THE SEARCH FOR THE EXOTIC – SUBFACTORS AND CONFORMAL FIELD THEORIES

David E. Evans, Terry Gannon

Key words: subfactors, conformal field theory

AMS Mathematics Subject Classification:. 46L37, 81T40

Abstract. We look at the construction of conformal field theories and their modular invariants via tools from subfactor theory.

1 Introduction

There is a hierachy or pyramid of understanding:

- conformal field theory
- statistical mechanical models
- subfactors, vertex operator algebras and twisted K-theory
- modular tensor categories, pre-projective algebras, Calabi-Yau algebras ...

The most basic algebraic structure here, namely that of a modular tensor category, may arise from subfactors, vertex operator algebras or twisted equivariant K-theory which in turn may give rise to statistical mechanical models which at criticality may produce conformal invariant field theories. That is to say, two-dimensional conformal field theories can be understood from the vantage point of conformal nets of subfactors or vertex operator algebras. In this paper we focus on the former setting, using von Neumann algebras of operators to understand modular invariant partition functions in statistical mechanics and conformal field theory. Our primary interest here is the search for integrable models or solvable models beyond what one can construct from loop groups and quantum groups or orbifolds from finite groups and related constructions like coset theories. For this purpose, subfactors are convenient. However, an alternative K-theoretic approach based on the twisted equivariant K-theoretic description of Verlinde algebras by [23] has been proposed by us in [13, 14, 16].

1.1 Operator algebras

Let us start with the basics of analytic and measure theoretic objects of operator algebras, i.e. with some fundamental examples of C^*-algebras and von Neumann algebras. We will begin with a fundamental example of an operator algebra – tensor powers of 2×2 matrices: $\bigotimes^n M_2 \simeq M_{2^n} \simeq End(\bigotimes^n \mathbb{C}^2)$. We can complete this under the embedding $x \to x \otimes 1$ in the norm topology to get a meaning for the infinite tensor product, called the Pauli algebra: $\bigotimes^\infty M_2 \simeq M_{2^\infty}$.

To get some idea of the algebra, suppose we compute dimensions of projections $e = e^* = e^2$ using the trace: $\dim(\mathrm{e}) = \mathrm{trace}(\mathrm{e})/\mathrm{trace}(1) \in \{0, \frac{1}{2^{\mathrm{n}}}, \frac{2}{2^{\mathrm{n}}}, \frac{3}{2^{\mathrm{n}}} \cdots, 1\}$. Here we have really normalized the trace to be 1 on the identity operator so that the possible values are these dyadic rationals. They generate the semigroup of positive dyadic rationals $\mathbb{N}[1/2]$ and hence taking the Grothendieck completion the group of dyadic rationals $\mathbb{Z}[1/2]$. This is the K-group of this operator algebra, namely $K_0(\otimes_{\mathbb{N}} M_2)$. If we repeated this exercise with 3×3 matrices we would get the triadic rationals and so the two algebras are very different – they are not isomorphic.

What we are interested in though are von Neumann algebras, which are not only closed in the norm topology but in the weak operator topology. Suppose we complete this infinite tensor product in a different way. First represent the finite tensors on a Hilbert space. This can be done by turning the matrices into a Hilbert space using the trace as an inner product $H_2 = M_2$, $\langle x, y\rangle = \mathrm{tr}\, y^*x/\mathrm{tr}\, 1$ and letting the algebra act on itself by left multiplication so that $\otimes^n M_2 \subset End(\otimes^n H_2)$. Using the normalized trace, this is compatible as we increase n. We can then take the weak completion and get a different algebra $R = \otimes^\infty M_2 \subset End(\otimes^\infty H_2)$. If we compute the K-group using the dimensions of projections, we find that the gaps get filled in in the dyadic rationals picture and we get the real numbers $K_0(R) \simeq \mathbb{R}$. Remarkably, an isomorphic algebra is obtained from 3×3 matrices.

To get our definitions set up – factors are von Neumann algebras which cannot be split as a sum. This is the same as having trivial centre $R' \cap R = \mathbb{C}$, if R' denotes the commutant, or requiring all non-zero representations to be faithful. Factors are of three kinds. First are those of type I, the matrices and their infinite dimensional counterpart of bounded linear operators on a Hilbert space. We are only going to be concerned with hyperfinite factors [11], i.e. ones which can be approximated by matrices – as naturally occurs in statistical mechanical transfer matrix constructions. There is an unique hyperfinite factor which has a finite trace, the hyperfinite type II_1 R constructed above, and if the algebra is not type I and has an infinite trace the algebra is type II_∞ and is isomorphic to $R \otimes B(H)$. If the factor has no trace at all then the algebra is type III. Consequently, we have:

I : M_n, $B(H)$; II : R, $R \otimes B(H)$; III

Indeed, all hyperfinite factors have been classified by Connes [11] with the III_1 case completed by Haagerup [27]. One construction of type III hyperfinite is to repeat the above construction of R with $H_2^j = M_2$, $\langle x, y\rangle_j = \mathrm{tr}\,(e^{-H_j}y^*x)/\mathrm{tr}\,e^{-H_j}$ and then complete $\otimes_{j=1}^n H_2^j$ with sufficiently non trivial Hamiltonians H_j. In the conformal field theory picture, type III (nets of) factors naturally arise from loop group representations.

1.2 Subfactors

A subfactor is an inclusion $N \subset M$ of one factor in another. Suppose to begin with that M is the hyperfinite II_1 factor, then by Connes [11] a subfactor is either a matrix algebra or (the case we are interested in) the hyperfinite II_1 and so isomorphic to M by $\rho : N \to M$. The larger algebra is a left module over the smaller one, and if this module is finitely generated and projective this yields an element of $[{}_N M] \in K_0(N) \simeq \mathbb{R}$. This is precisely when the Jones index $[N, M]$ is finite and equals this K-theoretic element $[{}_N M]$. The fundamental result of Jones [31] is that this index value is surprisingly constrained to be in $\{4cos^2(\pi/n)\} \cup [4, \infty)$.

