Disney, S.M., Gaalman, G. and Hosoda, T., (2012), "Review of stochastic cost functions for production and inventory control", Pre-prints of the 17th International Working Seminar of Production Economics, Innsbruck, Austria, February 20th – 24th, Vol. 1, pp117 – 128.

Review of stochastic cost functions for production and inventory control

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Abstract

We review a dozen cost functions that could be used to assign capacity related costs to a stochastic production rate. These cost functions compose of linear, step-wise and quadratic components. We assume demand is a normally distributed random variable. In some of the cases we are able to completely characterise the cost function and optimise the decision variables to minimise the defined cost function. In one instance there are no endogenous variables, so there is nothing to optimise. In all of the other cases we obtain insights into the convexity and limit behaviour of the cost function. This allows us to gain knowledge of the number of minimums and, in some cases, upper and lower bounds on optimal parameter settings and the costs incurred.

Keywords: Capacity costs, Bullwhip, Normal Distribution
Word Count: 3816

1. Introduction

The piecewise linear and convex cost function for characterising linear inventory holding and backlog costs is pervasive in the literature. However, an equivalent cost function for capacity costs, a.k.a. the bullwhip effect, is not so well established. Perhaps this is because real capacity costs can take several different forms depending upon context specific factors associated with an industrial setting. Factors such as the use of technology, labour practices, social norms and other factors associated with the product or service provided may necessitate a particular cost function. For example, some situations are dominated by capital intensive production technology; others are driven by labour considerations. Some companies / countries allow the use of over-time, others limit it, others use over-time, but don't pay a premium for over-time, and others use the "annualised hours" concept.

Several different cost functions already exist for characterising the long term, expected capacity costs in the literature. The purpose of this paper is to review, critique and analyse the various bullwhip cost functions in the literature as a single collection in a coherent manner. We also speculate about when each measure is relevant in different practical situations. This serves as a guide for both academics and practitioners as they are then able to match the appropriate objective function to a particular situation. The structure of this paper is as follows. Section 2 covers preliminary matter. Section 3 considers twelve cost functions for analysing capacity costs. Section 4 concludes.

2. Preliminaries: The normal distribution and other matters

Throughout the paper we consider the replenishment cycle to be one week long. This is because the natural cycle of working life is a week. It is unlikely that labour turning up for working in the morning will not know when they will be returning home. It is much more likely that over-time requirements are determined and communicated on a weekly basis. In this way labour may be informed of the need to do over-time near the beginning of the week, and the over-time is probably worked in the evenings, nights and / or weekends later in the week.

We also assume the production orders (or distribution / replenishment orders) are normally distributed. The probability density function (pdf) of the standard normal distribution and its inverse is

$$\left.\begin{array}{l} \varphi[x] = \overbrace{\tfrac{1}{\sqrt{2\pi}}\exp\left[-\tfrac{x^2}{2}\right]}^{\text{Mathematically}} = \overbrace{\text{NORMDIST}(x,0,1,\text{FALSE})}^{\text{In Excel}} \\ \varphi^{-1}[x] = \underbrace{\pm i\sqrt{\log[2\pi]+2\log[x]}}_{\text{Mathematically}} = \underbrace{\pm\text{SQRT}\Big(\text{ABS}\big(\text{LN}(2\times\text{PI}())+2\times\text{LN}(x)\big)\Big)}_{\text{In Excel}} \end{array}\right\}. \quad (1)$$

$\varphi[x] > 0\ \forall x$. $\varphi[x]$ is concave between $-1 < x < 1$ and convex when $x < -1$ and $x > 1$ from which we obtain the definition of the standard deviation. In (1) we have highlighted the mathematical definition and the formula to implement the equation in Excel. Figure 1 provides a visualisation of the pdf of the standard normal distribution.

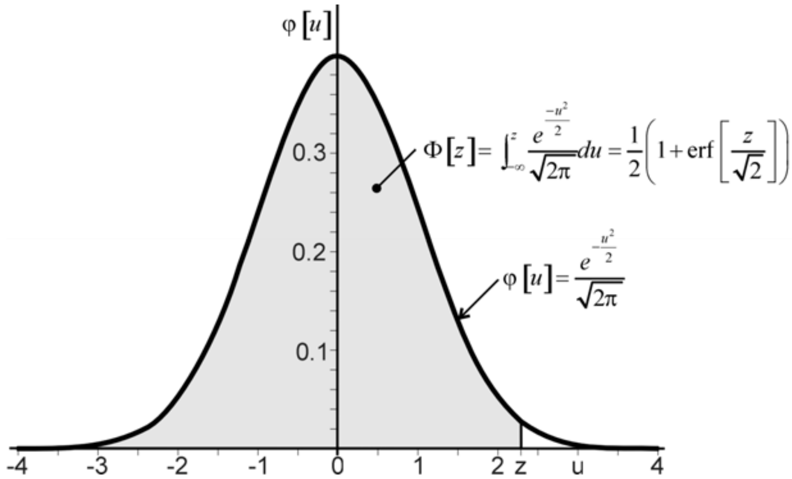

Figure 1. Probability density function of the standard normal distribution

The cumulative distribution function (cdf) and its inverse is given by

$$\left.\begin{array}{l} \Phi[z] = \int_{-\infty}^{z}\varphi[x]\,\mathrm{d}x = \tfrac{1}{2}\left(1+\text{erf}\left[\tfrac{z}{\sqrt{2}}\right]\right) = \text{NORMDIST}(z,0,1,\text{TRUE}) \\ \Phi^{-1}[z] = \sqrt{2}\,\text{erf}^{-1}[2z-1] = \text{NORMSINV}(z) \end{array}\right\}. \quad (2)$$

$\Phi[z]$ is an increasing function within the interval $0<\Phi[z]<1$ for $-\infty<z<\infty$. $\Phi[-z]=1-\Phi[z]$. $\Phi[z]$ is convex when $z<0$ and concave when $z>0$, see Figure 2.

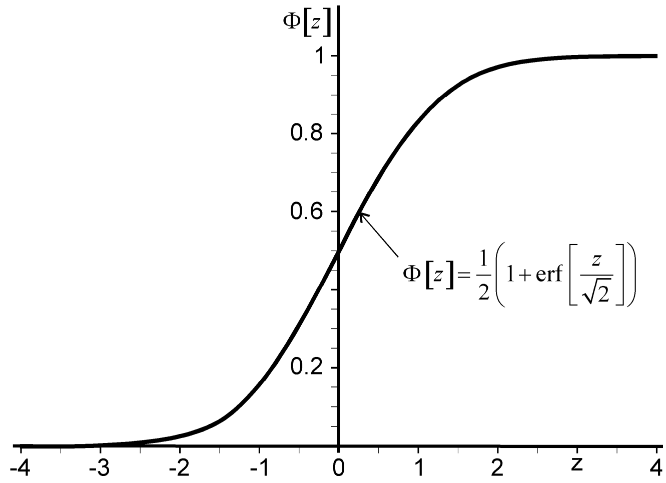

Figure 2. Cumulative Distribution Function of the Standard Normal Distribution

Another important relation that we use extensively is the so-called Error function, $\text{erf}[z]$,

$$\text{erf}[z]=2\Phi\left[z\sqrt{2}\right]-1 = \text{ERF(z)}. \quad (3)$$

Both $\Phi[z]$ and the $\text{erf}[z]$ can not be expressed in terms of finite additions, subtractions, multiplications and root extractions. So both must be either computed numerically or otherwise approximated [1]. The Loss function is given by

$$\begin{aligned} L[z] &= \int_z^\infty \varphi[x](x-z)\,\mathrm{d}x = \int_z^\infty (1-\Phi[x])\,\mathrm{d}x \\ &= \tfrac{1}{\sqrt{2\pi}}\exp\left[-\tfrac{z^2}{2}\right]-\tfrac{z}{2}\left(1-\text{erf}\left[\tfrac{z}{\sqrt{2}}\right]\right)=\varphi[z]-z(1-\Phi[z]) \\ &= \text{NORMDIST}(z,0,1,\text{FALSE})-z\times(1-\text{NORMDIST}(z,0,1,\text{TRUE})). \end{aligned} \quad (4)$$

$L[z]>0\ \forall z$ is a monotonically decreasing and convex function in *z*. Another important relation is $L[z]+z=L[-z]$. The inverse loss function, $L^{-1}[z]$ has no known solution. However, it can be approximated to an arbitrary level of accuracy using numerical search techniques. The Visual Basic code required to develop a Microsoft Excel Add-In to provide a numerical approximation to $L^{-1}[z]$ is given in Appendix A.

Appendix B provides numerical solutions for the pdf, cdf and the Loss function of the standard normal distribution for convenience. We also make extensive use of the maximum operator, $(x)^+=\max[x,0]$, the expectation operator $E[x]=\bar{x}$ and the sign operator, $\text{sgn}[x]=1$ if $x>0$, 0 if $x=0$ and -1 if $x<0$.

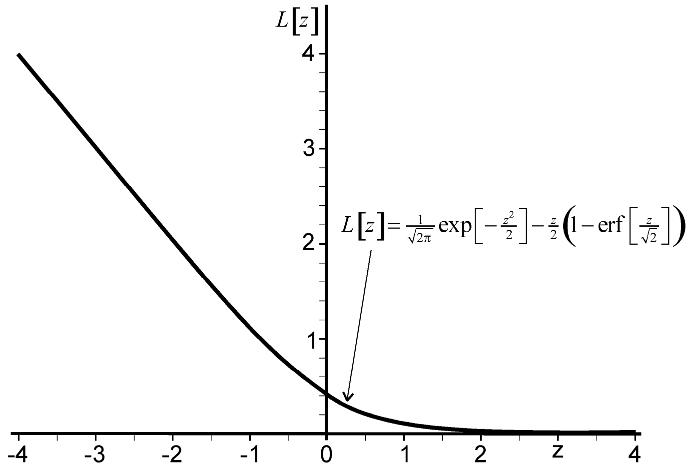

Figure 3. The Standard Normal Distribution Loss function

Motivation

We assume a linear periodic production and inventory control system is present. One such possibility for this is the so called Order-Up-To policy which is popular in both academic and industrial settings. The replenishment orders in time *t*, o_t, are normally distributed with a mean of μ and a standard deviation of σ and unless otherwise stated we assume the probability of negative orders is negligible. Thus $o_t \in N(\mu,\sigma)$, with $\mu \gg 4\sigma$. The mean orders will be linked to the mean demand, μ in linear systems. The linear assumption also implies that if demand is high, then over-time, or subcontracting is used to meet the peak demands, (or the probability of excessively high and unmet demands are negligible) and thus whatever is ordered that is what is delivered. The length of the periodic planning cycle is arbitrary, but is probably easiest to think of it being one week long in what follows. Also the lead-time is arbitrary, but again it probably easiest to think of what ever is ordered at the beginning of the week will be produced (or delivered) within the week.

