



# Spectral Analysis of Dirac Operators under Integral Conditions on the Potential

Thesis submitted to  
CARDIFF UNIVERSITY  
for the degree of  
DOCTOR OF PHILOSOPHY  
by  
DANIEL GORDON JOHN HUGHES

September 2012,  
School of Mathematics,  
Cardiff University

## DECLARATION

This work has not been submitted in substance for any other degree or award at this or any other university or place of learning, nor is being submitted concurrently in candidature for any degree or other award.

Signed \_\_\_\_\_ (candidate) Date \_\_\_\_\_

## STATEMENT 1

This thesis is being submitted in partial fulfilment of the requirements for the degree of PhD

Signed \_\_\_\_\_ (candidate) Date \_\_\_\_\_

## STATEMENT 2

This thesis is the result of my own independent work/investigation, except where otherwise stated. Other sources are acknowledged by explicit references. The views expressed are my own.

Signed \_\_\_\_\_ (candidate) Date \_\_\_\_\_

## STATEMENT 3

I hereby give consent for my thesis, if accepted, to be available for photocopying and for inter-library loan, and for the title and summary to be made available to outside organisations.

Signed \_\_\_\_\_ (candidate) Date \_\_\_\_\_

## STATEMENT 4: PREVIOUSLY APPROVED BAR ON ACCESS

I hereby give consent for my thesis, if accepted, to be available for photocopying and for inter-library loans after expiry of a bar on access previously approved by the Academic Standards & Quality Committee.

Signed \_\_\_\_\_ (candidate) Date \_\_\_\_\_

## Abstract

We show that the absolutely continuous part of the spectral function of the one-dimensional Dirac operator on a half-line with a constant mass term and a real, square-integrable potential is strictly increasing throughout the essential spectrum  $(-\infty, -1] \cup [1, \infty)$ . The proof is based on estimates for the transmission coefficient for the full-line scattering problem with a truncated potential and a subsequent limiting procedure for the spectral function. Furthermore, we show that the absolutely continuous spectrum persists when an angular momentum term is added, thus establishing the result for spherically symmetric Dirac operators in higher dimensions, too. Finally, with regard to this problem, we show that a sparse perturbation of a square integrable potential does not cause the absolutely continuous spectrum to become larger in the one-dimensional case.

The final problem considered is regarding bound states, where we show that if the electric potential obeys the asymptotic bound  $C := \limsup_{x \rightarrow \infty} x|q(x)| < \infty$  then the eigenvalues outside of the spectral gap  $[-m, m]$  must obey  $\sum_n (\lambda_n^2 - 1) < \frac{C^2}{2}$ , where  $m$  is the constant mass.

## Acknowledgements

It is a pleasure to thank the many people who made this thesis possible.

It is difficult to overstate my gratitude to my Ph.D. supervisor, Dr. Karl Michael Schmidt. With his enthusiasm, his inspiration, and his great efforts to explain things clearly and simply, he helped to make mathematics fun for me. Throughout my thesis-writing period, he provided encouragement, sound advice, good teaching, good company, and lots of good ideas. I also appreciate the efforts he made to always be the positive one.

I wish to thank all of my wonderful friends who have made getting up, travelling to university and getting through each day much more enjoyable. That is Angelico, Chris, Izabela, Julie, Lisa, Penny and more recently Lizzie, Mike, Rob and Tom.

I am grateful to the mathematics department of Cardiff for the help and support they have provided me from my undergraduate years right through until the completion of this thesis.

I wish to thank my parents. They bore me, raised me, supported me and loved me.

Lastly, and most importantly, I wish to thank my beautiful wife Emma. She knows everything she has done for me and without her I could never have survived the last three years.



# Contents

Chapter 1. Introduction	1
Chapter 2. Half Line Schrödinger and Dirac Operators with Square Integrable Potentials	13
1 Introduction	13
2 Compactly Supported Potentials and Convergence of Spectral Measures	16
3 Dirac Case: The Transmission Coefficient	19
4 Schrödinger Case: The Transmission Coefficient	34
5 Dirac Case: The Spectral Function	42
6 Schrödinger Case: The Spectral Function	46
Chapter 3. Spherically Symmetric Dirac Operators with Square Integrable Potentials	49
Chapter 4. The Dirac Operator with an $L^2$ -Sparse Potential	59
1 Introduction	59
2 Asymptotics of $\frac{v_2}{v_1}$	64
3 Estimates of $u(x, z)$ and $v(x, z)$ for $L^1$ -bounded Potentials	71
4 Estimate of $-\frac{f_2(x, z)}{f_1(x, z)}$ for Potentials Subject to an $L^2$ -type Condition	74
5 The Absolutely Continuous Spectrum and the Value Distribution	84
6 Spectral Analysis	89
Chapter 5. Bound States of the Dirac Operator for $O(\frac{1}{x})$ Potentials	91
Chapter 6. Further Research	99

Chapter A.	The Error Estimate $E_A(\delta)$	101
Bibliography		105

# Chapter 1

## Introduction

Quantum mechanics is part of, and in some sense the precursor to, the body of scientific principles that explains the behaviour of matter and its interactions with energy on the scale of atoms and atomic particles. Classical physics, including general relativity, explains matter and energy at the macroscopic level of the scale familiar to human experience, including the behaviour of astronomical bodies. It remains the key to measurement for much of modern science and technology; however, at the end of the 19th Century observers discovered phenomena in both the large and the small worlds that classical physics could not explain, for example the spectral distribution of thermal radiation from a black body or the low-temperature specific heats of solids. Coming to terms with these limitations led to the development of quantum mechanics, a major revolution in physics.

Spectral theory is an inclusive term for theories extending the eigenvector and eigenvalue theory of a single square matrix to a much broader theory of the structure of operators in a variety of mathematical spaces. The name spectral theory was introduced by David Hilbert in his original formulation of Hilbert space theory. For many years, there has been a strong connection between developments in spectral theory and the need to solve mathematical problems arising in mathematical physics. Indeed one of the milestones of spectral theory in the first half of the 20th century, the spectral representation theorem for general self-adjoint operators in Hilbert Space ([65], [60]) was motivated by von Neumann's and others' attempts to understand the mathematical structure of observables in quantum mechanics. Hamiltonian operators describing the total energy within a quantum system,



an example of a quantum observable, are often represented by differential operators. This thesis will be concerned with two such differential operators: the Schrödinger operator and the Dirac operator.

The one-dimensional time independent Schrödinger equation describes the non-relativistic motion of a particle in a conservative force field which can be represented by a potential energy; non-relativistic motion concerns bodies moving at less than a significant proportion of the speed of light. We will be considering the one-dimensional form of the equation

$$-\frac{d^2\psi}{dx^2} + q(x)\psi(x) = \lambda\psi(x) \quad (x \in I) \quad (1.1)$$

where  $I \subset \mathbb{R}$ ,  $q$  is a real valued potential function which is assumed to be locally integrable (in the Lebesgue sense) and  $\lambda \in \mathbb{C}$  is called the spectral parameter.

The one-dimensional Dirac equation is the relativistic counterpart of the one-dimensional Schrödinger equation; it is the Hamiltonian of a one-dimensional relativistic particle of (usually constant) mass  $m(x)$  moving in a conservative force field represented by potential  $q$ . It takes the form

$$-i\sigma_2 \frac{d\psi(x)}{dx} + \frac{k}{x}\sigma_1\psi(x) + m(x)\sigma_3\psi(x) + q(x)\psi(x) = \lambda\psi(x), \quad (x \in I) \quad (1.2)$$

where  $I \subset \mathbb{R}$ ,  $k \in \mathbb{R}$ ,  $m, q : I \rightarrow \mathbb{R}$  are given functions such that  $m, q \in L^1_{\text{loc}}(I)$ ,  $\lambda$  is the spectral parameter and  $\sigma_1, \sigma_2, \sigma_3$  are Pauli matrices. The term  $\frac{k}{x}\sigma_1$  is known as the angular momentum term. Note that the Dirac operator is a first-order matrix valued operator; the solutions of (1.2) consist of two components.

For fixed  $\lambda$  in the upper half plane, we say that the differential equation (1.1) resp. (1.2) is in the limit point case at a singular interval end-point if there exists only one linearly independent solution which is square integrable at that interval end-point and in the limit circle case if all solutions are square integrable at that interval end-point. This result is known as Weyl's alternative (cf. [69, Theorem 5.6]). For Sturm-Liouville expressions (of which the Schrödinger equation is an example) this result goes back to Weyl ([70]). For Dirac systems a first proof was given by Roos and Sangren ([47], [48], [49]) (we note that the result follows almost immediately from abstract facts about deficiency indices

of symmetric operators, c.f. [69] Theorems 5.6, 5.7 and 5.8). The limit point and limit circle cases are the only possibilities; this classification is independent of  $\lambda$ .

Let  $\mathcal{S}$  denote the set of all functions  $\xi$  such that

- (1)  $\xi$  is differentiable and  $\xi'$  is locally absolutely continuous <sup>1</sup>;
- (2)  $\xi, \left(-\frac{d^2}{dx^2} + q(x)\right)\xi \in L^2(0, \infty)$ ;
- (3)

$$\sin \alpha \xi(0) + \cos \alpha \xi'(0) = 0. \quad (1.3)$$

$S = -\frac{d^2}{dx^2} + q(x)$  can be defined as a self-adjoint operator in  $L^2(0, \infty)$  if functions in the domain of  $S$  are also in  $\mathcal{S}$  and if  $\infty$  is in the limit point case ([10, Chapter 9 Problem 13]);  $S$  is in the limit point case at  $\infty$  if for some  $c \in (0, \infty)$  and  $k \geq 0$ ,  $q(x) \geq -kx^2$  ( $x > c$ ) ([69, Theorem 6.6]).

On the other hand, let  $\mathcal{T}$  denote the set of all functions  $\eta$  such that

- (1)  $\eta_1, \eta_2$  are locally absolutely continuous;
- (2)  $\eta, \left(-i\sigma_2 \frac{d}{dx} + \frac{k}{x}\sigma_1 + m(x)\sigma_3 + q(x)\right)\eta \in L^2(0, \infty)^2$ ;
- (3)

$$\sin \alpha \eta_1(0) + \cos \alpha \eta_2(0) = 0. \quad (1.4)$$

and  $\widehat{\mathcal{T}}$  be the set of functions satisfying (1) and (2). Then  $T = -i\sigma_2 \frac{d}{dx} + \frac{k}{x}\sigma_1 + m(x)\sigma_3 + q(x)$  can be defined as self-adjoint operator in  $L^2(0, \infty)^2$  if:

- (i) in the case  $k = 0$ , functions in the domain of  $T$  are also in  $\mathcal{T}$  and  $\infty$  is in the limit point case; in fact  $\infty$  is always in the limit point case for both the operator with  $k = 0$  and with  $k \neq 0$  ([69, Theorem 5.7], [69, Theorem 6.8]);
- (ii) in the case  $k \neq 0$ , functions in the domain of  $T$  are also in  $\widehat{\mathcal{T}}$  and both 0 and  $\infty$  are in the limit point case ([69, Theorem 5.7]); 0 is in the limit point case if  $|k| \geq \frac{1}{2}$  and  $m, q \in L^1([0, \infty))$  (c.f. [16]).

In applications the limit point case is the most frequent; it is also the most important case for further development of the theory. If the singular end point at infinity is in

---

<sup>1</sup>A function  $g$  is described as being locally absolutely continuous, written  $g \in AC_{loc}(I)$ , if it can be written in the form

$$g(x) = \text{constant} + \int_0^x h(t) dt \quad (x \in I)$$

where  $h$  is locally integrable. The derivative of  $g$  is then almost everywhere equal to  $h$ .

the limit point case then no boundary condition is required there to define  $S$  or  $T$  as self-adjoint operators; in the limit circle case a boundary condition is required.

Let  $\Im\lambda > 0$  and consider complex-valued solutions of (1.1) and (1.2), whose coefficients satisfy the conditions required for both operators to be self-adjoint and, for this immediate discussion, we assume that  $k = 0$  in (1.2). Define two solutions of each equation,  $u_\alpha, v_\alpha$  such that for the Schrödinger equation

$$\begin{aligned} u_\alpha(0, \lambda) &= \cos \alpha & v_\alpha(0, \lambda) &= -\sin \alpha \\ u'_\alpha(0, \lambda) &= \sin \alpha & v'_\alpha(0, \lambda) &= \cos \alpha \end{aligned}$$

and for the Dirac equation

$$\begin{aligned} (u_\alpha)_1(0, \lambda) &= \cos \alpha & (v_\alpha)_1(0, \lambda) &= -\sin \alpha \\ (u_\alpha)_2(0, \lambda) &= \sin \alpha & (v_\alpha)_2(0, \lambda) &= \cos \alpha. \end{aligned}$$

As the singular end-point at infinity is in the limit point case we can, given any  $\lambda$  with  $\Im\lambda > 0$ , find a non-trivial solution  $f(\cdot, \lambda)$  of (1.1) or (1.2) such that  $f$  is square integrable. Further we know that  $f$  is unique up to a multiplicative constant. Thus  $f$  may be expressed as a linear combination of the two solutions  $u_\alpha, v_\alpha$  for any  $\alpha \in [0, \pi)$ . Writing  $f = A_\alpha u_\alpha + B_\alpha v_\alpha$  we can deduce that  $A_\alpha$  is non-zero. Indeed, if  $A_\alpha = 0$  then  $f$  is a constant multiple of  $v_\alpha$ , which satisfies the boundary condition at zero. Therefore  $f$  is an eigenvector and  $\lambda$  is an eigenvalue contradicting the fact that  $T$  or  $S$  respectively are self-adjoint operators which implies that all eigenvalues are real ([45] Theorem VI.8). Hence, without loss of generality, we may divide through by  $A_\alpha$  and assume that the square integrable solution of either equation (1.1) or (1.2) is of the form

$$f_\alpha(x, \lambda) = u_\alpha(x, \lambda) + m_\alpha(\lambda)v_\alpha(x, \lambda). \tag{1.5}$$

The coefficient  $m_\alpha$  is called the Weyl-Titchmarsh  $m$ -function; given  $\alpha$  and  $\lambda$  it is uniquely determined by the condition that  $f$  be square integrable. The  $m$ -function is a Herglotz function; this means that it is analytic in the upper half plane with positive imaginary part.

If  $\alpha = 0$  we will, for simplicity, ignore the  $\alpha$  dependent notation, i.e.  $m_\alpha$  will simply be written  $m$ . The  $m_\alpha(\lambda)$  functions for various  $\alpha$  are related to  $m(\lambda)$  through

$$m_\alpha(\lambda) = \frac{m(z) \cos \alpha + \sin \alpha}{\cos \alpha - m(z) \sin \alpha} \quad (1.6)$$

(see [41]). In view of this algebraic connection between the various  $m$ -functions we shall refer mainly to the properties of  $m(\lambda)$ ; however most results may easily be extended to the general  $\alpha$  case.

Herglotz functions, such as the  $m$ -function, admit an integral representation, known as the representation theorem for Herglotz functions ([1]); in the case of the  $m$ -function

$$m(\lambda) = \Re m(i) + \left( \lim_{s \rightarrow \infty} \frac{\Im m(is)}{s} \right) \lambda + \int_{\mathbb{R}} \left( \frac{1}{t - \lambda} - \frac{t}{t^2 + 1} \right) d\mu(t), \quad (1.7)$$

where  $\mu$  is a measure defined on Borel subsets of the real line. Further  $\int_{\mathbb{R}} \frac{d\mu(t)}{t^2 + 1} < \infty$ . For a given function  $m(\lambda)$ , (1.7) uniquely determines  $\mu$ . Define the function  $\rho(t)$  (up to an additive constant) to be such that the  $\mu$  measure of a finite interval  $\mu(a, b] = \rho(b) - \rho(a)$ ; then  $\rho$  is a monotonic non-decreasing and right continuous function. We shall call  $\mu$  the spectral measure for the differential operator associated with  $m(\lambda)$  and  $\rho$  will be called the spectral function for this operator.

The  $m$ -function,  $m_\alpha(\lambda)$  carries the complete spectral information of its related differential operator  $S$  or  $T$ , and the spectral measure  $\mu_\alpha$  is itself determined once the  $m$ -function is known.

The spectral measure can be decomposed into its absolutely continuous ( $\mu_{ac}$ ), singular continuous ( $\mu_{sc}$ ) and pure point ( $\mu_{pp}$ ) parts (see also [45, Theorem I.14, Lebesgue Decomposition Theorem])

$$\mu = \mu_{ac} + \mu_{sc} + \mu_{pp}.$$

To each of  $\mu_{ac}, \mu_{sc}, \mu_d$  we may define a corresponding spectral function  $\rho_{ac}, \rho_{sc}, \rho_{pp}$ . Investigating the support of each part of the spectral function's decomposition is the means by which we gain knowledge about the location of each part of the spectrum of the corresponding operator.

Consider again the one-dimensional Schrödinger operator  $S = -\frac{d^2}{dx^2} + q$ . It is well known that any self-adjoint realisation of  $S$  on  $[0, \infty)$  has essential spectrum  $[0, \infty)$  if  $q$  is integrable at 0 and  $q(x) \rightarrow 0$  ( $x \rightarrow \infty$ ). It was expected that for such decaying potentials the spectrum will be a discrete subset of  $\mathbb{R}$  for  $\lambda \leq 0$  and continuous for  $\lambda > 0$ ; this was an expectation born out of the study of the hydrogen atom.

Much early work considered conditions on the potential to ensure purely absolutely continuous spectrum for  $\lambda > 0$ . Under certain conditions, e.g. if  $q \in L^1([0, \infty))$  the spectrum can be shown to be purely absolutely continuous for  $\lambda > 0$  [69, Thm 15.3]. The same result holds for potentials which may be singular at the origin, assuming for example that  $\int_0^1 x|q_-(x)|dx < \infty$  ( $q_-$  is the negative part of  $q$ ). In this case we replace the integrability condition with  $q \in L^1(a, \infty)$  for some  $a > 0$  [69, Thm 15.3].

Such results and examples by no means exhaust the kinds of spectral behaviours that one may encounter; in fact they barely scratch the surface. Even for potentials satisfying bounds of the form  $x|q(x)| \leq \text{const}$  the point spectrum need not be confined to the negative half-line. As an example (see [43]) take  $\lambda = 1$  in (1.1) and set  $f(x, \lambda) = \frac{\sin(x)}{1+(2x-\sin 2x)^2}$ . It can easily be verified that  $q(x) \sim -\frac{8\sin 2x}{x}$  as  $x \rightarrow \infty$  and yet  $f$  is an eigenfunction for  $S$  with Dirichlet boundary condition having eigenvalue  $\lambda = 1$ . More strikingly, in a situation only slightly more general than the  $L^1$  case, the essential spectrum can be far from purely absolutely continuous; indeed Naboko [38] and Simon [58] constructed potentials such that  $x|q(x)| \rightarrow \infty$  ( $x \rightarrow \infty$ ) arbitrarily slowly and  $S$  has dense point spectrum in  $[0, \infty)$ . In these examples dense point spectrum is overlaid with absolutely continuous spectrum.

This can be seen from subsequent work focused on providing sufficient conditions on the potential to ensure the existence of (not necessarily purely) absolutely continuous spectrum. Important progress was made in a key paper of Kiselev ([31]). Using techniques from Fourier analysis, Kiselev considered the class of locally integrable potentials subject to a power bound  $|q(x)| \leq \text{const} \cdot x^{-(3/4+\delta)}$  for  $\delta > 0$ . Note that this class of potentials is defined by a pointwise bound; no assumptions on the smoothness or differentiability of  $q$  are implied. This was significant as at the time many results about the absolute continuity of the spectrum on the positive semi-axis for certain classes of decaying potentials, such as potentials of bounded variation [69, Thm 15.3] or specific oscillating potentials (see, e.g., [71, 37, 4, 26] for further references) were known but no general relations between

the rate of decay and spectral properties, apart from the absolutely integrable class, were known. Under this quite weak assumption, Kiselev was able to show that the entire positive half-line belongs to the absolutely continuous spectrum. Moreover he was able to define explicitly a subset of  $[0, \infty)$  on which the singular part of the measure for  $\lambda \geq 0$ , if it is non zero, is concentrated.

Further results of a similar nature subsequently started to appear; for example, Kiselev, Christ and Remling ([9], [8], [46]) showed that for potentials obeying  $|q(x)| \leq C(1 + |x|)^{-\frac{1}{2}-\varepsilon}$  for large  $x$  ( $\varepsilon > 0$ ), the absolutely continuous spectral measure of  $S$  is essentially supported on  $[0, \infty)$ ; note that the examples by Naboko [38] and Simon [58] above satisfy this condition. We also note that this decay condition is in a sense optimal; indeed there exist examples of potentials satisfying the bound  $|q(x)| \leq \text{const} \cdot x^{-1/2}$  for which the spectrum is purely singular ([32]).

Incidentally, further examples of exotic behaviour in the singular spectrum have involved the study of sparse potentials, that is potentials which are zero for 'most' values of  $x$ , but which may tend to infinity, be bounded below by some positive constant or perhaps decay very slowly to zero, on some sequence of intervals which become more and more separated at large distances from the origin. Such potentials may give rise to singular continuous spectrum (see [39]; also very explicit examples are given in [59] and [32]).

In their celebrated paper [11], Deift and Killip discovered that an integral-type condition on the potential is more natural than a pointwise bound, proving that the absolutely continuous spectrum of the Schrödinger operator is essentially supported on  $[0, \infty)$  whenever  $q \in L^2([0, \infty))$ . This result is optimal in terms of  $L^p$  decay, as there exist potentials belonging to  $L^p$ , for all  $p > 2$ , such that  $S$  has no absolutely continuous spectrum [32].

More recently, Killip and Simon have given an equivalent characterisation of the spectral measures of Schrödinger operators with square-integrable potentials which includes the Deift-Killip result [30].

Consider again the Dirac operator. The standard Dirac operator is the fundamental Hamiltonian in the relativistic quantum mechanics of a massive particle of spin 1/2. In its original form, the particle mass is a positive constant, but more recently a variable

mass, or scalar potential, has been used in relativistic models of quark confinement (c.f. references in [73], Thaller [61] p. 305).

What about the case  $m = 0$ ? This could, for example, describe the motion of a neutrino; a neutrino is an electrically neutral, weakly interacting elementary subatomic particle with half-integer spin. However, as a neutrino does not carry any electrical charge, the addition of an electric potential would not be of physical interest. In the 1980s, Fröhlich, Lieb, Loss and Yau ([21, 35]) showed that the zero eigenspace of a massless Dirac operator with magnetic potential plays a critical role in the question of stability of matter. More recently, massless Dirac operators with a plain electric-type potential, especially in two space dimensions, has aroused interest as a physically relevant object in its own right; it has been shown to give an effective description of electron movement in single-atom carbon layers (graphene) [3]. This has been a motivation for studying the massless Dirac operator in less than three dimensions, for example see [56].

Many results pertaining to the Dirac operator with non-constant mass term can be derived from the work on Krein systems (c.f. [33]); indeed Krein systems can be related to the so called canonical one-dimensional Dirac operator ([13])

$$D = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{d}{dx} + \begin{pmatrix} -b(r) & -a(r) \\ -a(r) & b(r) \end{pmatrix}.$$

Through this relation, many properties of Krein systems carry over to Dirac operators of this form.

Using harmonic analysis techniques, Martin [36] was able to prove, under the assumption  $a, b \in L^2(0, \infty)$ , that the whole real line was a support for the absolutely continuous part of the spectral measure, a result analogous to that in [11]. A weaker result was again later proved by Denisov ([12]) using wave operator techniques (c.f. [29]), requiring  $a \in L^2(0, \infty)$  and  $b \equiv 0$ . However, the full result with  $a, b \in L^2(0, \infty)$  was settled much earlier by Krein in [33] (see [13] for more details).

We note that any Dirac operator can be reduced to the canonical form by a suitable change of variables (c.f. [34] p. 48-50). Indeed, this even includes Dirac operators of the form  $T$  with constant mass; in this case, however, the conditions required in [36], [12] or [33] (i.e.  $a, b \in L^2(0, \infty)$ ) are never satisfied.

The subject of this thesis will be the standard one-dimensional Dirac operator, i.e. the Dirac operator with positive constant mass, which without loss of generality we set to 1

$$T = -i\sigma_2 \frac{d}{dx} + \frac{k}{x}\sigma_1 + \sigma_3 + q(x).$$

It is known that the spectrum of such an operator is never purely discrete (see [54] Proposition 5.1). Indeed, every real interval of length greater than 2 intersects the essential spectrum of  $T$ , which is therefore unbounded above and below; that the spectrum of the Dirac operator is unbounded below is a striking difference between the Dirac and Schrödinger operators. As an example, if one only assumes that  $q(x) \rightarrow 0$  ( $x \rightarrow \infty$ ), the essential spectrum of  $T$  is  $(-\infty, -1] \cup [1, \infty)$ . We note, however, that as in the case of the Schrödinger operator this essential spectrum need not be purely absolutely continuous. There are examples of potentials such that  $x|q(x)| \rightarrow \infty$  ( $x \rightarrow \infty$ ) arbitrarily slowly and the operator has a dense set of eigenvalues in the whole or part of its essential spectrum [54].

One of the most familiar Dirac systems is the Coulomb potential problem ([67] [64]); this is the case where  $k \in \mathbb{Z} \setminus 0$  and  $q = \frac{\alpha}{x}$ ,  $\alpha \in \mathbb{R}$  in  $T$ . It was noted by Plesset [42] and later Titchmarsh [64] that the spectrum of  $T$  for this choice of  $q$  continuously covers the whole real axis save for the interval  $(-1, 1)$  where it is discrete.

In [64], Titchmarsh actually considered the more general system where either  $q(x)$  or  $q(x) - \frac{\alpha}{x}$ , for some constant  $\alpha$ , is absolutely integrable towards zero and where  $q(x)$  is either "large" or "small" near infinity; further  $q \in C^1(0, \infty)$  such that  $q' \in AC_{\text{loc}}(0, \infty)$  was required. When  $q$  is small at infinity it was found that the spectrum is discrete in  $(-1, 1)$  and continuous elsewhere. If  $q$  is large at infinity then the continuous spectrum covers the whole real line. We note that the case for  $q$  polynomial had already been observed by Plesset [42]. Erdélyi [17] obtain the same result in the small potential case by considering  $q$  as the sum of short range and long range components,  $q(x) = q_1(x) + q_2(x)$ ,  $q_1(x) \rightarrow 0$  ( $x \rightarrow \infty$ ) and  $q'_1, q_2 \in L^1(x_0, \infty)$ . In the large potential case Erdélyi was able to weaken the requirements in [64] to essential  $q \in AC_{\text{loc}}$  and

$$\int_0^\infty \frac{|q'|}{q^2} < \infty$$

and still obtain the same result. It is implicit in both [17] and [64] that the continuous spectrum is absolutely continuous, indeed continuously differentiable (cf. remarks of



Evans and Harris [18]). In [55], Schmidt was able to extend Erdélyi's result to prove that if  $q(x)$  is infinite at infinity and  $1/q$  is of bounded variation then the whole real line is purely absolutely continuous; this extends Erdélyi's result because the local regularity of  $q$  is reduced from absolute continuity to bounded variation.

The aim of this thesis is to consider the absolutely continuous spectrum of the standard Dirac operator where the electric potential is small at infinity; indeed we study the Dirac operator where the electric potential satisfies some integrability condition. It can be proven that if  $q$  is absolutely integrable then the standard Dirac operator has purely absolutely continuous spectrum in the bands  $(-\infty, -1] \cup [1, \infty)$  [69, Thm 16.7]. It is our aim to progress from this to consider potentials which are square integrable, producing an analogous result to [11]. We will then progress on to considering the standard Dirac operator in three dimensions with a spherically symmetric potential by considering the one-dimensional Dirac operator with an angular momentum term. We will also study the effect of a sparse perturbation on the absolutely continuous spectrum.

This thesis is organised as follows:

Chapter 2 is motivated by the paper [11] of Deift and Killip. We show that the absolutely continuous part of the spectral function of the one-dimensional Dirac operator on a half-line with constant mass 1 and a real, square-integrable potential is strictly increasing throughout the essential spectrum  $(-\infty, -1] \cup [1, \infty)$ . This fact is proved using estimates for the transmission coefficient for the full-line scattering problem with a truncated potential. Indeed the majority of this proof is centred around the inequality

$$\int_{(-\infty, -1] \cup [1, \infty)} |\lambda| \sqrt{\lambda^2 - 1} \log |a(\lambda)| d\lambda \leq \frac{\pi}{2} \int_{\mathbb{R}} q^2(x) dx,$$

where  $a$  is the inverse of the transmission coefficient,  $\lambda$  is the spectral variable and  $q$ , the potential, is square integrable with compact support. A subsequent limiting procedure for the spectral function then completes the proof of the main result of this section. We also provide a proof of the main result for the Schrödinger case, giving details absent from the paper of Deift and Killip.

Chapter 3 draws on some well known results of spectral theory to consider the one-dimensional Dirac operator with constant mass 1, a real, bounded, square-integrable potential and an angular momentum term. Using the famous theory of Kato-Rosenblum concerning trace class perturbations and wave operators ([29] Chapter 10, [28], [50], [20]) together with the Gilbert-Pearson theory of subordinacy ([22], [24]) we show that the absolutely continuous spectrum of this operator is unitarily equivalent to the absolutely continuous spectrum of the operator considered in Chapter 3. En route to this result we also prove that the absolutely continuous part of the spectral function of a one-dimensional Dirac operator on the full line with constant mass and a real, square-integrable potential is strictly increasing throughout the essential spectrum  $(-\infty, -1] \cup [1, \infty)$ .

Chapter 4 extends the theory pertaining to valued distribution theory developed by Pearson et al. (see [5, 6, 7, 41, 40]) to incorporate the Dirac operator; these results are then used to show that the support of the absolutely continuous part of the spectral measure of the one-dimensional Dirac operator on a half-line with constant mass 1 and a real,  $L^2$ -sparse potential is contained within  $\mathbb{R} \setminus (-1, 1)$ . An  $L^2$ -sparse potential can be written as the sum of a square-integrable potential and a sparse perturbation; this chapter is thus complementary to the results contained in Chapter 2. Indeed, an  $L^2$ -sparse potential can be seen either as a square integrable perturbation of a sparse potential or as a sparse perturbation of a square integrable potential and from either viewpoint the results of this chapter show that the absolutely continuous spectrum does not encroach into the spectral gap  $(-1, 1)$ . Further, the strength of this result is the lack of hypothesis required to control the sparse part of the potential.

Although the logical steps employed in this chapter are similar to those in [5, 6, 7], significant adaptations had to be made to develop the theory to cover the Dirac operator, not least the need to incorporate the fact that the solutions have two components. We also expand on some of the details and steps for which the papers [5, 6, 7] are sketchy.

Chapter 5 is the first and only chapter within this thesis to consider bound states of the one-dimensional Dirac operator with constant mass 1. We take the result of Kiselev, Last and Simon ([32]) as motivation to prove that if the electric potential  $q$  obeys the asymptotic  $C := \limsup_{x \rightarrow \infty} x|q(x)| < \infty$  then, for eigenvalues outside of the gap

$[-1, 1]$ ,  $\sum_n (\lambda_n^2 - 1) \leq \frac{C^2}{2}$ . The main tool used to prove this result is a modified Prüfer transformation which was found in [54].

# Half Line Schrödinger and Dirac Operators with Square Integrable Potentials

## 1 Introduction

Consider the one-dimensional Schrödinger operator

$$S = -\frac{d^2}{dx^2} + q. \quad (2.1)$$

where  $q \in L^1_{\text{loc}}(\mathbb{R})$ . We assume that this operator is in the limit point case at  $\pm\infty$  ([69] Theorem 6.3) so that it has a unique self-adjoint realisation  $\tilde{S}$  in  $L^2(\mathbb{R})$ . In this chapter we will be interested in the self adjoint operator  $S$  on the half line  $[0, \infty)$  with the boundary condition

$$u(0) \cos \alpha + u'(0) \sin \alpha = 0, \quad (2.2)$$

for fixed  $\alpha \in \mathbb{R}$ . The spectral analysis of  $\tilde{S}$  and  $S$  is based upon the study of the corresponding Schrödinger eigenvalue equation

$$Su(x, \lambda) = -u''(x, \lambda) + q(x)u(x, \lambda) = \lambda u(x, \lambda) \quad (2.3)$$

with spectral parameter  $\lambda \in \mathbb{C}$ .

In the present chapter we also consider the relativistic counterpart of  $S$ , the Dirac operator

$$\tau = -i\sigma_2 \frac{d}{dx} + \sigma_3 + q(x), \quad (2.4)$$

where  $\sigma_2, \sigma_3$  are Pauli matrices and  $q \in L^1_{\text{loc}}(\mathbb{R})$ . It is the Hamiltonian of a one-dimensional relativistic particle of mass 1 moving in a force field of potential  $q$ . As this formal differential expression is always in the limit point case at  $\pm\infty$ , it has a unique self-adjoint realisation  $\tilde{T}$  in  $L^2(\mathbb{R})^2$ . We are mainly interested in the self-adjoint operator  $T$  realising  $\tau$  on the half-line  $[0, \infty)$  with the boundary condition

$$u_1(0) \cos \alpha + u_2(0) \sin \alpha = 0, \quad (2.5)$$

for fixed  $\alpha \in \mathbb{R}$ . The spectral analysis of  $\tilde{T}$  and  $T$  is based upon the study of the corresponding Dirac eigenvalue equation

$$\tau u(x, \lambda) = -i\sigma_2 u'(x, \lambda) + \sigma_3 u(x, \lambda) + q(x)u(x, \lambda) = \lambda u(x, \lambda), \quad \lambda \in \mathbb{C}. \quad (2.6)$$

At a superficial glance, one could be inclined to think that the question about the existence of absolutely continuous spectrum of  $T$  under the assumption of square-integrability of  $q$  was settled long ago by the work on Krein systems, which are closely related to the Dirac operator (cf. [33, eq. (15)]). Indeed, Denisov's extensive reworking of Krein's ideas includes the result that the wave operators for the half-line operator

$$-i\sigma_2 \frac{d}{dx} + a(x) \sigma_1 + b(x) \sigma_3 \quad (2.7)$$

with  $a, b \in L^2([0, \infty))$  relative to that with  $a = b = 0$  exist [12], [13, Thm 13.3]. Thus (2.7) with square-integrable coefficients will have absolutely continuous spectrum covering the whole real axis; this had been shown directly by Martin [36] using the method of [11]. Now  $\tau$  in (2.4) can be brought into the form of (2.7) by a pointwise unitary transformation; indeed, if  $Q' = q$ , then

$$e^{i\sigma_2 Q} \tau e^{-i\sigma_2 Q} = -i\sigma_2 \frac{d}{dx} + e^{2i\sigma_2 Q} \sigma_3,$$

which is (2.7) with  $a = -\sin 2Q$ ,  $b = \cos 2Q$ . But then  $|a|^2 + |b|^2 = 1$ , so the hypothesis that both  $a$  and  $b$  are square-integrable on  $[0, \infty)$  is never fulfilled. In fact, it would seem that a Dirac operator (2.7) with square-integrable  $a, b$  will arise very rarely, if ever, in

physical situations.

The main result of the present chapter is the following analogue of Deift and Killip's result:

**THEOREM 2.1.** *If  $q \in L^2([0, \infty))$ , then the absolutely continuous part of the spectral function of  $T$  is strictly increasing in  $(-\infty, -1] \cup [1, \infty)$ .*

Alongside this we will also present a detailed proof of Deift and Killip's result both for interest and comparison. To this end we will also prove

**THEOREM 2.2.** *If  $q \in L^2([0, \infty))$ , then the absolutely continuous part of the spectral function of  $S$  is strictly increasing in  $[0, \infty)$ .*

Without loss of generality, we are able to restrict our attention to the case  $\alpha = 0$  in both our boundary conditions, as the  $m$  functions for different  $\alpha$  are related by a Möbius transformation (1.6) (for further details see, for example, [40] Equation (4)). From this we can deduce that the absolutely continuous parts of the spectral function for two different values of  $\alpha$  have the same essential supports.

**REMARK 2.1.** Results about the stability of the absolutely continuous spectrum under finite rank and trace class perturbations (e.g. changes in boundary condition) are well known (c.f. [27] and [23] Section 3).

This chapter is organised as follows. In Section 2 we prove that the spectral measure for the operators  $\varsigma$  and  $\tau$  with potential  $q$  is the limit of the spectral measures for the operators  $\varsigma_n$  and  $\tau_n$  with truncated potentials, which we equate to  $q$  on  $[0, n]$  and to zero on  $[n, \infty)$ . This allows us to focus on the case of compactly supported potentials. Section 3 considers the transmission coefficient and gives a proof for the central inequality expressed in the following theorem for the Dirac case

**THEOREM 2.3.** *Let  $q$  be a real valued square integrable function on  $[0, \infty)$  with compact (essential) support. Then*

$$\int_{(-\infty, -1] \cup [1, \infty)} |\lambda| \sqrt{\lambda^2 - 1} \log |a(\lambda)| d\lambda \leq \frac{\pi}{2} \int_{\mathbb{R}} q^2(x) dx. \quad (2.8)$$

Theorem 4 presents an analogous analysis to that of Section 3 for the Schrödinger case and proves the following inequality (where  $k$  is the non-relativistic momentum variable defined in Section 4)

**THEOREM 2.4.** *Let  $q$  be a real valued square integrable function on  $[0, \infty)$  with compact (essential) support. Then*

$$\int_{\mathbb{R}} k^2 \log |a(k)| dk \leq \frac{\pi}{8} \int_{\mathbb{R}} q^2(x) dx \quad (2.9)$$

The function  $a(\lambda)$  is the inverse transmission coefficient; see Section 3/4 below for details. The underlying identities for inequalities (2.8) and (2.9) can be found in [19, Page 78]. A proof of both under additional smoothness assumptions on  $q$  is given in [72]. Finally, in Section 5 we use these inequalities together with the observations in Section 2 to prove Theorems 2.1 and 2.2.

