta^{1/2p_j} A \Delta^{1/2p_j}$ may not be equal as operators, Lemma 7.7 (2) in [4] ensures, that their composition as elements of the spaces $\mathcal{L}^*_p$ is well-defined: setting $B_1 = \Delta^{1/p_2}_{\phi_2,\Omega}$, $B_2 = -\Delta^{1/2p_2} A_2 \Delta^{1/2p_2}$ and $C_2 = \Delta^{1/p_1}_{\phi_1,\Omega}$, there holds $\sum_{i=1}^2 B_i = 0$ as elements in $L_{p_2}(\mathcal{M}, \Omega)$, and therefore, using the lemma cited,

$$\Delta^{1/p_2}_{\phi_2,\Omega} \Delta^{1/p_1}_{\phi_1,\Omega}\Omega \equiv \Delta^{1/2p_2} A_2 \Delta^{1/2p_2} \Delta^{1/p_1}_{\phi_1,\Omega}\Omega \tag{2.21}$$

as elements in $L_{r_1}(\mathcal{M}, \Omega) = L_{r'_1}(\mathcal{M}, \Omega)^*$, where $r_1^{-1} + r_1'^{-1} = 1$, $r_1'^{-1} = p_1'^{-1} + p_2'^{-1} - 1$ and $1 \le r'_1 \le 2$ (in comparison to [4] indices and primed indices have swapped places). Note that this means $r_1^{-1} = p_2^{-1} + p_1^{-1}$. Using the same lemma once more (with the appropriate choices of C_2 and B_3, B_4) gives

$$\Delta^{1/2p_2} A_2 \Delta^{1/2p_2} \Delta^{1/p_1}_{\phi_1,\Omega}\Omega \equiv \Delta^{1/2p_2} A_2 \Delta^{1/2p_2} \Delta^{1/2p_1} A_1 \Omega \tag{2.22}$$

as elements in $L_{r'_1}(\mathcal{M}, \Omega)^*$. Together (2.21) and (2.22) imply

$$\Delta^{1/p_2}_{\phi_2,\Omega} \Delta^{1/p_1}_{\phi_1,\Omega}\Omega \equiv \Delta^{1/2p_2} A_2 \Delta^{1/2p_2} \Delta^{1/2p_1} A_1 \Omega \tag{2.23}$$

as elements in $L_{r'_1}(\mathcal{M}, \Omega)^*$. Consequently,

$$\Delta^{1/p_2}_{\phi_2,\Omega} \Delta^{1/p_1}_{\phi_1,\Omega} \equiv \Delta^{1/2p_2} A_2 \Delta^{1/2p_2} \Delta^{1/2p_1} A_1 \Delta^{1/2p_1}, \tag{2.24}$$

as elements in $\mathcal{L}^*_{r'_1,0}(\mathcal{M}, \Omega)$. Iteration of this procedure results in

$$\Delta^{1/p_n}_{\phi_n,\Omega} \cdots \Delta^{1/p_1}_{\phi_1,\Omega}\Omega \equiv \Delta^{1/2p_n} A_n \Delta^{1/2p_n} \cdots \Delta^{1/2p_2} A_2 \Delta^{1/2p_2} \Delta^{1/2p_1} A_1 \Omega \tag{2.25}$$

as elements in $L_2(\mathcal{M}, \Omega)^*$, because of $\sum_{j=1}^n 1/p_j = 1/2$. But since $\mathcal{H} = \mathcal{H}^* = L_2(\mathcal{M}, \Omega)^*$ the proof is finished. □

Lemma 2.3.2. *Let $p \in \mathbb{N}$ be even and $A \in \mathcal{M}^+$. Then there exists $\phi \in \mathcal{M}^+_*$ such that*

$$\|\Delta^{1/2p} A \Omega\|_p = \phi(\mathbb{1})^{1/p} = \|\!| A |\!\|_p \tag{2.26}$$

Proof. As proved in Lemma 2.3.1, there exists $\phi \in \mathcal{M}_*^+$, such that$\|\Delta^{1/2p}A\Omega\|_p^p = \phi(\mathbb{1})$, and $\Delta^{1/2p}A\Delta^{1/2p} \equiv \Delta_{\phi,\Omega}^{1/p}$ as elements in $\mathcal{L}_{p,0}^*(\mathcal{M},\Omega)$. Thus, by (2.17) and (2.8), and because p is even,

$$\begin{aligned}
\||A\||_p^p &= \big((\mathrm{e}^{-\beta L/2p}\,A\,\mathrm{e}^{-\beta L/2p})^{p/2}\,\Omega,\ (\mathrm{e}^{-\beta L/2p}\,A\,\mathrm{e}^{-\beta L/2p})^{p/2}\,\Omega\big)\\
&= (\Delta_{\phi,\Omega}^{1/p}\cdots\Delta_{\phi,\Omega}^{1/p}\Omega, \Delta_{\phi,\Omega}^{1/p}\cdots\Delta_{\phi,\Omega}^{1/p}\Omega)\\
&\leq \phi(\mathbb{1}) = \|\Delta^{1/2p}A\Omega\|_p^p\,. \qquad (2.27)
\end{aligned}$$

(The scalar product contains $p/2$ factors of $\Delta_{\phi,\Omega}^{1/p}$ in each of its entries.) Since $\phi \in \mathcal{M}_*^+$, there exists [9] a vector $\xi \in \mathcal{P}^\sharp$ such that $\phi(X) = (\xi, X\xi)$ for $X \in \mathcal{M}$. Using $\xi = J_{\phi,\Omega}\Delta_{\phi,\Omega}^{1/2}\Omega = J_{\xi,\Omega}\Delta_{\xi,\Omega}^{1/2}\Omega$, there holds

$$\phi(X) = (\xi, X\xi) = (\Delta_{\phi,\Omega}^{1/2}\Omega, J_{\phi,\Omega}^*J_{\phi,\Omega}\Delta_{\phi,\Omega}^{1/2}X^*\Omega)\ ,$$

where $J_{\phi,\Omega}^*J_{\phi,\Omega} = s_{\mathcal{M}}(\xi)\,s_{\mathcal{M}'}(\Omega)$ is a projection [4, p. 396]. Therefore

$$\phi(\mathbb{1}) \leq (\Delta_{\phi,\Omega}^{1/2}\Omega, \Delta_{\phi,\Omega}^{1/2}\Omega) = \big((\mathrm{e}^{-\beta L/2p}\,A\,\mathrm{e}^{-\beta L/2p})^{p/2}\,\Omega,\ (\mathrm{e}^{-\beta L/2p}\,A\,\mathrm{e}^{-\beta L/2p})^{p/2}\,\Omega\big) = \||A\||_p^p\,,$$

which finishes the proof. ☐

Proof of Theorem 2.1.1. Assuming the requirements of Theorem 2.1.1, Lemma 2.3.1 together with inequality (2.8), relation (2.26) and $w_j = z_j - (2p_j)^{-1} - (2p_{j-1})^{-1}$ imply

$$\begin{aligned}
&\big|\big(\mathrm{e}^{-z_{m+1}L}\ A_{m+1}\cdots\mathrm{e}^{-z_n}\,A_n\,\Omega,\ \mathrm{e}^{-z_mL}\,A_m\cdots\mathrm{e}^{-z_1}\,A_1\,\Omega\big)\big|\\
&\quad= \Big|\big(\Delta^{w_{m+1}}\Delta^{1/2p_{m+1}}A_{m+1}\Delta^{1/2p_{m+1}}\cdots\Delta^{w_n}\Delta^{1/2p_n}A_n\,\Omega,\\
&\qquad\quad \Delta^{w_m}\Delta^{1/2p_m}A_m\Delta^{1/2p_m}\cdots\Delta^{w_1}\Delta^{1/2p_1}A_1\,\Omega\big)\Big|\\
&\quad= \Big|\Big(\Delta^{w_{m+1}}\Delta_{\phi_{m+1},\Omega}^{1/p_{m+1}}\cdots\Delta^{w_n}\Delta_{\phi_n,\Omega}^{1/p_n}\,\Omega,\ \Delta^{w_m}\Delta_{\phi_m,\Omega}^{1/p_m}\cdots\Delta^{w_1}\Delta_{\phi_1,\Omega}^{1/p_1}\,\Omega\Big)\Big|\\
&\quad\leq \omega_\beta(\mathbb{1})^{1-\sum_{j=1}^n(p_j)^{-1}}\prod_{j=1}^n\phi_j(\mathbb{1})^{1/p_j} = \prod_{j=1}^n |||A_j|||_{p_j}\,,
\end{aligned}$$

where the p_j are chosen according to (2.2). For the second equality Δ^{w_j} is to be understood as $\Delta^{w_j/2}\mathbb{1}\Delta^{w_j/2}$ and (2.17) is to be applied. ☐

Chapter 3

Construction of the Thermal $P(\phi)_2$ Model

3.1 Euclidean Fields on the Cylinder

In 1974 Høegh-Krohn [40] discovered that the Euclidean field theory on the cylinder allows to reconstruct *two* distinct quantum field theories. The vacuum model on a circle and - more importantly - the thermal model on the real line. This chapter is a streamlined, rigorous version of this construction based on the works [21, 22, 43]. Furthermore the relativistic KMS condition for the thermal model is proven at the same time.

The strategy for proving the above mentioned results is the following. At first an interacting measure on the cylinder is provided. Then the two different quantum field models are reconstructed and Nelson symmetry in the interacting case is established. Finally, information readily available for the vacuum model on the circle is carried over to the thermal model via Nelson symmetry.

3.1.1 The Interacting Measure on the Cylinder

The relevant measure for the interacting case is non-Gaussian. It formally results from adding a polynomial of the form $P(\phi)$, where $P(\lambda)$, $\lambda \in \mathbb{R}$, is a polynomial which is bounded from below, to the Hamiltonian of the free massive boson field.

In two dimensions, the singularities, which arise from taking powers of the Euclidean field ϕ at a point $(\alpha, x) \in S_\beta \times \mathbb{R}$, can be removed by first normal ordering $: \cdot :_c$ (see [29,64]) the monomials $\phi(f)^n$, $n \in \mathbb{N}$,

$$:\phi(f)^n:_c \; := \sum_{m=0}^{[n/2]} \frac{n!}{m!(n-2m)!} \, \phi(f)^{n-2m} \Big(-\frac{1}{2} \, c(f,f) \Big)^m \tag{3.1}$$

with respect to a covariance c, and then taking appropriate limits. $[\,.\,]$ denotes taking the integer part. We will normal order with respect to different covariances c, some of them being limiting cases of the covariance C defined in (1.58).

Normal-ordering of point-like fields is ill-defined (i.e. one cannot replace the test function $f \in \mathcal{S}_{\mathbb{R}}(S_\beta \times \mathbb{R})$ in (3.1) by a two dimensional Dirac δ-function), but integrals over normal-ordered point-like fields can be defined rigorously:

Theorem 3.1.1 (Ultraviolet renormalisation [22, 29]). *For*

$$f \in L^1(S_\beta \times \mathbb{R}) \cap L^2(S_\beta \times \mathbb{R}) \, ,$$

the following limit[1] *exists in* $\bigcap_{1 \le p < \infty} L^p(Q, \Sigma, \mathrm{d}\phi_C)$*:*

$$\lim_{k,\kappa \to \infty} \int_{S_\beta \times \mathbb{R}} f(\alpha, x) \, :\phi\big(\delta_k(.-\alpha) \otimes \delta_\kappa(.-x)\big)^n :_C \; \mathrm{d}\alpha \; \mathrm{d}x. \tag{3.2}$$

We denote it by $\int_{S_\beta \times \mathbb{R}} f(\alpha, x) \, :\phi(\alpha, x)^n :_C \; \mathrm{d}\alpha \, \mathrm{d}x$.

Remark 3.1.2. *This theorem, which follows from exactly the same arguments as in the vacuum case analysed by Glimm and Jaffe [29], establishes a crucial step forward in the construction of the $P(\phi)_2$ model in finite volume, as it takes care (see Eq. (3.3) below) of the* ultraviolet renormalisation*.*

Let $P(\lambda) = \sum_j c_j \lambda^j$ be a real valued polynomial, which is bounded from below. Replacing the function f in (3.2) by the characteristic function of the set $S_\beta \times [-l, l]$, $l \in \mathbb{R}^+$, and applying [64, Lemma V.5], we deduce that

$$\mathrm{e}^{-\int_{-\beta/2}^{\beta/2} \int_{-l}^{l} :P(\phi(\alpha,x)):_C \; \mathrm{d}\alpha \, \mathrm{d}x} \in L^1(Q, \Sigma, \, \mathrm{d}\phi_C). \tag{3.3}$$

The *Euclidean $P(\phi)_2$ model* on the cylinder with a spatial cut-off $l \in \mathbb{R}^+$ is specified by the measure

$$\mathrm{d}\mu_l := \frac{1}{Z_l} \, \mathrm{e}^{-\int_{-\beta/2}^{\beta/2} \int_{-l}^{l} :P(\phi(\alpha,x)):_C \; \mathrm{d}\alpha \, \mathrm{d}x} \, \mathrm{d}\phi_C. \tag{3.4}$$

[1] The approximations of Dirac deltas have been defined in (1.8).

The partition function Z_l is chosen such that $\int_Q \mathrm{d}\mu_l = 1$. By the Radon-Nikodym theorem, the measure $\mathrm{d}\mu_l$ is absolutely continuous with respect to the Gaussian measure $\mathrm{d}\phi_C$, as long as $l < \infty$. The limit of the functions in (3.3) fails to be in $L^1(Q, \Sigma, \mathrm{d}\phi_C)$ as $l \to \infty$, and therefore the formal limiting measure cannot be absolutely continuous with respect to the Gaussian measure. In fact, in order to show that a countably additive Borel measure exists in the limit $l \to \infty$, one has to show (see Theorem 3.1.4 below) that

$$\lim_{l \to +\infty} \int_Q \mathrm{e}^{i\phi(f)} \, \mathrm{d}\mu_l =: E_P(f), \qquad f \in \mathcal{S}_{\mathbb{R}}(S_\beta \times \mathbb{R}), \tag{3.5}$$

defines a generating functional on $\mathcal{S}'_{\mathbb{R}}(S_\beta \times \mathbb{R})$, which gives rise to a probability measure $d\mu$ via Minlos' Theorem 1.3.2.

3.1.2 Sharp-time Fields, Existence of the Euclidean Measure in the Thermodynamic Limit and Nelson Symmetry

Nelson symmetry results from replacing the product measure $\mathrm{d}\alpha \, \mathrm{d}x$ in the exponent in (3.4) by iterated integrals with respect to the two measures $\mathrm{d}\alpha$ and $\mathrm{d}x$, in different orders. The delicate point, which will now be addressed in some more detail, is that the integrand in (3.2) is rather singular in nature. In order to convey an intuitive understanding, the key steps for proving Nelson symmetry are shown in the sequel.

By (1.62), (1.63) and Theorem 1.3.3 the sequences of functions

$$\big\{ \phi\big(\delta_k(\cdot - \alpha) \otimes h\big) \big\}_{k \in \mathbb{N}} \quad \text{and} \quad \big\{ \phi\big(g \otimes \delta_\kappa(\cdot - x)\big) \big\}_{\kappa \in \mathbb{N}} \, , \tag{3.6}$$

for $h \in \mathcal{S}_{\mathbb{R}}(\mathbb{R})$, $g \in \mathcal{S}_{\mathbb{R}}(S_\beta)$ and $\alpha \in S_\beta$, $x \in \mathbb{R}$ fixed, are Cauchy sequences in

$$\bigcap_{1 \le p < \infty} L^p(Q, \Sigma, \mathrm{d}\phi_C) \, .$$

Therefore sharp-time fields can be defined by

$$\phi(\alpha, h) := \lim_{k \to \infty} \phi\big(\delta_k(\cdot - \alpha) \otimes h\big), \quad \phi(g, x) := \lim_{\kappa \to \infty} \phi\big(g \otimes \delta_\kappa(\cdot - x)\big). \tag{3.7}$$

Note that both $\phi(\alpha, h)$ and $\phi(g, x)$ belong to $\bigcap_{1 \le p < \infty} L^p(Q, \Sigma, \mathrm{d}\phi_C)$.

Lemma 3.1.3 (Integrals over sharp-time fields [22]). *(i) For $h \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$*

and $\alpha \in [0, 2\pi)$ *the limit*

$$\lim_{\kappa\to\infty} \int_{\mathbb{R}} h(x) :\phi(\alpha, \delta_\kappa(\cdot - x))^n:_{C_0} \,\mathrm{d}x \tag{3.8}$$

exists in $\bigcap_{1\le p<\infty} L^p(Q, \Sigma, \mathrm{d}\phi_C)$. *Denote it by* $\int_{\mathbb{R}} h(x) : \phi(\alpha, x)^n :_{C_0} \,\mathrm{d}x$. *Normal ordering in (3.8) is done with respect to the* temperature β^{-1} covariance *on* $\mathbb{R}$*: for* $h_1, h_2 \in \mathcal{S}(\mathbb{R})$

$$C_0(h_1, h_2) := \Big(h_1, \frac{(1 + \mathrm{e}^{-\beta\epsilon})}{2\epsilon(1 - \mathrm{e}^{-\beta\epsilon})} h_2\Big)_{L^2(\mathbb{R}, \mathrm{d}x)}. \tag{3.9}$$

(ii) For $g \in L^1(S_\beta) \cap L^2(S_\beta)$ *and* $x \in \mathbb{R}$ *the limit*

$$\lim_{k\to\infty} \int_{S_\beta} g(\alpha) :\phi(\delta_k(\cdot - \alpha), x)^n:_{C_\beta} \,\mathrm{d}\alpha \tag{3.10}$$

exists in $\bigcap_{1\le p<\infty} L^p(Q, \Sigma, \mathrm{d}\phi_C)$. *Denoted it by* $\int_{S_\beta} g(\alpha) : \phi(\alpha, x)^n :_{C_\beta} \,\mathrm{d}\alpha$. *Normal ordering in (3.10) is done w.r.t. the covariance*[2]

$$C_\beta(g_1, g_2) := \Big(g_1, \frac{1}{2\nu} g_2\Big)_{L^2(S_\beta, \mathrm{d}\alpha)}, \qquad g_1, g_2 \in \mathcal{S}(S_\beta). \tag{3.11}$$

The covariances C_0 and C_β are time zero limiting cases of the covariance C, cf. equations (1.62) and (1.63).

Returning to the integral in (3.2), we let f be the characteristic function on $S_\beta \times [-l, l]$. This enables us to rewrite (3.2) as $\lim_{k,\kappa\to\infty} F(k, \kappa)$, where

$$F(k, \kappa) = \sum_{m=0}^{[n/2]} \frac{n!\big(-\frac{1}{2} C(\delta^{(2)}_{k,\kappa}, \delta^{(2)}_{k,\kappa})\big)^m}{m!(n-2m)!} \int_{S_\beta\times[-l,l]} \phi\big(\delta_k(\cdot - \alpha) \otimes \delta_\kappa(\cdot - x)\big)^m \,\mathrm{d}\alpha \,\mathrm{d}x, \tag{3.12}$$

and $\delta^{(2)}_{k,\kappa}(\alpha, x) := \delta_k(\alpha) \otimes \delta_\kappa(x)$. Interchanging integrals and limits is permitted by the existence of (3.2), (3.8) and (3.10). Performing the two limits in different orders results in

$$\lim_{k,\kappa\to\infty} F(k, \kappa) = \lim_{\kappa\to\infty} \sum_{m=0}^{[n/2]} \frac{n!\big(-\frac{1}{2} C_0(\delta_\kappa, \delta_\kappa)\big)^m}{m!(n-2m)!} \int_{S_\beta} \int_{[-l,l]} \phi\big(\alpha, \delta_\kappa(\cdot - x)\big)^m \,\mathrm{d}x \,\mathrm{d}\alpha$$

[2]As mentioned in the introduction the inverse temperature β is strictly positive. Hence no notational clash between C_0 and C_β can arise.

and

$$\lim_{k,\kappa\to\infty} F(k,\kappa) = \lim_{k\to\infty} \sum_{m=0}^{[n/2]} \frac{n!\big(-\frac{1}{2}C_\beta(\delta_k,\delta_k)\big)^m}{m!(n-2m)!} \int_{[-l,l]} \int_{S_\beta} \phi\big(\delta_k(\cdot-\alpha),x\big)^m \,\mathrm{d}\alpha\,\mathrm{d}x.$$

Note that in the latter expression normal ordering is done w.r.t. the covariance C_β, whilst in the former normal ordering is done with respect to the temperature β^{-1} covariance C_0 on $\mathbb{R}$. It follows that the L^1-function (3.3) equals ($U(\alpha,x)$ has been defined in (1.56))

$$\mathrm{e}^{-\int_{-l}^{l} U(0,x)\left(\int_{-\beta/2}^{\beta/2} :P(\phi(\alpha,0)):_{C_\beta}\ \mathrm{d}\alpha\right)\mathrm{d}x} = \mathrm{e}^{-\int_{-\beta/2}^{\beta/2} U(\alpha,0)\left(\int_{-l}^{l} :P(\phi(0,x)):_{C_0}\ \mathrm{d}x\right)\mathrm{d}\alpha}. \tag{3.13}$$

A proof of this identity can be found in [22, Lemma 5.3]. The analog of (3.13) in the case $\beta=\infty$ is known as *Nelson symmetry* (see e.g. [64]). Interpreting x in (3.13) as imaginary time $\mathrm{d}\mu = \lim_{l\to\infty}\mathrm{d}\mu_l$ is the Euclidean measure of the vacuum $P(\phi)_2$ model on the circle, while interpreting α as the imaginary time $\mathrm{d}\mu$ is the Euclidean measure for the corresponding thermal model on $\mathbb{R}$. This argument can be made rigorous (see [22, Theorem 7.2], [40]) by exploiting various properties of a time dependent heat equation (see [22, Appendix A]).

Theorem 3.1.4. *Consider sharp-time fields as defined in (3.7), and integrals over normal ordered products as defined in (3.8) and (3.10).*

(i) (Thermodynamic limit of Euclidean measures). *The limiting generating functional $E_P(f)$ defined in (3.5) exists, satisfies the conditions of Minlos' theorem and thus defines a probability measure* $\mathrm{d}\mu$. *For $f\in C^\infty_{0\mathbb{R}}(S_\beta\times\mathbb{R})$*

$$E_P(f) = \lim_{l\to+\infty}\frac{1}{Z_l}\int_Q \mathrm{e}^{i\phi(f)}\,\mathrm{e}^{-\int_{-l}^{l} U(0,x)\left(\int_{-\beta/2}^{\beta/2} :P(\phi(\alpha,0)):_{C_\beta}\ \mathrm{d}\alpha\right)\mathrm{d}x}\,\mathrm{d}\phi_C\ . \tag{3.14}$$

(ii) (Nelson symmetry). *For $f\in C^\infty_{0\mathbb{R}}(S_\beta\times\mathbb{R})$*

$$E_P(f) = \lim_{l\to+\infty}\frac{1}{Z_l}\int_Q \mathrm{e}^{i\phi(f)}\,\mathrm{e}^{-\int_{-\beta/2}^{\beta/2} U(\alpha,0)\left(\int_{-l}^{l} :P(\phi(0,x)):_{C_0}\ \mathrm{d}x\right)\mathrm{d}\alpha}\,\mathrm{d}\phi_C\ . \tag{3.15}$$

The map $f\mapsto E_P(f)$ is continuous in some Schwartz semi-norm and thus extends to $\mathcal{S}(S_\beta\times\mathbb{R})$ [22, Theorem 7.2 (ii)].