We can extend the inclusion $N \subset M$ either upwards or downwards to a tower and a tunnel. There is a conjugate endomorphism $\bar\rho$ on M so that $\rho\bar\rho \succeq id_M$ just as for group representations or inverses of group elements. That allows us to continue the inclusion downwards. In the opposite direction we can extend upwards using a bi-module description or using a projection e of M onto N and adjoin:

$$\cdots \subset \rho\bar\rho M \subset \rho M \subset M \subset \langle M, e\rangle = M \otimes_N M \subset M \otimes_N M \otimes_N M \subset \cdots$$

$\leftarrow$ tunnel tower $\rightarrow$

The sequence of projections e_j constructed in this way describe a Temperley-Lieb algebra. We then have a doubly-infinite sequence of inclusions of factors: $M_k \subset M_l$, $k \leq l$, and in the finite index case, the relative commutants $(M_k)' \cap M_l$ are all finite dimensional and thus are sums of matrix algebras.

$$\begin{array}{ccccc} N' \cap M_k & \subset & N' \cap M_{k+1} & \to & A \\ \cup & & \cup & & \cup \\ M' \cap M_k & \subset & M' \cap M_{k+1} & \to & B \end{array} \tag{1}$$

An embedding between finite dimensional algebras , e.g. $N' \cap M_k \subset N' \cap M_{k+1}$, gives rise to a multiplicity graph. However due to periodicity $M_k \subset M_l \simeq M_{k+2} \subset M_{l+2}$, which is related to Pontryagin duality, only two bi-partite graphs really arise,

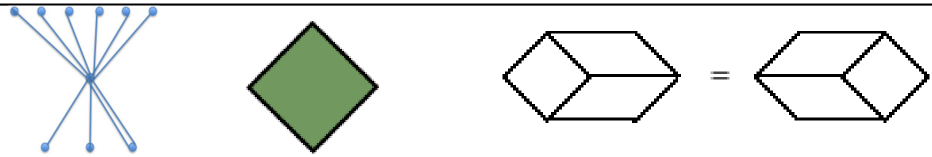


Figure 1: (i) Principal graphs examples (ii) Boltzmann diamond (iii) Yang-Baxter eqn.

called the principal and dual principal graphs, which adjoin as in the example of Figure 1 (i). There is however more information in the square by comparing two ways of embedding $M' \cap M_k \subset N' \cap M_{k+1}$. This is given by a connection in the terminology of Ocneanu [36], (see also [18]), an assignment of a complex number to each square whose edges are labelled by those of the two graphs. This is related to Boltzmann weights of statistical mechanical models with local configurations on the diamond of Figure 1 (ii). If we start with arbitrary graphs and try to set up subfactors by using the model of (1) with squares of finite dimensional approximations, we would need some integrability as in the Yang-Baxter equation of Figure 1 (iii) to ensure that the subfactor $B \subset A$ constructed in this way has the original graphs as their principal graphs. For example, E_7 does not appear in this way as a principal graph, and if we try to build up a subfactor from it in the natural way, then the principal graphs will both be D_{10}.

We can think of these relative commutants via decomposing endomorphisms or bimodules into irreducibles

$$(\rho\bar{\rho}\rho\bar{\rho}\cdots M)' \cap M \simeq End_N(M \otimes_N \cdots \otimes_N M)_{N\,or\,M}\,.$$

Going from one stage to the next is via multiplication by the fundamental object ρ or M in the endomorphism or bi-module descriptions respectively. This is illustrated in the Bratteli diagram examples of Figure 2, where the irreducibles ρ_i appear as one decomposes higher and higher powers of ρ and $\bar{\rho}$.

To give some concrete examples, suppose a finite group G acts outerly on a hyperfinite factor R. We can form the inclusion $R^G \subset R$ of fixed points which has the inclusion $R \subset R \rtimes G$ as the natural extension. We can iterate using dual actions, and the principal graphs in the case of the symmetric group S_3 is precisely as in Figure 1 (i). The upper vertices are labelled by group elements $g \in G$ and the lower ones by group representations $\pi \in \hat{G}$, with multiplicity $\dim(\pi)$.

There are of course two groups of cardinality 4, but at the integer index 4 we can construct examples, indeed all index 4 examples, via tensoring with 2×2 matrices and the natural adjoint $SU(2)$ group action. Taking the inclusion $R =$

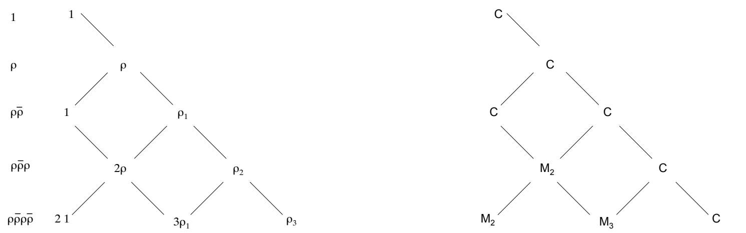


Figure 2: (i) Decomposing irreducibles (ii) Bratteli diagram

$\otimes_{\mathbb{N}} M_2 \subset R \otimes M_2$, the larger algebra is clearly 4 copies of the smaller as an R module and so the index is 4. The tower is the obvious one $R = \otimes_{\mathbb{N}} M_2 \subset R \otimes M_2 \subset R \otimes M_2 \otimes M_2 \subset R \otimes M_2 \otimes M_2 \otimes M_2 \subset \cdots$ with the relative commutants being finite dimensional tensors of two by two matrices. Taking fixed point actions under the product adjoint action of say $G = SU(2)$ the tower is $\mathbb{C} = M_2^G \subset (M_2 \otimes M_2)^G \subset (M_2 \otimes M_2 \otimes M_2)^G \subset \cdots$. The relative commutants are just the fixed point algebras $(\otimes^n M_2)^G$ generated through Weyl duality by transposition matrices in $End(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2)$ or a representation of the symmetric group. The eigen-projections of these transpositions are precisely the Temperley-Lieb projections at index 4. Comparing with the template of Figure 2, the graphs drawn there are precisely what appears for this $SU(2)$ example, and the irreducibles ρ_i of $SU(2)$ are the natural labelling. Deforming the action of $SU(2)$ to a quantum group reduces the index and yields certain representations of a Hecke algebra and related integrability or braid group Yang-Baxter type relations as in Figure 1 (iii).

At index four there is a classification of subfactors by affine ADE diagrams corresponding to subgroups of $SU(2)$ and twisted by cohomology. In the deformed case, with indices less than 4, there is an ADE classification but E_7 and D_{odd} do not appear. There is an analogous story for $SU(3)$, with subgroups of $SU(3)$ providing index 9 subfactors though the corresponding subfactors of index less than 9 are not so closely related. Figure 4, an embellishment of an atlas of [34] summarizes the possible values of indices but also maps other classifying graphs, namely the nimrep graphs of modular invariants which we will come to shortly.

The classification between 4 and 5 was recently completed by Izumi, Jones, Morrisson, Penneys, Peters [35, 30] following the fundamental work of Haagerup [28]. At index 5 there are certain group-like subactors but between 4 and 5 there are only 10 finite depth subfactors. The first is that of Haagerup [28] at index

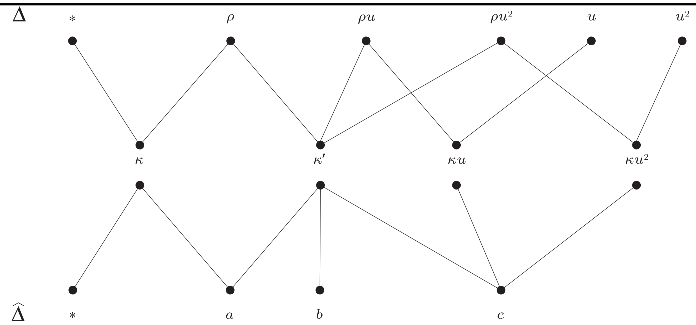


Figure 3: Principal and dual principal graphs for the Haagerup subfactor

$(5+\sqrt{13})/2$ and its dual, followed by that of Asaeda-Haagerup and its dual with index value a root of some cubic, the extended Haagerup $(5+\sqrt{17})/2$ and its dual whose existence was shown in [28]. The conformal embedding subfactor of $SU(2)_{10}$ in $SO(5)_1$ has principal graph the star shaped graph 3311 (where $n_1 n_2 \ldots n_m$ has m arms of length $n_1, n_2, \ldots n_m$) of index $3+\sqrt{3}$ and its dual. Finally there is the self dual Izumi subfactor 2221 of index $(5+\sqrt{21})/2$ and its opposite.