## 2 Compactly Supported Potentials and Convergence of Spectral Measures

For the proof of Theorems 2.1 and 2.2, we shall first prove the result for compactly supported potentials and then treat  $q \in L^2(0, \infty)$  as a limit of truncated potentials as the cut-off point moves to infinity. The spectral measures of the half-line operators with truncated potentials then converge vaguely to the spectral measure of  $T$  or  $S$  respectively, as our first lemma shows.

Let  $m, \rho$  be the Weyl-Titchmarsh  $m$ -function and the spectral function of  $T$ , respectively. For  $n \in \mathbb{N}$ , let  $q_n = \chi_{[0, n]} q$  and let  $T_n$  be the self-adjoint Dirac operator associated with the differential expression

$$-i\sigma_2 \frac{d}{dx} + \sigma_3 + q_n \quad (2.10)$$

on  $[0, \infty)$  with boundary condition (2.5). We denote its Weyl-Titchmarsh and spectral functions by  $m_n$  and  $\rho_n$ , respectively.

From the following Lemma we obtain the required result about the convergence of the spectral measures. The argument used in the proof of this Lemma follows the argument from the book of Coddington and Levinson for the derivation of the standard inversion

formula from Weyl theory [10, Chapter 9 Theorem 3.1]; however, rather than considering a singular self-adjoint problem on a finite interval and considering the limit as an interval end-point moves towards the singular limit we consider a problem with a truncated potential and send the truncation point towards the singular end-point.

LEMMA 2.1.  $\lim_{n \rightarrow \infty} \rho_n = \rho$  at all points of continuity of  $\rho$ .

PROOF. Let  $z \in \mathbb{C} \setminus \mathbb{R}$ , and let  $v : [0, \infty) \rightarrow \mathbb{C}^2$  be the solution of the initial-value problem  $\tau v = z v$ ,  $v(0) = \begin{pmatrix} -\sin \alpha \\ \cos \alpha \end{pmatrix}$ . Note that the differential equations with potentials  $q$  and  $q_n$  are identical on the interval  $[0, n]$ . Thus, from Weyl theory (see [10] Chapter 9 Section 2), it is known that the limit points  $m_n(z)$  and  $m(z)$  lie inside a complex circle of radius

$$r_n = \frac{1}{2\Im z \int_0^n |v|^2}.$$

Hence  $|m_n(z) - m(z)| \leq 2r_n(z) \rightarrow 0$  ( $n \rightarrow \infty$ ), as the Dirac equation is in the limit-point case at  $\infty$  and hence  $v \notin L^2(0, \infty)$ . This convergence is locally uniform; indeed,  $r_n$  depends continuously on  $z$ . Thus by Dini's Theorem ([52] Theorem 7.13), which states that a monotone sequence of continuous functions with continuous limit function is locally uniformly convergent, the locally uniform convergence of  $m_n$  to  $m$  follows.

We deduce from the Herglotz representation of  $m_n$  (see [40] Equations 5,5') and the boundedness of  $(m_n(i))_{n \in \mathbb{N}}$  that

$$\int_{\mathbb{R}} \frac{d\rho_n(\lambda)}{\lambda^2 + 1} = \Im m_n(i) \leq C$$

with a constant  $C$  independent of  $n$ . Thus,  $\forall x \in \mathbb{R}$ ,

$$|\rho_n(x)| = \left| \int_0^x d\rho_n \right| \leq (x^2 + 1) \left| \int_0^x \frac{1}{1 + \lambda^2} d\rho_n(x) \right| \leq C(x^2 + 1).$$

Hence, by Helly's First Theorem,  $(\rho_n)_{n \in \mathbb{N}}$  has a subsequence,  $(\rho_{n_j})_{j \in \mathbb{N}}$ , which converges pointwise to a non-decreasing function  $\tilde{\rho}$ . From this we can deduce, by Helly's Second Theorem, that  $\forall \Lambda > 0$ ,

$$\int_{-\Lambda}^{\Lambda} \frac{d\tilde{\rho}(\lambda)}{1 + \lambda^2} = \lim_{j \rightarrow \infty} \int_{-\Lambda}^{\Lambda} \frac{d\rho_{n_j}(\lambda)}{1 + \lambda^2} \leq C,$$

in other words

$$\int_{\mathbb{R}} \frac{d\tilde{\rho}(\lambda)}{1 + \lambda^2} \leq C.$$



If  $\mu \geq 1$  and  $\lambda \in \mathbb{R}$  such that  $|\lambda| \geq \mu$  then we have

$$\mu(1 + \lambda^2) = \mu + \mu\lambda^2 \leq \mu^3 + |\lambda|^3 \leq 2|\lambda|^3.$$

Hence

$$\left| \int_{|\lambda| \geq \mu} \frac{d\rho_{n_j}(\lambda)}{\lambda^3} \right| \leq \int_{|\lambda| \geq \mu} \frac{2}{\mu(1 + \lambda^2)} d\rho_{n_j}(\lambda) \leq \frac{2C}{\mu}.$$

Thus, for all  $\lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{R}$

$$\begin{aligned} & \frac{\Im m_{n_j}(\lambda_1)}{\Im \lambda_1} - \frac{\Im m_{n_j}(\lambda_2)}{\Im \lambda_2} \\ &= \int_{\mathbb{R}} \left( \frac{1}{|\lambda - \lambda_1|^2} - \frac{1}{|\lambda - \lambda_2|^2} \right) d\rho_{n_j}(\lambda) \\ &= \int_{|\lambda| \leq \mu} \left( \frac{1}{|\lambda - \lambda_1|^2} - \frac{1}{|\lambda - \lambda_2|^2} \right) d\rho_{n_j}(\lambda) + \int_{|\lambda| > \mu} \left( \frac{|\lambda_2|^2 - 2\Re \lambda \lambda_2 + 2\Re \lambda \lambda_1 - |\lambda_1|^2}{|\lambda - \lambda_1|^2 |\lambda - \lambda_2|^2} \right) d\rho_{n_j}(\lambda) \\ &= \int_{|\lambda| \leq \mu} \left( \frac{1}{|\lambda - \lambda_1|^2} - \frac{1}{|\lambda - \lambda_2|^2} \right) d\rho_{n_j}(\lambda) + \frac{2CK}{\mu} \\ &\rightarrow \int_{|\lambda| \leq \mu} \left( \frac{1}{|\lambda - \lambda_1|^2} - \frac{1}{|\lambda - \lambda_2|^2} \right) d\tilde{\rho}(\lambda) + \frac{2CK}{\mu} \\ &\rightarrow \int_{\mathbb{R}} \left( \frac{1}{|\lambda - \lambda_1|^2} - \frac{1}{|\lambda - \lambda_2|^2} \right) d\tilde{\rho}(\lambda), \end{aligned}$$

where the first step follows from the Herglotz representation for  $m_n$ , the  $j$  limit is calculated using Helly's Second Theorem and in the final step we sent  $\mu \rightarrow \infty$ .  $K$  is a constant arising in the estimate of the integrand for the integral over  $|\lambda| > \mu$  on the third line by  $\frac{K}{|\lambda|^3}$ . Sending  $j \rightarrow \infty$  on the left hand side

$$\frac{\Im m(\lambda_1)}{\Im \lambda_1} - \frac{\Im m(\lambda_2)}{\Im \lambda_2} = \int_{\mathbb{R}} \left( \frac{1}{|\lambda - \lambda_1|^2} - \frac{1}{|\lambda - \lambda_2|^2} \right) d\tilde{\rho}(\lambda),$$

and hence

$$\frac{\Im m(\mu)}{\Im \mu} = \int_{\mathbb{R}} \frac{d\tilde{\rho}}{|\lambda - \mu|^2} + k, \quad \mu \in \mathbb{C} \setminus \mathbb{R}, k \in \mathbb{R}.$$

Conversely, let  $\mu_1 < \mu_2$  be two points where  $\tilde{\rho}$  is continuous,  $\varepsilon > 0$ . Then

$$\begin{aligned}
\int_{\mu_1}^{\mu_2} \Im m(\nu + i\varepsilon) d\nu &= \varepsilon \int_{\mu_1}^{\mu_2} \int_{\mathbb{R}} \frac{d\tilde{\rho}(\lambda)}{|\lambda - \nu - i\varepsilon|^2} d\nu + \varepsilon k(\mu_2 - \mu_1) \\
&= \int_{\mathbb{R}} \int_{\mu_1}^{\mu_2} \frac{\varepsilon}{(\lambda - \nu)^2 + \varepsilon^2} d\nu d\tilde{\rho}(\lambda) + \varepsilon k(\mu_2 - \mu_1) \\
&= \int_{\mathbb{R}} \int_{\frac{\mu_1 - \lambda}{\varepsilon}}^{\frac{\mu_2 - \lambda}{\varepsilon}} \frac{ds}{1 + s^2} d\tilde{\rho}(\lambda) + \varepsilon k(\mu_2 - \mu_1) \\
&= \int_{\mathbb{R}} \left( \arctan \frac{\mu_2 - \lambda}{\varepsilon} - \arctan \frac{\mu_1 - \lambda}{\varepsilon} \right) d\tilde{\rho}(\lambda) + \varepsilon k(\mu_2 - \mu_1).
\end{aligned}$$

Now

$$\lim_{\varepsilon \rightarrow 0} \left( \arctan \frac{\mu_2 - \lambda}{\varepsilon} - \arctan \frac{\mu_1 - \lambda}{\varepsilon} \right) = \pi \chi_{(\mu_1, \mu_2)}(\lambda)$$

uniformly outside any neighbourhood of  $\{\mu_1, \mu_2\}$ . As  $\tilde{\rho}$  is continuous at  $\mu_1, \mu_2$  we can choose the neighbourhood of  $\{\mu_1, \mu_2\}$  sufficiently small so that its contribution to the integral becomes as small as we please. Thus

$$\lim_{\varepsilon \rightarrow 0} \int_{\mu_1}^{\mu_2} \Im m(\nu + i\varepsilon) d\nu = \pi \int_{\mu_1}^{\mu_2} d\tilde{\rho}(\lambda), \quad \tilde{\rho}(0) = 0, \quad \tilde{\rho} \text{ right continuous.}$$

As the  $m$ -function,  $m$ , and the spectral function,  $\rho$ , are linked by the Stieltjes Inversion formula ([10, Chapter 9 Theorem 3.1 (iv)]), we obtain that

$$\pi \int_{\mu_1}^{\mu_2} d\tilde{\rho}(\lambda) = \pi \int_{\mu_1}^{\mu_2} d\rho(\lambda), \quad \forall \mu_1, \mu_2 \in \mathbb{R} \setminus S,$$

where  $S$  is the set of points of discontinuity of  $\tilde{\rho}, \rho$ , which has measure zero. Thus  $\rho = \tilde{\rho}$  a.e.. As all subsequences of  $(\rho_n)_{n \in \mathbb{N}}$  have the same limit  $\tilde{\rho} = \rho$ , it follows that  $\rho_n \rightarrow \tilde{\rho} = \rho$  ( $n \rightarrow \infty$ ).  $\square$

Lemma 2.1 also holds for the Schrödinger case, with only minor cosmetic changes to the proof.

### 3 Dirac Case: The Transmission Coefficient

Throughout this section, we will assume that  $q$  is a square-integrable function with compact support in  $[0, \infty)$ . We shall use the function

$$\omega(\lambda) = \sqrt[3]{\lambda + 1} \sqrt{\lambda - 1} \quad (\lambda \in \mathbb{C}),$$

where  $\star\sqrt{\cdot}$  is the complex square root with branch cut along the negative real axis and  $\arg \star\sqrt{z} \in (-\frac{\pi}{2}, \frac{\pi}{2}]$  ( $z \in \mathbb{C}$ ), while  $\sqrt{\cdot}$  is the standard complex square root with branch cut along the positive real axis and  $\arg \sqrt{z} \in [0, \pi)$ . The function  $\omega$  is the relativistic substitute for the momentum variable  $k = \sqrt{\lambda}$  used in scattering analysis of the Schrödinger operator. Clearly  $\omega$  is analytic in  $\mathbb{C} \setminus \{(-\infty, -1] \cup [1, \infty)\}$  and satisfies  $\omega(\lambda)^2 = \lambda^2 - 1$  on  $\mathbb{C}$ . Moreover,  $\Im\omega(\lambda) > 0$  whenever  $\Im\lambda > 0$ , and for real  $\lambda$  we have

$$\omega(\lambda) = \begin{cases} -\sqrt{\lambda^2 - 1}, & \lambda \in (-\infty, -1] \\ i\sqrt{1 - \lambda^2}, & \lambda \in (-1, 1) \\ \sqrt{\lambda^2 - 1}, & \lambda \in [1, \infty). \end{cases} \quad (2.11)$$

By continuity, there is an open neighbourhood  $\Omega$  of  $(-1, 1)$ , encroaching into the lower half plane, in which  $\Im\omega > 0$ .  $\omega$  also satisfies several other useful properties which will be used extensively:

LEMMA 2.2. *Let  $q \in L^1([a, b])$ ,  $-\infty < a < b < \infty$ . Then the function  $\omega(\lambda)$  defined above satisfies*

- (i)  $(\omega + \lambda)(\omega - \lambda) = -1$ ;
- (ii)  $\omega(z) = -\overline{\omega(-\bar{z})}$ , ( $z \in \mathbb{C}^+$ );
- (iii)  $e^{\pm \frac{i}{\omega(\omega+\lambda)} \int_x^y q} = 1 + O\left(\frac{1}{|\lambda|^2}\right)$  ( $\lambda \rightarrow \infty$ ) uniformly in  $x, y \in [a, b]$ ;
- (iv)  $e^{\frac{i}{\omega(\omega-\lambda)} \int_x^y q} = e^{2i \int_x^y q} \left(1 + O\left(\frac{1}{|\lambda|^2}\right)\right)$  ( $\lambda \rightarrow \infty$ ) uniformly in  $x, y \in [a, b]$ .

PROOF. (i) is a simple calculation. For (ii), the case  $z \in \mathbb{R}$  follows by inspection.

Let  $z \in \mathbb{C}^+$ . Then

$$\begin{aligned} \omega(-\bar{z}) &= \star\sqrt{-\bar{z} + 1} \sqrt{-\bar{z} - 1} = \star\sqrt{-(z-1)} \sqrt{-(z+1)} \\ &= \left( \star\sqrt{-1} \cdot \star\sqrt{z-1} \right) \left( \sqrt{-1} \cdot \sqrt{z+1} \right) \\ &= -e^{i\frac{\pi}{2}} \star\sqrt{z-1} e^{-i\frac{\pi}{2}} \sqrt{z+1} \\ &= -\overline{\sqrt{z-1} \star\sqrt{z+1}} = -\overline{\omega(z)} \end{aligned}$$

For (iii) we estimate

$$\left| e^{\frac{i}{\omega(\omega+\lambda)} \int_x^y q} - 1 \right| \leq \sum_{j=1}^{\infty} \left| \frac{i}{\omega(\omega+\lambda)} \right|^j \frac{\|q\|_1^j}{j!} = O\left(\frac{1}{|\lambda|^2}\right), \quad (\lambda \rightarrow \infty)$$

uniformly in  $x, y \in [a, b]$ . Hence (iv) follows, as

$$e^{\frac{i}{\omega(\omega-\lambda)} \int_a^b q} = e^{-i(1+\frac{\lambda}{\omega}) \int_a^b q} = e^{-2i \int_a^b q} e^{-\frac{i}{\omega(\omega+\lambda)} \int_a^b q} = e^{-2i \int_a^b q} \left(1 + O\left(\frac{1}{|\lambda|^2}\right)\right)$$

( $\lambda \rightarrow \infty, x, y \in [a, b]$ )

□

Now consider the Dirac equation (2.6) on the whole real line, extending  $q$  by 0 to the negative half-line. For  $\lambda \in \mathbb{C} \setminus \{-1, 1\}$ , the functions

$$u(x, \lambda) = \begin{pmatrix} -\frac{i\omega}{\lambda-1} \\ 1 \end{pmatrix} e^{i\omega x}, \quad \tilde{u}(x, \lambda) = \begin{pmatrix} \frac{i\omega}{\lambda-1} \\ 1 \end{pmatrix} e^{-i\omega x} \quad (x \in I)$$

(where we write briefly  $\omega$  for  $\omega(\lambda)$ ) form a fundamental system of this equation on all intervals  $I$  where  $q$  vanishes. If  $\lambda \in \mathbb{R} \setminus [-1, 1]$ , then  $\omega(\lambda) \in \mathbb{R} \setminus \{0\}$ , and hence  $\tilde{u}(\cdot, \lambda) = \overline{u(\cdot, \lambda)}$ .

In particular, there is a solution  $y(\cdot, \lambda)$  for such  $\lambda$  with the property  $y(x, \lambda) = u(x, \lambda)$  for all  $x$  to the right of the support of  $q$ . For  $x$  to the left of the support of  $q$  this solution can be expressed as

$$y(x, \lambda) = a(\lambda)u(x, \lambda) + b(\lambda)\tilde{u}(x, \lambda) \quad (2.12)$$

with suitable constants  $a(\lambda)$  and  $b(\lambda)$ . Since  $e^{\pm i\omega x}$  represent right- and left-traveling waves, the solution  $y$  can be interpreted as describing a scattering process of a wave of amplitude  $a$  approaching from the left and split into a transmitted wave travelling to the right of amplitude 1 and a reflected wave traveling to the left of amplitude  $b$ . Correspondingly,  $t = \frac{1}{a}$  and  $r = \frac{b}{a}$  are called the *transmission* and *reflection coefficients*, respectively.

For  $\lambda \in \mathbb{R} \setminus [-1, 1]$ ,  $\overline{y(\cdot, \lambda)}$  is another solution of (2.6). Evaluating the constant Wronskian of  $y(\cdot, \lambda)$  and  $\overline{y(\cdot, \lambda)}$  both to the right and to the left of the support of  $q$ , we find that

$$|a(\lambda)|^2 = 1 + |b(\lambda)|^2. \quad (2.13)$$

Thus  $a(\lambda) \neq 0$ , and  $|t|^2 = 1 - |r|^2$ , expressing the conservation of the probability current in the quantum mechanical scattering process.

For points  $x$  to the left of the support of  $q$ , we obtain from (2.12) that

$$y_1(x, \lambda)\tilde{u}_2(x, \lambda) - y_2(x, \lambda)\tilde{u}_1(x, \lambda) = a(\lambda)(u_1(x, \lambda)\tilde{u}_2(x, \lambda) - u_2(x, \lambda)\tilde{u}_1(x, \lambda)) = \frac{2i\omega(\lambda)}{1 - \lambda} a(\lambda),$$

so

$$a(\lambda) = \frac{1 - \lambda}{2i\omega(\lambda)} (y_1(x, \lambda)\tilde{u}_2(x, \lambda) - y_2(x, \lambda)\tilde{u}_1(x, \lambda)) \quad (2.14)$$

with arbitrary  $x < 0$ . From this formula it is apparent that  $a$  is well defined for  $\lambda \in \mathbb{C} \setminus \{-1, 1\}$ , continuous for  $\lambda \in (\overline{\mathbb{C}^+} \cup \Omega) \setminus \{-1, 1\}$  and analytic in  $\mathbb{C}^+ \cup \Omega$ . Furthermore, it is immediate from (2.12) that

$$\begin{pmatrix} -\frac{i\omega}{\lambda-1} \\ 1 \end{pmatrix} a(\lambda) = \lim_{x \rightarrow -\infty} e^{-i\omega x} y(x, \lambda) \quad (\lambda \in \mathbb{C}^+ \cup \Omega).$$

The zeros of  $a$  in  $\mathbb{C}^+ \cup \Omega$  are exactly the eigenvalues of the full-line Dirac operator  $\tilde{T}$  and hence are all real. Indeed, for such  $\lambda$ ,  $y(\cdot, \lambda)$  is square integrable at  $\infty$ , and  $\tilde{u}(\cdot, \lambda)$  is square integrable at  $-\infty$  whilst  $u(\cdot, \lambda)$  is not. Thus  $y(\cdot, \lambda) \in L^2(\mathbb{R})^2$  if and only if  $a(\lambda) = 0$ . Further, (2.13) implies that  $a(\lambda)$  has no zeros for  $\lambda \in \mathbb{R} \setminus [-1, 1]$ . Moreover, we have the following information about the zeros of  $a$ .

**LEMMA 2.3.** *The number of zeros of  $a$  in  $\mathbb{C}^+ \cup \Omega$  is finite. On  $(-1, 1)$ ,  $a$  is real-valued and all its zeros are simple.*

**PROOF.** For the first statement, see [25, Cor. 3.2], bearing in mind that  $q$  has compact support. If  $\lambda \in (-1, 1)$ , then  $\omega(\lambda) = i\sqrt{1 - \lambda^2}$ , so

$$y(x, \lambda) = \begin{pmatrix} \frac{\sqrt{1-\lambda^2}}{\lambda-1} \\ 1 \end{pmatrix} e^{-\sqrt{1-\lambda^2}x} \in \mathbb{R}^2$$

for  $x$  to the right of the support of  $q$ . As all coefficients of the Dirac equation (2.6) are real,  $y(\cdot, \lambda)$  is real-valued throughout, in particular

$$a(\lambda) \begin{pmatrix} \frac{\sqrt{1-\lambda^2}}{\lambda-1} \\ 1 \end{pmatrix} + b(\lambda) \begin{pmatrix} -\frac{\sqrt{1-\lambda^2}}{\lambda-1} \\ 1 \end{pmatrix} = y(0, \lambda) \in \mathbb{R}^2,$$

which implies that  $a(\lambda), b(\lambda) \in \mathbb{R}$ . The last statement can be proved as [63, Lemma 2.12].  $\square$

DEFINITION 2.1 (Fourier Transform). Suppose  $f : \mathbb{R} \rightarrow \mathbb{C}$  is an integrable function. The Fourier transform of  $f$  is the function  $\hat{f} : \mathbb{R} \rightarrow \mathbb{C}$  given by

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i x \xi} f(x) dx \quad (\xi \in \mathbb{R}),$$

The inverse Fourier transform of  $f$ , denoted by  $\check{f}$  is the function

$$\check{f}(\lambda) = \int_{\mathbb{R}} e^{2\pi i x \lambda} f(x) dx \quad (\lambda \in \mathbb{R}).$$

In the proof of Theorem 2.3, we shall on several occasions use the following observation about the function  $\sin(2R|x|)/|x|$ , which is not absolutely integrable and has an  $R$ -independent envelope, but nevertheless turns out to generate an asymptotically diagonal integral kernel in a weak sense as  $R \rightarrow \infty$ . The square integral of  $q$  on the right-hand side of the inequality (2.8) arises in this way.

LEMMA 2.4. For  $x \in \mathbb{R}$  the function  $f : x \mapsto \begin{cases} \frac{\sin(x)}{x}, & x \in \mathbb{R} \setminus \{0\} \\ 1, & x = 0 \end{cases}$  is everywhere continuous and  $\int_{\mathbb{R}} f(x) dx = \pi$ . Furthermore, for compactly supported  $q \in L^2(\mathbb{R})$ ,

$$\lim_{R \rightarrow \infty} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\sin(2R|x-y|)}{\pi|x-y|} q(x) \overline{q(y)} dy = \int_{\mathbb{R}} |q|^2.$$

PROOF. It is clear that  $f$  is continuous for  $x \in \mathbb{R} \setminus \{0\}$ . For the case  $x = 0$  we consider the inequality

$$\cos(x) \leq \frac{\sin(x)}{x} \leq \frac{1}{\cos(x)} \quad (x \in \mathbb{R} \setminus \{0\})$$

Since  $\cos(x) \rightarrow 1$  as  $x \rightarrow 0$  we can infer that  $\frac{\sin(x)}{x}$  is continuous at  $x = 0$ .

For the integral, we first note that  $f$  is an even function, so it is enough to consider the integral on the half line. Let  $\alpha$  be an arbitrary real constant and define the function  $g$  by

$$g(\alpha) = \int_0^{\infty} e^{-\alpha w} \frac{\sin(w)}{w} dw.$$

Differentiating with respect to  $\alpha$  gives us

$$\frac{dg}{d\alpha} = - \int_0^{\infty} e^{-\alpha w} \frac{\sin(w)}{w} dw.$$

Now

$$\frac{d}{d\alpha} \int_0^\infty e^{-\alpha w} \frac{\sin(w)}{w} dw = \int_0^\infty \frac{\partial}{\partial \alpha} e^{-\alpha w} \frac{\sin(w)}{w} dw = - \int_0^\infty e^{-\alpha w} \sin(w) dw$$

Recalling that  $e^{iw} = \cos(w) + i \sin(w)$  we have that

$$- \Im \int_0^\infty e^{-\alpha w} e^{iw} dw = \Im \frac{1}{-\alpha + i} - \Im \frac{-\alpha - i}{\alpha^2 + 1} = \frac{-1}{\alpha^2 + 1}.$$

Thus

$$\frac{dg}{d\alpha} = \frac{-1}{\alpha^2 + 1} \implies \int_0^\infty \frac{dg}{d\alpha} d\alpha = \int_0^\infty \frac{-1}{\alpha^2 + 1} d\alpha,$$

giving us

$$\lim_{\rho \rightarrow \infty} g(\rho) - g(0) = - \lim_{\rho \rightarrow \infty} \arctan(\rho) + \arctan(0).$$

Now,

$$\lim_{\rho \rightarrow \infty} g(\rho) = \lim_{\rho \rightarrow \infty} \int_0^\infty e^{-\rho w} \frac{\sin(w)}{w} dw = 0$$

and

$$- \lim_{\rho \rightarrow \infty} \arctan(\rho) = -\frac{\pi}{2}.$$

Thus

$$g(0) = \int_0^\infty \frac{\sin(w)}{w} dw = \frac{\pi}{2}$$

and  $\int_{\mathbb{R}} \frac{\sin(x)}{x} dx = \pi$  follows.

Finally set  $q_R := q\left(\frac{\pi}{2R} \cdot\right)$  and  $f(t) = \frac{\sin \pi t}{\pi t}$  ( $t \in \mathbb{R} \setminus \{0\}$ ). Then

$$\begin{aligned} & \frac{1}{\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\sin(2R(y-x))}{(y-x)} q(x) dx \overline{q(y)} dy = \frac{\pi}{2R} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\sin(\pi(\xi-\eta))}{\pi(\xi-\eta)} q\left(\frac{\xi\pi}{2R}\right) d\xi \overline{q\left(\frac{\eta\pi}{2R}\right)} d\eta \\ & = \frac{\pi}{2R} \int_{\mathbb{R}} \left( f * q\left(\frac{\cdot}{2R}\right) \right) (\eta) \overline{q\left(\frac{\eta\pi}{2R}\right)} d\eta = \frac{\pi}{2R} (f * q_R, q_R) = \frac{\pi}{2R} (\widehat{f * q_R}, \widehat{q_R}) \\ & = \frac{\pi}{2R} (\widehat{f} \cdot \widehat{q_R}, \widehat{q_R}) = \frac{\pi}{2R} \int_{\mathbb{R}} \widehat{f}(\lambda) |\widehat{q_R}(\lambda)|^2 d\lambda. \end{aligned}$$

Now

$$\widehat{q_R}(\lambda) = \frac{2R}{\pi} \widehat{q}\left(\frac{2R\lambda}{\pi}\right),$$

and therefore continuing from above

$$\frac{\pi}{2R} \int_{\mathbb{R}} \widehat{f}(\lambda) |\widehat{q_R}(\lambda)|^2 d\lambda = \frac{2R}{\pi} \int_{\mathbb{R}} \widehat{f}(\lambda) \left| \widehat{q}\left(\frac{2R\lambda}{\pi}\right) \right|^2 d\lambda = \int_{\mathbb{R}} \widehat{f}\left(\frac{\pi\zeta}{2R}\right) |\widehat{q}(\zeta)|^2 d\zeta.$$

As  $\hat{f} = \chi_{[-\frac{1}{2}, \frac{1}{2}]}$ ,

$$\lim_{R \rightarrow \infty} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\sin(2R|x-y|)}{\pi|x-y|} q(x) dx \overline{q(y)} dy = \lim_{R \rightarrow \infty} \int_{-\frac{R}{\pi}}^{\frac{R}{\pi}} |\hat{q}(\zeta)|^2 d\zeta = \|\hat{q}\|^2 = \|q\|^2.$$

□

We now proceed to prove Theorem 2.3. Consider  $\lambda \in \mathbb{C}^+ \cup \Omega$ ; then  $\Im \omega > 0$ , where we write  $\omega$  briefly for  $\omega(\lambda)$ . Let  $y$  be as in (2.12), and define associated functions  $a(x, \lambda)$ ,  $b(x, \lambda)$  ( $x \in \mathbb{R}, \lambda \in \mathbb{C} \setminus \{-1, 1\}$ ) by setting

$$y(x, \lambda) = a(x, \lambda)u(x, \lambda) + b(x, \lambda)\tilde{u}(x, \lambda)$$

for all  $x \in \mathbb{R}$ . By comparison with (2.12),  $a(x, \lambda) = a(\lambda)$  and  $b(x, \lambda) = b(\lambda)$  to the left of the support of  $q$ , while  $a(x, \lambda) = 1$  and  $b(x, \lambda) = 0$  to the right of the support of  $q$ . We shall now derive an integral equation for the function  $a$ .

The function  $w(x, \lambda) := e^{-i \int_x^\infty q(t) dt - i\omega(\lambda)x} y(x, \lambda)$  satisfies the differential equation

$$w' = (\sigma_1 + i\sigma_2\lambda - i\omega)w + iq(1 - \sigma_2)w.$$

Indeed,

$$w' = [(iq - i\omega)y + y']e^{-i \int_x^\infty q - i\omega x} = (iq - i\omega)w + i\sigma_2(\lambda - q - \sigma_3)w = (\sigma_1 + i\sigma_2\lambda - i\omega)w + iq(1 - \sigma_2)w$$

Treating the potential term as a perturbation, we note that the equation in which the term involving  $q$  is dropped has the fundamental system

$$\varphi(x, \lambda) = e^{-i\omega x} (u(x, \lambda), \tilde{u}(x, \lambda)) = \begin{pmatrix} -\frac{i\omega(\lambda)}{\lambda-1} & \frac{i\omega(\lambda)}{\lambda-1} e^{-2i\omega(\lambda)x} \\ 1 & e^{-2i\omega(\lambda)x} \end{pmatrix} \quad (x \in \mathbb{R}, \lambda \in \mathbb{C} \setminus \{-1, 1\}).$$

Writing  $w = \varphi A$ , we find that  $A(x, \lambda) = e^{-i \int_x^\infty q} \begin{pmatrix} a(x, \lambda) \\ b(x, \lambda) \end{pmatrix}$ . In particular,  $A(x, \lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  for  $x$  to the right of the support of  $q$ , and  $A' = \varphi'(1 - \sigma_2)iq\varphi A$ . This yields the integral equation

$$A(x, \lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{1}{\omega(\lambda)} \int_x^\infty q(t) \Phi(t) A(t, \lambda) dt$$

with

$$\Phi(t) := \begin{pmatrix} i(\omega - \lambda) & ie^{-2i\omega t} \\ -ie^{2i\omega t} & i(\omega + \lambda) \end{pmatrix}.$$



Iterating this equation twice, we obtain the following identity for the top entry of  $A$ ,

$$\begin{aligned}
& e^{-i \int_x^\infty q} a(x, \lambda) \\
&= A_1(x, \lambda) \\
&= 1 + \frac{i}{\omega(\lambda + \omega)} \int_x^\infty q dt + \frac{1}{\omega^2} \int_x^\infty \int_t^\infty q(t)q(s) \left\{ e^{2i\omega(s-t)} - \frac{1}{(\lambda + \omega)^2} \right\} ds dt \\
&\quad - \frac{1}{\omega^3} \int_x^\infty \int_t^\infty \int_s^\infty q(t)q(s)q(r) e^{-i \int_r^\infty q} \\
&\quad \left\{ \left[ \frac{i}{(\omega + \lambda)^3} - \frac{i e^{2i\omega(s-t)}}{\omega + \lambda} - \frac{i e^{2i\omega(r-s)}}{\omega + \lambda} - \frac{i e^{2i\omega(r-t)}}{\omega - \lambda} \right] a(r, \lambda) \right. \\
&\quad \left. + \left[ i e^{2i\omega(s-t-r)} - \frac{i e^{2i\omega r}}{(\lambda + \omega)^2} + i e^{-2i\omega s} - \frac{i e^{-2i\omega t}}{(\omega - \lambda)^2} \right] b(r, \lambda) \right\} dr ds dt \quad (2.15)
\end{aligned}$$

Now, from the differential equation for  $A$ , we see that

$$\begin{aligned}
A_1'(x, \lambda) &= -\frac{i q(x) A_1(x, \lambda)}{\omega(\lambda)(\omega(\lambda) + \lambda)} + \frac{i q(x) e^{-2i\omega(\lambda)x} A_2(x, \lambda)}{\omega(\lambda)} \\
A_2'(x, \lambda) &= -\frac{i q(x) e^{2i\omega(\lambda)x} A_1(x, \lambda)}{\omega(\lambda)} - \frac{i q(x) A_2(x, \lambda)}{\omega(\lambda)(\omega(\lambda) - \lambda)}
\end{aligned}$$

and so, solving each as a first order differential equation,

$$\begin{aligned}
A_1(x, \lambda) &= e^{\frac{i}{\omega(\lambda)(\omega(\lambda)+\lambda)} \int_x^\infty q} - \int_x^\infty \frac{i q e^{-2i\omega(\lambda)t}}{\omega(\lambda)} e^{\frac{i}{\omega(\lambda)(\omega(\lambda)+\lambda)} \int_x^t q} A_2(t, \lambda) dt \\
A_2(x, \lambda) &= \int_x^\infty \frac{i q(t) e^{2i\omega(\lambda)t}}{\omega(\lambda)} e^{\frac{i}{\omega(\lambda)(\omega(\lambda)-\lambda)} \int_x^t q} A_1(t, \lambda) dt. \quad (2.16)
\end{aligned}$$

Hence eliminating  $A_2$ ,

$$\begin{aligned}
A_1(x, \lambda) &= e^{\frac{i}{\omega(\omega+\lambda)} \int_x^\infty q} \\
&\quad + \frac{1}{\omega^2} \int_x^\infty \int_t^\infty q(t)q(s) e^{2i\omega(s-t)} e^{\frac{i}{\omega(\omega+\lambda)} \int_x^t q} e^{\frac{i}{\omega(\omega-\lambda)} \int_t^s q} A_1(s, \lambda) ds dt. \quad (2.17)
\end{aligned}$$

We now assume that  $|\lambda|$ , and hence  $|\omega|$ , is large enough so that  $\left| e^{\frac{i}{\omega(\lambda)(\omega(\lambda)+\lambda)} \int_x^\infty q} \right| \leq 2$  and

$$\frac{1}{\omega(\lambda)^2} \int_x^\infty \int_t^\infty |q(t)| |q(s)| \left| e^{\frac{i}{\omega(\lambda)(\omega(\lambda)+\lambda)} \int_x^t q} \right| \left| e^{\frac{i}{\omega(\lambda)(\omega(\lambda)-\lambda)} \int_t^s q} \right| ds dt < \frac{1}{2};$$

this can be achieved in view of Lemma 2.2 (iii) and (iv), respectively. Hence, noting that

