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ON INEQUALITIES OF HARDY–SOBOLEV TYPE

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This paper is dedicated to Professor Josip E. Pečarić

Submitted by T. Riedel

ABSTRACT. Hardy–Sobolev–type inequalities associated with the operator \( L := x \cdot \nabla \) are established, using an improvement to the Sobolev embedding theorem obtained by M. Ledoux. The analysis involves the determination of the operator semigroup \( \{ e^{-tL^*L} \}_{t>0} \).

1. Introduction

The following inequalities of Hardy and Sobolev are well-known to play a fundamental role in Analysis:

Hardy’s inequality

\[
\int_{\mathbb{R}^n} |\nabla f|^p dx \geq C_H(n,p) \int_{\mathbb{R}^n} \frac{|f(x)|^p}{|x|^p} dx, \quad f \in C_0^\infty(\mathbb{R}^n \setminus \{0\}),
\]

with best possible constant \( C_H(n,p) = \{ (n-p)/p \}^p \);

Sobolev’s inequality for \( 1 \leq p < n \) and \( p^* := np/(n-p) \),

\[
\|f\|_{L^{p^*}(\mathbb{R}^n)} \leq C_S(n,p) \|\nabla f\|_{L^p(\mathbb{R}^n)}, \quad f \in C_0^\infty(\mathbb{R}^n),
\]

Date: Received: 11 April; Accepted: 20 June 2008.

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2000 Mathematics Subject Classification. Primary 46E35; Secondary 35K05.

Key words and phrases. Hardy’s inequality, Sobolev’s inequality, heat semigroup, Ledoux’s inequality.
with best possible constant
\[ C_S(n, p) = \pi^{-1/2} n^{-1/p} \left( \frac{p - 1}{n - p} \right)^{(p-1)/p} \left( \frac{\Gamma(1 + n/2) \Gamma(n)}{\Gamma(n/p) \Gamma(1 + n - n/p)} \right)^{1/n}, \]
for \( 1 < p < n \), and
\[ C_S(n, 1) = \pi^{-1/2} n^{-1} (\Gamma(1 + n/2))^{1/n}. \]

From (1.1) and (1.2) it follows that for \( 0 < \delta < C_H(n, p) \), \( 1 \leq p < n \),
\[
\| \nabla f \|_{L^p(\mathbb{R}^n)}^p - \delta \| f/| \|_{L^p(\mathbb{R}^n)}^p \\
\geq \left\{ 1 - \delta/C_H(n, p) \right\} \| \nabla f \|_{L^p(\mathbb{R}^n)}^p \\
\geq \left\{ 1 - \delta/C_H(n, p) \right\} \| f \|_{L^p(\mathbb{R}^n)}^p,
\]
and so
\[
\| f \|_{L^p(\mathbb{R}^n)}^p \leq C \left\{ \| \nabla f \|_{L^p(\mathbb{R}^n)}^p - \delta \| f/| \|_{L^p(\mathbb{R}^n)}^p \right\}, \tag{1.3}
\]
where \( C \geq C_S^p(n, p) \left\{ 1 - \delta/C_H(n, p) \right\}^{-1} \). In the case \( p = 2 \), Stubbe [8] shows that the optimal value of the constant \( C \) is
\[
C_S^2(n, 2) [1 - \delta/C_H(n, 2)]^{-(n-1)/n}.
\]

In Theorem 1 below we prove the inequality
\[
\int_{\mathbb{R}^n} |(x \cdot \nabla) f(x)|^p \, dx \geq (n/p)^p \int_{\mathbb{R}^n} |f(x)|^p \, dx, \quad f \in C_0^\infty(\mathbb{R}^n), \tag{1.4}
\]
which is satisfied (and non-trivial) for all values of \( n \), including \( n = p \), and show that this implies Hardy’s inequality for \( 1 \leq p \leq n \). The above argument leading to (1.3) does not work with the right-hand side \( \| \nabla f \|_{L^p(\mathbb{R}^n)}^p - \delta \| f/| \|_{L^p(\mathbb{R}^n)}^p \) replaced by \( \| (x \cdot \nabla) f \|_{L^p(\mathbb{R}^n)}^p - \delta \| f \|_{L^p(\mathbb{R}^n)}^p \) since, by scaling considerations, we don’t have a Sobolev–type inequality
\[
\| f \|_{L^q(\mathbb{R}^n)} \leq C \| (x \cdot \nabla) f \|_{L^p(\mathbb{R}^n)}
\]
for \( q \neq p \). It is natural to ask if there is some analogue of Stubbe’s inequality, and indeed of the \( L^p \) version (1.3), when \( \| \nabla f \| \) is replaced by \( \| (x \cdot \nabla) f \| \). This was the question which initiated this research. Our investigation makes use of the following result of Ledoux in [7] which, \textit{inter alia}, improves on the standard Sobolev inequality: for every \( 1 \leq p < q < \infty \) and every function \( f \) in the Sobolev space \( W^{1, q}(\mathbb{R}^n) \),
\[
\| f \|_{L^q(\mathbb{R}^n)} \leq C \| \nabla f \|_{L^p(\mathbb{R}^n)}^p \| f \|_{B^q(p; 0, q-1),}
\]
where \( \theta = p/q, C \) is a positive constant which depends only on \( p, q \) and \( n \), and \( B^q_{\infty, \infty} \) is the homogenous Besov space of indices \( (\alpha, \infty, \infty) \); see [9]. The latter is the space of tempered distributions for which the norm
\[
\| f \|_{B^q_{\infty, \infty}} := \sup_{t > 0} \{ t^{-\alpha/2} \| P_t f \|_{L^\infty(\mathbb{R}^n)} \}.
\]
is finite, where $P_t = e^{t\Delta}, t \geq 0$, is the heat semigroup on $\mathbb{R}^n$: recall that $\{P_t\}_{t \geq 0}$ is defined by $P_t f = f$ and

