Disney, S.M., Towill, D.R. and Van de Velde, W., (2004) "Variance amplification and the golden ratio in production and inventory control", International Journal of Production Economics, Vol. 90, No. 3, pp295–309.

# Variance amplification and the golden ratio in production and inventory control

S.M. Disney[#], D.R. Towill[#] and W. van de Velde[~]

[#] Logistics Systems Dynamics Group, Cardiff Business School, Cardiff University, Aberconway Building, Colum Drive, Cardiff, CF10 3EU, UK.

DisneySM@cardiff.ac.uk, ScottD1@cardiff.ac.uk (DR Towill),

Tel +44(0)29 2087 6083, Fax +44(0)29 2087 4301

[~]Procter and Gamble Services Company NV, Temselaan 100, B-1853 Strombeek-Bever, Belgium. E-mail: vandevelde.wm@pg.com.

## *Abstract*

A discrete linear control theory model of a generic model of a replenishment rule is presented. The replenishment rule, which we term a "Deziel Eilon - Automatic Pipeline, Inventory and Order Based Production Control System" (DE-APIOBPCS), is guaranteed to be stable. From a z-transform model of the policy, an analytical expression for bullwhip is derived that is directly equivalent to the common statistical measure often used in simulation, statistical and empirical studies to quantify the bullwhip effect. This analytical expression clearly shows that we can reduce bullwhip by taking a fraction of the error in between the target and actual inventory and pipeline (or Work In Progress or "orders placed but not yet received") positions. This is in contrast to the common situation where ordering policies account for all of the error every time an order is placed. Furthermore, increasing the average age of the forecast reduces bullwhip, as does reducing the production / distribution lead-time. We then derive an analytical expression for inventory variance using the same procedure to identify the closed form bullwhip expression.

We assume that a suitable objective function is linearly related to the bullwhip and inventory variance amplification ratios and then optimise the PIC system for different weightings of order rate and inventory level variance. We highlight two forms of the objective function, one where "the golden ratio" can be used to determine the optimal gain in the inventory and WIP feedback loop and another that allows the complete range of possible solutions to be visualised. It is interesting that the golden ratio, which commonly describes the optimum behaviour in the natural world, also describes the optimal feedback gain in a production and inventory control system.

## Key words

Bullwhip, inventory variance, production and inventory control, the golden ratio

## Introduction

This paper is concerned with the design of a linear production and inventory control system with the use of a z-transform model of a generic replenishment rule. The design of a production and inventory control (PIC) system is a worthy pursuit as this system is a major contributing factor to the dynamics of, and costs in, a production and/or distribution system. An inappropriately designed PIC system can result in huge fluctuations in the order rate propagating up the supply chain. Recently academics have termed this phenomenon as the "Bullwhip Effect" (Lee, Padmanabhan and Whang 1997a and b) and there are many real world supply chain examples in the literature, for example Holmström (1997) evaluates bullwhip in a retail supply chain.

Lee, Padmanabhan and Whang (1997a and b) posit that there are five fundamental causes of the bullwhip effect. These are non-zero lead-times, demand signal forecasting, order batching, gaming and promotions. Here we are concerned only with the first two causes. We explicitly neglect the effect of order batching, gaming and special promotions, thus our examinations here may be considered to be of a "lower bound" nature. That is, our bullwhip expressions are to be considered to be their best possible values, to which the other sources are to be added or preferably eliminated at source. This is exactly the same approach as that adopted by Chen, Drezner, Ryan and Simchi-Levi (2000) and many other contributions to tackling this problem.

The Lee, Padmanabhan and Whang (1997a and b) bullwhip papers have become very popular. They clearly state the nature of the problem in a very insightful manner, but somewhat surprisingly they fail to highlight known solutions to the bullwhip problem. These have been known for a very long time: specifically we refer to the work of Deziel and Eilon (1967), Sterman (1989), Wikner, Naim and Towill (1992), John, Naim and Towill (1994) although there are numerous other examples, arguably as far back as the work of Nobel prize winner for economics in 1978, Herbert Simon (1952) and Magee (1958). In the past the "non-zero lead-times" and "demand signal forecasting" have been rightly called the Forrester Effect after Jay Forrester (1961) and may be considered to be the fundamental structure of the ordering decision. "Order batching", has been called the Burbidge Effect, (Towill 1997) after its first prominent investigator. A pragmatic approach to "Gaming" was very elegantly described by Houlihan (1987).

The bullwhip effect has received much attention as it creates a business environment that can significantly add unnecessary costs, Metters (1997). Chen, Drezner, Ryan and Simchi-Levi (2000) have recently been using the ratio of the long-term variance of **OR**ders (OR) over the long-term variance of **D**emand (D) as a measure of the Bullwhip effect. ***Note that we consider here the long-term variance ratio, obtained in the limit when time, n, tends to*** $\infty$. This metric has been given the name here of the "Variance Ratio" (VR) and is described by Eq 1.

$$Bullwhip = VR_{OR} = \frac{\sigma_{OR}^2}{\sigma_D^2} \tag{1}$$

This metric can be applied to a single ordering decision or echelon in a supply chain, Disney and Towill (2003), or across many echelons in the supply chain, Dejonckheere, Disney, Lambrecht and Towill (2003a). When $VR_{OR}$>1 then we have bullwhip, when $VR_{OR}$ < 1 then we have smoothing and when $VR_{OR}$=1 we may (but not necessarily so) have a pure order-up-to or “pass-on-orders” policy. We use the subscript to denote the application of the $VR_{OR}$ to different parts of the system. For Eq 1 this is the **OR**der rate variance over the **D**emand variance. However we will also investigate the ratio between the **N**et **S**tock variance and Demand, i.e. the inventory variance.

In this paper we will highlight the fact that it is possible to avoid the bullwhip effect as defined by Eq 1 via the proper design of the production and inventory control system. Our approach draws on control theory and z-transform techniques. Transforms are particularly powerful as they allow the analyst to avoid complicated convolution in the time domain and instead to use simple vector manipulation in the complex frequency domain. There are also a number of fundamental mathematical theorems associated with both the Laplace and the z-transform available for the analyst to draw upon to investigate a particular model, some of which we will exploit here. We have elected to execute our analysis in discrete time because this exactly replicates the conditions of the Beer Game, many operational research models and much of industrial practice.