$|a| = |A_1|$ , we obtain from (2.17) that

$$\|a(\cdot, \lambda)\|_\infty \leq 2 + \frac{\|a(\cdot, \lambda)\|_\infty}{2},$$

which implies that  $\|a(\cdot, \lambda)\|_\infty \leq 4$ , for such values of  $\lambda$ . Thus, substituting (2.16) and (2.17) into (2.15)

$$\begin{aligned}
e^{-i \int_x^\infty q} a(x, \lambda) &= 1 + \frac{i}{\omega(\lambda + \omega)} \int_x^\infty q dt + \frac{1}{\omega^2} \int_x^\infty \int_t^\infty q(t)q(s) \left\{ e^{2i\omega(s-t)} - \frac{1}{(\lambda + \omega)^2} \right\} ds dt \\
&\quad - \frac{1}{\omega^3} \int_x^\infty \int_t^\infty \int_s^\infty q(t)q(s)q(r) \left[ \frac{i}{(\omega + \lambda)^3} - \frac{ie^{2i\omega(s-t)}}{\omega + \lambda} - \frac{ie^{2i\omega(r-s)}}{\omega + \lambda} - \frac{ie^{2i\omega(r-t)}}{\omega - \lambda} \right] \\
&\quad \times \left\{ e^{\frac{i}{\omega(\omega+\lambda)} \int_r^\infty q} + \frac{1}{\omega^2} \int_r^\infty \int_p^\infty q(r)q(p) e^{2i\omega(u-p)} e^{\frac{i}{\omega(\omega+\lambda)} \int_r^p q} e^{\frac{i}{\omega(\omega-\lambda)} \int_p^u q} A_1(u, \lambda) du dp \right\} dr ds dt \\
&\quad - \frac{1}{\omega^3} \int_x^\infty \int_t^\infty \int_s^\infty q(t)q(s)q(r) \left[ ie^{2i\omega(s-t-r)} - \frac{ie^{2i\omega r}}{(\lambda + \omega)^2} + ie^{-2i\omega s} - \frac{ie^{-2i\omega t}}{(\omega - \lambda)^2} \right] \\
&\quad \times \left\{ \int_r^\infty \frac{iq(p)e^{2i\omega p}}{\omega} e^{\frac{i}{\omega(\omega-\lambda)} \int_r^p q} A_1(p, \lambda) dp \right\} dr ds dt \\
&= 1 + \frac{i}{\omega(\lambda + \omega)} \int_x^\infty q dt + \frac{1}{\omega^2} \int_x^\infty \int_t^\infty q(t)q(s) e^{2i\omega(s-t)} ds dt \\
&\quad - \frac{i(\omega + \lambda)}{\omega^3} \int_x^\infty \int_t^\infty \int_s^\infty q(t)q(s)q(r) e^{2i\omega(r-t)} e^{\frac{i}{\omega(\omega+\lambda)} \int_r^\infty q} dr ds dt \\
&\quad - \frac{(\omega + \lambda)^2}{\omega^4} \int_x^\infty \int_t^\infty \int_s^\infty \int_r^\infty q(t)q(s)q(r)q(p) e^{2i\omega(p-t)} e^{\frac{i}{\omega(\omega-\lambda)} \int_r^p q} A_1(p, \lambda) dp dr ds dt \\
&\quad + O\left(\frac{1}{|\lambda|^4}\right) \quad (|\lambda| \rightarrow \infty).
\end{aligned}$$

We now take the limit  $x \rightarrow -\infty$ . Furthermore, using equation (2.17) we can substitute for  $A_1(x, \lambda)$  in the above equation. The 6-fold integral which arises can be moved directly into the asymptotic term. Using Lemma 2.2 (iii) to handle the first term from (2.17) we obtain:

$$\begin{aligned}
e^{-i \int_{-\infty}^\infty q} a(\lambda) &= \lim_{x \rightarrow -\infty} e^{-i \int_x^\infty q} a(x, \lambda) \\
&= 1 + \frac{i}{\omega(\lambda + \omega)} \int_{\mathbb{R}} q dt + \frac{1}{\omega^2} \int_{-\infty}^\infty \int_t^\infty q(t)q(s) e^{2i\omega(s-t)} ds dt \\
&\quad - \frac{i(\omega + \lambda)}{\omega^3} \int_{-\infty}^\infty \int_t^\infty \int_s^\infty q(t)q(s)q(r) e^{2i\omega(r-t)} dr ds dt \\
&\quad - \frac{(\omega + \lambda)^2}{\omega^4} \int_{-\infty}^\infty \int_t^\infty \int_s^\infty \int_r^\infty q(t)q(s)q(r)q(p) e^{2i\omega(p-t)} e^{\frac{i}{\omega(\omega-\lambda)} \int_r^p q} dp dr ds dt \\
&\quad + O\left(\frac{1}{|\lambda|^4}\right) \quad (|\lambda| \rightarrow \infty). \tag{2.18}
\end{aligned}$$

Consider the anticlockwise contour in the complex upper half-plane  $\gamma_R$  parametrised by  $\lambda(\theta) = \sqrt{R^2 e^{2i\theta} + 1}$  ( $\theta \in [0, \pi]$ ), with  $R > 1$ . This contour is chosen so that

$\omega(\lambda(\theta)) = Re^{i\theta}$ ; it follows that  $d\lambda = i\frac{\omega(\lambda)^2}{\lambda} d\theta = \frac{\omega}{\lambda} d\omega$ . Then

$$\begin{aligned} \int_{\gamma_R} \lambda \omega(\lambda) \log \left[ e^{i \int_{-\infty}^{\infty} q a(\lambda)} \right] d\lambda &= \int_{\gamma_R} \left\{ \frac{i\lambda}{\lambda + \omega} \int_{\mathbb{R}} q dt + \frac{\lambda}{\omega} \int_{-\infty}^{\infty} \int_t^{\infty} q(t)q(s)e^{2i\omega(s-t)} ds dt \right. \\ &+ \frac{i\lambda}{\omega^2(\omega - \lambda)} \int_{-\infty}^{\infty} \int_t^{\infty} \int_s^{\infty} q(t)q(s)q(r)e^{2i\omega(r-t)} dr ds dt \\ &- \frac{\lambda}{\omega^3(\omega - \lambda)^2} \int_{-\infty}^{\infty} \int_t^{\infty} \int_s^{\infty} \int_r^{\infty} q(t)q(s)q(r)q(p)e^{2i\omega(p-t)} e^{\frac{i}{\omega(\omega-\lambda)} \int_r^p q} dp dr ds dt \\ &\left. + O\left(\frac{1}{|\lambda|^2}\right) \right\} d\lambda \quad (|\lambda| \rightarrow \infty). \end{aligned} \quad (2.19)$$

We now consider each integral term on the right-hand side in turn. The first one evaluates to

$$\int_{\gamma_R} \left( \frac{i\lambda}{\lambda + \omega} \int_{\mathbb{R}} q dt \right) d\lambda = i \left( \int_{\mathbb{R}} q dt \right) \int_{\gamma_R} \lambda(\lambda - \sqrt{\lambda^2 - 1}) d\lambda = \left[ \frac{2iR^3}{3} - \frac{2i\sqrt{R^2 + 1}}{3} \right] \int_{\mathbb{R}} q dt,$$

which is purely imaginary.

To treat the second term, we apply the symmetrisation rule which states that

$$F \in L^1(\mathbb{R}^2), \quad F(x, y) = F(y, x) \quad ((x, y) \in \mathbb{R}^2) \quad \implies \quad \int_{-\infty}^{\infty} \int_x^{\infty} F(x, y) dy dx = \frac{1}{2} \int_{\mathbb{R}^2} F. \quad (2.20)$$

Indeed notice that we are only integrating over the upper left triangle of the smallest square of side length  $[\inf \text{supp}(q), \sup \text{supp}(q)]$ . Call this triangle  $T_1$  and the integral over the triangle  $\mathcal{IT}_1$ . Then, defining  $\frac{\lambda}{\omega} e^{2i\omega(z-y)} = f(z-y)$ ,

$$\begin{aligned} \mathcal{IT}_1 &= \int_x^{\infty} \int_y^{\infty} f(z-y)q(y)q(z) dz dy = \int_x^{\infty} \int_y^{\infty} f(|z-y|)q(y)q(z) dz dy \\ &= \int_x^{\infty} \int_z^{\infty} f(|y-z|)q(y)q(z) dz dy = \int_x^{\infty} \int_z^{\infty} f(y-z)q(y)q(z) dz dy = \mathcal{IT}_2 \end{aligned}$$

where  $T_2$  is the lower right triangle. Thus

$$\mathcal{IT}_1 = \frac{1}{2}(\mathcal{IT}_1 + \mathcal{IT}_2) = \frac{1}{2}(\mathcal{I}(T_1 \cup T_2)) = \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(|z-y|)q(y)q(z) dy dz.$$

We also make a change of variables in the contour integral, using  $\omega$  to denote the transformed variable by a slight abuse of notation.

The transformed contour is  $\gamma_R^\omega := \omega(\gamma_R)$ , in fact a simple semicircle. Since

$$\int_{\gamma_R} \frac{\lambda}{\omega(\lambda)} e^{2i\omega(\lambda)|s-t|} d\lambda = \int_{\gamma_R^\omega} e^{2i\omega|s-t|} d\omega = -\sin(2R|x-y|)/|x-y|,$$

$$\int_{\gamma_R} \frac{\lambda}{\omega} \int_{-\infty}^{\infty} \int_t^{\infty} q(t)q(s)e^{2i\omega(s-t)} ds dt d\lambda = \int_{\gamma_R} \frac{\lambda}{2\omega} \int_{\mathbb{R}} \int_{\mathbb{R}} q(t)q(s)e^{2i\omega|s-t|} ds dt d\lambda \rightarrow -\frac{\pi}{2} \int_{\mathbb{R}} q^2$$

( $R \rightarrow \infty$ ) by (2.20) and Lemma 2.4. For the third integral in (2.19) notice that

$$\frac{\lambda d\lambda}{\omega^2(\omega - \lambda)} = \frac{d\omega}{\omega(\omega - \lambda)} = -\frac{(\omega + \lambda) d\lambda}{\omega} = -\frac{(\lambda - \omega + 2\omega)d\omega}{\omega} = \left(-2 - \frac{1}{\omega(\lambda + \omega)}\right) d\omega.$$

Hence

$$\begin{aligned} & \int_{\gamma_R} \frac{i\lambda}{\omega^2(\omega - \lambda)} \int_{-\infty}^{\infty} \int_t^{\infty} \int_s^{\infty} q(t)q(s)q(r)e^{2i\omega(r-t)} dr ds dt d\lambda \\ &= \int_{\gamma_R} i \left(-2 - \frac{1}{\omega(\lambda + \omega)}\right) \int_{-\infty}^{\infty} \int_t^{\infty} \int_s^{\infty} q(t)q(s)q(r)e^{2i\omega(r-t)} dr ds dt d\omega \\ &= -2i \int_{-\infty}^{\infty} \int_t^{\infty} \int_s^{\infty} \frac{\sin(2R(r-t))}{(r-t)} q(t)q(s)q(r) dr ds dt + O\left(\frac{1}{R}\right) \quad (R \rightarrow \infty) \end{aligned}$$

noting that the length of the contour  $\gamma_R^\omega$  is  $O(R)$ . This is purely imaginary up to the error term. For the final integral term in (2.19), we have

$$\begin{aligned} & \int_{\gamma_R} \frac{\lambda}{\omega^3(\omega - \lambda)^2} \int_{-\infty}^{\infty} \int_t^{\infty} \int_s^{\infty} \int_r^{\infty} q(t)q(s)q(r)q(p)e^{2i\omega(p-t)} e^{\frac{i}{\omega(\omega-\lambda)} \int_r^p q} dp dr ds dt d\lambda \\ &= \int_{\gamma_R} \frac{\lambda [4\omega^2 + 4(\lambda - \omega)\omega + (\lambda - \omega)^2]}{\omega^3} \\ & \quad \times \int_{-\infty}^{\infty} \int_t^{\infty} \int_s^{\infty} \int_r^{\infty} q(t)q(s)q(r)q(p)e^{2i\omega(p-t)} e^{\frac{i}{\omega(\omega-\lambda)} \int_r^p q} dp dr ds dt d\lambda \\ &= 4 \int_{\gamma_R^\omega} \int_{-\infty}^{\infty} \int_t^{\infty} \int_s^{\infty} \int_r^{\infty} q(t)q(s)q(r)q(p)e^{2i\omega(p-t)} e^{-2i \int_r^p q} dp dr ds dt d\omega + O\left(\frac{1}{R}\right) \end{aligned}$$

( $R \rightarrow \infty$ ) where we used Lemma 2.2 (iv) in the last step. By an integration by parts,

$$\int_r^{\infty} q(p)e^{2i\omega(p-t)} e^{-2i \int_r^p q} dp = -\frac{i}{2} e^{2i\omega(r-t)} + \omega \int_r^{\infty} e^{-2i \int_r^p q} e^{2i\omega(p-t)} dp.$$

Indeed,

$$\begin{aligned} & \int_r^{\infty} q(p)e^{2i\omega(p-t)} e^{-2i \int_r^p q} dp \\ &= \frac{i}{2} \int_r^{\infty} \frac{d}{dp} \left( e^{-2i \int_r^p q} \right) e^{2i\omega(p-t)} dp \\ &= \frac{i}{2} \left[ \lim_{\alpha \rightarrow \infty} \left[ e^{-2i \int_r^p q} e^{2i\omega(p-t)} \right]_{p=r}^{p=\alpha} - 2i\omega \int_r^{\infty} e^{-2i \int_r^p q} e^{2i\omega(p-t)} dp \right] \\ &= \frac{i}{2} \lim_{\alpha \rightarrow \infty} \left\{ e^{-2i \int_r^\alpha q} e^{2i\omega(\alpha-t)} \right\} - \frac{i}{2} e^{2i\omega(r-t)} + \omega \int_r^{\infty} e^{-2i \int_r^p q} e^{2i\omega(p-t)} dp. \end{aligned}$$

As  $\Im\omega > 0$  and  $\alpha - t > 0$ ,

$$\left| e^{2i \int_{\alpha}^{\infty} q} e^{2i\omega(\alpha-t)} \right| = \left| e^{-2\Im\omega(\alpha-t)} \right| \rightarrow 0 \quad (\alpha \rightarrow \infty).$$

Thus

$$\begin{aligned} & 4 \int_{\gamma_R^{\omega}} \int_{-\infty}^{\infty} \int_t^{\infty} \int_s^{\infty} \int_r^{\infty} q(t)q(s)q(r)q(p) e^{2i\omega(p-t)} e^{-2i \int_r^p q} dp dr ds dt d\omega \\ &= -2i \int_{\gamma_R^{\omega}} \int_{-\infty}^{\infty} \int_r^{\infty} \int_s^{\infty} q(t)q(s)q(r) e^{2i\omega(r-t)} dr ds dt d\omega \\ & \quad + \int_{\gamma_R^{\omega}} 4\omega \int_{-\infty}^{\infty} \int_t^{\infty} \int_s^{\infty} q(t)q(s)q(r) \left[ \int_r^{\infty} e^{-2i \int_r^p q} e^{2i\omega(p-t)} dp \right] dr ds dt d\omega. \end{aligned} \quad (2.21)$$

This leaves us with two integrals to consider. Performing the contour integral first, we see that the first term is purely imaginary. The remaining integral can be resolved by repeated integrations by parts, starting from the innermost integral. Observe that for any  $z \in \mathbb{R}$  and  $x \geq v$  and by integrating by parts

$$\begin{aligned} & \int_z^{\infty} q(x) e^{-2i \int_x^{\infty} q} \left[ \int_x^{\infty} e^{2i \int_y^{\infty} q} e^{2i\omega(y-v)} dy \right] dx \\ &= \frac{i}{2} e^{-2i \int_z^{\infty} q} \int_z^{\infty} e^{2i \int_y^{\infty} q} e^{2i\omega(y-v)} dy + \frac{1}{4\omega} e^{2i\omega(z-v)}. \end{aligned} \quad (2.22)$$

Thus, by an integration by parts, the last term in (2.21) equals

$$\begin{aligned} & \int_{\gamma_R^{\omega}} \int_{-\infty}^{\infty} \int_t^{\infty} q(t)q(s) e^{2i\omega(s-t)} ds dt d\omega \\ & \quad + \int_{\gamma_R^{\omega}} 2i\omega \int_{-\infty}^{\infty} \int_t^{\infty} q(t)q(s) e^{-2i \int_s^{\infty} q} \left[ \int_s^{\infty} e^{2i \int_p^{\infty} q} e^{2i\omega(p-t)} dp \right] ds dt d\omega. \end{aligned} \quad (2.23)$$

Again we have two integrals to consider. Referring back to the treatment of the second integral (page 29), after symmetrisation the first term in (2.23) tends to  $-\frac{\pi}{2} \int_{\mathbb{R}} q^2$  as  $R \rightarrow \infty$  by Lemma 2.4. We can again apply (2.22) to the second integral in (2.23), then

integrate by parts in the innermost integral, giving

$$\begin{aligned}
& \int_{\gamma_R^\omega} 2i\omega \int_{-\infty}^{\infty} q(t) \int_t^{\infty} q(s) e^{-2i \int_s^\infty q} \left[ \int_s^{\infty} e^{2i \int_p^\infty q} e^{2i\omega(p-t)} dp \right] ds dt d\omega \\
&= - \int_{\gamma_R^\omega} \omega \int_{\mathbb{R}} q(t) e^{-2i \int_t^\infty q} \left[ \int_t^{\infty} e^{2i \int_p^\infty q} e^{2i\omega(p-t)} dp \right] dt d\omega + \frac{i}{2} \int_{-\infty}^{\infty} q \int_{\gamma_R^\omega} d\omega \\
&= - \int_{\gamma_R^\omega} \int_{-\infty}^{\infty} q(t) e^{-2i \int_t^\infty q} \int_t^{\infty} q(p) e^{2i \int_p^\infty q} e^{2i\omega(p-t)} dp dt d\omega \\
&= \int_{-\infty}^{\infty} \int_t^{\infty} q(t) q(p) e^{2i \int_p^t q} \frac{\sin(2R(p-t))}{(p-t)} dp dt.
\end{aligned}$$

Taking the real part, symmetrising and applying Lemma 2.5 twice, we find that

$$\begin{aligned}
& \Re \left( \int_{\gamma_R^\omega} 2i\omega \int_{-\infty}^{\infty} q(t) \int_t^{\infty} q(s) e^{-2i \int_s^\infty q} \left[ \int_s^{\infty} e^{2i \int_p^\infty q} e^{2i\omega(p-t)} dp \right] ds dt d\omega \right) \\
&= \int_{-\infty}^{\infty} \int_t^{\infty} q(t) q(p) \cos \left( 2 \int_p^t q \right) \frac{\sin(2R(p-t))}{(p-t)} dp dt \\
&= \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} q(t) q(p) \left( \cos \left( 2 \int_p^\infty q \right) \cos \left( 2 \int_t^\infty q \right) + \sin \left( 2 \int_p^\infty q \right) \sin \left( 2 \int_t^\infty q \right) \right) \frac{\sin(2R|p-t|)}{|p-t|} dp dt \\
&\rightarrow \frac{\pi}{2} \int_{-\infty}^{\infty} q^2(t) \left[ \cos^2 \left( 2 \int_t^\infty q \right) + \sin^2 \left( 2 \int_t^\infty q \right) \right] dt = \frac{\pi}{2} \int_{-\infty}^{\infty} q^2(t) dt
\end{aligned}$$

as  $R \rightarrow \infty$ . This cancels out the first term of (2.23). In summary, (2.19) comes down to

$$\lim_{R \rightarrow \infty} \Re \int_{\gamma_R} \lambda \omega(\lambda) \log \left[ e^{-i \int_{-\infty}^\infty q} a(\lambda) \right] d\lambda = -\frac{\pi}{2} \int_{\mathbb{R}} q^2(x) dx. \quad (2.24)$$

Let  $0 < \varepsilon < 1$  and consider the closed contour  $\tilde{\Gamma}_\varepsilon^R = \Gamma_{R,\varepsilon} \cup \Gamma_\varepsilon \cup \gamma_{R,\varepsilon}$  where

$$\gamma_{R,\varepsilon} = \gamma_R \cap \{\lambda : \varepsilon \leq \Im \lambda\}, \quad \Gamma_\varepsilon = [-1 + i\varepsilon, 1 + i\varepsilon], \quad \Gamma_{R,\varepsilon} = [\varkappa_-, -1 + i\varepsilon] \cup [1 + i\varepsilon, \varkappa_+].$$

Here  $\varkappa_\pm$  are the points where the contour  $\gamma_R$  intersects the line  $\Im \lambda = \varepsilon$ . In addition,

we consider the two-component contour  $\gamma_{R,\varepsilon}^c = \gamma_R \setminus \gamma_{R,\varepsilon}$ . Recalling that  $\lambda \omega(\lambda) \log \left[ e^{-i \int_{-\infty}^\infty q} a(\lambda) \right] = O(1)$  ( $|\lambda| \rightarrow \infty$ ) from (2.18), we see that

$$\Re \int_{\gamma_{R,\varepsilon}^c} \lambda \omega(\lambda) \log \left[ e^{-i \int_{-\infty}^\infty q} a(\lambda) \right] d\lambda = O(\varepsilon) \quad (\varepsilon \rightarrow 0).$$

On the other hand, we find using Cauchy's Integral Theorem that

$$\begin{aligned}
& -\Re \int_{\gamma_{R,\varepsilon}} \lambda \omega(\lambda) \log \left[ e^{-i \int_{-\infty}^{\infty} q} a(\lambda) \right] d\lambda \\
&= \Re \left( \int_{\Gamma_{R,\varepsilon}} \lambda \omega(\lambda) \log a(\lambda) d\lambda + \int_{\Gamma_{\varepsilon}} \lambda \omega(\lambda) \log a(\lambda) d\lambda + \log \left[ e^{-i \int_{-\infty}^{\infty} q} \right] \int_{\Gamma_{R,\varepsilon} \cup \Gamma_{\varepsilon}} \lambda \omega(\lambda) d\lambda \right).
\end{aligned} \tag{2.25}$$

By Lemma 2.2 (ii), it is clear that  $\Re \omega(-\mu + i\varepsilon) = -\Re \omega(\mu + i\varepsilon)$  and  $\Im \omega(-\mu + i\varepsilon) = \Im \omega(\mu + i\varepsilon)$ . Hence the imaginary part of the integrand of the last integral in (2.25) is odd, and the logarithmic factor is purely imaginary. Thus the real part of the last term in (2.25) vanishes.

The first integral in (2.25) can be rewritten as

$$\int_{\Gamma_{R,\varepsilon}} \lambda \omega(\lambda) \log a(\lambda) d\lambda = \int_{\Gamma_{R,\varepsilon}} \lambda \omega(\lambda) \log |a(\lambda)| d\lambda + i \int_{\Gamma_{R,\varepsilon}} \lambda \omega(\lambda) \arg a(\lambda) d\lambda.$$

Now it is clear that

$$\lim_{\varepsilon \rightarrow 0} \Im \int_{\Gamma_{R,\varepsilon}} \lambda \omega(\lambda) \arg a(\lambda) d\lambda = 0,$$

as  $\arg a(\lambda)$  is real and bounded and  $\Im(\lambda \omega(\lambda)) \rightarrow 0$  uniformly. Thus we need only consider

$$\lim_{\varepsilon \rightarrow 0} \int_{\Gamma_{R,\varepsilon}} \lambda \omega(\lambda) \log |a(\lambda)| d\lambda = \lim_{\varepsilon \rightarrow 0} \int_{(-\sqrt{R^2+1}, -1] \cup [1, \sqrt{R^2+1})} (t+i\varepsilon) \omega(t+i\varepsilon) \log |a(t+i\varepsilon)| dt.$$

In view of (2.14),

$$\lim_{\lambda \rightarrow \pm 1} \lambda \omega(\lambda) \log |a(\lambda)| = 0$$

for real  $\lambda$ ; also  $a$  is continuous and has no zeros in  $\mathbb{R} \setminus [-1, 1]$ . Thus  $(t+i\varepsilon) \omega(t+i\varepsilon) \log |a(t+i\varepsilon)|$  is bounded uniformly in  $\varepsilon$  on  $(-\sqrt{R^2+1}, -1] \cup [1, \sqrt{R^2+1})$ , and by dominated convergence

$$\lim_{\varepsilon \rightarrow 0} \Re \int_{\Gamma_{R,\varepsilon}} \lambda \omega(\lambda) \log a(\lambda) d\lambda = \int_{(-\sqrt{R^2+1}, -1] \cup [1, \sqrt{R^2+1})} \lambda \omega(\lambda) \log |a(\lambda)| d\lambda. \tag{2.26}$$

Finally we consider the second integral in (2.25),

$$\begin{aligned}
& \int_{\Gamma_{\varepsilon}} \lambda \omega(\lambda) \log a(\lambda) d\lambda \\
&= \int_{-1}^1 (t+i\varepsilon) \omega(t+i\varepsilon) \log |a(t+i\varepsilon)| dt + i \int_{-1}^1 (t+i\varepsilon) \omega(t+i\varepsilon) \arg a(t+i\varepsilon) dt.
\end{aligned} \tag{2.27}$$

For the first of these integrals, we note that  $a$  has a finite number of distinct zeros in the interval  $(-1, 1)$ , which we label  $\beta_1, \dots, \beta_M$  in increasing order. The (real) logarithm function is integrable at zero and so, by dominated convergence and (2.11), this integral tends to the purely imaginary limit

$$\int_{-1}^1 t \omega(t) \log |a(t)| dt = \int_{-1}^1 it \sqrt{1-t^2} \log |a(t)| dt$$

as  $\varepsilon \rightarrow 0$ . Concerning the second integral in (2.27), we note that, by Lemma 2.3,  $a(\lambda)$  is real for  $\lambda \in (-1, 1)$ . Therefore, between any two zeros of  $a$  on  $(-1, 1)$ , the argument of  $a$  is constant. Thus we need only consider the argument of  $a$  at a zero  $\beta_j$ . We write

$$a(\lambda) = (\lambda - \beta_j) b(\lambda),$$

where  $b$  is analytic and non-zero in some neighbourhood  $\Omega$  of  $\beta_j$ . The respective arguments satisfy

$$\arg a(\lambda) = \arg(\lambda - \beta_j) + \arg b(\lambda)$$

and  $\arg b(\lambda)$  is continuous at  $\beta_j$ . Therefore, if we consider  $a(\lambda)$  on the intersection of  $\{\lambda : \Im \lambda = \varepsilon\}$  with  $B_{\sqrt{\varepsilon}}(\beta_j)$ , the ball of radius  $\sqrt{\varepsilon}$  and centre  $\beta_j$ , the argument of  $b$  is almost constant and thus the change in the argument of  $a$  between the left and right ends of this interval is  $\sim -2 \arccos \sqrt{\varepsilon}$ , which tends to  $-\pi$  in the limit  $\varepsilon \rightarrow 0$ . The limiting values of the argument of  $a$  thus have the form

$$\arg a(\lambda) = \arg a(-1 + 0) - \pi \sum_{m=1}^M \chi_{(\beta_m, 1)}(\lambda) \quad (\lambda \in (-1, 1) \setminus \{\beta_i \mid i = 1 \dots M\}).$$

Thus, bearing in mind (2.11),

$$\begin{aligned} & i \lim_{\varepsilon \rightarrow 0} \int_{-1}^1 (\lambda + i\varepsilon) \omega(\lambda + i\varepsilon) \arg a(\lambda + i\varepsilon) d\lambda \\ &= -\arg a(-1 + 0) \int_{-1}^1 \lambda \sqrt{1-\lambda^2} d\lambda + \pi \sum_{m=1}^M \int_{\beta_m}^1 \lambda \sqrt{1-\lambda^2} d\lambda \\ &= -\frac{\pi}{3} \sum_m (1 - \beta_m^2)^{\frac{3}{2}}. \end{aligned} \tag{2.28}$$



Hence, by (2.24), (2.25), (2.26), (2.28) and (2.11),

$$\begin{aligned} \frac{\pi}{2} \int_{\mathbb{R}} q^2 &= - \lim_{R \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \Re \int_{\gamma_{R,\varepsilon}} \lambda \omega(\lambda) \log \left[ e^{i \int_{\mathbb{R}} q} a(\lambda) \right] d\lambda \\ &= \int_{(-\infty, -1] \cup [1, \infty)} |\lambda| \sqrt{\lambda^2 - 1} \log |a(\lambda)| d\lambda + \frac{\pi}{3} \sum_m (1 - \beta_m^2)^{\frac{3}{2}}. \end{aligned} \quad (2.29)$$

This completes the proof of Theorem 2.3.

## 4 Schrödinger Case: The Transmission Coefficient

The following is a more detailed exposition of the paper of Deift and Killip [11], which has served as a basis for the Dirac treatment in Section 3 (which is, however, more complicated). It is included here for the purposes of comparison and also because the paper for Deift-Killip is cursory on some of the details. We also take a slightly different approach here in that we consider truncated approximations to the potential rather than continuous approximations.

Throughout this section we shall again assume that  $q$  is square integrable and supported on a compact subset of  $[0, \infty)$ . For each such  $q$  it is well known that for every  $k \in \mathbb{C}$ , there exists a solution to

$$-\psi''(x) + q(x)\psi(x) = k^2\psi(x), \quad \forall x \in \mathbb{R} \quad (2.30)$$

such that  $\psi(x) = e^{ikx}$  for all  $x$  to the right of the support of  $q$ . Moreover, for each  $x$ ,  $\psi(x)$  and  $\psi'(x)$  are analytic functions of  $k$  (this follows from  $\psi$  being a solution of the above equation with an analytic boundary condition).

To the left of the support of  $q$ ,  $\psi$  must satisfy the free Schrödinger equation (i.e. (2.30) with  $q$  identically zero) and so take the form

$$\psi(x) = a(k)e^{ikx} + b(k)e^{-ikx}. \quad (2.31)$$

As  $\psi$  depends analytically on  $k$ , so do  $a$  and  $b$ . Since  $e^{\pm ikx}$  represents waves propagating to the right/left in the time dependent picture, it is natural to term  $t = \frac{1}{a}$  and  $r = \frac{b}{a}$  the transmission and reflection coefficients respectively. We are now in a position to prove Theorem 2.4.

PROOF. We begin by rewriting equation (2.30) as an integral equation. Let  $u, w$  be two linearly independent solutions of the homogeneous equation

$$\psi'' + k^2\psi = 0. \quad (2.32)$$

The inhomogeneous equation (2.30) can be written as a system of equations as follows:

$$\begin{pmatrix} \psi \\ \psi' \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -k^2 & 0 \end{pmatrix} \begin{pmatrix} \psi \\ \psi' \end{pmatrix} + \begin{pmatrix} 0 \\ q\psi \end{pmatrix}$$

We let  $S := \begin{pmatrix} \psi \\ \psi' \end{pmatrix} = \Phi P$  for  $\Phi = \begin{pmatrix} u & w \\ u' & w' \end{pmatrix}$  and some unknown  $P$ ; this is known as the variation of constants method. Then  $S' = \Phi'P + \Phi P'$ , allowing us to deduce that  $\Phi P' = \begin{pmatrix} 0 \\ q\psi \end{pmatrix}$ . Since  $u, w$  are linearly independent solutions, we know that  $\det \Phi$  is non zero; hence  $\Phi$  is invertible. Thus, as  $u, w$  are continuous  $q$  is integrable with compact support and  $\psi$  is continuous,

$$P(x) = -\frac{1}{uw' - wu'} \int_x^\infty \begin{pmatrix} -w \\ u \end{pmatrix} q\psi dt + C, \quad (x \geq 0).$$

Therefore

$$S = \Phi \left( \frac{1}{uw' - wu'} \int_x^\infty \begin{pmatrix} w \\ -u \end{pmatrix} q\psi dt + C \right)$$

yielding the integral equation

$$\psi(x) = c_1 u(x) + c_2 w(x) + \frac{1}{u(x)w'(x) - w(x)u'(x)} \int_x^\infty (u(x)w(t) - w(x)u(t)) q(t)\psi(t) dt. \quad (2.33)$$

We have assumed that  $u, w$  are any linearly independent solutions of (2.32); thus  $u(x) = e^{ikx}$  and  $w(x) = e^{-ikx}$  is sufficient. This gives from (2.33)

$$\psi(x) = c_1 e^{ikx} + c_2 e^{-ikx} + \frac{i}{2k} \int_x^\infty (e^{ikx-ikt} - e^{ikt-ikx}) q(t)\psi(t) dt.$$

Multiplying by  $e^{-ikx}$  and defining  $v(x) = e^{-ikx}\psi(x)$  we obtain

$$v(x) = c_1 + c_2 e^{-2ikx} + \frac{i}{2k} \int_x^\infty (1 - e^{2ik(t-x)}) q(t)v(t) dt.$$

Since we have chosen our solution to the right of the support of  $q$  such that  $\psi(x) = e^{ikx}$ ,  $v(x) = 1$  to the right of the support of  $q$  and so we have that  $c_1 = 1$  and  $c_2 = 0$ . Thus

$$v(x) = 1 + \frac{i}{2k} \int_x^\infty (1 - e^{2ik(t-x)}) q(t)v(t) dt, \quad (2.34)$$

is an integral form of the equation (2.30).