The Haagerup subfactor is the first finite depth subfactor of index bigger than 4. It can be regarded as a deformation of the symmetric group S_3, with even vertices satisfying the non-commutative fusion rules: $\alpha^3 = 1\,, \rho\alpha = \alpha^2\rho\,, \rho^2 = 1+\rho+\rho\alpha+\rho\alpha^2$. The statistical dimension $d_\rho = [M, \rho M]^{1/2}$ satisfies the relation $d_\rho^2 = 1 + 3d_\rho$ and so $d_\rho = (3+\sqrt{13})/2$. The index $d_\kappa^2 = d_\rho + 1$ of the Haagerup subfactor $\kappa M \subset M$ is then $(5+\sqrt{13})/2$. There are currently three ways to construct this subfactor. One is by bare hands – Haagerup constructed basically 6j-symbols or Boltzmann weights. Izumi showed the existence of this subfactor by constructing endomorphisms on Cuntz algebras satisfying these fusion rules [29]. More recently [5] found the Haagerup subfactor by constructing the planar algebra or relative commutants. Izumi [29] put the Haagerup in a potential series of subfactors for the graphs 33...3 ($2n+1$ arms) and an abelian group of order $2n+1$, and established existence and uniqueness for $\mathbb{Z}_3$ and $\mathbb{Z}_5$. We showed [15] that there are (respectively) 1, 2, 0 subfactors of Izumi type $\mathbb{Z}_7$, $\mathbb{Z}_9$ and $\mathbb{Z}_3^2$, and found strong numerical evidence for at least 2, 1, 1, 1, 2 subfactors of Izumi type $\mathbb{Z}_{11}, \mathbb{Z}_{13}, \mathbb{Z}_{15}, \mathbb{Z}_{17}, \mathbb{Z}_{19}$. We are confident there will be at least one subfactor of Izumi type, for the cyclic group $\mathbb{Z}_{2n+1}$ (any n), and more than one whenever $4n^2+4n+5$ is composite. More recently [17], we generalised Izumi's framework, weakening his equations and allowing solutions for even order abelian groups. In particular, we have constructed new subfactors at indices $3+\sqrt{5}$ and $4+\sqrt{10}$ and with graphs 3333 and 333333, and expect these to again fall into an infinite series.

2 Statistical Mechanical models at criticality

We are not only interested in subfactors but braided systems – in the type III setting with systems of endomorphism reproducing the Verlinde fusion ring. Before we indulge in the mathematical aspects of this, let us see how conformal field theories naturally throw up such structures, beginning with statistical mechanical models at criticality.

We can look in detail at the case of the Ising model which will exhibit many of the features we wish to highlight. Take the nearest neighbour Ising hamiltonian on the configuration space $\mathcal{P} = \{\pm\}^{\mathbb{Z}^2}$, $H(\sigma) = \Sigma_{\alpha,\beta} J\sigma_\alpha\sigma_\beta$ for $\sigma \in \mathcal{P}$. Then the partition function decomposes as $Z = \Sigma_\sigma exp(-H(\sigma)) = \Sigma\ \Pi$ Boltzmann weights for a Boltzmann weight involving local interactions on a plaquette. We can compute this by first taking the partition function $T_{\zeta\eta}$ of a column, with boundary distributions ζ, η. This can be computed using vertical and horizontal interactions in the nearest neighbour Hamiltonian:

$$T \;=\; V^{1/2}\, W\; V^{1/2} = e^{-\mathcal{H}}$$

Here $V = exp\, K\Sigma\sigma^x_j\sigma^x_{j+1}$ and $W = exp\, L^*\Sigma\sigma^z_j$ are the partition functions or transfer matrices for interactions along columns and rows respectively, in terms of Pauli matrices, where σ^x is the diagonal matrix with ±1 eigenvalues with eigenvectors $|\pm\rangle$, and σ^z interchanges these vectors, and K and K^* are temperature dependant interaction constants. At zero temperature K^* is zero and $T = V$ has a degenerate 2-dimensional largest eigenspace, whilst at infinite temperature K vanishes and $T = W$ has a non-degenerate largest eigenspace spanned. To relate this to the operator algebraic approach to the phase transition and subsequent algebraic conformal field theory, it is slicker to work with a half lattice $\mathbb{Z}\times\mathbb{N}$, but see [18] for a discussion of the full lattice. Then by this transfer matrix formalism, the classical one-dimensional lattice model is understood via a two-dimensional non-commutative quantum model $C\{+,-\}^{\mathbb{Z}\times\mathbb{N}} = \otimes_{\mathbb{Z}\times\mathbb{N}}(\mathbb{C}^2) \to M_2 \otimes M_2 \otimes \cdots$ where classicial expectation values are computed via quantum ones $\mu(F) = \phi_\mu(F_\beta)$ with time development α_t given by the quantum Hamiltonian $\mathcal{H} = \log T$,