Remark 3.1.5. *This result solves the* infrared problem *for the thermal field theory under consideration.*

As they will be needed at a later stage, two results [22, Propositions 7.3 and 7.5], regarding the L^p-spaces for the interacting measure $\mathrm{d}\mu$ are recalled:

Lemma 3.1.6. *(i)* (Sharp-time fields are in $L^p(Q,\Sigma,\mathrm{d}\mu)$). *Let $h \in \mathcal{S}_{\mathbb{R}}(\mathbb{R})$ and $\alpha \in S_\beta$. Then the sequence $\phi\big(\delta_k(\cdot-\alpha)\otimes h\big)$ is Cauchy in $\bigcap_{1\le p<\infty} L^p(Q,\Sigma,\mathrm{d}\mu)$ and hence*

$$\phi(\alpha,h) := \lim_{k\to\infty} \phi\big(\delta_k(\cdot-\alpha)\otimes h\big) \in \bigcap_{1\le p<\infty} L^p(Q,\Sigma,\mathrm{d}\mu).$$

Moreover, the map

$$\begin{array}{ccc} S_\beta & \to & \bigcap_{1\le p<\infty} L^p(Q,\Sigma,\mathrm{d}\mu) \\ \alpha & \mapsto & \phi(\alpha,h) \end{array}$$

is continuous for $h \in \mathcal{S}_{\mathbb{R}}(\mathbb{R})$ fixed.

(ii) (Convergence of sharp-time Schwinger functions, Part I). *Let $h_i \in C^\infty_{0\,\mathbb{R}}(\mathbb{R})$ and $\alpha_i \in S_\beta$, $1 \le i \le n$. Then*

$$\lim_{l\to\infty} \int_Q \prod_{j=1}^n \mathrm{e}^{i\phi(\alpha_j,h_j)}\,\mathrm{d}\mu_l = \int_Q \prod_{j=1}^n \mathrm{e}^{i\phi(\alpha_j,h_j)}\,\mathrm{d}\mu.$$

In Section 3.4.2 it is proven that products of Euclidean sharp-time fields are also elements of $\bigcap_{1\le p<\infty} L^p(Q,\Sigma,\mathrm{d}\mu)$.

3.2 The Osterwalder-Schrader Reconstruction

In this section the two models under consideration are explicitly reconstructed from the measure $\mathrm{d}\mu$ by providing a Hilbert space, time-zero field operators, a generator of time translations and a distinguished vector for each model.

The invariance of $\mathrm{d}\mu$ under rotations and translations $\mathfrak{t}_{(\alpha,x)}$ as well as the reflections $\mathfrak{r}$ and $\mathfrak{r}'$ (as defined in subsection 1.2.1) entails the following facts about $U(\alpha,x)$ as well as R and R'.

(i) $U(\alpha,x)$ defines a two-parameter group of measure-preserving $*$-automorphisms of $L^\infty(Q,\Sigma,\mathrm{d}\mu)$, strongly continuous in measure, and to a strongly continuous two-parameter group of isometries of $L^p(Q,\Sigma,\mathrm{d}\mu)$ for $1\le p<\infty$.

(ii) R and R' extend to two measure preserving $*$-automorphisms of $L^\infty(Q,\Sigma,\mathrm{d}\mu)$ and to isometries of $L^p(Q,\Sigma,\mathrm{d}\mu)$ for $1\le p<\infty$.

(iii) $U(\alpha, x)$ is unitary on the Hilbert space $L^2(Q, \Sigma, \mathrm{d}\mu)$.

Definition 3.2.1. *For $0 \leq \gamma \leq \beta$ (resp. $0 \leq y \leq \infty$) the sub σ-algebra of the Borel σ-algebra Σ generated by the functions $\mathrm{e}^{i\phi(f)}$ with supp $f \subset [0, \gamma] \times \mathbb{R}$ (resp. supp $f \subset S_\beta \times [0, y]$) are denoted by $\Sigma_{[0,\gamma]}$ (resp. $\Sigma^{[0,y]}$).*

Next define two scalar products:

$$\forall F, G \in L^2(Q, \Sigma_{[0,\beta/2]}, \mathrm{d}\mu): \quad (F, G) := \int_Q R(\overline{F}) G \; \mathrm{d}\mu\,, \tag{3.16}$$

and

$$\forall F, G \in L^2(Q, \Sigma^{[0,\infty)}, \mathrm{d}\mu): \quad (F, G)' := \int_Q R'(\overline{F}) G \; \mathrm{d}\mu\,. \tag{3.17}$$

The measure $\mathrm{d}\mu$ is *Osterwalder-Schrader positive* with respect to *both* reflections R and R' [36, 49]:

$$\forall F \in L^2(Q, \Sigma_{[0,\beta/2]}, \mathrm{d}\mu) : \, (F, F) \geq 0$$

and

$$\forall G \in L^2(Q, \Sigma^{[0,\infty)}, \mathrm{d}\mu) : \, (G, G)' \geq 0.$$

Let $\mathcal{N} \subset L^2(Q, \Sigma_{[0,\beta/2]}, \mathrm{d}\mu)$ be the kernel of the positive quadratic form $(\,.\, , \,.\,)$ and $\mathcal{N}' \subset L^2(Q, \Sigma^{[0,\infty)}, \mathrm{d}\mu)$ the kernel of the positive quadratic form $(\,.\, , \,.\,)'$. Set

$$\mathcal{H}_\beta := \overline{L^2(Q, \Sigma_{[0,\beta/2]}, \mathrm{d}\mu)/\mathcal{N}} \quad \text{and} \quad \mathcal{H}_C := \overline{L^2(Q, \Sigma^{[0,\infty)}, \mathrm{d}\mu)/\mathcal{N}'}.$$

The completions of the pre-Hilbert spaces are taken w.r.t. the norms induced by $(\,.\, , \,.\,)$ and by $(\,.\, , \,.\,)'$, respectively. The canonical projection from $L^2(Q, \Sigma_{[0,\beta/2]}, \mathrm{d}\mu)$ to $\mathcal{H}_\beta$ and from $L^2(Q, \Sigma^{[0,\infty)}, \mathrm{d}\mu)$ to $\mathcal{H}_C$ are denoted by $\mathcal{V}$ and $\mathcal{V}'$, respectively. The distinguished vectors

$$\Omega_\beta := \mathcal{V}(1), \qquad \Omega_C := \mathcal{V}'(1),$$

arise as the image of 1, the constant function equal to 1 on Q.

The Abelian algebra

(i) $L^\infty(Q, \Sigma_{\{0\}}, \mathrm{d}\mu)$ preserves $L^2(Q, \Sigma_{[0,\beta/2]}, \mathrm{d}\mu)$ and $\mathcal{N}$. Thus a representation π_β of $L^\infty(Q, \Sigma_{\{0\}}, \mathrm{d}\mu)$ on the Hilbert spaces $\mathcal{H}_\beta$ is given by

$$\pi_\beta(A)\mathcal{V}(F) := \mathcal{V}(AF), \quad F \in L^2(Q, \Sigma_{[0,\beta/2]}, \mathrm{d}\mu), \quad A \in L^\infty(Q, \Sigma_{\{0\}}, \mathrm{d}\mu);$$

(ii) $L^\infty(Q, \Sigma^{\{0\}}, \mathrm{d}\mu)$ preserves $L^2(Q, \Sigma^{[0,\infty)}, \mathrm{d}\mu)$ and $\mathcal{N}'$. Thus one obtains a representation π_C of $L^\infty(Q, \Sigma^{\{0\}}, \mathrm{d}\mu)$ on $\mathcal{H}_C$, specified by

$$\pi_C(B)\mathcal{V}'(G) := \mathcal{V}'(BG), \quad G \in L^2(Q, \Sigma^{[0,\infty)}, \mathrm{d}\mu), \quad B \in L^\infty(Q, \Sigma^{\{0\}}, \mathrm{d}\mu).$$

The von Neumann algebras $\pi_\beta(L^\infty(Q, \Sigma_{\{0\}}, \mathrm{d}\mu))$ and $\pi_C(L^\infty(Q, \Sigma^{\{0\}}, \mathrm{d}\mu))$ can be interpreted as the algebras generated by bounded functions of the thermal time-zero fields on the real line and the vacuum time-zero fields on the circle, respectively.

Using Stone's Theorem [59] the representations π_β and π_C are used to define the time-zero fields of the respective theories. For real-valued $h \in C_0^\infty(\mathbb{R})$ define

$$\phi_\beta(h) := -i\frac{\mathrm{d}}{\mathrm{d}s}\pi_\beta\left(\mathrm{e}^{i\phi(0,sh)}\right)\Big|_{s=0}. \tag{3.18}$$

And for real-valued $g \in \mathcal{S}_\mathbb{R}(S_\beta)$ define

$$\phi_C(g) := -i\frac{\mathrm{d}}{\mathrm{d}s}\pi_C(\mathrm{e}^{i\phi(sg,0)})\Big|_{s=0}. \tag{3.19}$$

Both ϕ_β and ϕ_C are self-adjoint operators on $\mathcal{H}_\beta$ and H_C, respectively.

The reconstruction of the dynamics requires a more pronounced distinction of the two cases under consideration, which in the thermal case relies on a remarkable result on local symmetric semi-groups by Fröhlich [19] and, independently, Klein and Landau [48]:

(i) The semi-group $\{U(\alpha, 0)\}_{\alpha>0}$ does *not* preserve $L^2(Q, \Sigma_{[0,\beta/2]}, \mathrm{d}\mu)$. But setting, for $0 \le \gamma \le \beta/2$,

$$\mathcal{D}_\gamma := \mathcal{V}\mathcal{M}_\gamma, \qquad \text{with} \quad \mathcal{M}_\gamma := L^2(Q, \Sigma_{[0,\beta/2-\gamma]}, \mathrm{d}\mu),$$

one can define, for $0 \le \alpha \le \gamma$, a linear operator $P(\alpha)\colon \mathcal{D}_\gamma \to \mathcal{H}_\beta$ with domain $\mathcal{D}_\gamma$ by setting

$$P(\alpha)\mathcal{V}\psi := \mathcal{V}U(\alpha, 0)\psi, \qquad \psi \in \mathcal{M}_\gamma.$$

The triple $(P(\alpha), \mathcal{D}_\alpha, \beta/2)$ forms a *local symmetric semi-group* (see [19] [48]):

(a) for each α, $0 \le \alpha \le \beta/2$, $\mathcal{D}_\alpha$ is a linear subset of $\mathcal{H}_\beta$ such that $\mathcal{D}_\alpha \supset \mathcal{D}_\gamma$ if $0 \le \alpha \le \gamma \le \beta/2$, and

$$\mathcal{D} := \bigcup_{0<\alpha\le\beta/2} \mathcal{D}_\alpha$$

is dense in $\mathcal{H}_\beta$;

(b) for each α, $0 \le \alpha \le \beta/2$, $P(\alpha)$ is a linear operator on $\mathcal{H}_\beta$ with domain $\mathcal{D}_\alpha$;

(c) $P(0) = 1$, $P(\alpha)\mathcal{D}_\gamma \subset \mathcal{D}_{\gamma-\alpha}$ for $0 \le \alpha \le \gamma \le \beta/2$, and

$$P(\alpha)P(\gamma) = P(\alpha + \gamma)$$

on $\mathcal{D}_{\alpha+\gamma}$ for $\alpha, \gamma, \alpha + \gamma \in [0, \beta/2]$;

(d) $P(\alpha)$ is symmetric, i.e.,

$$(\Psi, P(\alpha)\Psi') = (P(\alpha)\Psi', \Psi), \qquad 0 \le \alpha \le \beta/2,$$

for all $\Psi, \Psi' \in \mathcal{D}_\alpha$ and $0 \le \alpha \le \beta/2$;

(e) $P(\alpha)$ is weakly continuous, i.e., if $\Psi \in \mathcal{D}_\gamma$, then

$$\alpha \mapsto (\Psi, P(\alpha)\Psi)$$

is a continuous function of α for $0 \le \alpha \le \gamma$ and $0 \le \gamma \le \beta/2$.

By the results cited [19, 48] there exists a self-adjoint operator L on $\mathcal{H}_\beta$ such that for $0 \le \alpha \le \gamma$

$$\mathcal{V}(U(\alpha, 0)F) = \mathrm{e}^{-\alpha L}\mathcal{V}(F), \qquad F \in L^2(Q, \Sigma_{[0,\beta/2-\gamma]}, \mathrm{d}\mu). \tag{3.20}$$

The self-adjoint operator L is said to be *associated* to the local symmetric semi-group $(P(\alpha), \mathcal{D}_\alpha, \beta/2)$.

Lemma 3.2.2. *$\mathcal{D}_\gamma$ is dense in $\mathcal{H}_\beta$ for $0 < \gamma < \beta/2$.*

Proof. Assume that

$$\forall \Phi \in \mathcal{D}_\gamma : \qquad (\Psi, \Phi) = 0\,. \tag{3.21}$$

Now consider, for $h_1, h_2 \in \mathcal{D}_\mathbb{R}(\mathbb{R})$ fixed, the analytic function

$$z \mapsto (\Psi, \mathrm{e}^{i\phi_\beta(h_1)}\mathrm{e}^{-zL}\mathrm{e}^{i\phi_\beta(h_2)}\Omega_\beta), \qquad \{z \in \mathbb{C} \mid 0 < \Re z < \beta/2\}. \tag{3.22}$$

Clearly $\mathrm{e}^{i\phi_\beta(h_1)}\mathrm{e}^{-\Re z L}\mathrm{e}^{i\phi_\beta(h_2)}\Omega_\beta \in \mathcal{D}_\gamma$ for $0 < \Re z < \gamma$ and consequently, because of (3.21), the analytic function (3.22) vanishes on an open line segment in the interior of its domain, and is therefore identical zero. It follows that

$$(\Psi, \mathrm{e}^{i\phi_\beta(h_1)}\mathrm{e}^{-\frac{\beta}{2}L}\mathrm{e}^{i\phi_\beta(h_2)}\Omega_\beta) = 0 \qquad \forall h_1, h_2 \in \mathcal{D}_{\mathbb{R}}(\mathbb{R}). \tag{3.23}$$

The set $\{\mathrm{e}^{i\phi_\beta(h_1)}\mathrm{e}^{-\frac{\beta}{2}L}\mathrm{e}^{i\phi_\beta(h_2)}\Omega_\beta \mid h_1, h_2 \in \mathcal{D}_{\mathbb{R}}(\mathbb{R})\}$ is dense in $\mathcal{H}_\beta$ [49, Theorem 11.2], and therefore (3.23) implies $\Psi = 0$. In other words, $\mathcal{D}_\gamma$ is dense in $\mathcal{H}_\beta$. □

(ii) The semi-group $U(0,x)$, $x \geq 0$, preserves the half-space $L^2(Q, \Sigma^{[0,\infty)}, \mathrm{d}\mu)$ as $U(0,x)$ maps $L^2(Q, \Sigma_{[0,\infty)}, \mathrm{d}\mu)$ into itself. Following [46] one can therefore define a self-adjoint positive operator H_C on $\mathcal{H}_C$ such that for $G \in L^2(Q, \Sigma_{[0,\infty)}, \mathrm{d}\mu)$

$$\mathcal{V}'(U(0,x)G) = \mathrm{e}^{-xH_C}\mathcal{V}'(G), \qquad x > 0. \tag{3.24}$$

The operators e^{-xH_C}, $x > 0$, form a strongly continuous semi-group of contractions on $\mathcal{H}_C$.

The next step in the reconstruction program is to define non-Abelian von Neumann algebras $\mathcal{R}_\beta \subset \mathcal{B}(\mathcal{H}_\beta)$ and $\mathcal{R}_C \subset \mathcal{B}(\mathcal{H}_C)$, generated by the operators

$$\tau_t(\pi_\beta(A)) := \mathrm{e}^{itL}\pi_\beta(A)\mathrm{e}^{-itL}, \quad t \in \mathbb{R}, \quad A \in L^\infty(Q, \Sigma_{\{0\}}, \mathrm{d}\mu),$$

and

$$\tau'_s(\pi_C(A)) := \mathrm{e}^{isH_C}\pi_C(A)\mathrm{e}^{-isH_C}, \quad s \in \mathbb{R}, \quad A \in L^\infty(Q, \Sigma^{\{0\}}, \mathrm{d}\mu),$$

respectively. τ_t and τ'_s extend to *-automorphisms of $\mathcal{R}_\beta$ and $\mathcal{R}_C$, respectively.

The algebra $\mathcal{R}_\beta \subset \mathcal{B}(\mathcal{H}_\beta)$ has a cyclic and separating vector, namely Ω_β [49, Lemma 8.1]. The time-translation invariant state ω_β (a normalised positive linear functional) on $\mathcal{R}_\beta$ defined by

$$\omega_\beta(a) := (\Omega_\beta, a\,\Omega_\beta), \quad a \in \mathcal{R}_\beta, \tag{3.25}$$

is invariant under the spatial translations induced by $\mathfrak{t}_{(0,y)}$, $y \in \mathbb{R}$. Furthermore, it satisfies the *KMS condition* [49]: the functions

$$F_{h_1,\ldots,h_n}(t_1 - t_2, \ldots, t_{n-1} - t_n) := \big(\Omega_\beta\,, \tau_{t_1}(\mathrm{e}^{i\phi_\beta(h_1)}) \ldots \tau_{t_n}(\mathrm{e}^{i\phi_\beta(h_n)})\Omega_\beta\big) \tag{3.26}$$

extend to analytic functions in the domain

$$\big\{(z_1,\dots,z_{n-1})\in\mathbb{C}^{n-1}\mid \Im z_k<0,\ -\beta<\textstyle\sum_{k=1}^{n-1}\Im z_k\big\} \tag{3.27}$$

and satisfy the KMS boundary condition: for each $1\le k<n$

$$\begin{aligned}&F_{h_1,\dots,h_n}(t_1-t_2,\dots,t_{k-2}-t_{k-1},t_{k-1}-t_k-i\beta,t_k-t_{k+1},\dots,t_{n-1}-t_n)\\ &\quad=F_{h_k,\dots,h_n,h_1,\dots,h_{k-1}}(t_k-t_{k+1},\dots,t_{n-1}-t_n,t_n-t_1,t_1-t_2,\dots,t_{k-2}-t_{k-1})\end{aligned} \tag{3.28}$$

for all $t_1,\dots,t_n\in\mathbb{R}$ and $h_1,\dots,h_n\in C^\infty_{0\mathbb{R}}$.

The algebra $\mathcal{R}_C\subset\mathcal{B}(\mathcal{H}_C)$ has a cyclic vector, namely Ω_C. The state ω_C on $\mathcal{R}_C$,

$$\omega_C(a):=(\Omega_C,a\,\Omega_C),\quad a\in\mathcal{R}_C,$$

is invariant under the rotations induced by $\mathfrak{t}_{(\gamma,0)}$, $\gamma\in[0,2\pi)$, and satisfies the *spectrum condition* (see Theorem 3.3.2 below). Since ω_C is the unique vacuum state (see below), the commutant $\mathcal{R}_C'$ of $\mathcal{R}_C$ equals $\mathbb{C}\cdot 1$ and therefore $\mathcal{R}_C=\mathcal{B}(\mathcal{H}_C)$.

3.3 The Wightman Functions on the Circle

In this section the vacuum theory on the circle as defined in the previous section and as defined in the Hamiltonian approach will be identified and the relevant information on this theory will be collected.

Starting from the same Hilbert space $\mathcal{H}_C^{(0)}$ and time zero field $\phi_C^{(0)}$ as in the free case (Subsection 1.3.2), the interaction is introduced through a perturbation of the free Hamiltonian $H_C^{(0)}$. Define

$$V:=\int_{S_\beta}:P(\phi_C(\alpha)):_{C_\beta}\ \mathrm{d}\alpha\,. \tag{3.29}$$

The operator sum

$$\mathrm{d}\Gamma(\nu)+V-E_C \tag{3.30}$$

is essentially self-adjoint on its natural domain $\mathcal{D}(\,\mathrm{d}\Gamma(\nu))\cap\mathcal{D}(V)$ and bounded from below [56]. Its closure defines the Hamiltonian H_C' of the $P(\phi)_2$ model on the circle S_β. The additive constant E_C is chosen such that zero is the lowest eigenvalue, i.e.

$\inf \sigma(H'_C) = 0$. Denote the corresponding unique [64, Theorem V.17] eigenvector by Ω'_C.

The identification of this model with the one constructed in the previous section is done via the Gell'Mann-Low formula [64, Theorem V.19], which yields the equality of the Schwinger functions of the two models,

$$\begin{aligned}\Big(\Omega'_C\,,\ \phi^{(0)}_C(g_1)\mathrm{e}^{-s_1 H'_C}\cdots\phi^{(0)}_C(g_{n-1})\,\mathrm{e}^{-s_{n-1}H'_C}\,\phi^{(0)}_C(g_n)\,\Omega'_C\Big)&\\ = \int \mathrm{d}\mu\,\phi(g_1\otimes\delta)\,U(0,s_1)\cdots\phi(g_{n-1}\otimes\delta)\,U(0,s_{n-1})\,\phi(g_n\otimes\delta)&\\ = \big(\Omega_C,\ \phi_C(g_1)\,\mathrm{e}^{-s_1 H_C}\cdots\phi_C(g_{n-1})\,\mathrm{e}^{-s_{n-1}H_C}\,\phi_C(g_n)\,\Omega_C\big)\ .&\end{aligned}$$

Because of the uniqueness of the analytic continuation and in view of the uniqueness part of Theorem 1.2.4 it follows that $(\mathcal{H}^{(0)}_C, \Omega'_C, \phi^{(0)}_C, H'_C)$ and $(\mathcal{H}_C, \Omega_C, \phi_C, H_C)$ are unitarily equivalent. Henceforth there will be no notational distinction between the two theories, i.e. only $(\mathcal{H}_C, \Omega_C, \phi_C, H_C)$ will be used.

The rest of this section gives all the results on the vacuum model, which are used at a later stage.

At first there are the Glimm-Jaffe ϕ-bounds [17, 27, 31] (the employed version can be found in [22, Proposition 5.4]): for $c \gg 1$ and some $\mathtt{C} \in \mathbb{R}^+$ there holds

$$\pm\phi_C(g) \le \mathtt{C}\,\|g\|_{H^{-\frac12}(S_\beta)}(H_C + c)^{1/2}\,, \qquad g \in H^{-\frac12}(S_\beta)\,, \tag{3.31}$$

and

$$\pm\phi_C(g) \le \mathtt{C}\,\|g\|_{H^{-1}(S_\beta)}(H_C + c)\,, \qquad g \in H^{-1}(S_\beta)\,. \tag{3.32}$$

In Subsection 3.4.3 a ϕ-bound will be used, which interpolates between (3.31) and (3.32).

Proposition 3.3.1. *For $0 \le \epsilon \le 1$ fixed there exist constants $c_1, c_2 > 0$ such that*

$$\pm\phi_C(g) \le c_1\,\|g\|_{H^{-\frac12-\frac{\epsilon}{2}}(S_\beta)}(H_C + c_2)^{\frac12+\epsilon} \tag{3.33}$$

for all $g \in H^{-\frac12-\frac{\epsilon}{2}}(S_\beta)$.

Proof. Recall $H^{(0)}_0 = \mathrm{d}\Gamma(\nu)$. It is sufficient to prove that

$$A(g) := (H^{(0)}_C + 1)^{-\frac14-\frac{\epsilon}{2}}\phi_C(\nu^{\frac12+\frac{\epsilon}{2}}g)(H^{(0)}_C + 1)^{-\frac14-\frac{\epsilon}{2}} \tag{3.34}$$

is a bounded operator on Fock space, uniformly bounded for $\|g\|_2 \le 1$. The first order estimate (see, e.g., [61, Equ. (2.21)])

$$(H_C^{(0)} + 1) \leq c_3(H_C + c_2) \quad \text{for } c_2, c_3 \gg 1. \tag{3.35}$$

and operator monotonicity of the map $\lambda \mapsto \lambda^\alpha$ for $0 \leq \alpha \leq 1$ (see, e.g.. [44, Example 4.6.46]) then ensure the fractional ϕ-bound (3.33).