To proceed we first note that

$$\begin{aligned} v(x) &= 1 + \frac{i}{2k} \int_x^\infty [1 - e^{2ik(y-x)}] q(y)v(y) dy \\ &= \underbrace{\left(1 + \frac{i}{2k} \int_x^\infty q(y)v(y) dy\right)}_{E_1} + e^{-2ikx} \underbrace{\left(-\frac{i}{2k} \int_x^\infty e^{2iky} q(y)v(y) dy\right)}_{E_2}, \end{aligned}$$

where we notice that for  $x$  to the left of the support of  $q$ ,  $E_1$  and  $E_2$  are constant. This gives us by comparison with (2.31), noting that 1 and  $e^{-2ikx}$  are linearly independent,

$$a(x, k) = 1 + \frac{i}{2k} \int_x^\infty q(y)v(y) dy.$$

We solve this by repeated substitution for  $v(\cdot)$  from (2.34).

$$\begin{aligned} a(x, k) &= 1 + \frac{i}{2k} \int_x^\infty q(y) dy - \frac{1}{4k^2} \int_x^\infty \int_y^\infty [1 - e^{2ik(z-y)}] q(y)q(z) dz dy \\ &\quad - \frac{i}{8k^3} \int_x^\infty \int_y^\infty \int_z^\infty [1 - e^{2ik(z-y)}][1 - e^{2ik(w-z)}] q(y)q(z)q(w) dy dz dw \\ &\quad + \frac{1}{16k^4} \int_x^\infty \int_y^\infty \int_z^\infty \int_w^\infty [1 - e^{2ik(z-y)}][1 - e^{2ik(w-z)}][1 - e^{2ik(u-w)}] \\ &\quad \quad \quad \times q(y)q(z)q(w)q(u)v(u) du dy dz dw. \quad (2.35) \end{aligned}$$

We consider in more detail the four-fold integral in (2.35). Define

$$\begin{aligned} f(k) &:= \frac{1}{16} \int_x^\infty \int_y^\infty \int_z^\infty \int_w^\infty [1 - e^{2ik(z-y)}][1 - e^{2ik(w-z)}][1 - e^{2ik(u-w)}] \\ &\quad \quad \quad \times q(y)q(z)q(w)q(u)v(u) du dy dz dw. \end{aligned}$$

Then

$$|f(k)| \leq \frac{1}{2} \int_{\text{supp}(q)} \int_{\text{supp}(q)} \int_{\text{supp}(q)} \int_{\text{supp}(q)} |q(y)||q(z)||q(w)||q(u)||v(u)|dudydzdw.$$

Choosing  $k$  large enough so that  $\frac{1}{|k|} \int_{\text{supp}(q)} q(x)dx \leq \frac{1}{2}$ , (2.34) gives

$$\|v\|_\infty \leq 1 + \frac{1}{|k|} \int_{\text{supp}(q)} q(x)dx \|v\|_\infty,$$

giving  $\|v\|_\infty \leq 2$ . Thus  $|f(k)| \leq M$  for some  $M > 0$ . Thus,

$$\left| \frac{f(k)}{k^4} \right| \leq \frac{M}{|k|^4} \rightarrow 0, \quad k \rightarrow \infty$$

and so we can write  $I_1 = O(k^{-4})$ . Returning to the integral equation for  $a$  we have

$$\begin{aligned} a(x, k) &= 1 + \frac{i}{2k} \int_x^\infty q(y)dy - \frac{1}{4k^2} \int_x^\infty \int_y^\infty [1 - e^{2ik(z-y)}]q(y)q(z)dydz \\ &\quad - \frac{i}{8k^3} \int_x^\infty \int_y^\infty \int_z^\infty [1 - e^{2ik(z-y)}][1 - e^{2ik(w-z)}]q(y)q(z)q(w)dydzdw \\ &\quad + O(k^{-4}) \end{aligned}$$

It is now our aim to show, for  $\gamma_R$  an anticlockwise contour parametrised by  $k(\theta) = Re^{i\theta}$ ,  $\theta \in [0, \pi]$ , that

$$\lim_{R \rightarrow \infty} \int_{\gamma_R} k^2 \log a(k) dk = -\frac{\pi}{8} \int_{\mathbb{R}} q^2(x) dx$$

We know that for  $|u| < 1$ ,  $\log(1+u) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} u^n$ . Define  $u = a - 1$ . Then considering  $k$  initially large enough so that  $|a(x, k) - 1| < 1$

$$\begin{aligned} \log a(x, k) &= u - \frac{u^2}{2} + \frac{u^3}{3} + O(k^{-4}) \\ &= \frac{i}{2k} \int_x^\infty q(y)dy - \frac{1}{4k^2} \int_x^\infty \int_y^\infty [1 - e^{2ik(z-y)}]q(y)q(z)dydz \\ &\quad - \frac{i}{8k^3} \int_x^\infty \int_y^\infty \int_z^\infty [1 - e^{2ik(z-y)}][1 - e^{2ik(w-z)}]q(y)q(z)q(w)dydzdw \\ &\quad + \frac{1}{8k^2} \left( \int_x^\infty q(y)dy \right)^2 + \frac{i}{8k^3} \left( \int_x^\infty q(y)dy \right) \int_x^\infty \int_y^\infty [1 - e^{2ik(z-y)}]q(y)q(z)dzdy \\ &\quad - \frac{i}{24k^3} \left( \int_x^\infty q(y)dy \right)^3 + O(k^{-4}). \end{aligned}$$

If we consider the two-fold integral in more detail it is possible for us to perform a symmetrisation procedure in the same way as we did for the Dirac case on page 28, with the function  $f$  instead defined by  $1 - e^{2ik(z-y)} = f(z-y)$ . Incorporating this

symmetrisation into our expression for  $\log a(x, z)$  and introducing our chosen contour integral and limit,

$$\begin{aligned}
& \lim_{R \rightarrow \infty} \int_{\gamma_R} k^2 \log a(k) dk \\
&= \lim_{R \rightarrow \infty} \int_{\gamma_R} \left[ \frac{ik}{2} \int_{\mathbb{R}} q(x) dx + \frac{1}{8} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{2ik|x-y|} q(x)q(y) dx dy \right. \\
&\quad - \frac{i}{8k} \int_x^\infty \int_y^\infty \int_z^\infty [1 - e^{2ik(z-y)}][1 - e^{2ik(w-z)}] q(y)q(z)q(w) dy dz dw \\
&\quad + \frac{i}{8k} \int_{-\infty}^\infty q(x) dx \int_x^\infty \int_y^\infty [1 - e^{2ik(z-y)}] q(y)q(z) dy dz \\
&\quad \left. - \frac{i}{24k} \left( \int_x^\infty q(x) dx \right)^3 + O(k^{-2}) \right] dk. \\
&= \lim_{R \rightarrow \infty} \int_{\gamma_R} \left[ \frac{ik}{2} \int_{\mathbb{R}} q(x) dx + \frac{1}{8} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{2ik|x-y|} q(x)q(y) dx dy \right. \\
&\quad - \frac{i}{8k} \left\{ \int_x^\infty \int_y^\infty \int_z^\infty [1 - e^{2ik(z-y)}][1 - e^{2ik(w-z)}] q(y)q(z)q(w) dy dz dw \right. \\
&\quad \left. - \int_x^\infty q(x) dx \int_x^\infty \int_y^\infty [1 - e^{2ik(z-y)}] q(y)q(z) dy dz + \frac{1}{3} \left( \int_x^\infty q(x) dx \right)^3 \right\} \\
&\quad \left. + O(k^{-2}) \right] dk.
\end{aligned}$$

We will now consider each term in turn; note first that the asymptotic term vanishes in the limit and further that

$$\lim_{R \rightarrow \infty} \int_{\gamma_R} \frac{ik}{2} \int_{\mathbb{R}} q(x) dx dk = \lim_{R \rightarrow \infty} \left[ - \int_{-R}^R \frac{ik}{2} dk \right] \int_{\mathbb{R}} q(x) dx = 0$$

Recognising the term

$$\lim_{R \rightarrow \infty} \int_{\gamma_R} \frac{1}{8} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{2ik|x-y|} q(x)q(y) dx dy dk = -\frac{\pi}{8} \lim_{R \rightarrow \infty} \int_{\mathbb{R}} \left[ \int_{\mathbb{R}} \frac{\sin(2R|x-y|)}{\pi|x-y|} q(x) dx \right] q(y) dy.$$

we see that we can apply Lemma 2.4 from page 23 to obtain

$$\lim_{R \rightarrow \infty} \int_{\gamma_R} \frac{1}{8} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{2ik|x-y|} q(x)q(y) dx dy dk = -\frac{\pi}{8} \int_{\mathbb{R}} q^2(x) dx.$$

We now have only to consider

$$\begin{aligned} \lim_{R \rightarrow \infty} \int_{\gamma_R} \left[ -\frac{i}{8k} \left\{ \int_x^\infty \int_y^\infty \int_z^\infty [1 - e^{2ik(z-y)} - e^{2ik(w-z)} + e^{2ik(w-y)}] q(y)q(z)q(w) dydzdw \right. \right. \\ \left. \left. - \int_x^\infty q(x)dx \int_x^\infty \int_y^\infty [1 - e^{2ik(z-y)}] q(y)q(z) dydz + \frac{1}{3} \left( \int_x^\infty q(x)dx \right)^3 \right\} \right] dk \end{aligned} \quad (2.36)$$

Consider the integral  $\int_{\gamma_R} \frac{e^{i\alpha k}}{k} dk$ . As  $\frac{\partial}{\partial \alpha} \left( \frac{e^{i\alpha k}}{k} \right) = ie^{i\alpha k}$ , we can write

$$\frac{e^{i\alpha k}}{k} = \frac{1}{k} + i \int_0^\alpha e^{isk} ds$$

and so

$$\begin{aligned} \int_{\gamma_R} \frac{e^{i\alpha k}}{k} dk &= \int_{\gamma_R} \frac{1}{k} dk + i \int_{\gamma_R} \int_0^\alpha e^{isk} ds dk = \pi i + i \int_0^\alpha \int_{\gamma_R} e^{isk} dk ds \\ &= \pi i - i \int_0^\alpha \int_{-R}^R e^{isk} dk ds = \pi i - 2i \int_0^\alpha \frac{\sin(sR)}{s} ds \end{aligned}$$

Now  $\frac{\sin(sR)}{s}$  is an even function of  $s$ , and so

$$\begin{aligned} \int_{\gamma_R} \frac{e^{i\alpha k}}{k} dk &= \pi i - i \int_{-\alpha}^\alpha \frac{\sin(sR)}{s} ds \\ &= i \left[ \pi - \int_{-\alpha R}^{\alpha R} \frac{\sin(y)}{y} dy \right] \rightarrow 0, \quad \text{pointwise as } R \rightarrow \infty, \alpha > 0 \end{aligned}$$

Consider  $\alpha$  equal in turn to each of  $2|z-y|$ ,  $2|w-z|$  and  $2|w-y|$ . Then

$$\int_{\gamma_R} \frac{e^{i\alpha k}}{k} dk \rightarrow 0, \quad \text{almost everywhere as } R \rightarrow \infty$$

and  $\int_{\gamma_R} \frac{e^{i\alpha k}}{k} dk$  has a uniform bound for all  $R$  and  $\alpha$ . Thus uniform convergence follows by the dominated convergence theorem. Considering (2.36) again we are now left with

$$\begin{aligned} \lim_{R \rightarrow \infty} \int_{\gamma_R} \left[ -\frac{i}{k} \left\{ \int_x^\infty \int_y^\infty \int_z^\infty q(y)q(z)q(w) dydzdw \right. \right. \\ \left. \left. - \int_x^\infty q(x)dx \int_x^\infty \int_y^\infty q(y)q(z) dydz + \frac{1}{3} \left( \int_x^\infty q(x)dx \right)^3 \right\} \right] dk \end{aligned}$$

If we consider the integral

$$\begin{aligned} \int_x^\infty \int_y^\infty \int_z^\infty q(y)q(z)q(w)dydzdw &= \int_S \tilde{q}(\mathbf{x})d\mathbf{x} = \int_{\pi S} \underbrace{\tilde{q}(\pi^{-1}\mathbf{x})}_{=\tilde{q}(\mathbf{x})} d\mathbf{x}, \quad \text{where } \pi \in S^3 \\ &= \frac{1}{6} \sum_{\pi \in S^3} \int_{\pi S} \tilde{q}(\mathbf{x})d\mathbf{x} = \frac{1}{6} \int_{\text{cube}} \tilde{q}(\mathbf{x})d\mathbf{x} \end{aligned}$$

where we have used that

$$\begin{aligned} [\inf \text{supp}(q), \sup \text{supp}(q)]^3 &= \{(x, y, z) | \inf \text{supp}(q) \leq x, y, z \leq \sup \text{supp}(q)\} \\ &= \underbrace{\{\text{any two or three of } x, y, z \text{ are equal}\}}_{\text{set of measure 0}} \cup \underbrace{\{(x, y, z) | x \neq y \neq z \neq x\}}_{=:S} \end{aligned}$$

where

$$S = \{x < y < z\} \cup \{y < x < z\} \cup \{x < z < y\} \cup \{y < z < x\} \cup \{z < x < y\} \cup \{z < y < x\}.$$

Thus

$$\begin{aligned} \lim_{R \rightarrow \infty} \int_{\gamma_R} &\left[ -\frac{i}{k} \left\{ \int_x^\infty \int_y^\infty \int_z^\infty q(y)q(z)q(w)dydzdw \right. \right. \\ &\quad \left. \left. - \int_x^\infty q(x)dx \int_x^\infty \int_y^\infty q(y)q(z)dydz + \frac{1}{3} \left( \int_x^\infty q(x)dx \right)^3 \right\} \right] dk \\ &= \lim_{R \rightarrow \infty} \int_{\gamma_R} \left[ -i \left\{ \frac{1}{6} \left( \int_{-\infty}^\infty q(x)dx \right)^3 - \frac{1}{2} \left( \int_x^\infty q(x)dx \right)^3 + \frac{1}{3} \left( \int_x^\infty q(x)dx \right)^3 \right\} \right] dk \\ &= 0 \end{aligned}$$

Collecting the above results we see that all terms in the integral expansion for  $\log a(x, k)$  vanish except one. We have thus shown that

$$\lim_{R \rightarrow \infty} \int_{\gamma_R} k^2 \log a(k)dk = -\frac{\pi}{8} \int_{\mathbb{R}} q^2(x)dx$$

Since  $a$  may have zeros,  $\log(a)$  need not be analytic in the upper half plane. However, at any point  $\Im(k) > 0$  where  $a(k)$  vanishes, equation (2.30) must have a  $L^2(\mathbb{R})$  solution.

This follows because our solution  $\psi(x)$  outside of the support of  $q$  has the form

$$\psi(x) = \begin{cases} a(k)e^{ikx} + b(k)e^{-ikx}, & x < \inf \text{supp}(q) \\ e^{ikx}, & x > \sup \text{supp}(q) \end{cases}.$$

Further, since we are in the limit point case, only one such solution exists (see [10] page 231 Corollary 2). This solution is an eigenfunction. Thus  $a(k)$  vanishes precisely when  $k^2$  is an eigenvalue; these eigenvalues are negative real numbers. Indeed, as we take  $\Im k > 0$  and self-adjointness implies that the eigenvalues are real,  $k \in i\mathbb{R}$ . Since  $q$  is compactly supported, there are only finitely many such eigenvalues (indeed, as per [57], less than  $1 + \int_{\mathbb{R}} |xq_-(x)|dx$ ). Let  $\{i\beta_m\}$  enumerate the zeros of  $a$ . These zeros can only be of finite order. Indeed, since  $a$  is an analytic function of  $k$ , we can write its Taylor expansion around any point  $k_0$ ,

$$a(k) = \sum_{n=0}^{\infty} \alpha_n (k - k_0)^n, \quad \alpha_i = \frac{f^{(i)}(k_0)}{i!} \neq 0 \text{ for at least one } i.$$

Assume that  $a$  has a zero of infinite order at  $k = \varepsilon$ . Then, the Taylor expansion of  $a$  around  $\varepsilon$  would reduce to

$$a(k) = a(\varepsilon) = 0, \quad \forall k.$$

This is a contradiction, as  $a(k) = 0$  only when  $k^2$  is an eigenvalue, which is a finite set of  $k$  from above. Thus all zeros of  $a$  are of finite order. Define the corresponding Blaschke product for the upper half plane,

$$B = \prod_m \left( \frac{k - i\beta_m}{k + i\beta_m} \right)^{\alpha_m}, \quad \alpha_m \text{ the multiplicity of the zero } \beta_m.$$

Note that it can be shown that  $\alpha_m = 1$  for all  $m$ . Then  $\log(Bt) = \log\left(\frac{B}{a}\right) = -\log\left(\frac{a}{B}\right)$  is analytic in the upper half plane, where  $t$  is the transmission coefficient. This follows because  $a$  is analytic in the upper half plane, and  $B$  has zeros only at the points where  $a = 0$ . Thus  $\frac{a}{B}$  is analytic in the upper half plane, and the logarithm of an analytic function is an analytic function. Now  $a(-k) = \overline{a(k)}$  and  $B(-k) = \prod_m \left( \frac{-k - i\beta_m}{-k + i\beta_m} \right) = \overline{B(k)}$  (for  $k \in \mathbb{R}$ ). Thus

$$\Im[\log(B(-k)t(-k))] = \Im \left[ -\log \frac{a(-k)}{B(-k)} \right] = \Im \left[ -\log \frac{\overline{a(k)}}{\overline{B(k)}} \right] = \Im \left[ \log \frac{a(k)}{B(k)} \right] = \Im[-\log(B(k)t(k))],$$



i.e.  $\Im[\log(B(k)t(k))]$  is an odd function of  $k \in \mathbb{R}$ . Thus, using that  $|B| = 1$

$$\begin{aligned}
\int_{\mathbb{R}} k^2 \log |a(k)| dk &= \int_{\mathbb{R}} (\log |a| - \log |B|) k^2 dk = \int_{\mathbb{R}} \log \left| \frac{a}{B} \right| k^2 dk = \int_{\mathbb{R}} \Re \log \left( \frac{a}{B} \right) k^2 dk \\
&= \int_{\mathbb{R}} \left( \Re \log \left( \frac{a}{B} \right) + i \Im \log \left( \frac{a}{B} \right) \right) k^2 dk = \int_{\mathbb{R}} \log \left( \frac{a}{B} \right) k^2 dk \\
&= \int_{\mathbb{R}} (\log(a) - \log(B)) k^2 dk = \lim_{R \rightarrow \infty} \int_{\gamma_R} [-\log(a) + \log(B)] k^2 dk \\
&= \frac{\pi}{8} \int_{\mathbb{R}} q^2(x) dx + \lim_{R \rightarrow \infty} \int_{\gamma_R} \log(B) k^2 dk
\end{aligned}$$

where in the eighth equality we have used, in an essential way, the fact that  $a$  and  $B$  have exactly the same zeros. Now

$$\lim_{R \rightarrow \infty} \int_{\gamma_R} \log(B) k^2 dk = \lim_{R \rightarrow \infty} \sum_m \int_{\gamma_R} \log \left( \frac{k - i\beta_m}{k + i\beta_m} \right) k^2 dk$$

Thus, for  $k$  large enough, we have

$$\begin{aligned}
\lim_{R \rightarrow \infty} \int_{\gamma_R} \log(B) k^2 dk &= \lim_{R \rightarrow \infty} \int_{\gamma_R} \left[ \sum_m \alpha_m \left\{ k^2 \log \left( 1 - \frac{i\beta_m}{k} \right) - k^2 \log \left( 1 + \frac{i\beta_m}{k} \right) \right\} \right. \\
&= \lim_{R \rightarrow \infty} \int_{\gamma_R} \left[ \sum_m \alpha_m \left\{ \left( -i\beta_m k - \frac{k^2}{2} \left( -\frac{i\beta_m}{k} \right)^2 + \frac{k^2}{3} \left( -\frac{i\beta_m}{k} \right)^3 \right) \right. \right. \\
&\quad \left. \left. - \left( i\beta_m k - \frac{k^2}{2} \left( \frac{i\beta_m}{k} \right)^2 + \frac{k^2}{3} \left( \frac{i\beta_m}{k} \right)^3 \right) + O(k^{-2}) \right\} \right] dk \\
&= \lim_{R \rightarrow \infty} \sum_m \alpha_m \left[ \int_{\gamma_R} \left( -2i\beta_m k + \frac{2i\beta_m^3}{3k} \right) dk \right] \\
&= -\frac{2\pi}{3} \sum_m \alpha_m \beta_m^3
\end{aligned}$$

Now

$$\frac{2\pi}{3} \sum_m \alpha_m \beta_m^3 > 0$$

as  $\Im k \geq 0$  implies that  $\beta_m > 0$  for all  $m$ . This provides our desired estimate (2.9).  $\square$

## 5 Dirac Case: The Spectral Function

We now proceed to prove Theorem 2.1. We shall show that for all compact subsets  $K \subset \mathbb{R} \setminus [-1, 1]$  of positive Lebesgue measure,  $\rho(K) > 0$ . This implies the statement of Theorem 2.1; indeed, assume  $\lambda \in \mathbb{R} \setminus [-1, 1]$  is not a growth point of the absolutely

continuous part of the spectral function,  $\rho_{ac}$ . Then there is  $\varepsilon > 0$  such that  $\rho_{ac}([\lambda - \varepsilon, \lambda + \varepsilon]) = 0$ . Let  $B \subset \mathbb{R}$  be an open set of Lebesgue measure  $< \varepsilon$  such that  $\rho_{sing}(\mathbb{R} \setminus B) = 0$ . Then  $K := [\lambda - \varepsilon, \lambda + \varepsilon] \setminus B$  is compact and has positive Lebesgue measure, so by the above

$$0 < \rho(K) = \rho_{ac}(K) \leq \rho_{ac}([\lambda - \varepsilon, \lambda + \varepsilon]),$$

a contradiction.

The proof of the above statement will use the following estimate (see [11])

LEMMA 2.5. *Let  $A \subset \mathbb{R}$  be open and let  $w \in L^1_{loc}(\Omega), w > 0$ . Let  $(\rho_n)_{n \in \mathbb{N}}$  be a sequence of absolutely continuous non-decreasing functions which converge to a non-decreasing function  $\rho$  at the points of continuity of  $\rho$ . Let  $K \subset A$  be compact and of positive Lebesgue measure. Then*

$$\limsup_{n \rightarrow \infty} \int_K \log \left( \frac{\rho'_n}{w} \right) \frac{w}{w(K)} \leq \log \left( \frac{\rho(K)}{w(K)} \right), \quad (2.37)$$

where  $w(K) = \int_K w$ .

PROOF. Let

$$\phi_n(x) = \max\{0, 1 - n \cdot \text{dist}(x, K)\}, \quad (x \in \mathbb{R}, n \in \mathbb{N}).$$

Then  $\text{supp}(\phi_n(x)) \subset [\inf K - 1, \sup K + 1]$ . Further,  $(\phi_n)_{n \in \mathbb{N}}$  is a non increasing sequence converging to  $\chi_K$  pointwise as  $n \rightarrow \infty$ , the characteristic function of  $K$ . Thus

$$\rho(K) = \int_{\mathbb{R}} \chi_K(x) d\rho = \lim_{m \rightarrow \infty} \int_{\mathbb{R}} \phi_m(x) d\rho = \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \int_{\mathbb{R}} \phi_m(x) d\rho_n \geq \limsup_{n \rightarrow \infty} \int_K \rho'_n,$$

where the second equality follows from the monotone convergence theorem, the third by Helly's integration theorem and the inequality from the fact that  $\phi_n \geq \chi_K$ . Thus

$$\log \left( \frac{\rho(K)}{w(K)} \right) \geq \limsup_{n \rightarrow \infty} \log \int_K \frac{\rho'_n}{w} \frac{w}{w(K)} \geq \limsup_{n \rightarrow \infty} \int_K \log \left( \frac{\rho'_n}{w} \right) \frac{w}{w(K)},$$

where the last inequality follows from Jensen's Inequality (see [51] Theorem 3.3).  $\square$

From Lemma 2.1 and Lemma 2.5 we see that it is sufficient to prove that

$$\int_K \left( -\log \left( \frac{\rho'_n}{w} \right) w \right) = \int_K \left( -\log \left[ \frac{\Im m_n(\lambda + i0)}{\pi w(\lambda)} \right] w(\lambda) \right) d\lambda$$

is bounded above uniformly in  $n$  for some positive weight function  $w$ . As  $q_n$  is square integrable with compact support, the Titchmarsh-Weyl  $m$ -function for the Dirac equation

associated with (2.10) with boundary condition (2.5) can be expressed in terms of the solution  $y$  of (2.12),

$$m_n(\lambda) = \frac{y_{n,2}(0, \lambda)}{y_{n,1}(0, \lambda)} = i \frac{\lambda - 1}{\omega(\lambda)} \frac{a_n(\lambda) + b_n(\lambda)}{a_n(\lambda) - b_n(\lambda)} = i \frac{\lambda - 1}{\omega(\lambda)} \frac{1 + r_n(\lambda)}{1 - r_n(\lambda)},$$

denoting by  $a_n$  and  $b_n$  the coefficients of  $y_n$ , and by  $r_n$  the corresponding reflection coefficient. Conversely  $r_n(\lambda) = \frac{m_n(\lambda) - i(\lambda-1)/\omega(\lambda)}{m_n(\lambda) + i(\lambda-1)/\omega(\lambda)}$ . Thus

$$|t_n(\lambda)|^2 = 1 - |r_n(\lambda)|^2 = \frac{4\Re\left(\overline{m_n(\lambda + i0)} \frac{i(\lambda-1)}{\omega(\lambda)}\right)}{\left|m_n(\lambda + i0) + \frac{i(\lambda-1)}{\omega(\lambda)}\right|^2}$$

for a.e.  $\lambda \in \mathbb{R} \setminus [-1, 1]$ . Now the spectrum is purely absolutely continuous in this set because the potential has compact support, and so (see [24])  $0 < \lim_{\varepsilon \rightarrow 0} \Im m_n(\lambda + i\varepsilon) < \infty$  and  $(\lambda - 1)/\omega(\lambda) > 0$ . Consequently

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \left| m_n(\lambda + i\varepsilon) + \frac{i(\lambda + i\varepsilon - 1)}{\omega(\lambda + i\varepsilon)} \right|^2 &= \left( \frac{\lambda - 1}{\omega(\lambda)} + \Im \lim_{\varepsilon \rightarrow 0} m_n(\lambda + i\varepsilon) \right)^2 + (\Re \lim_{\varepsilon \rightarrow 0} m_n(\lambda + i\varepsilon))^2 \\ &\geq \left( \frac{\lambda - 1}{\omega(\lambda)} \right)^2 \end{aligned}$$

( $\lambda \in \mathbb{R} \setminus [-1, 1]$ ). Thus we can estimate

$$\frac{1}{|a_n(\lambda)|^2} = |t_n(\lambda)|^2 \leq \frac{4\omega(\lambda)}{\lambda - 1} \Im \lim_{\varepsilon \rightarrow 0} m_n(\lambda + i\varepsilon).$$

Now let  $\delta > 0$  and apply Lemma 2.5 with  $A := \mathbb{R} \setminus [-1 - \delta, 1]$  and  $w(\lambda) := |\lambda - 1|/(4\pi\sqrt{\lambda^2 - 1})$  ( $\lambda \in A$ ). For any compact set  $K \subset A$  of positive Lebesgue measure, we find using Theorem 2.3 and the facts that  $|a_n| \geq 1$  and

$$\frac{|\lambda - 1|}{\sqrt{\lambda^2 - 1}} = \frac{\sqrt{\lambda^2 - 1}}{|\lambda + 1|} \leq \frac{\sqrt{\lambda^2 - 1}}{\delta} \quad (\lambda \in A)$$

that

$$\begin{aligned}
-\int_K \log\left(\frac{\rho'_n}{w}\right) w &= -\int_K \log\left(\frac{\Im \lim_{\varepsilon \rightarrow 0} m_n(\lambda + i\varepsilon)}{\pi w(\lambda)}\right) w(\lambda) d\lambda \\
&= -\frac{1}{4\pi} \int_K \log\left(\frac{4\Im \lim_{\varepsilon \rightarrow 0} m_n(\lambda + i\varepsilon)\omega(\lambda)}{\lambda - 1}\right) \frac{|\lambda - 1|}{\sqrt{\lambda^2 - 1}} d\lambda \\
&\leq \frac{1}{2\pi} \int_K \log |a_n(\lambda)| \frac{|\lambda|}{\delta} \sqrt{\lambda^2 - 1} d\lambda \\
&\leq \frac{1}{2\pi\delta} \int_{\mathbb{R} \setminus (-1, 1)} \log |a_n(\lambda)| |\lambda| \sqrt{\lambda^2 - 1} d\lambda \\
&\leq \frac{1}{4\delta} \int_{\mathbb{R}} q_n^2 \leq \frac{1}{4\delta} \int_{\mathbb{R}} q^2.
\end{aligned} \tag{2.38}$$

Thus the integral in (2.37) is bounded below independently of  $n \in \mathbb{N}$ , and so  $\rho(K) > 0$ .

This concludes the proof of Theorem 2.1.

**Remark** The inequality (2.38) is rather a bad estimate for large values of  $\lambda$ ; indeed, the bounded factor  $\frac{|\lambda-1|}{\sqrt{\lambda^2-1}}$  is replaced with the upper bound  $\frac{|\lambda|}{\delta} \sqrt{\lambda^2-1}$ , which grows as  $\lambda^2$  for  $\lambda \rightarrow \pm\infty$ , in order to fit the estimate (2.8).

In fact, the assertion of theorem 2.1 will already follow if

$$\int_{(-\infty, -1] \cup [1, \infty)} \frac{\lambda\omega}{\lambda^2 + 1} \log |a_n(\lambda)| d\lambda$$

is bounded above. Estimating this integral by the method of Section 3 turns out to be easier due to the better decay properties of the integrand, and gives, instead of (2.8)

$$\begin{aligned}
\int_{(-\infty, -1] \cup [1, \infty)} \frac{\lambda\omega}{\lambda^2 + 1} \log(a_n(\lambda)) d\lambda &= -\sqrt{2}\pi \log |a_n(i)| - \pi \sum_{m=1}^M \int_{\beta_m}^1 \frac{\lambda\sqrt{1-\lambda^2}}{\lambda^2 + 1} \\
&\leq -\sqrt{2}\pi \log |a_n(i)|,
\end{aligned}$$

where, again, the  $\beta_m$ ,  $m \in \{1, 2, \dots, M\}$ , are the zeros of  $a$ . More generally

$$\int_{(-\infty, -1] \cup [1, \infty)} \frac{\lambda\omega}{\lambda^2 + \alpha^2} \log |a_n(\lambda)| d\lambda \leq -\sqrt{2}\pi \log |a_n(i\alpha)|$$

for any  $\alpha > 0$ . This means that to obtain an equivalent result to Theorem 2.1 (page 15) one only needs to show that there exists an  $\alpha > 0$  such that  $|a_n(i\alpha)| \rightarrow 0$  as the cut off point of the potential tends towards infinity. This seems to be a very weak condition and its relation to the  $L^2$  condition in Theorem 2.1 is somewhat obscure. Note that if we consider a constant potential, for which the assertion of Theorem 2.1 clearly does not

hold, then  $|a_n(i\alpha)| \rightarrow 0$  for all  $\alpha > 0$  ( $n \rightarrow \infty$ ).

## 6 Schrödinger Case: The Spectral Function

We now proceed to prove Theorem 2.2. Again it is enough to show that for all compact subsets  $K \subset \mathbb{R} \setminus [-1, 1]$  of positive Lebesgue measure,  $\rho(K) > 0$ . From Lemma 2.1 and Lemma 2.5 we again see that it is sufficient to prove that

$$\int_K \left( -\log \left( \frac{\rho'_n}{w} \right) w \right) = \int_K \left( -\log \left[ \frac{\Im m_n(\lambda + i0)}{\pi w(\lambda)} \right] w(\lambda) \right) d\lambda$$

is bounded above uniformly in  $n$  for some positive weight function,  $w$ . As  $q_n$  is square integrable with compact support, the Weyl-Titchmarsh function for the Schrödinger equation associated with (2.1) with boundary condition (2.2) can be expressed in the form

$$m(\lambda) = \frac{\psi'(0, \lambda)}{\psi(0, \lambda)}$$

and so (where  $\psi$  is given by (2.31))

$$m_n(k^2) = ik \left( \frac{a_n(k) - b_n(k)}{a_n(k) + b_n(k)} \right) = ik \left( \frac{1 - r_n(k)}{1 + r_n(k)} \right)$$

using  $r = \frac{b}{a}$ . We can rearrange this to give

$$r_n(k) = \frac{ik - m_n(k^2)}{ik + m_n(k^2)}.$$

Thus for  $k^2 \in K$ ,  $k > 0$  (i.e.  $\lambda = k^2 + i0$ )

$$\begin{aligned} |t_n|^2 &= 1 - |r_n^2| = 1 - \frac{ik - m_n}{ik + m_n} \frac{ik + \bar{m}_n}{ik - \bar{m}_n} \\ &= \frac{(ik + m_n)(ik - \bar{m}_n) - (ik - m_n)(ik + \bar{m}_n)}{(ik + m_n)(ik - \bar{m}_n)} \\ &= \frac{-ik\bar{m}_n + ikm_n - ik\bar{m}_n + ikm_n}{-|ik + m_n|^2} = \frac{\frac{2k}{i}(m_n - \bar{m}_n)}{|ik + m_n|^2} \\ &= \frac{4k\Im m_n}{|m_n + ik|^2} \\ &\leq \frac{4\Im m_n}{k}, \quad \text{as } |k - im_n|^2 = (k + \Im m_n)^2 + (\Re m_n)^2 \geq k^2 \end{aligned}$$

where the last two inequalities use the fact that  $\Im(m_n) \geq 0$  for such  $k$ , as it is a Herglotz function. Choosing  $w(\lambda) = \frac{\sqrt{\lambda}}{4\pi}$ ,

$$\begin{aligned}
& \int_K \left( -\log \left[ \frac{\Im m_n(\lambda + i0)}{\pi w(\lambda)} \right] w(\lambda) \right) d\lambda = \frac{1}{4\pi} \int_K \left( -\log \left( \frac{4\Im m_n}{\sqrt{\lambda}} \right) \sqrt{\lambda} \right) d\lambda \\
& = \frac{1}{4\pi} \int_{k^2 \in K, k > 0} \left( -\log \left( \frac{4\Im m_n}{k} \right) 2k^2 dk \right) \leq \frac{1}{2\pi} \int_{k^2 \in K, k > 0} \log \left| \frac{1}{t_n} \right|^2 k^2 dk \\
& \leq \frac{1}{\pi} \int_{k^2 \in K, k > 0} \log \left| \frac{1}{t_n} \right| k^2 dk \leq \frac{1}{\pi} \int_{\mathbb{R}} \log \left| \frac{1}{t_n} \right| k^2 dk \leq \frac{1}{8} \int_{\mathbb{R}} q_n^2 dx \\
& \leq \frac{1}{8} \int_{\mathbb{R}} q^2 dx,
\end{aligned}$$

which proves the required uniform boundedness.



# Spherically Symmetric Dirac Operators with Square Integrable Potentials

As discussed in the introduction, the Dirac operator is the fundamental object in relativistic quantum mechanics. Its classical form in three-dimensions is

$$T_3 = -i\alpha \cdot \nabla + \alpha_0 + V(x).$$

In the present chapter we are concerned the assumption that  $V$  is a spherically symmetric function; more precisely,  $V(x) = q(|x|)$  ( $x \in \mathbb{R}^3$ ). If this is the case then the operator  $T$  is spherically symmetric in the sense that rotations in space lead to unitarily equivalent operators.

In practice the one-dimensional Dirac operator most commonly arises from such a three-dimensional Dirac operator with a spherically symmetric potential by separation of variables in spherical polar coordinates (cf. [69, Appendix to Ch. 1]). On the other hand  $T_3$  is then unitarily equivalent to the direct sum of the countable family of these one-dimensional Dirac operators on the half line  $x > 0$ ,

$$T_k = -i\sigma_2 \frac{d}{dx} + \frac{k}{x}\sigma_1 + \sigma_3 + q(x)$$

where  $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  is the third Pauli matrix and  $k \in \mathbb{Z} \setminus \{0\}$  (In the case of a rotationally symmetric two-dimensional Dirac operator,  $k \in \mathbb{Z} - \frac{1}{2}$ ). The additional



angular momentum term  $\frac{k}{x}\sigma_1$  introduces a singularity at 0. This singular end-point is in the limit-point case if  $|k| \geq \frac{1}{2}$  and  $q$  is less singular at 0; indeed  $q \in L^1([0, *])$  (which follows from  $q \in L^2([0, \infty))$ ) is sufficient to ensure limit-point case at zero [16]. As the operator is always in the limit-point case at  $\infty$  (see [69, Thm 6.8]), this means that it has a unique self-adjoint realisation  $T_k$ .

The main result of this chapter is the following

**THEOREM 3.1.** *Let  $q \in L^2([0, \infty)) \cap L^\infty([c, \infty))$ , ( $c > 0$ ). Then the absolutely continuous part of the spectral function of  $T$  is strictly increasing in  $(-\infty, -1] \cup [1, \infty)$ .*

Let  $\mathcal{H}_1, \mathcal{H}_2$  be Hilbert spaces and  $T : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  be a bounded linear operator. Then there is a unique bounded linear operator  $T^* : \mathcal{H}_2 \rightarrow \mathcal{H}_1$ , called the *adjoint* of  $T$ , such that  $(Tx, y)_2 = (x, T^*y)_1$  for points  $x \in \mathcal{H}_1$  and  $y \in \mathcal{H}_2$  (see [68, Section 4.4]). This notion of an adjoint operator can also be extended to the unbounded case, in which case we require the operator  $T$  to be densely defined on  $\mathcal{H}_1$  (see [45, Section VIII] for more details).

A bounded linear operator  $A$  over a separable Hilbert space  $\mathcal{H}$  is said to be in the *trace class* if for some orthonormal basis  $\{\phi_n\}_{n=1}^\infty$  of  $\mathcal{H}$  the sum  $\text{tr}|A| = \sum_{n=1}^\infty ((A^*A)^{\frac{1}{2}}\phi_n, \phi_n)$  is finite.

Further, a bounded linear operator  $A$  over a separable Hilbert space  $\mathcal{H}$  is said to be a *Hilbert Schmidt operator* if  $\text{tr}A^*A < \infty$ .

In the following, we denote by  $S_1$  the space of trace-class operators and by  $S_2$  the space of Hilbert-Schmidt operators.

Amongst the set of all unbounded operators there are certain ones which admit a detailed treatment and are important for applications; these are the closed operators. Let  $T$  be an operator from  $\mathcal{H}_1$  to  $\mathcal{H}_2$ , two Hilbert spaces. A sequence  $u_n \in D(T)$  is said to be *T-convergent* to  $u \in \mathcal{H}_1$  if both  $\{u_n\}$  and  $\{Tu_n\}$  are Cauchy sequences and  $u_n \rightarrow u$ . We call  $T$  a *closed operator* if, for any sequence  $\{u_n\} \in D(T)$  such that  $u_n \rightarrow u$  and  $Tu_n \rightarrow v$  ( $n \rightarrow \infty$ ),  $u \in D(T)$  and  $Tu = v$ .

Let  $T$  be a closed operator on a Hilbert space  $\mathcal{H}$ . A complex number  $\lambda$  is said to be in the resolvent set  $\rho(T)$  of  $T$  if  $\lambda - T$  is a bijection of  $D(T)$  onto  $\mathcal{H}$  with a bounded inverse. For all  $\lambda$  in the resolvent set of  $T$  we call  $(\lambda - T)^{-1}$  the resolvent of  $T$  at  $\lambda$ .

The theory attached to wave operators gives a useful set of techniques for studying the absolutely continuous spectrum. Let  $T_1, T_2$  be any two self-adjoint operators on a complex Hilbert space  $\mathcal{H}$  and  $t \in \mathbb{R}$ . We then define  $\Omega_{\pm}(T_2, T_1)$  by the equalities

$$D(\Omega_{\pm}(T_2, T_1)) = \left\{ f \in \mathcal{H} : \lim_{t \rightarrow \pm\infty} e^{itT_2} e^{-itT_1} f \text{ exists} \right\}$$

$$\Omega_{\pm}(T_2, T_1)f = \lim_{t \rightarrow \pm\infty} e^{itT_2} e^{-itT_1} f \text{ for } f \in D(\Omega_{\pm}(T_2, T_1))$$

The operators  $\Omega_{\pm}(T_2, T_1)$  are isometric; indeed this follows since  $e^{itT_2} e^{-itT_1}$  is unitary for all  $t \in \mathbb{R}$ . Further,  $\Omega_{\pm}(T_2, T_1)$  are linear operators.  $\Omega_{\pm}$  are used to describe motion in a quantum mechanical system; they are called the *wave operators* and  $S = \Omega_+^* \Omega_-$  the *scattering operator*.

We denote by  $T_{k,ac}$ ,  $k = 1, 2$  the spectrally absolutely continuous parts of  $T_k$ , that is the part of  $T_k$  in the space  $\mathcal{H}_{k,ac}$  of absolute continuity for  $T_k$ . The orthogonal projection on  $\mathcal{H}_{k,ac}$  will be denoted by  $P_k$ ,  $k = 1, 2$ .

In general, wave operators will not exist unless  $T_1$  has purely continuous spectrum. However, it happens frequently that

$$W_{\pm} = W_{\pm}(T_2, T_1) = \text{s-lim}_{t \rightarrow \pm\infty} e^{itT_2} e^{-itT_1} P_1 \quad (3.1)$$

exist even when the wave operators do not exist. For this reason, it is more usual to consider the limits (3.1) rather than the proper wave operators.  $W_{\pm}$  are called the *generalised wave operators* associated with  $T_1$  and  $T_2$ . If  $W_+$  exists it is partially isometric with initial set  $\mathcal{H}_{1,ac}$  and final set  $\mathcal{M}_+ \subseteq \mathcal{H}_{2,ac}$  ([29, Theorem 3.2]). If  $\mathcal{M}_+ = \mathcal{H}_{2,ac}$ ,  $W_+$  is said to be complete. A similar definition applies to  $W_-$ . If  $W_+$  or  $W_-$  exist and are complete,  $T_1|_{\mathcal{H}_{1,ac}}$  is unitarily equivalent to  $T_2|_{\mathcal{H}_{2,ac}}$ ; this is useful if the absolutely continuous spectrum for  $T_2$  is known and results about the absolutely continuous spectrum of  $T_1$  are required.

