Contemporary Mathematics

Modular Invariants from Subfactors

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Abstract. In these lectures we explain the intimate relationship between modular invariants in conformal field theory and braided subfactors in operator algebras. A subfactor with a braiding determines a matrix Z which is obtained as a coupling matrix comparing two kinds of braided sector induction ("α-induction"). It has non-negative integer entries, is normalized and commutes with the S- and T-matrices arising from the braiding. Thus it is a physical modular invariant in the usual sense of rational conformal field theory. The algebraic treatment of conformal field theory models, e.g. $SU(n)_k$ models, produces subfactors which realize their known modular invariants. Several properties of modular invariants have so far been noticed empirically and considered mysterious such as their intimate relationship to graphs, as for example the A-D-E classification for $SU(2)_k$. In the subfactor context these properties can be rigorously derived in a very general setting. Moreover the fusion rule isomorphism for maximally extended chiral algebras due to Moore-Seiberg, Dijkgraaf-Verlinde finds a clear and very general proof and interpretation through intermediate subfactors, not even referring to modularity of S and T. Finally we give an overview on the current state of affairs concerning the relations between the classifications of braided subfactors and two-dimensional conformal field theories. We demonstrate in particular how to realize twisted (type II) descendant modular invariants of conformal inclusions from subfactors and illustrate the method by new examples.

1. Introduction and overview

A subfactor in its simplest guise arises from a group action $M^G \subset M$, the fixed point algebra M^G in the ambient von Neumann algebra M where a group G acts upon. If say the group is finite and acts outerly on M (equivalently $(M^G)' \cap M = \mathbb{C}\mathbf{1}$, where the prime denotes the commutant) and both the group and the algebra are amenable, then we can recover both the group and the action from the inclusion $M^G \subset M$. (If M is not amenable, i.e. hyperfinite, one may recover the group but not the action as in free group factors in free probability theory). However we will concentrate on (infinite-dimensional) hyperfinite von Neumann algebras M which are inductive limits of finite dimensional algebras and are factors i.e. have trivial

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center $M' \cap M = \mathbb{C}\mathbf{1}$. A subfactor $N \subset M$ is then an inclusion of one factor in another, which is thought to represent a deformation of a group, for us we will restrict to the case where we only think of those inclusions which are deviants of finite groups. (Cf. [**29**] as a general reference.)

Rather than a group of $*$-automorphisms of a von Neumann algebra M, we will more generally consider a system Δ of $*$-endomorphisms which is closed under composition

$$\lambda \circ \mu = \bigoplus_{\nu \in \Delta} N_{\lambda,\mu}^{\nu}\, \nu$$

for a suitable notion of addition of endomorphisms (for which we will need infinite von Neumann factors and consider endomorphisms up to inner equivalence, i.e. as sectors [**61**]) and non-negative integral coefficients $N_{\lambda,\mu}^{\nu}$. In our relationship with modular invariant partition functions in conformal field theory, our starting point will be a system of endomorphisms labelled by vertices of graphs as e.g. given in Fig. 1. Each $\lambda \in \Delta$ defines a matrix $N_\lambda = [N_{\lambda,\mu}^{\nu}]_{\mu,\nu}$ of multiplication by λ so that

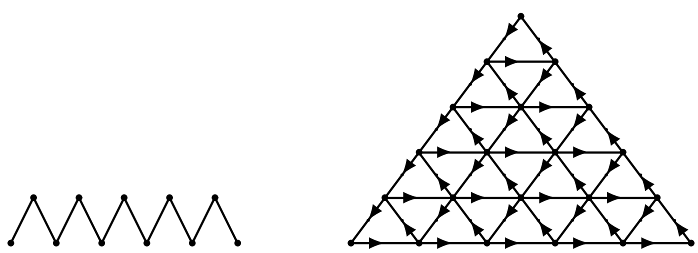

FIGURE 1. Fusion graphs of fundamental generators $\square$ of systems for $SU(2)_{10}$ and $SU(3)_5$

in the above setting the graph of $N_\square$ where $\square$ is the fundamental generator is as described in the figures. For example in the case of the Dynkin diagram A_3, as in Fig. 2. Here we labelled the vertices by b, s, v, and the graph represents the 'fusion'

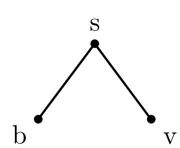

FIGURE 2. Dynkin diagram A_3 as fusion graph

by s, and so the multiplication by s gives the sum of nearest neighbors:

$$\mathrm{s} \cdot \mathrm{b} = \mathrm{s}\,, \qquad \mathrm{s} \cdot \mathrm{s} = \mathrm{b} \oplus \mathrm{v}\,, \qquad \mathrm{s} \cdot \mathrm{v} = \mathrm{s}\,.$$

(Here and in general it is understood that an unoriented edge represents an arrow in both directions.)

These are the well-known fusion rules of the conformal Ising model. A treatment of the Ising model in the framework of local quantum physics realizing these fusion rules in terms of endomorphisms on von Neumann factors was carried out in

[**4**], building on [**63**]. The transfer matrix formalism allows one to study classical statistical mechanical models via non-commutative operator algebras. A study of the Ising model in this framework was carried out in [**1, 27, 15**]. The fundamental example of this non-commutative framework for understanding the Ising model was the driving force towards the present work on understanding modular invariant partition functions via non-commutative operator algebras (cf. the lecture by the second author at the CBMS meeting in Eugene, Oregon, September 1993).

Using associativity of the fusion product one obtains for the fusion matrices

$$N_\lambda N_\mu = \sum\nolimits_\nu N_{\lambda,\mu}^\nu N_\nu \,,$$

i.e. the matrices N_λ themselves give a ("regular") representation of the fusion rules of Δ. Usually the system Δ will be closed under a certain conjugation $\lambda \mapsto \overline{\lambda}$ (generalizing the notion of inverse and conjugate representation in a group and group dual, respectively) which is anti-multiplicative and additive. This will mean that the transpose of N_λ is $N_{\overline{\lambda}}$. If we start with a system obeying commutative fusion rules (which will not always be the case), the collection $\{N_\lambda\}_{\lambda\in\Delta}$ will therefore constitute a family of normal commuting matrices, and hence be simultaneously diagonalizable, with spectra $\mathrm{spec}(N_\lambda) = \{\gamma_\rho^\lambda\}_\rho$. In fact their spectra will be labelled naturally by the entire set Δ itself, i.e. we will have $\rho \in \Delta$. In this diagonalization we have

$$\gamma_\rho^\lambda \gamma_\rho^\mu = \sum\nolimits_\nu N_{\lambda,\mu}^\nu \gamma_\rho^\nu \,, \tag{1.1}$$

i.e. the eigenvalues provide one-dimensional representations of the fusion rules. The matrix γ_ρ^λ is invertible and we can invert Eq. (1.1) to obtain the Verlinde formula [**81**]

$$N_{\lambda,\mu}^\nu = \sum\nolimits_\rho \frac{S_{\lambda,\rho}}{S_{0,\rho}} S_{\mu,\rho} S_{\nu,\rho}^* \,. \tag{1.2}$$

Here we write the eigenvalues of N_λ as $\gamma_\rho^\lambda = S_{\lambda,\rho}/S_{0,\rho}$, where the label "0" refers to the distinguished identity element ("vacuum") of the fusion rules, and $S_{0,\rho} = (\sum_\lambda |\gamma_\rho^\lambda|^2)^{1/2}$. (See [**36**] for fusion rules in the context of conformal field theory.)

In our subfactor approach to modular invariants we will have representations of the Verlinde fusion rules appearing naturally, with spectrum a proper subset of Δ and with multiplicities $Z_{\lambda,\lambda}$, $\lambda \in \Delta$, given by the diagonal part of a modular invariant. The representation matrices can be interpreted a adjacency matrices of graphs associated with modular invariants.

Modular invariant partition functions arise as continuum limits in statistical mechanics and play a fundamental role in conformal field theory. Recall that a modular invariant partition function is of the form (cf. Zuber's lectures, or see [**21, 35, 54, 20, 41**] for more details on these matters)

$$Z(\tau) = \sum\nolimits_{\lambda,\mu} Z_{\lambda,\mu} \chi_\lambda(\tau) \chi_\mu(\tau)^* \,.$$

Here $\chi_\lambda = \mathrm{tr}(q^{L_0 - c/24})$, $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$, is the trace in the irreducible representation of a chiral algebra, which for us will be a positive energy representation of a loop group with the conformal Hamiltonian L_0 being the infinitesimal generator of the rotation group on the circle. (More typically we would take un-specialized characters in order to have linearly independent characters. See for example [**18**] or [**29**, Sect. 8.3] for explicit computations with corner transfer matrices and derivations of the Virasoro characters in the context of the Ising model.) Then the action of the modular group $SL(2;\mathbb{Z})$ on $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$ via $\mathcal{S} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$: $\tau \mapsto -1/\tau$, and $\mathcal{T} = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$: $\tau \mapsto \tau+1$, transforms the family of characters $\{\chi_\lambda\}$ linearly. More precisely, there are matrices S and T such that

$$\chi_\lambda(-1/\tau) = \sum\nolimits_\mu S_{\lambda,\mu}\chi_\mu(\tau)\,, \qquad \chi_\lambda(\tau+1) = \sum\nolimits_\mu T_{\lambda,\mu}\chi_\mu(\tau)\,.$$

Note first that what is remarkable about the Verlinde formula, Eq. (1.2), is that the matrix which diagonalizes the fusion rules is the same as the modular matrix S which transforms the characters (e.g. the Kac-Peterson matrix for current algebra models, see [**54, 35**]). It is also remarkable that this matrix is symmetric: $S_{\lambda,\mu} = S_{\mu,\lambda}$.

From physical considerations we will require solutions to the matrix equations $ZS = SZ$, $ZT = TZ$, subject to the constraint $Z_{0,0} = 1$ ("uniqueness of the vacuum") and the "coupling matrix" Z having only non-negative integer entries (from multiplicities of the representations). There will always be at least one solution, the diagonal partition function

$$Z = \sum\nolimits_\lambda |\chi_\lambda|^2$$

(or $Z_{\lambda,\mu} = \delta_{\lambda,\mu}$), or more generally there may be permutation invariants

$$Z = \sum\nolimits_\lambda \chi_\lambda \chi^*_{\omega(\lambda)}\,,$$

whenever ω is a permutation of the labels which preserves the fusion rules, the vacuum, and the "conformal dimensions". Moore and Seiberg argue in [**66**] (see also [**24**]) that after a "maximal extension of the chiral algebra" (the hardest part is to make this mathematically precise) the partition function of a RCFT is at most a permutation matrix $Z^{\mathrm{ext}}_{\tau,\tau'} = \delta_{\tau,\omega(\tau')}$, where τ,τ' label the representations of the extended chiral algebra and now ω denotes a permutation of these with analogous invariance properties. Decomposing the extended characters χ^{ext}_τ in terms of the original characters χ_λ, we have $\chi^{\mathrm{ext}}_\tau = \sum_\lambda b_{\tau,\lambda}\chi_\lambda$ for some non-negative integral branching coefficients $b_{\tau,\lambda}$. The maximal extension yields the coupling matrix expression

$$Z_{\lambda,\mu} = \sum\nolimits_\tau b_{\tau,\lambda} b_{\omega(\tau),\mu}\,.$$

There is a distinction [**23**] between so-called type I invariants which arise from the diagonal invariant of the maximal extension, i.e. for which ω is trivial, and type II invariants corresponding to non-trivial automorphisms of the extended fusion rules. The coupling matrix of a type I invariant is in particular symmetric whereas type II invariants need not be so but still the "vacuum coupling" is symmetric: $Z_{0,\lambda} = Z_{\lambda,0}$ for all labels λ. To allow more generally for possibly non-symmetric vacuum coupling, $Z_{0,\lambda} \neq Z_{\lambda,0}$, one may need different extensions for the left and right chiral algebra [**9**] (see also [**58**] where this possibility is explicitly addressed in the context of simple current extensions), and then the distinction between type I and type II modular invariants does no longer make sense.[1]

A simple argument of Gannon [**38**] shows that there are at most finitely many solutions to our modular invariant problem. Since $d_\lambda = S_{\lambda,0}/S_{0,0}$ will be positive and at least 1 (the d_λ's will be the Perron-Frobenius weights of the graphs as in

[1]Surprisingly enough, all known modular invariants of $SU(n)_k$ models are entirely symmetric. Nevertheless there are known modular invariants of other models with non-symmetric ("heterotic") vacuum coupling — see Section 7.

Fig. 1, or indeed d_λ will be the statistical dimension of λ as a sector in the von Neumann algebra theory or the square root of the Jones index [**52**]), we obtain from $SZS^* = Z$ that $\sum_{\lambda,\mu} Z_{\lambda,\mu} \le \sum_{\lambda,\mu} d_\lambda Z_{\lambda,\mu} d_\mu = 1/S_{0,0}^2$. Consequently each integer $Z_{\lambda,\mu}$ must be bounded by $1/S_{0,0,}^2 = \sum_\lambda d_\lambda^2$ (from unitarity of the S-matrix), so that there are only finitely many solutions. Note that this bound will be our "global index" w, and this suggests a strong relation between Gannon's argument and Ocneanu's rigidity theorem (presented at a conference in January 1997 in Madras, India), the latter implying the finiteness of the number of subequivalent paragroups for a given paragroup.

Gannon's estimate can even be refined to the inequality

$$Z_{\lambda,\mu} \le d_\lambda d_\mu \tag{1.3}$$

for each individual entry of a modular invariant coupling matrix as follows. As by Verlinde's formula, Eq. (1.2), the eigenvalues of the non-negative fusion matrices N_λ are given by $\gamma_\rho^\lambda = S_{\lambda,\rho}/S_{0,\rho}$, Perron-Frobenius theory tells us that $|\gamma_\rho^\lambda|$ is bounded by the Perron-Frobenius eigenvalue γ_0^λ, so that $|S_{\lambda,\rho}| \le S_{\lambda,0} S_{0,\rho}/S_{0,0}$ (cf. [**41**]). Commutativity of Z with the unitary S then yields

$$Z_{\lambda,\mu} = \sum_{\rho,\nu} S_{\lambda,\rho} Z_{\rho,\nu} S_{\nu,\mu}^* \le \sum_{\rho,\nu} |S_{\lambda,\rho}| Z_{\rho,\nu} |S_{\nu,\mu}| \le d_\lambda Z_{0,0} d_\mu \,,$$

which provides Eq. (1.3) by the normalization $Z_{0,0} = 1$.

2. Operator algebraic input

We will study the classification of modular invariants and construction of maximal extensions through subfactors, in particular starting with braided systems of endomorphisms on loop group factors which are purely infinite factors with no traces, or more precisely type III_1. Recall that a factor is type I when there is a trace on the algebra taking discrete values on projections, type II when there is a trace that takes continuous values. (We hope that the classification of factors into types I, II and III will not be confused with the distinction of type I and type II modular invariants — it has nothing to do with it.) A trace on a von Neumann algebra M is a (possibly unbounded) linear functional τ satisfying $\tau(ab) = \tau(ba)$, $a, b \in M$, where the algebra or trace is finite if $\tau(\mathbf{1}) < \infty$, or infinite otherwise. Thus a finite type I factor is (isomorphic to) $\mathrm{Mat}(n) = \mathrm{End}(\mathbb{C}^n)$, the $n \times n$ complex matrices, the infinite factor is $B(H)$, the bounded linear operators on an infinite-dimensional Hilbert space. A factor is of type III (or purely infinite) otherwise, there is no trace and every non-zero projection p is equivalent to the unit in the sense that there is a partial isometry v in the algebra such that $v^*v = \mathbf{1}$ and $vv^* = p$. The factors relevant for RCFT are amenable, in the sense that they are hyperfinite, the completions of unions of finite-dimensional algebras. Murray and von Neumann showed that there is an unique hyperfinite II_1 (i.e. finite type II) factor which can be realized as for example the infinite tensor product of matrix algebras (arbitrarily chosen as long as they are non-commutative) completed with respect to the trace, i.e. use the trace τ (constructed as the tensor product of traces over the matrices) to define an inner product on $\mathcal{M}$ (the algebraic tensor product) $\langle a, b\rangle = \tau(b^*a)$. Letting $\Omega = \mathbf{1}$ regarded as a vector in the completion $\mathcal{H}$ of $\mathcal{M}$ with respect to this inner product, we can let $\mathcal{M}$ act on $\mathcal{H}$ by the induced left action of $\mathcal{M}$ on itself, and the hyperfinite II_1 factor is the von Neumann algebra generated by $\mathcal{M}$ in this representation. There is by Connes [**16**] an unique hyperfinite II_∞

(i.e. infinite type II factor) which is $R\otimes B(H)$ where R is the unique hyperfinite II_1 factor and $B(H)$ is type I_∞. There is a finer classification of type III factors into III_λ, $0\le\lambda\le 1$. For each $\lambda\in(0,1]$ there is by Connes an unique hyperfinite III_λ factor (the analysis completed by Haagerup [**47**] in the case $\lambda=1$). The type III_0 factors are classified by their flow of weights.

In the semi-finite case (I or II) where there is a trace τ, we can define a conjugation J, a conjugate linear map of the Hilbert space $\mathcal{H}$ of the trace, by $J: a\mapsto a^*$, $a\in M\subset\mathcal{H}$, or $Ja\Omega=a^*\Omega$. Then J is isometric because τ is a trace and interchanges left and right multiplication, indeed $JMJ=M'$. Thus M and M' are of comparable size. If $M=\mathrm{Mat}(n)=\mathbb{C}^n\otimes\overline{\mathbb{C}^n}$ is finite dimensional, then acting on itself (regarded as a Hilbert space) M becomes $M\otimes\mathbf{1}$ with commutant $\mathbf{1}\otimes M$. In general represent a factor M on a Hilbert space H with vector $\Phi\in H$ cyclic for M, i.e. $H=\overline{M\Phi}$, which is also cyclic for M', $H=\overline{M'\Phi}$. (Take a faithful normal state φ on M and the associated Hilbert space.) Then we can define $S: a\Phi\mapsto a^*\Phi$, $a\in M$, and take the polar decomposition $S=J\Delta^{1/2}$, where J is a conjugation and Δ the (possibly unbounded) Tomita-Takesaki modular operator. Then Tomita-Takesaki theory [**79**] tells us that $JMJ=M'$, and $\sigma_t=\mathrm{Ad}(\Delta^{\mathrm{i}t})$ defines a one-parameter automorphism group of M which describes how far the vector state $\varphi(\cdot)=\langle\cdot\Phi,\Phi\rangle$ is from being a trace, $\varphi(ab)=\varphi(b\sigma_{\mathrm{i}}(a))$ for analytic $a,b\in M$. In the case of a semi-finite algebra, and if φ is a trace, then $S=J$, $\Delta=\mathbf{1}$, and $\sigma_t=\mathrm{id}$, whilst for other choices of cyclic and separating vectors Φ, the Tomita-Takesaki modular group σ_t is at least inner.