We assume that demand is normally distributed, and in a linear production and inventory system, the orders will be normally distributed. The standard deviation of demand will probably be given, it is exogenous variable that is not controllable (at least not easily). However due to the bullwhip effect, the standard deviation of the orders is controllable as it is strongly influenced by the production planning and inventory control system that is used. Depending on lead-times, auto-correlation in the demand and how the forecasting and replenishment system is specified then $0 \le \text{Var}[\text{orders}]$ or $\text{Var}[\text{orders}] > \text{Var}[\text{demand}]$ is possible [2], [3].

By way of illustration consider Figure 4 which shows the production orders over time. Knowing the mean and variance of the normally distributed orders we may collect the realisations of demand into a probability density plot as shown. If there is a certain capacity of the production line of $(\mu + s)$ we may use the pdf of the normal distribution to determine a range of important characteristics of the system. This could include the proportion of periods that require over-time, the expected over-time per period, the optimal capacity level $(\mu + s)$, the amount of idle working time for example. Precisely how we formulate the cost function is a matter of debate. However, in this paper we will investigate 12 possible cost functions that could be used. Figure 5 illustrates these 12 different cost functions by way of introduction.

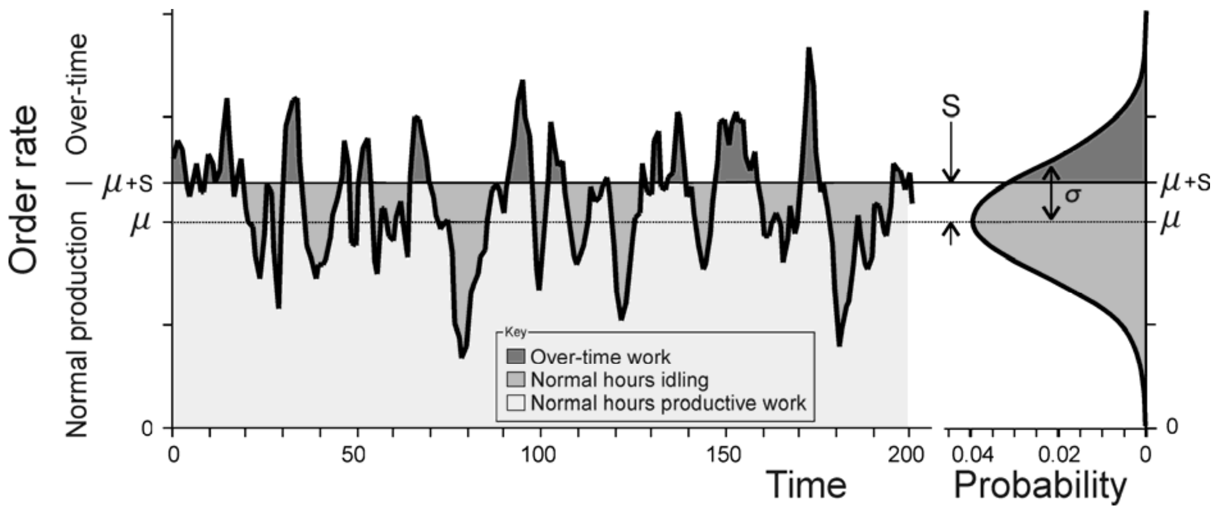

Figure 4. Visualisation of production orders over time

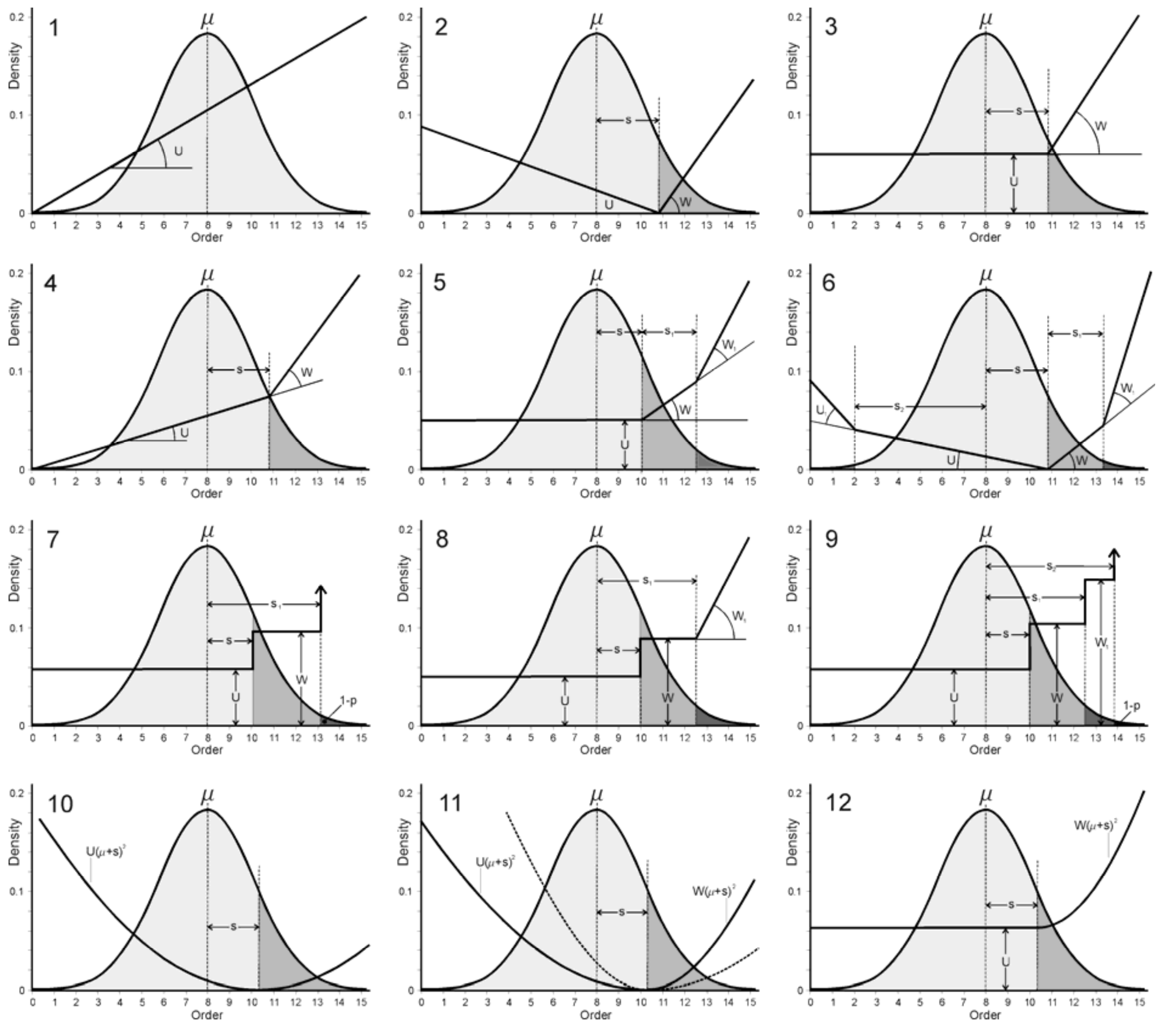

Figure 5. Visualisation of the 12 order rate cost functions studied

3. The twelve cost functions

3.1. Increasing

The simplest possible cost function is a simple linear function of the orders, o_t,

$$C_1 = E\left[C_{1,t}\right] = U\left(o_t\right)^+ . \tag{5}$$

Here *U* is the unit cost of production. This cost function may be suitable for situations that are dominated by material or purchasing costs. It may also be relevant in situations where the "annualised hours" concept is used. Annualised hours is a concept popular in the UK and some northern European countries where employees are contracted to work for a certain number of hours per year. On a week by week basis, working hours vary to suit the needs of the business, but the wage received by the employees remain stable. However when the total number of annual hours are reached, employees are given time off work whilst still receiving their standard wage. In this manner a company may gain some volume flexibility with labour costs.

If we assume that the probability of negative orders is negligible (i.e. $\mu \gg 4\sigma$) then $C_1 = \mu U$. This relation also holds if negative orders mean that an income of *U* per unit is received rather than a cost of *U* incurred. However, if the probability of negative orders is significant or negative demand is cost neutral then

$$\left.\begin{aligned} C_1 &= \int_0^{\infty}\left(\tfrac{Ux}{\sqrt{2\pi}\sigma}\exp\left[-\tfrac{(x-\mu)^2}{2\sigma^2}\right]\right)\mathrm{d}x = U\left(\tfrac{\sigma}{\sqrt{2\pi}}\exp\left[-\tfrac{\mu^2}{2\sigma^2}\right]+\tfrac{\mu}{2}\left(1+\operatorname{erf}\left[\tfrac{\mu}{\sqrt{2}\sigma}\right]\right)\right) \\ &= \int_0^{\infty}\tfrac{xU}{\sigma}\varphi\left[\tfrac{x-\mu}{\sigma}\right]\mathrm{d}x = U\left(\sigma\varphi\left[\tfrac{\mu}{\sigma}\right]+\mu\Phi\left[\tfrac{\mu}{\sigma}\right]\right)\end{aligned}\right\} \tag{6}$$

holds. C_1 is minimised by minimising $\{U,\mu,\sigma\}$, see Figure 6. It is fairly obvious that *U* should be reduced to decrease total costs. However, it is harder to accept that the mean orders (which presumably are the same as the mean demand) should be reduced. It is interesting that the standard deviation of the orders, σ should also be reduced when the possibility of negative demand is present. We have noticed that negative demand can be present in some industry settings. For example in the book publishing industry unsold books may be returned by retailers, resulting in negative demand. The consumer electronics industry also has a high rate of product return. Raw materials, when issued from a warehouse to the shop floor in bulk quantities, and left over materials are returned back to the warehouse after the production run is complete, can also result in ERP systems registering negative demands. (These considerations might also apply to the other cost functions as negative labour hours may not be possible). As all of the variables in C_1 are exogenous then no further analysis can be conducted.