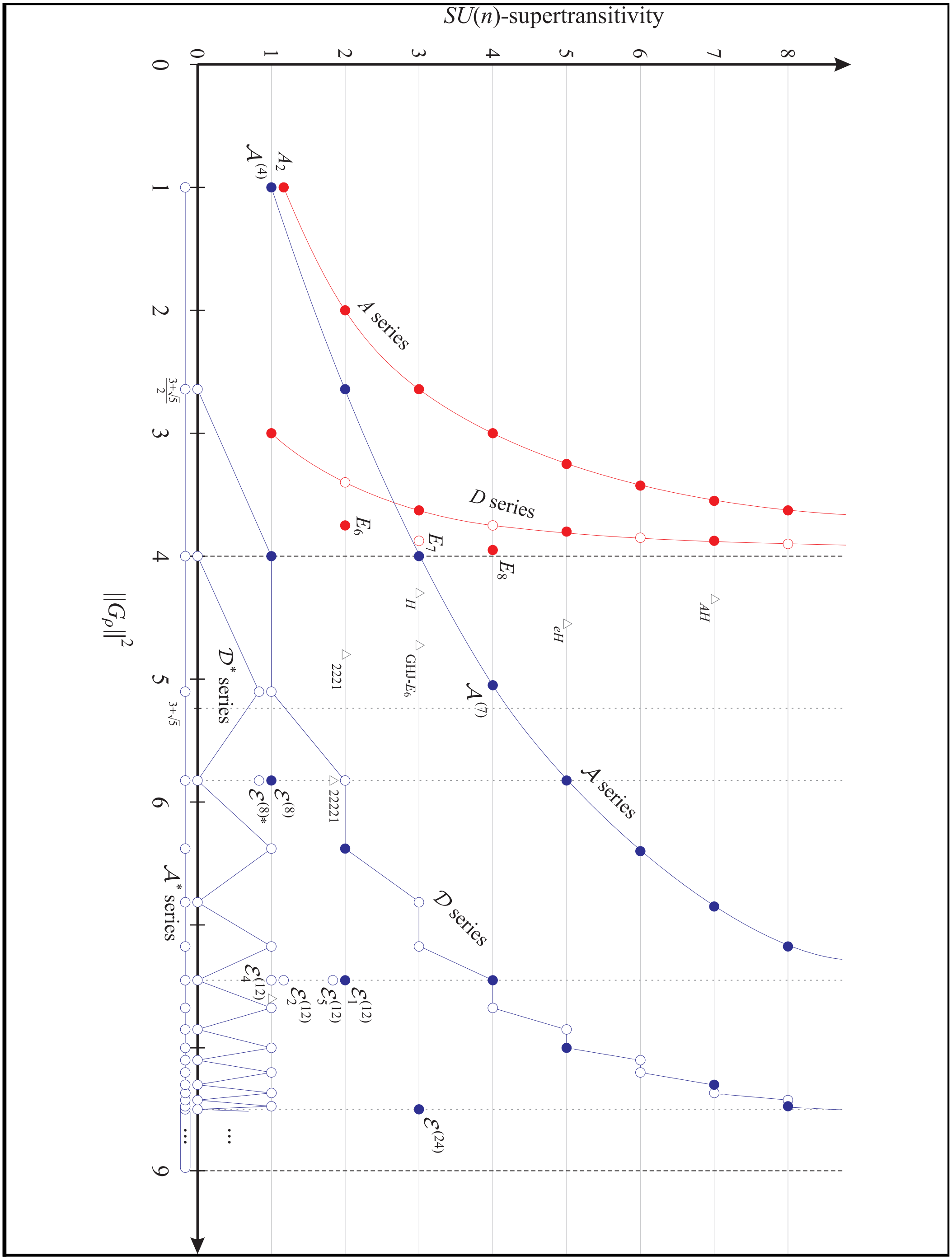


Figure 4: Plotting the $SU(n)$-supertransitivity and the norm squared of the $SU(n)$ N-M nimrep graphs G_ρ, $n = 2, 3$.

$\alpha_t = T^{it}(\ )T^{-it} = Ad\, e^{i\mathcal{H}t}$. Equilibrium states in the classical model correspond to ground states in the quantum model. At zero temperature, there are two extremal states given by $\phi_0^+ = \otimes_{\mathbb{Z}}\omega_+$ $\phi_0^- = \otimes_{\mathbb{Z}}\omega_-$; and at infinite temperature $\phi_\infty^+ = \otimes_{\mathbb{Z}}\omega$. Here $\omega_\pm A = \langle A\pm, \pm\rangle$ are the vector states on M_2 for the $\pm$ eigenspaces of σ^x and w is the vector state for the equi-distribution $(|+\rangle + |-\rangle)/\sqrt{2}$, the largest eigenspace for σ^z. What interests us here is that the Kramers-Wannier high temperature - low temperature duality, which interchanges the roles of V and W, relates the ground states at infinite and zero temperature $\phi_0^+ = \phi_\infty\nu$ if ν is the automorphism which switches $\sigma_j^x\sigma_{j+1}^x \leftrightarrow \sigma_j^z$. More precisely, define $\nu\sigma_j^x\sigma_{j+1}^x = \sigma_{j+1}^z$ and $\nu\sigma_j^z = \sigma_j^x\sigma_{j+1}^x$. Here ν is only defined on the even part of the Pauli algebra, if we grade σ^x as odd and σ^z as even. Squaring ν^2 is not the identity but a shift, the restriction of $\sigma_j^\gamma \to \sigma_{j+1}^\gamma$ to the even Pauli algebra. We can extend ν to the whole Pauli algebra by defining a Jordan-Wigner formulation it to be $\nu\sigma_j^x = \sigma_1^z\sigma_2^z\cdots\sigma_j^z$, but ν^2 is no longer the shift. To understand the key role of ν it is convenient to extend to a larger ambient algebra which is infinite with no trace - a Cuntz algebra $\mathcal{O}_2$ which is the semi-direct product of the Pauli algebra by the shift $\otimes^{\mathbb{N}}M_2 \rtimes \mathbb{N}$. The algebra $\mathcal{O}_2$ is generated by two isometries s_+, s_- with orthogonal ranges summing to the identity, $s_+s_+^* + s_-s_-^* = 1$. The Pauli algebra is naturally contained in the Cuntz algebra, e.g. $s_+s_+^*, s_-s_-^*, s_-s_+^*, s_+s_-^*$ form a copy of the matrix units of M_2.

This formalism enables amongst other things one to handle non-rectangular transfer matrices algebraically with for example s_+ on the right below:

We can extend ν to the Cuntz algebra with $\nu(s_+ \pm s_-) = \sqrt{2}(s_+s_\pm s_\pm^* + s_-s_\mp s_\mp^*)$ with the property on generators $\nu^2(s_\sigma) = s_+s_\sigma s_+^* + s_-s_{-\sigma}s_-^*$, and hence for any $x \in \mathcal{O}_2$ we get $\nu^2(x) = s_+xs_+^* + s_-\alpha x s_-^*$ if α denotes the automorphism of $\mathcal{O}_2$ which interchanges $+$ and $-$, i.e. $s_+ \leftrightarrow s_-$. What this means is that we can decompose the underlying Hilbert space $\mathcal{K}$ on which $\mathcal{O}_2$ acts by $\mathcal{K} = s_+\mathcal{K} + s_-\mathcal{K}$ so that $\nu^2(x)$ is represented as in Figure 5.(i), i.e. $\nu^2 = id + \alpha$. We are naturally led to systems of endomorphisms on infinite algebras (type III if completed appropriately) with fusion rules represented by Figure 5.(ii) so that vertices represent endomorphisms and edges multiplication by the fundamental generator ν.

Figure 5: (i) $\nu^2 = 1 + \alpha$ (ii) Ising endomorphisms

Taking lattice models, periodic in one direction, leads to cylinders with boundary conditions, and then a torus with defect lines.