Now consider the case of an infinite subfactor $N\subset M$, i.e. both factors N and M are infinite which means that they contain isometries with range projections being different from the identity. Then we can represent M on a Hilbert space H where there is a vector Φ which is cyclic and separating for both N and M. Taking the corresponding Tomita-Takesaki modular conjugations J_N and J_M where $J_NNJ_N=N'$, $J_MMJ_M=M'$, we define

$$\gamma=\mathrm{Ad}(J_NJ_M)|_M : M\to M'\subset N'\to N$$

called the canonical endomorphism [**60**] from M into N. Different choices of Hilbert spaces and cyclic and separating vectors only amount to a change $\gamma\to\mathrm{Ad}(u)\circ\gamma$ by a unitary $u\in N$, i.e. the N-M sector determined by γ is well-defined. (If $\rho\in\mathrm{Mor}(A,B)$ is a unital morphism from A to B, the B-A sector $[\rho]$ is the equivalence class of ρ where $\rho'\simeq\rho$ iff $\rho'=\mathrm{Ad}(u)\circ\rho$ for unitaries $u\in B$.) We then have an inclusion of factors:

$$\gamma(N)\subset\gamma(M)\subset N\subset M\subset M_1=\mathrm{Ad}(J_MJ_N)(N)\,. \tag{2.1}$$

We can continue upwards (called the Jones tower) or downwards (called the Jones tunnel) but the sequence is of period two, e.g. the inclusion $\gamma(N)\subset\gamma(M)$ is isomorphic to $N\subset M$, and $\gamma(M)=\mathrm{Ad}(J_N)(M')\subset N$ is isomorphic to $M=\mathrm{Ad}(J_M)(M')\subset\mathrm{Ad}(J_MJ_N)(N)=M_1$. This periodicity reduces to that between a group and its dual $G\leftrightarrow\hat{G}$ in the case of a group subfactor tower $M^G\subset M\subset M\rtimes G$. So there are basically two canonical endomorphisms, $\gamma\in\mathrm{End}(M)$ and $\theta\in\mathrm{End}(N)$, where $\theta=\gamma|_N$. We call γ the canonical endomorphism, and θ the dual canonical endomorphism for $N\subset M$.

The tower can be identified with the Jones extensions, in the case of finite index obtained by adjoining a sequence of projections satisfying the Temperley-Lieb relations. We could define the Jones index using the Pimsner-Popa inequality as follows. If $E : M \to N$ is a conditional expectation (a projection of norm one), then let $\mathrm{Ind}(E)$ be the best constant ξ such that $E(x^*x) \geq \xi^{-1}x^*x$ for all $x \in M$. Then the (Jones) index $[M : N]$ is the infimum of $\mathrm{Ind}(E)$ over all expectations E, and there is an unique expectation called the minimal expectation which realizes the index.

For us, all the relative commutants $N' \cap M_j$, $M' \cap M_j$, in the tower will be finite-dimensional and moreover $N' \cap M_j \subset N' \cap M_{j+1}$, $M' \cap M_j \subset M' \cap M_{j+1}$, will be described by *finite* graphs. Due to the periodicity of the tower, only two graphs appear here, the principal and dual principal graph. The finiteness of the graphs (equivalent to the finiteness in RCFT) will imply finite index and the Jones index will be the square of the norm of either graph.

One question which will engage us will be whether a particular endomorphism of a factor N should be a dual canonical endomorphism (of some subfactor $N \subset M$ without any a priori knowledge of what M should be). For example if Z is a modular invariant, we can consider $\bigoplus_{\lambda\in\Delta} Z_{0,\lambda}[\lambda]$, $\bigoplus_{\lambda\in\Delta} Z_{\lambda,0}[\lambda]$, $\bigoplus_{\lambda\in\Delta} Z_{\lambda,\mu}[\lambda \otimes \mu^{\mathrm{opp}}]$ as candidates for (the sectors of) dual canonical endomorphisms (on N, N, $N \otimes N^{\mathrm{opp}}$, respectively, if Δ is a system of endomorphisms of N).

Before we go any further let us formalize the notion of algebraic operations and sectors. Let A and B be type III von Neumann factors. A unital $*$-homomorphism $\rho : A \to B$ is called a B-A morphism, and we write $\rho \in \mathrm{Mor}(A, B)$. The positive number $d_\rho = [B : \rho(A)]^{1/2}$ is called the statistical dimension of ρ; here $[B : \rho(A)]$ is the Jones index of the subfactor $\rho(A) \subset B$. Now if $\sigma \in \mathrm{Mor}(B, C)$ (with C being another type III factor) then the multiplication or "fusion"

$$[\sigma][\rho] = [\sigma\rho]$$

is well defined on sectors. (We usually abbreviate $\sigma\rho \equiv \sigma \circ \rho$.) For $\tau_1, \tau_2 \in \mathrm{Mor}(A, B)$ take isometries $t_1, t_2 \in B$ such that $t_1t_1^* + t_2t_2^* = \mathbf{1}$ which we can find by infiniteness of B. Then define the sum

$$[\tau_1] \oplus [\tau_2] = [\tau]\,, \quad \text{where} \quad \tau(a) = t_1\tau_1(a)t_1^* + t_2\tau_2(a)t_2^*\,, \quad a \in A\,.$$

This is well-defined as if $s_1, s_2 \in B$ is another choice of isometries satisfying $s_1s_1^* + s_2s_2^* = \mathbf{1}$ then $u = s_1t_1^* + s_2t_2^*$ is a unitary in B, intertwining τ and τ' where $\tau'(a) = s_1\tau_1(a)s_1^* + s_2\tau_2(a)s_2^*$ for all $a \in A$. This notion of a sum is basically writing $[\tau_1] \oplus [\tau_2]$ as a 2×2 matrix $\begin{pmatrix} \tau_1(\cdot) & 0 \\ 0 & \tau_2(\cdot) \end{pmatrix}$ in B using the infiniteness of B to achieve the matrix decomposition. If ρ and σ are B-A morphisms with finite statistical dimensions, then the vector space of intertwiners

$$\mathrm{Hom}(\rho, \sigma) = \{t \in B : t\rho(a) = \sigma(a)t\,,\ a \in A\}$$

is finite-dimensional, and we denote its dimension by $\langle \rho, \sigma \rangle$. Note that for τ, τ_1, τ_2 and t_1, t_2 as above we have e.g. $t_1 \in \mathrm{Hom}(\tau_1, \tau)$. The impossibility of decomposing some $\rho \in \mathrm{Mor}(A, B)$ as $[\rho] = [\rho_1] \oplus [\rho_2]$ for some $\rho_1, \rho_2 \in \mathrm{Mor}(A, B)$, or irreducibility is then equivalent to the subfactor $\rho(A) \subset B$ being irreducible, i.e. $\rho(A)' \cap B = \mathbb{C}\mathbf{1}$.

For groups (and group duals) we have a notion of a conjugate of λ, namely the inverse λ^{-1} of λ (respectively the conjugate representation). There is a similar notion for sectors. For an irreducible $\lambda \in \mathrm{Mor}(A, B)$, an irreducible morphism $\overline{\lambda} \in \mathrm{Mor}(B, A)$ is a representative of the conjugate sector if $[\lambda\overline{\lambda}]$ or $[\overline{\lambda}\lambda]$ contain the identity sector ($[\mathrm{id}_A]$ or $[\mathrm{id}_B]$, respectively), and the multiplicity is then automatically one for both cases [**48**]. More generally, for an arbitrary morphism $\rho \in \mathrm{Mor}(A, B)$ of finite statistical dimension d_ρ, an A-B morphism $\overline{\rho}$ is a conjugate morphism if there are isometries $r_\rho \in \mathrm{Hom}(\mathrm{id}_A, \overline{\rho}\rho)$ and $\overline{r}_\rho \in \mathrm{Hom}(\mathrm{id}_B, \rho\overline{\rho})$ such that

$$\rho(r_\rho)^* \overline{r}_\rho = d_\rho^{-1} \mathbf{1}_B \qquad \text{and} \qquad \overline{\rho}(\overline{r}_\rho)^* r_\rho = d_\rho^{-1} \mathbf{1}_A \,. \tag{2.2}$$

Recall the tower-tunnel of Eq. (2.1). Suppose $E : M \to N$ is a conditional expectation and φ is a faithful normal state on N, and set $\omega = \varphi \circ E$. Then ω is a faithful normal state on M such that $\omega \circ E = \omega$. Take the GNS Hilbert space H of this state on M, with cyclic and separating vector Ω. We can identify this space with our previous Hilbert space (where there is vector Φ being cyclic and separating for both N and M) and the actions coincide. However Φ is not identified with Ω, as $\overline{N\Omega}$ is a proper subspace if $N \neq M$, with orthogonal Jones projection $e_N : \overline{M\Omega} \to \overline{N\Omega}$ such that $m\Omega \mapsto E(m)\Omega$, $m \in M$. Define $v' : n\Phi \mapsto n\Omega$, $n \in N$, on H so that $v' \in N'$, and $v'v'^* = e_N$. Then $v_1 = \mathrm{Ad}(J_M)(v')$ is an isometry in M_1, and also $v_1 v_1^* = e_N$. It is easily checked starting from $J_M v' = v' J_N$ that v_1 is an intertwiner in $\mathrm{Hom}(\mathrm{id}_{M_1}, \gamma_1)$, where $\gamma_1 = \mathrm{Ad}(J_{M_1} J_M)$ is the canonical endomorphism of $M \subset M_1$. Thus, by translating in the tunnel-tower the canonical and dual canonical endomorphism contain the identity sector.

Denoting by $\iota : N \hookrightarrow M$ the inclusion homomorphism we put $\overline{\iota} : M \to N$ as $\overline{\iota}(m) = \gamma(m)$, $m \in M$. Then $\gamma = \iota\overline{\iota}$ and $\theta = \overline{\iota}\iota$ both contain the identity sector so that $\overline{\iota}$ is in fact a conjugate morphism for ι. Similarly, if $\lambda \in \mathrm{End}(N)$, we can take γ_λ, θ_λ to be the canonical and dual canonical endomorphisms of the inclusion $\lambda(N) \subset N$. Then we can set $\overline{\lambda} = \lambda^{-1}\gamma_\lambda$, which is well defined so that $\lambda\overline{\lambda} = \gamma_\lambda$. In the group case, say if we have an outer action $\alpha : G \to \mathrm{Aut}(M)$ of a finite group G on a type III factor M and let N be the corresponding fixed point algebra, $N = M^G$, then γ decomposes as a sector into the group elements, $[\gamma] = \bigoplus_{g \in G} [\alpha_g]$, whereas the decomposition of θ is according to the group dual $\hat{G}$, i.e. $[\theta] = \bigoplus_{\pi \in \hat{G}} d_\pi [\rho_\pi]$, with multiplicities given by the dimensions d_π of the irreducible representations π of G.

Sometimes it is useful to use graphical expressions for formulae involving intertwiners. Roughly speaking, for an intertwiner $t \in \mathrm{Hom}(\rho, \sigma)$ we draw a picture as in Fig. 3, i.e. we represent morphisms by oriented "wires" and intertwiners by boxes. Reversing an arrow means replacing a label ρ by its conjugate $\overline{\rho}$, and taking ad-

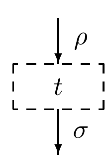

FIGURE 3. An intertwiner $t \in \mathrm{Hom}(\rho, \sigma)$

joints then corresponds to vertical reflection of the picture together with reversing all arrows. As $\mathrm{Hom}(\rho, \sigma) \subset \mathrm{Hom}(\rho\tau, \sigma\tau)$ we are allowed to add or remove straight wires on the right, i.e. we are free to pass from Fig. 3 to Fig. 4. On the other hand, the intertwiner $\mu(t)$ is in $\mathrm{Hom}(\mu\rho, \mu\sigma)$ and is represented graphically as in Fig. 5. With the convention that the identity morphism (of some factor) is labelled by "the invisible wire", the isometries r_ρ, $\overline{r}_\rho$ and r_ρ^*, $\overline{r}_\rho^*$ are represented as caps and cups, respectively, with different orientations of the wire labelled by ρ. Then, with certain

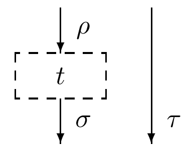

FIGURE 4. An intertwiner $t \in \mathrm{Hom}(\rho, \sigma) \subset \mathrm{Hom}(\rho\tau, \sigma\tau)$

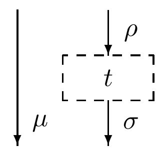

FIGURE 5. The intertwiner $\mu(t) \in \mathrm{Hom}(\mu\rho, \mu\sigma)$

normalization procedures taken care of in [**10**] (where the graphical framework is worked out in full detail – but see also [**65, 57, 84, 34, 32, 53**]), the relations of Eq. (2.2) become topological moves as in Fig. 6. The minimal conditional expecta-

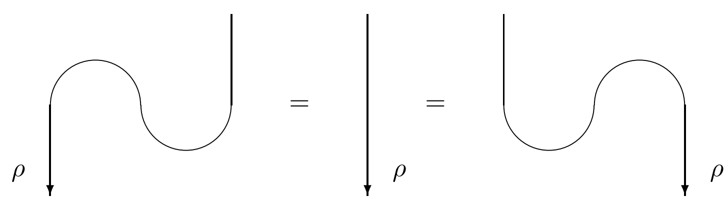

FIGURE 6. A topological invariance

tion is obtained as follows. First, the map $\phi_\rho : B \to A$, $b \mapsto r_\rho^* \overline{\rho}(b) r_\rho$, is the unique standard left inverse for ρ (as $\phi_\rho \circ \rho = \mathrm{id}_A$) and then $E_\rho = \rho \circ \phi_\rho : B \to \rho(A)$ is the minimal conditional expectation for the subfactor $\rho(A) \subset B$. In the graphical framework, Jones projections in the tunnel which were translates of $v_1 v_1^*$ appear as in Fig. 7. The Pimsner-Popa bound in the Kosaki-Jones index is realized by such

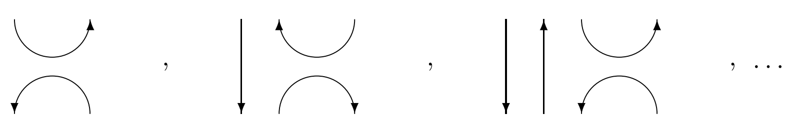

FIGURE 7. Jones projections in the tunnel

Jones projections so that the constant d_ρ in Eq. (2.2) is identified with $[B : \rho(A)]^{1/2}$, the square root of the Jones index.

Returning to our original subfactor $N \subset M$ with inclusion homomorphism $\iota : N \hookrightarrow M$, $\gamma = \iota\bar{\iota}$, $\theta = \bar{\iota}\iota$, where $\bar{\iota}$ is a conjugate for ι, we have isometries $w \equiv r_\iota \in \mathrm{Hom}(\mathrm{id}_N, \theta)$ and $v \equiv \overline{r}_\iota \in \mathrm{Hom}(\mathrm{id}_N, \gamma)$ satisfying the consistency relations

$w^*v = w^*\gamma(v) = d_\iota^{-1}\mathbf{1}$ with $d_\iota^2 = [M:N] = d_\gamma = d_\theta$. Note that we have pointwise equality $M = Nv$ as $m = [M:N]^{1/2} w^*\gamma(m)v$ where $[M:N]^{1/2}w^*\gamma(m) \in N$, $m \in M$, which means that v is a basis element for M as an N-module. The previous characterization of conjugates can be used to characterize which endomorphisms arise as a canonical endomorphism.

If $\gamma \in \mathrm{End}(M)$ where M is an infinite factor, then γ is a canonical endomorphism of some subfactor $N \subset M$ if and only if there exist isometries $v \in \mathrm{Hom}(\mathrm{id}_M, \gamma)$ and $w \in \mathrm{Hom}(\gamma, \gamma^2)$ such that

$$w^*\gamma(w) = ww^*\,, \qquad \gamma(w)w = w^2\,, \tag{2.3}$$

$$v^*w = w^*\gamma(v) = d^{-1}\mathbf{1}\,, \quad d > 0\,. \tag{2.4}$$

Note that if $v = \overline{r}_\iota$ and $w = r_\iota$ as before, then $w \in \mathrm{Hom}(\mathrm{id}_N, \theta) \subset \mathrm{Hom}(\gamma, \gamma^2)$. Conversely, if Eq. (2.3) and Eq. (2.4) hold then we can define $N = \{x \in M : wx = \gamma(x)w\,,\ wx^* = \gamma(x^*)w\}$, and then $E: M \to N$ defined by $E(x) = w^*\gamma(x)w$, $x \in M$, is a conditional expectation.

3. Subfactors arising from loop groups

We now turn to the actual algebras which we will use to describe our modular invariants, arising from loop groups. The loop group $LSU(n)$ consists of smooth maps $f: S^1 \to SU(n)$, the product being pointwise multiplication. The representations of interest will be projective representations of $LSU(n)$ which extend to positive energy representations of $LSU(n) \rtimes \mathrm{Rot}(S^1)$ where the rotation group acts on the maps of S^1 in a natural way so that the "Hamiltonian" or infinitesimal generator L_0 is positive. The ones of particular interest, the irreducible unitary positive energy representations are classified as follows. First there is a level k, a positive integer describing a cocycle because we are dealing with projective representations. The projective representation restricts to a genuine irreducible representation of the constant loops, identified with $SU(n)$ itself, the multiplier becomes irrelevant now since we are dealing with simply connected groups. In order to obtain positive energy, only finitely many irreducible representations are admissible, namely the vertices of (i.e. integrable weights in) the Weyl alcove $\mathcal{A}^{(n,k)}$. The (adjacency matrices of the) graphs N_λ, such as $N_\square$ itself, describe the fusion of positive energy representations.

Restricting to loops concentrated on an interval $I \subset S^1$ (proper, i.e. $I \neq S^1$ and non-empty), the corresponding subgroup denoted by

$$L_I SU(n) = \{f \in LSU(n) : f(z) = 1\,,\ z \notin I\}\,,$$

one finds that in each positive energy representation π_λ the sets of operators $\pi_\lambda(L_I SU(n))$ and $\pi_\lambda(L_{I^c} SU(n))$ commute where I^c is the complementary interval of I, again using that $SU(n)$ is simply connected. In turn we obtain a subfactor

$$\pi_\lambda(L_I SU(n))'' \subset \pi_\lambda(L_{I^c} SU(n))'\,, \tag{3.1}$$

involving hyperfinite type III_1 factors (see [**83**]). In the vacuum representation, labelled by $\lambda = 0$, we have Haag duality in that the inclusion collapses to a single factor $N(I) = N(I)$, and in general we obtain a subfactor. The level 1 representations of $LSU(n)$ are realized through the Fock state of the Hardy space projection P on $L^2(S^1; \mathbb{C}^n)$. Since $[f, P]$ is Hilbert-Schmidt for $f \in LSU(n)$ acting naturally on $L^2(S^1; \mathbb{C}^n)$, we have that $LSU(n)$ is implemented in the corresponding Fock space giving a positive energy representation.