Following ideas of Rosen (see, e.g. [61, Proof of Lemma 6.2]), it is shown in the sequel, that $A(g)$ is a bounded bilinear form in Fock space. The desired operator extension then follows from the Riesz representation theorem. It is sufficient to show that for $\|g\|_2 \leq 1$

$$|(\Phi, A(g)\Psi)| \leq c_4 \|\Phi\| \cdot \|\Psi\| \,, \tag{3.36}$$

for Φ, Ψ arbitrary vectors on Fock space and $c_4 > 0$ a constant. Now

$$\begin{aligned} |(\Phi, A(g)\Psi)| \qquad &\leq \tfrac{1}{\sqrt{2}} \Big(|(a^*(\nu^{\frac{\epsilon}{2}} \overline{g})(H_C^{(0)} + 1)^{-\frac{1}{4}-\frac{\epsilon}{2}}\Phi, (H_C^{(0)} + 1)^{-\frac{1}{4}-\frac{\epsilon}{2}}\Psi)| \\ &\qquad + |((H_C^{(0)} + 1)^{-\frac{1}{4}-\frac{\epsilon}{2}}\Phi, a^*(\nu^{\frac{\epsilon}{2}} g)(H_C^{(0)} + 1)^{-\frac{1}{4}-\frac{\epsilon}{2}}\Psi)| \Big) \,. \end{aligned} \tag{3.37}$$

Since $H_C^{(0)}$ commutes with the number operator, and both terms are of the same structure, it is sufficient to prove that for $\Phi_n \in \mathcal{H}_C^{(n)}$ and $\Psi_{n-1} \in \mathcal{H}_C^{(n-1)}$ with $\|\Phi_n\| \leq 1$ and $\|\Psi_{n-1}\| \leq 1$ one has that

$$\begin{aligned} &|((H_C^{(0)} + 1)^{-\frac{1}{4}-\frac{\epsilon}{2}}\Phi_n, a^*(\nu^{\frac{\epsilon}{2}} g)(H_C^{(0)} + 1)^{-\frac{1}{4}-\frac{\epsilon}{2}}\Psi_{n-1})| \\ &\qquad \leq \|(n+1)^{-\frac{1}{4}}\Phi_n\| \cdot \|a^*(\nu^{\frac{\epsilon}{2}} g)(H_C^{(0)} + 1)^{-\frac{1}{4}-\frac{\epsilon}{2}}\Psi_{n-1}\| \end{aligned} \tag{3.38}$$

is uniformly bounded in n. For simplicity it is assumed that the mass $m \geq 1$, so that $1 \leq \nu(k_j)$; otherwise one is left with yet another n-independent constant. Now

$$\begin{aligned} &(n+1)^{-1/2} \, \|\Phi_n\|^2 \, \|a^*(\nu^{\frac{\epsilon}{2}} g)(H_C^{(0)} + 1)^{-\frac{1}{4}-\frac{\epsilon}{2}}\Psi_{n-1}\|^2 \\ &\quad \leq \frac{n}{(n+1)^{1/2}} \, \|\Phi_n\|^2 \int \prod_{j=1}^{n} \frac{\mathrm{d}k_j}{\nu(k_j)} \left| \frac{\nu(k_n)^{\frac{\epsilon}{2}} \, g(k_n)}{\left(\sum_{i=1}^{n-1} \nu(k_i) + 1\right)^{1/4+\epsilon/2}} \, \Psi_{n-1}(k_1, k_2, \ldots k_{n-1}) \right|^2 \\ &\quad \leq \frac{n}{n+1} \, \|\Phi_n\|^2 \int \prod_{j=1}^{n} \frac{\mathrm{d}k_j}{\nu(k_j)} \left| \left(\frac{\nu(k_n)}{\sum_{i=1}^{n-1} \nu(k_i) + 1} \right)^{\frac{\epsilon}{2}} g(k_n) \, \Psi_{n-1}(k_1, k_2, \ldots k_{n-1}) \right|^2 \\ &\quad \leq \|g\|_2^2 \, \|\Phi_n\|^2 \, \|\Psi_{n-1}\|^2 \,, \end{aligned} \tag{3.39}$$

which establishes the claim. □

Denote the generator of space-translations in $\mathcal{H}_C$ by P_C. The following remarkable result is due to Heifets & Osipov [36].

Theorem 3.3.2 (Spectrum Condition [36]). *The joint spectrum of H_C and P_C is purely discrete and contained in the forward light cone $\{(E,p) \mid |p| < E\}$.*

The unitary operators $U_C(\alpha, s) \in \mathcal{B}(\mathcal{H}_C)$ given by

$$U_C(\alpha, s) := \mathrm{e}^{i(sH_C - \alpha P_C)}, \qquad \alpha \in [0, 2\pi),\ s \in \mathbb{R}, \tag{3.40}$$

implement the two parameter group of automorphisms $\tau'_{\alpha,s}$ of $\mathcal{R}_C$ on the Hilbert space $\mathcal{H}_C$. Let $g_i \in \mathcal{S}(S_\beta)$, $s_i \in \mathbb{R}$, $i \in \{1, \ldots, n\}$ and set

$$\phi_C(g_i, s_i) := \mathrm{e}^{is_i H_C} \phi_C(g_i) \mathrm{e}^{-is_i H_C} \,. \tag{3.41}$$

By Stone's theorem, the map $s \mapsto U_C(0, s)$ is strongly continuous. Together with the bound (3.31) this implies that

$$W_C^{(n)}(g_1, s_1, \ldots, g_n, s_n) := \big(\Omega_C, \phi_C(g_1, s_1) \cdots \phi_C(g_n, s_n)\Omega_C\big) \tag{3.42}$$

exists and is a separately continuous multi-linear functional of the arguments (g_i, s_i), $i \in \{1, \ldots, n\}$, as they vary over $\mathcal{S}_{\mathbb{R}}(S_\beta) \times \mathbb{R}$. It follows from the nuclear theorem that $\mathcal{W}_C^{(n)}$ can be viewed as a continuous multi-linear functional on $(\mathcal{S}_{\mathbb{R}}(S_\beta) \times \mathbb{R})^n$. Sometimes it will sloppily be denoted by

$$W_C^{(n)}(\alpha_1, s_1, \ldots, \alpha_n, s_n) = \big(\Omega_C, \phi_C(\alpha_1, s_1) \cdots \phi_C(\alpha_n, s_n)\Omega_C\big) \,. \tag{3.43}$$

Translation invariance implies (cf. equations (1.3) and (1.4)) that there exists a multi-linear continuous functional $\mathfrak{W}_C^{(n-1)}$ on $(\mathcal{S}_{\mathbb{R}}(S_\beta) \times \mathbb{R})^{n-1}$ such that on a formal level for coordinates

$$\xi_i = (\alpha_i - \alpha_{i+1}, s_i - s_{i+1}) \,, \qquad i \in \{1, \ldots, n-1\} \,,$$

there holds

$$\mathfrak{W}_C^{(n-1)}(\xi_1, \xi_2, \ldots, \xi_{n-1}) = W_C^{(n)}(\alpha_1, s_1, \alpha_2, s_2, \ldots, \alpha_n, s_n) \,. \tag{3.44}$$

$\mathfrak{W}_C^{(n-1)}$ is interpreted as a periodic generalised function. Consequently its continuous Fourier transform is a tempered distribution, which is a sum of point measures. The coefficients thereof can be identified with the discrete Fourier transform of $\mathfrak{W}_C^{(n-1)}$. No notational distinction will be made between these two points of view.

The above definitions seem unusual as the space and time entries are not in the common order. This choice is made so as not to have two different coordinate orderings appear for the cylinder $S_\beta \times \mathbb{R}$. Consequently, for the rest of this subsection only, the forward light cone V^+ on the cylinder will be $\{(\alpha, s) \subset (-\beta/2, \beta/2) \times \mathbb{R} \mid s > |\alpha|\}$.

The following lemma is required for the next theorem, which provides analytic continuations of $\mathfrak{W}_C^{(n-1)}$.

Lemma 3.3.3. *Let* $\widetilde{\mathfrak{W}}_C^{(n-1)}$ *denote the Fourier transform of* $\mathfrak{W}_C^{(n-1)}$. *Then*

$$\widetilde{\mathfrak{W}}_C^{(n-1)}\big((E_1, p_1), (E_2, p_2), \ldots, (E_{n-1}, p_{n-1})\big) = 0\,, \tag{3.45}$$

if $(E_i, p_i) \notin \mathrm{Sp}(H_C, P_C)$ *for some* $i \in \{1, \ldots, n-1\}$.

Proof. The Fourier transform of $\mathfrak{W}_C^{(n-1)}$ is[3]

$$\widetilde{\mathfrak{W}}_C^{(n-1)}\big((E_1, p_1), (E_2, p_2), \ldots, (E_{n-1}, p_{n-1})\big) =$$
$$= \int \mathrm{d}\xi_1 \cdots \mathrm{d}\xi_{n-1}\ \mathrm{e}^{i \sum_{j=1}^{n-1} \xi_j \cdot (E_j, p_j)}\ \mathfrak{W}_C^{(n-1)}(\xi_1, \ldots, \xi_{n-1})\ ,$$

where heuristically

$$\mathfrak{W}_C^{(n-1)}(\alpha_1 - \alpha_2, s_1 - s_2, \ldots, \alpha_{n-1} - \alpha_n, s_{n-1} - s_n) =$$
$$= \big(\Omega_C, \phi_C(\alpha_1)\mathrm{e}^{i(s_2 - s_1)H_C} \cdots \phi_C(\alpha_{n-1})\mathrm{e}^{i(s_n - s_{n-1})H_C}\phi_C(\alpha_n)\Omega_C\big)\ .$$

Next insert, as suggested in [65], a basis of common eigenfunctions $\Psi_{\epsilon,k}$ of the operators H_C, P_C: for all $\Phi \in \mathcal{H}_C$ the unitary operators $U_C(\alpha, s)$ defined in (3.40) can be expressed as

$$U_C(\alpha, s)\Phi = \sum_{(\epsilon,k)\in \mathrm{Sp}(H_C, P_C)} \mathrm{e}^{i(s\epsilon - \alpha k)}\ (\Psi_{\epsilon,k}, \Phi)\, \Psi_{\epsilon,k}\,.$$

[3]The earlier mentioned ordering conventions entail, that $\xi_j \cdot (E_j, p_j) = (\alpha_j, s_j) \cdot (E_j, p_j) = E_j s_j - p_j \alpha_j$.

Now consider, for $\Phi, \Phi' \in \mathcal{H}_C$ fixed, the map

$$\begin{aligned}(E,p) \mapsto & \int_0^{2\pi} \mathrm{d}\alpha \int_{\mathbb{R}} \mathrm{d}s \; \mathrm{e}^{-i(Es-p\alpha)} (\Phi', U(\alpha,s)\Phi) \\ &= \sum_{(\epsilon,k)\in \mathrm{Sp}(H_C,P_C)} \int_{\mathbb{R}} \mathrm{d}s \; \mathrm{e}^{-i(E-\epsilon)s} \int_{-\beta/2}^{\beta/2} \mathrm{d}\alpha \; \mathrm{e}^{i(p-k)\alpha} (\Psi_{\epsilon,k},\Phi)(\Phi',\Psi_{\epsilon,k}) \\ &= \sum_{(\epsilon,k)\in \mathrm{Sp}(H_C,P_C)} \delta_{\epsilon,E}\, \delta_{k,p}\, (\Psi_{\epsilon,k},\Phi)(\Phi',\Psi_{\epsilon,k}) \, ,\end{aligned}$$

where the Kronecker deltas $\delta_{\epsilon,E}$ and $\delta_{k,p}$ appear due to the discreteness of the spectrum. The sum on the r.h.s. vanishes, if $(E,p) \notin \mathrm{Sp}(H_C,P_C)$. This implies (3.45) unless each (E_i,p_i) lies in the forward light cone. □

Theorem 3.3.4. *For each $n \geq 1$, $\widetilde{\mathfrak{W}}_C^{(n-1)}$ has support in $(V^+)^{n-1}$ and $\mathfrak{W}_C^{(n-1)}$ is the boundary value of a polynomially bounded function $\mathcal{W}_+^{(n-1)}$ analytic in the forward tube $(S_\beta \times \mathbb{R} - iV^+)^{n-1}$.*

Proof. The support property of $\widetilde{\mathfrak{W}}_C^{(n-1)}$ was established in Lemma 3.3.3. By the Bros-Epstein-Glaser Lemma [60, Theorem IX.15] there exists a polynomial P and a polynomially bounded function $G^{(n-1)} \colon (S_\beta \times \mathbb{R})^{(n-1)} \to \mathbb{C}$ obeying

$$\mathrm{supp}\; G^{(n-1)} \subseteq \overline{(V^+)^{(n-1)}},$$

such that $\widetilde{\mathfrak{W}}_C^{(n-1)} = P(D) G^{(n-1)}$, with D a partial differential operator. Consequently an analytic continuation $\mathcal{W}_+^{(n-1)}$ of $\mathfrak{W}_C^{(n-1)}$ to $(S_\beta \times \mathbb{R} - iV^+)^{n-1}$ can be defined:

$$\begin{aligned}\mathcal{W}_+^{(n-1)}(\xi_1 - i\eta_1, \ldots, \xi_{n-1} - i\eta_{n-1}) &= (2\pi)^{-\frac{n-1}{2}} P\bigl(-i(\xi_1 - i\eta_1, \ldots, \xi_{n-1} - i\eta_{n-1})\bigr) \times \\ \times \int_{\mathbb{R}^{2(n-1)}} \prod_{j=1}^{n-1} \mathrm{d}E_j\, \mathrm{d}p_j \; \mathrm{e}^{-i(\xi_j - i\eta_j)\cdot(E_j,p_j)} & \, G^{(n-1)}\bigl((E_1,p_1), \ldots, (E_{n-1},p_{n-1})\bigr) \, . \qquad (3.46)\end{aligned}$$

If $\eta_j \in V^+$ for all $j \in \{1, \ldots, n-1\}$, this integral exists. Furthermore its boundary value for $(\eta_1, \ldots, \eta_{n-1}) \searrow 0$ is $\mathfrak{W}_C^{(n-1)}$. Polynomial boundedness of the analytic function $\mathcal{W}_+^{(n-1)}$ results from the following inequality [60, Theorem IX.16]:

$$\begin{aligned}&\Bigl|\mathcal{W}_+^{(n-1)}(\xi_1 - i\eta_1, \ldots, \xi_{n-1} - i\eta_{n-1})\Bigr| \\ &\qquad \leq C \; \bigl|P\bigl(-i(\xi_1 - i\eta_1, \ldots, \xi_{n-1} - i\eta_{n-1})\bigr)\bigr| \; \bigl(1 + d((\eta_1, \ldots, \eta_{n-1}))^{-N}\bigr) \, .\end{aligned}$$

C is a constant, $d((\eta_1, \ldots, \eta_{n-1}))$ is the distance of $(\eta_1, \ldots, \eta_{n-1})$ to $\partial(V^+)^{n-1}$ and N is a positive integer. □

The next lemma provides the germ for the analyticity domain of the relativistic KMS condition. It is a consequence of locality on the circle $S_\beta \equiv [0, 2\pi)$. Define W as the wedge $\{(\alpha, s) \in [0, \beta) \times \mathbb{R} \mid \alpha > |s|\}$ and $G_\beta := \{(\alpha, s) \in W \mid |\alpha| + |s| < \beta\}$.

Lemma 3.3.5. *Let $\lambda_i > 0$, $i \in \{1, \ldots, n-1\}$ and $\sum_{i=1}^{n-1} \lambda_i = 1$. The tempered distributions $W_C^{(n)}(\alpha_1, s_1, \ldots, \alpha_n, s_n)$ defined in (3.43) are real valued for $(\alpha_1, s_1, \ldots, \alpha_n, s_n) \in J^{(n)}$, where*

$$(\alpha_1, s_1, \ldots, \alpha_n, s_n) \in J^{(n)} \Leftrightarrow \begin{cases} (\alpha_i, s_i) \in S_\beta \times \mathbb{R}\,, \\ (\alpha_{i+1} - \alpha_i, s_{i+1} - s_i) \in \lambda_i G_\beta\,. \end{cases} \tag{3.47}$$

Proof. Assume that the space-time points (α_i, s_i) and (α_j, s_j) are space-like to each other for all choices of $i \neq j$ and $i, j \in \{1, \ldots, n\}$. Then, as a consequence of locality, all the field operators $\phi_C(\alpha_i, s_i)$ commute (as quadratic forms) with each other. Therefore $W_C^{(n)}(\alpha_1, s_1, \ldots, \alpha_n, s_n)$ equals

$$\begin{aligned} \big(\Omega_C, \phi_C(\alpha_1, s_1) \cdots \phi_C(\alpha_n, s_n)\Omega_C\big) &= \big(\Omega_C, \phi_C(\alpha_n, s_n) \cdots \phi_C(\alpha_1, s_1)\Omega_C\big) \\ &= \overline{W_C^{(n)}(\alpha_1, s_1, \ldots, \alpha_n, s_n)}. \end{aligned}$$

In other words, the tempered distributions $W_C^{(n)}(\alpha_1, s_1, \ldots, \alpha_n, s_n)$ are real valued. Thus the lemma follows, once it has been shown that the set $J^{(n)}$ consists of points, which are pairwise space-like to each other.

A point (α, s) on the cylinder is space-like to the origin $(0, 0)$ iff $(\alpha, s) \in G_\beta$. Space-likeness is a symmetric relation and therefore it suffices to prove that (α_i, s_i) is space-like to (α_j, s_j) for $i > j$, i.e.

$$(\alpha_j, s_j) - (\alpha_i, s_i) \in G_\beta \quad \text{for} \quad i > j. \tag{3.48}$$

Note that, for $0 < \lambda$, there holds $\lambda G_\beta = G_{\lambda\beta}$. The map $\mathfrak{n}\colon [0, \beta) \times \mathbb{R} \to \mathbb{R}^+$,

$$(\alpha, s) \mapsto |\alpha| + |s|,$$

defines a norm. Denote its restriction to the wedge W by $\mathfrak{n}_{|W}$. Equ. (3.48) now follows from the triangle inequality:

$$\begin{aligned}\mathfrak{n}_{|W}\left((\alpha_j, s_j) - (\alpha_i, s_i)\right) &= \mathfrak{n}_{|W}\big((\alpha_j - \alpha_{j-1}, s_j - s_{j-1}) + \ldots + (\alpha_{i+1} - \alpha_i, s_{i+1} - s_i)\big) \\ &\leq \mathfrak{n}_{|W}((\alpha_j - \alpha_{j-1}, s_j - s_{j-1})) + \ldots + \mathfrak{n}_{|W}((\alpha_{i+1} - \alpha_i, s_{i+1} - s_i)) \\ &< \lambda_{j-1}\beta + \lambda_{j-2}\beta + \ldots + \lambda_i\beta \leq \beta \sum_{k=1}^{n-1} \lambda_k = \beta,\end{aligned}$$

and therefore (3.47) implies (3.48). □

The following is an immediate consequence of the preceding results. Because the tempered distributions $W_C^{(n)}(\alpha_1, s_1, \ldots, \alpha_n, s_n)$ defined in (3.43) are real valued for

$$(\alpha_1, s_1, \ldots, \alpha_n, s_n) \in J^{(n)},$$

the Schwarz reflection principle can be applied. The function

$$\mathcal{W}_-^{(n-1)}(\xi_1 + i\eta_1, \ldots, \xi_{n-1} + i\eta_{n-1}) := \overline{\mathcal{W}_+^{(n-1)}(\xi_1 - i\eta_1, \ldots, \xi_{n-1} - i\eta_{n-1})} \tag{3.49}$$

is analytic on $(S_\beta \times \mathbb{R} + iV^+) \times \cdots \times (S_\beta \times \mathbb{R} + iV^+)$ and polynomially bounded as $\eta_i \searrow 0$. Furthermore, its boundary value for $(\eta_1, \ldots, \eta_{n-1}) \searrow 0$ is $\mathfrak{W}_C^{(n-1)}$. Accordingly there exist functions $\mathcal{W}_\pm^{(n-1)}$, which are holomorphic on

$$(\lambda_1 G_\beta \times \ldots \times \lambda_{n-1} G_\beta) \mp i(V^+ \times \ldots \times V^+),$$

and which coincide on $(\lambda_1 G_\beta \times \ldots \times \lambda_{n-1} G_\beta) + i\{0\}$. Since V^+ is a cone, $V^+ \times \ldots \times V^+$ is a cone (by definition). Applying the Edge-of-the-Wedge Theorem [65, Theorem 2-16] results in an analytic function $\mathcal{W}_C^{(n-1)}$ on

$$\big((\lambda_1 G_\beta \times \ldots \times \lambda_{n-1} G_\beta) + i((V^+)^{n-1} \cup (-V^+)^{n-1})\big) \cup \mathcal{N},$$

where $\mathcal{N}$ is a complex neighbourhood of $(\lambda_1 G_\beta \times \ldots \times \lambda_{n-1} G_\beta) + i\{0\}$. With this information $\mathcal{W}_C^{(n-1)}$ can be analytically continued to a larger domain.

Theorem 3.3.6. *There exists a function $\mathcal{W}_C^{(n-1)}$ defined and analytic in*

$$\mathcal{C}^{(n-1)} := \big((\lambda_1 G_\beta \times \ldots \times \lambda_{n-1} G_\beta) + i\,(V^- \cup V^+) \times \ldots \times (V^- \cup V^+)\big) \cup \mathcal{N}, \tag{3.50}$$

where $\lambda_j > 0$ and $\sum_{j=1}^{n-1} \lambda_j = 1$, which coincides with $\mathfrak{W}_C^{(n-1)}$ on $\lambda_1 G_\beta \times \ldots \times \lambda_{n-1} G_\beta$.

Proof. In order to show that $\mathcal{W}_C^{(n-1)}$ has an analytic continuation to (3.50) it will be shown that $\mathcal{W}_+^{(n-1)}$ has an analytic continuation to

$$(\lambda_1 G_\beta \times \ldots \times \lambda_{n-1} G_\beta) - i(V^+ \times \ldots \times (-V^+) \times \ldots \times V^+)\,, \tag{3.51}$$

i.e. with the light-cone flipped in the k-th coordinate pair. In the following calculation only the $\xi_j = (\alpha_j - \alpha_{j+1}, s_j - s_{j+1}) \in \lambda_j G_\beta$ for $j \in \{k-1, k, k+1\}$ will be shown as the other coordinates stay unchanged.

$$\begin{aligned}
&\mathcal{W}_C^{(n-1)}\big((\alpha_{k-1} - \alpha_k, s_{k-1} - s_k), (\alpha_k - \alpha_{k+1}, s_k - s_{k+1}), (\alpha_{k+1} - \alpha_{k+2}, s_{k+1} - s_{k+2})\big) \\
&= (\Omega_C, \phi_C(\alpha_1, s_1) \cdots \phi_C(\alpha_n, s_n)\, \Omega_C) \\
&= (\Omega_C, \phi_C(\alpha_1, s_1) \\
&\qquad \cdots \phi_C(\alpha_{k-1}, s_{k-1}) \phi_C(\alpha_{k+1}, s_{k+1}) \phi_C(\alpha_k, s_k) \phi_C(\alpha_{k+2}, s_{k+2}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdots \phi_C(\alpha_n, s_n)\, \Omega_C) \\
&= \mathfrak{W}_C^{(n-1)}\big((\alpha_{k-1} - \alpha_{k+1}, s_{k-1} - s_{k+1}), (\alpha_{k+1} - \alpha_k, s_{k+1} - s_k), (\alpha_k - \alpha_{k+2}, s_k - s_{k+2})\big)\,,
\end{aligned}$$

which can be written as

$$\mathcal{W}_C^{(n-1)}(\xi_{k-1}, \xi_k, \xi_{k+1}) = \mathfrak{W}_C^{(n-1)}(\xi_{k-1} + \xi_k, -\xi_k, \xi_k + \xi_{k+1})\,.$$

But the right hand side clearly has an analytic continuation to (3.51). Hence $\mathcal{W}_+^{(n-1)}$ has an analytic continuation to $(\lambda_1 G_\beta \times \ldots \times \lambda_{n-1} G_\beta) - i(V^+ \times \ldots \times (V^+ \cup -V^+) \times \ldots \times V^+)$. Here it is used that $(\lambda_1 G_\beta \times \ldots \times \lambda_{n-1} G_\beta) + i\{0\}$ is in the interior of $\mathcal{N}$. Since this procedure can be repeated for any coordinate pair, the domain of analyticity of $\mathcal{W}_C^{(n-1)}$ can be extended to (3.50). □

3.4 The Thermal Wightman Functions

In order to explore Nelson symmetry in its most concise form, interacting thermal Wightman functions and their analytic continuations have to be defined. Subsection 3.4.1 deals with analytic continuations of the Schwinger functions. The final two subsections deal with the domain of the thermal time-zero field and with the boundary values of the analytic functions constructed in Subsection 3.4.1, respectively.