Here we consider the cause of the class of bullwhip known as the Forrester Effect. We will show that, by slightly altering the way in which we incorporate feedback on inventory levels and Work In Progress (WIP) into the order being placed, we can actually eliminate the bullwhip problem, i.e. create a supply chain where the variance of the orders placed decreases as the order proceeds up the supply chain. We will also investigate the link between bullwhip and inventory variance. Conceptually, it is convenient to think of inventory variance and order variance as a trade-off. For example, at one extreme, follow demand exactly and hold minimum inventory (that is pass-on-orders) or in the other extreme, absorb the demand fluctuations in inventory and keep a level order rate (that is level schedule). However, because of lead-times, the trade-off is not that simple to evaluate. Furthermore, when placing the order, the best policy may be to absorb some of the fluctuations in inventory and follow some of the variation in demand. We refer to Towill, Lambrecht, Disney and Dejonckheere (2001) for a more conceptual discussion on this trade-off.

## *The production and inventory control policy*

This paper exploits a generic replenishment rule for controlling orders in a supply chain. We term this rule APIOBPCS or an Automatic Pipeline, Inventory and Order Based Production Control System after John, Naim and Towill (1994) who first placed the model into the IOBPCS family database (Towill, 1982). The importance of this latter reference is that it includes examples of best practice taken from a range of applications. This rule is not new; many versions of it can be seen in the literature, for example see Deziel and Eilon (1967) and Simon (1952). Hence, placing it into the IOBPCS database allows us to readily access previously known research results. APIOBPCS can be expressed in words as follows;

“Let the production targets (or replenishment orders) be equal to the sum of; average demand (exponentially smoothed over *Ta* time units), a fraction (1/*Ti*) of the inventory difference in actual Net Stock compared to target net stock and the same

fraction ($1/Tw$) of the difference between target Work In Progress (WIP) and actual WIP".
This generic replenishment rule is particularly powerful as it encompasses:

- the way people actually play the Beer Game, Sterman (1989), Naim and Towill (1995) and Riddalls and Bennett (2002)
- the capability of representing Lean, Suzaki (1987), Agile and Leagile, Naylor, Naim and Berry (1999) supply chains, Towill, Lambrecht, Disney and Dejonckheere (2001)
- a general case of order-up-to policies and many variants of it, Dejonckheere, Disney, Lambrecht and Towill (2003b)
- an approximation to (and an improvement on) the HMMS algorithm, Dejonckheere, Disney, Lambrecht and Towill (2003c)
- industrial usage by several of our industrial partners, for example by the real company with a fictitious name (WMC- World Class Manufacturer) reported in Lewis, Naim and Towill (1997)

We start our discussion of the design of a PIC system from the block diagram (Nise 1995) shown in Figure 1. We will not discuss the model building activities used to generate this z-transform block diagram of the replenishment rule due to the need for brevity; we refer to Disney and Towill (2002) and Disney (2001) for more details on this aspect.

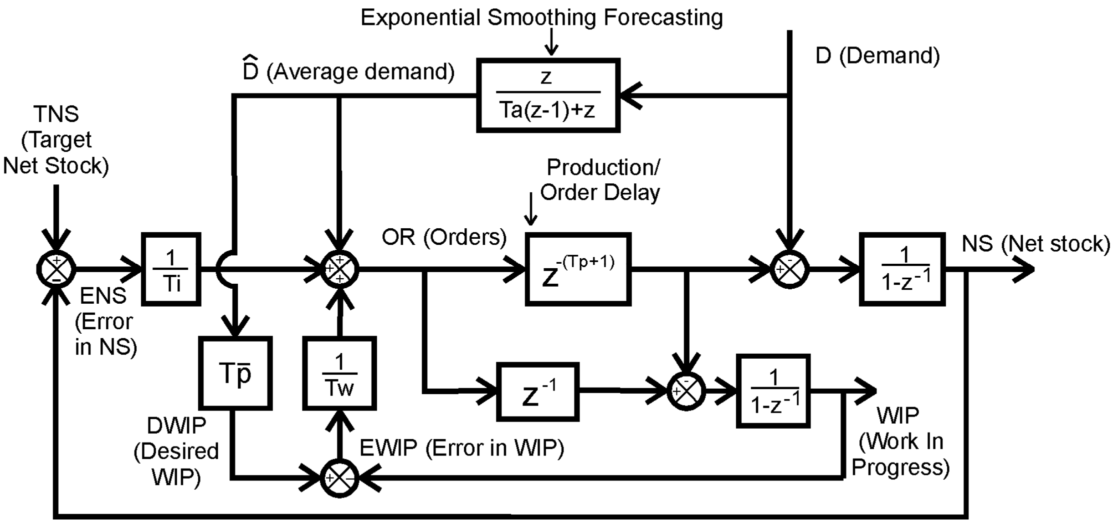


**Figure 1. Block diagram of APIOBPCS**

After manipulating the block diagram, we gain the following transfer functions (for the **OR**der rate and **N**et **S**tock levels as a function of the **D**emand) that completely describe the linear dynamic behaviour of our PIC system. Note that we will refer to a transfer function by capital letters followed by ($z$) and its time domain image by lower case letters followed by ($n$).

$$F_{OR}(z) = \frac{OR(z)}{D(z)} = a_1 \frac{z^{1+Tp}(z-a_2)}{(z-a_3)(z^{Tp}(z-a_4)-a_5)} \tag{2}$$

where;

$$a_1 = \frac{\overline{Tp}Ti + Tw + TaTw + TiTw}{(1+Ta)TiTw}$$

$$a_2 = \frac{\overline{Tp}Ti + TaTw + TiTw}{\overline{Tp}Ti + TaTw + TiTw + Tw} = 1 + \frac{Tw}{\overline{Tp}Ti + TaTw + TiTw + Tw}$$

$$a_3 = \frac{Ta}{1+Ta}$$

$$a_4 = 1 - \frac{1}{Tw}$$

$$a_5 = \frac{Ti - Tw}{TiTw} = \frac{1}{Tw} - \frac{1}{Ti}$$

and

$$F_{NS} = \frac{NS(z)}{D(z)} = \frac{z}{z-1}\frac{a_7(z-a_6)}{(z-a_3)\left(z^{Tp}(z-a_4)-a_5\right)} \tag{3}$$

where;

$$a_6 = \frac{Ta + \overline{Tp} + Tw}{1 + Ta + \overline{Tp} + Tw}$$

$$a_7 = \frac{1 + Ta + \overline{Tp} + Tw}{Tw(1+Ta)}$$

Inspection of Eq 2 and 3 allows us to reflect on the dimensions of the solution space. We have five parameters that describe the systems behaviour. These are;

- *Tp*, the production or distribution lead-time, is modelled here as a pure time delay. *Tp* is a positive real integer or zero,
- $\overline{Tp}$, our estimate of the average production lead-time, a positive real number or zero,
- *Ta*, the average age of the exponential forecast. Note, that we update our forecast every time period based on recent history, i.e. no knowledge of future events is required and we use this new forecast, not an old one. It is the average age of the data used in the forecast. *Ta* is a real number greater than -0.5 as this guarantees a stable response, Brown (1963).
- *Ti*, the fraction of the discrepancy between target net stock and the actual net stock that is incorporated into the order. *Ti* is a real number usually greater than 0.5.
- *Tw*, the fraction of the discrepancy between target WIP levels and the actual WIP levels that is incorporated into the order. *Tw* is a real number usually greater than 0.5.