The vacuum representation π_0 gives a clear geometric picture of the Tomita-Takesaki modular group action σ and modular conjugation J on the Fock vacuum vector, cyclic and separating for say $\pi_0(L_I SU(n))''$ for I being the upper half circle. The Tomita-Takesaki modular group is induced by the second quantization of the geometric action of $SU(1,1)$ on S^1, which is seen to be ergodic. Consequently the algebra must be type III_1 as the action σ is never ergodic otherwise (see e.g. [**3**, Cor. 1.10.8]). Similarly the conjugation J acts by flipping the circle, taking I into the complementary interval:

$$\pi_0(L_I SU(n))'' = J\pi_0(L_I SU(n))' J = \pi_0(L_{I^c} SU(n))' ,$$

so that Haag duality holds in the vacuum representation and is a consequence of Tomita-Takesaki theory. More generally the inclusion Eq. (3.1) can be read as providing an endomorphism λ (by abuse of notation denoted by the same symbol as the label) of the local algebra $N(I)$ such that Eq. (3.1) is isomorphic to $\lambda(N(I)) \subset N(I)$. By A. Wassermann's work [**83**] we obtain this way a system of endomorphisms $\Delta = \{\lambda\}$, the morphisms being labelled by the Weyl alcove $\mathcal{A}^{(n,k)}$, which is closed under sector fusion, and the fusion coefficients $N^{\nu}_{\lambda,\mu}$ match exactly the loop group fusion. Similar results have been obtained for minimal models [**59**] and (partially) $LSpin(2n)$ models [**80**]. (That the DHR morphisms of net of a conformal field theory model obey exactly the Verlinde fusion rules from the conformal character transformations was conjectured in [**33**]. Proofs for special cases can be found in [**83, 59, 80, 4, 5**]. Antony Wassermann has informed us that he has computed fusion for all simple, simply connected loop groups; and with Toledano-Laredo all but E_8 using a variant of the Dotsenko-Fateev differential equation considered in his thesis.) If we take the relative commutants for the tunnel

$$\cdots \subset \lambda\overline{\lambda}\lambda(N) \subset \lambda\overline{\lambda}(N) \subset \lambda(N) \subset N ,$$

we are decomposing products $\lambda\overline{\lambda}\lambda\cdots$ into irreducible components obtaining the same relative commutants as for the Jones-Wenzl type II_1 $SU(n)_k$ subfactors. More precisely, if λ is the fundamental representation $\square$ and $A \subset B$ is the hyperfinite type II_1 subfactor

$$\{g_i : i = 1,2,3,\dots\}'' \subset \{g_i : i = 0,1,2,\dots\}'' , \tag{3.2}$$

where the g_i's are the Hecke algebra generators obtained as explained below, then (using Popa [**69**]) the loop group subfactor $\lambda(N) \subset N$ is isomorphic to $A \otimes N \subset B \otimes N$.

The statistical mechanical models of [**17**] are generalizations of the Ising model. The configuration space of the Ising model, distributions of symbols "+" and "−" on the vertices of the square lattice $\mathbb{Z}^2$, can be thought of edges on the Dynkin diagram A_3 on the edges of a square lattice, where the end vertices are labelled by "+" and "−". This model can be generalized by replacing A_3 by other graphs Γ such as other Dynkin diagrams or indeed the Weyl alcove $\mathcal{A}^{(n,k)}$. A configuration is then a distribution of the edges of Γ over $\mathbb{Z}^2$, and associated to each local configuration is a Boltzmann weight satisfying the Yang-Baxter equation. The justification of $SU(n)$ models is as follows. By Weyl duality, the representation of the permutation group on $\bigotimes \mathrm{Mat}(n)$ is the fixed point algebra of the product action of $SU(n)$. Deforming this, there is a representation of the Hecke algebra in $\bigotimes \mathrm{Mat}(n)$, whose commutant is a representation of a deformation of $SU(n)$, the quantum group $SU(n)_q$ [**51**]. The Boltzmann weights lie in this algebra representation, and at critically reduce to the natural braid generators g_i, so that the Yang-Baxter equation satisfied by the Boltzmann weights reduces to the braid relation $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$. When $q = \mathrm{e}^{2\pi\mathrm{i}/(n+k)}$ is a root of unity, the irreducible representations of the corresponding Hecke algebra are labelled precisely by $\mathcal{A}^{(n,k)}$.

The graph Γ generates a von Neumann algebra by considering larger and larger matrices generated by larger and larger partition functions. A subfactor can be obtained by the adjoint action, placing the initial Boltzmann weights on the boundary. For $SU(n)_q$ subfactors this amounts to Eq. (3.2) because of the braid relations $\mathrm{Ad}(g_1 g_2 \cdots)(g_i) = g_{i+1}$. The principal graph of these inclusions is not the entire graph $N_\square$ (corresponding to $\mathcal{A}^{(n,k)}$ as in Fig. 1) but merely its zero-one part (the first two colours) as in Fig. 8. Nevertheless the entire graphs do have a meaning in

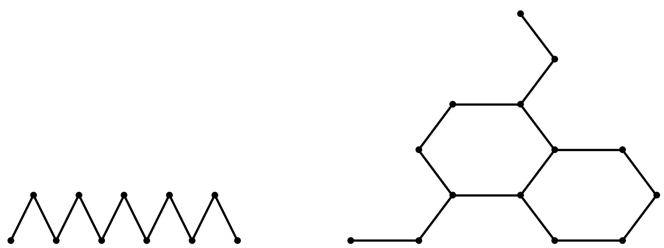

FIGURE 8. Colour zero-one part of the graphs in Fig. 1

subfactor theory simply as graphs encoding the fusion rules of associated systems of bimodules or sectors. Moreover, the center of $SU(n)$, namely $\mathbb{Z}_n$, induces an action on these subfactors and one may construct crossed product or orbifold subfactors $A^{\mathbb{Z}_n} \subset B^{\mathbb{Z}_n}$ [**28, 85**] which will in turn produce "orbifold" sector systems and graphs. As such graphs have been noticed to label certain modular invariants, this can be seen as a first indication that there is a relation between modular invariant partition functions and subfactors. Another strong indication is the special role of the Dynkin diagrams $\mathrm{D}_{\mathrm{odd}}$ and E_7: In the classification of $SU(2)_k$ modular invariants [**13, 14, 55**], the Dynkin diagrams A, $\mathrm{D}_{\mathrm{even}}$, E_6 and E_8 label the type I invariants whereas the invariants labelled by $\mathrm{D}_{\mathrm{odd}}$ and E_7 are type II, i.e. involve a non-trivial "twist". In subfactor theory it turned out that it is precisely the diagrams A, $\mathrm{D}_{\mathrm{even}}$, E_6 and E_8 which appear as principal graphs whereas $\mathrm{D}_{\mathrm{odd}}$ and E_7 are not allowed (see [**56**] and references therein).

4. Braiding, α-induction, and all that

The geometry on the circle together with Haag duality in the vacuum induces a braiding on the endomorphisms. The endomorphisms λ appearing above can be thought of as being defined on a global algebra $\mathcal{N}$ generated by the $N(J)$'s where J varies in the proper intervals on S^1, neither touching nor containing a fixed distinguished "point at infinity" $\zeta \in S^1$. Then λ will be localized on I in the sense that $\lambda(a) = a$ whenever $a \in N(J)$ with $J \cap I = \emptyset$, and transportable in the sense that for each interval J there is a unitary $u \in \mathcal{N}$ such that $\mathrm{Ad}(u) \circ \lambda$ is localized in J. Then if λ and μ are localized on disjoint intervals then they commute: $\lambda\mu = \mu\lambda$. If however λ and μ are localized in the same interval I, then we may choose a relatively disjoint interval J (whose closure does not contain ζ as well) and a unitary u such that $\mathrm{Ad}(u) \circ \mu$ is localized in J. Then λ and $\mathrm{Ad}(u) \circ \mu$ commute and in turn $\varepsilon_u(\lambda, \mu) = u^* \lambda(u)$ is a unitary intertwining $\lambda\mu$ and $\mu\lambda$. It turns out that this unitary is entirely independent on the choice of J and u, except that it may depend on the choice of J lying in the left or right connected complement of I with respect to the point at infinity ζ. (See e.g. [**46, 30, 31, 6**] for more detailed discussions of such matters.) Therefore we have in fact only two "statistics" or braiding operators $\varepsilon^+(\lambda, \mu)$ and $\varepsilon^-(\lambda, \mu)$, according to this choice. Indeed we have $\varepsilon^-(\lambda, \mu) = \varepsilon^+(\mu, \lambda)^*$, but $\varepsilon^+(\lambda, \mu)$ and $\varepsilon^-(\lambda, \mu)$ can be different. The statistics operators obey a couple of consistency equations which are called braiding fusion relations: Whenever $t \in \mathrm{Hom}(\lambda, \mu\nu)$ one has

$$\begin{aligned} \rho(t)\varepsilon^\pm(\lambda, \rho) &= \varepsilon^\pm(\mu, \rho)\mu(\varepsilon^\pm(\nu, \rho))t\,, \\ t\varepsilon^\pm(\rho, \lambda) &= \mu(\varepsilon^\pm(\rho, \nu))\varepsilon^\pm(\rho, \mu)\rho(t)\,. \end{aligned}$$

This in turn implies the braid relation (or "Yang-Baxter equation")

$$\rho(\varepsilon^\pm(\lambda, \mu))\varepsilon^\pm(\lambda, \rho)\lambda(\varepsilon^\pm(\mu, \rho)) = \varepsilon^\pm(\mu, \rho)\mu(\varepsilon^\pm(\lambda, \rho))\varepsilon^\pm(\lambda, \mu)\,.$$

These equations turn our system Δ of endomorphisms into a "braided C^*-tensor category" (cf. [**26**]).

The braiding operators can be nicely incorporated in our graphical intertwiner calculus. Namely, for $\varepsilon^+(\lambda, \mu)$ and $\varepsilon^-(\lambda, \mu)$ we draw over- and undercrossings, respectively, of wires λ and μ as in Fig. 9. Then the consistency relations are

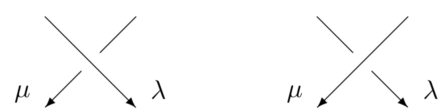

FIGURE 9. Braiding operators $\varepsilon^+(\lambda, \mu)$ and $\varepsilon^-(\lambda, \mu)$ as over- and undercrossings

translated into some kind of topological moves for the pictures, as e.g. the second braiding fusion relation for overcrossings is drawn graphically as in Fig. 10 whereas

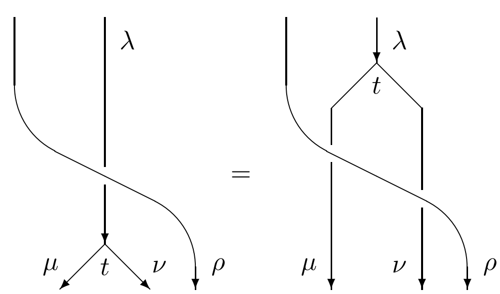

FIGURE 10. The second braiding fusion equation for over-crossings

the braid relation becomes a vertical Reidemeister move of type III, presented in Fig. 11. We would like to obtain generators of the modular group $SL(2; \mathbb{Z})$ (up to normalization) from the Hopf link and the twist, which is in fact possible if and only if the braiding is subject to a certain maximality condition, called "non-degeneracy", basically stating that $+$ and $-$ braiding operators are as different as

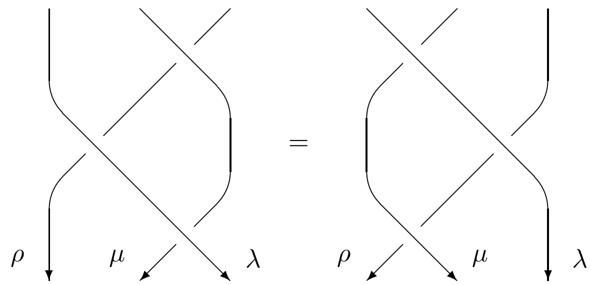

FIGURE 11. The braid relation as a vertical Reidemeister move of type III

possible [**72**]. For ρ irreducible we find $\varepsilon^+(\rho,\overline{\rho})^*\overline{r}_\rho = \omega_\rho r_\rho$ for some scalar $\omega_\rho \in \mathbb{T}$, thanks to the uniqueness of isometries in the one-dimensional $\mathrm{Hom}(\mathrm{id},\overline{\rho}\rho)$.

We will need net versions of canonical and dual canonical endomorphisms to handle inclusions $N(I) \subset M(I)$, where $N(I)$ are local, and which are standard in the sense that there is a single vector $\Omega \in H$ being cyclic and separating for all $M(I)$ on H and all $N(I)$ on a subspace $H_0 \subset H$, and such that there is a consistent family of conditional expectations $E_I : M(I) \to N(I)$ preserving Ω. In this case for each interval $I_\circ$, there is an endomorphism γ of the global algebra $\mathcal{M}$ associated to the net $\{M(I)\}$ such that $\gamma|_{M(I)}$ is a canonical endomorphism for $N(I) \subset M(I)$ whenever $I \supset I_\circ$. Moreover, the restricted $\theta = \gamma|_{\mathcal{N}}$ is localized and transportable and we have

$$\pi^0 \simeq \pi_0 \circ \gamma\,, \qquad \pi^0|_{\mathcal{N}} \simeq \pi_0 \circ \theta\,, \tag{4.1}$$

for π^0 denoting the defining representation of $\mathcal{M}$ on H and π_0 the representation of $\mathcal{N}$ on $\overline{\mathcal{N}\Omega}$. There is an isometry v intertwining the identity and γ, and then we have $\mathcal{M} = \mathcal{N}v$, indeed $M(I) = N(I)v$ whenever $I \supset I_\circ$. It is crucial to note that, though the net $\{N(I)\}$ satisfies locality by assumption, the net $\{M(I)\}$ is not local in general. In fact the latter is local if and only if the chiral locality condition holds

$$\varepsilon^+(\theta,\theta)v^2 = v^2\,,$$

(see the original work [**62**] as an excellent mathematical reference for these matters and [**76**] for a more physical discussion of local extensions) and locality of the extended net $\{M(I)\}$ is extremely constraining, e.g. this automatically implies that all the inclusions $N(I) \subset M(I)$ are irreducible, as shown in [**6**].

The statistics phase ω_ρ (ρ again irreducible) can also be obtained using the left inverse $\phi_\rho(\varepsilon^+(\rho,\rho)) = \omega_\rho/d_\rho$. Such formulae in algebraic quantum field theory (see [**46**] and references therein) predate subfactor theory. Graphically ω_ρ can be displayed as in Fig. 12. By a conformal spin and statistics theorem [**32, 31, 45**] one can identify

$$\omega_\rho = \mathrm{e}^{2\pi\mathrm{i}h_\rho}$$

where h_ρ is the lowest eigenvalue of the Hamiltonian L_0 in the superselection sector $[\rho]$. This will ensure that the statistics phase (and the modular T-matrix) in our subfactor context coincide with that in conformal field theory. Now note that for μ,ν irreducible the expression $d_\mu d_\nu \phi_\mu(\varepsilon(\nu,\mu)\varepsilon(\mu,\nu))^*$ must be a scalar (as it is in $\mathrm{Hom}(\nu,\nu)$) which we will denote by $Y_{\mu,\nu}$ and which is given graphically as in

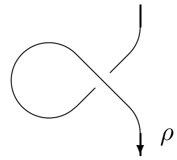

FIGURE 12. Statistics phase ω_ρ as a "twist"

Fig. 13. In case we are dealing with a closed system Δ of braided endomorphisms

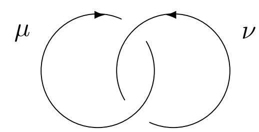

FIGURE 13. Matrix element $Y_{\lambda,\mu}$ of Rehren's Y-matrix as a "Hopf link"

it turns out [**72, 32, 31**] that

$$Y_{\mu,\nu} = \sum_{\lambda\in\Delta} \frac{\omega_\mu\omega_\nu}{\omega_\lambda} N_{\mu,\nu}^\lambda d_\lambda \,, \qquad \mu,\nu\in\Delta\,. \tag{4.2}$$

Normalizing the matrix Y will yield the (modular) S-matrix. Then from Eq. (4.2) it follows that if the ω's and N's coincide then so does the modular matrix S in the subfactor context and that in conformal field theory. Next define $z = \sum_{\lambda\in\Delta} d_\lambda^2 \omega_\lambda$. If $z\neq 0$ put $c = 4\arg(z)/\pi$, the central charge which is defined modulo 8, and set

$$S_{\lambda,\mu} = |z|^{-1} Y_{\lambda,\mu}\,, \qquad T_{\lambda,\mu} = \mathrm{e}^{-\pi \mathrm{i} c/12}\omega_\lambda \delta_{\lambda,\mu}\,.$$

Then the matrices S and T obey the partial Verlinde modular algebra

$$TSTST = S\,, \quad CTC = T\,, \quad CSC = S\,, \quad T^*T = \mathbf{1}\,,$$

where C is the conjugation matrix, i.e. $C_{\lambda,\mu} = \delta_{\lambda,\overline{\mu}}$. Moreover, the following conditions are equivalent [**72**]:

- The braiding is non-degenerate, i.e. $\varepsilon^+(\lambda,\mu) = \varepsilon^-(\lambda,\mu)$ for all $\mu\in\Delta$ only if $\lambda = \mathrm{id}$.
- We have $|z|^2 = w$ (recall that $w = \sum_{\lambda\in\Delta} d_\lambda^2$ is the global index of the system Δ) and S is invertible so that S and T obey the full Verlinde modular algebra, in particular $(ST)^3 = S^2 = C$, and S diagonalizes the fusion rules, i.e. the Verlinde formula of Eq. (1.2) holds.

In our setting of a subfactor $N\subset M$ with a system ${}_N\mathcal{X}_N$ of braided endomorphisms of N we will show how to induce endomorphisms of M. This method corresponds to Mackey-induction in the group-subgroup subfactor. The standard subfactor induction $\lambda\mapsto\iota\lambda\bar\iota$ will not be multiplicative on sectors as e.g. the statistical dimension is multiplied by d_θ — so that in some sense we need to divide out by θ. This is achieved by the notion of α-induction which goes back to Longo and Rehren [**62**] in the (nets of) subfactor setting, and it was studied in [**6, 7, 8, 10, 11, 9**] and in a similar framework (the relation is explained in [**87**]) also in [**86**].

Note that $\gamma(v) \in \mathrm{Hom}(\theta, \theta^2)$ as $v \in \mathrm{Hom}(\mathrm{id}, \gamma)$. Therefore the braiding fusion relations can be applied to obtain

$$\varepsilon^{\pm}(\lambda, \theta)\lambda\gamma(v)\varepsilon^{\pm}(\lambda, \theta)^* = \theta(\varepsilon^{\pm}(\lambda, \theta)^*)\gamma(v)\,,$$

and as

$$\varepsilon^{\pm}(\lambda, \theta)\lambda\gamma(n)\varepsilon^{\pm}(\lambda, \theta)^* = \theta\lambda(n)\,, \qquad n \in N\,,$$

we find by $M = Nv$ that $\mathrm{Ad}(\varepsilon^{\pm}(\lambda, \theta)) \circ \lambda\gamma$ maps M into $\gamma(M)$, and so, in a more stream-lined notation,

$$\alpha_\lambda^{\pm} = \bar{\iota}^{\,-1} \circ \mathrm{Ad}(\varepsilon^{\pm}(\lambda, \theta)) \circ \lambda \circ \bar{\iota}$$

is a well-defined endomorphism of M such that $\alpha_\lambda^{\pm}(v) = \varepsilon^{\pm}(\lambda, \theta)^* v$ and $\alpha_\lambda^{\pm}|_N = \lambda$. The maps α^+ and α^- are well-defined on sectors and are multiplicative, additive and preserve conjugates:

$$\alpha_{\lambda\mu}^{\pm} = \alpha_\lambda^{\pm}\alpha_\mu^{\pm}\,, \qquad \overline{\alpha_\lambda^{\pm}} = \alpha_{\bar{\lambda}}^{\pm}\,, \qquad [\alpha_\nu^{\pm}] = [\alpha_{\nu_1}^{\pm}] \oplus [\alpha_{\nu_2}^{\pm}]$$

for $[\nu] = [\nu_1] \oplus [\nu_2]$. In particular the sectors $[\alpha_\lambda^{\pm}]$ commute; indeed

$$\alpha_\mu^{\pm}\alpha_\lambda^{\pm} = \mathrm{Ad}(\varepsilon^{\pm}(\lambda, \mu)) \circ \alpha_\lambda^{\pm}\alpha_\mu^{\pm}\,.$$

In the restriction direction we write

$$\sigma_\beta = \bar{\iota} \circ \beta \circ \iota \equiv \gamma \circ \beta|_N$$

for β an endomorphism of M. Now σ is additive on sectors and preserves conjugates but it is not multiplicative (as e.g. $\sigma_{\mathrm{id}} = \theta$). In general we have

$$\langle \alpha_\lambda^{\pm}, \beta \rangle \le \langle \lambda, \sigma_\beta \rangle\,,$$

with equality in the case of chiral locality. One has to be careful though for which endomorphisms β of M one is considering in this formula. The inequality is true for any subsector of $[\alpha_\lambda^{\pm}]$, $\lambda \in {}_N\mathcal{X}_N$.