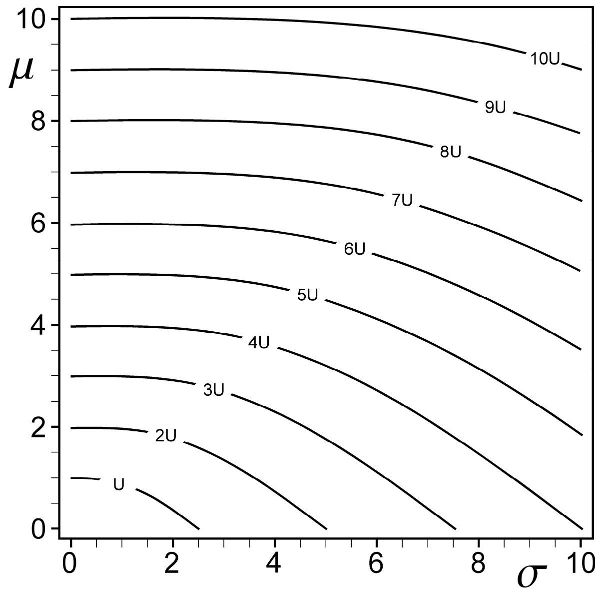

Figure 6. The costs , C_1, in scenario 1

3.2 Linear over-time and opportunity loss (*Piece-wise linear and convex, v-type costs*)
Assume that the we incur over-time (or subcontracting) cost at a rate of *W* for each unit produced above a nominal weekly capacity of $(\mu+s)$. We also penalise the failure to exploit the nominal weekly capacity with an opportunity loss of *U* for each unit of unused nominal capacity. This cost function may be important in situations where large investments have been made in (perhaps automated) production capacity and it is important to utilize them. The cost function is

$$C_2 = E\left[C_{2,t}\right];\ C_{2,t} = U\left(s+\mu-o_t\right)^+ + W\left(o_t-s-\mu\right)^+. \tag{7}$$

The expected costs when the o_t are normally distributed are then

$$\left.\begin{aligned} C_2 &= U\left(\int_{-\infty}^{\mu+s}\left(\tfrac{\mu+s-x}{\sqrt{2\pi}\sigma}\exp\left[-\tfrac{(x-\mu)^2}{2\sigma^2}\right]\right)\mathrm{d}x\right)+W\left(\int_{\mu+s}^{\infty}\left(\tfrac{x-\mu-s}{\sqrt{2\pi}\sigma}\exp\left[-\tfrac{(x-\mu)^2}{2\sigma^2}\right]\right)\mathrm{d}x\right)\\ &= U\left(\tfrac{\sigma}{\sqrt{2\pi}}\exp\left[-\tfrac{s^2}{2\sigma^2}\right]+\tfrac{s}{2}\left(1+\mathrm{erf}\left[\tfrac{s}{\sqrt{2}\sigma}\right]\right)\right)+W\left(\tfrac{\sigma}{\sqrt{2\pi}}\exp\left[-\tfrac{s^2}{2\sigma^2}\right]-\tfrac{s}{2}\left(1-\mathrm{erf}\left[\tfrac{s}{\sqrt{2}\sigma}\right]\right)\right)\\ &= \sigma(U+W)\varphi\left[\tfrac{s}{\sigma}\right]+s\left((U+W)\Phi\left[\tfrac{s}{\sigma}\right]-W\right) \end{aligned}\right\}. \tag{8}$$

Taking the derivative of (8) w.r.t. *s* yields

$$\frac{\mathrm{d}C_2}{\mathrm{d}s} = \tfrac{1}{2}\left(U-W+(U+W)\mathrm{erf}\left[\tfrac{s}{\sqrt{2}\sigma}\right]\right) = (U+W)\Phi\left[\tfrac{s}{\sigma}\right]-W\,, \tag{9}$$

and solving for zero gradient we find a stationary point at

$$s^* = \sigma\underbrace{\sqrt{2}\mathrm{erf}^{-1}\left[\tfrac{W-U}{W+U}\right]}_{\text{Constant}} = \sigma\underbrace{\Phi^{-1}\overbrace{\left[\tfrac{W}{W+U}\right]}^{p}}_{z} = \sigma\times\mathrm{NORMSINV}\left(\tfrac{W}{W+U}\right). \tag{10}$$

We note that s^* always exists as $\{U,W\}\in\Re$. When $W>U$ then $\frac{1}{2}<\frac{W}{W+U}<1$ and $s^*>0$. s^* can be negative which means that the nominal capacity $(\mu+s)$ is less than the average demand. By definition $P\{x>0\}=\Phi[x]$, which here is a measure of the probability, *p*, that over-time is not used. From (10) we find $p=\Phi\left[\frac{s^*}{\sigma}\right]=\frac{W}{W+U}$. As the second derivative is positive for all *s*, C_2 is strictly convex in *s* and so the minimum at s^* is a unique.

$$\frac{\mathrm{d}^2C_2}{\mathrm{d}s^2}=\exp\left[-\tfrac{s^2}{2\sigma^2}\right]\tfrac{(U+W)}{\sqrt{2\pi}\sigma}=\tfrac{(U+W)}{\sigma}\varphi\left[\tfrac{s}{\sigma}\right]>0\ \forall s. \quad (11)$$

Finally substituting (10) into (8) provides the minimised cost,

$$\left.\begin{aligned}C_2^* &= \underbrace{\sigma\tfrac{(U+W)}{\sqrt{2\pi}}\exp\left[-\mathrm{erf}^{-1}\left[\tfrac{W-U}{W+U}\right]^2\right]}_{\text{Constant}}=\sigma(U+W)\varphi[z]=\sigma\left(zU+(U+W)L[z]\right)\\ &=(U+W)\left(\sigma\varphi\left[\tfrac{\mu}{\sigma}\right]+\mu\Phi\left[\tfrac{\mu}{\sigma}\right]\right)-\mu W=(U+W)\tfrac{C_1}{U}+\mu W=\tfrac{\sigma U}{1-p}\varphi\left[\Phi^{-1}[p]\right]\end{aligned}\right\}. \quad (12)$$

It is interesting to note that this cost function behaves exactly the same way as the piecewise linear and convex inventory holding and backlog cost function. The solution is actually the same as the famous newsboy problem, [4]. Furthermore we can see that there is a structural link between C_1 and C_2. We further notice that the C_2 is linear in σ. The expected over-production per replenishment cycle, $EOPRC^*$, is given by

$$EOPRC^*=\sigma L[z]. \quad (13)$$

The percentage of products produced in over-time is given by $\left(EOPRC^*/\mu\right)\times 100\%$. We may wish to limit the expected over-time per replenishment period. Let $q\mu$ be the maximum expected number of products we wish to produce in overtime each period. We should then set the costs to satisfy $w/(w+u)=\Phi\left[L^{-1}[q\mu/\sigma]\right]$. The expected lost capacity cost per replenishment period, $ELPRC^*$, is given by

$$ELPRC^*=\sigma L[-z] \quad (14)$$

We note that both the $EOPRC^*$ and the $ELPRC^*$ are linear in σ_O. Finally, it is interesting to note that $C_2^*=\left(u\cdot ELPRC^*\right)+\left(w\cdot EOPRC^*\right)=\sigma\left(w\cdot L[z]+u\cdot L[-z]\right)$.

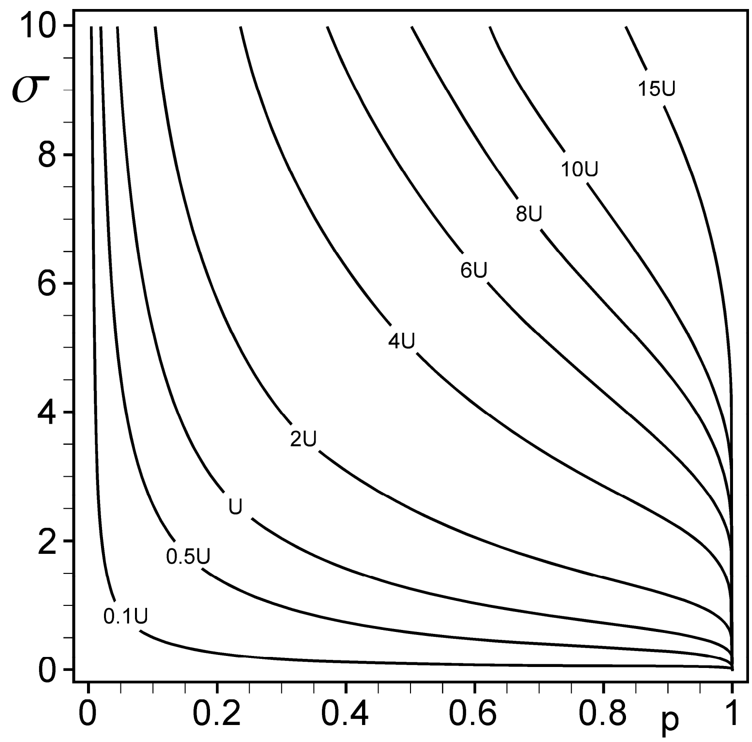

Figure 7. The minimised C_2^*

3.3 Guaranteed hours and linear over-time costs

Consider the case when a certain number of hours are guaranteed each week. This means that if less then the guaranteed hours of work is required to fulfil the weekly orders, labour stands idle (or undertakes housekeeping, personal development, or kaizen activities in their spare capacity), even if orders are negative. However, if more than 40 hours of work are required then the labour is available to work over-time. Over-time labour is flexible in that the volume of over-time can be varied to however much is required to complete the weeks order. The cost to produce one unit of work in normal hours is *u*. The amount of work that can be accomplished in normal hours is $(s+\mu)$, leaving $(o_t - s - \mu)^+$ to be produced in over-time at a unit cost of *w*. We assume $w \geq u$. This cost function maybe suitable in situations that are dominated by labour costs, for example, high variety / low volume warehouses with manual picking. The cost function is

$$C_3 = E\left[C_{3,t}\right];\ C_{3,t} = u(s+\mu) + w(o_t - s - \mu)^+ . \tag{15}$$