3.4.1 An Application of the Strong Disk Theorem

In the previous section it has been shown that the Wightman functions on the circle are the boundary values of a function $\mathcal{W}_C^{(n-1)}$ holomorphic in the region $\mathcal{C}^{(n-1)}$. For $\alpha_j > 0$ and $\sum_j \alpha_j < \beta$ a new function can be defined by

$$\mathcal{W}_\beta^{(n-1)}\big(-i\alpha_1, y_1, \ldots, -i\alpha_{n-1}, y_{n-1}\big) := \mathcal{W}_C^{(n-1)}\big(\alpha_1, -iy_1, \ldots, \alpha_{n-1}, -iy_{n-1}\big), \tag{3.52}$$

or alternatively by $\mathcal{W}_\beta^{(n-1)} := \mathcal{W}_C^{(n-1)} \circ \Xi^{-1}$, where Ξ is the coordinate transformation

$$(z_1, w_1, \ldots, z_{n-1}, w_{n-1}) \mapsto (iz_1, -iw_1, \ldots, iz_{n-1}, -iw_{n-1})$$

on $\mathbb{C}^{2(n-1)}$. Note that under Ξ the G_β in the real part of the domain become the V_β in the imaginary part of the new domain. Similarly, the light cones in the imaginary part become space-like wedges. That is, $\mathcal{W}_\beta^{(n-1)}$ is analytic in the domain $((Q^- \cup Q^+) - iV_\beta)^{n-1} \cup \Xi\mathcal{N}$, where the right and left wedges are

$$Q^\pm = \big\{(\tau, y) \in \mathbb{R}^2 \mid \pm y > |\tau|\big\}\,.$$

For mutually space-like points (t_i, x_i), $i \in \{1, \ldots, n\}$, equation (3.52) defines the thermal Wightman functions (as analytic functions)

$$\mathcal{W}_\beta^{(n-1)}\big(t_1 - t_2, x_1 - x_2, \ldots, t_{n-1} - t_n, x_{n-1} - x_n\big)\,. \tag{3.53}$$

This is not significant for the rest of the construction, but will be proven in Section 4.2.

As a next step the *strong disk theorem* [11–13,69] is applied to $\mathcal{W}_\beta^{(n-1)}$, which results in an extension of $\mathcal{W}_\beta^{(n-1)}$ into its holomorphic envelope. For the convenience of the reader, the strong disk theorem is stated in Appendix B.

Theorem 3.4.1. *The thermal Wightman functions $\mathcal{W}_\beta^{(n-1)}$ introduced in (3.52) are analytic in the product of domains*

$$(\lambda_1 \mathcal{T}_\beta) \times \cdots \times (\lambda_{n-1}\mathcal{T}_\beta), \qquad \mathcal{T}_\beta := \mathbb{R}^2 - iV_\beta, \qquad \sum_{j=1}^{n-1} \lambda_j = 1, \tag{3.54}$$

and $\lambda_j > 0$, $j \in \{1, \ldots, n-1\}$.

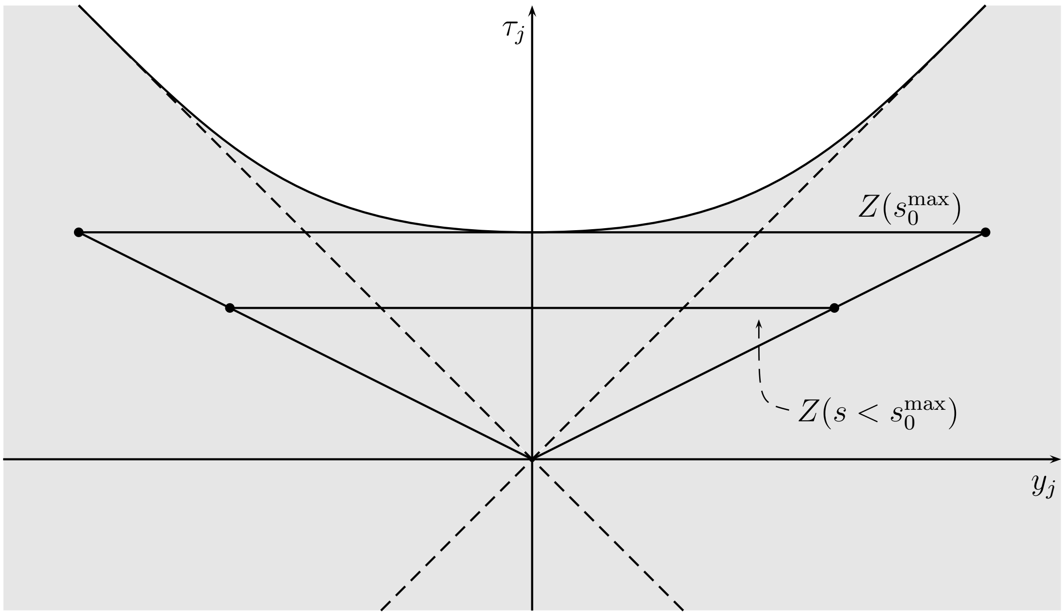

Figure 3.1: Geometric situation in the proof of Theorem 3.4.1. Only the real section is depicted. The grey area corresponds to G_j.

Proof. Let $G_j := (Q^- \cup Q^+ - i\lambda_j V_\beta) \cup \Pi_j(\Xi\mathcal{N})$, $j \in \{1, \ldots, n-1\}$, with Π_j the projection onto the j-th coordinate pair. Define maps $\mathcal{W}_\beta^{(n-1)\restriction j} \colon G_j \to \mathbb{C}$,

$$(z_j, w_j) \mapsto \mathcal{W}_\beta^{(n-1)}(z_1, w_1, \ldots, z_{n-1}, w_{n-1}) \,. \tag{3.55}$$

Since $\Pi_j(\Xi\mathcal{N})$ is an open set in $\mathbb{C}^2$ containing the subset $\{0\} - i\lambda_j V_\beta$, the domain G_j contains an open ball of radius[4] $r_{(\alpha,x)} > 0$ centred at

$$o_j := (0,0) - i(\alpha, x) \in G_j \,, \tag{3.56}$$

where (α, x) is an arbitrary point in $\lambda_j V_\beta$. In order to apply the strong disk theorem fix the curve γ in $\mathbb{C}$ defined by $\gamma_s = s - i\alpha$ for $s \geq 0$ and the disks

$$D(s) := \{w \in \mathbb{C} \mid -2s < \Re w < 2s,\ x - \epsilon < \Im w < x + \epsilon\}$$

for some sufficiently small ϵ to ensure that $\{(\alpha, x') \mid x - \epsilon < x' < x + \epsilon\} \subset \lambda_j V_\beta$. The

[4] Contrary to what was implicitly stated in [23], the radius of the open ball depends on the point (α, x). In fact, the radius $r_{(\alpha,x)} > 0$ has to shrink to zero as $(\alpha, x) \to \partial\lambda_j V_\beta$. Otherwise the edge of the wedge theorem would imply the existence of an open ball centred at the origin (which clearly would include both light-like and time-like points) for which the function $\mathcal{W}_\beta^{(n-1)\restriction j}$ would be analytic.

disks fulfill the conditions of the strong disk theorem. Define the sets

$$Z(s) := \{(\gamma_s, w) \in \mathbb{C}^2 \mid w \in D(s)\} .$$

There exists a maximal $s_0^{\max} > 0$ such that $\mathcal{W}_\beta^{(n-1)\restriction j}$ is analytic at all the points of the set $\bigcup_{0<s<s_0^{\max}} Z(s) \subset G_j$ and some points of the set $Z(s_0^{\max})$. Thus the strong disk theorem implies that $\mathcal{W}_\beta^{(n-1)\restriction j}$ can be analytically continued to all points of the set $Z(s_0^{\max})$ and consequently to an open neighbourhood of $\bigcup_{0<s\leq s_0^{\max}} Z(s)$. There now exists a $s_1^{\max} > s_0^{\max}$ such that $\mathcal{W}_\beta^{(n-1)\restriction j}$ is analytic at all of the points of $\bigcup_{0<s<s_1^{\max}} Z(s)$ and at some of the points of $Z(s_1^{\max})$. Thus the strong disk theorem can be applied again.

Iterating this procedure results in an increasing sequence of positive numbers $\{s_n^{\max}\}_{n\in\mathbb{N}}$. Now assume for the sake of contradiction that this sequence converges to some positive real number. Then the strong disk theorem can immediately be applied again, falsifying the assumption that the sequence converges and in every step $s_n^{\max}$ was maximal. Thus the sequence of positive numbers $\{s_n^{\max}\}_{n\in\mathbb{N}}$ is divergent. By the same line of arguments one constructs a decreasing, divergent sequence of negative numbers $\{s_n^{\min} \in \mathbb{R}^-\}_{n\in\mathbb{N}}$. Consequently $\mathcal{W}_\beta^{(n-1)\restriction j}$ is analytic in $\mathbb{R}^2 \cup \Pi_j(\Xi\mathcal{N})$, for each $j \in \{1, \ldots, n-1\}$ separately.

Applying Hartogs' theorem [69, p. 30] one concludes that $\mathcal{W}_\beta^{(n-1)}$ is analytic in the domain

$$\big((\mathbb{R}^2)^{n-1} - i(\lambda_1 V_\beta \times \ldots \times \lambda_{n-1} V_\beta)\big) \cup (\Xi\mathcal{N}) \supset (\mathbb{R}^2)^{n-1} - i\,(\lambda_1 V_\beta \times \ldots \times \lambda_{n-1} V_\beta)\,, \quad (3.57)$$

which equals the open set described in (3.54). □

The open set V_β does not include the origin, and thus $\mathbb{R}^{2(n-1)} + i\{0\}$ is not a subset of the open set (3.54). Thus it will be the main task of the proof of Theorem 3.4.6, to show that the boundary values of the analytic functions $\mathcal{W}_\beta^{(n-1)}$ as $\Im(z_j, w_j) \searrow 0$ yield tempered distributions.

3.4.2 Products of Sharp-time Fields and their Domains

In (3.18) the thermal time-zero field $\phi_\beta(h)$ has been defined via Stone's theorem. This provides little control on its domain. The results in this subsection provide the information about the domain of $\phi_\beta(h)$, which is necessary for the proof of Theorem 3.4.6.

Lemma 3.4.2. *(i)* (Products of sharp-time fields)*. Let* $h_i \in \mathcal{S}_{\mathbb{R}}(\mathbb{R})$ *for* $i \in \{1, \ldots, j\}$, $j \in \mathbb{N}$, *and* $0 \leq \alpha_1 \leq \ldots \leq \alpha_j < \beta$. *Then*

$$\phi(\alpha_j, h_j) \cdots \phi(\alpha_1, h_1) \in \bigcap_{1 \leq p < \infty} L^p(Q, \Sigma_{[0,\alpha_j]}, \mathrm{d}\mu)\,, \tag{3.58}$$

(ii) (Convergence of sharp-time Schwinger functions, Part II)*. Let* $h_i \in C^{\infty}_{0\,\mathbb{R}}(\mathbb{R})$ *and* $\alpha_i \in S_\beta$, $1 \leq i \leq n$. *Then*

$$\lim_{l \to \infty} \int_Q \phi(\alpha_n, h_n) \cdots \phi(\alpha_1, h_1)\ \mathrm{d}\mu_l = \int_Q \phi(\alpha_n, h_n) \cdots \phi(\alpha_1, h_1)\ \mathrm{d}\mu\,.$$

Proof. (i) Consider an approximation of the Dirac δ-function: $\delta_\kappa(x) := \kappa\chi(\kappa x)$, with χ a function in $C_0^\infty(\mathbb{R})$ and $\int \chi(x)\ \mathrm{d}x = 1$. It has been shown in [22, Proposition 7.3] that

$$\lim_{k \to \infty} \phi\left(\delta_k(\,.\, - \alpha_i) \otimes h_i\right) \in \bigcap_{1 \leq p < \infty} L^p(Q, \Sigma, \mathrm{d}\mu), \qquad h_i \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}).$$

As similar techniques are used in the proof of Lemma 3.4.4 it is sensible to recall the proof[5]:

$$\begin{aligned} &\textstyle\int_Q \big(\phi\left(\delta_k(\,.\, - \alpha_i) \otimes h_i\right)\big)^p\ \mathrm{d}\mu \\ &\qquad = (-i)^p \tfrac{\mathrm{d}^p}{\mathrm{d}\lambda^p} \left(\Omega_C\,,\, W_{[-\infty,+\infty]}\left(\lambda\big(\delta_k(\,.\, - \alpha_i) \otimes h_i\big)\right) \Omega_C\right)\Big|_{\lambda=0}\,, \end{aligned}$$

where $W_{[a,b]}(f)$ is a solution of the heat equation

$$\frac{\mathrm{d}}{\mathrm{d}b} W_{[a,b]}(f) = W_{[a,b]}(f)\big(-H_C + i\phi_C(f_b)\big), \qquad a \leq b,$$

with the boundary condition $W_{[a,a]}(f) = \mathbb{1}$ and with $f_b(\,.\,) := f(\,.\,, b) \in \mathcal{S}_{\mathbb{R}}(S_\beta)$ for $f \in \mathcal{S}_{\mathbb{R}}(S_\beta \times \mathbb{R})$. Now if $f = \delta_k(. - \alpha_i) \otimes h_i$, then the function $f_x \in \mathcal{S}_{\mathbb{R}}(S_\beta)$ is equal to $\delta_k(\,.\, - \alpha_i) h_i(x)$. It follows from (3.32), i.e. estimate (5.9) in [22, Proposition 5.4], that $h_i \in C^\infty_{0\,\mathbb{R}}(\mathbb{R})$ implies

$$\begin{aligned} \pm\phi_C\left(\delta_k(\,.\, - \alpha_i) h_i(x)\right) &\leq c\, \|\delta_k(\,.\, - \alpha_i)\, h_i(x)\|_{H^{-1}(S_\beta)} (H_C + 1) \\ &\leq c\, |h_i(x)|\, \|\delta_k\|_{H^{-1}(S_\beta)} (H_C + 1). \end{aligned}$$

[5] The equation corresponding to this one in the proof of [22, Proposition 7.3] is written only for even p. This is not necessary as is evident from [22, Lemma A.8]

Set $r_k(x) := c\,|h_i(x)|\,\|\delta_k\|_{H^{-1}(S_\beta)}$ and apply [22, Lemma A.8] to obtain

$$\left\| \frac{\mathrm{d}^p}{\mathrm{d}\lambda^p} W_{[-\infty,+\infty]} \left(\lambda \left(\delta_k(\,.\,-\alpha_i) \otimes h_i\right)\right) \right\| \leq p!\; \|r_k\|_\infty^p \; \mathrm{e}^{\|r_k\|_1 \|r_k\|_\infty^{-1}} . \tag{3.59}$$

Since $\delta_k(\,.\,-\alpha_i)$ converges to $\delta(\,.\,-\alpha_i)$ in $H^{-1}(S_\beta)$ and $h_i \in C^\infty_{0\,\mathbb{R}}(\mathbb{R})$ for $i = 1, \ldots, j$, we see that $\lim_{k\to\infty} \|r_k\|_1 < \infty$ and $\lim_{k\to\infty} \|r_k\|_\infty < \infty$. Thus

$$\int_Q (\phi(\alpha_i, h_i))^p \;\; \mathrm{d}\mu < \infty\,. \tag{3.60}$$

For $p \in \mathbb{N}$, part (i) then follows from the Hölder inequality

$$\int_Q |\phi(\alpha_j, h_j) \cdots \phi(\alpha_1, h_1)|^p \;\; \mathrm{d}\mu \leq \prod_{i=1}^{j} \left(\int_Q |\phi(\alpha_i, h_i)|^{p_j} \;\; \mathrm{d}\mu \right)^{1/p_j} .$$

$\Sigma_{[0,\alpha]}$-measurability follows from the fact that (a) for all k there is an ϵ_k (the δ_k were chosen to have compact support) such that

$$\phi\left(\delta_k(\,.\,-\alpha_i) \otimes h_i\right) \in \bigcap_{1\leq p<\infty} L^p(Q, \Sigma_{[0,\alpha_i+\epsilon_k]}, \,\mathrm{d}\mu)$$

and (b) the upper continuity of μ.

The general result follows from the Hölder inequality and the fact, that $\mathrm{d}\phi_C$ is a probability measure.

(ii) Now let $h_i \in C^\infty_{0\,\mathbb{R}}(\mathbb{R})$ and $\alpha_i \in S_\beta$, $1 \leq i \leq n$. Part (ii) follows the proof of [22, Proposition 7.5]. Differentiating equation (7.5) in [22] yields

$$\begin{aligned} &\lim_{l\to\infty} \int_Q \phi(\alpha_n, h_n) \cdots \phi(\alpha_1, h_1) \;\; \mathrm{d}\mu_l \\ &\quad = \lim_{k\to\infty} \frac{\mathrm{d}^j}{\mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_j} \Big(\Omega_C \,, W_{[-a,a]} \Big(\textstyle\sum_{i=1}^j \lambda_i \left(\delta_k(\,.\,-\alpha_i) \otimes h_i\right) \Big) \, \Omega_C \Big) \Big|_{\lambda_i=0} \\ &\quad = \int_Q \phi(\alpha_n, h_n) \cdots \phi(\alpha_1, h_1) \;\; \mathrm{d}\mu \,, \end{aligned}$$

for supp $\delta_k(\,.\,-\alpha_i) \otimes h_i \subset S_\beta \times [-a, a]$, $i \in \{1, \ldots, j\}$, as for $s \leq -a \leq a \leq t$ the map $(s, t) \mapsto (\Omega_C, W_{[s,t]}(f)\Omega_C)$ is constant. □

Taking advantage of their Euclidean heritage now allows for an investigation of the domain of the time-zero fields $\phi_\beta(h)$.

Proposition 3.4.3. *Let $h_i \in C^\infty_{0\,\mathbb{R}}(\mathbb{R})$, $1 \leq i \leq n$. Then*

(i) $\Omega_\beta \in \mathcal{D}(L)$ *and* $L\Omega_\beta = 0$*;*

(ii) If $\alpha_1, \dots, \alpha_n \geq 0$ *and* $\sum_{j=1}^n \alpha_j \leq \beta/2$*, then*

$$\mathrm{e}^{-\alpha_{n-1}L}\phi_\beta(h_{n-1}) \cdots \mathrm{e}^{-\alpha_1 L}\phi_\beta(h_1)\Omega_\beta \in \mathcal{D}\big(\phi_\beta(h_n)\big) \tag{3.61}$$

and

$$\phi_\beta(h_n)\mathrm{e}^{-\alpha_{n-1}L}\phi_\beta(h_{n-1}) \cdots \mathrm{e}^{-\alpha_1 L}\phi_\beta(h_1)\Omega_\beta \in \mathcal{D}\big(\mathrm{e}^{-\alpha_n L}\big). \tag{3.62}$$

Moreover, the linear span of such vectors is dense in $\mathcal{H}_\beta$ *and*

$$\begin{aligned}&\mathrm{e}^{-\alpha_n L}\phi_\beta(h_n)\mathrm{e}^{-\alpha_{n-1}L}\phi_\beta(h_{n-1}) \dots \mathrm{e}^{-\alpha_1 L}\phi_\beta(h_1)\Omega_\beta = \\ &\qquad = \mathcal{V}\Big(U(\alpha_n, 0)\phi(0, h_n)U(\alpha_{n-1}, 0)\phi(0, h_{n-1}) \cdots U(\alpha_1, 0)\phi(0, h_1)\Big);\end{aligned}$$

(iii) If $0 \leq \alpha_1 \leq \dots \leq \alpha_k \leq \beta/2$ *and* $\beta/2 \leq \alpha_{k+1} \leq \dots \leq \alpha_n \leq \beta$*, then*

$$\begin{aligned}&\int_Q \prod_{j=1}^n \phi(\alpha_j, h_j)\ \mathrm{d}\mu \\ &\quad = \big(\mathrm{e}^{(\alpha_n-\beta)L}\phi_\beta(h_n)\mathrm{e}^{(\alpha_{n-1}-\alpha_n)L}\phi_\beta(h_{n-1}) \cdots \mathrm{e}^{(\alpha_{k+1}-\alpha_{k+2})L}\phi_\beta(h_{k+1})\Omega_\beta\,, \\ &\qquad\quad \mathrm{e}^{-\alpha_1 L}\phi_\beta(h_1)\mathrm{e}^{(\alpha_1-\alpha_2)L}\phi_\beta(h_2) \cdots \mathrm{e}^{(\alpha_{k-1}-\alpha_k)L}\phi_\beta(h_k)\Omega_\beta\big)\,.\end{aligned} \tag{3.63}$$

(iv) $\|\mathrm{e}^{-(\beta/2)L}\phi_\beta(h_n) \cdots \phi_\beta(h_1)\Omega_\beta\| = \|\phi_\beta(h_n) \cdots \phi_\beta(h_1)\Omega_\beta\|$.

Proof. At first note that (3.63) formally results from differentiating the following identity, which is a consequence of $\mathrm{e}^{i\phi(0,h_j)} \in L^\infty(Q, \Sigma_{\{0\}}, \mathrm{d}\mu)$ for $h_i \in C^\infty_{0\,\mathbb{R}}(\mathbb{R})$ and the Osterwalder-Schrader reconstruction outlined in Section 3.2, in particular (3.16) and (3.20):

$$\begin{aligned}&\int_Q \prod_{j=1}^n \mathrm{e}^{i\phi(\alpha_j,h_j)}\,\mathrm{d}\mu = \int_Q R\left(\overline{U(\beta,0)\prod_{j=k+1}^n \mathrm{e}^{-i\phi(-\alpha_j,h_j)}}\right)\prod_{j=1}^k \mathrm{e}^{i\phi(\alpha_j,h_j)}\ \mathrm{d}\mu \\ &\quad = \Big(\mathcal{V}\big(U(\beta,0)\mathrm{e}^{-i\phi(-\alpha_n,h_n)} \dots \mathrm{e}^{-i\phi(-\alpha_{k+1},h_{k+1})}\big)\,, \mathcal{V}\big(\mathrm{e}^{i\phi(\alpha_k,h_k)} \cdots \mathrm{e}^{i\phi(\alpha_1,h_1)}\big)\Big) \\ &\quad = \big(\mathrm{e}^{(\alpha_n-\beta)L}\mathrm{e}^{-i\phi_\beta(h_n)}\mathrm{e}^{(\alpha_{n-1}-\alpha_n)L}\mathrm{e}^{-i\phi_\beta(h_{n-1})} \dots \mathrm{e}^{(\alpha_{k+1}-\alpha_{k+2})L}\mathrm{e}^{-i\phi_\beta(h_{k+1})}\Omega_\beta\,, \\ &\qquad\quad \mathrm{e}^{-\alpha_1 L}\mathrm{e}^{i\phi_\beta(h_1)}\mathrm{e}^{(\alpha_1-\alpha_2)L}\mathrm{e}^{i\phi_\beta(h_2)} \dots \mathrm{e}^{(\alpha_{k-1}-\alpha_k)L}\mathrm{e}^{i\phi_\beta(h_k)}\Omega_\beta\big)\,,\end{aligned} \tag{3.64}$$

for $1 \leq i \leq n$, and $0 \leq \alpha_1 \leq \cdots \leq \alpha_k \leq \beta/2$ and $\beta/2 \leq \alpha_{k+1} \leq \ldots \leq \alpha_n \leq \beta$. It has to be ensured, however, that (3.63) is well-defined.