To help compute such subsectors and their fusion rules, one has the relation

$$\langle \alpha_\lambda^{\pm}, \alpha_\mu^{\pm} \rangle \le \langle \theta\lambda, \mu \rangle\,,$$

again with equality in the case of chiral locality. Note that we really have divided out by θ, as in the case of standard sector induction $\lambda \mapsto \iota\lambda\bar{\iota}$ we would have $\langle \iota\lambda\bar{\iota}, \iota\mu\bar{\iota} \rangle = \langle \theta^2\lambda, \mu \rangle$ by Frobenius reciprocity [**49**].

We may also compare the two different "chiral" inductions α^+ and α^-. Then $\alpha_\lambda^+ = \alpha_\lambda^-$ is equivalent to the monodromy being trivial, i.e. $\varepsilon^+(\lambda, \theta)\varepsilon^+(\theta, \lambda) = \mathbf{1}$. Moreover, whenever chiral locality holds then we even have that the chiral induced sectors coincide, $[\alpha_\lambda^+] = [\alpha_\lambda^-]$, if and only if the monodromy is trivial [**6**]. Nevertheless one has quite generally that

$$\alpha_\mu^-\alpha_\lambda^+ = \mathrm{Ad}(\varepsilon^+(\lambda, \mu)) \circ \alpha_\lambda^+\alpha_\mu^-\,,$$

so that the sectors $[\alpha_\mu^-]$ and $[\alpha_\lambda^+]$ clearly commute. Indeed even their subsectors commute and this gives rise to a relative braiding symmetry between the chiral induced sectors [**8**].

5. Modular invariants, graphs and α-induction

The A-D-E classification of [**13, 14, 55**] associates a Dynkin diagram to each $SU(2)$ modular invariant in such a way that the multiplicities of the eigenvalues $S_{1,\lambda}/S_{0,\lambda}$ of the associated graphs match the diagonal entries $Z_{\lambda,\lambda}$ of the modular invariant. Here S is the modular S-matrix for $SU(2)$ at level k, and λ just takes the values in the $SU(2)_k$ spins $\lambda \in \{0,1,2,....,k\}$. For $SU(3)$, Di Francesco and Zuber [**22, 23, 20**] sought graphs to describe the modular invariants in an analogous way, guided partly by the principle that the affine A-D-E diagrams correspond to the finite subgroups of $SU(2)$, and so began with fusion or McKay graphs [**64**] of finite subgroups of $SU(3)$ and sought truncations with the correct eigenvalues — a science essentially based on trial and error. Nevertheless they found a lot of interesting and puzzling relations between graphs, fusion rules and coupling matrices, giving the impetus to further research. We illustrate our subfactor approach through analyzing one of the exceptional $SU(2)$ modular invariants which occurs at level $k=10$. The modular invariant is

$$Z_{\mathrm{E}_6} = |\chi_0+\chi_6|^2 + |\chi_4+\chi_{10}|^2 + |\chi_3+\chi_7|^2 \,. \tag{5.1}$$

This invariant was labelled by the Dynkin diagram E_6 by [**13**] since the diagonal part $\{\lambda : Z_{\lambda,\lambda} \neq 0\}$ of the invariant is $\{0,3,4,6,7,10\}$ in this case are the Coxeter exponents of E_6, i.e. the eigenvalues of the incidence (or adjacency) matrix of E_6 are precisely $\{S_{1,\lambda}/S_{0,\lambda} = 2\cos((\lambda+1)\pi/12) : \lambda = 0,3,4,6,7,10\}$. The E_6 modular invariant can be obtained from the conformal embedding $SU(2)_{10} \subset SO(5)_1$, i.e. an inclusion of $SU(2)$ in $SO(5)$ such that the level 1 positive energy representations of $LSO(5)$ decompose into the level 10 representations of $LSU(2)$, with finite multiplicity. The loop group $LSO(5)$ has three level 1 representations, the basic (b), vector (v) and spinor (s) representation, with characters χ_b, χ_v and χ_s, respectively, decomposing as

$$\chi_\mathrm{b} = \chi_0 + \chi_6\,, \quad \chi_\mathrm{v} = \chi_4 + \chi_{10}\,, \quad \chi_\mathrm{s} = \chi_3 + \chi_7\,,$$

on $LSU(2)$. The diagonal invariant $|\chi_\mathrm{b}|^2+|\chi_\mathrm{v}|^2+|\chi_\mathrm{s}|^2$ of $SO(5)_1$ then immediately produces the exceptional E_6 invariant of $SU(2)_{10}$ of Eq. (5.1). The positive energy representations of b, v, s of $SO(5)_1$ satisfy the Ising fusion rules with b being the identity and in particular fusion by s corresponds to the Dynkin diagram A_3 as in Fig. 2. Analogous to what we discussed for $LSU(n)$, they give rise to three endomorphisms in the loop group subfactor setting of $LSO(5)$ with the same Ising fusion rules [**5**]. The conformal embedding $SU(2)_{10} \subset SO(5)_1$ then gives in the vacuum representation a net of subfactors $\pi^0(L_I SU(2))'' \subset \pi^0(L_I SO(5))''$ or $N(I) \subset M(I)$. Over the net $\{N(I)\}$ we have a system of braided endomorphisms $\{\lambda_j\}$ labelled by vertices (enumerated by $j=0,1,...,10$, $\lambda_0 = \mathrm{id}$) of the Dynkin diagram A_{11}, and braided endomorphisms $\{\tau_\mathrm{b} = \mathrm{id}, \tau_\mathrm{v}, \tau_\mathrm{s}\}$ over $\{M(I)\}$ corresponding to the vertices of A_3, where the graphs A_{11} and A_3 represent fusion by λ_1 and τ_s.

We can put the Ising A_3 system to one side for the time being and focus on the A_{11} system of $\{N(I)\}$. Then (cf. [**62, 76**]) the dual canonical endomorphism θ of $\mathcal{N}$ is as a sector the sum $[\lambda_0] \oplus [\lambda_6]$ coming from the vacuum block — this is basically Eq. (4.1). We can thus perform first our α^+-induction to obtain 11 endomorphisms $\{\alpha_j^+ : j = 0,1,...,10\}$ (we abbreviate $\alpha_j^+ \equiv \alpha_{\lambda_j}^+$, but after decomposition into irreducible sectors, we only find six sectors $[\alpha_0^+] = [\mathrm{id}]$, $[\alpha_1^+]$, $[\alpha_2^+]$, $[\alpha_9^+]$, $[\alpha_{10}^+]$ and $[\varsigma]$, the latter appearing as a subsector of $[\alpha_3^+]$ which decomposes as $[\alpha_3^+] = [\varsigma] \oplus [\alpha_9^+]$.

The graph E_6 appears as fusion graph of $[\alpha_1^+]$ as in Fig. 14. We now turn to our

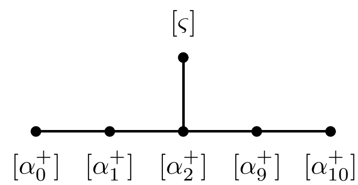

FIGURE 14. $SU(2)_{10} \subset SO(5)_1$: Fusion graph of $[\alpha_1^+]$ on the chiral system E_6^+

original braided system A_3 on $\{M(I)\}$. Reading off from the blocks of the modular invariant we find for σ-restriction:

$$[\sigma_{\rm b}] = [\lambda_0] \oplus [\lambda_6] \equiv [\theta]\,, \quad [\sigma_{\rm v}] = [\lambda_4] \oplus [\lambda_{10}]\,, \quad [\sigma_{\rm s}] = [\lambda_3] \oplus [\lambda_7]\,,$$

because σ-restriction reflects the restriction of (DHR) representations [**62**], which is basically Eq. (4.1) again. Since the net of loop group factors $M(I) = \pi^0(L_I SO(5))''$ satisfies locality we have $\alpha\sigma$-reciprocity

$$\langle \alpha_\lambda^\pm, \beta \rangle = \langle \lambda, \sigma_\beta \rangle$$

for λ in the A_{11} system and β representing any subsector of the induced system $\{[\alpha_j^+]\}$, here E_6. Then as σ-restriction takes us back into the A_{11} system, the sectors $[\tau_{\rm b}]$, $[\tau_{\rm v}]$, $[\tau_{\rm s}]$ must lie amongst the six E_6 sectors. They are identified as $[\alpha_0^+] = [\tau_{\rm b}]$, $[\alpha_{10}^+] = [\tau_{\rm v}]$, $[\varsigma] = [\tau_{\rm s}]$, and indeed satisfy the Ising fusion rules.

Other conformal inclusions and also simple current extension invariants (often also called orbifold invariants) can be handled similarly, the latter are realized by so-called crossed product subfactors using the simple current groups which represent the center $\mathbb{Z}_n$ of $SU(n)$ amongst the $SU(n)_k$ fusion rules [**7, 8**]. (See also Section 8.)

So far we have only considered "positive" α^+-induction, arising from the braiding ε^+. The same way we can use the opposite braiding ε^-, giving α^--induction with "negative" chirality. In either case, say for the conformal inclusion $SU(2)_{10} \subset SO(5)_1$, we have two induced systems E_6^+ and E_6^- of sectors on $M(I)$, but at least they intersect on the Ising sectors b, v, s of $LSO(5)$ at level 1, symbolically: $E_6^+ \cap E_6^- \supset A_3$. In fact they only coincide on these "marked vertices", in the terminology of Di Francesco and Zuber, b, v, s of $E_6^\pm$. Di Francesco and Zuber [**22, 23, 20**] had already empirically observed that the graphs which they sought to describe the diagonal part of a given modular invariant carried in the type I case fusion rule algebras with certain distinguished marked vertices forming fusion rule subalgebras describing the extended fusion rules. This now finds a clear explanation in terms of the game of induction and restriction of sectors.

More generally, in the case of conformal embedding subfactors, the following were shown to be equivalent [**8**]:

- $\mathcal{V}^+ \cap \mathcal{V}^- = \mathcal{T}$,
- $Z_{\lambda,\mu} = \langle \alpha_\lambda^+, \alpha_\mu^- \rangle$,
- The irreducible subsectors of $[\gamma]$ all lie in $\mathcal{V}^\alpha = \mathcal{V}^+ \vee \mathcal{V}^-$,
- $\sum_{\beta \in \mathcal{V}^\alpha} d_\beta^2 = w$.

Here $\mathcal{V}^{\pm}$ are the two chiral systems of induced irreducible sectors, $\mathcal{T} \subset \mathcal{V}^{\pm}$ is the subsystem of neutral or "ambichiral" sectors, arising from either induction and corresponding to the marked vertices, and finally $\mathcal{V}^{\alpha}$ is the system of irreducible sectors generated by products of the different chiral systems or equivalently obtained by decomposing sectors $[\alpha_\lambda^+ \alpha_\mu^-]$ into irreducibles. The second condition gives a nice interpretation of the modular invariant matrix Z as counting the coupling of the two chiral inductions. Note that it immediately produces the upper bound of Eq. (1.3) because the largest possible coupling occurs when $[\alpha_\lambda^+]$ and $[\alpha_\mu^-]$ both purely decompose into multiples of one and the same irreducible sector and the multiplicities are bounded by the statistical dimensions. In fact the bound $\langle \alpha_\lambda^+, \alpha_\mu^- \rangle \leq d_\lambda d_\mu$ even holds for degenerate braidings (i.e. with non-unitary S-matrices). The third and hence all completeness properties could be verified in a case by case analysis for e.g. all $SU(2)$ and $SU(3)$ conformal inclusion subfactors, and by a general proof for all simple current extensions of $SU(n)$ all levels in [**8**]. By adopting a graphical argument of Ocneanu [**68**] from the bimodule sectors to the sector framework, the generating property was proven in [**10**] to hold quite generally, provided the braiding is non-degenerate. (And this is the case for $SU(n)_k$ due to unitarity of the S-matrix.)

A more careful analysis in [**9**] using algebraic instead of graphical techniques shows that the "α-global index" $w_\alpha = \sum_{\beta \in \mathcal{V}^\alpha} d_\beta^2$ is in fact given by

$$w_\alpha = \frac{w}{\sum_{\lambda \text{ deg}} Z_{0,\lambda} d_\lambda}$$

with summation over degenerate elements λ for which $\varepsilon^+(\lambda,\mu) = \varepsilon^-(\lambda,\mu)$ for all μ. Thus the generating property can hold even for some degenerate systems. (An example is the conformal inclusion subfactor $SU(2)_{10} \subset SO(5)_1$ if we start only with the smaller system $\mathrm{A}_{11}^{\text{even}}$ of even spins.) Moreover, the methods of [**10, 11**] allow us to handle type II modular invariants as well as conformal embedding and simple current invariants.

We now turn to the general framework of [**10, 11, 9**]. We take a subfactor $N \subset M$ and a system ${}_N\mathcal{X}_N \subset \mathrm{End}(N)$ of endomorphisms by which we mean a collection of irreducible endomorphisms of finite statistical dimension, containing the identity morphism and closed under conjugation and irreducible decomposition of products. Then for $\iota : N \hookrightarrow M$ being the inclusion homomorphism and $\theta = \bar\iota\iota$ and $\gamma = \iota\bar\iota$ the dual canonical endomorphism and canonical endomorphism, respectively, we assume that θ lies in $\Sigma({}_N\mathcal{X}_N)$, the set of morphisms representing sector sums corresponding to the irreducibles in ${}_N\mathcal{X}_N$ — but make no assumption on γ. Moreover we assume that the system ${}_N\mathcal{X}_N$ is braided. We let ${}_M\mathcal{X}_M \subset \mathrm{End}(M)$ denote a system of endomorphisms consisting of a choice of representative of each irreducible subsector of sectors $[\iota\lambda\bar\iota]$, $\lambda \in {}_N\mathcal{X}_N$. We define ${}_M\mathcal{X}_M^\alpha \subset {}_M\mathcal{X}_M$ to be the subsystem of those endomorphisms which are representatives of some subsectors of $[\alpha_\lambda^+\alpha_\mu^-]$, $\lambda, \mu \in {}_N\mathcal{X}_N$. (Note that by $\alpha_\lambda^\pm \iota = \iota\lambda$, any subsector of $[\alpha_\lambda^+\alpha_\mu^-]$ will automatically be a subsector of $[\iota\lambda\mu\bar\iota]$ since $[\gamma]$ contains the identity sector.) Then we similarly define the chiral induced systems as the subsystems ${}_M\mathcal{X}_M^\pm \subset {}_M\mathcal{X}_M$ of irreducible sectors arising from positive/negative $\alpha^\pm$-induction, and the neutral system ${}_M\mathcal{X}_M^0 = {}_M\mathcal{X}_M^+ \cap {}_M\mathcal{X}_M^-$. Their global indices, i.e. sums over squares of statistical dimensions, are denoted by w, w_α, $w_\pm$, and w_0 (it follows from the assumptions that ${}_N\mathcal{X}_N$ and ${}_M\mathcal{X}_M$ have the same global index w) and fulfill $1 \leq w_0 \leq w_\pm \leq w_\alpha \leq w$.

Defining now a "coupling matrix" Z by setting

$$Z_{\lambda,\mu} = \langle \alpha^+_\lambda, \alpha^-_\mu \rangle \,, \qquad \lambda, \mu \in {}_N\mathcal{X}_N \,,$$

turns out to commute [**10**] with matrices Ω and Y, where $\Omega_{\lambda,\mu} = \delta_{\lambda,\mu}\omega_\lambda$ and Y is defined as in Eq. (4.2). (When the braiding is non-degenerate, we thus have a physical modular invariant Z which commutes with the modular S- and T-matrices, being the normalized matrices Y and Ω.) Moreover, the relative sizes of the various systems are encoded in Z, namely we have [**11**]

$$w_+ = \frac{w}{\sum_{\lambda \in {}_N\mathcal{X}_N} d_\lambda Z_{\lambda,0}} = \frac{w}{\sum_{\lambda \in {}_N\mathcal{X}_N} Z_{0,\lambda} d_\lambda} = w_- \tag{5.2}$$

as well as [**9**]

$$w_\alpha = \frac{w}{\sum_{\lambda \in {}_N\mathcal{X}_N^{\mathrm{deg}}} Z_{0,\lambda} d_\lambda} \,, \qquad w_0 = \frac{w_+^2}{w_\alpha} \,,$$

where ${}_N\mathcal{X}_N^{\mathrm{deg}} \subset {}_N\mathcal{X}_N$ denotes the subsystem of degenerate morphisms (i.e. ${}_N\mathcal{X}_N^{\mathrm{deg}} = \{\mathrm{id}\}$ in the non-degenerate case). Here the equality $\sum_\lambda d_\lambda Z_{\lambda,0} = \sum_\lambda Z_{0,\lambda} d_\lambda$ is due to the invariance $YZ = ZY$.

Although the original system ${}_N\mathcal{X}_N$ is braided, the induced systems ${}_M\mathcal{X}_M$ or even ${}_M\mathcal{X}_M^\pm$ need not even be commutative. Indeed if we complexify the fusion rules of ${}_M\mathcal{X}_M^\pm$ to obtain finite-dimensional C^*-algebras $\mathcal{Z}^\pm$ we find [**11**] (assuming non-degeneracy of the braiding)

$$\mathcal{Z}^\pm \simeq \bigoplus_{\tau \in {}_M\mathcal{X}_M^0} \bigoplus_{\lambda \in {}_N\mathcal{X}_N} \mathrm{Mat}(b^\pm_{\tau,\lambda}) \,, \tag{5.3}$$

where $b^\pm_{\tau,\lambda} = \langle \tau, \alpha^\pm_\lambda \rangle$ are the chiral branching coefficients for $\lambda \in {}_N\mathcal{X}_N$ and a neutral morphism $\tau \in {}_M\mathcal{X}_M^0$ — a marked vertex of Di Francesco and Zuber.