However, the problem is not as complicated as it appears at first sight. We can legitimately reduce the number of dimensions of the solution space via the following steps;

- Setting $\overline{Tp}$ equal to $Tp$. That is, in the physical situation we assume that our estimate of the production lead-time is actually correct. This is appropriate as John, Naim and Towill (1994) have shown that this is required in order to ensure inventory levels "lock-on" to their target values in response to step change in demand. They use the continuous time Laplace transforms for this analysis, but the same argument still holds in discrete time with z-transforms as shown in Disney (2001).
- Ensuring, via good design, that the system is stable. "Is the system stable?" is a fundamental question to ask with regard to the performance of a dynamic system. By a stable system is meant a system that will react to a disturbance in the input signal in a controlled manner, and after some time (with no input) will return to its initial conditions. By contrast an unstable system will oscillate, with ever increasing amplitude, in response to any disturbance in the input signal over time. Or it will immediately grow exponentially without bound. Thus it is essential that a PIC is fundamentally stable.

  It is well known that a system is stable if all its (often complex) roots lie within the unit circle in the z-domain, Jury (1964). It is possible to determine algebraically the nessecary conditions needed to guarantee stability. Disney and Towill (2002) have presented a procedure to identify that the stability boundary for APIOBPCS with a particular (pure) time delay in discrete time. Riddalls and Bennett (2002) have recently done similar work in continuous time. It can also be shown that for a given production lead-time ($Tp$) the stability of the system only depends on $Ti$ and $Tw$. Furthermore, setting $Tw$ to the same value as $Ti$ (that is recover WIP errors at the same rate as inventory errors) will always produce a stable dynamic response.

Thus, by setting $\overline{Tp} = Tp$ and $Tw=Ti$, we reduce the solution space from five to three dimensions, and have a system that has some extremely desirable properties. When we set $Tw=Ti$, we use the term DE-APIOBPCS to describe the systems structure. We use this term to acknowledge the fundamental breakthrough made by Deziel and Eilon (1967). They studied a variant of APIOBPCS (with a slightly different order of events that unfortunately resulted in inventory drift, which we have avoided here) when $Tw=Ti$. This sufficient condition for stability, $Tw=Ti$, can be readily seen from the characteristic equation

$$(z-a_3)\left(z^{Tp}(z-a_4)-a_5\right)=0 \tag{4}$$

in which $a_3$ is always less then unity, $a_5$ becomes zero when $Tw=Ti$ and $a_4$ is less than unity if $Ti=Tw>0.5$, putting all the roots of the characteristic equation inside the unit circle. *Thus the DE-APIOBPCS solutions (APIOBPCS with Tw=Ti) are guaranteed to be stable. Furthermore, it is known that time domain responses of DE-APIOBPCS will only contain exponential terms, which means we will avoid costly oscillations in the order rate (when $Ti \geq 1$).* This is a very desirable property to have in a PIC system. Thus in the rest of the paper we will focus exclusively on the Deziel-Eilon solutions to the APIOBPCS model.

## *Calculating the variance ratios*

We wish to understand how our ordering policy will react to a demand pattern that is an independently and identically distributed stationary stochastic variable. Disney and Towill (2003) have explored the relationship between the "long-run" variance ratio measure, noise bandwidth and the sum of the squares of the system's impulse response using the fundamental relationships identified by Tsypkin (1964) and previously exploited in a simulation based approach for production and inventory control by Deziel and Eilon (1967). A summary of Tsypkin's relationships between;

- the variance ratio,
- its statistical definition,
- the system transfer function via the area under the systems squared frequency response $F(j\omega)$; where $j$ is the imaginary number $\sqrt{-1}$ and $\omega$ is frequency of the input,
- the noise bandwidth $W_N$,
- and the sum of the systems squared impulse response, $f^2(n)$, in the time domain,

under the assumption that the demand (input) is an independent and identically distributed random variable (or pure white noise in control engineering terms), is as follows, Tsypkin (1964);

$$VR = \frac{\sigma^2_{Output}}{\sigma^2_{Input}} = \frac{1}{\pi}\int_0^{\pi} \left|F(j\omega)\right|^2 d\omega = \frac{W_N}{\pi} = \sum_{n=0}^{\infty} f^2(n) \tag{5}$$

Thus *the sum of the square of the discreta of the system order rate impulse response in the time domain is equal to the common bullwhip measure*. For simplicity, we also assume, without loss of generality, that the variance of the demand signal is unity.

The variance ratio expression can also be arrived at by other methods. For example, as Grubbström and Andersson (2002) have highlighted, the variance ratios may also be obtained by using Cauchy's contour integral. Alternatively summing all of the residues inside the unit circle also yields the variance ratio expressions. They show that it does not matter which distribution the random variable is drawn from. Grubbström and Andersson (2002) also highlight how the variance builds up over time using the z-transform multiplication theorem.

A much more pragmatic approach is to simply create a spreadsheet model of the ordering policy and define the input as an impulse and sum the square of the order rate to arrive at an accurate measure bullwhip in very little computational time. In a "bullwhip explorer" (Lambrecht and Dejonckheere, 1999a and b) this is a far easier model to explore than a model with stochastic demands.

### The bullwhip ratio

Our procedure for determining the bullwhip expression is as follows. Departing from the Order Rate transfer function (Eq 2), we set $\overline{Tp} = Tp$ and $Tw$=$Ti$ in order to reduce the complexity of the mathematics and ensure our system has some nice properties (such as stability) as discussed earlier. Under these conditions the constants in Eq 2 become;

$$a_1 = \frac{1 + Tp + Ta + Ti}{Ti(1 + Ta)}$$

$$a_2 = \frac{Tp + Ta + Ti}{1 + Tp + Ta + Ti}$$

$a_5$=0

which immediately simplifies Eq 2 to;

$$F_{OR}(z) = a_1 \frac{z(z - a_2)}{(z - a_3)(z - a_4)} \tag{6}$$

We then take the inverse z-transform to obtain the following time domain impulse response,

$$f_{OR}(n) = \frac{1}{(1 + Ta)(1 + Ta - Ti)Ti}\left((1 + Ta)\left(\frac{-1 + Ti}{Ti}\right)^n (1 + Ta + Tp) - \left(\frac{Ta}{1 + Ta}\right)^n Ti(Ti + Tp)\right) \tag{7}$$