(i) From [48, Lemma 8.4]: $1 \in \mathcal{M}_\alpha$. (As before $\mathcal{M}_\alpha = L^2(Q, \Sigma_{[0,\beta/2-\alpha]}, \mathrm{d}\mu)$.) Thus $\Omega_\beta \in \mathcal{D}_\alpha$ and $\mathrm{e}^{-\alpha L}\Omega_\beta = P(\alpha)\Omega_\beta = \Omega_\beta$ as $U(\alpha, 0)1 = 1$ for $0 \leq \alpha \leq \beta$;

(ii) The case $n = 1$, namely $\Omega_\beta \in \mathcal{D}\left(\phi_\beta(h_1)\right)$ and

$$\mathrm{e}^{-\alpha_1 L}\phi_\beta(h_1)\Omega_\beta \in \mathcal{H}_\beta \quad \text{for} \quad 0 \leq \alpha_1 \leq \beta/2$$

was proven in [23, Lemma 2]. Alternatively, it is a direct consequence of Lemma 3.1.6. In fact,

$$\mathrm{e}^{-\alpha_1 L}\phi_\beta(h_1)\Omega_\beta \in \mathcal{D}(\phi_\beta(h_2))\,,$$

as $\phi(0, h_2)$ acts as a multiplication operator on $\phi(\alpha_1, h_1)$ and

$$\phi(0, h_2)\phi(\alpha_1, h_1) \in \mathcal{M}_{\beta/2-\alpha_1}$$

by Lemma 3.4.2 (i). As $P(\alpha)\mathcal{D}_\gamma \subset \mathcal{D}_{\gamma-\alpha}$, it follows that

$$\mathrm{e}^{-\alpha_2 L}\phi_\beta(h_2)\mathrm{e}^{-\alpha_1 L}\phi_\beta(h_1)\Omega_\beta \in \mathcal{D}_{\beta/2-\alpha_1-\alpha_2}$$

and $\phi(0, h_3)\phi(\alpha_2, h_2)\phi(\alpha_1 + \alpha_2, h_1) \in \mathcal{M}_{\beta/2-\alpha_1-\alpha_2}$ implies

$$\mathrm{e}^{-\alpha_2 L}\phi_\beta(h_2)\mathrm{e}^{-\alpha_1 L}\phi_\beta(h_1)\Omega_\beta \in \mathcal{D}(\phi_\beta(h_3))\,.$$

Iterating this argument it follows that

$$\mathcal{V}\big(\phi(\alpha_k, h_k)\cdots\phi(\alpha_1 + \ldots + \alpha_k, h_1)\big) \in \mathcal{D}_{\beta/2-\gamma},$$

if $\sum_{i=1}^{k} \alpha_k \leq \gamma \leq \beta/2$. Thus (3.61) and (3.62) follow.

Next it has to be proven that vectors of the form

$$\mathrm{e}^{-\alpha_n L}\phi_\beta(h_n)\mathrm{e}^{-\alpha_{n-1} L}\phi_\beta(h_{n-1})\ldots\mathrm{e}^{-\alpha_1 L}\phi_\beta(h_1)\Omega_\beta$$

are dense in $\mathcal{H}_\beta$ for $\alpha_1, \ldots, \alpha_n \geq 0$ and $\sum_{j=1}^{n} \alpha_j \leq \beta/2$. Assume that for some $\Psi \in \mathcal{H}_\beta$,

$$\forall m, n \in \mathbb{N}: \qquad (\Psi, \phi_\beta(f)^n \mathrm{e}^{-\beta L/2}\phi_\beta(g)^m\Omega_\beta) = 0. \tag{3.65}$$

(Note that (3.65) is well-defined as a consequence of (3.62).) Then

$$(\Psi\,,\, \mathrm{e}^{i\phi_\beta(h_i)}\mathrm{e}^{-\beta L/2}\mathrm{e}^{i\phi_\beta(h_j)}\Omega_\beta) = 0\,. \tag{3.66}$$

But vectors of the form $\mathrm{e}^{i\phi_\beta(h_i)}\mathrm{e}^{-\beta L/2}\mathrm{e}^{i\phi_\beta(h_j)}\Omega_\beta$ are dense [49, Theorem 11.2] in $\mathcal{H}_\beta$, and therefore (3.66) implies $\Psi = 0$, establishing the claim.

(iii) If $0 \le \alpha_1 \le \ldots \le \alpha_k \le \beta/2$ and $\beta/2 \le \alpha_{k+1} \le \ldots \le \alpha_n \le \beta$, then according to (ii)

$$\Big(\mathrm{e}^{(\alpha_n-\beta)L}\phi_\beta(h_n)\mathrm{e}^{(\alpha_{n-1}-\alpha_n)L}\phi_\beta(h_{n-1})\ldots\mathrm{e}^{(\alpha_{k+1}-\alpha_{k+2})L}\phi_\beta(h_{k+1})\Omega_\beta\;,\\ \mathrm{e}^{-\alpha_1 L}\phi_\beta(h_1)\mathrm{e}^{-(\alpha_2-\alpha_1)L}\phi_\beta(h_2)\ldots\mathrm{e}^{-(\alpha_k-\alpha_{k-1})L}\phi_\beta(h_k)\Omega_\beta\Big)$$

is well-defined and equals

$$\begin{aligned}
&\Big(\mathcal{V}\big(\phi(\beta-\alpha_n,h_n)\ldots\phi(\beta-\alpha_{k+1},h_{k+1})\big)\;,\mathcal{V}\big(\phi(\alpha_k,h_k)\cdots\phi(\alpha_1,h_1)\big)\Big) = \\
&= \int_Q R\left(\overline{\textstyle\prod_{j=k+1}^n \phi(\beta-\alpha_j,h_j)}\right)\textstyle\prod_{j=1}^k \phi(\alpha_j,h_j)\;\mathrm{d}\mu \\
&= \int_Q R\left(\overline{U(\beta,0)\textstyle\prod_{j=k+1}^n \phi(-\alpha_j,h_j)}\right)\textstyle\prod_{j=1}^k \phi(\alpha_j,h_j)\;\mathrm{d}\mu \\
&= \int_Q R\left(\textstyle\prod_{j=k+1}^n \phi(-\alpha_j,h_j)\right)\textstyle\prod_{j=1}^k \phi(\alpha_j,h_j)\;\mathrm{d}\mu \\
&= \int_Q \left(\textstyle\prod_{j=k+1}^n \phi(\alpha_j,h_j)\right)\textstyle\prod_{j=1}^k \phi(\alpha_j,h_j)\;\mathrm{d}\mu \\
&= \int_Q \textstyle\prod_{j=1}^n \phi(\alpha_j,h_j)\;\mathrm{d}\mu\,.
\end{aligned} \tag{3.67}$$

It was used, that $U(\beta,0) = \mathbb{1}$, which holds by periodicity.

(iv) By (ii) we have $\phi_\beta(h_n)\phi_\beta(h_{n-1})\cdots\phi_\beta(h_1)\Omega_\beta \in \mathcal{D}\big(\mathrm{e}^{-\beta L/2}\big)$. Now

$$\begin{aligned}
&\left\|\mathrm{e}^{-\beta L/2}\phi_\beta(h_n)\phi_\beta(h_{n-1})\cdots\phi_\beta(h_1)\Omega_\beta\right\|^2 = \\
&= \left\|\mathcal{V}\big(U(\beta/2,0)\phi(0,h_n)\cdots\phi(0,h_1)\big)\right\|^2 \\
&= \int_Q \overline{U(\beta/2,0)\phi(0,h_n)\cdots\phi(0,h_1)}\; R\,U(\beta/2,0)\;\phi(0,h_n)\cdots\phi(0,h_1)\;\mathrm{d}\mu \\
&= \int_Q \overline{\phi(0,h_n)\cdots\phi(0,h_1)}\; U(-\beta/2,0)\,R\,U(\beta/2,0)\;\phi(0,h_n)\cdots\phi(0,h_1)\;\mathrm{d}\mu \\
&= \int_Q \phi(0,h_n)\cdots\phi(0,h_1)\;R\,U(\beta,0)\;\phi(0,h_n)\cdots\phi(0,h_1)\;\mathrm{d}\mu \\
&= \|\mathcal{V}\left(\phi(0,h_n)\cdots\phi(0,h_1)\right)\|^2 = \|\phi_\beta(h_n)\phi_\beta(h_{n-1})\cdots\phi_\beta(h_1)\Omega_\beta\|^2 \,,
\end{aligned}$$

again using $U(\beta,0)=1$. □

The extension of these results to real times is the next objective. Given the self-adjoint operator $\phi_\beta(h)$, $h \in C^\infty_{0\mathbb{R}}(\mathbb{R})$, set

$$\phi_\beta(t,h) := \mathrm{e}^{itL}\phi_\beta(h)\mathrm{e}^{-itL}, \qquad t \in \mathbb{R}\,. \tag{3.68}$$

The domain of the self-adjoint operator $\phi_\beta(t,h)$ is $\mathrm{e}^{itL}\mathcal{D}(\phi_\beta(h))$. That products of field operators smeared out in time can be applied to the distinguished vector Ω_β will be shown in the final subsection.

3.4.3 Temperedness of the Thermal Wightman Distributions

In this subsection the main theorem of this chapter is stated and proven. For the proof one more lemma is required.

Lemma 3.4.4. *For $h \in \mathcal{S}_\mathbb{R}(\mathbb{R})$ and even $p \in \mathbb{N}$ the expressions*

$$|\!|\!|\, h|\!|\!|_p := \max\nolimits_\pm \left[\, \int_Q \prod\nolimits_{k=1}^p \phi_\pm\!\left(\tfrac{k\beta}{p}, h\right)\;\mathrm{d}\mu\,\right]^{\frac{1}{p}} \tag{3.69}$$

are bounded from above by $\sqrt[p]{p!}\cdot|h|_\mathcal{S}$, for some Schwarz norm $|\,.\,|_\mathcal{S}$. $\phi_\pm(\alpha,h)$ denotes the positive and negative part[6] *of $\phi(\alpha,h)$, respectively.*

[6]That is, $\phi(\alpha,h) = \phi_+(\alpha,h) - \phi_-(\alpha,h)$ such that both $\phi_\pm(\alpha,h)$ are positive. $\phi_\pm(\alpha,h)$ necessarily have disjoint supports.

Proof. Let $p \in \mathbb{N}$ be even, $\alpha \in S_\beta$ and $h \in \mathcal{S}_{\mathbb{R}}(\mathbb{R})$. By the Hölder inequality on $L^p(Q, \Sigma, \mathrm{d}\mu)$ there holds

$$\||h\||_p^p \le \max_{\pm} \prod_{k=1}^{p} \int_Q \phi_\pm\Big(\delta(\,.\, - \tfrac{k\beta}{p}) \otimes h\Big)^p \,\mathrm{d}\mu\,. \tag{3.70}$$

Since

$$\begin{aligned} \int_Q \phi(\alpha,h)^p \,\mathrm{d}\mu &= \int_{\mathrm{supp}\,\phi_+(\alpha,h)} \phi_+(\alpha,h)^p \,\mathrm{d}\mu + \int_{\mathrm{supp}\,\phi_-(\alpha,h)} \phi_-(\alpha,h)^p \,\mathrm{d}\mu \\ &\ge \int_{\mathrm{supp}\,\phi_\pm(\alpha,h)} \phi_\pm(\alpha,h)^p \,\mathrm{d}\mu\,, \end{aligned} \tag{3.71}$$

it will be sufficient to prove the estimate for the l.h.s. of (3.71).

Taking advantage of the fractional ϕ-bound (3.33) the estimate [22, Lemma A.7, p. 167], for $R_1 = 0$ and $R_2 = \phi_C(\delta(\cdot - k\beta/p)h)$, can now be applied, which states that there is a constant $c' > 0$ such that

$$\left| \int \mathrm{d}\mu\; \mathrm{e}^{i\lambda\phi\left(\delta(\,\cdot\, - \frac{k\beta}{p})\otimes h\right)} \right| \le \mathrm{e}^{c'|\Im\lambda|^\gamma \|r\|_\gamma^\gamma} \tag{3.72}$$

for $\lambda \in \mathbb{C}$, $\gamma = (\frac{1}{2} - \epsilon)^{-1}$, $0 \le \epsilon < 1/2$, and $r(x) = \lim_{\ell\to\infty} r_\ell(x)$, with

$$r_\ell(x) := c\,|h(x)|\, \|\delta_\ell\|_{H^{-\frac{1}{2}-\frac{\epsilon}{2}}(S_\beta)}\,, \qquad c > 0\,. \tag{3.73}$$

This limit $\ell \to \infty$ exists, as the Dirac δ-function is in all Sobolev spaces H^q for $q < -1/2$.

From Proposition A.6 (i) and Theorem 7.2 (i) in [22] it follows that $\lambda \mapsto \int \mathrm{d}\mu\; \mathrm{e}^{i\lambda\,\phi(\delta\otimes h)}$ is entire (Note that $W_{[a,b]}(\lambda f) = U_\lambda(a,b)^*$). Applying Cauchy's formula on the circle of radius R centred around $\lambda \in \mathbb{R}$ yields

$$\int_Q \phi_\pm\left(\frac{k\beta}{p}, h\right)^p \,\mathrm{d}\mu \le \int_Q \phi\left(\frac{k\beta}{p}, h\right)^p \,\mathrm{d}\mu \le p!\, R^{-p} \mathrm{e}^{c' R^\gamma \|r\|_\gamma^\gamma}\,, \tag{3.74}$$

which can be improved by optimising with respect to R,

$$\begin{aligned} \left| \int_Q \phi_\pm\Big(\delta(\,.\, - \tfrac{k\beta}{p}) \otimes h\Big)^p \,\mathrm{d}\mu \right| &\le p! \left(\frac{c\gamma \mathrm{e}}{p}\right)^{p/\gamma} \|r\|_\gamma^p \\ &\le p!\, |h|_{\mathcal{S}}^p\,. \end{aligned} \tag{3.75}$$

Two things have been used here: $\sup_{p\in\mathbb{N}}\left(\frac{c\gamma \mathrm{e}}{p}\right)^{p/\gamma} < \infty$ and the fact that

$$\|r\|_\gamma = c\;\|\delta_\ell\|_{H^{-1/2-\epsilon}(S_\beta)}\left(\int |h(x)|^\gamma\,\mathrm{d}x\right)^{1/\gamma} \tag{3.76}$$

can be estimated by a Schwartz norm, if $h \in \mathcal{S}(\mathbb{R})$ and $\gamma > 2$. Finally, putting (3.70), (3.71) and (3.75) together results in

$$|\!|\!|\, h |\!|\!|_p^p \le p!\;|h|_{\mathcal{S}}^p\,, \tag{3.77}$$

which establishes the lemma. □

Remark 3.4.5. *Making use of the ϕ-bound (3.31) one can use the equation preceding [22, Equ. (A.9)] to arrive at Fröhlich's bound*

$$\int \mathrm{e}^{\pm\phi(g\otimes h)}\,\mathrm{d}\mu \le \mathrm{e}^{c\int_{\mathbb{R}}\,\mathrm{d}x\,|h(x)|^2 C_\beta(g,g)}\;,\quad g\in H^{-1/2}(S_\beta)\;,\;\; h\in L^2_{\mathbb{R}}(\mathbb{R})\;, \tag{3.78}$$

stated (for the special case $g = \mathbb{1}_{[0,l]}$ a characteristic function) in [18, Equ. (7)]. However, using only this bound, the author was unable to establish the existence of the products estimated in Lemma 3.4.4.

Theorem 3.4.6. *(i) The boundary values of the functions $\mathcal{W}_\beta^{(n-1)}$, which are analytic in*

$$(\lambda_1\mathcal{T}_\beta)\times\cdots\times(\lambda_{n-1}\mathcal{T}_\beta),\qquad \mathcal{T}_\beta := \mathbb{R}^2 - iV_\beta, \tag{3.79}$$

where $\lambda_i > 0$, $i\in\{1,\dots,n-1\}$ and $\sum_{i=1}^{n-1}\lambda_i = 1$, exist as tempered distributions denoted by

$$\mathfrak{W}_\beta^{(n-1)} \in \mathcal{S}'(\mathbb{R}^2)\,. \tag{3.80}$$

The $\mathfrak{W}_\beta^{(n-1)}$ are called thermal Wightman distributions of the $P(\phi)_2$ model.

(ii) The $\mathfrak{W}_\beta^{(n-1)}$ satisfy the KMS boundary condition: set $s_k = t_k - t_{k+1}$ and $y_k = x_k - x_{k+1}$, $1\le k < n$. In addition, set $s_n = t_n - t_1$ and $y_n = y_n - y_1$. Then

$$\begin{aligned}\mathfrak{W}_\beta^{(n-1)}\big(s_1,y_1,\dots,s_{k-1},y_{k-1},s_k - i\beta, y_k,\dots,s_{n-1},y_{n-1}\big) &=\\ = \mathfrak{W}_\beta^{(n-1)}\big(s_k,y_k,\dots,s_n,y_n,s_1,y_1,\dots,s_{k-2},y_{k-2}\big)&\end{aligned} \tag{3.81}$$

for all $(s_1, y_1 \dots, s_{n-1}, y_{n-1}) \in \mathbb{R}^{2(n-1)}$.

Remark 3.4.7. *For two space-time dimensions it is expected, that the Wightman distributions only have to be smeared out in the space coordinate, i.e. that* $\mathfrak{W}_\beta^{(n-1)}$ *is a continuous map on* $(\mathbb{R} \times \mathcal{S}(\mathbb{R}))^{n-1}$.

Proof. At first it is proven that the thermal Wightman functions are tempered distributions. Let $\{\phi_\ell(h) \in \mathcal{R}_\beta\}_{\ell \in \mathbb{N}}$ be a sequence of bounded elements approximating $\phi_\beta(h)$ in the strong topology. Denote the positive and negative parts[7] of $\phi_\ell(h)$ by $\phi_\ell(h)_\pm$. Then Theorem 2.1.1 and the linearity of ω_β implies, that each of the 2^n terms arising from the linear polar decomposition of the $\phi_\ell(h)$ can be estimated by

$$\begin{aligned}
&\lim_{\ell_i \to \infty} \left| \left(\Omega_\beta,\, \phi_{\ell_1}(h_1)_\pm\, \mathrm{e}^{-(\alpha_1 + it_1)L} \cdots \phi_{\ell_{n-1}}(h_{n-1})_\pm\, \mathrm{e}^{-(\alpha_{n-1}+it_{n-1})L} \phi_{\ell_n}(h_n)_\pm\, \Omega_\beta\right)\right| \\
&\leq\ |\!|\!| \phi_\beta(h_1)_\pm |\!|\!|_{p_1(\alpha_1)} \cdots |\!|\!| \phi_\beta(h_n)_\pm |\!|\!|_{p_n(\alpha_n)} \qquad (3.82)\\
&\leq\ \frac{p_1(\alpha_1)}{2} \cdots \frac{p_n(\alpha_n)}{2} \cdot |h_1|_{\mathcal{S}} \cdots |h_n|_{\mathcal{S}}\ , \qquad (3.83)
\end{aligned}$$

for $0 < \alpha_j$, $j \in \{1, \dots, n-1\}$, $\sum \alpha_j < \beta/2$ and with $p_i \equiv p_i(\alpha_i)$ the smallest even natural number such that

$$\frac{1}{p_i(\alpha_i)} < \frac{1}{\beta} \min\{\alpha_{i-1},\, \alpha_i\}\ , \qquad i \in \{1, \dots, n-1\}\,. \qquad (3.84)$$

(Setting $\alpha_0 = \alpha_1$.) Lemma 3.4.4 as well as Proposition 3.4.3 (iii) have been used in (3.82) to conclude that for p even and sufficiently large[8] there holds

$$|\!|\!| \phi_\beta(h)_\pm |\!|\!|_p = \left(\int_Q \prod_{k=1}^{p} \phi_\pm\!\left(\frac{k\beta}{p}, h\right)\ \mathrm{d}\mu \right)^{\frac{1}{p}} \leq |\!|\!|\, h |\!|\!|_p \leq \sqrt[p]{p!} \cdot |h|_{\mathcal{S}} < \frac{p}{2} \cdot |h|_{\mathcal{S}}\,. \qquad (3.85)$$

It will now be shown, following ideas in [60, p. 24], that this bound ensures that the boundary values exist as tempered distributions as $\alpha_j \searrow 0$: let $\underline{t} = (t_1, \dots, t_{n-1})$ and $\underline{\alpha} = (\alpha_1, \dots, \alpha_{n-1})$, and set

$$\mathbf{W}(\underline{t} - i\underline{\alpha}) := \lim_{\ell_i \to \infty} \left| \left(\Omega_\beta, \phi_{\ell_1}(h_1)\mathrm{e}^{-(\alpha_1+it_1)L} \cdots \phi_{\ell_{n-1}}(h_{n-1})\mathrm{e}^{-(\alpha_{n-1}+it_{n-1})L} \phi_{\ell_n}(h_n)\Omega_\beta\right)\right|$$

[7] In the sequel it will be important that for real $A \in L^\infty(Q, \Sigma_{\{0\}}, \mathrm{d}\mu)$ there holds $\pi_\beta(A_\pm) = \pi_\beta(A)_\pm$.
[8] Recall that $p! < (p/2)^p$ for $p \geq 6$.