In particular, the chiral systems are commutative only when $b^\pm_{\tau,\lambda} \le 1$ for all τ, λ. This explains the non-commutativity discovered by Feng Xu [**86**] with direct computations of some fusion rules for the conformal embedding $SU(4)_4 \subset SU(15)_1$ (and which lead to conceptual problems in the partially systematic approach to graphs from modular invariants of [**70**] based on certain assumptions) and provides now a whole series of non-commutative chiral fusion rules for $SU(n)_n \subset SU(n^2-1)_1$, $n \ge 4$. Moreover, by counting dimensions we find for the cardinality $\#_M\mathcal{X}_M^\pm$ of ${}_M\mathcal{X}_M^\pm$ that

$$\#_M\mathcal{X}_M^\pm = \sum_{\tau,\lambda} (b^\pm_{\tau,\lambda})^2 = \mathrm{tr}({}^{\mathrm{t}}b^\pm b^\pm) \,.$$

In the matrix algebra $\mathrm{Mat}(b^\pm_{\tau,\lambda})$, the induced $[\alpha^\pm_\nu]$ is scalar, being $S_{\lambda,\nu}/S_{\lambda,0}$. However, even in the degenerate case, the neutral elements always possess a braiding (hence have commutative fusion) arising as restriction of the relative braiding, and this braiding is non-degenerate if the original braiding on ${}_N\mathcal{X}_N$ is. In that case we have "extended " S- and T-matrices S^{ext} and T^{ext} from the neutral system, and in $\mathrm{Mat}(b^\pm_{\tau,\lambda})$ a neutral sector $[\tau']$ ($\tau' \in {}_M\mathcal{X}_M^0$) acts as a central element since it commutes with all subsectors of $[\alpha^\pm_\nu]$, as $S^{\mathrm{ext}}_{\tau,\tau'}/S^{\mathrm{ext}}_{\tau,0}$.

Even if the chiral systems are commutative, the full system ${}_M\mathcal{X}_M^\alpha = {}_M\mathcal{X}_M^+ \vee {}_M\mathcal{X}_M^-$ may not be. Although ${}_M\mathcal{X}_M^+$ and ${}_M\mathcal{X}_M^-$ relatively commute (thanks to the relative braiding), it may happen that "mixed" products of elements of ${}_M\mathcal{X}_M^+$ and

${}_M\mathcal{X}_M^-$ decompose into non-commuting irreducibles. Indeed (cf. Eq. (5.3) for the chiral fusion rules), if we complexify the fusion rules of ${}_M\mathcal{X}_M^\alpha = {}_M\mathcal{X}_M$ in the non-degenerate case to obtain a finite-dimensional C^*-algebra $\mathcal{Z}$, then we find [**10**]

$$\mathcal{Z} \simeq \bigoplus_{\lambda,\mu\in {}_N\mathcal{X}_N} \mathrm{Mat}(Z_{\lambda,\mu})\,.$$

(This particular decomposition has also been claimed by Ocneanu in his lectures [**68**] in case of A-D-E graphs and $SU(2)$ modular invariants.) Moreover, in the matrix algebra $\mathrm{Mat}(Z_{\lambda,\mu})$, the induced $[\alpha_\nu^+]$ and $[\alpha_\nu^-]$ are scalars, being $S_{\lambda,\nu}/S_{\lambda,0}$ and $S_{\mu,\nu}/S_{\mu,0}$, respectively. We have seen in the case of chiral locality (which holds e.g. for conformal embeddings) that we can obtain graphs with spectrum corresponding to the diagonal part of the modular invariant through the fusion graphs of $[\alpha_\lambda^\pm]$ on ${}_M\mathcal{X}_M^\pm$. In the general case where chiral locality may not necessarily hold, we instead look at the action of ${}_M\mathcal{X}_M \supset {}_M\mathcal{X}_M^\pm$ on the system ${}_M\mathcal{X}_N$ of M-N sectors. Here the system ${}_M\mathcal{X}_N$ is a choice of representatives of irreducible subsectors of the sectors $[\iota\lambda]$, $\lambda \in {}_N\mathcal{X}_N$. As M-N sectors cannot be multiplied among themselves there is no associated fusion rule algebra to decompose. (Nevertheless, when chiral locality does holds, ${}_M\mathcal{X}_N$ can be canonically identified with either ${}_M\mathcal{X}_M^\pm$ by $\beta \mapsto \beta\circ\iota$, $\beta \in {}_M\mathcal{X}_M^\pm$.) However, the left action of ${}_M\mathcal{X}_M$ on ${}_M\mathcal{X}_N$ defines a representation ϱ of the M-M fusion rule algebra, with matrix elements $[\varrho([\beta])]_{\xi,\xi'} = \langle \xi, \beta\xi'\rangle$, $\xi,\xi' \in {}_M\mathcal{X}_N$, and decomposes as [**10, 11**]

$$\varrho \simeq \bigoplus_{\lambda\in {}_N\mathcal{X}_N} \pi_{\lambda,\lambda}\,,$$

where $\pi_{\lambda,\lambda}$ is the irreducible representation corresponding to the matrix block $\mathrm{Mat}(Z_{\lambda,\lambda})$, so that $\pi_{\lambda,\lambda}([\alpha_\nu^\pm]) = S_{\lambda,\nu}/S_{\lambda,0}\mathbf{1}_{Z_{\lambda,\lambda}}$. In particular the spectrum is determined by the diagonal part of the modular invariant. Thus it is precisely this representation ϱ which provides an automatic connection between the modular invariant and fusion graphs (e.g. the representation matrix of some fundamental generator $\square$ corresponding to the left multiplication of $[\alpha_\square^\pm]$ on the M-N sectors) in such a way that (the multiplicities in) their spectra are canonically given by the diagonal entries of the coupling matrix. In fact, evaluation of ϱ on the $[\alpha_\lambda^\pm]$'s yields a "nimrep" of the original N-N fusion rules, i.e. a matrix representation where all the matrix entries are non-negative integers. Finally, by counting dimensions we see that $\#_M\mathcal{X}_N = \mathrm{tr}(Z)$.

We can illustrate this with the E_7 modular invariant of $SU(2)$:

$$\begin{aligned} Z_{\mathrm{E}_7} = {} & |\chi_0+\chi_{16}|^2 + |\chi_4+\chi_{12}|^2 + |\chi_6+\chi_{10}|^2 \\ & + |\chi_8|^2 + (\chi_2+\chi_{14})\chi_8^* + \chi_8(\chi_2+\chi_{14})^*\,. \end{aligned}$$

Instead of simply extending to a diagonal invariant, as in the E_6 case, we also insert a twist on the blocks. This is an example of the setting of Moore and Seiberg [**66**] (see also Dijkgraaf and Verlinde [**24**]) that taking a maximal extension of the "chiral algebra" $\mathcal{A}\subset\mathcal{B}$, a modular invariant of $\mathcal{A}$ is the restriction of some permutation invariant $Z^{\mathrm{ext}}_{\tau,\tau'} = \delta_{\tau,\omega(\tau')}$ where ω is a permutation of the sectors of the extended theory $\mathcal{B}$, defining an automorphism of their fusion rules and preserving the extended vacuum sector, $\omega(0)=0$. The E_7 invariant is a twist of the D_{10} invariant, the latter we can realize from a subfactor with the dual canonical endomorphism sector decomposing as $[\lambda_0]\oplus[\lambda_{16}]$, being a simple current extension [**7**]. As shown in [**11**], the E_7 modular invariant appears for a subfactor with dual canonical endomorphism sector $[\lambda_0]\oplus[\lambda_8]\oplus[\lambda_{16}]$. For either invariant we find $\mathrm{tr}({}^{\mathrm{t}}b^{\pm}b^{\pm})=10$ $(=\mathrm{tr}(Z_{\mathrm{D}_{10}}))$ so that indeed in either case the fusion graph of the generator $[\alpha_1^{\pm}]$ on ${}_M\mathcal{X}_M^{\pm}$ is D_{10}. However, $\mathrm{tr}(Z_{\mathrm{E}_7})=7$ and the fusion graph of $[\alpha_1^{\pm}]$ on ${}_M\mathcal{X}_N$ is E_7.

6. Type II modular invariants, extended fusion rule automorphisms, and all that

Now in our general setting we have

$$Z_{\lambda,\mu}=\langle\alpha_\lambda^+,\alpha_\mu^-\rangle=\sum_{\tau\in{}_M\mathcal{X}_M^0}b^+_{\tau,\lambda}b^-_{\tau,\mu}\,,$$

with chiral branching coefficients $b^{\pm}_{\tau,\lambda}=\langle\tau,\alpha^{\pm}_\lambda\rangle$. To write this in Moore-Seiberg form we would need $b^-_{\tau,\lambda}=b^+_{\omega(\tau),\lambda}$ for a permutation of the extended system, being identified as the neutral system ${}_M\mathcal{X}_M^0$, so that

$$Z_{\lambda,\mu}=\sum_{\tau\in{}_M\mathcal{X}_M^0}b^+_{\tau,\lambda}b^+_{\omega(\tau),\mu}\,.$$

Note that by $\omega(0)=0$ and $b^{\pm}_{\tau,0}=\delta_{\tau,0}$ (do not worry that we denote both the original and the extended "vacuum" i.e. identity morphism by the same symbol "0") we are automatically forced to have symmetric vacuum coupling $Z_{\lambda,0}=Z_{0,\lambda}$. To cover more general cases, which do occur as we shall see, we should consider instead of one maximal extension $\mathcal{A}\subset\mathcal{B}$ of the chiral algebra $\mathcal{A}$, but two different extensions $\mathcal{A}\subset\mathcal{B}_{\pm}$, yielding different labelling sets of extended fusion rules so that the extended modular invariant is

$$Z^{\mathrm{ext}}_{\tau_+,\tau_-}=\delta_{\tau_+,\omega(\tau_-)}\,,$$

where ω now is an isomorphism between the two sets of extended fusion rules, still subject to $\omega(0)=0$. Note that when we have two different labelling sets it makes no sense to ask whether a coupling matrix is symmetric or not.

When chiral locality does hold then

$$b^{\pm}_{\beta,\lambda}=\langle\alpha^{\pm}_\lambda,\beta\rangle=\langle\lambda,\sigma_\beta\rangle\,,$$

whenever $\beta\in{}_M\mathcal{X}_M^{\pm}$. In particular, when $\beta=\tau$ is neutral, i.e. lies in the intersection ${}_M\mathcal{X}_M^0={}_M\mathcal{X}_M^+\cap{}_M\mathcal{X}_M^-$, then

$$b^+_{\tau,\lambda}=b^-_{\tau,\lambda}\equiv b_{\tau,\lambda}\,,$$

and we have a block decomposition or "type I" modular invariant

$$Z_{\lambda,\mu}=\sum_{\tau\in{}_M\mathcal{X}_M^0}b_{\tau,\lambda}b_{\tau,\mu}\,.$$

Permutation invariants can be classified as follows. The following conditions are equivalent [**11**]:

- $Z_{\lambda,\mu}=\delta_{\lambda,\omega(\mu)}$ with ω a permutation of ${}_N\mathcal{X}_N$ with $\omega(0)=0$ and defining a fusion rule automorphism,
- $Z_{\lambda,0}=\delta_{\lambda,0}$,
- $Z_{0,\lambda}=\delta_{\lambda,0}$,
- $w_{\pm}=w$.

In this case the two inductions $\alpha^\pm$ are isomorphisms (i.e. each $[\alpha^\pm_\lambda]$ is irreducible) and $\omega = (\alpha^+)^{-1} \circ \alpha^-$. This result does not rely on non-degeneracy of the braiding.

We would like to decompose a modular invariant into its two parts, a type I part together with a twist, and in order to take care of heterotic vacuum coupling we will need to implement such a twist by an isomorphism rather than an automorphism. First we characterize chiral locality. If chiral locality holds, i.e. $\varepsilon^+(\theta,\theta)v^2 = v^2$, then $Z_{\lambda,0} = \langle \alpha^+_\lambda, \alpha^-_0 \rangle = \langle \alpha^+_\lambda, \mathrm{id} \rangle = \langle \lambda, \theta \rangle$, and similarly $Z_{0,\lambda} = \langle \lambda, \theta \rangle$. Indeed the following conditions are equivalent [**9**]:

- We have $Z_{\lambda,0} = \langle \theta, \lambda \rangle$ for all $\lambda \in {}_N\mathcal{X}_N$.
- We have $Z_{0,\lambda} = \langle \theta, \lambda \rangle$ for all $\lambda \in {}_N\mathcal{X}_N$.
- Chiral locality holds: $\varepsilon^+(\theta,\theta)v^2 = v^2$.

Thus chiral locality holds if and only if

$$[\theta] = \bigoplus_{\lambda \in {}_N\mathcal{X}_N} \langle \theta, \lambda \rangle [\lambda] = \bigoplus_{\lambda \in {}_N\mathcal{X}_N} Z_{\lambda,0}[\lambda] = \bigoplus_{\lambda \in {}_N\mathcal{X}_N} Z_{0,\lambda}[\lambda] \,.$$

In general we define sectors

$$[\theta_+] = \bigoplus_{\lambda \in {}_N\mathcal{X}_N} Z_{\lambda,0}[\lambda] \,, \qquad [\theta_-] = \bigoplus_{\lambda \in {}_N\mathcal{X}_N} Z_{0,\lambda}[\lambda] \,.$$

Note that $d_{\theta_+} = \sum_\lambda d_\lambda Z_{\lambda,0} = \sum_\lambda Z_{0,\lambda} d_\lambda = d_{\theta_-}$ (due to $(YZ)_{0,0} = (ZY)_{0,0}$) but in general $[\theta_+]$ and $[\theta_-]$ may be different. Using results on intermediate subfactors [**50**] it was shown in [**9**] that, starting with an arbitrary subfactor $N \subset M$ subject to our assumptions, both $[\theta_+]$ and $[\theta_-]$ are dual canonical endomorphism sectors of N, corresponding to intermediate subfactors

$$N \subset M_\pm \subset M \,,$$

and that $N \subset M_\pm$ satisfy chiral locality. We then can form the $\tilde\alpha^\pm_\delta$-inductions ($\lambda \mapsto \tilde\alpha^\pm_{\delta;\lambda}$), on $N \subset M_\delta$, $\delta = \pm$, and consider then the symmetric type I modular invariants $Z^\pm$,

$$Z^+_{\lambda,\mu} = \langle \tilde\alpha^+_{+;\lambda}, \tilde\alpha^-_{+;\mu} \rangle \,, \qquad Z^-_{\lambda,\mu} = \langle \tilde\alpha^+_{-;\lambda}, \tilde\alpha^-_{-;\mu} \rangle \,.$$

From the definition of $[\theta_\pm]$ we have $Z^+_{\lambda,0} = \langle \theta_+, \lambda \rangle = Z^+_{0,\lambda}$ as $N \subset M_+$ satisfies chiral locality, and so $Z^+_{\lambda,0} = Z^+_{0,\lambda} = Z_{\lambda,0}$ and similarly $Z^-_{\lambda,0} = Z^-_{0,\lambda} = Z_{0,\lambda}$. For $Z = Z_{\mathrm{E}_7}$ the E_7 modular invariant of $SU(2)$, $Z^\pm$ will both be $Z_{\mathrm{D}_{10}}$ but it is possible as we shall see that $Z^+ \neq Z^-$.

Next we argue that we can canonically identify ${}_M\mathcal{X}^+_M$ with ${}_{M_+}\mathcal{X}^+_{M_+}$ and ${}_M\mathcal{X}^-_M$ with ${}_{M_-}\mathcal{X}^-_{M_-}$. To do this it will be enough to find an injective map ${}_M\mathcal{X}^+_M \to {}_{M_+}\mathcal{X}^+_{M_+}$ (and ${}_M\mathcal{X}^-_M \to {}_{M_-}\mathcal{X}^-_{M_-}$) because the global indices w^+ are the same thanks to Eq. (5.2) and $Z_{\lambda,0} = Z^+_{\lambda,0}$. One can show that $\mathrm{Hom}(\mathrm{id}, \alpha^\pm_\nu) = \mathrm{Hom}(\mathrm{id}, \tilde\alpha^\pm_{\pm;\nu})$ which in turn implies $\mathrm{Hom}(\alpha^\pm_\lambda, \alpha^\pm_\mu) = \mathrm{Hom}(\tilde\alpha^\pm_{\pm;\lambda}, \tilde\alpha^\pm_{\pm;\mu})$. In particular $\mathrm{Hom}(\alpha^\pm_\lambda, \alpha^\pm_\mu) \subset M_\pm$. We can then move from intertwiners to endomorphisms. If $\beta \in {}_M\mathcal{X}^+_M$ represents a subsector of $[\alpha^+_\lambda]$, so that there is a $t \in M$ such that $\alpha^+_\lambda(\cdot) = t\beta(\cdot)t^* + \ldots$, then $tt^* \in \mathrm{Hom}(\alpha^\pm_\lambda, \alpha^\pm_\lambda) = \mathrm{Hom}(\tilde\alpha^\pm_{\pm;\lambda}, \tilde\alpha^\pm_{\pm;\lambda})$. We can then construct an endomorphism $\tilde\beta \in \mathrm{End}(M_+)$ representing a subsector of $[\tilde\alpha^\pm_{\pm;\lambda}]$ and such that $\beta|_{M_+} = \tilde\beta$. In this way we construct bijections $\vartheta_\pm : {}_M\mathcal{X}^\pm_M \to {}_{M_\pm}\mathcal{X}^\pm_{M_\pm}$ which [**9**]:

- preserve chiral branching rules $\langle \beta, \alpha^\pm_\lambda \rangle = \langle \vartheta(\beta), \tilde\alpha^\pm_{\pm;\lambda} \rangle$, $\beta \in {}_M\mathcal{X}^\pm_M$,
- preserve chiral fusion rules,

- and restrict to bijections of the neutral systems ${}_M\mathcal{X}_M^0 \to {}_{M_\pm}\mathcal{X}_{M_\pm}^0$.

This means that ${}_M\mathcal{X}_M^0$ can be used (rather than ${}_{M_\pm}\mathcal{X}_{M_\pm}^0$) to decompose the type I coupling matrices

$$Z^\pm_{\lambda,\mu} = \sum_{\tau \in {}_M\mathcal{X}_M^0} b^\pm_{\tau,\lambda} b^\pm_{\tau,\mu}$$

with chiral branching coefficients $b^\pm_\lambda = \langle \tau, \alpha^\pm_\lambda \rangle$, $\tau \in {}_M\mathcal{X}_M^0$, $\lambda \in {}_N\mathcal{X}_N$. If the two intermediate subfactors happen to be identical, $M_+ = M_-$ (so that the "parent" coupling matrices coincide, $Z^+ = Z^-$), then we can write

$$Z_{\lambda,\mu} = \sum_{\tau \in {}_M\mathcal{X}_M^0} b^+_{\tau,\lambda} b^+_{\omega(\tau),\mu}$$

for the (generically type II) coupling matrix Z. Here the permutation $\omega = \vartheta_+^{-1} \circ \vartheta_-$, satisfying $\omega(0) = 0$ clearly defines an automorphism of the neutral fusion rules.

In general when $M_+ \neq M_-$ we would write the extended coupling matrix as

$$Z^{\mathrm{ext}}_{\tau_+,\tau_-} = \delta_{\tau_-,\vartheta(\tau_+)}\,,$$

where $\tau_\pm \in {}_{M_\pm}\mathcal{X}_{M_\pm}^0$ and $\vartheta = \vartheta_- \circ \vartheta_+^{-1} : {}_{M_+}\mathcal{X}_{M_+}^0 \to {}_{M_-}\mathcal{X}_{M_-}^0$ is a bijection defining an isomorphism of the chiral fusion rules. We will illustrate that such heterotic situations do exist, in fact examples are already provided by certain $SO(n)_k$ current algebra models. We will deal with the simplest case at level $k = 1$ in Section 7.