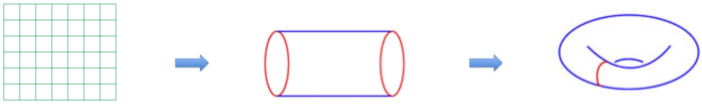


Figure 6: (i) lattice model (ii) cylinder with boundary conditions (iii) torus with defects

In the continuum limit we may expect to get a field theory with a partition function Z which decomposes as relative to some underlying symmetry (the underlying vertex operator algebra):

$$Z = \mathrm{tr}\, e^{2\pi i\tau(L_0 - c/24)} e^{-2\pi i\bar{\tau}(\overline{L_0} - c/24)} = \sum Z_{\lambda\mu}\chi_\lambda(\tau)\chi_\mu(\tau)^* \quad ,$$

where $\chi_\lambda = \mathrm{tr}\, q^{L_0 - c/24}$, $q = e^{2\pi i\tau}$, are the characters corresponding to irreducible λ. It was argued by Cardy that the parition function is invariant under reparameterisations of the torus: $Z(\tau) = Z((a\tau + b)/(c\tau + d))$. Since typically the characters themselves transform linearly under the action of $SL(2,\mathbb{Z})$, a modular invariant gives rise to a matrix of multiplicities $Z_{\lambda\mu} \in \{0, 1, 2, \ldots\}$, satisfying $Z = [Z_{\lambda\mu}] \in SL(2,\mathbb{Z})'$ and $Z_{00} = 1$, where 0 denotes the vacuum.

In the case of the two-dimensional Ising model, there are three irreducibles corresponding to the vertices of the A_3 Dynkin diagram of Figure 5 (ii), with $\pm$ labelling the end points and $\bullet$ the internal vertex. The transfer matrix formalism allows a description in terms of fermion operators $g_a : a \in \mathbb{N} - 1/2$ *or* $\mathbb{N}$ with half integer or integer labels and corresponding Hamiltonians and characters: $L_0 = \Sigma_{r\in\mathbb{N}-1/2} r g_r^* g_r \to \chi_\pm$, $L_0 = \Sigma_{n\in\mathbb{N}} n g_n^* g_n \to \chi_\bullet$. The half integer Hamiltonian is reducible according to a parity with corresponding characters:

$$\chi_+ \pm \chi_- = q^{-1/48}\Pi_{n\in\mathbb{N}}(1 \pm q^{n-1/2}), \quad \chi_\bullet = q^{1/24}\Pi_{n\in\mathbb{N}}(1 + q^n)$$

The corresponding action of $SL(2,\mathbb{Z})$ is given by:

$$\tau \to -1/\tau \qquad S = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}$$

$$\tau \to \tau + 1 \qquad T = \mathrm{diag}(e^{-\pi i/24}, e^{-\pi i/12}, e^{-\pi i 23/24})$$

What we need are braided systems of endomorphisms – not necessarily commutative but which commute up to an adjustment which can be chosen to satisfy the Yang-Baxter or braid relations and braiding fusion relations. Crossings represent intertwiners, from which one can form S and T matrices as scalar intertwiners from the Hopf link and a twist.

There are two principal sources of examples. The first arises from loop groups, e.g. that of $SU(n)$, developed by Wassermann and his students [42]. Restricting to loops concentrated on an interval $I \subset S^1$ (proper, i.e. $I \neq S^1$ and non-empty), the corresponding subgroup denoted by $L_I SU(n) = \{f \in LSU(n) : f(z) = 1\,, z \notin I\}$, one finds that in each positive energy representation π_λ we obtain hyperfinite type III_1 subfactors $\pi_\lambda(L_I SU(n))'' \subset \pi_\lambda(L_{I^c} SU(n))'$, where I^c denotes the complementary interval [42]. In the vacuum representation, labelled by $\lambda = 0$, we have Haag duality in that the inclusion collapses to a single factor $N(I) = N(I)$. More generally, the inclusion can be read as providing an endomorphism λ of the local algebra $N(I)$ such that the inclusion $\pi_\lambda(L_I SU(n))'' \subset \pi_\lambda(L_{I^c} SU(n))'$ is isomorphic to $\lambda(N(I)) \subset N(I)$. In this way we obtain systems of endomorphisms – which are braided from locality considerations where to compare two endomorphisms on the same interval we move one away to another disjoint interval, where commutativity holds, and then back again.

The second class comes from taking the double of systems of endomorphisms which themselves may not be braided nor even commutative, such as the quantum double of a finite group, Haagerup subfactor etc. If ${}_N\mathcal{X}_N$ denotes a system of endomorphisms on a type III factor, then there is a subfactor $\iota : A \subset N \otimes N^{opp}$, whose canonical endomorphism $\bar{\iota}\iota$ is expressible as $\Sigma_{\lambda \in \mathcal{X}} \lambda \otimes \lambda^{opp}$, with a non-degenerately braided system of endomorphisms on A. Thus doubles naturally come with braided inclusions.

2.1 Subfactor framework for modular invariants and RCFT

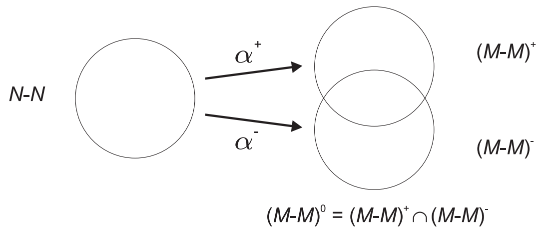
To understand modular invariants of the form $\sum Z_{\lambda\mu} \chi_\lambda \chi_\mu^*$, let us first consider the obvious one: the diagonal invariant $\sum \chi_\lambda \chi_\lambda^*$ or more generally $\sum \chi_\tau \chi_{\sigma\tau}^*$ for suitable permutations σ of the irreducibles. In some sense, made precise in [7], every modular invariant is of this form in some extended system. In subfactor language, the factor N which carries the Verlinde algebra $\mathcal{A}$ as a system of endomorphims is embedded in a larger von Neumann algebra with a system $\mathcal{B}$ of endomorphisms. When we restrict to the smaller system, σ-restriction on characters $\chi_\tau = \sum b_{\tau\lambda} \chi_\lambda$ should be interpreted as $\sigma_\tau = \sum b_{\tau\lambda} \lambda$ as endomorphisms. In particular this will certainly mean that ${}_N M_N$ thought of as an endomorphism decomposes as a sum of λ's.

Moreover the diagonal modular invariant for the ambient $\mathcal{B}$ system decomposes as

$$Z = \sum \chi_\tau \chi^*_{\sigma\tau} = \sum \left(\sum b_{\tau\lambda}\chi_\lambda\right)\left(\sum b_{\sigma\tau\,\lambda}\chi_\lambda\right)^* = \sum Z_{\lambda\mu}\,\chi_\lambda\chi^*_\mu$$

to yield a possibly non-trivial $Z_{\lambda\mu} = \sum b_{\tau\lambda} b_{\sigma\tau\,\mu}$.

However, in practice we will not be given the ambient extended system $\mathcal{B}$ but instead will start with an inclusion $N \subset M$ such that ${}_N M_N$ decomposes as a sum of λ's in $\mathcal{A}$. In such a situation we can induce the system on N to systems on M. Using the braiding and its opposite we get two ways of getting endomorphisms on M, namely $\alpha^\pm : \lambda \longrightarrow \alpha^\pm_\lambda$. What is important is their intersection.