And finally we exploit Tsypkin's relation (Eq 5) to yield the bullwhip expression,

$$\begin{aligned} Bullwhip = VR_{OR} = \frac{\sigma_{OR}^2}{\sigma_D^2} = \sum_{n=0}^{\infty} f_{OR}{}^2(n) = \\ \frac{2Ta^2 + 3Ti + 2Tp + 2(Ti + Tp)^2 + Ta(1 + 6Ti + 4Tp)}{(1 + 2Ta)(Ta + Ti)(-1 + 2Ti)} = \\ \frac{1}{2}\left(\frac{\left(Tp + \frac{1}{2}\right)^2}{\left(Ta + \frac{1}{2}\right)\left(Ti - \frac{1}{2}\right)(Ta + Ti)} + \frac{2Tp + 1}{\left(Ta + \frac{1}{2}\right)\left(Ti - \frac{1}{2}\right)} + \frac{1}{Ti - \frac{1}{2}} + \frac{1}{Ta + \frac{1}{2}} + \frac{1}{Ta + Ti}\right) \end{aligned} \tag{8}$$

Inspection of Equation 8 shows

- that to reduce the Bullwhip Effect the production lead-time should be made as small as possible, as it only occurs in the numerator with positive coefficients. This result verifies the value of the Time Compression Paradigm, Towill (1996).
- bullwhip is monotonically increasing in *Tp*. This can be seen from the expanded bullwhip expression, where for *Ti*>0.5 the coefficients of the two monotonically increasing expression in *Tp* are positive.
- bullwhip is symmetrical about *Ta*=*Ti*-1 and *Ti*=*Ta*+1 as exchanging *Ta*+1/2 for *Ti*-1/2 and vice versa changes nothing.
- bullwhip is monotonically decreasing in *Ta* and *Ti* as all five terms are monotonically decreasing in *Ta* and/or *Ti*.
- furthermore, there is only one minimum (of zero) which is at *Ta*=*Ti*=$\infty$, the level scheduling case.

Figure 3 enumerates Eq 8

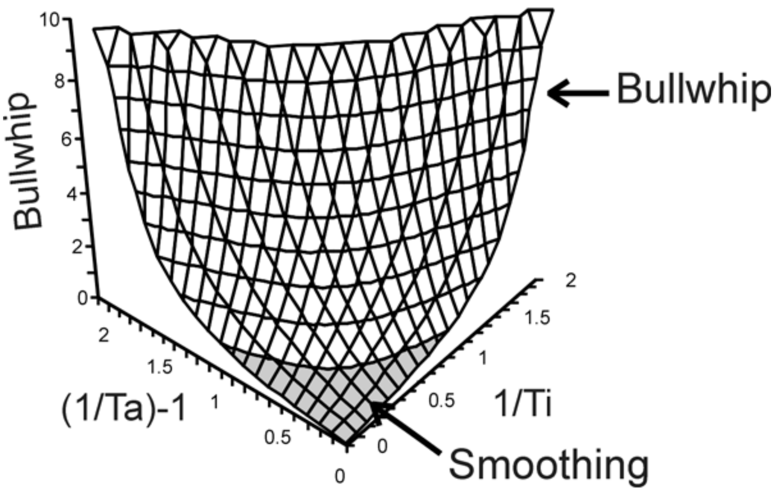


**Figure 2. Bullwhip produced when *Tp*=3**

Figure 2, confirms that bullwhip does reduce when *Ta* and *Ti* increase. Recognising that variance amplification exists if the $VR_{OR}>1$ it is easy to show, by solving $VR_{OR}=1$ for *Ti* and selecting the relevant root, that the following boundary exists for the bullwhip effect;

$$\text{Bullwhip exists if } Ti < \frac{\left(\begin{array}{l} 2+3Ta-2Ta^2+2Tp+\sqrt{1+2Ta} \\ \sqrt{\begin{array}{l} 4+4Ta+Ta^2+2Ta^3 \\ +8Tp+4TaTp+4Tp^2 \end{array}} \end{array}\right)}{4Ta} \tag{9}$$

This boundary may be shown graphically as in Figure 3 where the region of smoothing means $VR_{OR}<1$, and the region of Bullwhip means $VR_{OR}>1$.

## The inventory variance ratio

The variance of the net stock or inventory levels may be determined as follows. Departing from the actual inventory transfer function (Eq 3) and replacing $T\overline{p}=Tp$ and $Tw=Ti$ (so $a_7=a_1=\dfrac{1+Tp+Ta+Ti}{Ti(1+Ta)}$, $a_6=a_2=\dfrac{Tp+Ta+Ti}{1+Tp+Ta+Ti}$ and $a_5=0$) we have;

$$F_{NS}(z)=\frac{NS(z)}{D(z)}=\frac{z}{z-1}\left(\frac{a_1(z-a_2)zz^{-(Tp+1)}}{(z-a_3)(z-a_4)}-1\right)=\frac{z}{z-1}\left(F_{OR}z^{-(Tp+1)}-1\right) \tag{10}$$

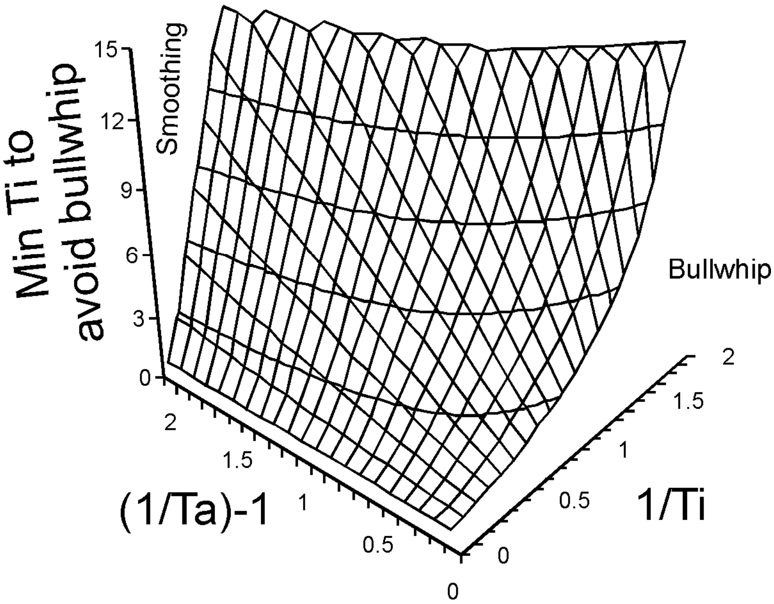


**Figure 3. The Bullwhip boundary for DE-APIOBPCS**

Here we can see the structure of the time domain response, $f_{NS}(n)$. There are two terms. The first is the inverse transform of an accumulation of the Order Rate (Eq 6), delayed by $Tp$+1 time units. The second term is a negative unit step function beginning at time $n$=0. The inverse transform of the accumulated Order Rate, the first term, $\frac{z}{z-1}\frac{a_1(z-a_2)z}{(z-a_3)(z-a_4)}$, after inserting the relevant parameter values for $a_1$, $a_2$, $a_3$, and $a_4$, is;