Now define, for $\lambda \in (0,1]$ fixed, a tempered distribution $T_{\underline{\alpha}}(\lambda) \in \mathcal{S}'(\mathbb{R}^{n-1})$ by

$$T_{\underline{\alpha}}(\lambda)(g) := \int_{\mathbb{R}^{n-1}} \mathrm{d}\underline{t}\; \mathbf{W}(\underline{t} - i\lambda\underline{\alpha})\, g(\underline{t})\;, \qquad g \in \mathcal{S}(\mathbb{R}^{n-1})\,. \tag{3.86}$$

Let $T^{(k)}_{\underline{\alpha}}(\lambda)$, $k = 1, 2, \ldots$, denote the k-th distributional derivatives

$$T^{(k)}_{\underline{\alpha}}(\lambda)(g) := \frac{\partial^k}{\partial\lambda^k} T_{\underline{\alpha}}(\lambda)(g) = \int_{\mathbb{R}^{n-1}} \mathrm{d}\underline{t}\; \mathbf{W}(\underline{t} - i\lambda\underline{\alpha}) \left(i\underline{\alpha} \cdot \frac{\partial}{\partial \underline{t}} \right)^k g(\underline{t})\;. \tag{3.87}$$

Thus, by the fundamental theorem of calculus,

$$\begin{aligned} T_{\underline{\alpha}}(\lambda) \;\; &= \;\; T_{\underline{\alpha}}(1) + \sum_{j=1}^{k-1} Q_j(\lambda)\, T^{(j)}_{\underline{\alpha}}(1) \\ &\quad - \int_{\lambda}^{1} \mathrm{d}\lambda_k \int_{\lambda_k}^{1} \mathrm{d}\lambda_{k-1} \cdots \int_{\lambda_2}^{1} \mathrm{d}\lambda_1\; T^{(k)}_{\underline{\alpha}}(\lambda_1)\;. \end{aligned} \tag{3.88}$$

The Q_j's in (3.88) are suitable polynomials. The limit $\lambda \downarrow 0$ in (3.88) can be taken, provided that

$$\lim_{\lambda\downarrow 0} \left| \int_{\lambda}^{1} \mathrm{d}\lambda_k \int_{\lambda_k}^{1} \mathrm{d}\lambda_{k-1} \cdots \int_{\lambda_2}^{1} \mathrm{d}\lambda_1\; T^{(k)}_{\underline{\alpha}}(\lambda_1) \right| < \infty\;. \tag{3.89}$$

This is done by estimating $T^{(k)}_{\underline{\alpha}}(\lambda)$ as given in (3.87): choose some $m \in \mathbb{N}$ large enough so that $\int_{\mathbb{R}^{n-1}} \mathrm{d}\underline{t}\,(1 + |\underline{t}|)^{-m} < \infty$. Then, for $\lambda \in (0,1]$,

$$\begin{aligned} \left|T^{j}_{\underline{\alpha}}(\lambda)(g)\right| &= C \sup_{\underline{t}\in\mathbb{R}^{n-1}} |(1 + |\underline{t}|)^m| \left| \left(i\underline{\alpha} \cdot \frac{\partial}{\partial \underline{t}} \right)^j g(\underline{t}) \right| p_1(\lambda\alpha_1) \cdots p_n(\lambda\alpha_n) \cdot |h_1|_{\mathcal{S}} \cdots |h_n|_{\mathcal{S}} \\ &\leq C' \cdot \lambda^{-n}\,, \qquad C, C' > 0\,. \end{aligned} \tag{3.90}$$

Note that

$$\lim_{\lambda\downarrow 0} \left| \int_{\lambda}^{1} \mathrm{d}\lambda_k \int_{\lambda_k}^{1} \mathrm{d}\lambda_{k-1} \cdots \int_{\lambda_2}^{1} \mathrm{d}\lambda_1\; \lambda_1^{-n} \right| < \infty \tag{3.91}$$

for k sufficiently large, i.e. $k > n + 2$. Combining (3.89), (3.90), and (3.91) one concludes that the limit of $T_{\underline{\alpha}}(\lambda)$ exists as $\lambda \downarrow 0$ and that each term in the limit is less than or equal to a constant times an $\mathcal{S}(\mathbb{R}^{n-1})$-semi-norm of g. Thus $\mathbf{W}(\underline{s} - i\underline{\alpha})$ converges in $\mathcal{S}'(\mathbb{R}^{n-1})$ as $\underline{\alpha} \downarrow 0$ to a tempered distribution, denoted by $W^{(n)}_{\beta}(h_1, t_1, h_2, t_2, \ldots, t_{n-1}, h_n)$. The limit does not depend on the direction at which the imaginary part goes to zero. This is completely analogous to the proof of [60, Theorem IX.16; p.25]

Translation invariance implies, that there exists a tempered distribution $\mathfrak{W}_\beta^{(n-1)}$ such that

$$\mathcal{W}_\beta^{(n)}(x_1, t_1, x_2, t_2, \ldots, t_{n-1}, x_n) = \mathfrak{W}_\beta^{(n-1)}(t_1, x_1 - x_2, \ldots, t_{n-1}, x_{n-1} - x_n) \,. \qquad (3.92)$$

The analyticity property stated in (i) equal those stated in Theorem 3.4.1. The KMS boundary condition follows by differentiating (see (3.18)) the boundary condition of the corresponding Weyl operators given in (3.28). □

Chapter 4

Properties of the Thermal $P(\phi)_2$ Model

4.1 Verification of the Wightman Axioms

For $f = g \otimes h \in \mathcal{S}(\mathbb{R}^2)$, h real-valued, define[1]

$$\phi_\beta(f) := \int \mathrm{d}t\, g(t)\, \phi_\beta(t, h)\,. \tag{4.1}$$

($\phi_\beta(t,h)$ has been defined in (3.68).) As a direct consequence of Theorem 3.4.6 the following limit exists for $f_k \in \mathcal{S}(\mathbb{R}^2)$, $k \in \{1, \ldots, n\}$ and $\alpha_j > 0$ small enough,

$$\begin{aligned} W_\beta^{(n)}(f_1 \otimes \cdots \otimes f_n) &:= \lim_{\alpha_j \to 0} \left(\Omega_\beta,\, \phi_\beta(f_1)\, \mathrm{e}^{-\alpha_1 L} \cdots \phi_\beta(f_{n-1})\, \mathrm{e}^{-\alpha_{n-1} L}\, \phi_\beta(f_n)\, \Omega_\beta\right) \\ &= \left(\Omega_\beta,\, \phi_\beta(f_1) \cdots \phi_\beta(f_n)\, \Omega_\beta\right)\,. \end{aligned} \tag{4.2}$$

By nuclearity it defines a distribution $W_\beta^{(n)} \in \mathcal{S}'(\mathbb{R}^{2n})$. It is the purpose of this section to establish, that the Wightman functions $W_\beta^{(n)}$ satisfy the Wightman axioms.

Axioms W1 (Wightman Distributions), W2 (Hermiticity) and W4 (Positive Definiteness) are a direct consequence of (4.2) (for positive definiteness cf. [65, Proof of Theorem 3-3]). Axiom W5 is a consequence of the translation invariance of the state ω_β defined in (3.25). The remaining axioms are treated in the following two lemmas.

At first, some notation has to be introduced. For an interval $I \subset \mathbb{R}$, let $\mathcal{U}_{AW}(I)$ be

[1] There is a notational collision with the thermal time-zero field. This will not be a problem since it will always be clear whether the test function is in $\mathcal{S}(\mathbb{R})$ or $\mathcal{S}(\mathbb{R}^2)$.

the Abelian algebra generated by

$$\{\mathrm{e}^{i\phi_\beta(h)} \mid h \in C^\infty(\mathbb{R}),\ \mathrm{supp}\, h \subset I,\ h \text{ real-valued}\}\,,$$

and let $\mathcal{B}_\alpha(I)$ denote the von Neumann algebra generated by

$$\{\tau_t(A) \,|\, A \in \mathcal{U}_{AW}(I),\ |t| < \alpha\}\,.$$

Define

$$\mathfrak{h}_I := \{h \in H^{-1/2}(\mathbb{R}) \mid \mathrm{supp}\, h \subset I\}\,, \tag{4.3}$$

and let $\mathcal{R}_{AW}(I)$ be the von Neumann algebra generated by

$$\{\mathrm{e}^{i\phi_{AW}(h)} \mid h \in \mathfrak{h}_I\}\,,$$

where $\phi_{AW}(h)$ is defined as in (1.84). $\mathcal{R}_{AW}(I)$ can be thought of as the algebra of observables of the free field in the double-cone with base I [1].

Lemma 4.1.1 (Locality). *The thermal Wightman functions $W_\beta^{(n)}$ of the $P(\phi)_2$ model satisfy Axiom W3.*

Proof. It has been shown in [22, p.146], that $\mathcal{B}_\alpha(I) \subset \mathcal{R}_{AW}(I+]-\alpha,\alpha[)$. Let $f_1 = g_1 \otimes h_1$ and $f_2 = g_2 \otimes h_2$ both in $\mathcal{S}(\mathbb{R}^2)$ such that h_1 and h_2 are real-valued and such that supp f_1 and supp f_2 are space-like separated with respect to each other. Then there holds for all A and B in $\mathcal{R}_\beta$

$$\begin{aligned}\int \mathrm{d}t_1\ \mathrm{d}t_2\, g_1(t_1)\, g_2(t_2)\, \omega_\beta\big(A\, \mathrm{e}^{i\phi_\beta(t_1,h_1)}\, \mathrm{e}^{i\phi_\beta(t_2,h_2)} B\big)& \\ = \int \mathrm{d}t_1\ \mathrm{d}t_2\, g_1(t_1)\, g_2(t_2)\, \omega_\beta\big(A\, \mathrm{e}^{i\phi_\beta(t_2,h_2)}\, \mathrm{e}^{i\phi_\beta(t_1,h_1)}\, B\big)&\,.\end{aligned} \tag{4.4}$$

Denote the algebra generated by elements of the form $\tau_t(\mathrm{e}^{i\phi_\beta(h)})$ by $\mathcal{C} \subset \mathcal{R}_\beta$. Let now $B \in \mathcal{C}$. Then $B\Omega_\beta \in \mathcal{D}(\phi_\beta(f_1)\phi_\beta(f_2)) \cap \mathcal{D}(\phi_\beta(f_2)\phi_\beta(f_1))$. Since there holds

$$\begin{aligned}&\big(\Omega_\beta,\, A\,\phi_\beta(f_1)\,\phi_\beta(f_2)\,B\,\Omega_\beta\big)\\ &= \partial_{s_1}\partial_{s_2}\,\omega_\beta\left(A\int \mathrm{d}t_1\, g_1(t_1)\, \mathrm{e}^{is_1\phi_\beta(t_1,h_1)} \int \mathrm{d}t_2\, g_2(t_2)\, \mathrm{e}^{is_2\phi_\beta(t_2,g_2)}\, B\right)\Bigg|_{s_1=s_2=0}\,,\end{aligned} \tag{4.5}$$

equation (4.4) implies

$$(\Omega_\beta,\, A\,[\phi_\beta(f_1), \phi_\beta(f_2)]\, B\ \Omega_\beta) = 0\,. \tag{4.6}$$

Since A and B can be freely chosen in $\mathcal{C}$, Axiom W3 can be proven by differentiating these Weyl operators. □

Lemma 4.1.2 (Space-like clustering). *The Wightman functions $W_\beta^{(n)}$ of the thermal $P(\phi)_2$ model fulfil Axiom W6.*

Proof. In [22, Lemma 7.7] it has been shown that the state ω_β is space-like clustering:

$$\forall A, B \in \mathcal{A}: \qquad \lim_{x\to\infty} \omega_\beta(A\,\sigma_x(B)) = \omega_\beta(A)\,\omega_\beta(B)\ . \tag{4.7}$$

Invoking the cyclicity of Ω_β for $\mathcal{R}_\beta$, elements of the form

$$\phi_\beta(f_1)\cdots\phi_\beta(f_n)\,\Omega_\beta\,, \tag{4.8}$$

where $f_j \in \mathcal{S}(\mathbb{R}^2)$, $j \in \{1,\ldots,n\}$, can be approximated by sequences in $\mathcal{R}_\beta\,\Omega_\beta$. The result follows. □

By Theorem 3.4.6 and the results of this section the following theorem is established.

Theorem 4.1.3. *The thermal Wightman distributions $W_\beta^{(n)}$ satisfy the Wightman axioms W1 to W6 and the relativistic KMS condition 1.2.11.*

At this stage it would be very satisfying to prove both equations of motion (1.50) and (1.49) in the infinite volume limit[2]. For Minkowski signature such proofs are conspicuously absent from the text books. The reason might be, that there is no control on the domain of the interaction, viewed as unbounded operator on $\mathcal{H}_\beta$. In the Euclidean measure space, however, the verification of the equation of motion is manageable.

Proposition 4.1.4 (Euclidean equation of motion). *For h_1 and h_2 both in $\mathcal{S}_\mathbb{R}(\mathbb{R})$, define*[3]

$$A(h_1, h_2) := \mathrm{e}^{i\phi(\delta\otimes h_1)}\ \big(U(\beta/2, 0)\mathrm{e}^{i\phi(\delta\otimes h_2)}\big)\ . \tag{4.9}$$

[2] For the vacuum field on the cylinder cf. [60, pp 224].

[3] The existence of functions of this form follows from Lemma 3.1.6. Recall that they are dense in $\mathcal{H}_\beta$.

For all $f \in \mathcal{S}_{\mathbb{R}}\left((0,\beta/2)\times\mathbb{R}\right)$ *there holds*

$$\int \mathrm{d}\mu\, A(h_1,h_2)\left(\phi\left((-\triangle+m^2)f\right)+\int \mathrm{d}\alpha\ \mathrm{d}x\, f(\alpha,x)\ :P'(\phi(\alpha,x)):_C\right)=0\,. \tag{4.10}$$

Proof. At first observe that

$$C\left((-\triangle+m^2)f,g\right)=(f,g)_{L^2(S_\beta\times\mathbb{R})}\,. \tag{4.11}$$

For a function g on $S_\beta\times\mathbb{R}$, which implements the spatial cutoff in the interaction, set

$$V(g)=\int_{S_\beta\times\mathbb{R}} \mathrm{d}x\, g(x)\ :P(\phi(x)):_C\ . \tag{4.12}$$

For $l>0$ let χ_l be the characteristic function on $S_\beta\times[-l,l]$. Then $\mathrm{d}\mu_l=Z_l^{-1}\mathrm{e}^{-V(\chi_l)}\,\mathrm{d}\phi_C$ (as in (3.4)). By the general integration by parts identity [29, 9.1.32],

$$\int A\,\phi(f)\ \mathrm{d}\mu_l=\int C\left(f,\frac{\delta A}{\delta\phi}-A\frac{\delta V}{\delta\phi}\right)\ \mathrm{d}\mu_l\,, \tag{4.13}$$

for $A\in L^2(Q,\Sigma,\,\mathrm{d}\mu_l)$, there holds

$$\int \mathrm{d}\mu_l\, A(h_1,h_2)\,\phi\left((-\triangle+m^2)f\right)=\int \mathrm{d}\mu_l\,\left(f,\,\frac{\delta A}{\delta\phi}-A\frac{\delta V(\chi_l)}{\delta\phi}\right)\,, \tag{4.14}$$

for $f\in\mathcal{S}(S_\beta\times\mathbb{R})$, $\operatorname{supp} f\subset(0,\beta/2)\times[-l,l]$. Because of the support properties of A and f

$$\left(f,\,\frac{\delta A}{\delta\phi}\right)=0\,. \tag{4.15}$$

The remaining term can be calculated,

$$\begin{aligned}\left(f,\,\frac{\delta}{\delta\phi}V(\chi_l)\right)&=\int \mathrm{d}x\, f(x)\,\frac{\delta}{\delta\phi(x)}\int \mathrm{d}y\,\chi_l(y)\ :P(\phi(y)):_C\\&=\int \mathrm{d}x\, f(x)\int \mathrm{d}y\,\chi_l(y)\,\delta(x-y)\ :P'(\phi(y)):_C\\&=\int \mathrm{d}x\, f(x)\,\chi_l(x)\ :P'(\phi(x)):_C\\&=\int \mathrm{d}x\, f(x)\ :P'(\phi(x)):_C\,,\end{aligned} \tag{4.16}$$

which is precisely the second term in equation (4.10). Since the right hand side of (4.16) is independent of l, taking the limit $l \to \infty$ establishes equation (4.10). $\square$

4.2 Exponential Decay of Correlation Functions

Lemma 4.1.2 can be strengthened by exploiting basic properties of the vacuum model on the circle and Nelson symmetry. Denote the right space-like wedge by $Q := \{(t,x) \in \mathbb{R}^2 \mid |t| < x\}$ and let $(\tau_j, \xi_j) \in Q$, $j \in \{1,\ldots,n\}$ as well as $\alpha_j \in S_\beta$. By Nelson symmetry (3.52) there holds (at first formally)

$$\begin{aligned}\mathcal{W}_\beta^{(n)}(-\tau_1 - i\alpha_1, \xi_1, \ldots, -\tau_n - i\alpha_n, \xi_n) &= \mathcal{W}_C^{(n)}(\alpha_1 - i\tau_1, -i\xi_1 \ldots, \alpha_n - i\tau_n, -i\xi_n) \\ &= (\Omega_C,\, \phi_C(\delta)\, \mathrm{e}^{-(\xi_1 H_C - (\tau_1 + i\alpha_1)P_C)} \ldots \phi_C(\delta)\, \mathrm{e}^{-(\xi_n H_C - (\tau_n + i\alpha_n)P_C)}\, \phi_C(\delta)\Omega_C)\,. \end{aligned} \tag{4.17}$$

As Lemma 4.2.1 below shows, (4.17) exists and it is possible to take the limit $\lim_{\alpha_j \to 0}$ in this equality.

Lemma 4.2.1. *For $(\tau_j, \xi_j) \in Q$ and $s_j \in S_\beta$, $j \in \{1,\ldots,n\}$, there holds*

$$\begin{aligned}&\|\mathrm{e}^{-(\xi_1 H_C - (\tau_1 + is_1)P_C)}\, \phi_C(\delta) \ldots \mathrm{e}^{-(\xi_n H_C - (\tau_n + is_n)P_C)}\, \phi_C(\delta)\, \Omega_C\| \\ &\qquad \le D_1\, \|\delta\|^n_{H^{-1}(S_\beta)} \prod_{j=1}^n \left((\xi_j - |\tau_j|)^{-1} + 1\right)\,, \end{aligned} \tag{4.18}$$

where D_1 is a constant independent of (τ_j, ξ_j).

Two ingredients are necessary for the proof of this lemma. Firstly the fact, that the Dirac delta is an element in the Sobolev space of order -1 over S_β, which is denoted by $H^{-1}(S_\beta)$. And secondly the H-bounds

$$\|(H_C + 1)^{-1/2} \phi_C(f) (H_C + 1)^{-1/2}\| \le \mathtt{C}'\, \|f\|_{H^{-1}(S_\beta)}\,, \tag{4.19}$$

$$\|\mathrm{e}^{-\epsilon H_C}(H_C + 1)\| \le \epsilon^{-1} + 1\,, \qquad \epsilon > 0\,, \tag{4.20}$$

for the vacuum model on the circle. Inequality (4.19) is a consequence of the ϕ-bound (3.32). (The constant c can be removed, by substituting a larger constant $\mathtt{C}'$ for $\mathtt{C}$.)

Inequality (4.20) follows from continuous functional calculus and the estimate[4]

$$\mathrm{e}^{-\epsilon x}(x+1) \leq \epsilon^{-1} + 1\,, \qquad \epsilon > 0,\ x \geq 0\,. \tag{4.21}$$

Proof. At first use the spectrum condition [36], $|P_C| \leq H_C$, to get

$$-\xi H_C + \tau P_C \leq -\xi H_C + |\tau|\,|P_C| \leq -\xi H_C + |\tau| H_C \leq -(\xi - |\tau|) H_C\,. \tag{4.22}$$

Then, inserting $(H_C+1)^{1/2}(H_C+1)^{-1/2}$ between the fields on the l.h.s. of (4.18), the H-bounds are sufficient to prove the lemma. □

Taking the limit in (4.17) therefore results in

$$\begin{aligned}\mathfrak{W}_\beta^{(n)}(\tau_1,\xi_1,\ldots,\tau_n,\xi_n) &= \mathfrak{W}_C^{(n)}(i\xi_1, i\tau_1,\ldots,i\xi_n,i\tau_n)\\ &= \left(\Omega_C,\ \phi_C(\delta)\,\mathrm{e}^{-(\xi_1 H_C - \tau_1 P_C)}\ldots\phi_C(\delta)\,\mathrm{e}^{-(\xi_n H_C - \tau_n P_C)}\,\phi_C(\delta)\Omega_C\right),\end{aligned} \tag{4.23}$$

for $(\tau_j,\xi_j) \in Q$, $j \in \{1,\ldots,n\}$, not only as distributions but also as smooth functions. The following lemma is a technical result for the vacuum theory on the circle. (4.23) will subsequently be used to carry it over to the thermal case, yielding an exponential decay result.

Lemma 4.2.2. *Let $m > 0$ denote the mass-gap of H_C. Then, for $x \geq 0$ and $(\tau_j,\xi_j) \in Q$, $j \in \{1,\ldots,n\}$,*

$$\begin{aligned}&\Bigg|\Big(\left(\Omega_C,\ \phi_C(\delta)\ \mathrm{e}^{-(\xi_1 H_C - \tau_1 P_C)}\cdots\phi_C(\delta)\,\mathrm{e}^{-((\xi_k + x)H_C - \tau_k P_C)}\,\phi_C(\delta)\cdots\phi_C(\delta)\,\Omega_C\right)\\ &\qquad - \left(\Omega_C,\ \phi_C(\delta)\,\mathrm{e}^{-(\xi_1 H_C - \tau_1 P_C)}\,\phi_C(\delta)\cdots\mathrm{e}^{-(\xi_{k-1} H_C - \tau_{k-1} P_C)}\,\phi_C(\delta)\,\Omega_C\right)\\ &\qquad\qquad \times \left(\Omega_C,\ \phi_C(\delta)\,\mathrm{e}^{-(\xi_{k+1} H_C - \tau_{k+1} P_C)}\cdots\phi_C(\delta)\,\Omega_C\right)\Big)\Bigg|\\ &\qquad \leq D_2\,\mathrm{e}^{-mx}\,,\end{aligned} \tag{4.24}$$

where D_2 is dependent on (τ_j,ξ_j) but independent of x.

[4]There holds

$$\sup_{x\geq 0}\mathrm{e}^{-\epsilon x}(x+1) = \begin{cases}\mathrm{e}^{\epsilon-1}/\epsilon\,, & 0 < \epsilon \leq 1\\ 1 & \epsilon > 1\end{cases}\,.$$

Remark 4.2.3. *As was proven in [36] the spectrum of H_C is discrete so H_C has a mass-gap. To the knowledge of the author there are no rigorous results on whether the size of this mass-gap is comparable to the mass parameter of the theory. But in view of the result by Glimm and Jaffe for the vacuum theory on $\mathbb{R}^2$ [28, Theorem 1.1.2], that the mass gap approaches the free one for small couplings, this is also to be expected for periodic boundary conditions.*

Proof. Let P_{λ_E} denote the spectral projection onto the eigenspace corresponding to the eigenvalue λ_E of H_C. Then the left hand side of (4.24) yields,

$$\Bigg| \Bigg(\Omega_C, \phi_C(\delta)\, \mathrm{e}^{-(\xi_1 H_C - \tau_1 P_C)} \cdots \mathrm{e}^{-(\xi_{k-1} H_C - \tau_{k-1} P_C)}\, \phi_C(\delta)\, \mathrm{e}^{-(\xi_k H_C - \tau_k P_C)/2}$$
$$\sum_{\lambda_E \in \sigma(H_C)\setminus\{0\}} \mathrm{e}^{-\lambda_E x} P_{\lambda_E}\, \mathrm{e}^{-(\xi_k H_C - \tau_k P_C)/2} \phi_C(\delta) \mathrm{e}^{-(\xi_{k+1} H_C - \tau_{k+1} P_C)} \cdots \phi_C(\delta)\, \Omega_C \Bigg) \Bigg| .$$

By an application of the Cauchy-Schwarz inequality this can be estimated by

$$\begin{aligned} &\left\| \mathrm{e}^{-(\xi_k H_C - \tau_k P_C)/2}\, \phi_C(\delta) \ldots \mathrm{e}^{-(\xi_1 H_C - \tau_1 P_C)}\, \phi_C(\delta)\, \Omega_C \right\| \\ &\quad \times \left\| \sum_{\lambda_E \in \sigma(H_C)\setminus\{0\}} \mathrm{e}^{-\lambda_E x} P_{\lambda_E}\, \mathrm{e}^{-(\xi_k H_C - \tau_k P_C)/2} \phi_C(\delta) \cdots \mathrm{e}^{-(\xi_{k_n} H_C - \tau_{k_n} P_C)}\, \phi_C(\delta)\, \Omega_C \right\| . \end{aligned} \tag{4.25}$$

Since the P_{λ_E} are *orthogonal* projections summing to $\mathbb{1}$, there holds

$$\left\| \sum_{\lambda_E \in \sigma(H_C)\setminus\{0\}} \mathrm{e}^{-\lambda_E x}\, P_{\lambda_E} \right\| \leq \mathrm{e}^{-mx} \tag{4.26}$$

as a bounded operator. Therefore (4.25) can be estimated by

$$\begin{aligned} \mathrm{e}^{-mx} \; &\left\| \mathrm{e}^{-(\xi_k H_C - \tau_k P_C)/2}\, \phi_C(\delta) \ldots \mathrm{e}^{-(\xi_1 H_C - \tau_1 P_C)}\, \phi_C(\delta)\, \Omega_C \right\| \\ &\times \left\| \mathrm{e}^{-(\xi_k H_C - \tau_k P_C)/2} \phi_C(\delta) \cdots \mathrm{e}^{-(\xi_{k_n} H_C - \tau_{k_n} P_C)}\, \phi_C(\delta)\, \Omega_C \right\| , \end{aligned} \tag{4.27}$$

which, together with (4.18), implies the desired result. □

Nelson symmetry (4.23) and the preceding lemma directly imply the following result.