What is the connection between the two chiral inductions and the picture of left- and right-chiral algebras in conformal field theory? An appropriate notion of chiral algebras in the setting of algebraic quantum field theory are "chiral observables" [**74**], and one can show that our coupling matrices describe in fact a Hilbert space decomposition of the vacuum sector of a two-dimensional quantum field theory upon restriction to the action of a tensor product of left- and right-chiral observables [**75**]. Suppose that our factor N is obtained as a local factor $N = N(I_\circ)$ of a quantum field theoretical net of factors $\{N(I)\}$ indexed by proper intervals $I \subset \mathbb{R}$ on the real line, and that the system ${}_N\mathcal{X}_N$ is obtained as restrictions of DHR-morphisms (cf. [**46**]) to N. This is in fact the case in our examples arising from conformal field theory where the net is defined in terms of local loop groups in the vacuum representation. Taking two copies of such a net and placing the real axes on the light cone, then this defines a local net $\{A(\mathcal{O})\}$, indexed by double cones $\mathcal{O}$ on two-dimensional Minkowski space (cf. [**74**] for such constructions). Given a subfactor $N \subset M$, determining in turn two subfactors $N \subset M_\pm$ obeying chiral locality, will provide two local nets of subfactors $\{N(I) \subset M_\pm(I)\}$ as a local subfactor basically encodes the entire information about the net of subfactors [**62**]. Arranging $M_+(I)$ and $M_-(J)$ on the two light cone axes defines a local net of subfactors $\{A(\mathcal{O}) \subset A_{\mathrm{ext}}(\mathcal{O})\}$ in Minkowski space. Rehren has recently proven [**75**] (see also [**12**] for a different but less general derivation) that there is a (type III) factor B such that we have an irreducible inclusions $N \otimes N^{\mathrm{opp}} \subset B$ such that the dual canonical endomorphism Θ of the inclusion $N \otimes N^{\mathrm{opp}} \subset B$ decomposes as

$$[\Theta] = \bigoplus_{\lambda,\mu \in {}_N\mathcal{X}_N} Z_{\lambda,\mu}\,[\lambda \otimes \mu^{\mathrm{opp}}]\,.$$

(Here the superscript "opp" just denotes the opposite algebra, i.e. $N^{\rm opp}$ is N as a linear space, with reversed multiplication. There is a canonical way of identifying $N(I)^{\rm opp}$ with the CPT reflection of $N(I)$ [**44**] which is involved in the two-dimensional construction.) Refining this result it has been shown [**9**] that our local extensions $M_\pm$ produce an intermediate subfactor

$$N \otimes N^{\rm opp} \subset M_+ \otimes M_-^{\rm opp} \subset B$$

such that moreover the dual canonical endomorphism $\Theta_{\rm ext}$ of the inclusion $M_+ \otimes M_-^{\rm opp} \subset B$ decomposes as

$$[\Theta_{\rm ext}] = \bigoplus_{\tau \in {}_M\mathcal{X}_M^0} [\vartheta_+(\tau) \otimes \vartheta_-(\tau)^{\rm opp}]\,.$$

The embedding $M_+ \otimes M_-^{\rm opp} \subset B$ gives rise to another net of subfactors $\{A_{\rm ext}(\mathcal{O}) \subset B(\mathcal{O})\}$, and a condition which ensures that the net $\{B(\mathcal{O})\}$ obeys local commutation relations can be established. The existence of the local net was already proven in [**75**], and now the decomposition of $[\Theta_{\rm ext}]$ tells us that the chiral extensions $N(I) \subset M_+(I)$ and $N(I) \subset M_-(I)$ for left and right chiral nets are indeed maximal (in the sense of [**74**]), following from the fact that the coupling matrix for $\{A_{\rm ext}(\mathcal{O}) \subset B(\mathcal{O})\}$ is a bijection. This shows that the inclusions $N \subset M_\pm$ should in fact be regarded as the subfactor version of left- and right maximal extensions of the chiral algebra.

7. Heterotic examples

Let us now consider the $SO(n)$ loop group models at level 1, where n is a multiple of 16, $n = 16\ell$, $\ell = 1, 2, 3, \ldots$. These theories have four sectors, the basic (0), vector (v), spinor (s) and conjugate spinor (c) module, corresponding to highest weights 0, $\Lambda_{(1)}$, $\Lambda_{(r-1)}$ and $\Lambda_{(r)}$, respectively; here $r = n/2 = 8\ell$ is the rank of $SO(n)$. The conformal dimensions are given as $h_0 = 0$, $h_{\rm v} = 1/2$, $h_{\rm s} = h_{\rm c} = \ell$, and the sectors obey $\mathbb{Z}_2 \times \mathbb{Z}_2$ fusion rules. The Kac-Peterson matrices are given explicitly as

$$S = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \qquad T = \mathrm{e}^{-2\pi \mathrm{i} \ell/3} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{7.1}$$

It is easy to check that there are exactly six modular invariants, $Z = \mathbf{1}$, W, $X_{\rm s}$, $X_{\rm c}$, Q, ${}^{\rm t}Q$. Here

$$W = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad X_{\rm s} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and $X_{\rm c} = W X_{\rm s} W$. (Note that $Q = X_{\rm s} W$ and ${}^{\rm t}Q = W X_{\rm s}$.) The matrix Q and its transpose ${}^{\rm t}Q$ are two examples for modular invariants with non-symmetric vacuum coupling. Such "heterotic" invariants seem to be extremely rare and have not enjoyed particular attention in the literature, perhaps because they were erroneously dismissed as being spurious in the sense that they would not correspond to a physical partition function. Examples for truly spurious modular invariants were given in [**78, 82, 37**] and found to be "coincidental" linear combinations of proper physical invariants. Note that although there is a linear dependence here, namely

$$\mathbf{1} - W - X_{\mathrm{s}} - X_{\mathrm{c}} + Q + {}^{\mathrm{t}}Q = 0\,,$$

we cannot express Q (or ${}^{\mathrm{t}}Q$) alone as a linear combination of the four symmetric invariants. This may serve as a first indication that Q and ${}^{\mathrm{t}}Q$ are not spurious. We will now demonstrate that they can be realized from subfactors.

The $\mathbb{Z}_2 \times \mathbb{Z}_2$ fusion rules for these models were proven in the DHR framework in [**5**], and together with the conformal spin and statistics theorem [**32, 31, 45**] we conclude that there is a net of type III factors on S^1 with a system $\{\mathrm{id}, \rho_{\mathrm{v}}, \rho_{\mathrm{s}}, \rho_{\mathrm{c}}\}$ of localized and transportable, hence braided endomorphisms, such that the statistics S- and T-matrices are given by Eq. (7.1). Because the statistics phases are second roots of unity as $\omega_{\mathrm{v}} = -1$ and $\omega_{\mathrm{s}} = \omega_{\mathrm{c}} = 1$, we can by [**73**] choose the morphisms in the system such that obey the $\mathbb{Z}_2 \times \mathbb{Z}_2$ fusion rules even by individual multiplication,

$$\rho_{\mathrm{v}}^2 = \rho_{\mathrm{s}}^2 = \rho_{\mathrm{c}}^2 = \mathrm{id}\,, \qquad \rho_{\mathrm{v}}\rho_{\mathrm{s}} = \rho_{\mathrm{s}}\rho_{\mathrm{v}} = \rho_{\mathrm{c}}\,.$$

This is enough to proceed with the DHR construction of the field net [**25**], as already carried out similarly for simple current extensions with cyclic groups in [**7, 8**]. In fact, all we need to do here is to pick a single local factor $N = N(I)$ such that the interval $I \subset S^1$ contains the localization region of the morphisms, and then we construct the cross product subfactor $N \subset N \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$. Then the corresponding dual canonical endomorphism θ decomposes as a sector as

$$[\theta] = [\mathrm{id}] \oplus [\rho_{\mathrm{v}}] \oplus [\rho_{\mathrm{s}}] \oplus [\rho_{\mathrm{c}}]\,.$$

Checking $\langle \iota\lambda, \iota\mu \rangle = \langle \theta\lambda, \mu \rangle = 1$ for $\lambda, \mu = \mathrm{id}, \rho_{\mathrm{v}}, \rho_{\mathrm{s}}, \rho_{\mathrm{c}}$, we find that there is only a single M-N sector, namely $[\iota]$. From $\mathrm{tr} Z = \#_M \mathcal{X}_N$ we conclude that the modular invariant coupling matrix Z arising from this subfactor must fulfill $\mathrm{tr} Z = 1$. This leaves only the possibility that Z is Q or ${}^{\mathrm{t}}Q$. We may and do assume that $Z = Q$, otherwise we exchange braiding and opposite braiding. It is easy to determine the intermediate subfactors $N \subset M_\pm \subset M$. Namely, we have $M_+ = N \rtimes_{\rho_{\mathrm{s}}} \mathbb{Z}_2$ and $M_- = N \rtimes_{\rho_{\mathrm{c}}} \mathbb{Z}_2$ with dual canonical endomorphism sectors $[\theta_+] = [\mathrm{id}] \oplus [\rho_{\mathrm{s}}]$ and $[\theta_-] = [\mathrm{id}] \oplus [\rho_{\mathrm{c}}]$, respectively. That both extensions are local also follows from $\omega_{\mathrm{s}} = \omega_{\mathrm{c}} = 1$. We therefore find $Z^+ = X_{\mathrm{s}}$ and $Z^- = X_{\mathrm{c}}$. Finally, the permutation invariant W is obtained from the non-local extension $M_{\mathrm{v}} = N \rtimes_{\rho_{\mathrm{v}}} \mathbb{Z}_2$.

8. Realization of modular invariants from subfactors

In our general setting, we have the following situation: For a given type III von Neumann factor N equipped with a braided system of endomorphism ${}_N\mathcal{X}_N$, any embedding $N \subset M$ of N in a larger factor M which is compatible with the system ${}_N\mathcal{X}_N$ (in the sense that the dual canonical endomorphism decomposes in ${}_N\mathcal{X}_N$) defines a coupling matrix Z through α-induction. This matrix Z commutes with the matrices Y and Ω arising from the braiding and in turn is a "modular invariant mass matrix" whenever the braiding is non-degenerate. Suppose we start with a system corresponding to the RCFT data of $SU(n)_k$. Then the following question is natural, but difficult to answer:

> **Can any physical modular invariant be realized from some subfactor $N \subset M$?**

The first problem with this question is that one needs to specify what the term "physical" means. Quite often in the literature, any modular invariant matrix (i.e. $ZS = SZ$, $ZT = TZ$) subject to the constraint that all entries are non-negative integers and with normalization $Z_{0,0} = 1$ is called a physical invariant. Well, with this interpretation of "physical" the answer to the question is clearly negative. Namely, our general theory says that there is always some associate extended theory carrying another representation of the modular group $SL(2;\mathbb{Z})$ which is compatible with the chiral branching rules. As mentioned above, it is however known [**78, 82, 37**] that there are "spurious" modular invariants satisfying the above constraints but which do not admit an extended modular S-matrix. But even with this relatively simple specification we have another problem: Complete classifications of such modular invariant matrices are known only for very few models, not much more than $\mathbb{Z}_n$ conformal field theories [**19**], $SU(2)$ all levels [**14, 55**], $SU(3)$ all levels [**39**], and some classifications for affine partition functions at low levels [**40**].

Another specification of "physical" (but unfortunately mathematically harder to reach) would be that Z arises from "the existence of some 2D conformal field theory". A promising way of making this precise seems for us to be the concept of chiral observables as light-cone nets built in an observable net over 2D Minkowski space [**74**]. As mentioned in Section 6, Rehren has shown [**75**] that any subfactor $N \subset M$ of our kind which arises as an extension of a local factor $N = N(I_\circ)$ of a Möbius covariant net $\{N(I)\}$ over $\mathbb{R}$ (or equivalently $S^1 \setminus \zeta$) determines an entire 2D conformal field theory over Minkowski space. The converse direction, however, is an open problem: Does any 2D conformal field theory with chiral building blocks containing $\{N(I)\}$ determine a subfactor $N \subset M$ producing the modular invariant matrix Z which describes the coupling between left- and right-chiral sectors? (In particular in the case that the coupling matrix is type II.) Nevertheless there are partial answers to this question. First of all the trivial invariants, $Z_{\lambda,\mu} = \delta_{\lambda,\mu}$, are obtained from the trivial subfactor $N \subset M$ with $M = N$. Next, any conformal inclusion determines a subfactor which in turn produces a modular invariant, being the type I exceptional invariant which arises from the diagonal invariant of the extended theory, here the level 1 representation theory of the larger affine Lie algebra (e.g. of $SO(5)$ for the embedding $SU(2)_{10} \subset SO(5)_1$ as treated above). The situation is even better for simple current invariants, which in a sense produce the majority of non-trivial modular invariants. Simple currents [**77**] are primary fields with unit quantum dimension and appear in our framework as sectors with statistical dimension one, hence its representatives are automorphisms. They form a closed abelian group G under fusion which is hence a product of cyclic groups. Simple currents give rise to modular invariants, and all such invariants have been classified [**42, 58**].

If we take generators $[\sigma_i]$ for each cyclic subgroup $\mathbb{Z}_{n_i}$ then we can construct the crossed product subfactor $N \subset M = N \rtimes G$ whenever we can choose a representative σ_i in each such simple current sector such that we have exact cyclicity $\sigma_i^{n_i} = \mathrm{id}$ (and not only as sectors). As we are starting with a chiral quantum field theory (e.g. from loop groups), Rehren's lemma [**73**] applies which states that such a choice is possible if and only if the statistics phase is an n_i-th root of unity, or in the conformal context if and only if the conformal weight h_{σ_i} is an integer multiple of $1/n_i$. Sometimes this may only be possible for a simple current subgroup $H \subset G$, but any non-trivial subgroup ($H \neq \{0\}$) gives rise to a non-trivial subfactor and in turn to a modular invariant. In fact one can check by our methods that all simple current invariants are realized this way. For example, for $SU(n)_k$ the simple current group is just $\mathbb{Z}_n$, corresponding to weights $k\Lambda_{(j)}$, $j = 0, 1, ..., n-1$. The conformal dimensions are $h_{k\Lambda_{(j)}} = kj(n-j)/2n$ which allow for extensions except when n is even and k and j are odd. (This reflects the fact that e.g. for $SU(2)$ there are no D-invariants at odd levels.) An extension by a simple current subgroup $\mathbb{Z}_m \subset \mathbb{Z}_n$, i.e. m is a divisor of n, is moreover local, if the generating current (and hence all in the $\mathbb{Z}_m$ subgroup) has integer conformal weight, $h_{k\Lambda_{(q)}} \in \mathbb{Z}$, where $n = mq$. This happens exactly if $kq \in 2m\mathbb{Z}$ if n is even, or $kq \in m\mathbb{Z}$ if n is odd. For $SU(2)$ this corresponds to the $\mathrm{D_{even}}$ series whereas the $\mathrm{D_{odd}}$ series are non-local extensions. For $SU(3)$, there is a simple current extension at each level, but only those at $k \in 3\mathbb{Z}$ are local. Clearly, the cases with chiral locality match exactly the type I simple current modular invariants. Our results imply that the system ${}_M\mathcal{X}_M^0$ of neutral morphisms, which is obtained by decomposing $[\alpha_\lambda^\pm]$'s with colour zero mod m, carries a non-degenerate braiding. This nicely reflects a general fact about non-degenerate extensions of degenerate (sub-) systems conjectured by Rehren [**72**] and proven by Müger [**67**].

For the exceptional modular invariants arising from conformal inclusions, the corresponding subfactor comes (almost) for free. A conformal inclusion means that the level 1 representations of some loop group of a Lie group restrict in a finite manner to the positive energy representations of a certain embedded loop group of an embedded (simple) Lie group at some level. As discussed for the E_6 example, a subfactor is obtained by taking this embedding as a local subfactor in the vacuum representation. Since the embedding level one theory is always local, the modular invariant will necessarily be type I. For $SU(2)$, the modular invariants arising from conformal embeddings are, besides E_6, the E_8 and the D_4 ones, corresponding to embeddings $SU(2)_{28} \subset (\mathrm{G}_2)_1$ and $SU(2)_4 \subset SU(3)_1$, respectively, the latter happens to be a simple current invariant at the same time. For $SU(3)$, the invariants from conformal embeddings are $\mathcal{D}^{(6)}$, $\mathcal{E}^{(8)}$, $\mathcal{E}^{(12)}$ and $\mathcal{E}^{(24)}$, corresponding to $SU(3)_3 \subset SO(8)_1$, $SU(3)_5 \subset SU(6)_1$, $SU(3)_9 \subset (\mathrm{E}_6)_1$, $SU(3)_{21} \subset (\mathrm{E}_7)_1$, respectively.

With these techniques we can obtain a huge amount of modular invariants from subfactors. Nevertheless we still do not have a systematic procedure to get all physical invariants. The more problematic cases are typically the exceptional type II invariants. We did realize the E_7 invariant of $SU(2)$ by some subfactor, namely we used the existence of a certain Goodman-de la Harpe-Jones subfactor [**43**] for this case, however, this method will not apply to general invariants of $SU(n)$. It seems to follow from Ocneanu's recent announcement (see his lectures) that there are subfactors realizing all $SU(3)$ modular invariants, but also his methods relying on the "$SU(3)$ wire model" (as well as on Gannon's classification of modular invariants) do not solve the general problem. Nevertheless a large class of exceptional type II invariants can be dealt with quite generally, namely those which are type II descendants of conformal embeddings. Since the embedding level 1 theories are typically (whenever simply laced Lie groups are worked with) $\mathbb{Z}_n$ theories, i.e. pure simple current theories, the subfactors producing their modular invariants can be constructed by simple current methods, and in turn we will obtain the relevant subfactors for the embedded theories, say $SU(n)$.

For a while we will be looking at the so-called $\mathbb{Z}_n$ conformal field theories as treated in [**19**], which have n sectors, labelled by $\lambda = 0,1,2,...,n-1 \,(\mathrm{mod}\, n)$, obeying $\mathbb{Z}_n$ fusion rules, and conformal dimensions of the form $h_\lambda = a\lambda^2/2n \;(\mathrm{mod}\, 1)$, where a is an integer $\mathrm{mod}\, 2n$, a and n coprime and a is even whenever n is odd. The modular invariants of such models have been classified [**19**]. They are labelled by the divisors δ of $\tilde{n}$, where $\tilde{n} = n$ if n is odd and $\tilde{n} = n/2$ if n is even. Explicitly, the modular invariants $Z^{(\delta)}$ are given by

$$Z^{(\delta)}_{\lambda,\mu} = \begin{cases} 1 & \text{if} \quad \lambda,\mu = 0 \,\mathrm{mod}\, \alpha \quad \text{and} \quad \mu = \omega(\delta)\lambda \,\mathrm{mod}\, n/\alpha \,, \\ 0 & \text{otherwise}\,, \end{cases}$$

where $\alpha = \gcd(\delta, \tilde{n}/\delta)$ so that there are numbers $r, s \in \mathbb{Z}$ such that $r\tilde{n}/\delta\alpha - s\delta/\alpha = 1$ and then $\omega(\delta)$ is defined as $\omega(\delta) = r\tilde{n}/\delta\alpha + s\delta/\alpha$. The trivial invariant corresponds to $\delta = \tilde{n}$, i.e. $Z^{(\tilde{n})} = \mathbf{1}$ and $\delta = 1$ gives the charge conjugation matrix, $Z^{(1)} = C$.