In order to prove Theorem 3.1 we shall use the following corollary to the Kato-Rosenblum perturbation theorem (c.f. [29, Thm 4.4]).

**THEOREM 3.2** ([29, Thm 4.12]). *Let  $H_1$  and  $H_2$  be self-adjoint operators in a Hilbert space such that*

$$(H_2 - z)^{-1} - (H_1 - z)^{-1} \in S_1$$

for some non-real  $z$ . Then the generalised wave operators  $W_{\pm}(H_2, H_1)$  exist and are complete. In particular, the absolutely continuous parts of  $H_1$  and  $H_2$  are unitarily equivalent.

The theory of subordinacy was first developed for one-dimensional Schrödinger operators by Gilbert and Pearson in their papers [22]/[24]; it was extended to the Dirac equation with spherically symmetric potentials (respectively the separated Dirac operator) by Behncke ([4]), and further to the one-dimensional Dirac operator with locally integrable potential by Amar ([2]).

The method of subordinacy is advantageous in several respects. In the first place, only very general requirements need to be met, for example that the electric potential is locally integrable and the operator is in the limit point case at the singular end points. Moreover, in principle a complete analysis of the spectrum can be achieved by considering only real values of the spectral parameter  $\lambda$ . Further, and most interestingly, in order to identify the absolutely continuous spectrum of the operator it is only necessary to consider the behaviour of solutions at the limit end points.

Let  $T$  be an operator defined on an interval  $[c, \infty)$ ,  $c > 0$ , which is regular at  $c$  and singular at  $\infty$ , with the further requirement that the operator is in the limit point case at  $\infty$ . Further, we impose a boundary condition at  $c$ . Then a non-trivial solution  $u_s(x, \lambda)$  of  $Tu = \lambda u$ ,  $\lambda \in \mathbb{R}$ , is said to be subordinate at infinity if for every linearly independent solution  $u(x, \lambda)$  of  $Tu = \lambda u$

$$\lim_{N \rightarrow \infty} \frac{\|u_s(x, \lambda)\|_N}{\|u(x, \lambda)\|_N} = 0,$$

where  $\|\cdot\|_N$  denotes the  $L^2[c, N]^d$  norm, where  $d = 1$  in the Schrödinger case and  $d = 2$  in the Dirac case. In the case where  $T$  is singular at both end points, the definition for a solution to be subordinate at  $c$  is equivalent, except the  $L^2[c, N]^d$  norm is replaced by the  $L^2[c + \frac{1}{N}, \infty]^d$  norm.

From the Gilbert-Pearson theory of subordinacy ([24], [22]), as well as its extension to Dirac operators ([4], [2]), it is known that a minimal support of the absolutely continuous

spectral measure of a self-adjoint Dirac operator  $L$  on  $(\alpha, \beta)$  is given by

$$\mathcal{M}_{ac}(L) = \{\lambda \in \mathbb{R} : \text{no solution of } Lu = \lambda u \text{ is subordinate at } \beta\} \quad (3.2)$$

if  $\alpha$  is a regular,  $\beta$  a singular end-point, and

$$\begin{aligned} \mathcal{M}_{ac}(L) &= \{\lambda \in \mathbb{R} : \text{no solution of } Lu = \lambda u \text{ is subordinate at } \beta\} \\ &\cup \{\lambda \in \mathbb{R} : \text{no solution of } Lu = \lambda u \text{ is subordinate at } \alpha\} \end{aligned} \quad (3.3)$$

if both end-points are singular. We recall that a subset  $S$  of  $\mathbb{R}$  is said to be a *minimal support* of a measure  $\nu$  if  $\nu(\mathbb{R} \setminus S) = 0$  and  $\nu(S_0) = 0 \Rightarrow \text{mes } S_0 = 0$  ( $S_0 \subset S$ ), where  $\text{mes}$  denotes the Lebesgue measure. Further define

DEFINITION 3.1 (Essential Closure). We define the essential closure of a set  $\Sigma$  as

$$\overline{\Sigma}^{\text{ess}} = \{\lambda \in \mathbb{R} : \forall \varepsilon > 0, \text{mes}((\lambda - \varepsilon, \lambda + \varepsilon) \cap \Sigma) > 0\},$$

where  $\text{mes}$  represents the Lebesgue measure.

It follows immediately that if  $\Sigma_1 \subset \Sigma_2$ , then  $\overline{\Sigma}_1^{\text{ess}} \subset \overline{\Sigma}_2^{\text{ess}}$ .

LEMMA 3.1. *The set of growth points of  $\rho_{ac}$  is given by  $\overline{\mathcal{M}}_{ac}^{\text{ess}}$ .*

PROOF. Let  $\lambda$  be a growth point of  $\rho_{ac}$ . Then, recalling the definition of a minimal support, for all  $\varepsilon > 0$

$$0 < \rho_{ac}((\lambda - \varepsilon, \lambda + \varepsilon)) = \rho_{ac}((\lambda - \varepsilon, \lambda + \varepsilon) \cap \mathcal{M}_{ac}).$$

As  $\rho_{ac}$  is absolutely continuous with respect to the Lebesgue measure, this implies that

$$\text{mes}((\lambda - \varepsilon, \lambda + \varepsilon) \cap \mathcal{M}_{ac}) > 0,$$

and hence  $\lambda \in \overline{\mathcal{M}}_{ac}^{\text{ess}}$ . We now let  $\lambda \in \overline{\mathcal{M}}_{ac}^{\text{ess}}$ . Then for all  $\varepsilon > 0$

$$\text{mes}((\lambda - \varepsilon, \lambda + \varepsilon) \cap \mathcal{M}_{ac}) > 0.$$

As  $[(\lambda - \varepsilon, \lambda + \varepsilon) \cap \mathcal{M}_{ac}] \subset \mathcal{M}_{ac}$ ,

$$0 < \rho_{ac}((\lambda - \varepsilon, \lambda + \varepsilon) \cap \mathcal{M}_{ac}) = \rho_{ac}((\lambda - \varepsilon, \lambda + \varepsilon)),$$

and hence  $\lambda$  is a growth point of  $\rho_{ac}$ . □

(3.2) and (3.3) imply that

$$\mathcal{M}_{ac}(H) = \mathcal{M}_{ac}(H_c^-) \cup \mathcal{M}_{ac}(H_c^+). \quad (3.4)$$

Thus

THEOREM 3.3 ([22]).

$$\sigma_{ac}(H) = \sigma_{ac}(H_c^\alpha) \cup \sigma_{ac}(H_c^\beta) \quad (3.5)$$

where  $c \in (\alpha, \beta)$  and  $H_c^\beta$  is the operator restricted to  $[c, \beta)$  and  $H_c^\alpha$  that to  $(\alpha, c]$  with some boundary condition at  $c$

PROOF. Let  $A, B$  be sets and let  $\lambda \in \overline{A \cup B}^{\text{ess}}$ . Then, for all  $\varepsilon > 0$

$$\begin{aligned} 0 < \text{mes}((\lambda - \varepsilon, \lambda + \varepsilon) \cap (A \cup B)) &= \text{mes}(\{(\lambda - \varepsilon, \lambda + \varepsilon) \cap A\} \cup \{(\lambda - \varepsilon, \lambda + \varepsilon) \cap B\}) \\ &\leq \text{mes}(\{(\lambda - \varepsilon, \lambda + \varepsilon) \cap A\}) + \text{mes}(\{(\lambda - \varepsilon, \lambda + \varepsilon) \cap B\}). \end{aligned}$$

Thus for all  $\varepsilon > 0$ ,  $\text{mes}(\{(\lambda - \varepsilon, \lambda + \varepsilon) \cap A\}) > 0$  or  $\text{mes}(\{(\lambda - \varepsilon, \lambda + \varepsilon) \cap B\}) > 0$ . By the monotonicity of measure, this implies that either  $\text{mes}(\{(\lambda - \varepsilon, \lambda + \varepsilon) \cap A\}) > 0$  for all  $\varepsilon > 0$  or  $\text{mes}(\{(\lambda - \varepsilon, \lambda + \varepsilon) \cap B\}) > 0$  for all  $\varepsilon > 0$ . Thus  $\lambda \in \overline{A}^{\text{ess}}$  or  $\overline{B}^{\text{ess}}$ . Hence  $\lambda \in \overline{A}^{\text{ess}} \cup \overline{B}^{\text{ess}}$ . On the other hand, as  $\overline{A}^{\text{ess}} \subset \overline{A \cup B}^{\text{ess}}$  and  $\overline{B}^{\text{ess}} \subset \overline{A \cup B}^{\text{ess}}$  we know that  $\overline{A}^{\text{ess}} \cup \overline{B}^{\text{ess}} \subset \overline{A \cup B}^{\text{ess}}$ . Hence  $\overline{A \cup B}^{\text{ess}} = \overline{A}^{\text{ess}} \cup \overline{B}^{\text{ess}}$ . This, together with Equation (3.4) and Lemma 3.1, gives the result.  $\square$

Thus we can draw the following conclusion from Theorem 2.1.

COROLLARY 3.1. Consider the self-adjoint Dirac operator on  $\mathbb{R}$

$$\tilde{T} = -i\sigma_2 \frac{d}{dx} + \sigma_3 + q \quad (x \in \mathbb{R}).$$

If  $q \in L^2(\mathbb{R})$ , then the absolutely continuous part of the spectral function of  $\tilde{T}$  is strictly increasing in  $(-\infty, -1] \cup [1, \infty)$ .

We now consider

$$T_k = -i\sigma_2 \frac{d}{dx} + \sigma_3 + \frac{k}{x}\sigma_1 + q(x)$$

in  $L^2((0, \infty))$ , where  $|k| \geq \frac{1}{2}$  and  $q \in L^2([0, \infty)) \cap L^\infty([0, \infty))$ . Then, by [53, Lemma 3], the operators  $T_k$  and

$$\tilde{T}_k = -i\sigma_2 \frac{d}{dx} + \sigma_3 + \mu(x)\sigma_3 + \tilde{q}(x),$$

with  $\mu(x) = \sqrt{1 + \frac{k^2}{x^2}} - 1$  and  $\tilde{q}(x) = q(x) + \frac{k}{2(x^2+k^2)}$  ( $x > 0$ ), are unitarily equivalent. Obviously  $\tilde{q} \in L^2([0, \infty)) \cap L^\infty([c, \infty))$  and  $\mu \in L^1((c, \infty)) \cap L^2((c, \infty))$ , where  $c$  is fixed by the hypothesis on  $q$  imposed in Theorem 3.1.

Consider also the operator on  $\mathbb{R}$ ,

$$H = -i\sigma_2 \frac{d}{dx} + \sigma_3 + \hat{\mu}(x)\sigma_3 + \hat{q}(x) \quad (x \in \mathbb{R}),$$

where  $\hat{q}$  is the even extension of  $\chi_{[c, \infty)}\tilde{q}$  to the whole real line and  $\hat{\mu}$  is the even extension of  $\chi_{[c, \infty)}\mu$  to the whole real line. The transformation  $u(x) = \sigma_3 v(-x)$  then turns  $Hu = \lambda u$  into  $Hv = \lambda v$  (follows as both  $\hat{q}$  and  $\hat{\mu}$  are even functions of  $x$ ). Because of this symmetry, the sets

$$\{\lambda \in \mathbb{R} : \text{no solution of } Hu = \lambda u \text{ is subordinate at } \infty\}$$

and

$$\{\lambda \in \mathbb{R} : \text{no solution of } Hu = \lambda u \text{ is subordinate at } -\infty\}$$

coincide. As the differential expressions for  $H$  and  $\tilde{T}_k$  are the same near  $+\infty$ , (3.3) then implies that

$$\mathcal{M}_{ac}(H) \subset \mathcal{M}_{ac}(\tilde{T}_k). \quad (3.6)$$

Define two further operators on  $\mathbb{R}$ , namely

$$H_0 = -i\sigma_2 \frac{d}{dx} + \sigma_3 + \hat{q}(x), \quad H_{00} = -i\sigma_2 \frac{d}{dx} + \sigma_3.$$

As both  $H$  and  $H_0$  have the form  $H_{00} + F$ , where  $F$  is a bounded perturbation, all three operators have the same domain. From [44, Thm XI.20] and a simple modification of its proof, we obtain the following statement.

LEMMA 3.2. *Let  $\varphi \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ . Then*

$$\varphi(H_{00} - \lambda)^{-1} \in \mathcal{S}_2, \quad (H_{00} - \lambda)^{-1}\varphi \in \mathcal{S}_2.$$

PROOF. Let  $\mathcal{F} : L_x^2(\mathbb{R}) \rightarrow L_\xi^2(\mathbb{R})$  be the Fourier transform. Then  $\mathcal{F}(-i\frac{d}{dx}) = \xi\mathcal{F}$ , where  $\xi$  is the operator of multiplication with the variable in  $L_\xi^2(\mathbb{R})$ . Further  $\mathcal{F}H_{00}\mathcal{F}^{-1} = \sigma_2\xi + \sigma_3$ , and thus  $\mathcal{F}H_{00}^{-1}\mathcal{F}^{-1} = \frac{\sigma_2\xi + \sigma_3}{\xi^2 + 1}$ . Hence

$$H_{00}^{-1} = \mathcal{F}^{-1} \left( \frac{\sigma_2\xi + \sigma_3}{\xi^2 + 1} \right) \mathcal{F}.$$

Therefore

$$\begin{aligned} (\varphi H_{00}^{-1} f)(y) &= \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(y) e^{-2\pi i(y-x)\xi} \frac{\sigma_2 \xi + \sigma_3}{|\xi|^2 + 1} f(x) dx d\xi \\ &= \int_{\mathbb{R}} \varphi(y) \check{\mathcal{G}}(y-x) f(x) dx, \end{aligned}$$

where  $\mathcal{G}(\xi) = \frac{\sigma_2 \xi + \sigma_3}{|\xi|^2 + 1}$ , ( $\xi \in \mathbb{R}$ ), and  $\check{\cdot}$  represents the inverse Fourier transform. As  $\mathcal{G}(\xi) \in L^2_{\xi}(\mathbb{R})$ ,  $\check{\mathcal{G}}(x) \in L^2_x(\mathbb{R})$ . Thus

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |\varphi(y)|^2 |\check{\mathcal{G}}(x-y)|^2 dx dy = \int_{\mathbb{R}} |\varphi(y)|^2 \left( \int_{\mathbb{R}} |\check{\mathcal{G}}(x-y)|^2 dx \right) dy \leq \|\mathcal{G}\|_2^2 \|\varphi\|_2^2 < \infty.$$

Hence, by [45, Theorem VI.23],  $\varphi(H_{00} - \lambda)^{-1} \in \mathcal{S}_2$ . To show that  $(H_{00} - \lambda)^{-1} \varphi \in \mathcal{S}_2$ , note that as  $\varphi \in L^\infty(\mathbb{R})$ ,  $\varphi f \in L^2(\mathbb{R})$  and so  $\mathcal{F}(\varphi f)$  is well defined. Thus

$$\begin{aligned} (H_{00}^{-1} \varphi f)(y) &= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-2\pi i(y-x)\xi} \frac{\sigma_2 \xi + \sigma_3}{|\xi|^2 + 1} \varphi(x) f(x) dx d\xi \\ &= \int_{\mathbb{R}} \check{\mathcal{G}}(y-x) \varphi(x) f(x) dx, \end{aligned}$$

where  $\mathcal{G}$  is defined as before. Also, as before, the integral kernel  $\check{\mathcal{G}}(y-x)\varphi(x)$  is square integrable and hence the result follows.  $\square$

Thus, taking  $\lambda \in \mathbb{C} \setminus \mathbb{R}$  and using the Second Resolvent Identity [68, Theorem 5.1], we find

$$\begin{aligned} (H - \lambda)^{-1} - (H_0 - \lambda)^{-1} &= (H - \lambda)^{-1} (-\hat{\mu}\sigma_3)(H_0 - \lambda)^{-1} \\ &= (H - \lambda)^{-1} (-\hat{\mu}\sigma_3)(H_{00} - \lambda)^{-1} - (H - \lambda)^{-1} (-\hat{\mu}\sigma_3)(H_{00} - \lambda)^{-1} \hat{q}(H_0 - \lambda)^{-1} \\ &= (H_{00} - \lambda)^{-1} (-\sqrt{\hat{\mu}}\sigma_3\sqrt{\hat{\mu}})(H_{00} - \lambda)^{-1} + (H - \lambda)^{-1} (\hat{\mu}\sigma_3 + \hat{q})(H_{00} - \lambda)^{-1} (-\hat{\mu}\sigma_3)(H_{00} - \lambda)^{-1} \\ &\quad + (H - \lambda)^{-1} \hat{\mu}\sigma_3(H_{00} - \lambda)^{-1} \hat{q}(H_{00} - \lambda)^{-1} \\ &\quad + (H - \lambda)^{-1} \hat{\mu}\sigma_3(H_{00} - \lambda)^{-1} \hat{q}(H_{00} - \lambda)^{-1} \hat{q}(H_0 - \lambda)^{-1} \in \mathcal{S}_1. \end{aligned}$$

Here we used Lemma 3.2 together with the facts that  $S_2 S_2 \subset S_1$  and that  $S_1$  and  $S_2$  are invariant under multiplication with bounded operators [45, Section VI.6]. Thus, by Theorem 3.2, the absolutely continuous parts of  $H$  and  $H_0$  are unitarily equivalent. By Corollary 3.1, this implies that  $H$  has absolutely continuous spectrum on  $(-\infty, -1] \cup [1, \infty)$ . Thus (3.6) and Lemma 3.1 give

$$(-\infty, -1] \cup [1, \infty) \subset \sigma_{ac}(H) = \overline{\mathcal{M}}_{ac}^{\text{ess}}(H) \subset \overline{\mathcal{M}}_{ac}^{\text{ess}}(\tilde{T}_k) = \sigma_{ac}(\tilde{T}_k),$$

and Theorem 3.1 follows.

REMARK 3.1. We note here that the analogous result for Schrödinger operators follows immediately from the result of Deift and Killip ([11]); indeed, the Schrödinger operator with angular momentum term has the form

$$\varsigma = -\frac{d^2}{dx^2} + q(x) + \frac{C}{x^2}, \quad x \in (0, \infty)$$

where  $C$  is a constant. If  $q$  is square integrable then, by using Gilbert-Pearson subordinacy to sidestep the singularity at zero, as we did in this chapter for the Dirac case, and absorbing the angular momentum term  $\frac{C}{x^2}$  into the potential, the result follows.





# The Dirac Operator with an $L^2$ -Sparse Potential

## 1 Introduction

A real valued, locally integrable function  $Q$  defined on the half line  $[0, \infty)$  is said to be an  $L^2$ -sparse potential if, given any  $\delta, N > 0$ , there exists a subinterval  $(a, b)$  of  $[0, \infty)$  such that  $b - a = N$  and  $\int_a^b Q(x)^2 dx < \delta$ . In other words, if  $Q$  is  $L^2$ -sparse, one can find arbitrarily long intervals on which the  $L^2$  norm of  $Q$  is arbitrarily small. Any  $L^2$ -sparse potential is a sum  $Q_1 + Q_2$ , where  $Q_1$  is a sparse potential and  $Q_2 \in L^2$ . Here sparse means that arbitrarily long intervals exist on which the function is identically zero. An  $L^2$ -sparse potential can thus be viewed as a perturbation of a sparse.

Given an  $L^2$ -sparse potential  $q$  one can define the one-dimensional Dirac operator

$$\tau = -i\sigma_2 \frac{d}{dx} + \sigma_3 + q(x), \tag{4.1}$$

where  $\sigma_2, \sigma_3$  are again Pauli matrices. We again note that as this formal differential expression is always in the limit point case at  $\pm\infty$ , it has a unique self-adjoint realisation  $\tilde{T}$  in  $L^2(\mathbb{R})^2$ . In the present chapter we are interested in the self adjoint operator  $T$  on the half-line  $[0, \infty)$  with a boundary condition at zero.

In the paper [5] it was shown that the spectral theory for the Schrödinger operator

$$\varsigma = -\frac{d^2}{dx^2} + q, \quad (x \in [0, \infty), q \in L^2\text{-sparse})$$

can be closely linked to the theory of value distributions for real-valued functions, and in particular value distributions for functions which are defined as boundary values of Herglotz functions. It was further shown that the support of the absolutely continuous part of the spectral measure of  $\varsigma$  is contained within  $[0, \infty)$ . This was the culmination of the papers [6] and [7], in which the results seen in [41] and [40] were extended to show how spectral theory for Herglotz functions and differential operators is related to and dependent on the geometrical properties of the upper half plane.

The following definition of a value distribution appears in [6], and it is the one which we will use here. We also recall that  $\text{mes}(\cdot)$  is used to represent Lebesgue measure.

DEFINITION 4.1. Let  $A, B \subset \mathbb{R}$  be Borel subsets. Let

$$\mathcal{M} : (A, B) \rightarrow \mathcal{M}(A, B),$$

be a mapping satisfying the properties:

- (i)  $A \mapsto \mathcal{M}(A, B)$  defines a measure on Borel subsets of  $\mathbb{R}$ , for fixed  $B$   
 $B \mapsto \mathcal{M}(A, B)$  defines a measure on Borel subsets of  $\mathbb{R}$ , for fixed  $A$
- (ii)  $\mathcal{M}(A, \mathbb{R}) = \text{mes}(A)$ . Hence the measure  $A \mapsto \mathcal{M}(A, B)$  is absolutely continuous with respect to Lebesgue measure. Indeed, if  $\text{mes}(A) = 0$  then  $\mathcal{M}(A, \mathbb{R}) = \text{mes}(A) = 0$ . The result then follows from  $\mathcal{M}(A, B) \leq \mathcal{M}(A, \mathbb{R}) = 0$ ;
- (iii) The measure  $B \mapsto \mathcal{M}(A, B)$  is absolutely continuous with respect to Lebesgue measure.

Then  $\mathcal{M}$  will be called a value distribution function.

Let  $G_+ : \mathbb{R} \rightarrow \mathbb{R}$  be a Lebesgue measurable function. Let  $A, B \subset \mathbb{R}$  be Borel subsets. Let the map  $\mathcal{V} : (A, B) \rightarrow \mathbb{R} \cup \{\infty\}$  be defined by

$$\mathcal{V}(A, B) = \text{mes}(A \cap G_+^{-1}(B))$$

where  $G_+^{-1}(B) = \{\lambda \in \mathbb{R}; G_+(\lambda) \in B\}$ . It is clear that  $\mathcal{V}$  satisfies the requirements of the Definition 4.1 and thus we call  $\mathcal{V}$  the value distribution of  $G_+$ . Indeed property (i) follows from the properties of the Lebesgue measure; property (ii) is clear as  $G_+$  is an everywhere

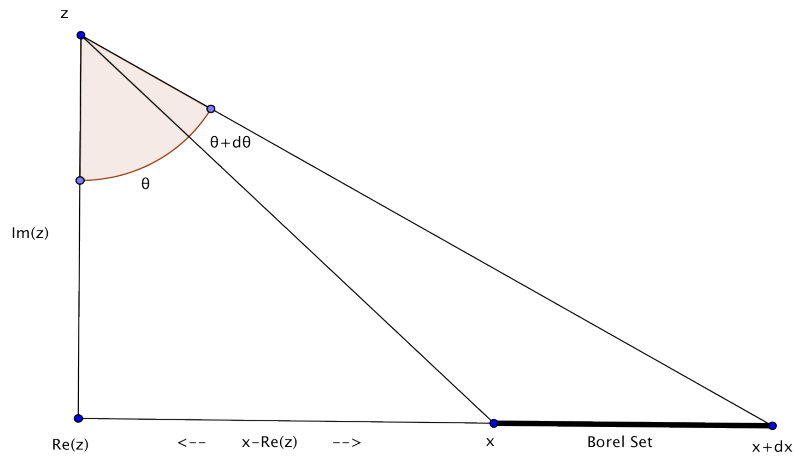
real function; property (iii) is equivalent to  $\text{mes}(G_+^{-1}(S)) = 0$  whenever  $\text{mes}(S) = 0$ . We are chiefly concerned with the important case that  $G_+$  is almost everywhere the boundary value of a Herglotz function; in this case we can write ([6] Equations (9), (10))

$$\mathcal{V}(A, B) = \frac{1}{\pi} \int_A \lim_{\varepsilon \rightarrow 0^+} \theta(G(\lambda + i\varepsilon), B) d\lambda,$$

where  $\theta(z, B)$  denotes the angle subtended at a point  $z \in \mathbb{C}^+$  by the Borel subset  $B$  of  $\mathbb{R}$ . We note that we define the angle subtended by a Borel subset  $B \subset \mathbb{R}$  at a point  $z \in \mathbb{C}^+$  by

$$\theta(z, B) = \int_B \Im \left( \frac{1}{\alpha - z} \right) d\alpha. \quad (4.2)$$

Indeed, considering this idea geometrically for an interval  $B$ ,



$$\theta = \arctan \left( \frac{x - \Re z}{\Im z} \right), \quad \theta + d\theta = \arctan \left( \frac{x + dx - \Re z}{\Im z} \right).$$

Thus

$$\begin{aligned}
d\theta &= \arctan\left(\frac{x + dx - \Re z}{\Im z}\right) - \arctan\left(\frac{x - \Re z}{\Im z}\right) \\
&= \frac{1}{\Im z} \arctan'\left(\frac{x - \Re z}{\Im z}\right) dx \\
&= \frac{dx}{\Im z \left(1 + \left(\frac{x - \Re z}{\Im z}\right)^2\right)} = \frac{\Im z \, dx}{(\Im z)^2 + (x - \Re z)^2} \\
&= \frac{\Im z \, dx}{|x - z|^2} = \Im\left(\frac{x - \bar{z}}{|x - z|^2}\right) dx = \Im\left(\frac{1}{x - z}\right) dx,
\end{aligned}$$

which gives us the result above.

Further, for  $\lambda \in \mathbb{R}$ , we define  $\theta(\lambda, B) = \pi\chi_B(\lambda)$  where  $\chi_B$  is the characteristic function of  $B$ . For complex argument  $z \in \mathbb{C}^+$  we define  $\omega(\cdot, B, F)$  by

$$\omega(z, B, F) = \frac{1}{\pi}\theta(F(z), B). \quad (4.3)$$

Unless  $z_1, z_2 \in \mathbb{C}^+$  are two points close to the real axis,  $\theta(z_1, B)$  will be close to  $\theta(z_2, B)$  if  $z_1$  is close to  $z_2$ . Defining an estimate of separation  $\gamma(\cdot, \cdot)$  of points in the upper half plane by

$$\gamma(z_1, z_2) = \frac{|z_1 - z_2|}{\sqrt{\Im(z_1)}\sqrt{\Im(z_2)}}, \quad (z_1, z_2 \in \mathbb{C}^+), \quad (4.4)$$

we can give a quantitative expression to how close  $\theta(z_1, B), \theta(z_2, B)$  are. Indeed, the following is given in [7] (Proposition 2)

LEMMA 4.1. *The estimate of separation  $\gamma(\cdot, \cdot)$  given in (4.4), may be expressed in terms of angle subtended, given by (4.2), as*

$$\gamma(z_1, z_2) = \sup_B \frac{|\theta(z_1, B) - \theta(z_2, B)|}{\sqrt{\theta(z_1, B)\theta(z_2, B)}}. \quad (4.5)$$

We note, however, that this measure of separation is not a metric as it does not satisfy the triangle inequality. It can, however, be related to a metric; indeed [7, Proposition 1] tells us that

$$\gamma(z_1, z_2) = 2 \sinh\left(\frac{1}{2}D(z_1, z_2)\right),$$

where  $D(z_1, z_2)$  is the hyperbolic distance defined for  $z_1, z_2$  in the upper half plane,  $\mathbb{H}$ , to be

$$D(z_1, z_2) = \inf\{\text{length}_{\mathbb{H}}(\sigma) : \sigma \text{ is a piecewise differentiable path with end points } z_1, z_2\}$$

and where  $\text{length}_{\mathbb{H}}$  is defined for a path  $\sigma : [a, b] \in \mathbb{R} \rightarrow \mathbb{H}$  to be

$$\text{length}_{\mathbb{H}}(\sigma) = \int_{\sigma} \frac{1}{\Im(z)} = \int_a^b \frac{|\sigma'(t)|}{\Im(\sigma(t))} dt.$$

Value distribution for boundary values of Herglotz functions is closely connected with the geometric properties of the upper half plane, regarded as a hyperbolic space. To see this if  $F_1, F_2$  are two Herglotz functions satisfying

$$\gamma(F_1(z), F_2(z)) < \varepsilon.$$

for all  $z$  such that  $\Im z = d$  and  $\Re z \in A$  then the value distribution associated with  $F_2$ ,  $\mathcal{V}_2(A, B)$ , is then a good approximation to the value distribution  $\mathcal{V}_1(A, B)$  associated with  $F_1(z)$ . Indeed

$$|\mathcal{V}_1(A, B) - \mathcal{V}_2(A, B)| \leq \varepsilon \text{mes}(A) + 2E_A(d). \quad (4.6)$$

(see [5] Equation (4)).  $E_A(d)$  is an error estimate and, as shown in [6] and [7],  $E_A(d)$  is an increasing function of  $d$  and  $\lim_{d \rightarrow 0} E_A(d) = 0$  for a fixed Borel set  $A$  (see Appendix A for some further details concerning  $E_A(d)$ ).

Let  $\varphi(\cdot, \lambda)$  be a solution of

$$\tau y = \lambda y(x, \lambda) \quad (4.7)$$

for  $\Im \lambda > 0$ . We will henceforth use  $z$  to denote points in the upper half plane;  $\lambda$  will be reserved for real values. Let  $u(\cdot, z), v(\cdot, z)$  be two further solutions of (4.7) forming a canonical fundamental system at 0, i.e.

$$\begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} (0, z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (4.8)$$

where  $v$  satisfies the boundary condition endowed to the operator at 0. As we are in the limit point case, we can use these solutions to uniquely define the Weyl m-function by

$$u(\cdot, z) + m(z)v(\cdot, z) \in L^2(\mathbb{R})^2$$

or, for  $f(\cdot, z) \in L^2(\mathbb{R})^2$  a non trivial solution

$$m(z) = \frac{f_2(0, z)}{f_1(0, z)}. \quad (4.9)$$

It is the aim of this chapter to demonstrate that the spectral theory of the Dirac operator can be linked to the theory of value distribution. We provide a link between the value distribution and the  $m$ -function for points in a subset of the absolutely continuous spectrum. This result is analogous to Theorem 1 of [6].

**THEOREM 4.1.** *Let  $A$  be a Borel subset of an essential support of the absolutely continuous part  $\rho_{ac}$  of the spectral measure  $\rho$  for the Dirac operator,  $T$ , acting in  $L^2(0, \infty)^2$  with  $|A| < \infty$ . Then we have, for any Borel subset  $B$  of  $\mathbb{R}$*

$$\lim_{N \rightarrow \infty} \left[ \text{mes} \left( \left\{ \lambda \in A; \frac{v_2}{v_1}(N, \lambda) \in B \right\} \right) - \frac{1}{\pi} \int_A \theta(m_+^N(\lambda), B) d\lambda \right] = 0.$$

This link is then used to prove

**THEOREM 4.2.** *Suppose that  $q$  is  $L^2$ -sparse. Then the support of the absolutely continuous part of the spectral measure of  $T$  is contained within  $(-\infty, -1] \cup [1, \infty)$ .*

This chapter is organised as follows: in Section 2 we define value distributions and the measure of separation which becomes our method of estimation. The fact that the  $m$ -function can be defined as the ratio of solution components motivates Section 3 in which we consider the asymptotics of  $\frac{v_2}{v_1}$ . In Section 4 we consider estimates of the canonical fundamental system for  $L^1$ -bounded potentials, comparing them to solutions with zero potential. In Section 5 we consider another ratio of solutions,  $\frac{f_2}{f_1}$ , which is closely related to the  $m$ -function for the problem with an  $L^2$ -sparse potential. We are able to relate this ratio to  $\frac{v_2}{v_1}$  which we have in turn related to the  $m$ -function for the free operator. In Section 6 we prove some results about value distributions and the absolutely continuous spectrum, culminating in a proof of Theorem 4.1. In Section 7 we prove Theorem 4.2.

## 2 Asymptotics of $\frac{v_2}{v_1}$

As discussed in the introduction and in more detail in Section 6 (see Equation (4.9)) it is clear that the  $m$ -function is dependent on the ratio of the components of a square integrable solution  $f(\cdot, z)$  of Equation (4.7). We will not approach this ratio directly; we will instead relate it to solutions we have a greater knowledge of. Thus, we begin our analysis by considering such a ratio of solution components as the potential is

varied, initially the solution  $v(\cdot, z)$  defined above i.e. the solution satisfying the boundary condition at zero. Bearing in mind our need to consider Herglotz functions it will become evident that we must use the ratio  $-\frac{v_2}{v_1}$  rather than  $\frac{v_2}{v_1}$ . Before proceeding, however, we require the following result; we use the notation for the Liouville bracket,  $[f, g](x) = f_1(x)g_2(x) - f_2(x)g_1(x)$ .

LEMMA 4.2. *Let  $f$  be a solution of the Dirac equation (4.7). Then*

$$[f, \bar{f}](x) = 2i\Im z \int_0^x f^T(t)\bar{f}(t)dt + [f, \bar{f}](0),$$

Further, if  $f_1 \neq 0$

$$\Im \left( -\frac{f_2}{f_1} \right) (x) = \frac{[f, \bar{f}](x)}{2i|f_1|^2}.$$

PROOF. The first result is a standard calculation using the fact that the components of a solution of the Dirac equation (4.7) must satisfy

$$\varphi_1' = (z - q + 1)\varphi_2$$

$$\varphi_2' = (1 + q - z)\varphi_1.$$

The second result follows easily. □

LEMMA 4.3. *Let  $z \in \mathbb{C}^+$  and let  $v, \tilde{v}$  be solutions of (4.7) for potentials  $q, \tilde{q}$  respectively, satisfying the initial condition  $v(0, z), \tilde{v}(0, z) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ . Then, for  $x \in (0, \infty)$ ,  $-\frac{v_2}{v_1}(x, z)$  and  $-\frac{\tilde{v}_2}{\tilde{v}_1}(x, z)$  are Herglotz functions and*

$$\gamma \left( -\frac{v_2}{v_1}(x, z), -\frac{\tilde{v}_2}{\tilde{v}_1}(x, z) \right) \leq \frac{(\int_0^x |q(t) - \tilde{q}(t)|^2 |\tilde{v}(t)|^2 dt)^{\frac{1}{2}}}{\Im z (\int_0^x |\tilde{v}(t)|^2 dt)^{\frac{1}{2}}}. \quad (4.10)$$

Further, if we assume that  $q \in L^2(\mathbb{R})$  and we set  $\tilde{q} = 0$  and denote by  $v^0$  the solution of (4.7) with zero potential then, for any  $L \geq \frac{1}{\sqrt{|z^2 - 1|}}$  we have the bound

$$\gamma \left( -\frac{v_2}{v_1}(L, z), -\frac{v_2^0}{v_1^0}(L, z) \right) \leq \frac{C|z^2 - 1|^{\frac{1}{4}}}{\Im z} \left( \int_0^L q^2 \right)^{\frac{1}{2}}, \quad (4.11)$$

where  $C$  is a positive constant.

PROOF. Let  $z \in \mathbb{C}^+$ . We begin by noting that, for all  $x_0 > 0$ ,  $v_1(x_0) \neq 0$ . Indeed, if this were not the case,  $v_1(x)$  would be a Dirichlet eigenfunction for the complex eigenvalue  $z$  of a regular boundary value problem on  $[0, x_0]$  with a boundary condition at



zero. Further, by Lemma 4.2

$$\Im \left( -\frac{v_2}{v_1} \right) (x) = \frac{[v, \bar{v}](x)}{2i|v_1|^2} = \frac{2i\Im z \int_0^x v^T(t)\bar{v}(t)dt + [v, \bar{v}](0)}{2i|v_1|^2} = \frac{2i\Im z \int_0^x v^T(t)\bar{v}(t)dt}{2i|v_1|^2} > 0 \quad (4.12)$$

( $x \in [0, \infty)$ ) with a similar result for  $\tilde{v}$ . Thus, as  $\Im \left( -\frac{v_2}{v_1} \right), \Im \left( -\frac{\tilde{v}_2}{\tilde{v}_1} \right) > 0$  we can consider

$$\gamma \left( -\frac{v_2}{v_1}(x, z), -\frac{\tilde{v}_2}{\tilde{v}_1}(x, z) \right) = \frac{\left| \frac{v_2}{v_1} - \frac{\tilde{v}_2}{\tilde{v}_1} \right|}{\sqrt{\Im \left( -\frac{v_2}{v_1} \right) \Im \left( -\frac{\tilde{v}_2}{\tilde{v}_1} \right)}} = \frac{\left| \frac{v_2\tilde{v}_1 - v_1\tilde{v}_2}{v_1\tilde{v}_1} \right|}{\sqrt{\Im \left( -\frac{v_2}{v_1} \right) \Im \left( -\frac{\tilde{v}_2}{\tilde{v}_1} \right)}}.$$

By a similar calculation to that of Lemma 4.2

$$[\tilde{v}, v](x) = \int_0^x (q(t) - \tilde{q}(t))v(t)^T \tilde{v}(t)dt, \quad (x \in [0, \infty))$$

and thus we have (using equation (4.12))

$$\begin{aligned} \gamma \left( -\frac{v_2}{v_1}(x, z), -\frac{\tilde{v}_2}{\tilde{v}_1}(x, z) \right) &= \frac{\left| \int_0^x (q - \tilde{q})v^T \tilde{v} \right|}{\Im z \left( \int_0^x |v|^2 \right)^{\frac{1}{2}} \left( \int_0^x |\tilde{v}|^2 \right)^{\frac{1}{2}}} \\ &\leq \frac{\left( \int_0^x |q - \tilde{q}|^2 |\tilde{v}|^2 \right)^{\frac{1}{2}}}{\Im z \left( \int_0^x |\tilde{v}|^2 \right)^{\frac{1}{2}}} \quad (x \in [0, \infty)) \end{aligned}$$

on application of the Cauchy-Schwarz Inequality, proving (4.10).