By inspection, the expected costs per period with this cost function is

$$C_3 = u(s+\mu) + \int_{\mu+s}^{\infty} \tfrac{w(x-s-\mu)}{\sqrt{2\pi}\sigma}\varphi\left[-\tfrac{x-\mu}{\sigma}\right]\mathrm{d}x = u(\mu+s) + w\left(\sigma\varphi\left[\tfrac{s}{\sigma}\right] + s\left(\Phi\left[\tfrac{s}{\sigma}\right]-1\right)\right). \tag{16}$$

Taking the derivative w.r.t. *s* produces

$$\frac{\mathrm{d}C_3}{\mathrm{d}s} = u - w\left(1 - \Phi\left[s/\sigma\right]\right). \tag{17}$$

Setting (17) to zero and solving for *s* reveals a stationary point at

$$s^* = \sigma\Phi^{-1}\left[\left(w-u\right)/w\right]. \tag{18}$$

The optimal slack capacity, s^*, results in $\left(1-\left(w-u\right)/w\right)\times 100\% = \left(u/w\right)\times 100\%$ of periods exploiting over-time. This is interesting because it shows if the cost of over-time is twice the cost of normal working, then 50% of periods (weeks) use over-time. As $w \to u$ then the proportion of work completed in over-time approaches 100%. This is intuitive as when $w \to u$ the overtime is not more expensive than the normal production cost, but due to the greater volume flexibility does not incur idling loss. This ratio is easy to understand and conceptually attractive. (19) shows that C_3 is convex in s and s^* is a global minimum. We note that s^* with C_3 is less than s^* with C_2.

$$\frac{\mathrm{d}^2 C_3}{\mathrm{d}s^2} = \left(w/\sigma\right)\varphi\left[s/\sigma\right] > 0 \ \forall s \tag{19}$$

The minimised costs, C_3^*, is obtained by substituting (18) into (16),

$$C_3^* = \mu u + \sigma w \varphi\left[z\right]. \tag{20}$$

(20) is linear in σ but the costs do not originate from the origin, there is an offset of μu. The expected over-time production per replenishment cycle (which may be limited by law) when an optimal slack capacity is present, $EOPRC^*$, is

$$EOPRC^* = \sigma L\left[z\right]. \tag{21}$$

The percentage of products produced in over-time is given by $\left(EOPRC^*/\mu\right)\times 100\%$. The expected idling per replenishment period with an optimal slack capacity, $EIPRC^*$, is given by

$$EIPRC^* = \sigma L\left[-z\right] \tag{22}$$

We note that both $EOPRC^*$ and $EIPRC^*$ are linear in σ_O. Finally we note that the minimised costs can also be expressed as $C_3^* = u(\mu + \sigma z) + w\sigma L\left[z\right]$.

3.4 Material and over-time costs

Consider the case when material and labour costs u per unit of o_t are incurred. Note unit labour costs that vary with volume requirements may also be included in u. A premium for over-time work, over and above the labour costs already included in u, of w per unit of o_t, $w > 0$ is also incurred for orders above the nominal capacity, $\mu + s$. This was the cost function adopted by [5] and [6]. The cost function in the time domain then becomes

$$C_4 = E\left(C_{4,t}\right);\ C_{4,t} = U\left(o_t\right)^+ + W\left(o_t - \mu - s\right)^+. \tag{23}$$

The expected cost per period are given by

$$\left.\begin{aligned} C_4 &= \int_0^\infty \tfrac{Ux}{\sqrt{2\pi}\sigma}\exp\left[-\tfrac{(x-\mu)^2}{2\sigma^2}\right]\mathrm{d}x+\int_{\mu+s}^\infty \tfrac{W(x-s-\mu)}{\sqrt{2\pi}\sigma}\exp\left[-\tfrac{(x-\mu)^2}{2\sigma^2}\right]\mathrm{d}x \\ &= U\left(\tfrac{\sigma}{\sqrt{2\pi}}\exp\left[-\tfrac{\mu^2}{2\sigma^2}\right]+\tfrac{\mu}{2}\left(1+\mathrm{erf}\left[\tfrac{\mu}{\sqrt{2}\sigma}\right]\right)\right)+W\left(\tfrac{\sigma}{\sqrt{2\pi}}\exp\left[-\tfrac{s^2}{2\sigma^2}\right]-\tfrac{s}{2}\left(1-\mathrm{erf}\left[\tfrac{s}{\sqrt{2}\sigma}\right]\right)\right) \\ &= U\left(\sigma\varphi\left[\tfrac{\mu}{\sigma}\right]+s\Phi\left[\tfrac{\mu}{\sigma}\right]\right)+W\left(\sigma\varphi\left[\tfrac{s}{\sigma}\right]+s\Phi\left[\tfrac{s}{\sigma}\right]\right) \end{aligned}\right\}. \tag{24}$$

Here we can see that the material costs are the same as C_1 - we assumed the negative demands are cost neutral. The over-time costs are the same as the over-time cost in C_2 and C_3. The first order differential is

$$\frac{\mathrm{d}C_4}{\mathrm{d}s}=\tfrac{W}{2}\left(\mathrm{erf}\left[\tfrac{s}{\sqrt{2}\sigma}\right]-1\right)=W\left(\Phi\left[\tfrac{s}{\sigma}\right]-1\right), \tag{25}$$

which shows us that $s^*=\infty$ as $\left(\mathrm{erf}\left[s/\sqrt{2}\sigma\right]-1\right)$ approaches zero monotonically from below as $s\to\infty$. C_4 is strictly convex in s as $\mathrm{d}^2C_4/\mathrm{d}s^2=\tfrac{W}{\sqrt{2\pi}\sigma}\exp\left[-\tfrac{s^2}{2\sigma^2}\right]>0\ \forall s$. However as $s^*=\infty$ then C_4 is monotonically decreasing in s, implying that it is best to avoid over-time if possible. The minimised cost of $C_4^*=C_1$.

3.5 Guaranteed hours with double v-type over-time costs

Suppose that we have a number of guaranteed hours in each planning period (even with negative orders) and the possibility of two types of over-time, with one type of over-time more expensive than the other. Perhaps Saturday working is time and a half, Sunday working is double time and both types of over-time have volume flexibility. The costs per period are then

$$C_5=E\left[C_{5,t}\right];\ C_{5,t}=U\left(s+\mu\right)+W\left(o_t-s-\mu\right)^+ +W_1\left(o_t-s-s_1-\mu\right)^+. \tag{26}$$

Any extra capacity requirement must be made up on Sunday (or during the night / evening sometime and costs $W_P=W+W_1$ per unit (thus $W_1=W_P-W$)) is the incremental increase in the unit cost in Sunday over-time, above the Saturday over-time.

$$\left.\begin{aligned} C_5 &= U(s+\mu)+\int_{s+\mu}^\infty \tfrac{W(x-s-\mu)}{\sqrt{2\mu}\sigma}\exp\left[-\tfrac{(x-\mu)^2}{2\sigma^2}\right]\mathrm{d}x+\int_{s+s_1+\mu}^\infty \tfrac{W_1(x-s-s_1-\mu)}{\sqrt{2\mu}\sigma}\exp\left[-\tfrac{(x-\mu)^2}{2\sigma^2}\right]\mathrm{d}x \\ &= U(s+\mu)+\tfrac{W\sigma}{\sqrt{2\pi}}\exp\left[-\tfrac{s^2}{2\sigma^2}\right]+\tfrac{sW}{2}\left(\mathrm{erf}\left[\tfrac{s}{\sqrt{2}\sigma}\right]-1\right)+\tfrac{W_1\sigma}{\sqrt{2\pi}}\exp\left[-\tfrac{(s+s_1)^2}{2\sigma^2}\right]+\tfrac{(s+s_1)W_1}{2}\left(\mathrm{erf}\left[\tfrac{s+s_1}{\sqrt{2}\sigma}\right]-1\right) \\ &= U(s+\mu)+W\left(\sigma\varphi\left[\tfrac{s}{\sigma}\right]+s\left(\Phi\left[\tfrac{s}{\sigma}\right]-1\right)\right)+W_1\left(\sigma\varphi\left[\tfrac{s+s_1}{\sigma}\right]+\left(s+s_1\right)\left(\Phi\left[\tfrac{s+s_1}{\sigma}\right]-1\right)\right) \end{aligned}\right\}. \tag{27}$$

The derivative of C_5 w.r.t. s is given by

$$\frac{\mathrm{d}C_5}{\mathrm{d}s}=U-\tfrac{W}{2}\left(1-\mathrm{erf}\left[\tfrac{s}{\sqrt{2}\sigma}\right]\right)-\tfrac{W_1}{2}\left(1-\mathrm{erf}\left[\tfrac{s+s_1}{\sqrt{2}\sigma}\right]\right)=U+W\left(\Phi\left[\tfrac{s}{\sigma}\right]-1\right)+W_1\left(\Phi\left[\tfrac{(s+s_1)}{\sigma}\right]-1\right). \tag{28}$$

We are unable to find an explicit closed form solution for its stationary point. However, we note that s^* must be independent of μ. There are many known approximations of $\Phi\left[x\right]$ that may be useful, [1], [7]. For example we find the approximation

$$\Phi_a\left[x\right]=\begin{cases}\frac{1}{10}\left(5+x\left(\frac{22}{5}+x\right)\right) & \text{for } -2.2\le x\le 0\\ \frac{1}{10}\left(5+x\left(\frac{22}{5}-x\right)\right) & \text{for } 0\le x\le 2.2\end{cases} \tag{29}$$

gives an approximation to $\Phi\left[x\right]$ that is accurate within 2 decimal places and allows us to obtain second order solutions for s^* that are reasonably accurate. Furthermore it is easy to verify that the lower bound $s^*\ge\sigma\sqrt{2}\text{erf}^{-1}\left[1-\frac{2U}{W}\right]$ exists. Together with the approximation in (29) this lower bound would lead to quite an efficient numerical search routine. The second derivative is always positive when $W_1\ge 0$ so we know C_5 is a convex function in s with single minimum as

$$\begin{aligned}\frac{\text{d}^2C_5}{\text{d}s^2}&=\tfrac{1}{\sqrt{2\pi}\sigma}\exp\left[-\tfrac{2s^2+2ss_1+s_1^2}{2\sigma^2}\right]\left(W\exp\left[-\tfrac{(s+s_1)^2}{2\sigma^2}\right]+W_1\exp\left[-\tfrac{s^2}{2\sigma^2}\right]\right)\\&=\tfrac{\sqrt{2\pi}}{\sigma}\varphi\left[\tfrac{\sqrt{(s+s_1)^2+s^2}}{\sigma}\right]\left(W_1\varphi\left[\tfrac{s+s_1}{\sigma}\right]+W\varphi\left[\tfrac{s}{\sigma}\right]\right)>0\ \forall s.\end{aligned} \tag{30}$$