Theorem 4.2.4 (Exponential decay). *Let $m > 0$ denote the mass-gap of H_C and let $(\tau_j, \xi_j) \in Q$, $j \in \{1, \ldots, n\}$. Then there holds for $x \geq 0$*

$$\begin{aligned} \Big| \mathfrak{W}_\beta^{(n)} & (\tau_1, \xi_1, \ldots, \tau_k, \xi_k + x \ldots, \tau_n, \xi_n) \\ & - \mathfrak{W}_\beta^{(k-1)}(\tau_1, \xi_1, \ldots, \tau_{k-1}, \xi_{k-1}) \times \mathfrak{W}_\beta^{(n-k)}(\tau_{k+1}, \xi_{k+1}, \ldots, \tau_n, \xi_n) \Big| \\ & \leq C\, \mathrm{e}^{-mx} . \end{aligned} \tag{4.28}$$

Remark 4.2.5. *(i) There is no information on whether the parameter m in Theorem (4.2.4) is related to any notion of mass in the thermal theory.*

(ii) Since $\mathfrak{W}_\beta^{(n)}$ is symmetric under spatial reflections, the conditions of the theorem can be replaced by: $(\tau_j, \xi_j) \in -Q$, $j \in \{1, \ldots, n\}$, and $x \leq 0$.

(iv) The constant C is independent of m and x but dependent on the initial configuration (τ_j, ξ_j). With the presently used techniques this is unavoidable, as $\mathfrak{W}_\beta^{(n)}$ ceases to be a function, if for any j the relative coordinates (τ_j, ξ_j) tend to zero. However, from a physical point of view, the requirement that only space-like initial configurations are allowed seems superfluous.

(v) In [36, Lemma 3.2] it is proven that the Lorentz rotated Hamiltonian $H_C(\beta) := \beta_0 H_C + \beta P_C$, $\beta_0^2 - \beta^2 = 1$, is bounded below and has discrete spectrum. Therefore it is possible to derive an exponential decay result like (4.28) for every space-like direction. However, there do not seem to be any results on the dependence of the mass gap of $H_C(\beta)$ on β. Accordingly the rate of decay a priori depends on β and may therefore be different for different space-like directions.

4.3 On the Källén-Lehmann Representation

It is the purpose of this section to give some partial results on the damping factor in the Källén-Lehmann representation in both models. Naturally, the same strategy of deriving results for the vacuum model and then using Nelson symmetry will be employed.

For the proof of the Källén-Lehmann representation of either of the models the *commutator function* is an important object. For any two-point function $\mathfrak{W}$ it is defined by

$$C(f) := \mathfrak{W}(f) - \mathfrak{W}(f_r), \qquad f \in \mathcal{S}(X),\ f_r(x) = f(-x) . \tag{4.29}$$

Locality implies, that the support of C is contained in the closure of the union of the forward and backward light cones (For the vacuum model on the cylinder the light cones are defined in (4.30) below.).

In the next two subsections the Källén-Lehmann representation for a general vacuum two-point function on the circle will be proven and some elementary properties of the damping factors will be derived. The consequences thereof in the present situation will be laid out in Subsection 4.3.3.

4.3.1 The Källén-Lehmann Representation for Vacuum Models on a Circle

As this and the next subsection treat vacuum models on a circle only, some special conventions will be used. Because it is customary in Quantum Field Theory to have the time coordinate as the first entry of functions of space time, the cylinder will here be viewed as $\mathbb{R} \times S_\beta$. The circle S_β will always be parametrised by the interval $(-\beta/2, \beta/2]$. Note that only then the time-like and space-like regions w.r.t. the origin are characterised by $t^2 - \alpha^2 > 0$ and $t^2 - \alpha^2 < 0$, respectively. And only then the definitions,

$$V^+ := \{(t, \alpha) \in \mathbb{R} \times S_\beta \mid t^2 - \alpha^2 > 0\} \quad \text{and} \quad V^- := -V^+ \tag{4.30}$$

of the light-cones on the cylinder make sense. The partial Fourier transformation in the angular coordinate

$$\widetilde{f}(n) := \frac{1}{\beta} \int_{-\beta/2}^{\beta/2} \mathrm{d}\alpha \, f(\alpha) \, \mathrm{e}^{-in\alpha} \tag{4.31}$$

and its inverse

$$f(\alpha) = \sum_{n \in 2\pi\mathbb{Z}/\beta} \widetilde{f}(n) \, \mathrm{e}^{in\alpha} \tag{4.32}$$

are indicated by tildes. The Fourier transformation in both variables is indicated by a hat,

$$F(t, \alpha) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{d}\omega \sum_{n \in 2\pi\mathbb{Z}/\beta} \mathrm{e}^{-i(\omega t - n\alpha)} \, \widehat{F}(\omega, n) \, . \tag{4.33}$$

The Fourier transform of the commutator function of the *free* scalar field with mass

m on the cylinder is given by

$$\widehat{C}_{C,m}(\omega,n) = \frac{\sqrt{2\pi}}{\beta}\,\epsilon(\omega)\,\delta(\omega^2 - n^2 - m^2)\,, \qquad n \in \frac{2\pi\mathbb{Z}}{\beta}\,, \tag{4.34}$$

where $\epsilon(\omega) = \Theta(\omega) - \Theta(-\omega)$. This can be verified easily with the explicit expression for the free two-point function $\mathfrak{W}_{C,\text{free}}$. The next theorem is the analog of [15, Section 4] for the vacuum case on $\mathbb{R} \times S_\beta$.

Theorem 4.3.1. *For any odd (w.r.t. the chart $\mathbb{R} \times (-\beta/2, \beta/2]$) tempered distribution F on the cylinder $\mathbb{R} \times S_\beta$ there exists a distribution $D_F(\alpha, m)$ such that $\widehat{F}$ and F can be represented as*

$$\widehat{F}(\omega,n) = \frac{\sqrt{2\pi}}{\beta}\,\epsilon(\omega)\int_0^\infty \mathrm{d}s \sum_{n'\in 2\pi\mathbb{Z}/\beta} \delta\big(\omega^2 - (n-n')^2 - s\big)\,\widetilde{\rho}_F(n', s)\,, \tag{4.35}$$

where

$$\widetilde{\rho}_F(n', s) = \frac{1}{2\beta\sqrt{s}}\int_{-\beta/2}^{\beta/2} \mathrm{d}\alpha\, \mathrm{e}^{-in'\alpha}\, D_F(\alpha, \sqrt{s})\,, \tag{4.36}$$

and

$$F(t,\alpha) = \int_0^\infty \mathrm{d}m\, C_{C,m}(t,\alpha)\, D_F(\alpha, m)\,, \tag{4.37}$$

respectively.

Remark 4.3.2. *(i) Equation* (4.35) *is a* Jost-Lehmann-Dyson *integral representation. Its Fourier transformation* (4.37) *is called the* Källén-Lehmann *representation.*

(ii) The proof of the theorem only makes use of the anti-symmetry of F. Support properties are not needed.

Proof. The method of the proof is essentially the one from [15] with a minor adjustment necessitated by the altered geometry. The functions Ψ and Φ, which will be defined in this proof, are all to be viewed as distribution-valued analytic functions in λ for $\Re\lambda > 0$.

$$\Psi(\alpha,\lambda) := i\int_{\mathbb{R}} \mathrm{d}t\, \mathrm{e}^{-\frac{t^2}{4\lambda}}\, t\, F(t,\alpha)\,, \tag{4.38}$$

$$\widetilde{\Psi}(n,\lambda) = (2\lambda)^{3/2}\int_{\mathbb{R}} \mathrm{d}\omega\, \mathrm{e}^{-\lambda\omega^2}\,\omega\,\widehat{F}(\omega,n)\,. \tag{4.39}$$

Equation (4.39) can be interpreted as a Laplace transformation in the variable ω^2. The assumption, that F is antisymmetric enters here [15, p. 507], [58, Theorem A.3]. Now let $f(\alpha,\lambda)$ be such, that its Fourier series $\widetilde{f}(n,\lambda)$ is

$$(2\lambda)^{3/2}\,\mathrm{e}^{-\lambda n^2}\,.$$

(For the properties of f cf. Appendix C.) Then set

$$\Phi(\alpha,\lambda) := f(\alpha,\lambda)^{-1}\,\Psi(\alpha,\lambda)\,. \tag{4.40}$$

This is possible since f has no zeros (Appendix C). As Ψ and f are analytic for $\Re\lambda > 0$, so is Φ. Then, since the convolution theorem also holds for Abelian groups [38, Theorem 19.6],

$$\widetilde{\Psi}(n,\lambda) = (2\lambda)^{3/2} \sum_{n'\in 2\pi\mathbb{Z}/\beta} \mathrm{e}^{-\lambda(n-n')^2}\,\widetilde{\Phi}(n',\lambda)\,. \tag{4.41}$$

$\Phi(\alpha,\cdot)$ can be represented as the Laplace transform of some function $\rho_F(\alpha,\cdot)$, here displayed in Fourier space,

$$\widetilde{\Phi}(n,\lambda) = \int_0^\infty \mathrm{d}s\,\mathrm{e}^{-\lambda s}\,\widetilde{\rho}_F(n,s)\,, \tag{4.42}$$

with the inverse given by

$$\rho_F(\alpha,s) = \frac{1}{2\pi}\int_{\mathbb{R}-i\epsilon} \mathrm{d}\nu\,\mathrm{e}^{i\nu s}\,\Phi(\alpha,i\nu), \qquad \epsilon > 0\,. \tag{4.43}$$

Due to the analyticity of Φ the left hand side is independent of ϵ. While these results are well-known for functions [70], the generalisation to distributions can be found in [58, Appendix A]. Now (4.41) and (4.42) together imply

$$\widetilde{\Psi}(n,\lambda) = (2\lambda)^{3/2}\int_0^\infty \mathrm{d}s \sum_{n'\in 2\pi\mathbb{Z}/\beta} \mathrm{e}^{-\lambda\left((n-n')^2+s\right)}\,\widetilde{\rho}_F(n',s) \tag{4.44}$$

$$= (2\lambda)^{3/2}\int_0^\infty \mathrm{e}^{-\lambda\omega^2}\,\mathrm{d}\omega^2 \int_0^\infty \mathrm{d}s \sum_{n'\in 2\pi\mathbb{Z}/\beta} \delta\bigl(\omega^2-(n-n')^2-s\bigr)\widetilde{\rho}_F(n',s)\,. \tag{4.45}$$

Comparing (4.39) and (4.45) establishes, that $\Theta(\omega)\widehat{F}(\omega, n)$ and

$$\int_0^\infty \mathrm{d}s \sum_{n' \in 2\pi\mathbb{Z}/\beta} \delta\big(\omega^2 - (n-n')^2 - s\big)\widetilde{\rho}_F(n', s)$$

have the same Laplace-transform and are thus equal. Setting $D_F(\alpha, m) = 2m\, \rho_F(\alpha, m^2)$, (4.37) follows as the Fourier series of (4.35). □

There also holds an inversion formula akin to [15, Equation (20)]. Equations (4.43), (4.40) and (4.38) together imply

$$\rho_F(\alpha, s) = \frac{i}{2\pi} \int_{\mathbb{R}} \mathrm{d}t\, F(t, \alpha)\, t \int_{\mathbb{R}-i\epsilon} \mathrm{d}\nu\, \frac{\mathrm{e}^{-\frac{t^2}{4i\nu} + i\nu s}}{f(\alpha, i\nu)}\,. \tag{4.46}$$

The r.h.s. is again independent of ϵ due to analyticity.

Definition 4.3.3. *A distribution $\mathfrak{W} \in \mathcal{S}'(\mathbb{R} \times S_\beta)$ is called a* local, spectrally positive two-point function on $\mathbb{R} \times S_\beta$*, if*

(i) $\mathfrak{W}(-t, -x) = \mathfrak{W}(t, x)$ for $t^2 - x^2 < 0$ (space-like points), and

(ii) supp $\widehat{\mathfrak{W}} \subset V^+$.

Remark 4.3.4. *Any quantum field theory satisfying Axioms W1 to W5 and the Spectrum Condition 1.2.8 has a local, spectrally positive two-point function. But Axioms W2 and W4 are not needed for Theorem 4.3.5 below.*

A local, spectrally positive two-point function is connected with its commutator function by $\widehat{\mathfrak{W}}(\omega, n) = \Theta(\omega)\widehat{C}(\omega, n)$ [15, Equation (6)]. Denote $\widehat{\mathfrak{W}}_{C,m}(\omega, n) = \Theta(\omega)\widehat{C}_{C,m}$, i.e. $\mathfrak{W}_{C,m}$ is the two-point function of the free scalar field on $\mathbb{R} \times S_\beta$. Since C associated to any $\mathfrak{W}$ is anti-symmetric, Theorem 4.3.1 immediately implies

Theorem 4.3.5. *Let $\mathfrak{W} \in \mathcal{S}'(\mathbb{R} \times S_\beta)$ be a local, spectrally positive two-point function on $\mathbb{R} \times S_\beta$. Then there exists a distribution $D_C(\alpha, m)$ such that it can be represented by*

$$\widehat{\mathfrak{W}}(\omega, n) = \frac{1}{2\pi}\, \Theta(\omega) \int_0^\infty \mathrm{d}s \sum_{n' \in 2\pi\mathbb{Z}/\beta} \delta\big(\omega^2 - (n-n')^2 - s\big)\, \widetilde{\rho}_C(n', s)\,, \tag{4.47}$$

and by

$$\mathfrak{W}(t, \alpha) = \int_0^\infty \mathrm{d}m\, \mathfrak{W}_{C,m}(t, \alpha)\, D_C(\alpha, m)\,, \tag{4.48}$$

where ρ_C and D_C are connected by (4.36).

4.3.2 Properties of Weight Functions for Vacuum Models on the Circle

From the discreteness of the spectrum and locality for a vacuum theory on the circle some basic conclusions can be drawn. This is the content of the following proposition.

Proposition 4.3.6. *Let $\mathfrak{W} \in \mathcal{S}'(\mathbb{R} \times S_\beta)$ be a local, spectrally positive two-point function on the cylinder $\mathbb{R} \times S_\beta$, for which supp $\widehat{\mathfrak{W}} \subset V^+$ is a discrete set and C its associated commutator function. Then*

(i) $D_C(\alpha, m)\,\mathrm{d}m$ is a sum of point measures in m;

(ii) for a fixed $n \in 2\pi\mathbb{Z}/\beta$ the summation over s and n' in (4.35) *only includes values satisfying $n'^2 - 2n'n + s \geq 0$;*

(iii) for $(\omega, n) \in$ supp $\widehat{C}$ fixed the summation over s and n' in (4.35) *only includes values $n' \in [n - \omega, n + \omega]$.*

Remark 4.3.7. *Next to the $P(\phi)_2$ model discreteness of the spectrum has been shown for the Yukawa$_2$ model in [37].*

Proof. (i) If $\widehat{\mathfrak{W}}$ has discrete support this is also true for $\widehat{C}$. From equation (4.35) it follows[5], that $\widehat{C}(\omega, n)$ is discrete in ω, if and only if the solutions for ω of the equation $\omega^2 = l^2 + s$, where $l \in \mathbb{Z}$ and $s \geq 0$, form a discrete set. This is only possible if s is restricted to a discrete set, which implies that $\widetilde{\rho}_C(n, s)\,\mathrm{d}s$ is a sum of point measures in s.

(ii) Using the result (i) the Jost-Lehmann-Dyson representation can be rewritten as

$$\widehat{C}(\omega, n) = \frac{1}{2\pi}\epsilon(\omega) \sum_{s \in S} \sum_{n' \in 2\pi\mathbb{Z}/\beta} \delta\big(\omega^2 - (n - n')^2 - s\big)\, \widetilde{\rho}_C(n', s), \tag{4.49}$$

where S is some discrete subset of $\mathbb{R}_0^+$, the δ is now a Kronecker-Delta and $\widetilde{\rho}_C$ now denotes the coefficients of the point measure sum $\widetilde{\rho}_C(\cdot, s)\,\mathrm{d}s$. Furthermore,

[5] For a technically impeccable version of this argument the notation of [58, Section 3.1] should be used. In particular consider equation (3.2) therein.

evaluating the δ, equation (4.49) becomes

$$\widehat{C}(\omega, n) = \frac{1}{2\pi}\epsilon(\omega) \sum_{(n',s)\in I_{(\omega,n)}} \widetilde{\rho}_C(n', s), \tag{4.50}$$

where $I_{(\omega,n)} := \{(n', s) \in \frac{2\pi\mathbb{Z}}{\beta} \times S \mid \omega^2 = (n - n')^2 + s\}$. Since the support of $\widehat{C}$ is characterised by $\omega^2 - n^2 \geq 0$, only values $(n', s) \in 2\pi\mathbb{Z}/\beta \times S$ which satisfy $(n - n')^2 + s \geq n^2$ can contribute.

(iii) Consider again the equation $\omega^2 = (n - n')^2 + s$ which characterises $I_{(\omega,n)}$. Since $s \geq 0$, there must hold $\omega^2 - (n - n')^2 \geq 0$. This inequality can be solved for n' with solutions $n' \in [n - \omega, n + \omega]$. □

In the situation of item (ii) of the preceding proposition, if there exists an $M > 0$, such that the support of $\widehat{C}$ is contained in $\omega^2 \geq n^2 + M^2$, then the excluded regions are

$$n'^2 - 2n'n + s < M^2. \tag{4.51}$$

This is evident from the proof. In the following figures the excluded regions of $I_{(n',s)}$ are depicted for various values of the parameters ω, n and M. As can be seen from Figure 4.3 only for $|\omega| = n$ a mass gap of size M in $D_C(\alpha, m)$ can be guaranteed. For (ω, n)-values away from the boundary of the light cones the excluded mass range shrinks to zero.

4.3.3 Ramifications for the Thermal $P(\phi)_2$ Model

In this subsection the mass parameter m will be reflected in the notation for the free two-point functions,

$$\mathfrak{W}_{\beta,m} := \mathfrak{W}^{(1)}_{\beta,\text{free}} \qquad \text{and} \qquad \mathfrak{W}_{C,m} := \mathfrak{W}^{(1)}_{C,\text{free}}\,, \tag{4.52}$$

where $\mathfrak{W}^{(1)}_{\beta,\text{free}}$ and $\mathfrak{W}^{(1)}_{C,\text{free}}$ are the distributions in relative coordinates, which are associated to $W^{(2)}_{\beta,\text{free}}$ and $W^{(2)}_{C,\text{free}}$, respectively (cf. equations (1.3) and (1.4)). For $(\alpha_j, x_j) \in S_\beta \times \mathbb{R}$, $j \in \{1, 2\}$,

$$S_m(\alpha_2 - \alpha_1, x_2 - x_1) = S^{(2)}_{\text{free}}((\alpha_1, x_1), (\alpha_2, x_2)) \tag{4.53}$$

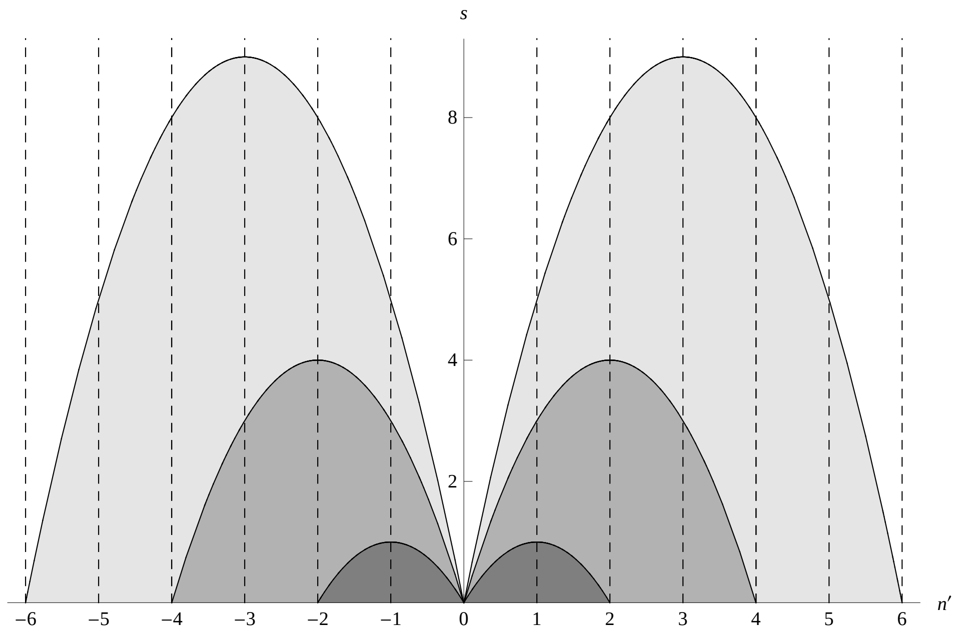

Figure 4.1: The regions excluded by Proposition 4.3.6 (ii) for $|n| = 1, 2, 3$ ($|n| = 1$ darkest, positive n-values on the right, negative n-values on the left, the maxima of the respective curves are at n).

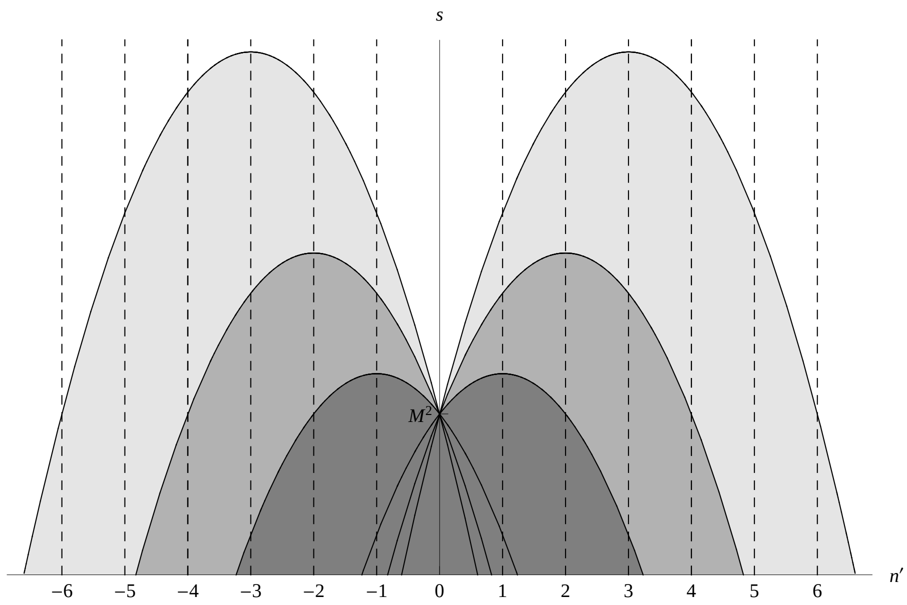

Figure 4.2: The regions excluded by Equation (4.51) for mass M and $|n| = 1, 2, 3$ ($|n| = 1$ darkest, positive n-values on the right, negative n-values on the left, the maxima of the respective curves are at n).