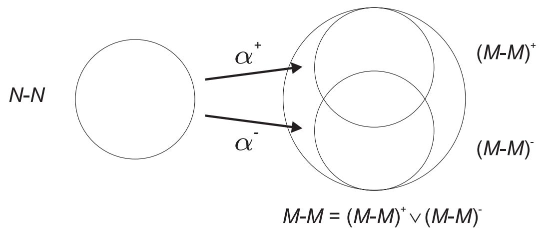


When we decompose $\alpha^+\lambda, \alpha^-\mu$ into irreducibles, we count the number of common sectors and get a multiplicity $Z_{\lambda\mu} = \langle \alpha^+_\lambda, \alpha^-_\mu \rangle$. The resulting $\sum Z_{\lambda\mu}\chi_\lambda\chi^*_\mu$ is a modular invariant. By associativity, we can regard the multiplication of the N-N system on itself as a representation of the Verlinde algebra $\lambda\mu = \sum_\nu N^\nu_{\lambda\mu}\nu$ by commuting matrices $N_\lambda = [N^\nu_{\lambda\mu}]_{\mu\nu}$.

Such a family of commuting matrices can be straightforwardly diagonalised: $N_\lambda = \sum_\kappa S_{\lambda\kappa}/S_{0\kappa}|S_\kappa\rangle\langle S_\kappa|$. What is not straightforward is that the diagonalising matrix is the same as the S matrix in the representation of $SL(2,\mathbb{Z})$.

We can form a system of N-M sectors ${}_N\mathcal{X}_M$ from $\iota\lambda$, where $\lambda \in {}_N\mathcal{X}_N$ and $\iota : N \subset M$. Now, multiplication of N-N on N-M gives a nimrep – a representation of the Verlinde algebra by positive integer matrices $G_\lambda = [G^b_{\lambda a}]_{ab}$. These can likewise be diagonalised: $G_\lambda = \sum_\kappa S_{\lambda\kappa}/S_{0\kappa}|\psi_\kappa\rangle\langle\psi_\kappa|$, with spectrum $\sigma(G_\lambda) = \{S_{\lambda\mu}/S_{0\lambda} :$ multiplicity $Z_{\lambda\lambda}\}$ coinciding precisely with the diagonal part of the modular invariant. In the case of $SU(2)$ modular invariants, this is the conceptual explanation of the ADE classification of Capelli-Itzykson-Zuber [10]. All $SU(2)$ [10] and $SU(3)$ [25] modular invariants can be realised by subfactors following work of Ocneanu, Feng-Xu, Böckenhauer, Evans, Kawahigashi and Pugh. We refer to the review article [21] for precise references.

The map of Figure 4 describes a map of nimrep index values, i.e. the squares of the norms of nimrep generators λ =fundamental weight, for $SU(2)$ (roman) and $SU(3)$ (script). The $SU(n)$-supertransitivity measures how far the nimrep graph remains alike to the identity nimrep graph before diverging following Jones [32] in the bi-partite or $SU(2)$ case, with a precise definition in the review [21].



The larger family ${}_M\mathcal{X}_M$ of M-M sectors is obtained from the irreducibles of $\iota\lambda\bar{\iota}$ and co-incides with those generated by the images of the two inductions by decomposing $\alpha^+_\lambda\alpha^-_\nu$ when the braiding is non-degenerate. Remarkably, this can be identified with the nimrep graph for the (usually non-normalized) modular invariant ZZ^*.

In the cases we are interested in, the factor N is obtained as a local factor $N = N(I_\circ)$ of a conformally covariant quantum field theoretical net of factors $\{N(I)\}$ indexed by proper intervals $I \subset \mathbb{R}$ on the real line arising from current algebras defined in terms of local loop group representations, and the N-N system is obtained as restrictions of Doplicher-Haag-Roberts morphisms (cf. [26]) to N. Taking two copies of such a net and placing the real axes on the light cone, then this defines a local conformal net $\{A(\mathcal{O})\}$, indexed by double cones $\mathcal{O}$ on two-dimensional Minkowski space (cf. [40] for such constructions). A braided subfactor $N \subset M$, determining in turn two subfactors $N \subset M_\pm$ obeying chiral locality, will provide two local nets of subfactors $\{N(I) \subset M_\pm(I)\}$. Arranging $M_+(I)$ and $M_-(J)$ on the two light cone axes defines a local net of subfactors $\{A(\mathcal{O}) \subset A_{\mathrm{ext}}(\mathcal{O})\}$ in Minkowski space. The embedding $M_+ \otimes M_-^{\mathrm{op}} \subset B$ gives rise to another net of subfactors $\{A_{\mathrm{ext}}(\mathcal{O}) \subset B(\mathcal{O})\}$, where the conformal net $\{B(\mathcal{O})\}$ satisfies locality. As shown in [40], there exist a local conformal two-dimensional quantum field theory such that the coupling matrix Z describes its restriction to the tensor products of its chiral building blocks $N(I)$. There are chiral extensions $N(I) \subset M_+(I)$ and $N(I) \subset M_-(I)$ for left and right chiral nets which are indeed maximal and should be regarded as the subfactor version of left- and right maximal extensions of the chiral algebra.

2.2 Exotic possibilities

The most natural place to look for exotic possibilities of subfactors and hence of conformal field theories is with the Haagerup subfactor and its siblings. However to get braided systems we need to take the doubles. The upper part of Fig. 7 shows the double of the even part of the principal graph Δ of the Haagerup subfactor, computed by Izumi, and the lower part comes from the double of the even part of the dual principal graph, computed in [15].

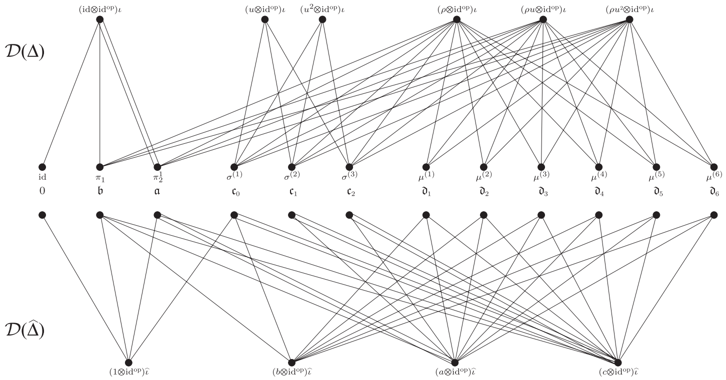


Figure 7: Dual principal graphs of the double of the Haagerup subfactor

This was the first time the dual graph was computed – using the theory of modular invariants for the double which as we have noted come equipped with canonical braided inclusions and hence canonical modular invariants. These should be compared with the corresponding objects for the doubles of the symmetric group and its dual. Note how we can recover the graph and dual graph for S_3 from this diagram by tracing from the vacuum sector on the bottom to the top, and vice versa respectively.