Furthermore the second derivative w.r.t s_1 is

$$\frac{\text{d}^2C_5}{\text{d}s_1^2}=\tfrac{W_1}{\sqrt{2\pi}\sigma}\exp\left[-\tfrac{(s+s_1)^2}{2\sigma^2}\right]>0\ \forall s_1. \tag{31}$$

The optimal $s_1^*=\infty$ and in the limit as $s_1\to\infty$ then $s^*\to\sigma\Phi^{-1}\left[\frac{W-U}{W}\right]$ and $C_5^*\to\mu U+\sigma W\varphi\left[\Phi^{-1}\left[\frac{W-U}{W}\right]\right]$. As an interesting aside, as $s_1\to 0$ then $s^*\to\sigma\Phi^{-1}\left[\frac{W_1-U}{W_1}\right]$ and $C_5^*\to\mu U+\sigma W_1\varphi\left[\Phi^{-1}\left[\frac{W_1-U}{W_1}\right]\right]$, highlighting the similarities between C_5 and C_3.

3.6 Double v-type costs

Consider an extension to the piece-wise linear and convex, v-type costs given by case 2 and case 5, where there is a double v, both in the opportunity costs and then over-time costs [8]. That is, costs are given by

$$C_6=E\left[C_{6,t}\right];\ C_{6,t}=W_1\left(o_t-\mu-s-s_1\right)^++W\left(o_t-\mu-s\right)^++U\left(\mu+s_1-o_t\right)^++U_1\left(\mu+s_1-o_t-s_2\right)^+. \tag{32}$$

Taking the expectations yields

$$C_6 = \frac{1}{2}\begin{pmatrix} s(U+U_1) - s_2U_1 + \sqrt{\tfrac{2}{\pi}}\left(U_1\exp\left[-\tfrac{(s-s_2)^2}{2\sigma^2}\right] + (U+W)\exp\left[-\tfrac{s^2}{2\sigma^2}\right] + W_1\exp\left[-\tfrac{(s+s_1)^2}{2\sigma^2}\right]\right)\sigma + \\ s\left((U+W)\operatorname{erf}\left[\tfrac{s}{\sqrt{2}\sigma}\right] - W\right) + W_1(s+s_1)\left(\operatorname{erf}\left[\tfrac{s+s_1}{\sqrt{2}\sigma}\right]-1\right) + U_1(s-s_2)\operatorname{erf}\left[\tfrac{s-s_2}{\sqrt{2}\sigma}\right] \end{pmatrix}$$
$$= \begin{pmatrix} \sigma(U+W)\varphi\left[\tfrac{s}{\sigma}\right] + \sigma W_1\varphi\left[\tfrac{s+s_1}{\sigma}\right] + \sigma U_1\varphi\left[\tfrac{s-s_2}{\sigma}\right] + s(U+W)\Phi\left[\tfrac{s}{\sigma}\right] + \\ (s+s_1)W_1\Phi\left[\tfrac{s+s_1}{\sigma}\right] + (s-s_2)U_1\Phi\left[\tfrac{s-s_2}{\sigma}\right] - sW - (s+s_1)W_1 \end{pmatrix} \tag{33}$$

The derivative of C_6 w.r.t. s is

$$\begin{aligned}\frac{\mathrm{d}C_6}{\mathrm{d}s} &= \tfrac{1}{2}\left(U\left(1+\operatorname{erf}\left[\tfrac{s}{\sqrt{2}\sigma}\right]\right) + U_1\left(1+\operatorname{erf}\left[\tfrac{s\text{-}s_2}{\sqrt{2}\sigma}\right]\right) - W\left(1-\operatorname{erf}\left[\tfrac{s}{\sqrt{2}\sigma}\right]\right) - W_1\left(1-\operatorname{erf}\left[\tfrac{s+s_1}{\sqrt{2}\sigma}\right]\right)\right) \\ &= (U+W)\Phi\left[\tfrac{s}{\sigma}\right] + W_1\Phi\left[\tfrac{s+s_1}{\sigma}\right] + U_1\Phi\left[\tfrac{s+s_2}{\sigma}\right] - W - W1\end{aligned} \tag{34}$$

The first order conditions for s_1 imply that $s_1^* = \infty$ and those for s_2 imply that $s_2^* = -\infty$. When this happens then $s^* = \sigma\Phi^{-1}\left[\frac{W}{W+U}\right]$ and $C_6^* = \sigma(U+W)\varphi\left[\Phi^{-1}\left[\frac{W}{W+U}\right]\right]$. Also as $s_1 = 0$ and $s_2 = 0$ then $s^* \to \sigma\Phi^{-1}\left[\frac{W_1}{W_1+U}\right]$ and $C_6^* \to \sigma(U+W)\varphi\left[\Phi^{-1}\left[\frac{W_1}{W_1+U}\right]\right]$. If $\{s_1, s_2\}$ are not set optimally then there is only one minimum for s in C_6 as

$$\frac{\mathrm{d}^2C_6}{\mathrm{d}s^2} = \tfrac{1}{\sqrt{2\pi}\sigma}\exp\left[-\tfrac{(s+s_1)^2+s_2^2}{2\sigma^2}\right]\left(U_1\exp\left[\tfrac{s_1^2+2s(s_1+s_2)}{2\sigma^2}\right] + \exp\left[\tfrac{s_2^2}{2\sigma^2}\right]\left((U+W)\exp\left[\tfrac{s_1(2s+s_1)}{2\sigma^2}\right] + W_1\right)\right) > 0\ \forall s \tag{35}$$

and in this case we find that the approximation for the cdf in (29) yields adequate starting points for numerical search routines when $\left\{\frac{s}{\sigma}, \frac{s+s_1}{\sigma}, \frac{s-s_2}{\sigma}\right\}$ are sufficiently close to the origin.

3.7 Step-wise guaranteed hours and guaranteed over-time

Consider the situation where hours within the normal working week are guaranteed and over-time comes in a discrete block if it is needed. The cost function then becomes

$$C_7 = E\left[C_{7,t}\right];\ C_{7,t} = U(s+\mu) + W(s_1 - s - \mu)\left((o_t - s - \mu)^+ / (o_t - s - \mu)\right). \tag{36}$$

As the normal distribution is defined from $-\infty$ to ∞ it is necessary that an upper limit, $s_1 + \mu$, is placed on the orders that can be completed in a week in (36). If the orders are less then or equal to the nominal capacity then the cost of the guaranteed hours is incurred. If the orders are greater than the nominal capacity then over-time is used to produce $(s_1 - s)$ orders at a unit cost of W. s_1 could be set to ensure that only a certain proportion (p) of periods end without producing the demanded orders. Perhaps, in the case where orders are set to the last observed demand and lead-times are smaller than the customer's expectation, this can be linked to the service level agreement a company has with its customer. In this case, $s_1 = \sqrt{2}\sigma\operatorname{erf}^{-1}[2p-1] = \Phi^{-1}[p]$.

$$C_7 = U\left(s+\mu\right) + W\overbrace{\tfrac{1}{2}\left(1-\text{erf}\left[\tfrac{s}{\sqrt{2}\sigma}\right]\right)}^{P(o_t>\mu+s)}\overbrace{\left(\sqrt{2}\sigma\text{erf}^{-1}\left[2p-1\right]-s\right)}^{s_1-s} = U\left(s+\mu\right)+W\left(1-\Phi\left[\tfrac{s}{\sigma}\right]\right)\left(s_1-s\right). \quad (37)$$

As $s \to \infty$ then $C_7 \to \infty$ and $\left.\frac{\text{d}C_7}{\text{d}s}\right|_{s\to\infty} = U$. As $s \to -\infty$ then $C_7 \to \text{sgn}\left[W-U\right]\infty$ and $\left.\frac{\text{d}C_7}{\text{d}s}\right|_{s\to-\infty} = U-W$. When $W=U$ then as $s \to -\infty$, $C_7 \to U\left(\mu + s_1\right)$. The derivative of C_7,

$$\frac{\text{d}C_7}{\text{d}s} = U - \tfrac{W}{2}\left(1-\text{erf}\left[\tfrac{s}{\sqrt{2}\sigma}\right]\right) - \tfrac{W}{\sqrt{2\pi}\sigma}\exp\left[-\tfrac{s^2}{2\sigma^2}\right]\left(s_1-s\right) = U - W\left(1-\Phi\left[\tfrac{s}{\sigma}\right]+\tfrac{(s_1-s)}{\sigma}\varphi\left[\tfrac{s}{\sigma}\right]\right), \quad (38)$$

has no closed form solution. We note that (38) has a structure that is very similar to the Loss function. The second derivative of C_7 w.r.t. *s*,

$$\frac{\text{d}^2C_7}{\text{d}s^2} = \frac{W\left(2\sigma^2+s_1s-s^2\right)}{\sqrt{2\pi}}\exp\left[-\tfrac{s^2}{2\sigma^2}\right] = \frac{W\left(2\sigma^2+s_1s-s^2\right)}{\sigma^3}\varphi\left[\tfrac{s}{\sigma}\right] > 0\ \forall s, \quad (39)$$

shows us that C_7 is convex in *s* between

$$s = \tfrac{1}{2}\left(s_1 \pm \sqrt{s_1^2 + 8\sigma^2}\right). \quad (40)$$

This means that there will be a minimum between these two points. Numerical explorations and (38) seem to suggest that if $W > U$ then this minimum will be unique, but we have no formal proof of this fact. Numerical search techniques (for example Solver in Excel) should use a starting point between the two bounds given in (40).