$$P_t f(x) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} f(y) e^{-|x-y|^2/4t} \, dy$$

for $t > 0, x \in \mathbb{R}^n$. Cases of (1.5) were earlier established in [2], [3] and [4]. The inequality (1.5) is easily seen to include the classical Sobolev inequality (1.2). Ledoux’s technique requires specific information on the heat semi-group $e^{t\Delta}$ in $L^2(\mathbb{R}^n)$. Our first task therefore was to determine the operator semi-group associated with the inequality (1.4), namely $e^{-tL^*L}$, where $L = x \cdot \nabla$. This is done in section 3. We show that the analogue of (1.5) is in fact a consequence of Ledoux’s result. Corollaries of this analogue in the case $p = 2$, contain the following inequalities:

$$\|r f(r \omega)\|_{L^{2^*}(\mathbb{S}^{n-1})}^2 \leq C \left\{ \|Lf\|_{L^2(\mathbb{R}^n)}^2 - \frac{n^2}{4} \|f\|_{L^2(\mathbb{R}^n)}^2 \right\}^{1/n} \times \sup_{\omega \in \mathbb{S}^{n-1}} \|f\|_{L^2(\mathbb{R}^n)}^{2(1-1/n)},$$

$$\|r F(r)\|_{L^{2^*}(\mathbb{R}^n; d\mu)}^2 \leq C \left\{ \|Lf\|_{L^2(\mathbb{R}^n)}^2 - \frac{n^2}{4} \|f\|_{L^2(\mathbb{R}^n)}^2 \right\}^{1/n} \times \|f\|_{L^2(\mathbb{R}^n)}^{2(1-1/n)},$$

where $2^* = 2n/(n-2), d\mu(r) = r^{n-1}dr, C$ is a positive constant depending only on $n$ and, in polar co-ordinates $x = r \omega, F(r)$ is the integral mean of $f$ over the unit sphere $\mathbb{S}^{n-1}$, that is,

$$F(r) := \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} f(r \omega) d\omega.$$

These have a number of consequences. One is a Hardy–Sobolev type inequality (Corollary 4) which is an analogue of the type we set out to establish of Stubbe’s inequality: that if $f, Lf \in L^2(\mathbb{R}^n), n \geq 3$, then, for $\delta \in [0, n^2/4],$

$$\|r F\|_{L^{2^*}(\mathbb{R}^n; d\mu)}^2 \leq C \left[ \frac{n^2}{4} - \delta \right]^{-\frac{(n-1)}{n}} \left\{ \|Lf\|_{L^2(\mathbb{R}^n)}^2 - \delta \|f\|_{L^2(\mathbb{R}^n)}^2 \right\}.$$

It also follows from (1.6) that, for $\delta \in [0, (n-2)^2/4],$

$$\|F\|_{L^{2^*}(\mathbb{R}^n; d\mu)}^2 \leq C \left[ \frac{(n-2)^2}{4} - \delta \right]^{-\frac{(n-1)}{n}} \left\{ \|\nabla f\|_{L^2(\mathbb{R}^n)}^2 - \delta \|f\| \cdot \|f\|_{L^2(\mathbb{R}^n)} \right\}.$$  \hfill (1.7)

Since $\|F\|_{L^{2^*}(\mathbb{R}^n; d\mu)} \leq |\mathbb{S}^{n-1}|^{-1/2} \|f\|_{L^{2^*}(\mathbb{R}^n)},$ by Hölder’s inequality, (1.7) is implied by the case $p = 2$ of (1.3).

We also establish the following local Hardy–Sobolev type inequalities (see Corollaries 6 and 7): if $f$ is supported in the annulus $A_R := \{ x \in \mathbb{R}^n : 1/R \leq |x| \leq R \}$, then

$$\|r F(r)\|_{L^{2^*}(\mathbb{R}^n; d\mu)}^2 \leq C (\ln R)^{2(n-1)/n} \left\{ \|Lf\|_{L^2(\mathbb{R}^n)}^2 - \frac{n^2}{4} \|f\|_{L^2(\mathbb{R}^n)}^2 \right\};$$
\[ \| F \|_2^2 \leq C(\ln R)^{2(n-1)/n} \left\{ \| \nabla f \|_2^2 \left( \frac{n-2}{2} \right)^2 \left\| \frac{f}{|x|} \right\|_2^2 \right\}. \] (1.8)

The inequality (1.8) is reminiscent of the case \( s = 1 \) of (2.6) in [6] (proved in section 6.4); this is also proved in [1]. To be specific, it is that if \( f \in C_0^\infty(\Omega) \) and \( 2 \leq q < 2^* \),

\[ \| f \|_q^2 \leq C|\Omega|^{2(1/q-1/2^*)} \left\{ \| \nabla f \|_2^2 \left( \frac{n-2}{2} \right)^2 \left\| \frac{f}{|x|} \right\|_2^2 \right\}, \] (1.9)

where \( |\Omega| \) denotes the volume of \( \Omega \). It is noted in [6], Remark 2.4, that, in contrast to (1.8), the \( q \) in (1.9) must be strictly less than the critical Sobolev exponent \( 2^* = 2n/(n-2) \) if \( \Omega \) includes the origin.

The authors are grateful to Rupert Frank, Elliot Lieb and Robert Seiringer for some valuable comments.