We now claim that

$$Z^{(\delta)}_{\lambda,\lambda} = \begin{cases} 1 & \text{if} \quad \lambda = 0 \,\mathrm{mod}\, \tilde{n}/\delta, \\ 0 & \text{otherwise}\,. \end{cases} \tag{8.1}$$

Notice that $\omega(\delta) - 1 = 2s\delta/\alpha$. Assume first that $\lambda = x\tilde{n}/\delta$, $x \in \mathbb{Z}$. Then clearly $\lambda = 0 \,\mathrm{mod}\, \alpha$ since α divides $\tilde{n}/\delta$, and we have $(\omega(\delta) - 1)\lambda = 2sx\tilde{n}/\alpha$, implying $\lambda = \omega(\delta)\lambda \,\mathrm{mod}\, n/\alpha$, thus $Z^{(\delta)}_{\lambda,\lambda} = 1$. Conversely, assume $Z^{(\delta)}_{\lambda,\lambda} = 1$ so that $\lambda = y\alpha$ and $(\omega(\delta)-1)\lambda = zn/\alpha$ with $y, z \in \mathbb{Z}$. This gives $2sy\delta = zn/\alpha$, hence $2sy = zn/\delta\alpha$. Now s is coprime to $\tilde{n}/\delta\alpha$, and therefore it follows that y is a multiple of $\tilde{n}/\delta\alpha$ (as we see that z must be even if n is odd) which implies in fact $\lambda = 0 \,\mathrm{mod}\, \tilde{n}/\delta$.

From Eq. (8.1) we obtain the following trace property of $Z^{(\delta)}$:

$$\mathrm{tr}(Z^{(\delta)}) = \epsilon\delta \,, \qquad \text{where} \quad \epsilon = \frac{n}{\tilde{n}} = \begin{cases} 2 & \text{if } n \text{ is even}, \\ 1 & \text{if } n \text{ is odd}. \end{cases}$$

Now suppose that for such a $\mathbb{Z}_n$ theory at hand we have corresponding braided endomorphisms ρ_λ of some type III factor N, such that their statistical phases are given by $\mathrm{e}^{2\pi \mathrm{i} h_\lambda}$ with conformal weights h_λ as above (as is the case for level 1 loop group theories). As we are dealing with $\mathbb{Z}_n$ fusion rules, all our morphisms ρ_λ will in fact be automorphisms. Note that if n is odd then we can always assume that $\rho_1^n = \mathrm{id}$ as morphisms (and our system can be chosen as $\{\rho_1^\lambda\}_{\lambda=0}^{n-1}$). However, if n is even, then we cannot choose a representative of the sector $[\rho_1]$ such that its n-th power gives the identity, nevertheless we can always assume that $\rho_\epsilon^{\tilde{n}} = \mathrm{id}$. Thus we have a simple current (sub-) group $\mathbb{Z}_{\tilde{n}}$, for which we can form the crossed product subfactor $N \subset M = N \rtimes \mathbb{Z}_{\tilde{n}/\delta}$ for any divisor δ of $\tilde{n}$. It is quite easy to see that $N \subset M = N \rtimes \mathbb{Z}_{\tilde{n}/\delta}$ indeed realizes $Z^{(\delta)}$: The crossed product by $\mathbb{Z}_{\tilde{n}/\delta}$ gives the dual canonical endomorphism sector $[\theta] = [\mathrm{id}] \oplus [\rho_{\epsilon\delta}] \oplus [\rho_{\epsilon\delta}^2] \oplus \ldots \oplus [\rho_{\epsilon\delta}^{\tilde{n}/\delta - 1}]$. The formula $\langle \iota\rho_\lambda, \iota\rho_\mu \rangle = \langle \theta\rho_\lambda, \rho_\mu \rangle$ then shows that the system of M-N morphisms is labelled by $\mathbb{Z}_n/\mathbb{Z}_{\tilde{n}/\delta} \simeq \mathbb{Z}_{\epsilon\delta}$, i.e. $\#_M\mathcal{X}_N = \epsilon\delta$. Therefore our general theory implies that the modular invariant arising from $N \subset M = N \rtimes \mathbb{Z}_{\tilde{n}/\delta}$ has trace equal to $\epsilon\delta$, and thus must be $Z^{(\delta)}$. Thus all modular invariants classified in [**19**] are realized from subfactors.

It is instructive to apply the above results to descendant modular invariants of conformal inclusions. Let us consider the conformal inclusion $SU(4)_6 \subset SU(10)_1$.

The associated modular invariant, which can be found in [**77**], reads

$$Z=\sum_{j\in\mathbb{Z}_{10}}|\chi^j|^2$$

with $SU(10)_1$ characters decomposing into $SU(4)_6$ characters as

$$\begin{aligned}
\chi^0 &= \chi_{0,0,0}+\chi_{0,6,0}+\chi_{2,0,2}+\chi_{2,2,2}, & \chi^5 &= \chi_{0,0,6}+\chi_{6,0,0}+\chi_{0,2,2}+\chi_{2,2,0},\\
\chi^1 &= \chi_{0,0,2}+\chi_{2,4,0}+\chi_{2,1,2}\,, & \chi^6 &= \chi_{4,0,0}+\chi_{0,2,4}+\chi_{1,2,1}\,,\\
\chi^2 &= \chi_{0,1,2}+\chi_{2,3,0}+\chi_{3,0,3}\,, & \chi^7 &= \chi_{3,0,1}+\chi_{1,2,3}+\chi_{0,3,0}\,,\\
\chi^3 &= \chi_{1,0,3}+\chi_{3,2,1}+\chi_{0,3,0}\,, & \chi^8 &= \chi_{0,3,2}+\chi_{2,1,0}+\chi_{3,0,3}\,,\\
\chi^4 &= \chi_{0,0,4}+\chi_{4,2,0}+\chi_{1,2,1}\,, & \chi^9 &= \chi_{2,0,0}+\chi_{0,4,2}+\chi_{2,1,2}\,.
\end{aligned}$$

As usual, this invariant can be realized from the conformal inclusion subfactor

$$N=\pi^0(L_ISU(4))''\subset\pi^0(L_ISU(10))''=M_+\,,$$

with π^0 denoting the level 1 vacuum representation of $LSU(10)$. The dual canonical endomorphism sector corresponds to the vacuum block,

$$[\theta_+]=[\lambda_{0,0,0}]\oplus[\lambda_{0,6,0}]\oplus[\lambda_{2,0,2}]\oplus[\lambda_{2,2,2}]\,.$$

Proceeding with α-induction $\lambda_{p,q,r}\mapsto\alpha^\pm_{+;p,q,r}\in\mathrm{End}(M_+)$, it is a straightforward calculation that the graphs describing left multiplication by fundamental generators $[\alpha^\pm_{+;1,0,0}]$ and $[\alpha^\pm_{+;0,1,0}]$ (which is the same as right multiplication by $[\lambda_{1,0,0}]$ and $[\lambda_{0,1,0}]$, respectively) on the system of M_+-N sectors gives precisely the graphs found by Petkova and Zuber [**71**, Figs. 1 and 2] by their more empirical procedure to obtain graphs with spectrum matching the diagonal part of some given modular invariant. In our framework, the graph [**71**, Fig. 1] obtains the following meaning: Take the outer wreath, pick a vertex with 4-ality 0 and label it by $[\iota_+]\equiv[\tau_0\iota_+]$, where $\iota_+:N\hookrightarrow M_+$ denotes the injection homomorphism, as usual. Going around in a counter-clockwise direction the vertices will then be the marked vertices labelled by the $\mathbb{Z}_{10}$ sectors $[\tau_1\iota_+]$, $[\tau_2\iota_+]$, ... , $[\tau_9\iota_+]$ of $SU(10)_1$. Passing to the next inner wreath the 4-ality 1 vertex adjacent to $[\iota_+]$ is then the sector $[\alpha^\pm_{+;1,0,0}\iota_+]=[\iota_+\lambda_{1,0,0}]$, and the others its $\mathbb{Z}_{10}$ translates. Similarly the inner wreath consists of the $\mathbb{Z}_{10}$ translates of $[\iota_+\lambda_{0,1,0}]$. The remaining two vertices in the center correspond to subsectors of the reducible $[\iota\lambda_{1,1,0}]$ and $[\iota\lambda_{0,1,1}]$. The graph itself then represents left (right) multiplication by $[\alpha^\pm_{+;1,0,0}]$ ($[\lambda_{1,0,0}]$).

As for $LSU(10)$ at level 1 we are in fact dealing with a $\mathbb{Z}_n$ conformal field theory, we have $n=10$ and $\tilde n=5$, we thus know that there are only two modular invariants: The diagonal one which in restriction to $LSU(4)$ gives exactly the above type I invariant $Z\equiv Z^{(5)}$, but there is also the charge conjugation invariant $Z^{(1)}$, written as

$$Z^{(1)}=\sum_{j\in\mathbb{Z}_{10}}\chi^j(\chi^{-j})^*\,.$$

Whereas $Z^{(5)}$ can be thought of as the trivial extension $M_+\subset M_+$, the conjugation invariant $Z^{(1)}$ can be realized from the crossed product $M_+\subset M=M_+\rtimes\mathbb{Z}_5$ which has dual canonical endomorphism sector

$$[\theta^{\mathrm{ext}}]=[\tau_0]\oplus[\tau_2]\oplus[\tau_4]\oplus[\tau_6]\oplus[\tau_8]\,.$$

So far we have considered the situation on the "extended level", but we may now descend to the level of $SU(4)_6$ sectors and characters. Namely we may consider the subfactor $N \subset M = M_+ \rtimes \mathbb{Z}_5$. Its dual canonical endomorphism sector $[\theta]$ is obtained by σ-restriction of $[\theta^{\rm ext}]$ which can now be read off from the character decomposition,

$$\begin{aligned}[\theta] \quad &= \bigoplus_{j=0}^4 [\sigma_{\tau_{2j}}] = [\lambda_{0,0,0}] \oplus [\lambda_{0,6,0}] \oplus [\lambda_{2,0,2}] \oplus [\lambda_{2,2,2}] \oplus [\lambda_{0,1,2}] \\ &\oplus [\lambda_{2,3,0}] \oplus [\lambda_{3,0,3}] \oplus [\lambda_{0,0,4}] \oplus [\lambda_{4,2,0}] \oplus [\lambda_{1,2,1}] \oplus [\lambda_{4,0,0}] \oplus [\lambda_{0,2,4}] \\ &\oplus [\lambda_{1,2,1}] \oplus [\lambda_{0,3,2}] \oplus [\lambda_{2,1,0}] \oplus [\lambda_{3,0,3}] \,.\end{aligned}$$

This subfactor produces the conjugation invariant $Z^{(1)}$ written in $SU(4)_6$ characters which is the same as taking the original $SU(4)_6$ conformal inclusion invariant and conjugating on the level of the $SU(4)_6$ characters. Note that this invariant has only 16 diagonal entries.

Also note that we will still have entries $Z_{\lambda,\mu} \geq 2$, for instance the diagonal entry corresponding to the weight $(2,1,2)$ is 2 as $|\chi_{2,1,2}|^2$ appears in $\chi^1(\chi^9)^*$ and in $\chi^9(\chi^1)^*$. Hence the system of M-M sectors will have non-commutative fusion rules (as had the M_+-M_+ system). When passing from M_+ to $M = M_+ \rtimes \mathbb{Z}_5$, the M_+-N system will change to the M-N system in such a way that all sectors which are translates by τ_{2j}, $j = 0,1,2,3,4$, have to be identified, and similarly fixed points split. Thus our new system of M-N morphisms will be some kind of orbifold of the old one. To see this, we first recall that all the irreducible M_+-N morphisms are of the form $\beta\iota_+$ with $\beta \in {}_{M_+}\mathcal{X}_{M_+}^{\pm}$. To such an irreducible M_+-N morphism $\beta\iota_+$ we can now associate an M-N morphism $\iota^{\rm ext}\beta\iota_+$ which may no longer be irreducible; here $\iota^{\rm ext}$ is the injection homomorphism $M_+ \hookrightarrow M$. Then the reducibility can be controlled by Frobenius reciprocity as we have

$$\langle \iota^{\rm ext}\beta\iota_+, \iota^{\rm ext}\beta'\iota_+ \rangle = \langle \theta^{\rm ext}\beta\iota_+, \beta'\iota_+ \rangle \,,$$

and $\theta^{\rm ext} = \overline{\iota}^{\rm ext}\iota^{\rm ext}$. Carrying out the entire computation we find that there are 16 M-N sectors, and the right multiplication by $[\lambda_{1,0,0}]$ is displayed graphically as in Fig. 15. Here the 4-alities 0,1,2,3 of the vertices are indicated by solid circles of decreasing size. The $[\iota]$ vertex (with $\iota = \iota^{\rm ext}\iota_+$ denoting the injection homomorphism $N \hookrightarrow M$ of the total subfactor $N \subset M = M_+ \rtimes \mathbb{Z}_5$) is the 4-ality 0 vertex in the center of the picture, and the 4-ality 1 vertex above corresponds to $[\iota\lambda_{1,0,0}]$. Each group of five vertices on the top and the bottom of the picture arise from the splitting of the two central vertices of the graphs in [**71**] as they are $\mathbb{Z}_5$ fixed points. That our orbifold graph inherits the 4-ality of the original graph is due to the fact that all entries in $[\theta]$ have 4-ality zero which in turn comes from the fact that all even marked vertices (corresponding to the subgroup $\mathbb{Z}_5 \subset \mathbb{Z}_{10}$) of the graph of Petkova and Zuber have 4-ality zero. We also display the graph corresponding to the second fundamental representation, namely the right multiplication by $[\lambda_{0,1,0}]$ in Fig. 15.

The conformal inclusion $SU(3)_5 \subset SU(6)_1$ can be treated along the same lines. The associated $SU(3)_5$ modular invariant, i.e. the one which is the specialization of the diagonal $SU(6)_1$ invariant,

$$Z = \sum_{j \in \mathbb{Z}_6} |\chi^j|^2 \,,$$

with $SU(6)_1$ characters decomposing in $SU(3)_5$ variables as

$$\begin{aligned}&\chi^0 = \chi_{0,0} + \chi_{2,2} \,, \qquad \chi^1 = \chi_{2,0} + \chi_{2,3} \,, \qquad \chi^2 = \chi_{2,1} + \chi_{0,5} \,, \\ &\chi^3 = \chi_{3,0} + \chi_{0,3} \,, \qquad \chi^4 = \chi_{1,2} + \chi_{5,0} \,, \qquad \chi^5 = \chi_{0,2} + \chi_{3,2} \,,\end{aligned}$$

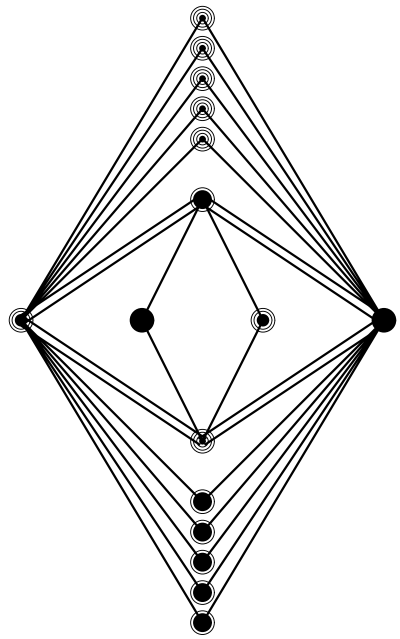

FIGURE 15. Graph G_1 associated to the conjugation invariant of the conformal inclusion $SU(4)_6 \subset SU(10)_1$

is labelled by the graph $\mathcal{E}^{(8)}$. Besides this diagonal invariant $Z \equiv Z^{(3)}$, the extended $SU(6)_1$ theory, being a $\mathbb{Z}_6$ theory, possesses only the conjugation invariant $Z^{(1)} = \sum_{j\in\mathbb{Z}_6} \chi^j (\chi^{-j})^*$, corresponding to the divisors 3 and 1 of 3, respectively. Writing again the conformal inclusion subfactor as $N \subset M_+$, the conjugation invariant can be realized from the extension $N \subset M = M_+ \rtimes \mathbb{Z}_3$ with canonical endomorphism sector

$$[\theta] = [\lambda_{0,0}] \oplus [\lambda_{2,2}] \oplus [\lambda_{2,1}] \oplus [\lambda_{0,5}] \oplus [\lambda_{1,2}] \oplus [\lambda_{5,0}]\,,$$

which arises as $\theta = \sigma_{\theta^{\mathrm{ext}}}$ where $[\theta^{\mathrm{ext}}] = [\tau_0] \oplus [\tau_2] \oplus [\tau_4]$. Whereas the M_+-N system is labelled by the vertices of the graph $\mathcal{E}^{(8)}$ and can be given by $\{\beta\iota\}$ where β runs through the chiral M_+-M_+ system determined in [**7**, Subsect. 2.3 (iv)], the M-N system will now be obtained from this one by identification of all $\mathbb{Z}_3$ translations (corresponding to the vertices labelled by $[\alpha_{(0,0)}]$, $[\alpha_{(5,5)}]$ and $[\alpha_{(5,0)}]$ in [**7**, Fig. 11]). We have no fixed points here so that the 12 vertices of $\mathcal{E}^{(8)}$ collapse to 4 vertices, and it is easy to see that the new M-N fusion graph is exactly the graph $\mathcal{E}^{(8)^*}$ in the list of Di Francesco and Zuber (see Zuber's lectures or [**2**]). Note that this time the orbifold graph $(\mathcal{E}^{(8)^*})$ looses the triality of the original graph $(\mathcal{E}^{(8)})$ because the even marked vertices (corresponding to the subgroup $\mathbb{Z}_3 \subset \mathbb{Z}_6$) of $\mathcal{E}^{(8)}$ are not exclusively of colour zero.

This way we understand why the descendants of modular invariants of conformal inclusions (where the extended theory has $\mathbb{Z}_n$ fusion rules) are in fact labelled

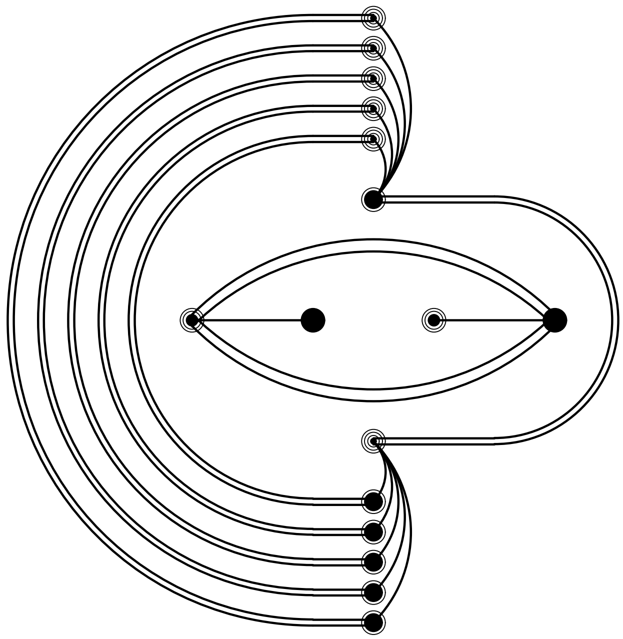

FIGURE 16. Graph G_2 associated to the conjugation invariant of the conformal inclusion $SU(4)_6 \subset SU(10)_1$

by orbifold graphs of the graph labelling the original, block-diagonal conformal inclusion invariant, and why the conjugation invariant corresponds to the maximal $\mathbb{Z}_{\tilde{n}}$ orbifold.