3.8 Guaranteed hours and over-time with additional linear over-time

Here we consider the case where there are guaranteed working hours each week, guaranteed blocks of over-time if required and in addition to that there are linear over-time costs related to the volume requirements to meet exceptionally large orders. This could represent the case where Monday to Friday to unit cost per unit of output at full capacity is *U*. Then on Saturday the workforce is given a days worth of over-time and the available weekly capacity with Saturday working is s_1. The Saturday capacity costs *W* per unit of output. In addition to this, workers are offered volume flexible over-time to meet any additional orders on Sundays. (41) details this situation

$$C_8 = E\left[C_{8,t}\right];\ C_{8,t} = U\left(\mu+s\right) + W\left(s_1-\mu-s\right)\left(\left(o_t-\mu-s\right)^+ / \left(o_t-\mu-s\right)\right) + W_1\left(o_t-\mu-s_1\right). \quad (41)$$

Taking the expectation we have

$$\left.\begin{aligned} C_8 &= U\left(s+\mu\right) + \tfrac{W(s_1-s)}{2}\left(1-\text{erf}\left[\tfrac{s}{\sqrt{2}\sigma}\right]\right) + \int_{\mu+s_1}^{\infty}\tfrac{W_1(x-s_1-\mu)}{\sqrt{2\pi}\sigma}\exp\left[-\tfrac{(x-\mu)^2}{2\sigma^2}\right]\text{d}x \\ &= U(s+\mu) + \tfrac{(s-s_1)W}{2}\left(\text{erf}\left[\tfrac{s}{\sqrt{2}\sigma}\right]-1\right) + \tfrac{\sigma W_1}{\sqrt{2\pi}}\exp\left[-\tfrac{s_1^2}{2\sigma^2}\right] + \tfrac{s_1W_1}{2}\left(\text{erf}\left[\tfrac{s_1}{\sqrt{2}\sigma}\right]-1\right) \\ &= U\left(s+\mu\right) + W\left(s-s_1\right)\left(\Phi\left[\tfrac{s}{\sigma}\right]-1\right) + W_1\left(\sigma\phi\left[\tfrac{s_1}{\sigma}\right] + s_1\left(\Phi\left[\tfrac{s_1}{\sigma}\right]-1\right)\right) \end{aligned}\right\}. \quad (42)$$

The derivative of C_8 w.r.t. s_1 is

$$\frac{dC_8}{ds_1}=\tfrac{W_1}{2}\left(\text{erf}\left[\tfrac{s_1}{\sqrt{2}\sigma}\right]-1\right)-\tfrac{W}{2}\left(\text{erf}\left[\tfrac{s}{\sqrt{2}\sigma}\right]-1\right)=W_1\left(\Phi\left[\tfrac{s_1}{\sigma}\right]-1\right)-W\left(\Phi\left[\tfrac{s}{\sigma}\right]-1\right), \quad (43)$$

that reveals a stationary point exists at

$$s_1^*=\sqrt{2}\sigma\text{erf}^{-1}\left[1+\tfrac{W}{W_1}\left(\text{erf}\left[\tfrac{s}{\sqrt{2}\sigma}\right]-1\right)\right]=\sigma\Phi^{-1}\left[1+\tfrac{W}{W_1}\left(\Phi\left[\tfrac{s}{\sigma}\right]-1\right)\right], \quad (44)$$

which is a minimum as the second derivative, $\frac{\text{d}^2C_8}{\text{d}s_1^2}=\frac{W_1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{s_1^2}{2\sigma^2}\right]>0\ \forall s_1$. As $s\to\infty$ then $C_8\to\infty$ and $\frac{\text{d}C_8}{\text{d}s}\Big|_{s\to\infty}=U$. As $s\to-\infty$ then $C_8\to\text{sgn}\left[W-U\right]\infty$ and $\frac{\text{d}C_8}{\text{d}s}\Big|_{s\to-\infty}=U-W$. When $W=U$ then as $s\to-\infty$,

$$C_8\to U\left(s_1+\mu\right)+W\left(\tfrac{\sigma}{\sqrt{2\pi}}\exp\left[-\tfrac{s_1^2}{2\sigma^2}\right]-\tfrac{s_1}{2}\left(1-\text{erf}\left[\tfrac{s_1}{\sqrt{2}\sigma}\right]\right)\right)=U\left(s_1+\mu\right)+W\sigma\varphi\left[\tfrac{s_1}{\sigma}\right]-s_1\left(1-\Phi\left[\tfrac{s_1}{\sigma}\right]\right). \quad (45)$$

The derivative of C_8 w.r.t. s is

$$\frac{\text{d}C_8}{\text{d}s}=U+\tfrac{W(s-s_1)}{\sqrt{2\pi}\sigma}\exp\left[-\tfrac{s^2}{2\sigma^2}\right]+\tfrac{W}{2}\left(\text{erf}\left[\tfrac{s}{\sqrt{2}\sigma}\right]-1\right)=U+\tfrac{W(s-s_1)}{\sigma}\varphi\left[\tfrac{s}{\sigma}\right]+W\left(\Phi\left[\tfrac{s}{\sigma}\right]-1\right), \quad (46)$$

(46) has no known solution. However, the second derivative $\frac{\text{d}^2C_8}{\text{d}s^2}=\frac{\text{d}^2C_7}{\text{d}s^2}$ in (39). So C_8 is convex in s between the two points given by (40).

3.9 Guaranteed hours and double guaranteed over-time

Consider the case when there are guaranteed hours and over-time come in two discrete blocks. The upper limit, s_2, is to be set in a manner similar to case 7, to place a limit on the maximum order that will be attempted to be accomplished as otherwise there is no solution. That is, $s_2=\Phi^{-1}\left[p\right]$, where p is the target probability of orders that will be completed. The following difference equation details the costs in the time domain

$$C_9=E\left[C_{9,t}\right];C_{9,t}=U\left(\mu+s\right)+W\left(s_1-s\right)\tfrac{(o_t-\mu-s)^+}{o_t-\mu-s}+W_1\left(s_2-s_1-s\right)\tfrac{(o_t-\mu-s_1-s)^+}{o_t-\mu-s_1-s}. \quad (47)$$

which has the following expected value

$$\begin{aligned}C_9&=U\left(s+\mu\right)+\tfrac{W(s_1-s)}{2}\left(1-\text{erf}\left[\tfrac{s}{\sqrt{2}\sigma}\right]\right)+\tfrac{W_1(s_2-s_1-s)}{2}\left(1-\text{erf}\left[\tfrac{s+s_1}{\sqrt{2}\sigma}\right]\right)\\&=U\left(s+\mu\right)+W(s_1-s)\left(1-\Phi\left[\tfrac{s}{\sigma}\right]\right)+W_1(s_2-s_1-s)\left(1-\Phi\left[\tfrac{s+s_1}{\sigma}\right]\right)\end{aligned}. \quad (48)$$

In the limit as $s\to\infty$ then $C_9\to\infty$ and as $s\to-\infty$, $C_9\to\text{sgn}\left[W+W_1-U\right]\infty$. C_9 is linear in s_2. The derivatives of C_9 w.r.t. s are obtainable, but are rather hard to analyse further. The derivatives of C_9 w.r.t. s_1 lead to

$$\frac{\mathrm{d}^2 C_9}{\mathrm{d}s_1^2}=\frac{\exp\left[-\frac{(s+s_1)^2}{2\sigma^2}\right]\left(2W_1\sigma^2-(s+s_1)(s+s_1-s_2)W_1\right)}{\sqrt{2\pi}\sigma^3}=\frac{\varphi\left[\frac{s+s_1}{\sigma}\right]\left(2W_1\sigma^2-(s+s_1)(s+s_1-s_2)W_1\right)}{\sigma^3} \tag{49}$$

which show that C_9 is convex in s_1 between the points

$$s_1=\frac{s_2-2s\pm\sqrt{s_2^2+8\sigma^2}}{2}. \tag{50}$$

Numerical investigations reveal that C_9 seems to have one minimum between these two points, no minimums above them, but $C_9\to-\infty$ if $W+W_1-U$ is negative.

3.10 Quadratic costs

[9] discuss the cost of overtime. They argue that the cost function given by (15) "can be expected only if there are no discontinuities and random disturbances in production process"… "since workers are each somewhat specialized in function, it is likely that a small increase in production will require only a few employees who work in bottleneck functions to work over. As production increases, more and more employees are required"… "the effect of this is to smooth the overtime cost curve". If this is so, then HMMS argue that rather then (piece-wise) linear costs, perhaps (piece-wise) quadratic cost functions are more appropriate. [10] also present a similar argument.

Suppose that small deviations from the nominal capacity result in small costs, but large deviation cause larger costs. We could model this situation with capacity costs that are proportional to the square of the deviation from a nominal capacity of $(\mu+s)$,

$$C_{10}=E\left[C_{10,t}\right];\ E\left[C_{10,t}\right]=U\left(o_t-\mu-s\right)^2. \tag{51}$$

The expected per period cost are then given by

$$C_{10}=\int_{-\infty}^{\infty}\frac{U(x-\mu-s)^2}{\sqrt{2\pi}\sigma}\exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]\mathrm{d}x=\int_{-\infty}^{\infty}\frac{U(x-\mu-s)^2}{\sigma}\varphi\left[\frac{x-\mu}{\sigma}\right]\mathrm{d}x=U\left(s^2+\sigma^2\right). \tag{52}$$