2. THE HARDY-TYPE INEQUALITY (1.4)

**Theorem 2.1.** Let \( n \geq 1 \) and \( 1 \leq p < \infty \). Then for all \( f \in C_0^\infty(\mathbb{R}^n) \)

\[ \int_{\mathbb{R}^n} |(x \cdot \nabla) f|^p dx \geq \left( \frac{n}{p} \right)^p \int_{\mathbb{R}^n} |f|^p dx. \] (2.1)

**Proof.** On integration by parts and the application of Hölder’s inequality we have

\[ n \int_{\mathbb{R}^n} |f(x)|^p dx = \int_{\mathbb{R}^n} \text{div}(x)|f(x)|^p dx \]

\[ = -p \text{Re} \int_{\mathbb{R}^n} (x \cdot \nabla)f(x)|f(x)|^{p-2} \overline{f(x)} dx \]

\[ \leq p \left( \int_{\mathbb{R}^n} |(x \cdot \nabla) f(x)|^p dx \right)^{1/p} \left( \int_{\mathbb{R}^n} |f(x)|^p dx \right)^{(p-1)/p} \]

which yields (2.1). \[ \square \]

**Remark 2.2.** The inequality (2.1) implies (1.1) for \( 1 \leq p \leq n \). For we have from

\[ \nabla(|x|f) = \frac{x}{|x|}f + |x|\nabla f \]

that

\[ \| \nabla(|x|f) \|_{L^p(\mathbb{R}^n)} \geq \| |x|\nabla f \|_{L^p(\mathbb{R}^n)} - \| f \|_{L^p(\mathbb{R}^n)} \]

\[ \geq \| (x \cdot \nabla)f \|_{L^p(\mathbb{R}^n)} - \| f \|_{L^p(\mathbb{R}^n)} \]

\[ \geq \left( \frac{n-p}{p} \right) \| f \|_{L^p(\mathbb{R}^n)} \]

whence (1.1) on replacing \( f(x) \) by \( f(x)/|x| \).
3. Calculation of the semigroup \( e^{-tL^*L} \)

**Theorem 3.1.** Let \( L = x \cdot \nabla, x = r \omega, r = |x| \). Then the semigroup \( e^{-tL^*L} \) is given by

\[
(e^{-tL^*L} \psi)(x) = \frac{e^{-tn^2/4}}{\sqrt{4\pi t}} r^{-n/2} \int_0^\infty e^{-\frac{(ln r - ln s)^2}{4t}} s^{-n/2} \psi(s \omega) s^{n-1} ds.
\]

**(3.1)**

**Proof.** Before embarking on the proof, some preliminary remarks and results might be helpful. The gist of the proof is that after a change of co-ordinates, \( L^*L \) is seen to be related to the Laplacian in \( \mathbb{R}^n \), and this then yields the result. The co-ordinate change is determined by the map \( \Phi : L^2(\mathbb{R}^n) \to L^2(\mathbb{R} \times S^{n-1}) \) defined by

\[
(\Phi \psi)(s, \omega) := e^s \psi(e^s \omega)
\]

for \( \omega \in S^{n-1} \) and \( s \in \mathbb{R} \). Note that we equip \( \mathbb{R} \times S^{n-1} \) with the usual one dimensional Lebesgue measure on \( \mathbb{R} \) and the usual surface measure on \( S^{n-1} \). Thus \( \Phi \) preserves the \( L^2 \) norm. The inverse of \( \Phi \) satisfies \( \Phi^{-1} : L^2(\mathbb{R} \times S^{n-1}) \to L^2(\mathbb{R}^n) \) and is given by

\[
(\Phi^{-1} \varphi)(x) = r^{-n/2} \varphi(\ln r, \omega).
\]

**(3.3)**

The dilations \( U(t) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \) given by

\[
U(t) \psi(x) := e^{tn/2} \psi(e^t x)
\]

form a group of unitary operators with generator \( U(t) = e^{iAt} \), where \( A \) is given by

\[
iA \psi = \frac{\partial}{\partial t} U(t) \psi \big|_{t=0} = (x \cdot \nabla + \frac{n}{2}) \psi = \frac{1}{2} (x \cdot \nabla + \nabla \cdot x) \psi.
\]

Thus

\[
A = \frac{1}{i} (x \cdot \nabla + \frac{n}{2}) = -iL - \frac{n}{2},
\]

and so

\[
L = iA - \frac{n}{2},
\]

where \( A \) is the self-adjoint generator of dilations in \( L^2(\mathbb{R}^n) \). In particular,

\[
L^*L = (-iA - \frac{n}{2})(iA - \frac{n}{2}) = A^2 + \frac{n^2}{4}.
\]

Since

\[
(\Phi \psi)(s, \omega) = (U(s) \psi)(\omega)
\]

for \( \omega \in S^{n-1} \) and \( s \in \mathbb{R} \), it follows from the group property of the dilations \( U(\cdot) \) that

\[
(\Phi(U(t) \psi))(s, \omega) = (U(s)(U(t) \psi))(\omega) = (U(s + t) \psi)(\omega) = (\Phi \psi)(s + t, \omega).
\]

In particular, in the new co-ordinates given by \( \Phi \), the dilations \( U(t) \) act simply as shifts by \( t \) and should be diagonalizable with the help of a Fourier transform! We now proceed to confirm this prediction.
Define $M : L^2(\mathbb{R}^n) \to L^2(\mathbb{R} \times S^{n-1})$ by

$$(M\psi)(\tau, \omega) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ist}\Phi\psi(s, \omega) \, ds,$$  

(3.4)

so that $M = \mathcal{F} \circ \Phi$, where $\mathcal{F}$ is the Fourier transform on $\mathbb{R}$. Then

$$(MU(t)\psi)(\tau, \omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ist}\Phi\psi(s + t, \omega) \, ds$$

$$= e^{it\tau} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ist}\Phi\psi(s, \omega) \, ds = e^{it\tau}(M\psi)(\tau, \omega).$$  

(3.5)

The map $M = \mathcal{F} \circ \Phi$ is the Mellin transformation and has an explicit representation using the group structure of $\mathbb{R}^+$ under multiplication: it is the Fourier transform on this group.