Proceeding to the second statement of the Lemma, we know from the first that

$$\gamma \left( -\frac{v_2}{v_1}(x), -\frac{v_2^0}{v_1^0}(x) \right) \leq \frac{\left( \int_0^x |q|^2 |v^0|^2 \right)^{\frac{1}{2}}}{\Im z \left( \int_0^x |v^0|^2 \right)^{\frac{1}{2}}} \quad (x \in [0, \infty)).$$

The solution of (4.7) with  $q = 0$  can be expressed as:

$$\varphi(x, z) = \begin{pmatrix} \alpha(z) \\ i\beta(z) \end{pmatrix} e^{i\sqrt{z^2-1}x} + \begin{pmatrix} \alpha(z) \\ -i\beta(z) \end{pmatrix} e^{-i\sqrt{z^2-1}x} \quad (4.13)$$

where the ratio

$$\frac{\beta(z)}{\alpha(z)} = \frac{\sqrt{z^2-1}}{z+1}$$

is fixed by the equation and we are considering the square root which takes values in the upper half plane (as in Chapter 2 Section 3).

Applying the initial conditions (4.8), we obtain an explicit expression (up to a constant multiplier) for  $v^0$ :

$$v^0(x) = \frac{1}{2} \begin{pmatrix} \frac{z+1}{i\sqrt{z^2-1}} \\ 1 \end{pmatrix} e^{i\sqrt{z^2-1}x} - \frac{1}{2} \begin{pmatrix} \frac{z+1}{i\sqrt{z^2-1}} \\ -1 \end{pmatrix} e^{-i\sqrt{z^2-1}x}, \quad (x \in [0, \infty)) \quad (4.14)$$

We write  $\sqrt{z^2-1} = a + ib$ , where  $a, b$  are real and  $b > 0$ . We also write  $A(z) = \frac{z+1}{i\sqrt{z^2-1}}$ .

Thus

$$\begin{aligned} |v^0|^2(x) &= \frac{1}{4} \begin{pmatrix} A(z)(e^{iax}e^{-bx} - e^{-iax}e^{bx}) \\ (e^{-iax}e^{bx} + e^{iax}e^{-bx}) \end{pmatrix} \begin{pmatrix} \overline{A(z)}(e^{-iax}e^{-bx} - e^{iax}e^{bx}) \\ (e^{iax}e^{bx} + e^{-iax}e^{-bx}) \end{pmatrix} \\ &= \frac{1}{4} \left[ |A(z)|^2 (e^{2bx} - e^{-2iax} - e^{2iax} + e^{-2bx}) + (e^{2bx} + e^{-2iax} + e^{2iax} + e^{-2bx}) \right] \\ &= \frac{1}{2} \left[ |A(z)|^2 (\cosh(2bx) - \cos(2ax)) + (\cosh(2bx) + \cos(2ax)) \right] \\ &\begin{cases} = \left( \frac{|A(z)|^2+1}{2} \right) \left[ \cosh(2bx) - \left( \frac{|A(z)|^2-1}{|A(z)|^2+1} \right) \cos(2ax) \right] \\ \leq \left( \frac{|A(z)|^2+1}{2} \right) (\cosh(2bx) + 1). \end{cases} \end{aligned}$$

Thus, using Lemma 4.3

$$\gamma \left( -\frac{v_2}{v_1}(L), -\frac{v_2^0}{v_1^0}(L) \right) \leq \frac{\left( \int_0^L |q|^2(t) (\cosh(2bt) + 1) dt \right)^{\frac{1}{2}}}{\Im z \left( \int_0^L \left[ \cosh(2bt) - \left( \frac{|A(z)|^2-1}{|A(z)|^2+1} \right) \cos(2at) \right] dt \right)^{\frac{1}{2}}}, \quad (L \in [0, \infty))$$

Considering the numerator, we find

$$\int_0^L q^2(t) (\cosh(2bt) + 1) dt \leq (\cosh(2bL) + 1) \int_0^L q^2(s) ds.$$

Considering the denominator for  $a \neq 0$ , we attain

$$\begin{aligned} &\int_0^L \left[ \cosh(2bt) - \left( \frac{|A(z)|^2-1}{|A(z)|^2+1} \right) \cos(2at) \right] dt \\ &= \left( \frac{\sinh(2bt)}{2b} - \left( \frac{|A(z)|^2-1}{|A(z)|^2+1} \right) \frac{\sin(2at)}{2a} \right) \Big|_0^L \\ &= \frac{\sinh(2bL)}{2b} - \left( \frac{|A(z)|^2-1}{|A(z)|^2+1} \right) \frac{\sin(2aL)}{2a}. \end{aligned} \quad (4.15)$$

On the other hand, if  $a = 0$ , we have to consider

$$\begin{aligned} & \int_0^L \left[ \cosh(2bt) - \left( \frac{|A(z)|^2 - 1}{|A(z)|^2 + 1} \right) \right] dt \\ &= \frac{\sinh(2bL)}{2b} - \left( \frac{|A(z)|^2 - 1}{|A(z)|^2 + 1} \right) L \end{aligned} \quad (4.16)$$

We shall assume that  $L \geq \frac{1}{\sqrt{|z^2-1|}}$ . This condition, along with  $\sqrt{z^2-1} = a + ib$ , implies that  $L \geq \frac{1}{\sqrt{2}\eta}$ ,  $\eta = \max\{|a|, b\}$ . Indeed, should  $L < \frac{1}{\sqrt{2}a}$  and  $L < \frac{1}{\sqrt{2}b}$  then  $|z^2 - 1| = a^2 + b^2 < \frac{1}{L^2}$  a contradiction. We will consider each case in turn.

First consider the case  $L \geq \frac{1}{\sqrt{2}|a|}$ . This case can only occur for  $a \neq 0$  (as  $|a| > b > 0$ ).

Now

$$\left| \left( \frac{|A(z)|^2 - 1}{|A(z)|^2 + 1} \right) \frac{\sin(2aL)}{2a} \right| \leq \left| \frac{\sin(2aL)}{2a} \right| \leq \frac{1}{2a} \leq \frac{L}{\sqrt{2}} \leq \frac{1}{\sqrt{2}} \frac{\sinh(2bL)}{2b}$$

and it follows that

$$\frac{\sinh(2bL)}{2b} - \left( \frac{|A(z)|^2 - 1}{|A(z)|^2 + 1} \right) \frac{\sin(2aL)}{2a} > \left( 1 - \frac{1}{\sqrt{2}} \right) \frac{\sinh(2bL)}{2b}$$

We now consider the case  $L \geq \frac{1}{\sqrt{2}|b|}$ , which occurs when  $|a| \leq b$ . As  $\frac{\sinh(x)}{x}$  is an increasing function,

$$\frac{\sinh(2bL)}{2b} \geq \frac{L \sinh(\sqrt{2})}{\sqrt{2}} \quad (4.17)$$

whereas

$$\left| \left( \frac{|A(z)|^2 - 1}{|A(z)|^2 + 1} \right) \frac{\sin(2aL)}{2a} \right| \leq \left| \frac{\sin(2aL)}{2a} \right| \leq L \quad (a \neq 0)$$

follows from  $\frac{\sin(x)}{x} \leq 1$ . On the other hand, for  $a = 0$ , it is easy to see that

$$\left| \left( \frac{|A(z)|^2 - 1}{|A(z)|^2 + 1} \right) L \right| < L.$$

Thus, using equation (4.17) it follows that

$$L \leq \frac{\sqrt{2} \sinh(2bL)}{2b \sinh(\sqrt{2})}$$

and so we obtain as a bound for (4.15) (the  $a \neq 0$  case)

$$\begin{aligned} \frac{\sinh(2bL)}{2b} - \left( \frac{|A(z)|^2 - 1}{|A(z)|^2 + 1} \right) \frac{\sin(2aL)}{2a} &\geq \frac{\sinh(2bL)}{2b} - \frac{\sqrt{2} \sinh(2bL)}{2b \sinh(\sqrt{2})} \\ &= \left( 1 - \frac{\sqrt{2}}{\sinh(\sqrt{2})} \right) \frac{\sinh(2bL)}{2b}, \end{aligned} \quad (4.18)$$

and in fact obtain the same lower bound for (4.16), the  $a = 0$  case.

Noting that  $\sinh \sqrt{2} < 2$  we see that the bound for  $L \geq \frac{1}{\sqrt{2b}}$  is also sufficient for the  $L \geq \frac{1}{\sqrt{2a}}$  case and thus we have the estimate

$$\begin{aligned} \gamma \left( -\frac{v_2}{v_1}(L), -\frac{v_2^0}{v_1^0}(L) \right) &\leq \frac{1}{\Im z} \left( \frac{(1 + \cosh(2bL)) \int_0^L q^2}{\left(1 - \frac{\sqrt{2}}{\sinh(\sqrt{2})}\right) \frac{\sinh(2bL)}{2b}} \right)^{\frac{1}{2}} \\ &= \frac{1}{\Im z} \left( \int_0^L q^2 \right)^{\frac{1}{2}} \left( 1 - \frac{\sqrt{2}}{\sinh(\sqrt{2})} \right)^{-\frac{1}{2}} \left( 2b \frac{(1 + \cosh(2bL))}{\sinh(2bL)} \right)^{\frac{1}{2}}. \end{aligned}$$

Now

$$\begin{aligned} 2b \frac{(1 + \cosh(2bL))}{\sinh(2bL)} &= 2b \frac{(2 + \cosh(2bL) - 1)}{\sinh(2bL)} = 2b \left( \frac{2}{\sinh(2bL)} + \tanh(bL) \right) \\ &< 2 \left( \frac{2b}{\sinh(2bL)} + b \right) \leq \frac{2}{L} + 2b \leq 2\sqrt{|z^2 - 1|} + 2b \leq 4\sqrt{|z^2 - 1|}, \end{aligned}$$

and (4.11) follows.  $\square$

Using Equation (4.14) it is easy to see that

$$\lim_{L \rightarrow \infty} -\frac{v_2^0}{v_1^0}(L) = -\lim_{L \rightarrow \infty} \frac{e^{2i\sqrt{z^2-1}L} + 1}{\frac{z+1}{i\sqrt{z^2-1}} (e^{2i\sqrt{z^2-1}L} - 1)} = i \frac{\sqrt{z^2-1}}{z+1}.$$

Thus we can find a bound for the difference between  $-\frac{v_2^0}{v_1^0}$  and its asymptotic limit, in particular a bound for the separation  $\gamma$ . This leads us to the following result:

LEMMA 4.4. *With  $z \in \mathbb{C}^+$  and  $v^0(x, z)$  defined as in Lemma 4.3, for any  $L \geq \frac{1}{\sqrt{|z^2-1|}}$  we have the bound*

$$\gamma \left( -\frac{v_2^0}{v_1^0}(L), i \frac{\sqrt{z^2-1}}{z+1} \right) \leq \frac{\sqrt{2}|z+1|\sqrt{a^2+b^2}e^{-bL}}{|a+a\Re z+b\Im z| \sqrt{\sinh(2bL) + \sin(2aL)} \left( \frac{(a\Im z - b\Re z - b)}{(a+a\Re z + b\Im z)} \right)}$$

PROOF. Explicitly, we have

$$\begin{aligned} -\frac{v_2^0}{v_1^0}(x) &= -\frac{i\sqrt{z^2-1}}{z+1} \left( \frac{e^{-bx}e^{iax} + e^{-iax}e^{bx}}{e^{-bx}e^{iax} - e^{-iax}e^{bx}} \right) \\ &= -\frac{(ia-b)}{z+1} \left( \frac{e^{-2bx} - e^{2iax} + e^{-2iax} - e^{2bx}}{|e^{-bx}e^{iax} - e^{-iax}e^{bx}|^2} \right) \\ &= \frac{2(ia-b)}{z+1} \left( \frac{\sinh(2bx) + i\sin(2ax)}{|e^{-bx}e^{iax} - e^{-iax}e^{bx}|^2} \right), \quad (x \in [0, \infty)) \end{aligned}$$

Thus we have

$$\begin{aligned}
\Im \left( -\frac{v_2^0}{v_1^0}(x) \right) &= \Im \left[ \frac{2(ia - b)}{z + 1} \left( \frac{\sinh(2bx) + i \sin(2ax)}{|e^{-bx}e^{iax} - e^{-iax}e^{bx}|^2} \right) \right] \\
&= \frac{2}{|e^{-bx}e^{iax} - e^{-iax}e^{bx}|^2} \Im \left[ \frac{(ia - b)}{z + 1} (\sinh(2bx) + i \sin(2ax)) \right] \\
&= \frac{2}{|e^{-bx}e^{iax} - e^{-iax}e^{bx}|^2} \left[ \Im \left( \frac{ia \sinh(2bx)}{z + 1} \right) - \Im \left( \frac{a \sin(2ax)}{z + 1} \right) \right. \\
&\quad \left. - \Im \left( \frac{b \sinh(2bx)}{z + 1} \right) - \Im \left( \frac{ib \sin(2ax)}{z + 1} \right) \right] \\
&= \frac{2}{|e^{-bx}e^{iax} - e^{-iax}e^{bx}|^2} \left[ \Im \left( \frac{ia \sinh(2bx)(\bar{z} + 1)}{|z + 1|^2} \right) - \Im \left( \frac{a \sin(2ax)(\bar{z} + 1)}{|z + 1|^2} \right) \right. \\
&\quad \left. - \Im \left( \frac{b \sinh(2bx)(\bar{z} + 1)}{|z + 1|^2} \right) - \Im \left( \frac{ib \sin(2ax)(\bar{z} + 1)}{|z + 1|^2} \right) \right] \\
&= \frac{2}{|e^{-bx}e^{iax} - e^{-iax}e^{bx}|^2} \left\{ \frac{a \sinh(2bx)}{|z + 1|^2} - \frac{b \sin(2ax)}{|z + 1|^2} + \frac{a \Re z \sinh(2bx)}{|z + 1|^2} \right. \\
&\quad \left. + \frac{a \Im z \sin(2ax)}{|z + 1|^2} + \frac{b \Im z \sinh(2bx)}{|z + 1|^2} - \frac{b \Re z \sin(2ax)}{|z + 1|^2} \right\} \\
&= \frac{2}{|z + 1|^2 |e^{-bx}e^{iax} - e^{-iax}e^{bx}|^2} [(\sinh(2bx)(a + a \Re z + b \Im z) + \sin(2ax)(a \Im z - b \Re z - b))].
\end{aligned}$$

Also

$$\Im \left( i \frac{\sqrt{z^2 - 1}}{z + 1} \right) = \Im \left( \frac{ia - b}{z + 1} \right) = \Im \left( \frac{(ia - b)(\bar{z} + 1)}{|z + 1|^2} \right) = \frac{(a + a \Re z + b \Im z)}{|z + 1|^2}.$$

Lastly

$$\begin{aligned}
\left| -\frac{v_2^0}{v_1^0}(x) - i \frac{\sqrt{z^2 - 1}}{z + 1} \right| &= \left| -\frac{v_2^0}{v_1^0}(x) - i \frac{a + ib}{z + 1} \right| \\
&= \left| -\frac{i(a + ib)}{z + 1} \left( \frac{e^{-bx}e^{iax} + e^{-iax}e^{bx}}{e^{-bx}e^{iax} - e^{-iax}e^{bx}} \right) - \frac{2i(a + ib)}{z + 1} \right| \\
&= \frac{\sqrt{a^2 + b^2}}{|z + 1|} \left| \left( \frac{e^{-bx}e^{iax} + e^{-iax}e^{bx}}{e^{-bx}e^{iax} - e^{-iax}e^{bx}} \right) \right| \\
&= \frac{2\sqrt{a^2 + b^2}e^{-bx}}{|z + 1| |e^{-bx}e^{iax} - e^{-iax}e^{bx}|} \quad (x \in [0, \infty)).
\end{aligned}$$

Putting these results together at  $x = L$  we find

$$\begin{aligned}
& \gamma \left( -\frac{v_2^0}{v_1^0}(L), i \frac{\sqrt{z^2 - 1}}{z + 1} \right) \\
&= \frac{\frac{4\sqrt{a^2 + b^2}e^{-bL}}{|z+1||e^{-bL}e^{iaL} - e^{-iaL}e^{bL}|}}{\sqrt{\frac{8(a+a\Re z + b\Im z)}{|z+1|^4|e^{-bL}e^{iaL} - e^{-iaL}e^{bL}|^2}[(\sinh(2bL)(a + a\Re z + b\Im z) + \sin(2aL)(a\Im z - b\Re z - b)]}} \\
&= \frac{\sqrt{2}|z + 1|\sqrt{a^2 + b^2}e^{-bL}}{|a + a\Re z + b\Im z|\sqrt{\sinh(2bL) + \sin(2aL)} \left( \frac{(a\Im z - b\Re z - b)}{(a + a\Re z + b\Im z)} \right)}.
\end{aligned}$$

□

### 3 Estimates of $u(x, z)$ and $v(x, z)$ for $L^1$ -bounded Potentials

In this section we consider solutions  $u(x, z)$  and  $v(x, z)$  of Equation (4.7) on a fixed interval  $0 \leq x \leq N$ , subject to initial conditions (4.8) at  $x = 0$ . We compare these solutions to the corresponding solutions  $u^0(x, z)$ ,  $v^0(x, z)$  with zero potential, again satisfying (4.8) at  $x = 0$ , which we know explicitly.

In order to carry out this comparison, we will have need of the following Gronwall type inequality:

LEMMA 4.5. *Let  $f, g : [0, \cdot) \rightarrow [0, \infty)$ . Let  $c > 0$  be constant and let  $f, g$  satisfy*

$$g(x) \leq c \int_0^x f + \int_0^x fg, \quad (x \geq 0). \quad (4.19)$$

Then

$$g(x) \leq c(e^{\int_0^x f} - 1), \quad (x \geq 0).$$

PROOF.  $f e^{-\int_0^x f} > 0$ , and so multiplying both sides of (4.19) by this expression and integrating gives us

$$\begin{aligned}
\int_0^x \left\{ f(s) e^{-\int_0^s f(t) dt} g(s) \right\} ds &\leq c \int_0^x \left\{ f(s) e^{-\int_0^s f(t) dt} \int_0^s f(t) dt \right\} ds \\
&\quad + \int_0^x \left\{ f(s) e^{-\int_0^s f(t) dt} \int_0^s f(t) g(t) dt \right\} ds.
\end{aligned}$$

Integrating the last term by parts:

$$\int_0^x \left\{ f(s)e^{-\int_0^s f(t)dt} \int_0^s f(t)g(t)dt \right\} ds = \int_0^x f(s)e^{-\int_0^s f(t)dt} g(s)ds - e^{-\int_0^x f(t)dt} \int_0^x f(t)g(t)dt,$$

and so the above becomes

$$\int_0^x f(t)g(t)dt \leq ce^{\int_0^x f(t)dt} \int_0^x \left\{ f(s)e^{-\int_0^s f(t)dt} \int_0^s f(t)dt \right\} ds.$$

Integrating the right hand side by parts twice gives

$$\int_0^x f(t)g(t)dt \leq c[e^{\int_0^x f(t)dt} - 1] - c \int_0^x f(t)dt$$

and again using (4.19)

$$g(x) - c \int_0^x f(t)dt \leq c[e^{\int_0^x f(t)dt} - 1] - c \int_0^x f(t)dt,$$

and the result follows.  $\square$

We will also require the following result about matrix norms.

LEMMA 4.6. *The Frobenius Norm*

$$\|A\|_F^2 = \sum_{i=1}^n \sum_{j=1}^n |A_{ij}|^2,$$

is sub-multiplicative, i.e.  $\|AB\|_F \leq \|A\|_F \|B\|_F$ .

PROOF. Let  $A \in \mathbb{C}^{m \times n}$ ,  $B \in \mathbb{C}^{n \times r}$  and  $x \in \mathbb{C}^n$ . Then, using  $\|\cdot\|_2$  to denote the Euclidean norm and  $A_{i*}$  to denote the  $i$ -th row of  $A$ ,

$$\|Ax\|_2^2 = \sum_i |A_{i*}x|^2 \leq \sum_i \|A_{i*}\|_2^2 \|x\|_2^2 = \|A\|_F^2 \|x\|_2^2,$$

and thus the Frobenius norm is compatible with the Euclidean norm. Thus it follows that

$$\|AB\|_F^2 = \sum_j \|[AB]_{*j}\|_2^2 = \sum_j \|AB_{*j}\|_2^2 \leq \sum_j \|A\|_F^2 \|B_{*j}\|_2^2 = \|A\|_F^2 \sum_j \|B_{*j}\|_2^2 = \|A\|_F^2 \|B\|_F^2$$

where  $A_{*j}$  represents the  $j$ -th column of  $A$ .  $\square$

We now use the notation  $|\cdot|$  to represent either the Euclidean norm or Frobenius norm.

We now present a proof of the following relatively standard result which is essentially an exercise from [10].

LEMMA 4.7. Let  $K$  be a fixed compact subset of  $\mathbb{C}^+$ , and let  $N > 0$  be fixed. Let  $u(x, z)$  and  $v(x, z)$  be solutions of Equation (4.7) on the fixed interval  $0 \leq x \leq N$ , subject to initial conditions (4.8) at  $x = 0$  and let  $u^0(x, z)$ ,  $v^0(x, z)$  be the corresponding solutions with zero potential, again satisfying (4.8) at  $x = 0$ . Then, given any  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that for any potential function  $q$  satisfying  $\int_0^N |q(t)| dt < \delta$ , we have, for all  $z \in K$  and for all  $x \in [0, N]$ ,

$$|u(x, z) - u_0(x, z)| < \varepsilon, \quad |v(x, z) - v_0(x, z)| < \varepsilon.$$

PROOF. Let  $M$  be the  $2 \times 2$  matrix given by  $M(x, z) = [u(x, z), v(x, z)]$  and let  $M_0(x, z) = [u_0(x, z), v_0(x, z)]$ . Now consider

$$\begin{aligned} \frac{d}{dx} M_0^{-1} M &= -M_0^{-1} \frac{dM_0}{dx} M_0^{-1} M + M_0^{-1} \frac{dM}{dx} \\ &= M_0^{-1} \left[ \frac{dM}{dx} - \frac{dM_0}{dx} M_0^{-1} M \right] = q M_0^{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} M \end{aligned}$$

We note that  $\det M = 1$ . Indeed,  $\det M$  is the Wronskian of two linearly independent solutions of (4.7) forming a canonical fundamental system at 0. Hence we can show that

$$q M_0^{-1} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} M = q A M_0^{-1} M, \quad q M_0^{-1} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} M = q B M_0^{-1} M$$

where  $A = (-v_1^0, u_1^0)^T (u_1^0, v_1^0)$  and  $B = (-v_2^0, u_2^0)^T (u_2^0, v_2^0)$ . On the other hand

$$M_0^{-1} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -v_2^0 \\ 0 & u_2^0 \end{pmatrix}.$$

Thus

$$\frac{d}{dx} M_0^{-1} M = q(A + B) M_0^{-1} M,$$

which implies that

$$M_0^{-1} M(x) = \mathbb{I} + \int_0^x q(t)(A + B)(t) M_0^{-1} M(t) dt.$$

Thus, for  $x \geq 0$ ,

$$|M_0^{-1} M(x) - \mathbb{I}| \leq \int_0^x |q(t)| \cdot |A + B|(t) dt + \int_0^x |q(t)| \cdot |A + B|(t) \cdot |M_0^{-1} M(t) - \mathbb{I}| dt.$$



An application of Lemma 4.5 now implies that

$$|M_0^{-1}M(x) - \mathbb{I}| \leq \exp\left(\int_0^N |q(t)| \cdot |A + B|(t)dt\right) - 1, \quad (x \in [0, N])$$

Thus,  $\forall x \in [0, N]$

$$\begin{aligned} |M(x) - M_0(x)| &\leq |M_0(x)| \cdot |M_0^{-1}M(x) - \mathbb{I}| \\ &< |M_0(x)| \left( \exp\left(\int_0^N |q(t)| \cdot |A + B|(t)dt\right) - 1 \right), \end{aligned}$$

and the result follows by noting that  $|A|, |B|$  and  $|M_0|$  are bounded on  $[0, N]$  and  $z \in K$ , and using that

$$|u - u_0|(x) \leq |M(x) - M_0(x)|, \quad |v - v_0|(x) \leq |M(x) - M_0(x)| \quad (x \in [0, N]).$$

□

The following corollary is a straightforward consequence of the previous Lemma.

**COROLLARY 4.1.** *Let  $K$  be a fixed compact subset of  $\mathbb{C}^+$ , and let  $N > 0$  be fixed. Then, given any  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that for all potential functions  $q$  satisfying  $\int_0^N |q(t)|dt < \delta$ , we have for all  $z \in K$  and all solutions  $u, v, u_0, v_0$  defined as in Lemma 4.7.*

$$\left| \int_0^N \Im(\bar{u}^T(t, z)v(t, z))dt - \int_0^N \Im(\bar{u}_0^T(t, z)v_0(t, z))dt \right| < \varepsilon.$$

## 4 Estimate of $-\frac{f_2(x, z)}{f_1(x, z)}$ for Potentials Subject to an $L^2$ -type Condition

Before proceeding with the main result of this section, we first prove the following lemma

**LEMMA 4.8.** *Let  $z \in \mathbb{C}^+$ . Further let  $u^0, v^0$  be solutions of (4.7) with  $q = 0$  subject to the initial conditions (4.8). Then*

$$\int_0^\infty \Im([\bar{u}^0]^T v^0)dt = \infty$$

PROOF. Using (4.13) and the initial conditions we can deduce that

$$u^0(x, z) = \frac{1}{2} \begin{pmatrix} 1 \\ i\frac{\sqrt{z^2-1}}{z+1} \end{pmatrix} e^{i\sqrt{z^2-1}x} + \frac{1}{2} \begin{pmatrix} 1 \\ -i\frac{\sqrt{z^2-1}}{z+1} \end{pmatrix} e^{-i\sqrt{z^2-1}x}.$$

and so using (4.14)

$$\begin{aligned} [\bar{u}^0(x, z)]^T v^0(x, z) &= \frac{1}{4} \begin{pmatrix} e^{i\sqrt{z^2-1}x} + e^{-i\sqrt{z^2-1}x} \\ \frac{i\sqrt{z^2-1}}{z+1} [e^{i\sqrt{z^2-1}x} - e^{-i\sqrt{z^2-1}x}] \end{pmatrix}^T \begin{pmatrix} \frac{z+1}{i\sqrt{z^2-1}} [e^{i\sqrt{z^2-1}x} - e^{-i\sqrt{z^2-1}x}] \\ e^{i\sqrt{z^2-1}x} + e^{-i\sqrt{z^2-1}x} \end{pmatrix} \\ &= \frac{1}{2} \left( \frac{z+1}{\sqrt{z^2-1}} - \frac{\sqrt{z^2-1}}{z+1} \right) \sin(2x\Re\sqrt{z^2-1}) \\ &\quad + \frac{i}{2} \left( \frac{z+1}{\sqrt{z^2-1}} + \frac{\sqrt{z^2-1}}{z+1} \right) \sinh(2x\Im\sqrt{z^2-1}). \end{aligned}$$

Let  $\sqrt{z-1} = \alpha + i\beta$ ,  $\sqrt[4]{z+1} = \gamma + i\delta$ , where these square roots are as in Chapter 2 Section 3. Then, as  $z \in \mathbb{C}^+$ ,  $\alpha, \beta, \gamma, \delta > 0$ . Then

$$\frac{z+1}{\sqrt{z^2-1}} = \frac{\sqrt[4]{z+1}}{\sqrt{z-1}} = \frac{(\alpha\gamma + \delta\beta) + i(\alpha\delta - \beta\gamma)}{|\sqrt{z-1}|^2},$$

and similarly

$$\frac{\sqrt{z^2-1}}{z+1} = \frac{(\alpha\gamma + \delta\beta) + i(\alpha\delta - \beta\gamma)}{|\sqrt[4]{z+1}|^2}.$$

Thus, for large  $x$ ,

$$\begin{aligned} \Im([\bar{u}^0]^T(x, z)v^0(x, z)) &= \frac{1}{2} \left[ \frac{(\alpha\delta - \beta\gamma)}{|\sqrt{z-1}|^2} - \frac{(\alpha\delta - \beta\gamma)}{|\sqrt[4]{z+1}|^2} \right] \sin(2x\Re\sqrt{z^2-1}) \\ &\quad + \frac{1}{2} \left[ \frac{(\alpha\gamma + \delta\beta)}{|\sqrt{z-1}|^2} + \frac{(\alpha\gamma + \delta\beta)}{|\sqrt[4]{z+1}|^2} \right] \sinh(2x\Im\sqrt{z^2-1}) \\ &\sim C(z) \sinh(2x\Im\sqrt{z^2-1}) \end{aligned}$$

where  $C(z) > 0$ , and thus  $\Im([\bar{u}^0]^T v^0)$  is not integrable.  $\square$

We also require a further lemma, which calls upon the following results. These follow by a simple calculation using Lemma 4.2 and that the Wronskians of  $u, v$  and  $\bar{u}, \bar{v}$  are 1 at the point  $N$ .

LEMMA 4.9. *Let  $u(\cdot, z), v(\cdot, z)$  be solutions of Equation (4.7) subject to initial conditions (4.8). Let  $N > 0$  and  $z \in \mathbb{C}^+$ . Then*

$$(1) \quad \int_0^N \Im(\bar{u}^T v) dx = -\frac{1 - \Re[u, \bar{v}](N)}{2\Im z}; \quad (4.20)$$

$$(2) \quad |[u, \bar{v}](N)|^2 = 1 - \mathbb{W}(u, \bar{u})\mathbb{W}(v, \bar{v}) \quad (4.21)$$

where for simplicity  $u, v$  represent the solutions  $u(\cdot, z)$ ,  $v(\cdot, z)$  and  $\mathbb{W}$  represents the Wronskian.

We now use these identities to prove the following convergence lemma, analogous to the result proved for the Schrödinger equation in [6].

LEMMA 4.10. *Let  $u(\cdot, z)$ ,  $v(\cdot, z)$  be solutions of Equation (4.7) subject to initial conditions (4.8). Let  $m^{(1)}$  be any constant such that  $\Im m^{(1)} \geq 0$ . Then, for any  $N > 0$  and for all  $z \in \mathbb{C}^+$  we have the estimate*

$$\gamma \left( -\frac{v_2}{v_1}(N, z), -\frac{u_2(N, z) + \overline{m^{(1)}}v_2(N, z)}{u_1(N, z) + \overline{m^{(1)}}v_1(N, z)} \right) \leq \frac{1}{\sqrt{I(I+1)}}$$

where the integral  $I$  is defined by

$$I(N, z) = (\Im z) \int_0^N \Im(\bar{u}^T(x, z)v(x, z)) dx.$$

PROOF. Using equation (4.4) we have at  $x = N$

$$\gamma^2 \left( -\frac{v_2}{v_1}, -\frac{u_2 + \overline{m^{(1)}}v_2}{u_1 + \overline{m^{(1)}}v_1} \right) = \frac{\left| -\frac{v_2}{v_1} + \frac{u_2 + \overline{m^{(1)}}v_2}{u_1 + \overline{m^{(1)}}v_1} \right|^2}{\Im \left( -\frac{v_2}{v_1} \right) \Im \left( -\frac{u_2 + \overline{m^{(1)}}v_2}{u_1 + \overline{m^{(1)}}v_1} \right)}.$$

Thus, using Lemma 4.2 and  $\mathbb{W}(u, v) = 1$ ,

$$\begin{aligned} \gamma^2 \left( -\frac{v_2}{v_1}, -\frac{u_2 + \overline{m^{(1)}}v_2}{u_1 + \overline{m^{(1)}}v_1} \right) &= -\frac{4 \left[ \left| -\frac{v_2}{v_1} + \frac{u_2 + \overline{m^{(1)}}v_2}{u_1 + \overline{m^{(1)}}v_1} \right| \cdot |v_1| \cdot |u_1 + \overline{m^{(1)}}v_1| \right]^2}{[v, \bar{v}](N)[u + \overline{m^{(1)}}v, \bar{u} + \overline{m^{(1)}}\bar{v}](N)} \\ &= -\frac{4|\mathbb{W}(u, v)|^2}{[v, \bar{v}](N)[u + \overline{m^{(1)}}v, \bar{u} + \overline{m^{(1)}}\bar{v}](N)} \\ &= -\frac{4}{[v, \bar{v}](N)[u + \overline{m^{(1)}}v, \bar{u} + \overline{m^{(1)}}\bar{v}](N)}. \end{aligned}$$

As  $m^{(1)} \in \overline{\mathbb{C}^+}$ , we can write  $m^{(1)} = \Re m^{(1)} + iY$ ,  $Y \geq 0$ . Then,

$$\begin{aligned} & -[v, \bar{v}][u + \overline{m^{(1)}}v, \bar{u} + m^{(1)}\bar{v}] \\ &= -[v, \bar{v}][u, \bar{u}] - 2i[v, \bar{v}]\Im[u, \bar{v}]\Re m^{(1)} \\ & \quad - 2iY[v, \bar{v}]\Re[u, \bar{v}] - ([v, \bar{v}])^2[\Re m^{(1)}]^2 - ([v, \bar{v}])^2Y^2. \end{aligned}$$

This is of the form  $A + B\Re m^{(1)} + C[\Re m^{(1)}]^2 - \alpha(Y)$  with  $A, C \geq 0$  and  $B \in \mathbb{R}$ . Indeed, as we have seen,

$$\begin{aligned} A &= -[v, \bar{v}][u, \bar{u}] = -\left[\int_0^N 2i\Im z|v|^2 dx\right]\left[\int_0^N 2i\Im z|u|^2 dx\right] = 4(\Im z)^2 \int_0^N |u|^2 dx \int_0^N |v|^2 dx \\ B &= -2i[v, \bar{v}]\Im[u, \bar{v}] = -2i\left[\int_0^N 2i\Im z|v|^2 dx\right]\Im[u, \bar{v}] = 2\left[\int_0^N 2\Im z|v|^2 dx\right]\Im[u, \bar{v}] \\ C &= -([v, \bar{v}])^2 = -\left[\int_0^N 2i\Im z|v|^2 dx\right]^2 = 4(\Im z)^2 \left(\int_0^N |v|^2 dx\right)^2 \end{aligned}$$

We can bound  $A + B\Re m^{(1)} + C[\Re m^{(1)}]^2$  below by  $A - \frac{B^2}{4C}$ , and so we obtain

$$\gamma^2 \left( -\frac{v_2}{v_1}, -\frac{u_2 + \overline{m^{(1)}}v_2}{u_1 + \overline{m^{(1)}}v_1} \right) \leq -\frac{4}{[u, \bar{u}][v, \bar{v}] + 2iY[v, \bar{v}]\Re[u, \bar{v}] + ([v, \bar{v}])^2Y^2 + [\Im[u, \bar{v}]]^2}.$$

Since

$$\begin{aligned} [u - iYv, \overline{u - iYv}] &= [u, \bar{u}] + Y^2[v, \bar{v}] + 2iY[u, \bar{v}] \\ \Im[u - iYv, \bar{v}] &= \Im[u, \bar{v}] - \Im(iY[v, \bar{v}]) = \Im[u, \bar{v}] \end{aligned}$$

we can re-write the above in the form

$$\gamma^2 \left( -\frac{v_2}{v_1}, -\frac{u_2 + \overline{m^{(1)}}v_2}{u_1 + \overline{m^{(1)}}v_1} \right) \leq -\frac{4}{[u - iYv, \overline{u - iYv}][v, \bar{v}] + [\Im[u - iYv, \bar{v}]]^2}.$$

Considering equation (4.21) applied to the functions  $u - iYv, v$ , subtracting  $[\Re[u - iYv, \bar{v}]]^2$  from both sides, and substituting for  $[\Im[(u - iYv, \bar{v})]]^2$  we obtain

$$\gamma^2 \left( -\frac{v_2}{v_1}, -\frac{u_2 + \overline{m^{(1)}}v_2}{u_1 + \overline{m^{(1)}}v_1} \right) \leq -\frac{4}{1 - [\Re[u - iYv, \bar{v}]]^2} = \frac{4}{[\Re[u - iYv, \bar{v}]]^2 - 1}.$$

Now,

$$\begin{aligned}
[\Re[u - iYv, \bar{v}]]^2 - 1 &= [\Re[u, \bar{v}] + \Re[-iYv, \bar{v}]]^2 - 1 \\
&= \left[ 2\Im z \int_0^N \Im(\bar{u}^T v) dx + 1 + Y \Im z \int_0^N |v|^2 dx \right]^2 - 1 \\
&= \left( \Im z \int_0^N [2\Im(\bar{u}^T v) + Y|v|^2] dx \right)^2 + 2 \left( \Im z \int_0^N [2\Im(\bar{u}v) + Y|v|^2] dx \right) \\
&\geq 4I^2 + 4I,
\end{aligned}$$

and the result then follows.  $\square$

The following three lemmas will also be required for the proof of Theorem 4.3 to follow.