$$f(n)=1+\frac{\left(\frac{Ta}{1+Ta}\right)^{1+n}(Ti+Tp)-\left(\frac{Ti-1}{Ti}\right)^{1+n}(1+Ta+Tp)}{1+Ta-Ti} \tag{11}$$

after accounting for the delay function ($z^{-(Tp+1)}$), (11) becomes

$$f(n)=\begin{cases}0,\ for\ n\le Tp\\ 1+\frac{\left(\frac{Ta}{1+Ta}\right)^{n-Tp}(Ti+Tp)-\left(\frac{Ti-1}{Ti}\right)^{n-Tp}(1+Ta+Tp)}{1+Ta-Ti},\ for\ n>Tp\end{cases} \tag{12}$$

This accounts for the first term of Eq 10 and adding in the negative unit step yields;

$$f_{NS}(n)=\begin{cases}-1,\ for\ n\le Tp\\ \frac{\left(\frac{Ta}{1+Ta}\right)^{n-Tp}(Ti+Tp)-\left(\frac{Ti-1}{Ti}\right)^{n-Tp}(1+Ta+Tp)}{1+Ta-Ti},\ for\ n>Tp\end{cases} \tag{13}$$

Thus we may build up the long run variance amplification ratio between the Net Stock and Demand as follows;

$$VR_{NS} = \sum_{n=0}^{Tp}(-1)^2 + \sum_{n=Tp+1}^{\infty}\left(\left(\left(\frac{Ta}{1+Ta}\right)^{n-Tp}(Tp+Ti)-(Tp+1+Ta)\left(\frac{-1+Ti}{Ti}\right)^{n-Tp}\right)/(1+Ta-Ti)\right)^2 \tag{14}$$

(14) converges to the following closed form;

$$VR_{NS} = 1+Tp+\frac{\begin{pmatrix}2Ta^2(Ti-1)^2+Ti(1+Tp)^2\\+Ta(1+Tp)(1+(2Ti-1)Tp)\end{pmatrix}}{(1+2Ta)(Ta+Ti)(2Ti-1)} = Tp+\frac{2(1+Ta)^2Ti^2-TaTp^2+TiTp(2+Tp+2Ta(1+Tp))}{(1+2Ta)(Ta+Ti)(2Ti-1)} \tag{15}$$

Inspection of Eq 15 shows that;

- the inventory variance is always greater than 1 since *Ti*>0.5, i.e. inventory levels will always vary more than the demand signal
- as the coefficients of *Tp* are positive, the inventory variance increases as the production lead-time increases
- inventory variance is symmetrical about *Ta*=*Ti*-1 and *Ti*=*Ta*+1
- numerical investigations reveal that minimum inventory variance occurs when $Ta=\infty$ and *Ti*=1 or when *Ta*=0 and $Ti=\infty$

Eq 15 may be shown graphically as in Figure 4.

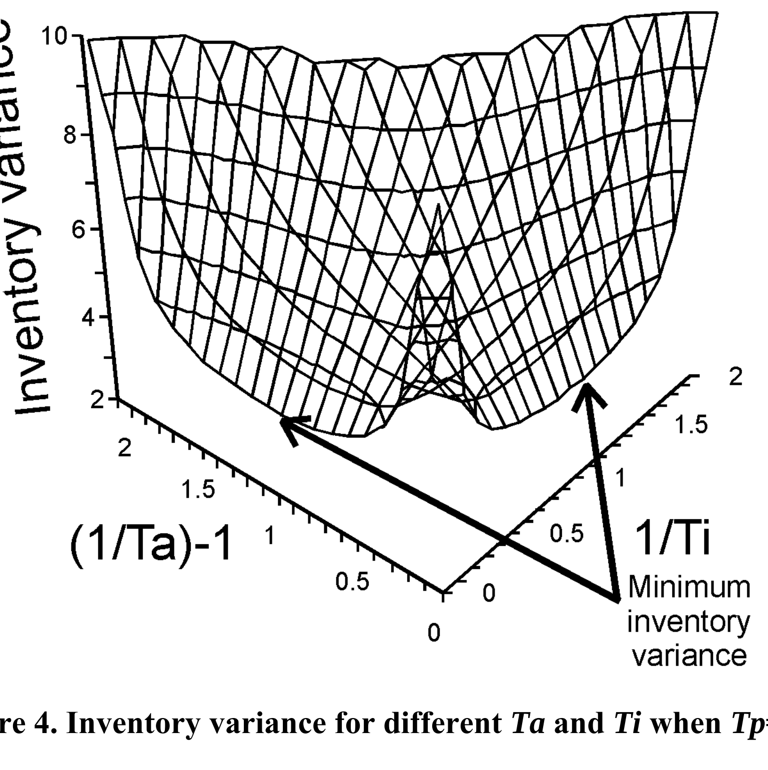


**Figure 4. Inventory variance for different *Ta* and *Ti* when *Tp*=3**

## *The Golden Ti*

The variance ratios we have identified in the previous sections are particularly useful. For example the inventory variance is often used inside probability density functions in inventory literature to investigate aspects such as the expected inventory holding and backlog costs per period, Zipkin (2000). Alternatively the probability density function may be used to gain understanding of customer service metrics such as the "fill-rate" or "expected shortages", Disney, Farasyn, Lambrecht, Towill and van de Velde (2003).

The same probability density function approach can be used with the order rate variance expression to investigate measures such as the proportion of products manufactured in a 40-hour working week against those that have been subcontracted or produced in over-time working at a premium. This type of approach has recently been exploited by Disney and Grubbström (2003) and Chen and Disney (2003). However, most probability density functions are very complex and essentially non-algebraic. So although results obtained through them are completely analytical and exact they are often very hard to manipulate in further analysis. Hence, we elect here to concern ourselves with a cost function that solely consists of the variance ratios. In other words we assume here for simplicity that the costs in a particular situation are a linear function of the bullwhip and inventory variance expressions.