The next step is to show that

$$(MA\psi)(\tau, \omega) = \tau(M\psi)(\tau, \omega)$$  

(3.6)

for $\psi$ in the domain $\mathcal{D}(A)$: it follows that $\psi \in \mathcal{D}(A)$ if and only if $(\tau, \omega) \mapsto \tau(M\psi)(\tau, \omega) \in L^2(\mathbb{R} \times S^{n-1})$. To see (3.6) we note that $iAe^{itA} = \partial_tU(t)$ and so, from (3.5)

$$(MiAe^{itA}\psi)(\tau, \omega) = (M\partial_tU(t)\psi)(\tau, \omega) = \partial_t(MU(t)\psi)(\tau, \omega)$$

$$= \partial_t e^{it\tau}(M\psi)(\tau, \omega) = i\tau e^{it\tau}(M\psi)(\tau, \omega).$$

Setting $t = 0$ yields (3.6).

We are now in a position to complete the proof of the theorem. We have $e^{-tL^*L} = e^{-tn^2/4}e^{-tA^2}$ and by (3.4)

$$(Me^{-tA^2}\psi)(\tau, \omega) = e^{-tr^2}(M\psi)(\tau, \omega).$$

So

$$e^{-tA^2} = M^{-1}e^{-tr^2}M.$$  

Since $M = \mathcal{F} \circ \Phi$, we see that

$$e^{-tA^2} = \Phi^{-1} \circ \mathcal{F}^{-1}(e^{-tr^2}\mathcal{F} \circ \Phi).$$

Of course,

$$\mathcal{F}^{-1}(e^{-tr^2}M\psi)(\lambda, \omega) = \mathcal{F}^{-1}(e^{-tr^2}\mathcal{F} \circ \Phi)(\lambda, \omega)$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i\lambda\tau} e^{-tr^2}e^{-ist}\Phi\psi(s, \omega) \, ds \, d\tau$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} \left( \int_{\mathbb{R}} e^{-tr^2 + i(\lambda - s)\tau} \, d\tau \right)(\Phi\psi)(s, \omega) \, ds$$

The integral in big parentheses is a Gaussian integral which gives

$$\int_{\mathbb{R}} e^{-tr^2 + i(\lambda - s)\tau} \, d\tau = \sqrt{\frac{\pi}{t}} e^{-\frac{(\lambda - s)^2}{4t}}.$$
Thus
\[
\mathcal{F}^{-1}\left(e^{-tr^2}M\psi\right)(\lambda,\omega) = \frac{1}{\sqrt{4\pi t}} \int e^{-\frac{(\lambda-s)^2}{4t}} (\Phi\psi)(s,\omega) \, ds =: \varphi_t(\lambda,\omega)
\]
and, with \(x = r\omega\),
\[
\left(e^{-tA^2}\psi\right)(r\omega) = (\Phi^{-1}\varphi_t)(r\omega)
\]
\[= \frac{1}{\sqrt{4\pi t}} \int e^{-\frac{(\ln r-s)^2}{4t}} (\Phi\psi)(s,\omega) \, ds.
\]
Since \((\Phi\psi)(s,\omega) = \exp\left\{\frac{n}{2}\psi(e^{s}\omega)\right\}\), we get from the change of variables \(z = e^s\),
\[
\left(e^{-tA^2}\psi\right)(r\omega) = \frac{1}{\sqrt{4\pi t}} \int \exp\left\{-\frac{(\ln r-\ln z)^2}{4t}\right\} \frac{n}{2}\psi(z\omega) \, dz.
\]
which is (3.1).

Once it is realised that \(A\) is simply multiplication by \(\tau\) in the sense of (3.6), it is clear that \(A\) is the momentum operator on \(\mathbb{R}\), that is, \(\Phi A\Phi^{-1}\) is given by
\[
\Phi A\Phi^{-1} = -i\partial_s \otimes 1_{S^{n-1}}
\]
on using this and the functional calculus we get
\[
\Phi L^* L\Phi^{-1} = (\Phi A\Phi^{-1})^2 + \frac{n^2}{4} = -\partial^2_s \otimes 1_{S^{n-1}} + \frac{n^2}{4}.
\]
Thus, \(L^* L = -\Phi^{-1}\partial^2_s \otimes 1_{S^{n-1}}\Phi + \frac{n^2}{4}\) and
\[
e^{-tL^* L} = e^{-tn^2/4}e^{-t\Phi^{-1}\partial^2_s \otimes 1_{S^{n-1}}\Phi} = e^{-tn^2/4}\Phi^{-1}e^{-t\partial^2_s \otimes 1_{S^{n-1}}\Phi} (3.7)
\]
which is a convenient way of expressing (3.1).

On substituting (3.2) and (3.3) and making an obvious change of variables, we obtain from (3.1) the following representation for \(e^{-tA^2}\); see also (3.7).