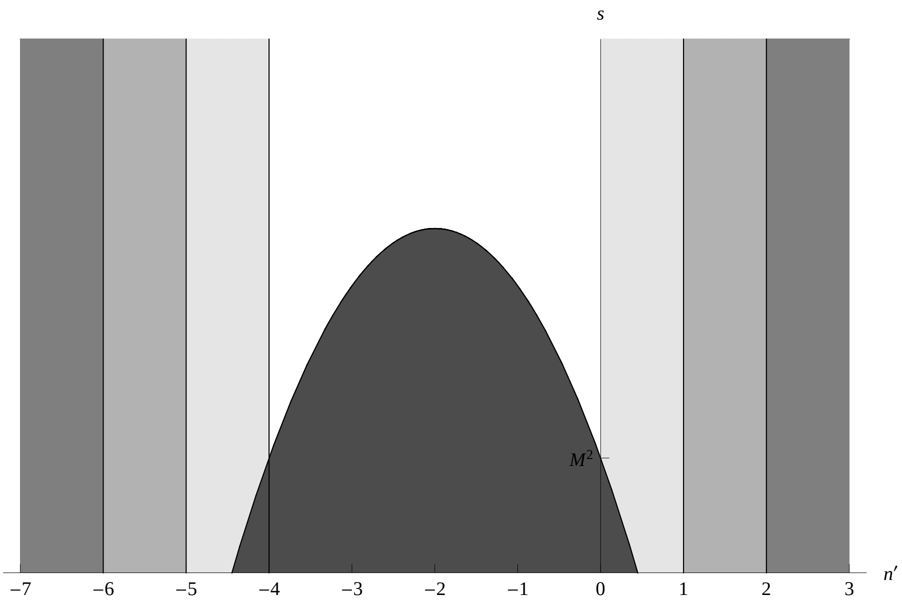

Figure 4.3: The regions excluded by Proposition 4.3.6 for mass M, $n = -2$ and $|\omega| = 2, 3, 4$ ($|\omega| = 2$ narrowest).

defines the free Schwinger two-point function in relative coordinates. The Fourier transform of the free commutator function on $\mathbb{R}^2$ is given by

$$\widehat{C}_m(\omega, k) = \frac{1}{2\pi}\,\epsilon(\omega)\,\delta(\omega^2 - k^2 - m^2)\,. \tag{4.54}$$

In [15] Bros and Buchholz have proven the following representation for thermal two-point functions.

Theorem 4.3.8. *Let a quantum field theory on $\mathbb{R}^2$ satisfy Axioms W1 to W5 as well as the KMS condition 1.2.9. There exists a distribution $D_\beta(x, m)$ such that the corresponding thermal commutator function C_β and its Fourier transformation can be represented by*

$$C_\beta(t, x) = \int_0^\infty \mathrm{d}m\, D_\beta(x, m)\, C_m(t, x)\,, \tag{4.55}$$

and

$$\widehat{C}(\omega, p) = \frac{1}{\sqrt{2\pi}}\,\epsilon(\omega) \int_{\mathbb{R}} \mathrm{d}u \int_0^\infty \mathrm{d}s\, \delta\big(\omega^2 - (p-u)^2 - s\big)\widetilde{\rho}_\beta(u, s)\,, \tag{4.56}$$

where

$$\widetilde{\rho}_\beta(u,s) = \frac{1}{2\sqrt{2\pi s}} \int_{\mathbb{R}} \mathrm{d}x \, \mathrm{e}^{-iux} \, D_\beta(x, \sqrt{s}) \,. \tag{4.57}$$

For thermal theories C_β and $\mathfrak{W}_\beta$ are connected by $(1-\mathrm{e}^{-\beta\omega})\,\widehat{\mathfrak{W}_\beta}(\omega,p) = \widehat{C}_\beta(\omega,p)$ [15, Equation (33)]. Therefore the additional assumption of time-like clustering carries this theorem over to the Wightman function. Time-like clustering, however, has not been proven for the thermal $P(\phi)_2$ model, yet[6]. Therefore the corresponding Källén-Lehmann representation receives an additional term [15, p. 500].

Theorem 4.3.9. *There exists a distribution $D_\beta(x,m)$ and a distribution $\int$ such that the two-point function of the thermal $P(\phi)_2$ model can be represented by*

$$\mathfrak{W}^{(1)}_\beta(t,x) = \int_0^\infty \mathrm{d}m \, D_\beta(x,m) \, \mathfrak{W}_{\beta,m}(t,x) \; + \; \int(x). \tag{4.58}$$

In both (4.48) and (4.58) all the objects can be analytically continued in the respective time variable. Using Nelson symmetry,

$$\mathfrak{W}^{(1)}_\beta(i\alpha,x) = \mathfrak{W}^{(1)}_C(ix,\alpha) \tag{4.59}$$

results in the following equation,

$$\int_0^\infty \mathrm{d}m \, D_C(\alpha,m) \, S_m(\alpha,x) = \int_0^\infty \mathrm{d}m \, D_\beta(x,m) \, S_m(\alpha,x) \; + \; \int(x). \tag{4.60}$$

It has been used, that Nelson symmetry holds in particular for the free fields,

$$\mathfrak{W}_{\beta,m}(i\alpha,x) = S_m(\alpha,x) = \mathfrak{W}_{C,m}(ix,\alpha) \,. \tag{4.61}$$

Unfortunately it does not seem straightforward to extract information about $D_\beta(x,m)$ from equation (4.60).

[6] It should be pointed out, that time-clustering would follow from the uniqueness of the KMS state. The exponential decay result in Section 4.2 is a strong indicator in favour of uniqueness.

Chapter 5

Conclusion

In the present work the thermal Wightman functions of the $P(\phi)_2$ model have been constructed and it has been shown, that they satisfy the relativistic KMS condition. The Hölder inequality for KMS states (proven in Chapter 2) and Nelson symmetry were the key tools for the construction. While Nelson symmetry allows to exploit information available for the relatively simpler vacuum $P(\phi)_2$ model on the circle, the Hölder inequality is a basic, new inequality for thermal systems. Further results on the $P(\phi)_2$ model include the verification of the Wightman axioms and a spatial exponential decay result for the thermal model.

Additionally the Källén-Lehmann representation for commutator functions on the circle is proven. Like in the thermal case, the weight function therein is space dependent. Some basic conclusions for the weight function can be drawn from the requirement of locality. Suppose a quantum field theory on the Einstein cylinder has a mass gap in the common spectrum of time and space generators. Then, for energy-momentum on the boundary of the light-cones, the weight function in the Jost-Lehmann-Dyson representation exhibits the same mass gap. For energy-momentum away from the boundary this gap cannot be guaranteed only by the requirement of locality and of having a mass gap.

It has been attempted to gain information on the thermal damping factor D_β using the available information on the weight function D_C and Nelson symmetry. The resulting equation (4.60) looks promising. However the author was yet unable to exploit this equation in a proper way.

Another open problem is the proof of the time-like clustering property. It is connected to the aforementioned one as it would simplify (4.60) by eliminating the additional term $\int$. A slightly weaker general result can be found in [41].

Lastly there is the question of uniqueness of the KMS state. The spatial exponential decay of the Wightman functions are a strong indicator, that the KMS state constructed here, is in fact unique. In this context it seems worth to mention the work of Araki [3] on the uniqueness of KMS states for lattice systems. There is no principle obstruction to generalising the techniques developed therein to prove the uniqueness in the continuous, relativistic case. The uniqueness of the KMS state would imply the time-like clustering property.

From the author's point of view the problem of gaining rigorous information on the damping factor D_β in the Källén-Lehmann representation for the thermal field is the most important one. This is the most promising way of concepting a notion of particle in thermal field theories. The first result, which seems to be within reach, is the existence of a mass gap in the Källén-Lehmann representation. The mass gap can be made plausible by the following piece of heuristics.

For time 0 the free, thermal two-point function can be calculated explicitly. For $x > 0$ there holds [30, 8.432 9.]

$$\mathfrak{W}_{\beta,m}(0,x) = (\Omega_C^{(0)}, \phi_C(\delta)\mathrm{e}^{-xH_C^{(0)}}\phi_C(\delta)\Omega_C^{(0)}) = \int_{\mathbb{R}} \mathrm{d}k \, \frac{\mathrm{e}^{-x\sqrt{k^2+m^2}}}{2\sqrt{k^2+m^2}} = K_0(mx)\,. \tag{5.1}$$

K_n denotes the modified Bessel functions of the second kind. Their asymptotic behaviour is (for all n)

$$K_n(x) = \sqrt{\frac{\pi}{2x}}\,\mathrm{e}^{-x}\left(1 + O\left(\tfrac{1}{x}\right)\right). \tag{5.2}$$

Now for $0 < m < m_\beta$ Theorem 4.2.4 yields

$$\lim_{x\to\infty} \frac{\mathfrak{W}_\beta(0,x)}{\mathfrak{W}_{\beta,m}(0,x)} \leq \text{const.} \lim_{x\to\infty} \frac{\mathrm{e}^{-m_\beta x}}{\mathrm{e}^{-mx}/\sqrt{x}} \leq \text{const.} \lim_{x\to\infty} \sqrt{x}\mathrm{e}^{-(m_\beta - m)x} = 0\,, \tag{5.3}$$

i.e. the interacting two-point function decays faster than the free one of mass m, even though it should contribute to the Källén-Lehmann representation (4.58). This is impossible unless $D_\beta(x,m)$ decays exponentially in x or is rapidly oscillating in m. This seems unlikely, and thus it can be conjectured that the interval $[0, m_\beta)$ is excluded from the integral (4.58):

Conjecture (Mass gap at positive temperature).

$$\mathfrak{W}_\beta(t,x) = \int_{m_\beta}^{\infty} \mathrm{d}m\, D_\beta(x,m)\, \mathfrak{W}_{\beta,m}(t,x), \tag{5.4}$$

where $m_\beta > 0$ *is the rate of exponential decay appearing in Theorem 4.2.4.*

Appendix A

Relative Modular Theory in Finite Dimensions

The purpose of this appendix is to exhibit relative modular operators in finite dimensions and to show, that (2.8) is the natural generalisation of the finite-dimensional case. A more elaborate introduction for modular theory in finite dimensions can be found in the lecture notes[1] by Jaksic, Ogata, Pautrat and Pillet entitled “Entropic Fluctuations in Quantum Statistical Mechanics”.

Consider the space of complex $n \times n$-matrices $M_n(\mathbb{C}) = \mathcal{B}(M_n(\mathbb{C}))$, which is a Hilbert space with scalar product

$$(\xi, \eta) = \mathrm{Tr}\ \xi^* \eta\,, \qquad (\xi, \eta) \in (M_n(\mathbb{C}))^2\,. \tag{A.1}$$

Define $\mathcal{D} := \{\omega \in M_n(\mathbb{C}) \mid \omega > 0, \mathrm{Tr}\ \omega = 1\}$ and let $\omega \in \mathcal{D}$. The corresponding expectation (state) is denoted by

$$\langle A \rangle_\omega := \mathrm{Tr}\ \omega\, A = \mathrm{Tr}\ \omega^{1/2}\, A\, \omega^{1/2} = (\omega^{1/2}, A\, \omega^{1/2})\,, \tag{A.2}$$

for $A \in M_n(\mathbb{C})$. For a second matrix $\nu \in \mathcal{D}$ define the *relative modular operator* by

$$\Delta_{\nu,\omega}\, \xi = \nu\, \xi\, \omega^{-1}\,, \qquad \xi \in M_n(\mathbb{C})\,. \tag{A.3}$$

Proposition A.10. *Let ω and ν be in $\mathcal{D}$.*

(i) $\Delta_{\nu,\omega}$ is positive self-adjoint.

[1]Available online under http://pillet.univ-tln.fr.

(ii) For $p \in \mathbb{N}$ there holds

$$\Delta_{\nu,\omega}^{1/p}\,\xi = \nu^{1/p}\,\xi\,\omega^{-1/p}\,. \tag{A.4}$$

(iii) For $A \in M_n(\mathbb{C})$ there holds

$$\langle A\rangle_\nu = \langle \Delta_{\nu,\omega}^{1/2}\,A\,\Delta_{\nu,\omega}^{1/2}\rangle_\omega\,. \tag{A.5}$$

Proof. Let $\omega \in \mathcal{D}$ and $\nu \in \mathcal{D}$.

(i) For $\xi \in M_n(\mathbb{C})$ and $\eta \in M_n(\mathbb{C})$ there holds

$$(\xi, \Delta_{\nu,\omega}\eta) = \mathrm{Tr}\ \xi^*\nu\eta\omega^{-1} = \mathrm{Tr}\ \omega^{-1}\xi^*\nu\eta = \mathrm{Tr}\ (\nu\xi\omega^{-1})^*\eta = (\Delta_{\nu,\omega}\eta, \xi)\,.$$

Let e_λ, $\lambda \in \sigma(\nu)$ denote the eigenvectors of ν. Then, because $\xi\omega^{-1}\xi^*$ is positive,

$$\begin{aligned}(\xi, \Delta_{\nu,\omega}\,\xi) &= \mathrm{Tr}\,\xi^*\,\nu\,\xi\,\omega^{-1} = \mathrm{Tr}\,\nu\,\xi\,\omega^{-1}\,\xi^* = \sum_{\lambda\in\sigma(\nu)} \lambda\; e_\lambda^*\,\xi\,\omega^{-1}\,\xi^*\,e_\lambda \\ &\geq \left(\inf_{\lambda\in\sigma(\nu)} \lambda\right)\ \mathrm{Tr}\,\xi\,\omega^{-1}\,\xi^* > 0\,.\end{aligned}$$

(ii) For $p \in \mathbb{N}$ define $\Delta_{\nu,\omega}^{(1/p)}\xi := \nu^{1/p}\xi\omega^{-1/p}$. Then for $\xi \in M_n(\mathbb{C})$,

$$\begin{aligned}\left(\Delta_{\nu,\omega}^{(1/p)}\right)^p \xi &= \left(\Delta_{\nu,\omega}^{(1/p)}\right)^{p-1} \nu^{1/p}\,\xi\,\omega^{-1/p} = \left(\Delta_{\nu,\omega}^{(1/p)}\right)^{p-2} \nu^{2/p}\,\xi\,\omega^{-2/p} \\ &= \ldots = \nu\,\xi\,\omega^{-1} = \Delta_{\nu,\omega}\,\xi\,.\end{aligned}$$

But roots of positive operators are positive and unique.

(iii) For $A \in M_n(\mathbb{C})$

$$\begin{aligned}\langle A\rangle_\nu &= (\nu^{1/2}, A\nu^{1/2}) = (\nu^{1/2}\omega^{1/2}\omega^{-1/2}, A\nu^{1/2}\omega^{1/2}\omega^{-1/2}) \\ &= (\Delta_{\nu,\omega}^{1/2}\omega^{1/2}, A\Delta_{\nu,\omega}^{1/2}\omega^{1/2}) = \langle \Delta_{\nu,\omega}^{1/2}\,A\,\Delta_{\nu,\omega}^{1/2}\rangle_\omega\,.\end{aligned}$$

□

The *modular conjugation* on $M_n(\mathbb{C})$ is the anti-linear involution defined by

$$J\colon \xi \mapsto \xi^*\,. \tag{A.6}$$

Theorem A.11. *For $A \in M_n(\mathbb{C})$ there holds*

$$J\,\Delta_{\nu,\omega}^{1/2}\,A\,\omega^{1/2} = A^*\,\nu^{1/2}\,. \tag{A.7}$$

Furthermore

$$JM_n(\mathbb{C})J = M_n(\mathbb{C})'\,. \tag{A.8}$$

Proof. For $A \in M_n(\mathbb{C})$ there holds

$$J\Delta_{\nu,\omega}^{1/2}A\omega^{1/2} = J\nu^{1/2}A\omega^{1/2}\omega^{-1/2} = A^*\nu^{1/2}\,.$$

If also $B \in M_n(\mathbb{C})$,

$$(JAJ)B\xi = (A(B\xi)^*)^* = (A\xi^*B^*)^* = B\xi A^* = B(JAJ)\xi\,,$$

which implies (A.8). □

Define $\|A\|_{\omega,p}^p \doteq \mathrm{Tr}\ (\omega^{1/2p}|A|\omega^{1/2p})^p$ for $1 \le p < \infty$. The following Hölder trace inequality has been proven in [52],

$$|\mathrm{Tr}\ (\omega AB)| = \|AB\|_{\omega,1} \le \|A\|_{\omega,p}\,\|B\|_{\omega,q}\,, \qquad p^{-1}+q^{-1}=1\,. \tag{A.9}$$

Calculate the p-norm of a p-root of a relative modular operator,

$$\begin{aligned}\|\Delta_{\nu_j,\omega}^{1/p}\|_{\omega,p} &= \mathrm{Tr}\ (\Delta_{\nu_j,\omega}^{1/p}\omega^{1/p})^p = \mathrm{Tr}\ (\Delta_{\nu_j,\omega}^{1/p}\omega^{1/p})^{p-1}\nu_j^{1/p}\omega^{1/p}\omega^{-1/p} = \mathrm{Tr}\ \nu^{1/p}(\Delta_{\nu_j,\omega}^{1/p}\omega^{1/p})^{p-1}\\ &= \mathrm{Tr}\ \nu^{1/p}(\Delta_{\nu_j,\omega}^{1/p}\omega^{1/p})^{p-2}\nu_j^{1/p}\omega^{1/p}\omega^{-1/p} = \ldots = \mathrm{Tr}\ \nu = \langle \mathbb{1}\rangle_\nu\,.\end{aligned}$$

For ω, ν_1 and ν_2 in $\mathcal{D}$, matrices $A_j \in M_n(\mathbb{C})$, $j \in \{0,1,2\}$ and for $1/p+1/q=1$, $1 \le p < \infty$, there holds

$$\begin{aligned}|\langle A_2\Delta_{\nu_2,\omega}^{1/p}A_1\Delta_{\nu_1,\omega}^{1/q}A_0\rangle_\omega| &\le \Big(\prod_{j=0}^{2}\|A_j\|_\infty\Big)\,\|\Delta_{\nu_2,\omega}^{1/p}\|_{\omega,p}\,\|\Delta_{\nu_1,\omega}^{1/q}\|_{\omega,q}\\ &= \Big(\prod_{j=0}^{2}\|A_j\|_\infty\Big)\,\langle\mathbb{1}\rangle_{\nu_2}^{1/p}\,\langle\mathbb{1}\rangle_{\nu_1}^{1/q}\,,\end{aligned} \tag{A.10}$$

which is completely analogous to Araki's proto-Hölder inequality (2.8).

Appendix B

The Strong Disk Theorem

The *strong disk theorem* is attributed to Bremermann (see also [13]). For the convenience of the reader the theorem is restated here in a version similar to [69, p. 151].

Theorem B.12. *Consider a Jordan curve in* $\mathbb{C}^{n-1}$ *of the form*

$$(z_2(t), \ldots, z_n(t)) = (z_2(0), \ldots, z_n(0)) + \lambda(t) \cdot (b_2, \ldots, b_n), \qquad 0 \leq t \leq 1, \tag{B.1}$$

where $(b_2, \ldots, b_n) \in \mathbb{C}^{n-1}$ *and* $\lambda(t) \in \mathbb{R}$ *with* $\lambda(0) = 0$*. Suppose also that the family of domains* $D(t) \subset \mathbb{C}$*,* $0 \leq t \leq 1$*, lying in the* z_1*-plane, possess the property that for every compact set* $K \subset D(0)$*, there exists a number* $\eta \equiv \eta(K) \in (0, 1]$ *such that* $K \subset D(t)$ *for all* t *in* $[0, \eta)$*.*

If the function $f(z)$ *is holomorphic at all points of the disks*

$$\{(z_1, z_2(t), \ldots, z_n(t)) \in \mathbb{C}^n \mid z_1 \in D(t)\}, \qquad 0 < t \leq 1, \tag{B.2}$$

and at least at one point of the limit disk

$$\{(z_1, z_2(0), \ldots, z_n(0) \in \mathbb{C}^n \mid z_1 \in D(0)\}, \tag{B.3}$$

then this function is holomorphic at all points of the limit disk.

Appendix C

An Auxiliary Function

This appendix is devoted to the following function on $S_\beta \times \mathbb{C}^+$ defined in terms of its Fourier series

$$f(\alpha, \lambda) := (2\lambda)^{3/2} \sum_{n \in 2\pi\mathbb{Z}/\beta} \mathrm{e}^{-\lambda n^2} \, \mathrm{e}^{in\alpha}, \tag{C.1}$$

so that

$$\widetilde{f}(n, \lambda) = (2\lambda)^{3/2} \, \mathrm{e}^{-\lambda n^2}. \tag{C.2}$$

(C.1) can be rewritten as

$$f(\alpha, \lambda) = (2\lambda)^{3/2} \sum_{n \in \mathbb{Z}} \mathrm{e}^{-\lambda \left(\frac{2\pi}{\beta}\right)^2 n^2} \, e^{i \frac{2\pi n}{\beta} \alpha}. \tag{C.3}$$

The sum converges, iff $\Re\lambda > 0$. Accordingly $f(\alpha, \cdot)$ is holomorphic on $\mathbb{C}^+$ for every α; and $f(\cdot, \lambda)$ is a C^∞ function. Furthermore f can be expressed in terms of the Theta function ϑ_3, which is defined as [30, p. 877]

$$\vartheta_3(u, q) = \sum_{n \in \mathbb{Z}} q^{n^2} \, \mathrm{e}^{2nui} = 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \cos 2nu \, . \tag{C.4}$$

Thus

$$f(\alpha, \lambda) = (2\lambda)^{3/2} \, \vartheta_3 \left(\frac{\pi\alpha}{\beta}, \mathrm{e}^{-\lambda \left(\frac{2\pi}{\beta}\right)^2} \right) \, . \tag{C.5}$$

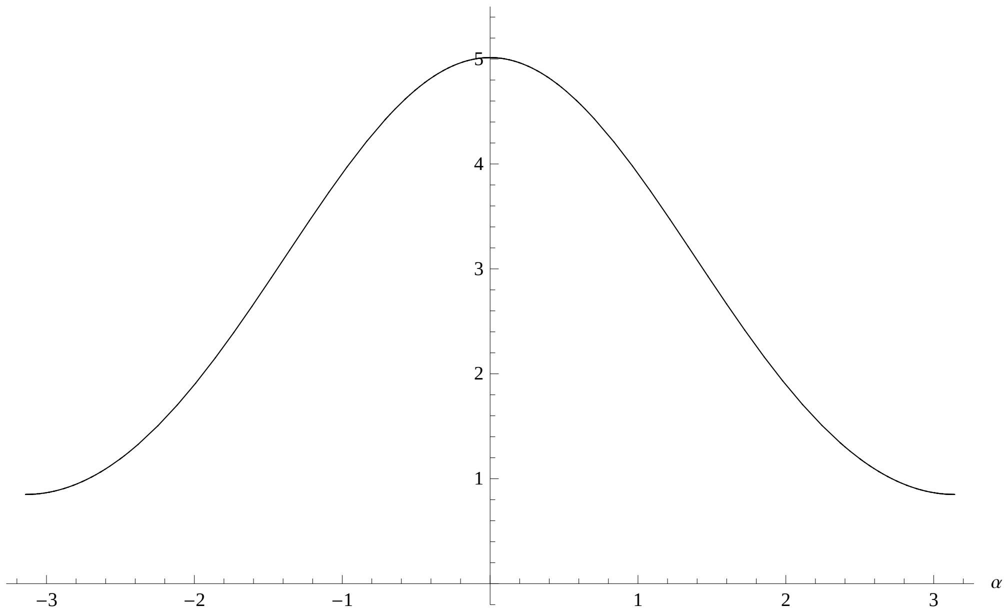

Figure C.1: A plot of $f(\alpha, 1)$ for $\beta = 2\pi$.

This can be used to show, that f has no zeros.

Lemma C.13. *The equation $f = 0$ has no solutions.*

Proof. Only the ϑ_3-term in (C.5) can have zeros in the domain of f. The zeros of $\vartheta_3(u, q)$ are given by [30, p. 879]

$$u = (2m-1)\frac{\pi}{2} + (2n-1)\frac{\pi\tau}{2}\,, \tag{C.6}$$

where m and n are integers and where $q = \mathrm{e}^{i\pi\tau}$. In the present case

$$u = \frac{\pi\alpha}{\beta} \qquad \text{and} \qquad \tau = i\frac{\lambda}{\pi}\left(\frac{2\pi}{\beta}\right)^2 .$$

Plugging this into equation (C.6) and solving for λ gives

$$\lambda = i\ \underbrace{\beta\frac{(2m-1)\beta - 2\alpha}{4\pi(2n-1)}}_{\in\mathbb{R}}\,,$$

which is not in the domain of f. ☐

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