## Forecasting with minimum mean squared error

We have shown that the bullwhip is a monotonically decreasing function of *Ta* and *Ti*. We also know from enumeration of Eq 15 that the minimum inventory variance occurs when $Ta = \infty$ and *Ti*=1 or when *Ta*=0 and $Ti = \infty$. It is easy to realise that if we were to add these two variances together then the optimum values of *Ta* and *Ti* would be either $Ta = \infty$ and *Ti* would be something greater than 1 or *Ta* something greater than 0 and $Ti = \infty$. This is verified in Figure 5 for the case when *Tp*=3. As it is also well known that $Ta = \infty$ will also give the minimum mean squared error forecast of future demand (as this setting will give its conditional expectation), thus it makes sense to exploit this fact and simplify our bullwhip and inventory variance expressions as follows;

$$VR_{OR@Ta=\infty} = \frac{1}{2Ti-1} \tag{16}$$

$$VR_{NS@Ta=\infty} = \frac{Ti^2 - Tp + 2TiTp}{2Ti-1} = 1 + Tp + \frac{(Ti-1)^2}{2Ti-1} \tag{17}$$

These expressions have been previously presented by Dejonckheere, Disney, Farasyn, Janssen, Lambrecht, Towill and Van de Velde (2002) where these bullwhip and inventory variance expressions were coupled to the fill-rate customer service metric. They showed that 90% of bullwhip could be eliminated with only a quarter of a period's worth of inventory holding to maintain the fill-rate.

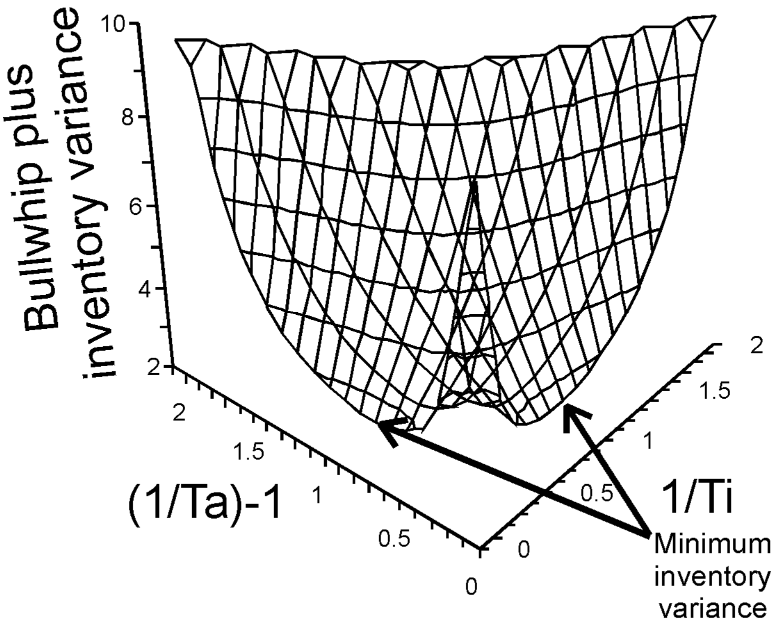


**Figure 5. Bullwhip plus inventory variance**

Now let us consider the following objective function;

$$OF = \left(wVR_{OR\,@\,Ta=\infty}\right) + \left(xVR_{NS\,@\,Ta=\infty}\right) = \frac{\left(Ti^2 - Tp + 2TiTp\right)x + w}{2Ti - 1} \tag{18}$$

Differentiating with respect to *Ti* gives us the gradient (23). Notice that the lead-time, *Tp*, has now dropped out of the equation.

$$\frac{dOF}{dTi} = \frac{2Ti\left(Ti - 1\right)x - 2w}{\left(1 - 2Ti\right)^2} \tag{19}$$

Solving for zero gradient and selecting the relevant root yields the optimum *Ti* to minimise the weighted sum of bullwhip and inventory variance. It is,

$$OptTi = \frac{\sqrt{x} + \sqrt{4w + x}}{2\sqrt{x}} \tag{20}$$

Now when $x$=1, so unit Net Stock variance is added to a weighted bullwhip variance, the optimal *Ti* to minimise the objective function is given by,

$$OptTi_1 = \frac{1 + \sqrt{1 + 4w}}{2} \tag{21}$$

that we will recognise as having the same form as the irrational fraction, "The Golden Ratio". So, the optimal *Ti* has many previously known mathematical properties, such as; $OptTi_1{}^2 = OptTi_1 + w$ and if $w$ is an integer then $OptTi_1{}^2$ has the same decimal points as $OptTi_1$. We find it interesting that the number that describes the optimal behaviour in the natural world also turns up in the design of a production planning and inventory control policy. Indeed the Golden Ratio, or the Golden Section as it is also called, is found in a large number of places, Knott (2003). Knott (2003), has developed and maintains a very extensive website on the subject and notes that the Golden ratio describes;

- the optimal placement of seeds, petals and leaves in growing plants
- the optimal ratio of male and female bees,
- the logarithmic spiral of a snail's shell,
- many aspects of trigonometry,
- many forms of architecture and art,

and is claimed to be the most irrational number as it has the simplest continued fraction, 1+(1/(1+(1/(1+(1/(1+(1/(….

For illustration we have simulated the "golden" response (i.e. *Ti* set to the Golden Ratio) to an i.i.d. random demand pattern when *Tp*=1. The frequency histograms refer to the simulation of 10,000 time periods. We can see that after 10,000 the statistical process is reasonably close to the theoretical values of Bullwhip (0.447) and inventory variance (2.171). Simulating for a longer time period will obviously reduce this error.

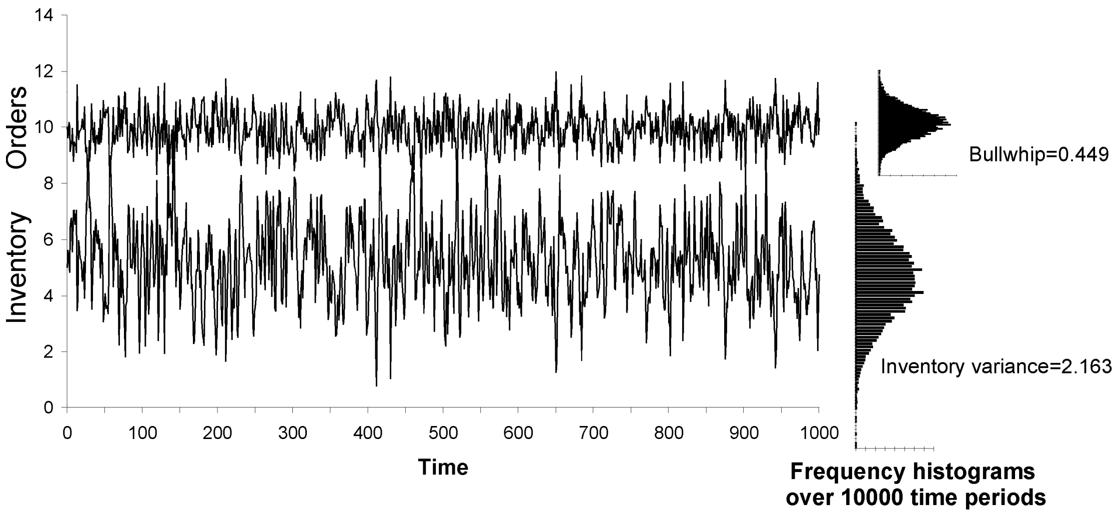


**Figure 6. Sample simulation of the "golden" solution**

## An alternative objective function

Another way to look at the objective function (Eq 18) is to consider the trade-off as a convex combination of a single weight, *w*. This may be achieved by replacing *x* with (1-*w*) in Eq (20), producing;