LEMMA 4.11. *Let  $f(\cdot, z)$  be a solution of (4.7) satisfying  $\Im\left(-\frac{f_2(0, z)}{f_1(0, z)}\right) > 0$  with  $f_1(x, z) \neq 0$  ( $x \geq 0$ ). Then for all  $x > 0$ ,  $\Im\left(-\frac{f_2(x, z)}{f_1(x, z)}\right) > 0$ .*

PROOF. Follows by Lemma 4.2.  $\square$

LEMMA 4.12. *For any  $\varepsilon > 0$  and any  $K \subset \mathbb{C}^+$  compact, we can find an  $N > 0$  such that*

$$\gamma\left(-\frac{v_2}{v_1}(N, z), -\frac{f_2}{f_1}(N, z)\right) \leq \frac{1}{\sqrt{I(I+1)}}$$

for all  $z \in K$  and solutions  $f$  of the Dirac equation (4.7) satisfying the conditions

$$\Im\left(-\frac{f_2(0, z)}{f_1(0, z)}\right) > 0 \text{ and } f_1(x, z) \neq 0, (x \geq 0).$$

PROOF. By Lemma 4.11,  $\Im\left(-\frac{f_2(x, z)}{f_1(x, z)}\right) > 0$ . We can write

$$-\frac{f_2(x, z)}{f_1(x, z)} = -\frac{u_2(x, z) + \bar{m}^{(1)}v_2(x, z)}{u_1(x, z) + \bar{m}^{(1)}v_1(x, z)}$$

where  $m^{(1)}$  is given by  $-\bar{m}^{(1)} = -\frac{f_2(0, z)}{f_1(0, z)}$ . Thus  $\Im m^{(1)} \geq 0$  and the result follows from Lemma 4.10.  $\square$

LEMMA 4.13. *Let  $z \in \mathbb{C}^+$  and  $u(\cdot, z), v(\cdot, z)$  be solutions of Equation (4.7) subject to initial conditions (4.8). Then*

$$\Psi(N) = \int_0^N \Im(\bar{u}^T(x, z)v(x, z)) dx$$

is an increasing function of  $N$ .

PROOF. Since we are in the limit point case, we have only one square integrable solution for a given spectral parameter;  $v(\cdot, z)$  cannot be this solution else  $z$  would be a non-real eigenvalue in violation to the Dirac equation we are considering being self-adjoint. Thus,  $v(\cdot, z) \notin L^2([0, \infty))^2$  and so

$$\lim_{N \rightarrow \infty} \int_0^N |v(x, z)|^2 dx = \infty,$$

uniformly over compact subsets of  $\mathbb{C}^+$ . Consider the ratio  $\frac{\int_0^N \Im(\bar{u}^T v) dx}{\int_0^N |v|^2 dx}$  and its derivative. Using Lemma 4.9 and dropping the dependence on  $x, z$  in the notation for brevity

$$\begin{aligned} \frac{d}{dN} \frac{\int_0^N \Im(\bar{u}^T v) dx}{\int_0^N |v|^2 dx} &= \frac{\frac{d}{dN} \left( \int_0^N \Im(\bar{u}^T v) dx \right) \int_0^N |v|^2 dx - \frac{d}{dN} \left( \int_0^N |v|^2 dx \right) \int_0^N \Im(\bar{u}^T v) dx}{\left( \int_0^N |v|^2 dx \right)^2} \\ &= \frac{\Im(\bar{u}^T v) \int_0^N |v|^2 dx - |v|^2 \int_0^N \Im(\bar{u}^T v) dx}{\left( \int_0^N |v|^2 dx \right)^2} \\ &= \frac{\left( \frac{\bar{u}^T v - u^T \bar{v}}{2i} \right) \frac{1}{2i\Im z} [v, \bar{v}] + |v|^2 \frac{1}{2\Im z} (1 - \Re[u, \bar{v}])}{\left( \int_0^N |v|^2 dx \right)^2} \\ &= \frac{(u^T \bar{v} - \bar{u}^T v)[v, \bar{v}] + |v|^2 (2 - 2\Re[u, \bar{v}])}{4\Im z \left( \int_0^N |v|^2 dx \right)^2} \\ &= \frac{(u^T \bar{v} - \bar{u}^T v)[v, \bar{v}] + |v|^2 (2 - [u, \bar{v}] - [\bar{u}, v])}{4\Im z \left( \int_0^N |v|^2 dx \right)^2} \\ &= \frac{(u_1 \bar{v}_1 - \bar{u}_1 v_1) \mathbb{W}(v, \bar{v}) + v_1 \bar{v}_1 (2 - \mathbb{W}(u, \bar{v}) - \mathbb{W}(\bar{u}, v))}{4\Im z \left( \int_0^N |v|^2 dx \right)^2} \\ &\quad + \frac{(u_2 \bar{v}_2 - \bar{u}_2 v_2)[v, \bar{v}] + v_2 \bar{v}_2 (2 - [u, \bar{v}] - [\bar{u}, v])}{4\Im z \left( \int_0^N |v|^2 dx \right)^2}. \end{aligned}$$

Now, using  $\mathbb{W}(u(N), v(N)) = \mathbb{W}(\overline{u(N)}, \overline{v(N)}) = 1$

$$\begin{aligned} &(u_1 \bar{v}_1 - \bar{u}_1 v_1)[v, \bar{v}] + v_1 \bar{v}_1 (2 - [u, \bar{v}] - [\bar{u}, v]) \\ &= (u_1 \bar{v}_1 - \bar{u}_1 v_1)(v_1 \bar{v}_2 - v_2 \bar{v}_1) + (v_1 \bar{v}_1 (2 - u_1 \bar{v}_2 + u_2 \bar{v}_1 - \bar{u}_1 v_2 + \bar{u}_2 v_1)) \\ &= (u_2 v_1 - u_1 v_2) \bar{v}_1^2 + (\bar{u}_2 \bar{v}_1 - \bar{u}_1 \bar{v}_2) v_1^2 + 2v_1 \bar{v}_1 \\ &= -v_1^2 - \bar{v}_1^2 + 2v_1 \bar{v}_1 = -(v_1 - \bar{v}_1)^2, \end{aligned}$$

with a similar result for the other numerator. Thus

$$\begin{aligned} \frac{d}{dN} \frac{\int_0^N \Im(\bar{u}^T v) dx}{\int_0^N |v|^2 dx} &= \frac{-(v_1 - \bar{v}_1)^2 - (v_2 - \bar{v}_2)^2}{4\Im z \left( \int_0^N |v|^2 dx \right)^2} \\ &= \frac{(\Im v_1(N, z))^2 + (\Im v_2(N, z))^2}{\Im z \left( \int_0^N |v(x, z)|^2 dx \right)^2} \geq 0, \end{aligned}$$

from which it follows that the ratio of integrals cannot decrease with  $N$ .  $\square$

We can now state an estimate of convergence of  $-\frac{f_2(x, z)}{f_1(x, z)}$  to  $i\frac{\sqrt{z^2-1}}{z+1}$  based on an  $L^2$  type condition on the potential.

**THEOREM 4.3.** *Let  $K$  be any fixed compact subset of  $\mathbb{C}^+$ . Then, given any  $\varepsilon > 0$ , there exist  $\delta > 0$  and  $N > 0$  such that for all  $L \geq N$  and for all potential functions  $q$  satisfying the  $L^2$  bound*

$$\int_0^L |q(t)|^2 dt < \delta,$$

*the corresponding solution of (4.7),  $f(x, z)$ , satisfying the condition  $\Im\left(-\frac{f_2(0, z)}{f_1(0, z)}\right) > 0$ , also satisfies the estimate*

$$\gamma\left(-\frac{f_2(L, z)}{f_1(L, z)}, i\frac{\sqrt{z^2-1}}{z+1}\right) < \varepsilon \quad (4.22)$$

*for all  $z \in K$ .*

**PROOF.** Note that the separation  $\gamma(z_1, z_2)$  between two points  $z_1, z_2 \in \mathbb{C}^+$  does not satisfy the triangle inequality. However, we have the following substitute: If  $z_1, z_2, z_3 \in \mathbb{C}^+$  and it is given that

$$\gamma(z_1, z_2) < \alpha, \quad \gamma(z_2, z_3) < \beta, \quad 0 < \alpha, \beta \leq 2$$

then it follows easily (see [5] Theorem 1) that

$$\gamma(z_1, z_3) < \sqrt{2}(\alpha + \beta).$$

As a simple consequence, the three inequalities  $\gamma(z_1, z_2) < \frac{\varepsilon}{6}, \gamma(z_2, z_3) < \frac{\varepsilon}{6}$  and  $\gamma(z_3, z_4) < \frac{\varepsilon}{6}$ , with  $0 < \varepsilon < 1$ , together imply that  $\gamma(z_1, z_4) < \varepsilon$ . If we define  $u, v, u^0, v^0$  as in the previous lemmas, then in order to verify (4.22) it will be sufficient to show that if  $z \in K$ , we can find an  $N$  large enough so that we have the following three inequalities at  $x = L$ ,

$L \geq N$ ,

$$\gamma\left(-\frac{f_2}{f_1}(L), -\frac{v_2}{v_1}(L)\right) < \frac{\varepsilon}{6}, \quad \gamma\left(-\frac{v_2}{v_1}(L), -\frac{v_2^0}{v_1^0}(L)\right) < \frac{\varepsilon}{6}, \quad \gamma\left(-\frac{v_2^0}{v_1^0}(L), i\frac{\sqrt{z^2-1}}{z+1}\right) < \frac{\varepsilon}{6}. \quad (4.23)$$

Given any  $\varepsilon > 0$  and compact subset  $K \subset \mathbb{C}^+$ , we fix  $N$  to satisfy, for all  $z \in K$ , the three inequalities

$$\int_0^N \Im(\bar{u}_0^T v_0) dt > \frac{12}{\varepsilon \Im z}, \quad (4.24)$$

$$\frac{\sqrt{2}|z+1|\sqrt{a^2+b^2}e^{-bN}}{|a+a\Re z+b\Im z|\sqrt{\sinh(2bN)+\sin(2aN)}\left(\frac{a\Im z-b\Re z-b}{a+a\Re z+b\Im z}\right)} < \frac{\varepsilon}{6}, \quad (4.25)$$

$$N > \frac{1}{\sqrt{|z^2-1|}}. \quad (4.26)$$

That  $N$  may be chosen to satisfy the first of these inequalities for  $z \in K$  follows from the fact that  $\int_0^\infty \Im(\bar{u}_0^T v_0) dt = \infty$ , by Lemma 4.8, and that, for fixed  $N$ , the integral  $\int_0^N \Im(\bar{u}_0^T v_0) dt$  depends continuously on  $z$  for  $\Im z > 0$ . In the second inequality we have  $\sqrt{z^2-1} = a + ib$ , where both  $a$  and  $b$  are bounded for  $z \in K$  (indeed  $a = b = 0$  iff  $z \in [-1, 1]$ ). We also note that  $\frac{1}{\sqrt{|z^2-1|}}$  is bounded for  $z \in K$ . From Corollary 4.1 we know that, for  $z \in K$ , the integral  $\int_0^N \Im(\bar{u}^T v) dt$  is close to  $\int_0^N \Im(\bar{u}_0^T v_0) dt$  provided that  $\int_0^N |q(t)| dt$  is sufficiently small. In particular, inequality (4.24) implies that there exists  $\delta_0 > 0$  such that, for all  $z \in K$ , we have

$$\int_0^N |q(t)| dt < \delta_0 \implies \int_0^N \Im(\bar{u}^T v) dt \geq \int_0^N \Im(\bar{u}_0^T v_0) - \left| \int_0^N (\Im(\bar{u}^T v) - \Im(\bar{u}_0^T v_0)) \right| > \frac{6}{\varepsilon \Im z}. \quad (4.27)$$

Having fixed the values of  $N$  and  $\delta_0$ , define  $\delta > 0$  to satisfy

$$\delta < \frac{\delta_0^2}{N}, \quad (4.28)$$

$$\frac{C|z^2-1|^{\frac{1}{4}}\sqrt{\delta}}{\Im z} < \frac{\varepsilon}{6}, \quad \forall z \in K. \quad (4.29)$$

Here  $C$  is defined in Lemma 4.3. Now suppose that  $L \geq N$  and  $\int_0^L |q(t)|^2 dt < \delta$ . Using the Cauchy-Schwarz inequality

$$\int_0^N |q(t)| dt \leq \left( N \int_0^N |q(t)|^2 dt \right)^{\frac{1}{2}} < (\delta N)^{\frac{1}{2}} < \delta_0.$$



Hence (4.27) implies that

$$\int_0^N \Im(\bar{u}^T v) dt > \frac{6}{\varepsilon \Im z}.$$

By Lemma 4.10, 4.12 and 4.13 we have, for any solution  $f$  of (4.7) satisfying  $\Im\left(-\frac{f_2(0, z)}{f_1(0, z)}\right) > 0$ ,

$$\gamma\left(-\frac{f_2}{f_1}(L), -\frac{v_2}{v_1}(L)\right) \leq \frac{1}{\Im z \int_0^L \Im(\bar{u}^T v) dt} < \frac{\varepsilon}{6},$$

our first inequality from (4.23). The second follows from Lemma 4.3 combined with (4.29), using  $\int_0^N |q(t)|^2 dt < \delta$ . Lemma 4.4 combined with inequality (4.25) completes the proof of the inequalities (4.23), and the proof of the theorem.  $\square$

REMARK 4.1. Generally  $L^2$  smallness is seen as a less severe restriction than  $L^1$  smallness, but here it is used to prove  $L^1$  smallness for a particular case in order to prove that  $\gamma\left(-\frac{f_2}{f_1}(L), -\frac{v_2}{v_1}(L)\right) < \frac{\varepsilon}{6}$  ( $\varepsilon > 0$ ). This gives the impression that the results of this chapter can be proven for the case with  $L^1$ -sparse potential (and, incidentally, this is true and requires fewer preliminary results).

We now explore some consequences of Theorem 4.3 in the case of  $L^2$ -sparse potentials. Let  $q$  be an  $L^2$ -sparse potential. Then a sequence of subintervals  $\{(a_k, b_k) \subset \mathbb{R}^+\}_{k \in \mathbb{N}}$  can be found such that, with  $L_k = b_k - a_k$ ,

$$\lim_{k \rightarrow \infty} L_k = \infty, \quad \lim_{k \rightarrow \infty} \int_{a_k}^{b_k} (q(t))^2 dt = 0.$$

Given  $A \subset \mathbb{R}$ , fixed bounded and measurable, having closure  $\bar{A}$ , and given any  $\varepsilon > 0$ , we first of all find  $d > 0$  such that  $E_A(d) < \frac{\varepsilon \text{mes}(A)}{2}$ .  $E_A(d)$  is the error estimate defined in (4.6), and from (4.6) we deduce that

$$|\mathcal{V}_1(A, S) - \mathcal{V}_2(A, S)| < 2\varepsilon \text{mes}(A) \tag{4.30}$$

provided  $\gamma(F_1, F_2) < \varepsilon$  for all  $z \in K$ , where  $K \subset \mathbb{C}^+$  is compact and defined by  $\Im z = d$ ,  $\Re z \in \bar{A}$ .

By Lemma 4.11 we can use Theorem 4.3 to define  $\delta, N$  such that for all  $L \geq N$  and for all potentials  $q$  satisfying  $\|q\|_{L^2(0, L)} < \delta^{\frac{1}{2}}$ , we have

$$\gamma\left(-\frac{f_2(L, z)}{f_1(L, z)}, i \frac{\sqrt{z^2 - 1}}{z + 1}\right) < \varepsilon, \tag{4.31}$$

where  $f(\cdot, z)$  is a solution of the Dirac equation (4.7) for which

$$\Im \left( -\frac{f_2(0, z)}{f_1(0, z)} \right) > 0.$$

Take  $k_0$  sufficiently large so that for  $k > k_0$  we have  $L_k \geq N$  and such that the bound  $\|q\|_{L^2(a_k, a_k + L_k)} < \delta^{\frac{1}{2}}$  is satisfied by our sparse potential  $q$ .

We now apply (4.31) with  $L = L_k$ , where  $f$  is a suitably chosen solution of (4.7), but with potential modified by an appropriate change of  $x$ -coordinate. There are two separate cases to be considered:

- (1) Firstly, define  $f(x, z) = v(x + a_k, z)$  for  $0 \leq x \leq L_k$ . Then for  $x \in [0, L_k]$ ,  $f(\cdot, z)$  satisfies (4.7) with potential  $q(x + a_k)$ . Moreover, we have

$$\int_0^{L_k} (q(t + a_k))^2 dt = \int_{a_k}^{b_k} (q(t))^2 dt < \delta.$$

Hence (4.31) is satisfied in this case, and we have

$$\gamma \left( -\frac{v_2(b_k, z)}{v_1(b_k, z)}, i \frac{\sqrt{z^2 - 1}}{z + 1} \right) < \varepsilon.$$

From (4.30) we now deduce that the respective value distributions for the Herglotz functions  $-\frac{v_2(b_k, z)}{v_1(b_k, z)}$  and  $i \frac{\sqrt{z^2 - 1}}{z + 1}$  differ by at most  $2\varepsilon \operatorname{mes}(A) \forall k > k_0$ .

- (2) Secondly, let  $F(\cdot, z)$  be a non-trivial solution in  $L^2(0, \infty)^2$  of (4.7), with sparse potential  $q$ . Then the  $m$ -function  $m^{a_k}(z)$  for the Dirac operator  $H = -i\sigma_2 \frac{d}{dx} + \sigma_3 + q$  acting in  $L^2(a_k, \infty)^2$  is then given by

$$m^{a_k}(z) = \frac{F_2(a_k, z)}{F_1(a_k, z)}.$$

We can now define  $f(\cdot, z)$  by

$$f(x, z) = \sigma_2 F(b_k - x, z), \quad 0 \leq x \leq L_k,$$

so that  $f(\cdot, z)$  satisfies the Dirac equation with potential  $q(b_k - x)$ . Since  $\Im \left( \frac{F_2(b_k, z)}{F_1(b_k, z)} \right) > 0$ , we also have

$$\Im \left( -\frac{f_2(0, z)}{f_1(0, z)} \right) > 0.$$

In this case, an application of (4.22) with  $L = L_k$  results in the estimate

$$\gamma(m^{a_k}, i \frac{\sqrt{z^2 - 1}}{z + 1}) < \varepsilon,$$

and it follows as before that the respective value distributions for the Herglotz functions  $m^{a_k}$  and  $i \frac{\sqrt{z^2 - 1}}{z + 1}$  differ by at most  $2\varepsilon \text{mes}(A)$  for all  $k > k_0$ .

The following theorem summarises the situation regarding asymptotic value distribution in the case of  $L^2$ -sparse potentials.

**THEOREM 4.4.** *Let  $v(\cdot, \lambda)$  be the solution of the Dirac equation at real spectral parameter  $\lambda$ , subject to initial conditions  $v(0, \lambda) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ , in the case of an  $L^2$ -sparse potential  $q$ . Let  $\{(a_k, b_k) \subset \mathbb{R}^+\}$  be a sequence of subintervals for which  $\lim_{k \rightarrow \infty} (b_k - a_k) = \infty$  and  $\lim_{k \rightarrow \infty} \int_{a_k}^{b_k} |q(t)|^2 dt = 0$ . Then, for Borel subsets  $A, B$  of  $\mathbb{R}$ ,  $|A| < \infty$ , we have*

$$\lim_{k \rightarrow \infty} \frac{1}{\pi} \int_A \theta(m_+^{a_k}(\lambda), B) d\lambda = \frac{1}{\pi} \int_A \theta\left(i \frac{\sqrt{\lambda^2 - 1}}{\lambda + 1}, B\right) d\lambda$$

$$\text{and } \lim_{k \rightarrow \infty} \text{mes} \left( \left\{ \lambda \in A : \frac{v_2(b_k, \lambda)}{v_1(b_k, \lambda)} \in B \right\} \right) = \frac{1}{\pi} \int_A \theta\left(i \frac{\sqrt{\lambda^2 - 1}}{\lambda + 1}, -B\right) d\lambda$$

## 5 The Absolutely Continuous Spectrum and the Value Distribution

In this section we prove Theorem 4.1, giving a relation between the value distribution and the  $m$ -function for points in subsets of the absolutely continuous spectrum.

For any Herglotz function,  $F$ , we define a translated Herglotz function  $F^\delta = F(z + i\delta)$  ( $\delta > 0$ ) and set

$$\omega^\delta(\lambda, B, F) = \omega(\lambda, B, F^\delta) = \frac{1}{\pi} \theta(F(\lambda + i\delta), B).$$

where  $\omega, \theta$  are defined in (4.3) (page 62) and (4.2) (page 61) respectively. Before proceeding with the main result of this section we require two preliminary results; the first appears in [7] (Theorem 1)

LEMMA 4.14. *Let  $F(z)$  be a arbitrary Herglotz function, and let  $A$  be a set of finite measure. Let  $B$  be an arbitrary Borel subset of  $\mathbb{R}$ . Then we have*

$$\left| \int_A \omega(\lambda, B; F) d\lambda - \int_A \omega^\delta(\lambda, B; F) d\lambda \right| \leq E_A(\delta) = \frac{1}{\pi} \int_A \theta(\lambda + i\delta, A^c) d\lambda, \quad (4.32)$$

where  $E_A(\delta) \rightarrow 0$  for  $\delta \rightarrow 0$ . Further  $E_A(\delta)$  is a non-decreasing function of  $\delta$ . Since  $E_A(\delta)$  is independent of  $B, F$ , the bound is uniform over all sets  $B$  and all Herglotz functions  $F$ .

LEMMA 4.15. *Let  $A$  be a Borel subset of an essential support of the absolutely continuous part  $\rho_{ac}$  of the spectral measure  $\rho$  for the Dirac operator,  $T$ , acting in  $L^2(0, \infty)^2$ . Let  $A_0^{R,r} \subset A$  be defined by*

$$A_0^{R,r} = \{\lambda \in A \mid \Im m_+(\lambda) < r \text{ or } |m_+(\lambda)| > R\},$$

where  $m_+(\lambda) = \lim_{\delta \downarrow 0} m(\lambda + i\delta)$  and  $m(z)$  is the Weyl-Titchmarsh  $m$ -function. Then  $\lim_{r \rightarrow 0, R \rightarrow \infty} \text{mes}(A_0^{R,r}) = 0$ .

PROOF.  $A_0^{R,r}$  can be written as

$$A_0^{R,r} = \{\lambda \in A \mid \Im m_+(\lambda) < r\} \cup \{\lambda \in A \mid |m_+(\lambda)| > R\},$$

and so it is enough to prove that each of these two sets can be made small independently. Starting with the first set, assume the contrary, i.e.

$$\exists \varepsilon > 0 : \forall r > 0 \quad |\{\lambda \in A \mid \Im m_+(\lambda) < r\}| \geq \frac{\varepsilon}{2}.$$

In particular this must be true for the sequence  $r_n = \frac{1}{n}$ . Let  $\alpha_n = \{\lambda \in A \mid \Im m_+(\lambda) < r_n\}$ . Then  $\exists \varepsilon > 0 : \forall n \in \mathbb{N} \quad |\alpha_n| \geq \frac{\varepsilon}{2}$ . Further, for all  $n \geq 1$ ,  $\alpha_{n+1} \subset \alpha_n$ . Thus

$$\exists \varepsilon > 0 : |\bigcap_{n \in \mathbb{N}} \alpha_n| \geq \frac{\varepsilon}{2}.$$

Now

$$\bigcap_{n \in \mathbb{N}} \alpha_n \subset \{\lambda \in A \mid \Im m_+(\lambda) = 0\}.$$

An essential support for  $\rho_{ac}$ , the absolutely continuous part of the spectral measure, is all  $\lambda \in \mathbb{R}$  at which the boundary value  $m_+(\lambda)$  of  $m(z)$  exists with strictly positive imaginary part (see [24] Proposition 1). Hence we can assume that  $\Im m_+(\lambda) > 0$  for all  $\lambda \in A$ , providing our contradiction.

The fact that we can choose  $R > 0$  such that

$$|\{\lambda \in A \mid |m_+(\lambda)| > R\}| < \frac{\varepsilon}{2},$$

follows by a similar process to the above combined with Lemma 1 of [24].  $\square$

We now have the machinery to tackle the proof of Theorem 4.1

Let  $m(z)$  denote the Weyl-Titchmarsh  $m$  function for  $T$ . As we know, an essential support for  $\rho_{ac}$  is given as the set of all  $\lambda \in \mathbb{R}$  at which the boundary value  $m_+(\lambda)$  of  $m(z)$  exists and has strictly positive imaginary part (see [24] Proposition 1). Hence we can assume that  $\Im m_+(\lambda) > 0$  for all  $\lambda \in A$ .

Let  $\varepsilon > 0$ . We wish to form a finite partition of the set  $A$

$$A = A_0^{R,r} \cup A_1 \cup A_2 \cup \dots \cup A_n,$$

where  $A_i \cap A_j = \emptyset$ ,  $i \neq j$ ,  $A_0^{R,r} \cap A_i = \emptyset$  for all  $i$ ,  $\text{mes}(A_0^{R,r}) \leq \varepsilon \text{mes}(A)$  and  $\text{mes}(A_j) \leq \infty$ ,  $j \geq 1$ . Further, we would like to associate to each  $j \geq 1$  a constant  $m^{(j)}$  with positive imaginary part such that  $\gamma(m_+(\lambda), m^{(j)}) \leq \varepsilon$  ( $\lambda \in A_j$ ). The way in which we do this is as follows: Points  $\lambda \in A$  at which  $|\lambda|$  or  $|m_+(\lambda)|$  are large, or at which  $\Im(m_+(\lambda))$  are small put into  $A_0^{R,r}$ . More precisely, by Lemma 4.15 we know that  $\exists R, r > 0$  such that we can make  $\text{mes}(A_0^{R,r}) \leq \varepsilon \text{mes}(A)$ . The range  $m_+(\lambda)$ , as  $\lambda$  runs over  $A \setminus A_0^{R,r}$ , is contained in  $C \subset \mathbb{C}^+$ .  $C = \{\lambda \in A \mid \Im m_+(\lambda) \geq r \text{ or } |m_+(\lambda)| \leq R\}$  and thus, as  $|\lambda|$  is bounded because  $A$  is bounded,  $C$  is bounded. The Heine-Borel Theorem then tells us that  $C$  is compact and therefore it has a finite cover. Indeed,  $\forall z \in C$ ,  $\exists r_z > 0 : \forall y \in B_{r_z}(z)$ ,  $\gamma(y, z) \leq \varepsilon$ . Then  $\{B_{r_z}(z) \mid z \in C\}$  is an open cover of  $C$ . The fact that  $C$  is compact then implies that there is a finite subcover  $\{B_{r_{z_j}}(z_j) \mid z_j \in C, j \in \{1, \dots, n\}\}$ . Thus we can find a partition, with  $n < \infty$ ,

$$C = \bigcup_{i=1}^n C_i,$$

with  $C_1 = B_{r_{z_1}}(z_1) \cap C$ ,  $C_j = (B_{r_{z_j}}(z_j) \cap C) \setminus \bigcup_{i=1}^{j-1} C_i$  (i.e.  $C_i \cap C_j = \emptyset$  ( $i \neq j$ )). Further we have  $\forall j = 1, 2, \dots, n$

$$z_1, z_2 \in C_j \Rightarrow \gamma(z_1, z_2) \leq \varepsilon.$$

Finally, take  $A_j = (A \setminus A_0) \cap m_+^{-1}(C_j)$  and  $m^{(j)} = m_+(\lambda_j)$  for any fixed  $\lambda_j \in A_j$ . This defines a partition satisfying the above properties which we desired.

By Lemma 4.14 we know that there exists a  $\delta_0 > 0$  such that for an arbitrary Herglotz function  $F$ , Borel set  $B \subset \mathbb{R}$ ,  $\delta \in (0, \delta_0)$  and  $j = 1, 2, \dots, n$  we have the bound

$$\left| \int_{A_j} \omega^\delta(\lambda; B; F) d\lambda - \int_{A_j} \omega(\lambda; B; F) d\lambda \right| \leq \varepsilon |A_j|, \quad (4.33)$$

where  $\omega^\delta(\lambda; B; F) = \frac{1}{\pi} \theta(F(\lambda + i\delta), B)$ . That  $\delta_0$  can be chosen independently of  $j$  follows from the fact that there are only finitely many sets  $A_j$ , so we just choose  $\delta_0 = \min \delta_0^j$ .

We now complete the proof by showing that, for  $j \geq 1$

- i)  $\frac{1}{\pi} \int_{A_j} \theta(m_+^N(\lambda), B) d\lambda$  is close to the integral  $\frac{1}{\pi} \int_{A_j} \theta\left(\frac{u_2(N, \lambda) + m^{(j)} v_2(N, \lambda)}{u_1(N, \lambda) + m^{(j)} v_1(N, \lambda)}, B\right) d\lambda$ ;
- ii)  $\text{mes}\left(\left\{\lambda \in A_j; \frac{v_2}{v_1}(N, \lambda) \in B\right\}\right)$  is close to  $\frac{1}{\pi} \int_{A_j} \theta\left(\frac{u_2(N, \lambda) + m^{(j)} v_2(N, \lambda)}{u_1(N, \lambda) + m^{(j)} v_1(N, \lambda)}, B\right) d\lambda$  for all  $N$  sufficiently large.

We begin with the proof of (i):

It can be shown that

$$m_+^N(\lambda) = \frac{u_2(N, \lambda) + m_+(\lambda) v_2(N, \lambda)}{u_1(N, \lambda) + m_+(\lambda) v_1(N, \lambda)},$$

([6] Equation (5)). Hence, for fixed  $N, \lambda$ , the mapping from  $m_+(\lambda)$  to  $m_+^N(\lambda)$  is a Möbius transformation with real coefficients and discriminant one. By Lemma 2 of [6], the Gamma separation is invariant under Möbius transformations, and so

$$\gamma\left(m_+^N(\lambda), \frac{u_2(N, \lambda) + m^{(j)} v_2(N, \lambda)}{u_1(N, \lambda) + m^{(j)} v_1(N, \lambda)}\right) = \gamma(m_+(\lambda), m^{(j)}) \leq \varepsilon, (\lambda \in A_j, j \geq 1).$$

Using Equation (4.5) we can deduce that

$$\sup_S \frac{\left| \theta(m_+^N(\lambda), S) - \theta\left(\frac{u_2(N, \lambda) + m^{(j)} v_2(N, \lambda)}{u_1(N, \lambda) + m^{(j)} v_1(N, \lambda)}, S\right) \right|}{\sqrt{\theta(m_+^N(\lambda), S)} \sqrt{\theta\left(\frac{u_2(N, \lambda) + m^{(j)} v_2(N, \lambda)}{u_1(N, \lambda) + m^{(j)} v_1(N, \lambda)}, S\right)}} \leq \varepsilon,$$

which implies that

$$\frac{1}{\pi} \left| \theta(m_+^N(\lambda), B) - \theta\left(\frac{u_2(N, \lambda) + m^{(j)} v_2(N, \lambda)}{u_1(N, \lambda) + m^{(j)} v_1(N, \lambda)}, B\right) \right| \leq \varepsilon.$$

Integrating with respect to  $\lambda$  over  $A_j$  leads us to the bound

$$\frac{1}{\pi} \left| \int_{A_j} \theta(m_+^N(\lambda), B) d\lambda - \int_{A_j} \theta\left(\frac{u_2(N, \lambda) + m^{(j)} v_2(N, \lambda)}{u_1(N, \lambda) + m^{(j)} v_1(N, \lambda)}, B\right) d\lambda \right| \leq \varepsilon \text{mes}(A_j) (j \geq 1).$$

We now proceed with the proof of (ii):

For  $j \geq 1$  define  $A_j^\delta = A_j + i\delta \subset \mathbb{C}^+$ . As  $A_j$  is bounded, we know that  $A_j^\delta$  is a compact subset of  $\mathbb{C}^+$ . Hence Lemma 4.10 and Lemma 4.13 implies that we can find an  $N_0$  such that  $\forall j \geq 1, N > N_0$  and  $z \in A_j^\delta$

$$\gamma \left( -\frac{v_2(N, z)}{v_1(N, z)}, -\frac{u_2(N, z) + \overline{m}^{(j)}v_2(N, z)}{u_2(N, z) + \overline{m}^{(j)}v_2(N, z)} \right) \leq \varepsilon.$$

In a similar way to part (i) we can now deduce that

$$\frac{1}{\pi} \left| \int_{A_j} \theta \left( -\frac{v_2(N, \lambda + i\delta)}{v_1(N, \lambda + i\delta)}, -B \right) d\lambda - \int_{A_j} \theta \left( -\frac{u_2(N, \lambda + i\delta) + \overline{m}^{(j)}v_2(N, \lambda + i\delta)}{u_2(N, \lambda + i\delta) + \overline{m}^{(j)}v_2(N, \lambda + i\delta)}, -B \right) d\lambda \right| \leq \varepsilon \text{mes}(A_j).$$

This bound is valid for all  $j \geq 1$  and  $N > N_0$ . Noting that both  $-\frac{v_2}{v_1}$  and  $-\frac{u_2(N, \lambda + i\delta) + \overline{m}^{(j)}v_2(N, \lambda + i\delta)}{u_2(N, \lambda + i\delta) + \overline{m}^{(j)}v_2(N, \lambda + i\delta)}$

are Herglotz functions, we see that we can use inequality (4.33) to compare these two

integrals in the limit  $\delta \rightarrow 0^+$ . We know that

$$\begin{aligned} \frac{1}{\pi} \int_{A_j} \lim_{\delta \rightarrow 0^+} \theta \left( -\frac{v_2(N, \lambda + i\delta)}{v_1(N, \lambda + i\delta)}, -B \right) d\lambda &= \text{mes} \left( \left\{ \lambda \in A_j; -\frac{v_2(N, \lambda)}{v_1(N, \lambda)} \in -B \right\} \right) \\ &= \text{mes} \left( \left\{ \lambda \in A_j; \frac{v_2(N, \lambda)}{v_1(N, \lambda)} \in B \right\} \right), \end{aligned}$$

and this, together with (4.33), allows us to arrive at the bound (for a suitably small  $\delta_0 > \delta$ )

$$\begin{aligned} & \left| \text{mes} \left( \left\{ \lambda \in A_j; \frac{v_2(N, \lambda)}{v_1(N, \lambda)} \in B \right\} \right) - \frac{1}{\pi} \int_{A_j} \theta \left( -\frac{u_2(N, \lambda) + \overline{m}^{(j)}v_2(N, \lambda)}{u_2(N, \lambda) + \overline{m}^{(j)}v_2(N, \lambda)}, -B \right) d\lambda \right| \\ & \leq \left| \text{mes} \left( \left\{ \lambda \in A_j; \frac{v_2(N, \lambda)}{v_1(N, \lambda)} \in B \right\} \right) - \frac{1}{\pi} \int_{A_j} \theta \left( -\frac{v_2(N, \lambda + i\delta)}{v_1(N, \lambda + i\delta)}, -B \right) d\lambda \right| \\ & + \left| \frac{1}{\pi} \int_{A_j} \theta \left( -\frac{v_2(N, \lambda + i\delta)}{v_1(N, \lambda + i\delta)}, -B \right) d\lambda - \frac{1}{\pi} \int_{A_j} \theta \left( -\frac{u_2(N, \lambda + i\delta) + \overline{m}^{(j)}v_2(N, \lambda + i\delta)}{u_2(N, \lambda + i\delta) + \overline{m}^{(j)}v_2(N, \lambda + i\delta)}, -B \right) d\lambda \right| \\ & + \left| \frac{1}{\pi} \int_{A_j} \theta \left( -\frac{u_2(N, \lambda + i\delta) + \overline{m}^{(j)}v_2(N, \lambda + i\delta)}{u_2(N, \lambda + i\delta) + \overline{m}^{(j)}v_2(N, \lambda + i\delta)}, -B \right) d\lambda \right. \\ & \quad \left. - \frac{1}{\pi} \int_{A_j} \theta \left( -\frac{u_2(N, \lambda) + \overline{m}^{(j)}v_2(N, \lambda)}{u_2(N, \lambda) + \overline{m}^{(j)}v_2(N, \lambda)}, -B \right) d\lambda \right| \\ & \leq 3\varepsilon \text{mes}(A_j). \end{aligned}$$

Using the identity  $\theta(-\alpha, -B) = \theta(\bar{\alpha}, B)$  to obtain

$$\left| \text{mes} \left( \left\{ \lambda \in A_j; \frac{v_2}{v_1}(N, \lambda) \in B \right\} \right) - \frac{1}{\pi} \int_{A_j} \theta \left( \frac{u_2(N, \lambda) + m^{(j)}v_2(N, \lambda)}{u_2(N, \lambda) + m^{(j)}v_2(N, \lambda)}, B \right) d\lambda \right| \leq 3\varepsilon \text{mes}(A_j),$$

(as our solutions  $u, v$  are real) which holds for all  $j \geq 1$  and  $N \geq N_0$  and completes the proof of (ii). Combining the results of (i) and (ii) now yields

$$\left| \text{mes} \left( \left\{ \lambda \in A_j; \frac{v_2}{v_1}(N, \lambda) \in B \right\} \right) - \frac{1}{\pi} \int_{A_j} \theta(m_+^N(\lambda), B) d\lambda \right| \leq 4\varepsilon \text{mes}(A_j), (j \geq 1, N \geq N_0).$$

We chose  $\text{mes}(A_0^{R,r}) \leq \varepsilon \text{mes}(A)$ , and so we have (for all  $N \geq N_0$ )

$$\begin{aligned} & \left| \text{mes} \left( \left\{ \lambda \in A; \frac{v_2}{v_1}(N, \lambda) \in B \right\} \right) - \frac{1}{\pi} \int_A \theta(m_+^N(\lambda), B) d\lambda \right| \\ & \leq \sum_{j=0}^n \left| \text{mes} \left( \left\{ \lambda \in A_j; \frac{v_2}{v_1}(N, \lambda) \in B \right\} \right) - \frac{1}{\pi} \int_{A_j} \theta(m_+^N(\lambda), B) d\lambda \right| \\ & \leq 2 \text{mes}(A_0^{R,r}) + 4\varepsilon \sum_{j=1}^n \text{mes}(A_j) \leq 6\varepsilon \text{mes}(A). \end{aligned}$$

This concludes the proof of Theorem 4.1

## 6 Spectral Analysis

Having extended the theory pertaining to value distributions to incorporate the Dirac operator, we can now prove Theorem 4.2.