In the above examples, the trivial and conjugation invariant of the extended theory still remained distinct when written in terms of the $SU(4)_6$ characters. This need not be the case in general. Let us look at a familiar modular invariant of $SU(3)$ at level 9, namely

$$Z_{\mathcal{E}^{(12)}} = |\chi_{0,0} + \chi_{9,0} + \chi_{0,9} + \chi_{4,1} + \chi_{1,4} + \chi_{4,4}|^2 + 2\,|\chi_{2,2} + \chi_{5,2} + \chi_{2,5}|^2 \,,$$

which arises from the conformal embedding $SU(3)_9 \subset (\mathrm{E}_6)_1$. Now E_6 at level 1 gives a $\mathbb{Z}_3$ theory and in terms of the extended characters the above invariant is the trivial extended invariant

$$Z_{\mathcal{E}_1^{(12)}} = |\chi^0|^2 + |\chi^1|^2 + |\chi^2|^2 \,,$$

using obvious notation. Here both the $(\mathrm{E}_6)_1$ characters χ^1 and χ^2 specialize to $\chi_{2,2} + \chi_{5,2} + \chi_{2,5}$ in terms of $SU(3)_9$ variables. Let $N \subset M_+$ denote the conformal inclusion subfactor obtained by analogous means as in the previous example. It has been treated in [**8**] and produces the graph $\mathcal{E}_1^{(12)}$ of the list of Di Francesco and Zuber as chiral fusion graphs — and in turn as M_+-N fusion graph, thanks to chiral locality.

Corresponding to the two divisors 3 and 1 of 3, we know that besides the trivial there is only the conjugation invariant of our $\mathbb{Z}_3$ theory. It is given as

$$Z_{\mathcal{E}_2^{(12)}} = |\chi^0|^2 + \chi^1(\chi^2)^* + \chi^2(\chi^1)^*$$

but this distinct invariant restricts to the same invariant $Z_{\mathcal{E}^{(12)}}$ when specialized to $SU(3)_9$ variables. Nevertheless we will obtain a different subfactor $N \subset M$ since the conjugation invariant of our $\mathbb{Z}_3$ theory is realized from the extension $M_+ \subset M = M_+ \rtimes \mathbb{Z}_3$. In particular, the subfactor $N \subset M$ has dual canonical endomorphism sector

$$[\theta] = [\lambda_{0,0}] \oplus [\lambda_{9,0}] \oplus [\lambda_{0,9}] \oplus [\lambda_{4,1}] \oplus [\lambda_{1,4}] \oplus [\lambda_{4,4}] \oplus 2[\lambda_{2,2}] \oplus 2[\lambda_{5,2}] \oplus 2[\lambda_{2,5}]\,,$$

determined by σ-restriction of

$$[\theta^{\mathrm{ext}}] = [\tau_0] \oplus [\tau_1] \oplus [\tau_2]\,.$$

As before, the M-N system can be obtained from the M_+-N system by dividing out the cyclic symmetry carried by $[\theta^{\mathrm{ext}}]$. In terms of graphs, the cyclic $\mathbb{Z}_3$ symmetry corresponds to the three wings of the graph $\mathcal{E}_1^{(12)}$ which are transformed into each other by translation through the $[\tau_j]$'s, and dividing out this symmetry gives exactly the graph $\mathcal{E}_2^{(12)}$ as the wings are identified whereas each vertex on the middle axis splits into three nodes of identical Perron-Frobenius weight. This way we understand the graph $\mathcal{E}_2^{(12)}$ as the label for the conjugation invariant $Z_{\mathcal{E}_2^{(12)}}$ of $Z_{\mathcal{E}_1^{(12)}}$ which accidentally happens to be the same as the selfconjugate $Z_{\mathcal{E}^{(12)}}$ when specialized to $SU(3)_9$ variables.

Though here the same modular invariant, possessing a second interpretation as its own conjugation, gave rise to two different graphs, it often happens that an exceptional self-conjugate invariant is labelled by only one and the same graph which is its own orbifold. The very simplest case is the conformal inclusion $SU(2)_4 \subset SU(3)_1$, giving rise to the D_4 invariant which is self-conjugate for $SU(2)$ though the non-specialized diagonal $SU(3)_1$ invariant is not. We could proceed as above, passing from the conformal inclusion subfactor $N \subset M_+$ to $N \subset M = M_+ \rtimes \mathbb{Z}_3$, collapsing the M_+-N fusion graph D_4 into its $\mathbb{Z}_3$ orbifold. However, identifying the three external vertices and splitting the $\mathbb{Z}_3$ fixed point into 3 nodes gives us again D_4: The Dynkin diagram D_4 is its own $\mathbb{Z}_3$ orbifold.

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References

[1] Araki, H., Evans, D.E.: *On a C^*-algebra approach to phase transition in the two dimensional Ising model.* Commun. Math. Phys. **91**, 489-503 (1983)
[2] Behrend, R.E., Pearce, P.A., Petkova, V.B., Zuber, J.-B.: *Boundary conditions in rational conformal field theories.* Nucl. Phys. **B570**, 525-589 (2000)
[3] Baumgärtel, H.: *Operatoralgebraic methods in quantum field theory.* Berlin: Akademie Verlag 1995
[4] Böckenhauer, J.: *Localized endomorphisms of the chiral Ising model.* Commun. Math. Phys. **177**, 265-304 (1996)
[5] Böckenhauer, J.: *An algebraic formulation of level one Wess-Zumino-Witten models.* Rev. Math. Phys. **8**, 925-947 (1996)
[6] Böckenhauer, J., Evans, D.E.: *Modular invariants, graphs and α-induction for nets of subfactors. I.* Commun. Math. Phys. **197**, 361-386 (1998)
[7] Böckenhauer, J., Evans, D.E.: *Modular invariants, graphs and α-induction for nets of subfactors. II.* Commun. Math. Phys. **200**, 57-103 (1999)
[8] Böckenhauer, J., Evans, D.E.: *Modular invariants, graphs and α-induction for nets of subfactors. III.* Commun. Math. Phys. **205**, 183-228 (1999)

[9] Böckenhauer, J., Evans, D.E.: *Modular invariants from subfactors: Type I coupling matrices and intermediate subfactors.* Preprint math.OA/9911239, to appear in Commun. Math. Phys.

[10] Böckenhauer, J., Evans, D.E., Kawahigashi, Y.: *On α-induction, chiral generators and modular invariants for subfactors.* Commun. Math. Phys. **208**, 429-487 (1999)

[11] Böckenhauer, J., Evans, D.E., Kawahigashi, Y.: *Chiral structure of modular invariants for subfactors.* Commun. Math. Phys. **210**, 733-784 (2000)

[12] Böckenhauer, J., Evans, D.E., Kawahigashi, Y.: *Longo-Rehren subfactors arising from α-induction.* Preprint math.OA/0002154

[13] Cappelli, A., Itzykson, C., Zuber, J.-B.: *Modular invariant partition functions in two dimensions.* Nucl. Phys. **B280**, 445-465 (1987)

[14] Cappelli, A., Itzykson, C., Zuber, J.-B.: *The A-D-E classification of minimal and $A_1^{(1)}$ conformal invariant theories.* Commun. Math. Phys. **113**, 1-26 (1987)

[15] Carey, A.L., Evans, D.E.: *The operator algebras of the two dimensional Ising model.* In: Birman, J. *et al* (eds.): *Braids.* Contemp. Math. **78**, 117-165 (1988)

[16] Connes, A.: *Classification of injective factors.* Ann. Math. **104**, 73-115 (1976)

[17] Date, E., Jimbo, M., Miwa, T., Okado, M.: *Solvable lattice models.* In: *Theta functions – Bowdoin 1987, Part 1.* Proc. Symp. Pure Math. **49**, Providence, R.I.: American Math. Soc., pp. 295-332 (1987)

[18] Davies, E.B., Pearce, P.A.: *Conformal invariance and critical spectrum of corner transfer matrices.* J. Phys. **A23**, 1295-1312 (1990)

[19] Degiovanni, P.: *Z/NZ Conformal field theories.* Commun. Math. Phys. **127**, 71-99 (1990)

[20] Di Francesco, P.: *Integrable lattice models, graphs and modular invariant conformal field theories.* Int. J. Mod. Phys. **A7**, 407-500 (1992)

[21] Di Francesco, P., Mathieu, P., Sénéchal, D.: *Conformal field theory.* New York: Springer-Verlag 1996

[22] Di Francesco, P., Zuber, J.-B.: *$SU(N)$ lattice integrable models associated with graphs.* Nucl. Phys. **B338**, 602-646 (1990)

[23] Di Francesco, P., Zuber, J.-B.: *$SU(N)$ lattice integrable models and modular invariance.* In: *Recent Developments in Conformal Field Theories.* Trieste 1989, Singapore: World Scientific 1990, pp. 179-215

[24] Dijkgraaf, R., Verlinde, E.: *Modular invariance and the fusion algebras.* Nucl. Phys. (Proc. Suppl.) **5B**, 87-97 (1988)

[25] Doplicher, S., Haag, R., Roberts, J.E.: *Fields, observables and gauge transformations. II.* Commun. Math. Phys. **15**, 173-200 (1969)

[26] Doplicher, S., Roberts, J.E.: *A new duality theory for compact groups.* Invent. Math. **98**, 157-218 (1989)

[27] Evans, D.E., Lewis, J.T.: *On a C^*-algebra approach to phase transition in the two dimensional Ising model. II.* Commun. Math. Phys. **102**, 521-535 (1986)

[28] Evans, D.E., Kawahigashi, Y.: *Orbifold subfactors from Hecke algebras.* Commun. Math. Phys. **165**, 445-484 (1994)

[29] Evans, D.E., Kawahigashi, Y.: *Quantum symmetries on operator algebras.* Oxford: Oxford University Press 1998

[30] Fredenhagen, K., Rehren, K.-H., Schroer, B.: *Superselection sectors with braid group statistics and exchange algebras. I.* Commun. Math. Phys. **125**, 201-226 (1989)

[31] Fredenhagen, K., Rehren, K.-H., Schroer, B.: *Superselection sectors with braid group statistics and exchange algebras. II.* Rev. Math. Phys. **Special issue**, 113-157 (1992)

[32] Fröhlich, J., Gabbiani, F.: *Braid statistics in local quantum theory.* Rev. Math. Phys. **2**, 251-353 (1990)

[33] Fröhlich, J., Gabbiani, F.: *Operator algebras and conformal field theory.* Commun. Math. Phys. **155**, 569-640 (1993)

[34] Fröhlich, J., King, C.: *Two-dimensional conformal field theory and three-dimensional topology.* Int. J. Mod. Phys. **A4**, 5321-5399 (1989)

[35] Fuchs, J.: *Affine Lie algebras and quantum groups.* Cambridge: Cambridge University Press 1992

[36] Fuchs, J.: *Fusion rules in conformal field theory.* Fortschr. Phys. **42**, 1-48 (1994)

[37] Fuchs, J., Schellekens, A.N., Schweigert, C.: *Galois modular invariants of WZW models.* Nucl. Phys. **B437**, 667-694 (1995)

[38] Gannon, T.: *WZW commutants, lattices and level–one partition functions.* Nucl. Phys. **B396**, 708-736 (1993)

[39] Gannon, T.: *The classification of affine $SU(3)$ modular invariants.* Commun. Math. Phys. **161**, 233-264 (1994)

[40] Gannon, T.: *The level two and three modular invariants of $SU(n)$.* Lett. Math. Phys. **39**, 289-298 (1997)

[41] Gannon, T.: *Monstrous moonshine and the classification of CFT.* Lectures given in Istambul, August 1998, math.QA/9906167

[42] Gato-Rivera, B., Schellekens A.N.: *Complete classification of modular invariants for RCFT's with a center $(\mathbf{Z}_p)^k$.* Commun. Math. Phys. **145**, 85-121 (1992)

[43] Goodman, F., de la Harpe, P., Jones, V.F.R.: *Coxeter graphs and towers of algebras.* MSRI publications 14, Berlin: Springer, 1989

[44] Guido, D., Longo, R.: *Relativistic invariance and charge conjugation in quantum field theory.* Commun. Math. Phys. **148**, 521-551 (1992)

[45] Guido, D., Longo, R.: *The conformal spin and statistics theorem.* Commun. Math. Phys. **181**, 11-35 (1996)

[46] Haag, R.: *Local Quantum Physics.* Berlin: Springer-Verlag 1992

[47] Haagerup, U.: *Connes' bicentralizer problem and the uniqueness of the injective factor of type III_1.* Acta Math. **158**, 95-148 (1987)

[48] Izumi, M.: *Application of fusion rules to classification of subfactors.* Publ. RIMS, Kyoto Univ. **27**, 953-994 (1991)

[49] Izumi, M.: *Subalgebras of infinite C^*-algebras with finite Watatani indices II: Cuntz-Krieger algebras.* Duke Math. J. **91**, 409-461 (1998)

[50] Izumi, M., Longo, R., Popa, S.: *A Galois correspondence for compact groups of automorphisms of von Neumann algebras with a generalization to Kac algebras.* J. Funct. Anal. **155**, 25-63 (1998)

[51] Jimbo, M.: *A q-analogue of $U(N+1)$, Hecke algebra and the Yang-Baxter equation.* Lett. Math. Phys. **11**, 247-252 (1986)

[52] Jones, V.F.R.: *Index for subfactors.* Invent. Math. **72**, 1-25 (1983)

[53] Jones, V.F.R.: *Planar algebras.* Preprint math.QA/9909027

[54] Kač, V.G.: *Infinite dimensional Lie algebras*, 3rd edition, Cambridge: Cambridge University Press, 1990

[55] Kato, A.: *Classification of modular invariant partition functions in two dimensions.* Mod. Phys. Lett. **A2**, 585-600 (1987)

[56] Kawahigashi, Y.: *On flatness of Ocneanu's connections on the Dynkin diagrams and classification of subfactors.* J. Funct. Anal. **127**, 63-107 (1995)

[57] Kirillov, A.N., Reshetikhin, N.Yu.: *Representations of the algebra $U_q(sl_2)$, q-orthogonal polynomials and invariants for links.* In: Kač, V.G. (ed.): *Infinite dimensional Lie algebras and groups.* Advanced Series in Mathematical Physics, Vol. **7** (1988) pp. 285-339

[58] Kreuzer, M., Schellekens A.N.: *Simple currents versus orbifolds with discrete torsion – a complete classification.* Nucl. Phys. **B411**, 97-121 (1994)

[59] Loke, T.: *Operator algebras and conformal field theory of the discrete series representations of* $\mathrm{Diff}(S^1)$. Dissertation Cambridge (1994)

[60] Longo, R.: *Solution of the factorial Stone-Weierstrass conjecture.* Invent. Math. **76**, 145-155 (1984)

[61] Longo, R: *Index of subfactors and statistics of quantum fields II.* Commun. Math. Phys. **130**, 285–309 (1990)

[62] Longo, R., Rehren, K.-H.: *Nets of subfactors.* Rev. Math. Phys. **7**, 567-597 (1995)

[63] Mack, G., Schomerus, V.: *Conformal field algebras with quantum symmetry from the theory of superselection sectors.* Commun. Math. Phys. **134**, 139-196 (1990)

[64] McKay, J.: *Graphs, singularities and finite groups.* Proc. Symp. in Pure Math. AMS **37**, 183-186 (1980)

[65] Moore, G., Seiberg, N.: *Polynomial equations for rational conformal field theories.* Phys. Lett.**B212**, 451-460 (1988)

[66] Moore, G., Seiberg, N.: *Naturality in conformal field theory.* Nucl. Phys. **B313**, 16-40 (1989)

[67] Müger, M.: *On charged fields with group symmetry and degeneracies of Verlinde's matrix S.* Ann. Inst. H. Poincaré (Phys. Théor.) **71**, 359-394 (1999)

[68] Ocneanu, A.: *Paths on Coxeter diagrams: From Platonic solids and singularities to minimal models and subfactors.* (Notes recorded by S. Goto) In: Rajarama Bhat, B.V. et al. (eds.), Lectures on operator theory, The Fields Institute Monographs, Providence, Rhode Island: AMS publications 2000, pp. 243-323

[69] Popa, S.: *Classification of subfactors of finite depth of the hyperfinite type* III_1 *factor.* Comptes Rendus de l'Académie des Sciences, Série I, Mathémathiques, **318**, 1003-1008 (1994)

[70] Petkova, V.B., Zuber, J.-B: *From CFT to graphs.* Nucl. Phys. **B463**, 161-193 (1996)

[71] Petkova, V.B., Zuber, J.-B.: *Conformal field theory and graphs.* In: Proceedings Goslar 1996 "Group 21", hep-th/9701103

[72] Rehren, K.-H.: *Braid group statistics and their superselection rules.* In: Kastler, D. (ed.): *The algebraic theory of superselection sectors.* Palermo 1989, Singapore: World Scientific 1990, pp. 333-355

[73] Rehren, K.-H.: *Space-time fields and exchange fields.* Commun. Math. Phys. **132**, 461-483 (1990)

[74] Rehren, K.-H.: *Chiral observables and modular invariants.* Commun. Math. Phys. **208**, 689-712 (2000)

[75] Rehren, K.-H.: *Canonical tensor product subfactors.* Commun. Math. Phys. **211**, 395-406 (2000)

[76] Rehren, K.-H., Stanev, Y.S., Todorov, I.T.: *Characterizing invariants for local extensions of current algebras.* Commun. Math. Phys. **174**, 605-633 (1996)

[77] Schellekens A.N., Yankielowicz, S.: *Extended chiral algebras and modular invariant partition functions.* Nucl. Phys. **B327**, 673-703 (1989)

[78] Schellekens A.N., Yankielowicz, S.: *Field identification fixed points in the coset construction.* Nucl. Phys. **B334**, 67-102 (1990)

[79] Takesaki, M.: *Tomita's theory of modular Hilbert algebras and its applications.* Heidelberg: Springer-Verlag 1970

[80] Toledano Laredo, V.: *Fusion of positive energy representations of* LSpin_{2n}. PhD Thesis, Cambridge 1997

[81] Verlinde, E.: *Fusion rules and modular transformations in 2D conformal field theory.* Nucl. Phys. **B300**, 360-376 (1988)

[82] Verstegen, D.: *New exceptional modular invariant partition functions for simple Kac-Moody algebras.* Nucl. Phys. **B346**, 349-386 (1990)

[83] Wassermann, A.: *Operator algebras and conformal field theory III: Fusion of positive energy representations of* $LSU(N)$ *using bounded operators.* Invent. Math. **133**, 467-538 (1998)

[84] Witten, E.: *Gauge theories and integrable lattice models.* Nucl. Phys. **B322**, 629-697 (1989)

[85] Xu, F.: *Orbifold construction in subfactors.* Commun. Math. Phys. **166**, 237-254 (1994)

[86] Xu, F.: *New braided endomorphisms from conformal inclusions.* Commun. Math. Phys. **192**, 347-403 (1998)

[87] Xu, F.: *3-Manifold invariants from cosets.* Preprint math.GT/9907077

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