$$OptTi_2 = \frac{1}{2}\left(1+\frac{\sqrt{1+3w}}{\sqrt{1-w}}\right) = \frac{w-1-\sqrt{1+2w-3w^2}}{2w-2} \qquad (22)$$

This form is especially convenient because we can see the $OptTi_2$ across all weighting functions (and all lead-times and all values of *Ti*) as shown in Figure 7.

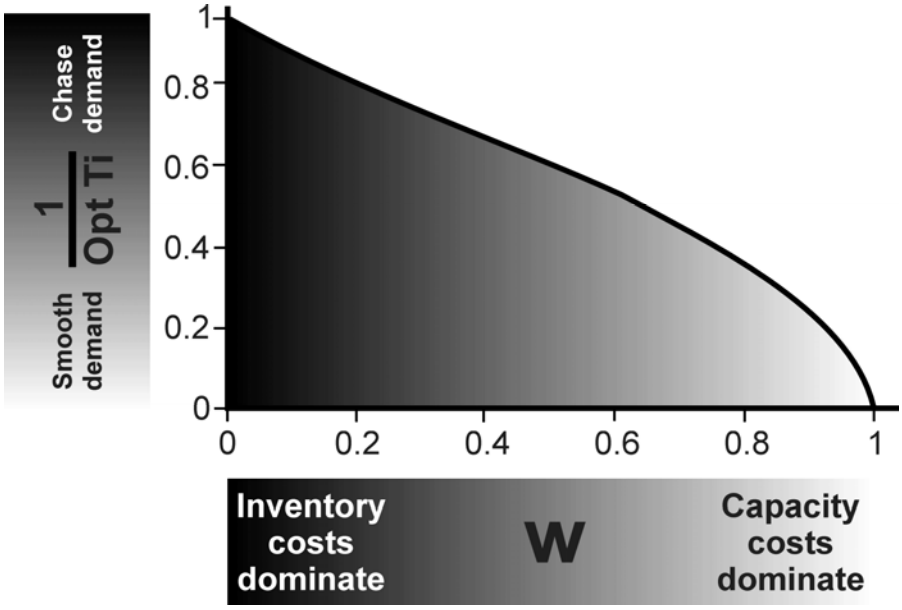


**Figure 7. Optimal *Ti* for different weights on objective function form 2**

From Figure 7 we can see that the optimal *Ti* remains low until $w$>0.9. After which the optimal *Ti* rises very quickly. Furthermore, plotting optimal *Ti* as a reciprocal illustrates the complete range of recommended *Ti* (from 1 to $\infty$). Figure 7 is also very nearly linear in the range $w$=0.2 to 0.8, where the following equation may be used (it resulted from a regression analysis, $R^2$=0.999);

$$\frac{1}{OptTi_2} \approx 0.974 - 0.719w \qquad (23)$$

We can see that as the capacity related costs become more important, the more *Ti* is needed to minimise the linear variance related cost structure we have defined. However if to minimise inventory costs is more important than to minimise bullwhip costs, a lower *Ti* is required. This translates into the following management insight; "*if your business suffers from bullwhip related costs to a greater extent than inventory related costs then use a proportional controller (Ti) in your replenishment rule. Use the Golden Ratio to tune this controller*".

## Conclusions

We have studied a generic PIC system. We selected a subset of a generic replenishment rule with some "nice" mathematical properties that meant that the PIC was guaranteed to be stable. We derived analytical expressions for the bullwhip and inventory variance produced by the PIC system.

Analysis of these variance ratios showed us that Bullwhip could be avoided with our policy. We have highlighted the bullwhip boundary as a function of the feedback gain, *Ti*. However a zero inventory scenario cannot be achieved with our policy. We then assumed that costs in a given setting were a linear function of the variance ratios and proceeded to analyse two forms of the cost function. In the first form a simple weight was given to the bullwhip expression. We then found that the closed form expression for the optimum *Ti* ("the golden *Ti*") that minimises the sum of the weighted variance ratio was of the same form as the Golden Ratio. With the second form of cost function we could visualise the complete design space at once, but the closed form was not so elegant.

There are clearly many other criteria that could be used to design PIC systems. Immediately obvious ideas are;

- minimising expected costs in the next period by using the variance ratios in probability density functions and assigning costs to inventory holding and backlogs,
- maximising the Net Present Value of the economic consequences of the cash flows through time created by an ordering system. There is a long history of work emanating from Linköping Institute of Technology (Grubbström 1967 and 1991) that has observed and exploited the fact that replacing the complex frequency (*s* or *z*) with the discount rate (or one plus the discount rate in discrete time) in the transfer function of the cash flow yields the Net Present Value of the cash flow.
- incorporating customer service measures such as availability (or the probability of inventory being available from the shelf at the end of each period) and fill rates (percentage of demand shipped on time) into the bullwhip and inventory variance trade-off.

Of course, the natural progression of this paper would be to compare the "Golden *Ti*" to situations with more complicated and arguably more realistic (but still not necessarily complete) cost functions to test its performance. These ideas will be the subject of future research.

## Acknowledgements

We would like to thank the referees for their very useful suggestions. It has helped us to provide much more rigour to, and gain insight from, the math. We would also like to thank the EURONiL project (EPSRC. Grant No. GR/N66162/01) for sponsoring this research.

## *References*

Brown, R.G., (1963) "Smoothing, forecasting and prediction of discrete time series", Prentice-Hall, Inc., Englewood Cliffs, New Jersey.

Chen, Y.F., Drezner, Z., Ryan, J.K. and Simchi-Levi, D., (2000) "Quantifying the Bullwhip effect in a simple supply chain: The impact of forecasting, lead-times and information". Management Science 46 (3) 436-443.

Chen, Y.F. and Disney, S.M., (2003) "The order-up-to policy "sweet spot" - Using proportional controllers to eliminate the bullwhip problem", EUROMA POMS Conference, Como Lake, Italy, 16-18 June, Vol. 2, 551-560.