It is clear to see that C_{10} is convex in s and has a minimum at $s=0$. It is also interesting to note that C_{10} is linear in the order variance and convex in the standard deviation of the orders. Notice that in (52) we penalised negative orders. If negative orders are cost neutral and $C_{10,t}=U\left(\left(o_t-\mu-s\right)^+\right)^2$ then we should consider the costs of

$$\left.\begin{aligned}C_{10}&=\int_{0}^{\infty}\frac{U(x-\mu-s)^2}{\sqrt{2\pi}\sigma}\exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]\mathrm{d}x=\int_{-\infty}^{\infty}\frac{U(x-\mu-s)^2}{\sigma}\varphi\left[\frac{x-\mu}{\sigma}\right]\mathrm{d}x\\&=\frac{U}{2}\left(s^2+\sigma^2\right)\left(1+\operatorname{erf}\left[\frac{\mu}{\sqrt{2}\sigma}\right]\right)-\frac{\sigma U(2s+\mu)}{\sqrt{2\pi}}\exp\left[-\frac{\mu^2}{2\sigma^2}\right]=U\left(s^2+\sigma^2\right)\Phi\left[\frac{\mu}{\sigma}\right]-\sigma U\left(2s+\mu\right)\varphi\left[\frac{\mu}{\sigma}\right]\end{aligned}\right\} \tag{53}$$

which has the following derivative,

$$\frac{\mathrm{d}C_{10}}{\mathrm{d}s}=U\left(s\left(1+\operatorname{erf}\left[\frac{\mu}{\sqrt{2}\sigma}\right]\right)-\sigma\sqrt{\frac{2}{\pi}}\exp\left[-\frac{\mu^2}{2\sigma^2}\right]\right)=2U\left(s\Phi\left[\frac{\mu}{\sigma}\right]-\sigma\phi\left[\frac{\mu}{\sigma}\right]\right) \tag{54}$$

and the first order conditions for a stationary point are

$$s^* = \sigma\sqrt{\tfrac{2}{\pi}}\exp\left[-\tfrac{\mu^2}{2\sigma^2}\right]\Big/\left(1+\text{erf}\left[\tfrac{\mu}{\sqrt{2}\sigma}\right]\right) = \sigma\varphi\left[\tfrac{\mu}{\sigma}\right]\Big/\Phi\left[\tfrac{\mu}{\sigma}\right]. \tag{55}$$

s^* is a global minimum as $\frac{\text{d}C_{10}^2}{\text{d}s^2} = U\left(1+\text{erf}\left[\tfrac{\mu}{\sqrt{2}\sigma}\right]\right) > 0\ \forall s$. Interestingly, s^* is not influenced by the cost (gain) factor U. The minimised costs are

$$\left.\begin{aligned} C_{10}^* &= U\left(\tfrac{\sigma^2}{2}\left(1+\text{erf}\left[\tfrac{\mu}{\sqrt{2}\sigma}\right]\right) - \tfrac{\mu\sigma}{\sqrt{2\pi}}\exp\left[\tfrac{\mu^2}{2\sigma^2}\right] - \left(\sigma^2\exp\left[-\tfrac{\mu^2}{\sigma^2}\right]\Big/\pi\left(1+\text{erf}\left[\tfrac{\mu}{\sqrt{2}\sigma}\right]\right)\right)\right) \\ &= U\left(\sigma^2\Phi\left[\tfrac{\mu}{\sigma}\right] - \mu\sigma\varphi\left[\tfrac{\mu}{\sigma}\right] - \left(\sigma^2\varphi\left[\tfrac{\mu}{\sigma}\right]\Big/\sqrt{2\pi}\Phi\left[\tfrac{\mu}{\sigma}\right]\right)\right)\end{aligned}\right\}. \tag{56}$$

3.11 Piecewise quadratic costs

The natural extension of C_{10} is to consider the case when order over and under the nominal capacity are penalised at different rates. That is, suppose

$$C_{11} = E\left[C_{11,t}\right];\ C_{11,t} = U\left(\left(o_t-\mu-s\right)^+\right)^2 + W\left(\left(\mu+s-o_t\right)^+\right)^2, \tag{57}$$

which has the following expectation,

$$\left.\begin{aligned} C_{11} &= \int_{-\infty}^{\mu+s}\tfrac{U(x-s-\mu)^2}{\sqrt{2\pi}\sigma}\exp\left[-\tfrac{(x-\mu)^2}{2\sigma^2}\right]\text{d}x + \int_{\mu+s}^{\infty}\tfrac{W(x-s-\mu)^2}{\sqrt{2\pi}\sigma}\exp\left[-\tfrac{(x-\mu)^2}{2\sigma^2}\right]\text{d}x \\ &= \tfrac{s(U-W)\sigma}{\sqrt{2\pi}}\exp\left[-\tfrac{s^2}{2\sigma^2}\right] + \tfrac{\left(s^2+\sigma^2\right)}{2}\left(U+W+(U-W)\text{erf}\left[\tfrac{s}{\sqrt{2}\sigma}\right]\right) \\ &= s\sigma\left(U-W\right)\varphi\left[\tfrac{s}{\sigma}\right] + \left(s^2+\sigma^2\right)\left(W+\left(U-W\right)\Phi\left[\tfrac{s}{\sigma}\right]\right)\end{aligned}\right\}. \tag{58}$$

The first order derivative w.r.t. s is

$$\left.\begin{aligned} \frac{\text{d}C_{11}}{\text{d}s} &= \sqrt{\tfrac{2}{\pi}}(U-W)\sigma\exp\left[-\tfrac{s^2}{2\sigma^2}\right] + s\left(U+W+(U-W)\text{erf}\left[\tfrac{s}{\sqrt{2}\sigma}\right]\right) \\ &= 2\left(U-W\right)\sigma\varphi\left[\tfrac{s}{\sigma}\right] + 2s\left(W+\left(U-W\right)\Phi\left[\tfrac{s}{\sigma}\right]\right)\end{aligned}\right\}. \tag{59}$$

(59) has no solution, but we know that there is only one minimum in s as (60) shows that C_{11} is convex in s. This means that there is a single minimum in the cost function.

$$\frac{\text{d}^2C_{11}}{\text{d}s^2} = U+W+(U-W)\text{erf}\left[\tfrac{s}{\sqrt{2}\sigma}\right] = 2\left(W+(U-W)\left(\Phi\left[\tfrac{s}{\sigma}\right]\right)\right) > 0\ \forall s \tag{60}$$

3.12 Guaranteed hours and quadratic over-time

Finally we consider the situation with guaranteed hours each week and the quadratic over-time costs based on volume. Perhaps this is representative of a Nagare Line where costs are dominated by labour and there is a marginal additional rate of output from increases in labour.

$$C_{12} = E\left[C_{12,t}\right] = U\left(\mu + s\right) + W\left(\left(o_t - \mu - s\right)^+\right)^2 \tag{61}$$

The expected costs per period are given by

$$\left.\begin{aligned} C_{12} &= U(s+\mu) + \int_{\mu+s}^{\infty} \tfrac{W(x-\mu-s)^2}{\sqrt{2\pi}\sigma} \exp\left[-\tfrac{(x-\mu)^2}{2\sigma^2}\right] \mathrm{d}x \\ &= U(s+\mu) - W\left(\tfrac{s\sigma}{\sqrt{2\pi}} \exp\left[-\tfrac{s^2}{2\sigma^2}\right] + \tfrac{\left(s^2+\sigma^2\right)}{2}\left(1 - \operatorname{erf}\left[\tfrac{s}{\sqrt{2}\sigma}\right]\right)\right) \\ &= U\left(s+\mu\right) - W\left(s\sigma\varphi\left[\tfrac{s}{\sigma}\right] + \left(s^2+\sigma^2\right)\left(1 - \Phi\left[\tfrac{s}{\sigma}\right]\right)\right) \end{aligned}\right\}. \tag{62}$$

The first order derivative w.r.t. s is

$$\frac{\mathrm{d}C_{12}}{\mathrm{d}s} = U + W\left(s\left(1 - \operatorname{erf}\left[\tfrac{s}{\sqrt{2}\sigma}\right]\right) - \sigma\sqrt{\tfrac{2}{\pi}} \exp\left[-\tfrac{s^2}{2\sigma^2}\right]\right) = U - 2W\left(\sigma\varphi\left[\tfrac{s}{\sigma}\right] + s\left(1 - \Phi\left[\tfrac{s}{\sigma}\right]\right)\right). \tag{63}$$

Again (63) is quite similar to the Loss function but has now known solution. However as

$$\frac{\mathrm{d}^2C_{12}}{\mathrm{d}s^2} = W\left(1 - \operatorname{erf}\left[\tfrac{s}{\sqrt{2}\sigma}\right]\right) = 2W\left(1 - \Phi\left[\tfrac{s}{\sigma}\right]\right) > 0 \ \forall s \tag{64}$$

then C_{12} has a single minimum.

4. Concluding remarks

We have explored twelve different cost functions that could be used to assign costs to a stochastic production and inventory control policy. What has become evident is that a few building blocks, linear, step-wise and quadratic components, can lead to a rich set of objective functions. A seemingly simple and straight forward modification / addition to the objective function can lead to an objective function that has no closed form solution. However, in most cases we are able to obtain insights into the convexity and limit behaviour. Knowing the number of, and location of minimums, we are able to assign confidence to the results of numerical search routines. Further work could involve exploring how these cost functions behave when approximations to $\varphi[x]$ and $\Phi[x]$ are used.

5. References

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Appendix A. Visual Basic for Excel Add-In for determining the Inverse Loss function

Visual Basic Code required to calculate the Inverse Loss Function

```
Function InvLossFun(x As Double)
Dim u As Double
Dim l As Double
Dim z As Double
Dim p As Double
Dim b As Double

u = 6.1
l = -6.1

For g = 1 To 100000000
    z = l + (( u - l) / 2)
    p = (z / 2) * (1 - (2 * WorksheetFunction.NormDist(z, 0, 1, True) - 1)) + x
    b = Exp( - (z ^ 2) / 2) / 2.506628274631
    If p > b Then
        upper = z
    Else
        lower = z
   End If
   If (u - l) < 0.000000000000001 Then
        g = 100000000
   End If
Next g

InvLossFun = l + (( u - l) / 2)

End Function
```

Table 1. Visual Basic code required to create an Excel Add-In for the Inverse Loss function