**Corollary 3.2.** Let \(P_t\) denote \(e^{-tA^2}\). Then
\[
\Phi P_t\Phi^{-1}\varphi(r,\omega) = \frac{1}{\sqrt{4\pi t}} \int \exp\left\{-\frac{1}{4t}(r-s)^2\right\} \varphi(s\omega) \, ds. \tag{3.8}
\]
4. The main inequalities

The fact that $\Phi e^{-tA^2}\Phi^{-1}$ in (3.8) is essentially radial means that the analogue of (1.5) derived by Ledoux’s technique is a consequence of the one-dimensional case of (1.5). Defining $B^\alpha$ to be the space of all tempered distributions $g$ on $\mathbb{R} \times S^{n-1}$ for which the norm

$$\|g\|_{B^\alpha} := \sup_{t>0}\{t^{-\alpha/2}\|\Phi e^{-tA^2}\Phi^{-1}g\|_{L^\infty(\mathbb{R} \times S^{n-1})}\} < \infty,$$  \hfill (4.1)

one obtains from the $n = 1$ case of (1.5), that for any $\omega \in S^{n-1},$

$$\int_\mathbb{R} |g(r,\omega)|^q dr \leq C^q \int_\mathbb{R} \left| \frac{\partial g(r,\omega)}{\partial r} \right|^p dr \times \left( \sup_{t>0,r\in\mathbb{R}} t^{\theta/2(1-\theta)} \left| \frac{1}{\sqrt{4\pi t}} \int e^{-(r-s)^2/4t} g(s,\omega) ds \right| \right)^{q(1-\theta)}\right) \quad \|g\|_{B^\theta(\theta-1)}^q \right) \leq C^q \int_\mathbb{R} \left| \frac{\partial g(r,\omega)}{\partial r} \right|^p dr \|g\|_{B^\theta(\theta-1)}^q \right),$$

On integrating with respect to $\omega$ over $S^{n-1}$ we obtain

**Theorem 4.1.** Let $1 \leq p < q < \infty$ and suppose that $g$ is such that $\Phi A\Phi^{-1}g = -i(\partial/\partial r)g \in L^p(\mathbb{R} \times S^{n-1})$ and $g \in B^{\theta/(\theta-1)}, \theta = p/q$. Then there exists a positive constant $C$, depending on $p$ and $q$, such that

$$\|g\|_{L^q(\mathbb{R} \times S^{n-1})} \leq C\|\partial/\partial r g\|_{L^p(\mathbb{R} \times S^{n-1})}^{1-\theta} \|g\|_{B^{\theta/(\theta-1)}}^{1-\theta}. \hfill (4.2)$$

The theorem has two natural corollaries featuring the Hardy-type inequality (2.1), the first an inequality of Sobolev type, and the second of Gagliardo-Nirenberg type.

**Corollary 4.2.** (i) Let $p^* := np/(n-p), 1 \leq p \leq n-1$, and suppose $(\partial/\partial r)g \in L^p(\mathbb{R} \times S^{n-1})$ and $\sup_{\omega \in S^{n-1}}\|g(\cdot, \omega)\|_{L^p(\mathbb{R})} < \infty$. Then

$$\|g\|_{L^{p^*}(\mathbb{R} \times S^{n-1})} \leq C\|\partial/\partial r g\|_{L^p(\mathbb{R} \times S^{n-1})}^{1/n} \sup_{\omega \in S^{n-1}}\|g(\cdot, \omega)\|_{L^p(\mathbb{R})}^{(n-1)/n}. \hfill (4.3)$$

(ii) If $G = \mathcal{M}(g)$ denotes the integral mean of $g$, namely,

$$G(r) = \mathcal{M}(g)(r) := \frac{1}{|S^{n-1}|} \int_{S^{n-1}} g(r,\omega) d\omega,$$

then if $g, (\partial/\partial r)g \in L^p(\mathbb{R} \times S^{n-1}),$

$$\|G\|_{L^{p^*}(\mathbb{R})} \leq C\|\partial/\partial r g\|_{L^p(\mathbb{R} \times S^{n-1})}^{1/n} \|g\|_{L^p(\mathbb{R} \times S^{n-1})}^{(n-1)/n}. \hfill (4.4)$$
If $g$ is supported in $[-\Lambda, \Lambda] \times \mathbb{S}^{n-1}$, then
$$
\|g\|_{L^p(\mathbb{R} \times \mathbb{S}^{n-1})} \leq C \Lambda^{(n-1)/n^2} \|\partial/\partial r\|_{L^p(\mathbb{R} \times \mathbb{S}^{n-1})} \sup_{\omega \in \mathbb{S}^{n-1}} \|g(\cdot, \omega)\|_{L^{p^*}(\mathbb{R})}^{(n-1)/n}; \tag{4.5}
$$
also
$$
\|G\|_{L^p(\mathbb{R})} \leq C \Lambda^{(n-1)/n} \|\partial/\partial r\|_{L^p(\mathbb{R} \times \mathbb{S}^{n-1})}. \tag{4.6}
$$

**Proof.** From (3.8), it follows that, for any $s \in [1, \infty)$,
$$
t^{-\theta/2(\theta-1)} \|\Phi P_1 \Phi^{-1} g\|_{L^\infty(\mathbb{R} \times \mathbb{S}^{n-1})} \leq Ct^{-\theta/2(\theta-1)-1/2s} \sup_{\omega \in \mathbb{S}^{n-1}} \|g\|_{L^s(\mathbb{R})}.
$$

If $1 \leq p < n - 1$ set $\theta = p/q$, $q = p(p+1)$ and $s = p$. Then, from Theorem 4.1
$$
\|g\|_{L^p(p+1)(\mathbb{R} \times \mathbb{S}^{n-1})} \leq C \|\partial/\partial r\|_{L^p(\mathbb{R} \times \mathbb{S}^{n-1})} \sup_{\omega \in \mathbb{S}^{n-1}} \|g/p(p+1)\|_{L^p(\mathbb{R})}. \tag{4.7}
$$
Thus $g \in L^p(p+1)(\mathbb{R} \times \mathbb{S}^{n-1}) \cap L^p(\mathbb{R} \times \mathbb{S}^{n-1})$, and since
$$
\frac{np}{n-p} = \frac{p(p+1)}{n-p} + \frac{p(n-p-1)}{n-p},
$$
we have by Hölder’s inequality,
$$
\int_{\mathbb{R} \times \mathbb{S}^{n-1}} |g|^{p^*} d\lambda \leq \left( \int_{\mathbb{R} \times \mathbb{S}^{n-1}} |g|^{(p(p+1)} d\lambda \right)^{1/(n-p)} \left( \int_{\mathbb{R} \times \mathbb{S}^{n-1}} |g|^p d\lambda \right)^{(n-p-1)/(n-p)}.
$$
Hence, from (4.7),
$$
\|g\|_{L^p(\mathbb{R} \times \mathbb{S}^{n-1})} \leq \|g\|_{L^p(p+1)(\mathbb{R} \times \mathbb{S}^{n-1})} \|g\|_{L^p(\mathbb{R} \times \mathbb{S}^{n-1})}^{(n-p-1)/n}
\leq C \|\partial/\partial r\|_{L^p(\mathbb{R} \times \mathbb{S}^{n-1})} \sup_{\omega \in \mathbb{S}^{n-1}} \|g(\cdot, \omega)\|_{L^p(\mathbb{R})}^{(n-1)/n}.
$$