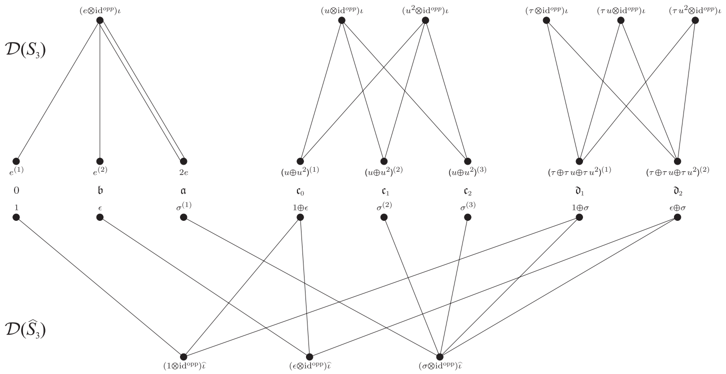


Figure 8: Dual principal graphs of the double of the S_3 subfactor

The Haagerup modular data was computed by Izumi [29], with T being the diagonal matrix $\mathrm{diag}(1,1,1,1,\xi_3,\overline{\xi_3},\xi_{13}^6,\xi_{13}^{-2},\xi_{13}^2,\xi_{13}^5,\xi_{13}^{-6},\xi_{13}^{-5})$. His S matrix though was obscure and involved a complicated rational function in $e^{2\pi ß/13}$ and $(1+ß\sqrt{5+2\sqrt{13}})/(1+\sqrt{13})$. We derived an explicit simple description for the S matrix:

$$S=\frac{1}{3}\begin{pmatrix}
x & 1-x & 1 & 1 & 1 & 1 & y & y & y & y & y & y\\
1-x & x & 1 & 1 & 1 & 1 & -y & -y & -y & -y & -y & -y\\
1 & 1 & 2 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 1 & -1 & 2 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 1 & -1 & -1 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 1 & -1 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\
y & -y & 0 & 0 & 0 & 0 & c(1) & c(2) & c(3) & c(4) & c(5) & c(6)\\
y & -y & 0 & 0 & 0 & 0 & c(2) & c(4) & c(6) & c(5) & c(3) & c(1)\\
y & -y & 0 & 0 & 0 & 0 & c(3) & c(6) & c(4) & c(1) & c(2) & c(5)\\
y & -y & 0 & 0 & 0 & 0 & c(4) & c(5) & c(1) & c(3) & c(6) & c(2)\\
y & -y & 0 & 0 & 0 & 0 & c(5) & c(3) & c(2) & c(6) & c(1) & c(4)\\
y & -y & 0 & 0 & 0 & 0 & c(6) & c(1) & c(5) & c(2) & c(4) & c(3)
\end{pmatrix},$$

for $x=(13-3\sqrt{13})/26$, $y=3/\sqrt{13}$ and $c(j)=-2y\cos(2\pi j/13)$. That this bears some relation with the double of S_3 may not be surprising given the relations between the Haagerup fusion rules and those of S_3 and $\widehat{S}_3$. There is however also a striking relationship with the affine algebra modular data $B_{6,2}$ which has central charge $c=12$, and 10 primaries. The T-matrix is $\mathrm{diag}(-1,-1;-ß,ß;-\xi_{13}^{6l^2})$, while the S-matrix is [33]

$$S=\frac{1}{3}\begin{pmatrix}
y/2 & y/2 & 3/2 & 3/2 & y & y & y & y & y & y\\
y/2 & y/2 & -3/2 & -3/2 & y & y & y & y & y & y\\
3/2 & -3/2 & 3/2 & -3/2 & 0 & 0 & 0 & 0 & 0 & 0\\
3/2 & -3/2 & -3/2 & 3/2 & 0 & 0 & 0 & 0 & 0 & 0\\
y & y & 0 & 0 & -c(1) & -c(2) & -c(3) & -c(4) & -c(5) & -c(6)\\
y & y & 0 & 0 & -c(2) & -c(4) & -c(6) & -c(5) & -c(3) & -c(1)\\
y & y & 0 & 0 & -c(3) & -c(6) & -c(4) & -c(1) & -c(2) & -c(5)\\
y & y & 0 & 0 & -c(4) & -c(5) & -c(1) & -c(3) & -c(6) & -c(2)\\
y & y & 0 & 0 & -c(5) & -c(3) & -c(2) & -c(6) & -c(1) & -c(4)\\
y & y & 0 & 0 & -c(6) & -c(1) & -c(5) & -c(2) & -c(4) & -c(3)
\end{pmatrix},$$

where y and $c(j)$ is as before. Ignoring the first 4 primaries, the only difference with the Haagerup modular data are some signs.

The question then arises as to whether there is a corresponding conformal field theory. Some evidence in support of this is that characters $\chi_\lambda(\tau)$ (with nonnegative integer coefficients) which transform among themselves according to this $SL(2,\mathbb{Z})$ representation were found following the procedures developed by [3].

References

1. M. Asaeda and U. Haagerup. *Exotic subfactors of finite depth with Jones indices* $(5+\sqrt{13})/2$ *and* $(5+\sqrt{17})/2$. Comm. Math. Phys. **202**, 1999. Pp. 1-63.
2. M. Asaeda and S. Yasuda. *On Haagerup's list of potential principal graphs of subfactors.* Comm. Math. Phys. **286**, 2009. Pp. 1141-1157.
3. P. Bantay and T. Gannon. *Vector-valued modular functions for the modular group and the hypergeometric equation.* Commun. Number Theory Phys. **1**, 2008. Pp. 637-666.
4. R. E. Behrend, P. A. Pearce, V. B. Petkova and J.-B. Zuber. *Boundary conditions in rational conformal field theories.* Nuclear Phys. B **579**, 2000. Pp. 707-773.
5. S. Bigelow, S. Morrison, E. Peters and N. Snyder. *Constructing the extended Haagerup planar algebra.* 2009. arXiv:0909.4099 [math.OA].
6. J. Böckenhauer and D. E. Evans. *Modular invariants, graphs and* α*-induction for nets of subfactors. I.* Comm. Math. Phys. **197**, 1998. Pp. 361-386; *II.* Comm. Math. Phys. **200**, 1999. Pp. 57-103; *III.* Comm. Math. Phys. **205**, 1999. Pp. 183-228.
7. J. Böckenhauer and D. E. Evans. *Modular invariants from subfactors: Type I coupling matrices and intermediate subfactors.* Comm. Math. Phys. **213**, 2000. Pp. 267-289.
8. J. Böckenhauer, D. E. Evans and Y. Kawahigashi. *On* α*-induction, chiral generators and modular invariants for subfactors.* Comm. Math. Phys. **208**, 1999. Pp. 429-487.
9. J. Böckenhauer, D. E. Evans and Y. Kawahigashi. *Chiral structure of modular invariants for subfactors.* Comm. Math. Phys. **210**, 2000. Pp. 733-784.
10. A. Cappelli, C. Itzykson, C. and J.-B. Zuber. *The* A-D-E *classification of minimal and* $A_1^{(1)}$ *conformal invariant theories.* Comm. Math. Phys. **113**, 1987. Pp. 1-26.
11. A. Connes. *Classification of injective factors.* Annals of Mathematics **104**, 1976. Pp. 73-115.
12. P. Di Francesco and J.-B. Zuber. SU(N) *lattice integrable models associated with graphs.* Nuclear Phys. B **338**, 1990. Pp. 602-646.
13. D. E. Evans. *Twisted K-theory and modular invariants: I. Quantum doubles of finite groups.* In: O. Bratteli et al (eds.) Operator Algebras: The Abel Symposium 2004. Berlin-Heidelberg: Springer, 2006. Pp. 117-144.
14. D. E. Evans and T. Gannon. *Modular invariants and twisted equivariant*