Appendix B. Standard Normal Distribution Table

z	$\varphi[z]$	$\Phi[z]$	$L[z]$
-6	6.07E-09	9.86E-10	6
-5	1.48E-06	2.86E-07	5
-4	0.000134	3.17E-05	4.000007
-3.95	0.000163	3.91E-05	3.950009
-3.9	0.000199	4.81E-05	3.900011
-3.85	0.000241	5.91E-05	3.850014
-3.8	0.000292	7.23E-05	3.800017
-3.75	0.000353	8.84E-05	3.750021
-3.71902	0.000396	0.0001	3.71904
-3.7	0.000425	0.000108	3.700026
-3.65	0.00051	0.000131	3.650032
-3.6	0.000612	0.000159	3.600039
-3.55	0.000732	0.000193	3.550048
-3.5	0.000873	0.000233	3.500058
-3.45	0.001038	0.00028	3.450071
-3.4	0.001232	0.000337	3.400087
-3.35	0.001459	0.000404	3.350105
-3.3	0.001723	0.000483	3.300127
-3.25	0.002029	0.000577	3.250154
-3.2	0.002384	0.000687	3.200185
-3.15	0.002794	0.000816	3.150223
-3.1	0.003267	0.000968	3.100267
-3.09023	0.003367	0.001	3.09051
-3.05	0.00381	0.001144	3.05032
-3	0.004432	0.00135	3.000382
-2.95	0.005143	0.001589	2.950455
-2.9	0.005953	0.001866	2.900542
-2.85	0.006873	0.002186	2.850643
-2.8	0.007915	0.002555	2.800761
-2.75	0.009094	0.00298	2.750899
-2.7	0.010421	0.003467	2.70106
-2.65	0.011912	0.004025	2.651247
-2.6	0.013583	0.004661	2.601464
-2.57583	0.01446	0.005	2.57741
-2.55	0.015449	0.005386	2.551715
-2.5	0.017528	0.00621	2.502004
-2.45	0.019837	0.007143	2.452337
-2.4	0.022395	0.008198	2.40272
-2.35	0.025218	0.009387	2.353159
-2.32635	0.026652	0.01	2.32974
-2.3	0.028327	0.010724	2.303662
-2.25	0.03174	0.012224	2.254235
-2.2	0.035475	0.013903	2.204887
-2.17009	0.03787	0.015	2.17541
-2.15	0.03955	0.015778	2.155628
-2.1	0.043984	0.017864	2.106468
-2.05375	0.048418	0.02	2.06109
-2.05	0.048792	0.020182	2.057418
-2	0.053991	0.02275	2.008491
-1.95996	0.058445	0.025	1.96941
-1.95	0.059595	0.025588	1.959698
-1.9	0.065616	0.028717	1.911054
-1.88079	0.068042	0.03	1.89241
-1.85	0.072065	0.032157	1.862575
-1.8	0.07895	0.03593	1.814276
-1.75069	0.086174	0.04	1.76683
-1.75	0.086277	0.040059	1.766174
-1.7	0.094049	0.044565	1.718288
-1.65	0.102265	0.049471	1.670637
-1.64485	0.103136	0.05	1.66575
-1.6	0.110921	0.054799	1.623242
-1.55477	0.119123	0.06	1.58061
-1.55	0.120009	0.060571	1.576124
-1.5	0.129518	0.066807	1.529307
-1.47579	0.134268	0.07	1.50675
-1.45	0.139431	0.073529	1.482813
-1.43953	0.141555	0.075	1.47312
-1.40507	0.148666	0.08	1.44133
-1.4	0.149727	0.080757	1.436668
-1.35	0.160383	0.088508	1.390898
-1.34076	0.162391	0.09	1.38248
-1.3	0.171369	0.0968	1.345528
-1.28155	0.175498	0.1	1.32889
-1.25	0.182649	0.10565	1.300587
-1.2	0.194186	0.11507	1.256102
-1.15035	0.205854	0.125	1.21241

z	$\varphi[z]$	$\Phi[z]$	$L[z]$
-1.15	0.205936	0.125072	1.212104
-1.1	0.217852	0.135666	1.16862
-1.05	0.229882	0.146859	1.12568
-1.03643	0.233159	0.15	1.11413
-1	0.241971	0.158655	1.083315
-0.95	0.254059	0.171056	1.041556
-0.93458	0.257775	0.175	1.02881
-0.9	0.266085	0.18406	1.000431
-0.85	0.277985	0.197663	0.959972
-0.84162	0.279962	0.2	0.953259
-0.8	0.289692	0.211855	0.920207
-0.75541	0.299913	0.225	0.885359
-0.75	0.301137	0.226627	0.881167
-0.7	0.312254	0.241964	0.842879
-0.67449	0.317777	0.25	0.823644
-0.65	0.322972	0.257846	0.805372
-0.6	0.333225	0.274253	0.768673
-0.59776	0.333672	0.275	0.767048
-0.55	0.342944	0.29116	0.732806
-0.52440	0.347693	0.3	0.714773
-0.5	0.352065	0.308538	0.697797
-0.45376	0.359915	0.325	0.666204
-0.45	0.360527	0.326355	0.663667
-0.43073	0.363599	0.333333	0.650753
-0.4	0.36827	0.344578	0.630439
-0.35	0.37524	0.363169	0.598131
-0.31863	0.379195	0.375	0.578345
-0.3	0.381388	0.382089	0.566761
-0.25334	0.386343	0.4	0.538351
-0.25	0.386668	0.401294	0.536345
-0.2	0.391043	0.42074	0.506895
-0.18911	0.391871	0.425	0.500615
-0.15	0.394479	0.440382	0.478422
-0.12566	0.395805	0.45	0.464919
-0.1	0.396953	0.460172	0.450935
-0.06270	0.398159	0.475	0.43108
-0.05	0.398444	0.480061	0.424441
0	0.398942	0.5	0.398942
0.05	0.398444	0.519939	0.374441
0.062706	0.398159	0.525	0.368373
0.1	0.396953	0.539828	0.350935
0.125661	0.395805	0.55	0.339257
0.15	0.394479	0.559618	0.328422
0.189118	0.391871	0.575	0.311496
0.2	0.391043	0.57926	0.306895
0.25	0.386668	0.598706	0.286345
0.253347	0.386343	0.6	0.285004
0.3	0.381388	0.617911	0.266761
0.318639	0.379195	0.625	0.259705
0.35	0.37524	0.636831	0.248131
0.38532	0.370399	0.65	0.235537
0.4	0.36827	0.655422	0.230439
0.43073	0.363599	0.666666	0.220023
0.45	0.360527	0.673645	0.213667
0.453762	0.359915	0.675	0.212442
0.5	0.352065	0.691462	0.197797
0.524401	0.347693	0.7	0.190372
0.55	0.342944	0.70884	0.182806
0.59776	0.333672	0.725	0.169288
0.6	0.333225	0.725747	0.168673
0.65	0.322972	0.742154	0.155372
0.67449	0.317777	0.75	0.149154
0.7	0.312254	0.758036	0.142879
0.75	0.301137	0.773373	0.131167
0.755415	0.299913	0.775	0.129944
0.8	0.289692	0.788145	0.120207
0.841621	0.279962	0.8	0.111638
0.85	0.277985	0.802337	0.109972
0.9	0.266085	0.81594	0.100431
0.934589	0.257775	0.825	0.094222
0.95	0.254059	0.828944	0.091556
1	0.241971	0.841345	0.083315
1.03643	0.233159	0.85	0.077694
1.05	0.229882	0.853141	0.07568
1.1	0.217852	0.864334	0.06862
1.15	0.205936	0.874928	0.062104

z	$\varphi[z]$	$\Phi[z]$	$L[z]$
1.15035	0.205854	0.875	0.06206
1.2	0.194186	0.88493	0.056102
1.25	0.182649	0.89435	0.050587
1.28155	0.175498	0.9	0.047343
1.3	0.171369	0.9032	0.045528
1.34076	0.162391	0.91	0.041723
1.35	0.160383	0.911492	0.040898
1.4	0.149727	0.919243	0.036668
1.40507	0.148666	0.92	0.03626
1.43953	0.141555	0.925	0.033590
1.45	0.139431	0.926471	0.032813
1.47579	0.134268	0.93	0.030962
1.5	0.129518	0.933193	0.029307
1.55	0.120009	0.939429	0.026124
1.55477	0.119123	0.94	0.025836
1.6	0.110921	0.945201	0.023242
1.64485	0.103136	0.95	0.020893
1.65	0.102265	0.950529	0.020637
1.7	0.094049	0.955435	0.018288
1.75	0.086277	0.959941	0.016174
1.75069	0.086174	0.96	0.016146
1.8	0.07895	0.96407	0.014276
1.85	0.072065	0.967843	0.012575
1.88079	0.068042	0.97	0.011618
1.9	0.065616	0.971283	0.011054
1.95	0.059595	0.974412	0.009698
1.95996	0.058445	0.975	0.009446
2	0.053991	0.97725	0.008491
2.05	0.048792	0.979818	0.007418
2.05375	0.048418	0.98	0.007343
2.1	0.043984	0.982136	0.006468
2.15	0.03955	0.984222	0.005628
2.17009	0.03787	0.985	0.005319
2.2	0.035475	0.986097	0.004887
2.25	0.03174	0.987776	0.004235
2.3	0.028327	0.989276	0.003662
2.32635	0.026652	0.99	0.003389
2.35	0.025218	0.990613	0.003159
2.4	0.022395	0.991802	0.00272
2.45	0.019837	0.992857	0.002337
2.5	0.017528	0.99379	0.002004
2.55	0.015449	0.994614	0.001715
2.57583	0.01446	0.995	0.001581
2.6	0.013583	0.995339	0.001464
2.65	0.011912	0.995975	0.001247
2.7	0.010421	0.996533	0.00106
2.75	0.009094	0.99702	0.000899
2.8	0.007915	0.997445	0.000761
2.85	0.006873	0.997814	0.000643
2.9	0.005953	0.998134	0.000542
2.95	0.005143	0.998411	0.000455
3	0.004432	0.99865	0.000382
3.05	0.00381	0.998856	0.00032
3.09023	0.003367	0.999	0.000277
3.1	0.003267	0.999032	0.000267
3.15	0.002794	0.999184	0.000223
3.2	0.002384	0.999313	0.000185
3.25	0.002029	0.999423	0.000154
3.3	0.001723	0.999517	0.000127
3.35	0.001459	0.999596	0.000105
3.4	0.001232	0.999663	8.67E-05
3.45	0.001038	0.99972	7.13E-05
3.5	0.000873	0.999767	5.85E-05
3.55	0.000732	0.999807	4.79E-05
3.6	0.000612	0.999841	3.91E-05
3.65	0.00051	0.999869	3.19E-05
3.7	0.000425	0.999892	2.59E-05
3.71902	0.000396	0.9999	2.39E-05
3.75	0.000353	0.999912	2.1E-05
3.8	0.000292	0.999928	1.7E-05
3.85	0.000241	0.999941	1.37E-05
3.9	0.000199	0.999952	1.11E-05
3.95	0.000163	0.999961	8.91E-06
4	0.000134	0.999968	7.15E-06
5	1.48E-06	1-(2.86E-07)	5.34E-08
6	6.07E-09	1-(9.86E-10)	1.56E-10

Table 2. Standard normal table