If $p = n - 1$, we choose $s = n - 1, q = p^* = n(n-1)$ and $\theta = 1/n$. Then Theorem 3 gives (4.3) immediately. The inequality (4.5) follows on applying Hölder’s inequality to $\|g(\cdot, \omega)\|_{L^p(\mathbb{R})}$. The inequalities (4.4) and (4.6) follow from (4.3) and (4.5) respectively, on substituting $G$ for $g$ and noting that
$$
\|G\|_{L^p(\mathbb{R} \times \mathbb{S}^{n-1})} \leq \|\partial/\partial r\|_{L^p(\mathbb{R} \times \mathbb{S}^{n-1})},
\|G\|_{L^p(\mathbb{R})} \leq \|\mathbb{S}^{n-1}\|^{-1/p} \|g\|_{L^p(\mathbb{R} \times \mathbb{S}^{n-1})},
$$

**Corollary 4.3.** (i) Let $1 \leq p < q < \infty, m = (q/p) - 1$, and suppose that $\partial/\partial r g \in L^p(\mathbb{R} \times \mathbb{S}^{n-1})$ and $\sup_{\omega \in \mathbb{S}^{n-1}} \|g(\cdot, \omega)\|_{L^m(\mathbb{R})} < \infty$. Then
$$
\|g\|_{L^q(\mathbb{R} \times \mathbb{S}^{n-1})} \leq C \|\partial/\partial r\|_{L^p(\mathbb{R} \times \mathbb{S}^{n-1})} \sup_{\omega \in \mathbb{S}^{n-1}} \|g(\cdot, \omega)\|_{L^m(\mathbb{R})}^{1-p/q}. \tag{4.8}
$$

(ii) If $\partial/\partial r g \in L^p(\mathbb{R} \times \mathbb{S}^{n-1})$ and $g \in L^m(\mathbb{R} \times \mathbb{S}^{n-1})$, then, with $G = \mathcal{M}(g)$,
$$
\|G\|_{L^q(\mathbb{R})} \leq C \|\partial/\partial r\|_{L^p(\mathbb{R} \times \mathbb{S}^{n-1})} \|g\|_{L^m(\mathbb{R} \times \mathbb{S}^{n-1})}^{1-p/q}. \tag{4.9}
$$
Proof. From (3.8), with \( \theta = p/q \) and \( m = q/p - 1 \), we deduce that
\[
\left\| \Phi P_t \Phi^{-1} g \right\|_{L^\infty(\mathbb{R} \times \mathbb{S}^{n-1})} \leq C t^{-\theta/(\theta - 1) - 1/2m} \sup_{\omega \in \mathbb{S}^{n-1}} \| g(\cdot, \omega) \|_{L^m(\mathbb{R})}
\]
\[
\leq C \sup_{\omega \in \mathbb{S}^{n-1}} \| g(\cdot, \omega) \|_{L^m(\mathbb{R})}
\]
and this yields (4.8). The inequality (4.9) follows from (4.8) on substituting \( G \) for \( g \).

The case \( p = 2 \) of Corollary 4.2 is of special interest.

**Corollary 4.4.** (i) Let \( f \) be such that \( Lf \in L^2(\mathbb{R}^n) \), \( L = x \cdot \nabla \), and
\[
\sup_{\omega \in \mathbb{S}^{n-1}} \| f(\cdot, \omega) \|_{L^2(\mathbb{R}^n; d\mu)} < \infty.
\]
Then, for \( n \geq 3 \),
\[
\| r f(r) \|_{L^{2^*}(\mathbb{R}^n)}^2 \leq C \left\{ \left( \left\| Lf \right\|_{L^2(\mathbb{R}^n)}^2 - \frac{n^2}{4} \left\| f \right\|_{L^2(\mathbb{R}^n)}^2 \right) \right\}^{1/n} \times \sup_{\omega \in \mathbb{S}^{n-1}} \| f(\cdot, \omega) \|_{L^2(\mathbb{R}^{r+}; d\mu)}^{2(1 - 1/n)}.
\]
where \( 2^* = 2n/(n - 2) \) and \( d\mu = r^{n-1} dr \).