K-theory. Commun. Number Theory Phys. **3**, 2009. Pp. 209-296.
15. D. E. Evans and T. Gannon. *The exoticness and realisability of the Haagerup-Izumi modular data.* Comm. Math. Phys. **307**, 2011. Pp. 463-512.
16. D. E. Evans and T. Gannon. *Modular invariants and twisted equivariant K-theory II: Dynkin diagram symmetries.* arXiv:1012.1634v1 [math.KT].
17. D. E. Evans and T. Gannon. *Izumi's generalized Haagerup and E_6 subfactors and their modular data.* In preparation.
18. D. E. Evans and Y. Kawahigashi. *Quantum symmetries on operator algebras.* Oxford Mathematical Monographs. Oxford University Press, New York, 1998.
19. D. E. Evans and P. R. Pinto. *Modular invariants and the double of the Haagerup subfactor.* In *Advances in Operator Algebras and Mathematical Physics. (Sinaia 2003)*, F.-P. Boca et al eds, The Theta Foundation, Bucharest, 2006. Pp. 67-88.
20. D. E. Evans and P. R. Pinto. *Subfactor realisation of modular invariants II.* Internat. J. Math., to appear.
21. D. E. Evans and M. Pugh. *Braided subfactors, spectral measures, planar algebras and Calabi-Yau algebras associated to $SU(3)$ modular invariants.* .
22. K. Fredenhagen, K.-H. Rehren and B. Schroer. *Superselection sectors with braid group statistics and exchange algebras. II. Geometric aspects and conformal covariance.* Rev. Math. Phys. **Special issue**, 1992. Pp. 113-157.
23. D. S. Freed, M. J. Hopkins, C. Teleman. *Twisted equivariant K-theory with complex coefficients.* J. Topol. **1**, 2008. Pp. 16-44.
24. J. Fuchs, I. Runkel and C. Schweigert. *Twenty five years of two-dimensional rational conformal field theory.* J. Math. Phys. **51**, 015210, 2010. 19 pages.
25. T. Gannon. *The classification of affine* SU(3) *modular invariant partition functions.* Comm. Math. Phys. **161**, 1994. Pp. 233-263.
26. R. Haag. *Local Quantum Physics.* Springer, Berlin, 1992.
27. U. Haagerup. *Connes' bicentralizer problem and the uniqueness of the injective factor of type III_1.* Acta Mathematica **158**, 1987. Pp. 95-148.
28. U. Haagerup. *Principal graphs of subfactors in the index range $4 < [M : N] < 3 + \sqrt{2}$.* In *Subfactors (Kyuzeso, 1993).* World Sci. Publ., 1994. Pp. 1-38.
29. M. Izumi. *The structure of sectors associated with Longo-Rehren inclusions. II. Examples.* Rev. Math. Phys. **13**, 2001. Pp. 603-674.
30. M. Izumi, V. F. R. Jones, S. Morrison and N. Snyder. *Subfactors of index less than 5, part 3: quadruple points.* arXiv:1109.3190v1 [math.OA]
31. V. F. R. Jones. *Index for subfactors.* Invent. Math. **72**, 1983. Pp. 1-25.
32. V. F. R. Jones. *The annular structure of subfactors.* Monogr. Enseign. Math. **38**, 2001. Pp. 401-463.

33. V.G. Kac and M. Wakimoto. *Modular and conformal invariance constraints in representation theory of affine algebras.* Adv. Math. **70**, 1988. Pp. 156-236.
34. S. Morrison. *Classifying small index subfactors.* talk given at GPOTS 2011. http://tqft.net/math/GPOTS-2011.pdf
35. S. Morrison, D. Penneys, E. Peters and N. Snyder. *Subfactors of index less than 5, part 2: triple points.* Internat. J. Math., to appear.
36. A. Ocneanu. *Quantized groups, string algebras and Galois theory for algebras.* In *Operator algebras and applications, Vol. 2*, London Math. Soc. Lecture Note Ser. **136**, Cambridge Univ. Press, 1988. Pp. 119-172.
37. A. Ocneanu. *Paths on Coxeter diagrams: from Platonic solids and singularities to minimal models and subfactors. (Notes recorded by S. Goto).* in *Lectures on operator theory*, (ed. B. V. Rajarama Bhat et al.), Fields Institute Monographs, Amer. Math. Soc., 2000. Pp. 243-323.
38. A. Ocneanu. *Higher Coxeter Systems (2000).* talk given at MSRI. http://www.msri.org/publications/ln/msri/2000/subfactors/ocneanu.
39. A. Ocneanu. *The classification of subgroups of quantum* $\mathrm{SU}(N)$. In *Quantum symmetries in theoretical physics and mathematics (Bariloche, 2000)*, Contemp. Math. **294**, Amer. Math. Soc., 2002. Pp. 133-159.
40. K.-H. Rehren. *Chiral observables and modular invariants.* Comm. Math. Phys. **208**, 2000. Pp. 689-712.
41. A. Wassermann. *Coactions and Yang-Baxter equations for ergodic actions and subfactors.* In *Operator algebras and applications, Vol. 2*, London Math. Soc. Lecture Note Ser. **136**, Cambridge Univ. Press, 1988. Pp. 203-236.
42. A. Wassermann. *Operator algebras and conformal field theory. III. Fusion of positive energy representations of* $\mathrm{LSU}(N)$ *using bounded operators.* Invent. Math. **133**, 1998. Pp. 467-538.
43. F. Xu. *New braided endomorphisms from conformal inclusions.* Comm. Math. Phys. **192**, 1998. Pp. 349-403.
44. J.-B. Zuber. *CFT, BCFT, ADE and all that.* In *Quantum symmetries in theoretical physics and mathematics (Bariloche, 2000)*, Contemp. Math. **294**, Amer. Math. Soc., 2002. Pp. 233-266.

David E. Evans
School of Mathematics, Cardiff University, Senghennydd Road, Cardiff CF24 4AG, Wales, U.K. e-mail: `EvansDE@cf.ac.uk`

Terry Gannon
Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1 e-mail: `tgannon@math.ualberta.ca`