Dejonckheere, J., Disney, S.M., Lambrecht, M.R. and Towill, D.R., (2003a) "The impact of information enrichment on the bullwhip effect in supply chains: A control engineering perspective", Forthcoming in the European Journal of Operational Research.

Dejonckheere, J. Disney, S.M. Lambrecht M.R. and Towill, D.R., (2003b) "Measuring the Bullwhip Effect: a control theoretic approach to analyse forecasting induced Bullwhip in order-up-to policies". European Journal of Operations Research, 147, (3) 567-590.

Dejonckheere, J., Disney, S.M., Lambrecht M.R. and Towill, D.R., (2003c) The dynamics of aggregate planning. Forthcoming in Production Planning and Control.

Dejonckheere, J., Disney, S.M., Farasyn, I., Janssen, F., Lambrecht, M., Towill, D.R. and Van de Velde, W., (2002) "Production and Inventory Control; The variability trade-off", Proceedings of the 9th EUROMA Conference, June 2-4, Copenhagen, Denmark.

Deziel, D.P. and Eilon, S., (1967) "A linear production – inventory control rule". The Production Engineer 43 93-104.

Disney S.M. and Towill, D.R., (2002) "A discrete linear control theory model to determine the dynamic stability of Vendor Managed Inventory". International Journal of Production Research 40 (1) 179-204.

Disney S.M. and Towill, D.R., (2003) "Solving the Bullwhip problem in supply chains". Forthcoming in Omega: The International Journal of Management Science.

Disney, S.M. and Grubbström, R.W., (2003) "The economic consequences of a production and inventory control policy", 17th International conference on Production Research, Virginia, USA, 3-7 August.

Disney, S.M., (2001) "The production and inventory control problem in Vendor Managed Inventory supply chains". PhD Thesis, Cardiff Business School, Cardiff University, UK.

Disney, S.M., Farasyn, I., Lambrecht, M., Towill, D.R. and Van de Velde, W., (2003), "Creating win-win scenarios from the bullwhip problem - By design not accident", EUROMA POMS Conference, Como Lake, Italy, 16-18 June, Vol. 2, 561-570.

Forrester, J., (1961) "Industrial dynamics", Cambridge MA, MIT Press.

Grubbström, R.W. and Andersson, L.-E., (2002) "The Multiplication Theorem of The z-Transform", Working Paper No. 304, Linköping Institute of Science and Technology, Sweden.

Grubbström, R.W., (1967) "On the application of the Laplace Transform to certain economic problems". Management Science 13 (7) 558-567.

Grubbström, R.W., (1991) "A closed-form expression for the net present value of a time-power cash-flow function". Managerial and Decision Economics 12 377-381.

Holmström, J., (1997) "Product range management: a case study of supply chain operations in the European grocery industry". Supply Chain Management 2 (3) 107-115.

Houlihan, J.B., (1987) "International supply chain management". International Journal of Physical Distribution and Materials Management 17 (2) 51-66.

John, S., Naim, M.M. and Towill, D.R., (1994) "Dynamic analysis of a WIP compensated decision support system". International Journal of Manufacturing Systems Design 1 (4) 283-297.

Jury, E.I., (1964) "Theory and application of the z-transform method", Robert E. Krieger Publishing Company, New York.

Knott, R., (2003) "Fibonacci Numbers and the Golden Section", http://www.mcs.surrey. ac.uk/Personal/R.Knott/Fibonacci/fib.html, Accessed 28/04/2003.

Lambrecht, M.R. and Dejonckheere, J., (1999a) “A Bullwhip Effect explorer”, Research Report 9910, Department of Applied Economics, Katholieke Universiteit, Leuven, Belgium,

Lambrecht, M.R. and Dejonckheere, J., (1999b) “Extending the Beer Game to include real-life supply chain characteristics”, Proc. EUROMA Int. Conf. On Managing Operations Networks, 237-243.

Lee, H.L., Padmanabhan P. and Whang, S., (1997a) “The Bullwhip Effect in supply chains”. Sloan Management Review, Spring 93-102.

Lee, H.L., Padmanabhan, P. and Whang, S., (1997b) “Information distortion in a supply chain: the Bullwhip Effect”. Management Science 43 543-558.

Lewis, J.C., Naim M.M. and Towill, D.R., (1997) “An integrated approach to re-engineering material and logistics control”. International Journal of Physical Distribution and Logistics Management 27 (3) 197-209.

Magee, J.F., (1958) “Production planning and inventory control”, McGraw-Hill Book Company, New York, pp80-83.

Metters, R., (1997) “Quantifying the bullwhip effect in supply chains”. Journal of Operations Management 15 89-100.

Naim, M.M. and Towill, D.R. (1995) “What’s in the Pipeline?” 2nd International Symposium on Logistics, Nottingham, 11-12 July, 135-142.

Naylor, J.B., Naim M.M. and Berry, D., (1999) “Leagility: Integrating the lean and agile manufacturing paradigms in the total supply chain”. International Journal of Production Economics 62, 107-188.

Nise, N.S., (1995) “Control systems engineering”, The Benjamin Cummings Publishing Company, Inc., California.

Riddalls, C.E. and Bennett, S., (2002) “The stability of supply chains”. International Journal of Production Research 40 459-475.

Simon, H.A., (1952) “On the application of servomechanism theory to the study of production control”. Econometrica 20 247-268.

Sterman, J., (1989) “Modelling managerial behaviour: Misperceptions of feedback in a dynamic decision making experiment”. Management Science 35 (3) 321-339.

Suzaki, K., (1987) “The new manufacturing challenge”, The Free Press, New York.

Towill, D.R., (1997) “Forridge - Principles of good material flow”. Production Planning and Control 8 (7) 622-632.

Towill, D.R., (1982) “Optimisation of an inventory and order based production control system”. Proceedings of the Second International Symposium on Inventories, Budapest, Hungary.

Towill, D.R., (1996) “Time compression and supply chain management - a guided tour”. Supply Chain Management 1 (1) 15-17.

Towill, D.R., Lambrecht, M.R., Disney, S.M. and Dejonckheere, J., (2001) “Explicit filters and supply chain design. Proceedings of the EUROMA Conference, Bath, UK, 401-411.

Tsypkin, Y.Z., (1964) “Sampling systems theory and its application”, Pergamon Press, Oxford.

Wikner, J., Naim, M.M. and Towill, D.R., (1992) "The system simplification approach in understanding the dynamic behaviour of a manufacturing supply chain". Journal of Systems Engineering, Vol. 2, No. 2, pp164-178.

Zipkin, P.H., (2000) “Foundations of inventory management”, McGraw-Hill, New York.