(ii) If \( f, Lf \in L^2(\mathbb{R}^n) \), then, with \( F := \mathcal{M}(f) \),
\[
\| r F(r) \|_{L^{2^*}(\mathbb{R}^n; d\mu)}^2 \leq C \left\{ \left( \left\| Lf \right\|_{L^2(\mathbb{R}^n)}^2 - \frac{n^2}{4} \left\| f \right\|_{L^2(\mathbb{R}^n)}^2 \right) \right\}^{1/n} \times \left\| f \right\|_{L^2(\mathbb{R}^n)}^{2(1 - 1/n)}.
\]
For \( 0 \leq \delta < n^2/4 \), we have
\[
\| r F(r) \|_{L^{2^*}(\mathbb{R}^n; d\mu)}^2 \leq C \left( \frac{n^2}{4} - \delta \right)^{-(n-1)/n} \left\{ \left\| Lf \right\|_{L^2(\mathbb{R}^n)}^2 - \delta \left\| f \right\|_{L^2(\mathbb{R}^n)}^2 \right\}.
\]

**Proof.** On using the facts that \( \Phi : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R} \times \mathbb{S}^{n-1}) \) is an isometry and, with \( g := \Phi f \),
\[
\left\| (\partial/\partial r) g \right\|_{L^2(\mathbb{R} \times \mathbb{S}^{n-1})}^2 = \left\| \Phi A \Phi^{-1} g \right\|_{L^2(\mathbb{R} \times \mathbb{S}^{n-1})}^2
\]
\[
= \left\| Af \right\|_{L^2(\mathbb{R}^n)}^2
\]
\[
= \left\| Lf \right\|_{L^2(\mathbb{R}^n)}^2 - \frac{n^2}{4} \left\| f \right\|_{L^2(\mathbb{R}^n)}^2
\]
since \( A^2 = L^* L - (n^2/4) \) from (3.6), it follows from (4.3) that
\[
\| \Phi f \|_{L^{2^*}(\mathbb{R} \times \mathbb{S}^{n-1})}^2 \leq C \left\{ \left\| Lf \right\|_{L^2(\mathbb{R}^n)}^2 - \frac{n^2}{4} \left\| f \right\|_{L^2(\mathbb{R}^n)}^2 \right\}^{1/n} \times \sup_{\omega \in \mathbb{S}^{n-1}} \| f(\cdot, \omega) \|_{L^2(\mathbb{R}^{r+}; d\mu)}^{2(1 - 1/n)}.
\]
Then (4.10) follows since
\[
\| \Phi f \|_{L^{2^*}(\mathbb{R} \times \mathbb{S}^{n-1})} = \| r f(r, \omega) \|_{L^{2^*}(\mathbb{R}^n)}.
\]
The inequality (4.11) follows in a similar way from (4.4) since
\[
\| \mathcal{M}(\Phi f) \|_{L^2^*(\mathbb{R})} = \| rF(r) \|_{L^2^*(\mathbb{R}^+; d\mu)},
\]
From Young’s inequality we have for any \( \varepsilon > 0 \) that
\[
n[\varepsilon/(n - 1)]^{1 - 1/n}ab \leq a^n + \varepsilon b^{n/(n-1)}.
\]
On applying this to (4.11) we get
\[
\varepsilon^{1 - 1/n} \| rF(r) \|_{L^2^*(\mathbb{R}^+; d\mu)}^2 \leq C \{ \| \nabla h \|_{L^2(\mathbb{R}^n)}^2 - \left( \frac{n - 2}{2} \right)^2 \| h/\cdot \|_{L^2(\mathbb{R}^n)}^2 \} \frac{1}{n}.
\]
This yields (4.12) on setting \( \varepsilon = n^2/4 - \delta \).

\[\square\]

**Corollary 4.5.** (i) Let \( \nabla h \in L^2(\mathbb{R}^n), n \geq 3, \) and
\[
\sup_{\omega \in \mathbb{S}^{n-1}} \| h(\cdot, \omega)/\cdot \|_{L^2(\mathbb{R}^n)}^2 < \infty.
\]
Then
\[
\| h \|_{L^2^*(\mathbb{R}^n)}^2 \leq C \left\{ \| \nabla h \|_{L^2(\mathbb{R}^n)}^2 - \left( \frac{n - 2}{2} \right)^2 \| h/\cdot \|_{L^2(\mathbb{R}^n)}^2 \right\}^{1/n} \times \sup_{\omega \in \mathbb{S}^{n-1}} \left\{ \| h(\cdot, \omega)/\cdot \|_{L^2(\mathbb{R}^n)}^2 \right\}^{1 - 1/n}.
\]

(ii) If \( h, \nabla h \in L^2(\mathbb{R}^n) \) then, with \( H := \mathcal{M}(h), \)
\[
\| H \|_{L^2^*(\mathbb{R}^+, d\mu)}^2 \leq C \left\{ \| \nabla h \|_{L^2(\mathbb{R}^n)}^2 - \left( \frac{n - 2}{2} \right)^2 \| h/\cdot \|_{L^2(\mathbb{R}^n)}^2 \right\}^{1/n} \times \left\{ \| h/\cdot \|_{L^2(\mathbb{R}^n)}^2 \right\}^{1 - 1/n}.
\]

For \( 0 \leq \delta < (n - 2)^2/4 \), we have
\[
\| H \|_{L^2^*(\mathbb{R}^+, d\mu)}^2 \leq C \left( (n - 2)^2/4 - \delta \right)^{(n-1)/n} \left\{ \| \nabla h \|_{L^2(\mathbb{R}^n)}^2 \right\} - \delta \| h/\cdot \|_{L^2(\mathbb{R}^n)}^2 \right\}.
\]

**Proof.** Since \( n \geq 3 \), we have that \( f := h/\cdot \in L^2(\mathbb{R}^n) \). We claim that \( Lf \in L^2(\mathbb{R}^n) \). For
\[
|\nabla(|x|f)|^2 = \left| \frac{x}{|x|}f + |x|\nabla f \right|^2
= |f|^2 + (|x||\nabla f|)^2 + 2\text{Re}[\bar{f}(x \cdot \nabla)f]
\]
and, on integration by parts, initially for \( f \in C_0^\infty(\mathbb{R}^n) \) and then by the usual continuity argument,
\[
\int_{\mathbb{R}^n} \bar{f}(x \cdot \nabla)f \, dx = \sum_{j=1}^n \int_{\mathbb{R}^n} x_j \bar{f} \frac{\partial f}{\partial x_j} \, dx
= -\sum_{j=1}^n \int_{\mathbb{R}^n} f \left( \bar{f} + x_j \frac{\partial \bar{f}}{\partial x_j} \right) \, dx
= -\int_{\mathbb{R}^n} \{ n|f|^2 + f(x \cdot \nabla)f \} \, dx.
\]
This gives
\[ 2\text{Re} \int_{\mathbb{R}^n} [\bar{f}(\mathbf{x} \cdot \nabla)f]d\mathbf{x} = -n \int_{\mathbb{R}^n} |f|^2d\mathbf{x} \]
and hence
\[ \int_{\mathbb{R}^n} |\nabla(|f|)|^2d\mathbf{x} = \int_{\mathbb{R}^n} (|x||\nabla f|)^2d\mathbf{x} - (n - 1) \int_{\mathbb{R}^n} |f|^2d\mathbf{x} \geq \int_{\mathbb{R}^n} |Lf|^2d\mathbf{x} - (n - 1) \int_{\mathbb{R}^n} |f|^2d\mathbf{x} \] \hspace{1cm} (4.16)
which confirms our claim. On substituting (4.16) and \( f = h/| \cdot | \) in (4.10), we get
\[ \|h\|^2_{L^2(\mathbb{R}^n)} \leq C \left\{ \|\nabla h\|^2_{L^2(\mathbb{R}^n)} + (n - 1)\|h/| \cdot ||^2_{L^2(\mathbb{R}^n)} - (n^2/4)\|h/| \cdot ||^2_{L^2(\mathbb{R}^n)} \right\}^{1/n} \sup_{\omega \in \mathbb{S}^{n-1}} \|h/| \cdot ||^{2(1-1/n)}_{L^2(\mathbb{R}^n; d\mu)} \]
which yields (4.13); (4.14) follows similarly from (4.11) and (4.14) yields (4.15).

If in (4.6) \( g = \Phi f \), where \( f \) is supported in the annulus \( A_R := \{ \mathbf{x} \in \mathbb{R}^n : 1/R \leq |\mathbf{x}| \leq R \} \), then \( G \) is supported in the interval \([ - \ln R, \ln R ] \) and we have as in the proof of Corollary 4

**Corollary 4.6.** Let \( f \in C_0^\infty(A_R) \). Then, with \( F := M(f) \),
\[ \|rF(r)\|^2_{L^2_\ast(\mathbb{R}^n; d\mu)} \leq C(\ln R)^{2(n-1)/n} \left\{ \|Lf\|^2_{L^2(\mathbb{R}^n)} - \frac{n^2}{4} \|f\|^2_{L^2(\mathbb{R}^n)} \right\} . \] \hspace{1cm} (4.17)

On putting \( f = h/| \cdot | \) in (4.17) and using (4.16), we have

**Corollary 4.7.** Let \( h \in C_0^\infty(A_R) \). Then, with \( H := M(h) \),
\[ \|H\|^2_{L^2_\ast(\mathbb{R}^n; d\mu)} \leq C(\ln R)^{2(n-1)/n} \left\{ \|\nabla h\|^2_{L^2(\mathbb{R}^n)} - \frac{(n-2)^2}{4} \|h/| \cdot ||^2_{L^2(\mathbb{R}^n)} \right\} . \]

Finally we have the following \( p = 2 \) case of Corollary 3(ii).

**Corollary 4.8.** Let \( 2 < q < \infty \) and \( m = q/2 - 1 \). Then, if \( f \) is such that \( f, Lf \in L^2(\mathbb{R}^n) \) and \( \int_{\mathbb{S}^{n-1}} |f(\mathbf{s}, \omega)|^m\mathbf{s}^{q(n-1)/2}d\mathbf{s}d\omega < \infty \), we have that \( \int_{\mathbb{R}^+} |F(s)|^q\mathbf{s}^{q(n-1)/2}dsd\omega < \infty \) and
\[ \int_{\mathbb{R}^+} |F(s)|^q\mathbf{s}^{q(n-1)/2}ds \leq C \left\{ \|Lf\|^2_{L^2(\mathbb{R}^n)} - \frac{n^2}{4} \|f\|^2_{L^2(\mathbb{R}^n)} \right\}^2 \times \left\{ \int_{\mathbb{R}^+} \int_{\mathbb{S}^{n-1}} |f(\mathbf{s}, \omega)|^m\mathbf{s}^{q(n-1)/2}d\mathbf{s}d\omega \right\}^2 \]
Proof. Corollary 4.3(ii) with $p = 2$ yields
\[
\|\mathcal{M}(\Phi f)\|_{L^q(R^n)} \leq C \left\{ \|Lf\|_{L^2(R^n)}^2 - \frac{n^2}{4} \|f\|_{L^2(R^n)}^2 \right\}^{2/q} \times \|\Phi f\|_{L^m(R \times S^{n-1})}^{1-2/q}.
\]
Since
\[
\|\mathcal{M}(\Phi f)\|_{L^q(R^n)}^q = \int_{R^+} |F(s)|^q s^{\left(\frac{m}{2} - 1\right)} ds
\]
and
\[
\|\Phi f\|_{L^m(R \times S^{n-1})}^m = \int_{R^+} \int_{S^{n-1}} |f(s, \omega)|^m s^{\left(\frac{2m}{r} - 1\right)} ds d\omega
\]
the corollary follows. 

Acknowledgements: The second author (WDE) gratefully acknowledges the hospitality and support of the Isaac Newton Institute, University of Cambridge, during June, 2007, when some of this work was done. The third author (DH) thanks the US National Science Foundation for financial support from grant DMS-0400940.

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