Suppose the contrary, i.e. that we can find a subset  $A$  of  $[-1, 1]$  for which  $\rho_{ac}(A) > 0$ , where  $\rho_{ac}$  is the absolutely continuous part of the spectral measure, . Then  $\text{mes}(A) > 0$ , and we may also suppose that  $A$  is a subset of the essential support of  $\rho_{ac}$ .

Now, as in Theorem 4.4, we define intervals  $(a_k, b_k)$  and set  $N_k = \frac{(a_k + b_k)}{2}$ . Then  $N_k$  can be seen as either the left hand endpoint of an interval  $(N_k, b_k)$  or as the right hand endpoint of an interval  $(a_k, N_k)$ . Theorem 4.4 then implies that

$$\lim_{k \rightarrow \infty} \frac{1}{\pi} \int_A \theta(m_+^{N_k}(\lambda), B) d\lambda = \frac{1}{\pi} \int_A \theta \left( i \frac{\sqrt{\lambda^2 - 1}}{\lambda + 1}, B \right) d\lambda, \quad (4.34)$$

whereas

$$\lim_{k \rightarrow \infty} \text{mes} \left( \left\{ \lambda \in A : \frac{v_2(N_k, \lambda)}{v_1(N_k, \lambda)} \in B \right\} \right) = \frac{1}{\pi} \int_A \theta \left( i \frac{\sqrt{\lambda^2 - 1}}{\lambda + 1}, -B \right) d\lambda. \quad (4.35)$$



Since  $A$  is a subset of the essential support of  $\rho_{ac}$ , we also have by Theorem 4.1 that

$$\lim_{k \rightarrow \infty} \left[ \text{mes} \left( \left\{ \lambda \in A : \frac{v_2(N_k, \lambda)}{v_1(N_k, \lambda)} \in B \right\} \right) - \frac{1}{\pi} \int_A \theta \left( i \frac{\sqrt{\lambda^2 - 1}}{\lambda + 1}, B \right) d\lambda \right] = 0. \quad (4.36)$$

Equations (4.34)-(4.36) now imply that

$$\frac{1}{\pi} \int_A \theta \left( i \frac{\sqrt{\lambda^2 - 1}}{\lambda + 1}, B \right) d\lambda = \frac{1}{\pi} \int_A \theta \left( i \frac{\sqrt{\lambda^2 - 1}}{\lambda + 1}, -B \right) d\lambda \quad (4.37)$$

However,  $i \frac{\sqrt{\lambda^2 - 1}}{\lambda + 1} \in \mathbb{R}^-$  for  $\lambda \in A$ , and taking  $B = \mathbb{R}^-$  we see that the left hand side of (4.37) is strictly positive, whilst the right hand side is zero. Hence we have a contradiction and the theorem is proved.

# Bound States of the Dirac Operator for $O(\frac{1}{x})$ Potentials

Consider the one-dimensional Schrödinger operator

$$\varkappa = -\frac{d^2}{dx^2} + q(x). \quad (5.1)$$

It is well known that any self-adjoint realisation of  $\varkappa$  on  $[0, \infty)$  has essential spectrum  $[0, \infty)$  if  $q(x) \rightarrow 0$  ( $x \rightarrow \infty$ ). Many results have been proven about the absolutely continuous spectrum of this operator (see Chapter 3 Introduction for references); however, it is also an interesting area of study to consider the bound states of this operator, in particular finding results either bounding the maximum eigenvalue, the number of eigenvalues or finding a bound for a sum involving the eigenvalues.

Von Neumann and Wigner [66] considered Schrödinger operators of the form (5.1) as a means of constructing an example of a Schrödinger operator acting in  $L^2(\mathbb{R}^3)$  with specially symmetric potential which decays to zero at infinity for which there is a positive eigenvalue, i.e. an eigenvalue embedded in the continuous spectrum; the significance of this example was that it contradicted the physical intuition at the time. It was believed that if the potential decayed to zero at infinity, no positive bound states could occur. Eastham-Kalf subsequently showed that if  $|q(x)| = o(x^{-1})$  ( $x \rightarrow \infty$ ),  $\varkappa$  has no positive eigenvalues; further, if  $\limsup_{x \rightarrow \infty} x|q(x)| = C < \infty$  any eigenvalue  $\lambda$  must obey  $\lambda \leq C^2$

(c.f. [14, Section 3.2], the original result is due to Wallach). On the other hand, Eastham-McLeod [15], with further developments by Thurlow [62], showed how to construct potentials  $q(x)$  of the type  $q(x) = \frac{f(x)}{x}$ , with  $f(x) \rightarrow \infty$  as  $x \rightarrow \infty$  such that a prescribed countable set of isolated points represents embedded positive eigenvalues of  $\mathcal{H}$ . Later, Naboko ([38]) described a technique which, given an arbitrary countable (possibly dense) set  $S$  of rationally independent numbers in  $\mathbb{R}^+$ , allows the construction of a potential  $q(x)$  satisfying  $|q(x)| \leq \frac{C(x)}{x}$  with  $C(x) \rightarrow \infty$  ( $x \rightarrow \infty$ ) monotonously at an arbitrarily slow given rate such that the corresponding Schrödinger operator has  $S$  among its eigenvalues. More recently Simon ([58]) described a different construction which does not require the rational independence assumption. Kiselev, Last and Simon proved the borderline case in [32, Theorem 4.1], showing that if  $q(x)$  obeys  $C = \limsup_{x \rightarrow \infty} x|q(x)| < \infty$  then  $\mathcal{H}u = \lambda u$  has at most countably many positive eigenvalues,  $\lambda \in \mathbb{R}$ . They also showed that  $\sum_n \lambda_n \leq \frac{C^2}{2}$ .

We are again going to consider the relativistic counterpart of  $\mathcal{H}$ , the Dirac operator

$$\tau = -i\sigma_2 \frac{d}{dx} + \sigma_3 + q(x) \quad (x \in (0, \infty)) \quad (5.2)$$

with the boundary condition

$$u_1(0) \cos \alpha + u_2(0) \sin \alpha = 0,$$

for fixed  $\alpha \in \mathbb{R}$ .

It is known that the spectrum of this operator is never purely discrete [54]. However, as shown by K. M. Schmidt considered in [54], there is a suitable potential  $q$  satisfying

$$q(x) < \frac{C(x)}{x}, \quad (x \in (0, \infty)),$$

with  $\lim_{x \rightarrow \infty} C(x) = \infty$  for which the operator has a prescribed set of eigenvalues dense in the whole or part of its essential spectrum.

Our aim is to prove what could be thought of as a boundary case to [54] using a similar method to that of Kiselev, Last and Simon [32]. We consider the Dirac operator (5.2) where  $q$  is real valued, locally integrable and  $C := \limsup_{x \rightarrow \infty} x|q(x)| < \infty$  and aim to prove

THEOREM 5.1. *Assume that  $C := \limsup_{x \rightarrow \infty} x|q(x)| < \infty$ . Then  $\sum_n (\lambda_n^2 - 1) \leq \frac{C^2}{2}$  ( $|\lambda_n| > 1$ ).*

REMARK 5.1. We note that it is the intention that the summation is taken only over eigenvalues in the essential spectrum.

As in [32], there exists an  $x_0 \in \mathbb{R}^+$  such that, by increasing  $C$ ,

$$|q(x)| \leq \frac{C}{1+|x|} \quad (x \in [x_0, \infty)). \quad (5.3)$$

Indeed,

$$C = \limsup_{x \rightarrow \infty} x|q(x)| = \lim_{x \rightarrow \infty} \left( \sup_{y \geq x} y|q(y)| \right)$$

implies that for all  $\varepsilon > 0$  there exists an  $x_0$  such that  $|\sup_{y \geq x_0} y|q(y)| - C| \leq \varepsilon$ . Thus  $|q(x)| \leq \frac{C+\varepsilon+\sup_x |q(x)|}{1+x} \leq \frac{C'}{1+x}$  ( $x \in [x_0, \infty)$ ).

Before proceeding with the proof of the result we need to state three standard estimates; we must also define a modified Prüfer transformation which is integral to the proof.

LEMMA 5.1 ([32] Lemma 4.3). *Let  $f, g \in C^1[1, \infty)$  such that*

$$|g'f| + |f'| \in L^1.$$

*Then*

$$\int_0^N f(x) e^{i(kx+g(x))} dx$$

*is bounded as  $N \rightarrow \infty$  for any  $k \neq 0$ .*

LEMMA 5.2 ([32] Lemma 4.4). *Let  $\{e_i\}_{i=1}^N$  be a set of unit vectors in a Hilbert space  $\mathcal{H}$  so that*

$$\alpha = N \sum_{i \neq j} (e_i, e_j) < 1,$$

*where  $(\cdot, \cdot)$  is the inner product on  $\mathcal{H}$ . Then*

$$\sum_{i=1}^N |(g, e_i)|^2 \leq (1 + \alpha) \|g\|_{\mathcal{H}}^2,$$

*for any  $g \in \mathcal{H}$ .*

LEMMA 5.3. *Let  $f \in L^1(\cdot, \infty)$ . Then  $\liminf_{x \rightarrow \infty} x|f(x)| = 0$ .*

We define the standard Prüfer transformation of a solution of  $\tau u = \lambda u$  by

$$u(x, \lambda) = \tilde{R}(x, \lambda) \begin{pmatrix} \sin(\tilde{\vartheta}(x, \lambda)) \\ \cos(\tilde{\vartheta}(x, \lambda)) \end{pmatrix}, \quad (5.4)$$

where  $R$  is called the Prüfer radius and  $\vartheta$  the Prüfer angle (c.f. [69, Section 5.16]). It can easily be shown that

$$\tilde{\vartheta}'(x, \lambda) = \lambda - q(x) + \cos(2\tilde{\vartheta}(x, \lambda)), \quad (5.5)$$

and

$$(\log \tilde{R})'(x, \lambda) = \sin(2\tilde{\vartheta}(x, \lambda)), \quad (5.6)$$

where  $'$  denotes differentiation with respect to  $x$ . However, this usual Prüfer transformation falls short of our requirements for the proof. Indeed, we wish the equation for the Prüfer radius to be homogeneous in  $q$ . We therefore consider a slightly modified transformation with  $|\lambda| > 1$

$$u(x, \lambda) = R(x, \lambda) \begin{pmatrix} \cos(\vartheta(x, \lambda) + \frac{\pi}{4}) \\ -\frac{\kappa}{1+\lambda} \sin(\vartheta(x, \lambda) + \frac{\pi}{4}) \end{pmatrix}, \quad (5.7)$$

where  $R$  is called the modified Prüfer radius,  $\vartheta$  the modified Prüfer angle and  $\kappa = \sqrt{\lambda^2 - 1}$ .

It can be shown that (see [54] Section 2)

$$\vartheta'(x, \lambda) = \kappa + q(x) \frac{\lambda + \sin(2\vartheta(x, \lambda))}{\kappa}, \quad (5.8)$$

and

$$(\log R)'(x, \lambda) = -\frac{q(x)}{\kappa} \cos(2\vartheta(x, \lambda)), \quad (5.9)$$

With these tools we can now proceed with the proof of Theorem 5.1; it suffices to show that for each  $N < \infty$

$$\sum_{n=1}^N (\lambda_n^2 - 1) \leq \frac{C^2}{2}.$$

Assume that  $\tau u = \lambda u$  has  $N$  distinct eigenvalues  $\lambda_1, \dots, \lambda_N$ . Define  $R_n(x, \lambda)$  to be the modified Prüfer radius corresponding to the  $L^2$  solution  $u(x, \lambda_n)$ ; we normalise  $u$  so that  $R_n(0, \lambda) = 1$ . Then, by (5.7),  $R_n(\cdot, \lambda_n) \in L^2(0, \infty)$ . Thus

$$\sum_{i=1}^N |R_n(\cdot, \lambda_n)|^2 \in L^1(0, \infty),$$

which implies, by Lemma 5.3, that  $\liminf_{x \rightarrow \infty} x \sum_{n=1}^N |R_n(x, \lambda_n)|^2 = 0$ . Thus there exists a positive sequence  $\{B_j\}_{j \in \mathbb{N}}$  such that  $B_j \rightarrow \infty$  ( $j \rightarrow \infty$ ) and

$$R_n(B_j, \lambda_n) \leq B_j^{-\frac{1}{2}}, \quad (n \in \{1, \dots, N\}, j \in \mathbb{N}).$$

Thus

$$\int_0^{B_j} \frac{d}{dy} \log R_n(y, \lambda_n) dy = \log R_n(B_j) \leq -\frac{1}{2} \log B_j.$$

By (5.9), this means that

$$\int_0^{B_j} \left( -\frac{q(y)}{\kappa_n} \cos(2\vartheta_n(y, \lambda_n)) \right) dy \leq -\frac{1}{2} \log B_j. \quad (5.10)$$

Consider the Hilbert spaces  $\mathcal{H}_j$  defined by

$$\mathcal{H}_j = L^2\{(0, B_j), (1+x)dx\}.$$

Then in  $\mathcal{H}_j$  we have using (5.3)

$$\|q\|_{\mathcal{H}_j}^2 = \int_0^{B_j} q^2(x)(1+x)dx \leq \int_0^{B_j} \frac{C^2}{1+x} dx = C^2 \log(B_j + 1) = C^2 \log B_j + O(1). \quad (5.11)$$

Consider

$$e_n^{(j)}(y) = -\frac{1}{\sqrt{N_n^{(j)}}} \frac{\cos(2\vartheta_n(y, \lambda_n))}{1+y} \chi_{[0, B_j]}(y), \quad (n \in \{1, \dots, N\})$$

where  $\chi$  is the characteristic function and  $N_n^{(j)} = \int_0^{B_j} \frac{\cos^2(2\vartheta_n(y, \lambda_n))}{1+|y|} dy$ . Then

$$\|e_n^{(j)}\|_{\mathcal{H}_j}^2 = \int_0^{B_j} \frac{1}{N_n^{(j)}} \frac{\cos^2(2\vartheta_n(y, \lambda_n))}{(1+|y|)^2} (1+y) dy = 1.$$

Thus  $\{e_n^{(j)}\}_{n=1}^N$  are a set of unit vectors in  $\mathcal{H}_j$ .

Notice that  $\vartheta_n(x) - \kappa_n x$  and  $2(\vartheta_n \pm \vartheta_m) - 2(\kappa_n \pm \kappa_m)x$  have derivatives that are  $O(x^{-1})$ , ( $x \rightarrow \infty$ ). Indeed

$$[\vartheta_n(x) - \kappa_n x]' = q(x) \frac{\lambda_n + \sin(2\vartheta_n(x, \lambda_n))}{\kappa_n}, \quad (x \in (0, \infty)).$$

Thus

$$\begin{aligned}
& \int_0^{B_j} \frac{\cos(2\vartheta_n(y)) \cos(2\vartheta_m(y)) - \frac{1}{2}\delta_{nm}}{1+y} dy \\
&= \frac{1}{2} \int_0^{B_j} \frac{\cos(2(\vartheta_n(y) - \vartheta_m(y))) + \cos(2(\vartheta_n(y) + \vartheta_m(y))) - \delta_{nm}}{1+y} dy \\
&= \frac{1}{2} \Re \int_0^{B_j} \frac{e^{i(2(\kappa_n - \kappa_m)y + 2(\vartheta_n - \vartheta_m) - 2(\kappa_n - \kappa_m)y)} + e^{i(2(\kappa_n + \kappa_m)y + 2(\vartheta_n + \vartheta_m) - 2(\kappa_n + \kappa_m)y)} - \delta_{nm}}{1+y} dy \\
&= O(1) \quad (j \rightarrow \infty)
\end{aligned}$$

by Lemma 5.1. Hence, as

$$\begin{aligned}
\sqrt{N_n^{(j)} N_m^{(j)}} (e_n^{(j)}, e_m^{(j)})_{\mathcal{H}_j} &= \int_0^{B_j} \frac{\cos(2\vartheta_n(y)) \cos(2\vartheta_m(y))}{1+y} dy \\
&= \frac{1}{2} \delta_{nm} \int_0^{B_j} \frac{dy}{1+y} + O(1) \\
&= \frac{1}{2} \delta_{nm} \log(B_j + 1) + O(1), \quad (j \rightarrow \infty)
\end{aligned}$$

we can conclude that

$$N_i^{(j)} = \frac{1}{2} \log B_j + O(1), \quad (j \rightarrow \infty) \quad (5.12)$$

$$(e_i^{(j)}, e_k^{(j)}) = O((\log B_j)^{-1}), \quad i \neq k \quad (j \rightarrow \infty). \quad (5.13)$$

Thus, by (5.10)

$$\begin{aligned}
(q, e_n^{(j)})_{\mathcal{H}_j} &= \int_0^{B_j} -\frac{q(y) \cos(2\vartheta_n(y, \lambda_n))}{\sqrt{N_n^{(j)}}} dy \leq -\frac{\kappa_n \log B_j}{\sqrt{N_n^{(j)}}} = -\frac{\kappa_n \log B_j}{\sqrt{\frac{1}{2} \log B_j + O(1)}} \\
&= -\frac{\kappa_n \log B_j}{\sqrt{\frac{1}{2} \log B_j}} \left(1 + \frac{2O(1)}{\log B_j}\right)^{-\frac{1}{2}} = -\frac{\kappa_n \log B_j}{\sqrt{\frac{1}{2} \log B_j}} \left(1 + O\left(\frac{1}{\log B_j}\right)\right)^{-\frac{1}{2}} \\
&= -\frac{\kappa_n \log B_j}{\sqrt{\frac{1}{2} \log B_j}} \left(1 - \frac{1}{2} O\left(\frac{1}{\log B_j}\right)\right) = -\frac{\sqrt{2} \kappa_n \log B_j}{\sqrt{\log B_j}} + O\left(\left(\frac{1}{\log B_j}\right)^{\frac{1}{2}}\right) \\
&= -\sqrt{2} \kappa_n (\log B_j)^{\frac{1}{2}} + O(1). \quad (5.14)
\end{aligned}$$

As  $N$  is fixed and  $B_j \rightarrow \infty$  as  $j \rightarrow \infty$  we may choose  $j$  large enough, by (5.13), so that

Lemma 5.2 applies. Thus

$$\sum_{n=1}^N |(q, e_n^{(j)})_{\mathcal{H}_j}|^2 \leq (1 + O((\log B_j)^{-1})) \|q\|_{\mathcal{H}_j}^2.$$

Thus, by (5.11) and (5.14)

$$\begin{aligned}
\sum_{n=1}^N |\sqrt{2}\kappa_n(\log B_j)^{\frac{1}{2}} + O(1)|^2 &\leq \sum_{n=1}^N |-(q, e_n^{(j)})_{\mathcal{H}_j}|^2 \\
&\leq (1 + O((\log B_j)^{-1})) \|q\|_{\mathcal{H}_j}^2 \\
&\leq (1 + O((\log B_j)^{-1})) (C^2 \log B_j + O(1)).
\end{aligned}$$

and hence

$$2 \left( \sum_{n=1}^N \kappa_n^2 \right) \log B_j \leq C^2 \log B_j + O(1),$$

so

$$\sum_{n=1}^N \kappa_n^2 \leq \frac{C^2}{2}.$$

□





# Chapter 6

## Further Research

The results of this thesis have settled the question regarding the absolutely continuous spectrum for a one-dimensional Dirac operator with a square integrable potential and extended this to the three-dimensional case, albeit with an additional boundedness assumption. Further, the techniques of value distribution theory have been modified and applied to the one-dimensional Dirac operator to allow the study of square integrable perturbation of sparse electric potentials. A bound on the sum of the relativistic equivalent of the momentum variable has also been described. However, some further prospective areas of research have been opened.

An interesting question which arises from Chapter 2 is whether Theorem 2.1 is optimal in terms of  $L^p$  decay. Is it the case that for all  $L^p$  spaces with  $p > 2$  there exist examples of electric potentials for which there is no absolutely continuous spectrum? This question has already been answered in the affirmative by both Pearson [39] and Kiselev, Last and Simon [32] for the Schrödinger case as detailed in Chapter 2.

Another interesting question arises from the remark at the end of Chapter 2 Section 5; although it is possible to prove the result of Chapter 2 using this rather rough estimate, it raises a question: is it possible to relate the transmission coefficient and potential more effectively, and if so, could this lead to either a simpler proof or a different result?

In [30], Simon and Killip give necessary and sufficient conditions for a positive measure to be the spectral measure of a half-line Schrödinger operator with square integrable potential. By doing this they were able to improve the result of Deift and Killip discussed in Chapter 2 and bring it into the context of sum rules. It would be interesting to produce an analogous result for the Dirac operator case.

In Chapter 3 the need to introduce an  $L^\infty$  condition on the potential to proceed with the chosen method was unfortunate. However, it is my belief that this condition is not a requirement and thus it should be possible to have a complete analysis of the Dirac operator with square integrable potential in up to three dimensions.

Finally in Chapter 4 the ratio of solution components,  $\frac{v_2(x,\lambda)}{v_1(x,\lambda)}$ , which occurs throughout the chapter is a tantalising motivation towards a potential simplification of the results of this chapter, or at least the possibility of a different approach. Indeed, if we were to consider only real values for the spectral parameter  $\lambda$ , this ratio is the tangent of the Prüfer angle and so many of the results reduce to deducing the rate of increase of the Prüfer angle as well as the number of passes through the interval  $[0, \pi]$ . If some progress could be made in this direction it would greatly simplify the theory not only by simplifying the results themselves but by reducing the need to move into the complex plane.

## The Error Estimate $E_A(\delta)$

The aim of this note is to give some further details about the error estimate  $E_A(\delta)$  which appears in Chapter 4 as part of a bound for the difference between value distributions for two different Herglotz functions. These details are due to Breimesser and Pearson and appear in [7, 6].

Let  $G(z)$  be a Herglotz function with boundary value  $G_+(\lambda)$  defined as in Chapter 4. Further, let  $\mathcal{V}(A, B; G)$  be the associated value distribution function for  $G$  as in Chapter 4.

For practical applications it is difficult to estimate  $\mathcal{V}(A, B; G)$ ; indeed, if we were to use the formula

$$\mathcal{V}(A, B; G) = \int_A \omega(\lambda, B; G) d\lambda$$

where

$$\omega(\lambda, B, G) = \lim_{\delta \rightarrow 0^+} \omega(\lambda + i\delta, B; G), \quad (\lambda \in \mathbb{R}), \quad (\text{A.1})$$

(see Chapter 4 for details) there would be difficulties because the determination of  $\omega(\lambda, B; G)$  through (A.1) requires knowledge of the behaviour of the Herglotz function close to the real axis, where precise bounds are not easy to obtain.

In order to broach this problem, we consider a translation away from the real axis. Indeed, define the translated Herglotz function  $G^\delta$  by  $G^\delta(z) := G(z + i\delta)$ , with  $\delta > 0$ . Further,

set

$$\omega^\delta(\lambda, B; G) := \omega(\lambda, B; G^\delta) = \frac{1}{\pi} \theta(G(\lambda + i\delta), B),$$

as per Equation (4.3).

An application of the Lebesgue dominated convergence theorem shows that (for  $\text{mes}(A) < \infty$ )

$$\mathcal{V}(A, B; G) = \lim_{\delta \rightarrow 0^+} \int_A \omega^\delta(\lambda, B; G) d\lambda = \int_A \omega(\lambda, B; G) d\lambda. \quad (\text{A.2})$$

The following theorem has the surprising consequence that, for any fixed  $A$ , this limit is uniform over all Borel sets  $B$  and over all Herglotz functions  $G$ .

**THEOREM A.1** ([7] Theorem 1). *Let  $G$  be an arbitrary Herglotz function and let  $A$  be a set of finite measure. Let  $B$  be an arbitrary Borel subset of  $\mathbb{R}$ . Then we have*

$$\left| \int_A \omega^\delta(\lambda, B; G) d\lambda - \int_A \omega(\lambda, B; G) d\lambda \right| \leq E_A(\delta) \quad (\text{A.3})$$

where  $E_A(\delta) \rightarrow 0$  for  $\delta \rightarrow 0$  and  $E_A(\delta)$  is a nondecreasing function of  $\delta$ . Since  $E_A(\delta)$  is independent of  $B$  and  $G$ , the bound is uniform over all sets  $B$  and all Herglotz functions  $G$ .

The theorem shows that the convergence of the integral used in (A.2) to determine the value distribution  $\mathcal{V}$  is uniform over  $B, G$ . As remarked in [6], and proven in [7], an explicit bound in (A.3) is obtained by setting

$$E_A(\delta) = \frac{1}{\pi} \int_A \theta(\lambda + i\delta, A^c) d\lambda \quad (\text{A.4})$$

where  $A^c$  is the complement of  $A$ . By making this choice for  $E_A(\delta)$  it is possible to show that (A.3) is optimal; indeed, equality can be attained with the choice  $B = A^c$  and  $G(z) = z$ . Further, as per [7] Corollary 1, if  $A$  is chosen to be an interval,  $A = (a, b)$ , it is straightforward to deduce using (A.4) that

$$E_A(\delta) = \frac{2(b-a)}{\pi} \arctan\left(\frac{\delta}{b-a}\right) + \frac{\delta}{\pi} \log\left(1 + \frac{(b-a)^2}{\delta^2}\right).$$

If we consider both Theorem A.1 and Equation (A.2) together, it is clear that we are able to carry out an estimate, the the order  $E_A(\delta)$ , of the value distribution function  $\mathcal{V}$

through the evaluation of  $\int_A \omega^\delta(\lambda, B; G) d\lambda$ .

As a final remark, using (A.4), we see that  $E_A(\delta) \leq \text{mes}(A)$  and, by symmetry,  $E_A(\delta) < \text{mes}(A^c)$ . Hence  $E_A(\delta) \leq \min(\text{mes}(A), \text{mes}(A^c))$ . As detailed in [7] Lemma 1, it can then be deduced that  $E_A(\delta)$  is finite if and only if either  $\text{mes}(A) < \infty$  or  $\text{mes}(A^c) < \infty$ .



# Bibliography

- [1] N. I. Akhiezer and I. M. Glazman. *Theory of Linear Operators in Hilbert Space*. Pitman: London, 1981.
- [2] E. Amar. Spectrum and Solutions of Dirac Systems. *Math. Nachr.*, 233/234:5–18, 2002.
- [3] J. H. Bardarson, M. Titov, and P. W. Brouwer. Electrostatic Confinement of Electrons in an Integrable Graphene Quantum Dot. *Phys. Rev. Lett.*, 102, 2009.
- [4] H. Behncke. Absolute Continuity of Hamiltonians with von Neumann-Wigner Potentials. *Proc. Amer. Math. Soc.*, 111:373–384, 1991.
- [5] S.V. Breimesser, J.D.E. Grant, and D.B. Pearson. Value Distribution and Spectral Theory of Schrödinger Operators with  $\mathcal{L}^2$ -Sparse Potentials. *J. Comput. Appl. Math.*, 148(1):307–322, 2002.
- [6] S.V. Breimesser and D.B. Pearson. Asymptotic Value Distribution for Solutions of the Schrödinger Equation. *Math. Phys. Anal. Geom.*, 3:285–403, 2000.
- [7] S.V. Breimesser and D.B. Pearson. Geometrical Aspects of Spectral Theory and Value Distribution for Herglotz Functions. *Math. Phys. Anal. Geom.*, 6:29–57, 2003.
- [8] M. Christ and A. Kiselev. Absolutely Continuous Spectrum for One-Dimensional Schrödinger Operators with Slowly Decaying Potentials: Some Optimal Results. *J. Amer. Math. Soc.*, 11(4):771–797, 1998.
- [9] M. Christ, A. Kiselev, and C. Remling. The Absolutely Continuous Spectrum of One-Dimensional Schrödinger Operators with Decaying Potentials. *Math. Res. Lett.*, 4(5):719–723, 1997.
- [10] Coddington and N. E. Levinson. *Theory of Ordinary Differential Equations*. New York-Toronto-London McGraw-Hill Book Company, 1955.
- [11] P. Deift and R. Killip. On the Absolutely Continuous Spectrum of One-Dimensional Schrödinger Operators with Square Summable Potentials. *Comm. Math. Phys.*, 203(2):341–347, 1999.
- [12] S. A. Denisov. On the Existence of Wave Operators for Some Dirac Operators with Square Summable Potentials. *Geom. Funct. Anal.*, 14(3):529–534, 2004.
- [13] S. A. Denisov. Continuous Analogs of Polynomials Orthogonal on the Unit Circle and Krein Systems. *IMRS Int. Math. Res. Surv.*, 2006.



- [14] M. S. P. Eastham and H Kalf. *Schrödinger-type Operators with Continuous Spectra*. Pitman, Boston, 1982.
- [15] M. S. P. Eastham and J. B. McLeod. The Existence of Eigenvalues Embedded in the Continuous Spectrum of Ordinary Differential Operators. *Proc. Roy. Soc. Edinburgh Sect. A*, 79(1-2):25–34, 1977/78.
- [16] M. S. P. Eastham and K. M. Schmidt. Asymptotics of the Spectral Density for Radial Dirac Operators with Divergent Potentials. *Publ. RIMS*, 44:107–129, 2008.
- [17] A. Erdelyi. Note on a Paper by Titchmarsh. *Quart. J. Math. Oxford*, 14:147–152, 1963.
- [18] W. D. Evans and B. J. Harris. Bounds for the Point Spectra of Separated Dirac Operators. *Proc. Roy. Soc. Edinburgh Sect. A*, 88:1–15, 1981.
- [19] L. D. Faddeev and L. A. Takhtajan. *Hamiltonian Methods in the Theory of Solitons*. Berlin Springer, 1987.
- [20] K. Friedrichs. On The Perturbation of Continuous Spectra. *Commun. on Appl. Math.*, 1:361–401, 1948.
- [21] J. Fröhlich, E. H. Lieb, and M. Loss. Stability of Coulomb Systems with Magnetic Fields. I. The One-Electron Atom. *Commun. Math. Phys.*, 104:251–270, 1986.
- [22] D. J. Gilbert. On Subordinacy and Analysis of the Spectrum of Schrödinger Operators with Two Singular Endpoints. *Proc. Roy. Soc. Edinburgh Sect. A*, 112:213–229, 1989.
- [23] D. J. Gilbert. Asymptotic Methods in the Spectral Analysis of Sturm-Liouville Operators. In *Sturm-Liouville Theory*, pages 121–136. Birkhäuser, Basel, 2005.
- [24] D. J. Gilbert and D.B. Pearson. On Subordinacy and Analysis of the Spectrum of One-Dimensional Schrödinger Operators. *Journal of Mathematical Analysis and Applications*, 128(1):30–56, 1987.
- [25] D. B. Hinton, A. B. Mingarelli, T. T. Read, and J. K. Shaw. On the Number of Eigenvalues in the Spectral Gap of a Dirac System. *Proceedings of the Edinburgh Mathematical Society*, 29:367–378, 1986.
- [26] D. B. Hinton and J. K. Shaw. Absolutely Continuous Spectra of Second-Order Differential Operators with Short and Long Range Potentials. *SIAM J Math. Anal.*, 17:182–196, 1986.
- [27] T. Kato. On Finite Dimensional Perturbations of Self-adjoint Operators. *J. Math. Soc. Japan*, 9:239–249, 1957.
- [28] T. Kato. Perturbation of Continuous Spectra by Trace Class Operators. *Proc. Japan Acad.*, 33:260–264, 1957.
- [29] T. Kato. *Perturbation Theory for Linear Operators*. Springer, 1995.
- [30] R. Killip and B. Simon. Sum Rules and Spectral Measures of Schrödinger Operators with  $\mathcal{L}^2$  Potentials. *Ann. Math.*, 170(2):739–782, 2009.
- [31] A. Kiselev. Absolutely Continuous Spectrum of One-Dimensional Schrödinger Operators with Slowly Decreasing Potentials. *Communications in Mathematical Physics*, 179:377–400, 1996.

- [32] A. Kiselev, Y. Last, and B. Simon. Modified Prüfer and EFGP Transforms and the Spectral Analysis of One Dimensional Schrödinger Operators. *Communications in Mathematical Physics*, 194:1–45, 1998.
- [33] M. G. Krein. Continuous Analogues of Propositions on Polynomials Orthogonal on the Unit Circle. (*Russian*) *Dokl. Akad. Nauk SSSR (N.S.)*, 105:637–640, 1955.
- [34] B. M. Levitan and I. S. Sargsjan. *Introduction to Spectral Theory: Self-Adjoint Ordinary Differential Operators*, volume 39 of *Translations of Mathematical Monographs*. American Mathematical Society: Providence, 1975.
- [35] M. Loss and H. T. Yau. Stability of Coulomb Systems with Magnetic Fields. III. Zero Energy Bound States of the Pauli Operators. *Commun. Math. Phys.*, 104:283–290, 186.
- [36] M. Martin. Spectral Analysis of One-Dimensional Dirac Operators with Slowly Decreasing Potentials. *Math. Phys. Anal. Geom.*, 6(4):385–398, 2003.
- [37] V. B. Mateev. Wave Operators and Positive Eigenvalues for a Schrödinger Equation with Oscillating Potential. *Theoret. and Math. Phys.*, 15:574–583, 1973.
- [38] S. N. Naboko. Dense Point Spectrum of Schrödinger and Dirac Operators. *Theor. Math. Phys.*, 68(1):18–28, 1986.
- [39] D. B. Pearson. Singular Continuous Measures in Scattering Theory. *Communications in Mathematical Physics*, 60:13–36, 1978.
- [40] D. B. Pearson. Value Distribution and Spectral Analysis of Differential Operators. *J. Phys. A*, 26(16):4067–4080, 1993.
- [41] D. B. Pearson. Value Distribution and Spectral Theory. *Proc. London Math. Soc. (3)*, 68(1):127–144, 1994.
- [42] M. S. Plesset. The Dirac Equation in Simple Fields. *Phys. Rev.*, 42(2):278–290, 1932.
- [43] M. Reed and B. Simon. *Methods of Modern Mathematical Physics Vol 4: Analysis of Operators*. Academic Press Limited, 1978.
- [44] M. Reed and B. Simon. *Methods of Modern Mathematical Physics Vol 3: Scattering Theory*. Academic Press Limited, 1979.
- [45] M. Reed and B. Simon. *Methods of Modern Mathematical Physics Vol 1: Functional Analysis*. Academic Press Limited, 1980.
- [46] C. Remling. The Absolutely Continuous Spectrum of One-Dimensional Schrödinger Operators with Decaying Potentials. *Commun. Math. Phys.*, 193(1):151–170, 1998.
- [47] B. W. Roos and W. C. Sangren. Spectra for a Pair of Singular First Order Differential Equations. *Proc. Amer. Math. Soc.*, 12:468–476, 1961.
- [48] B. W. Roos and W. C. Sangren. Spectral Theory of Dirac’s Radial Relativistic Wave Equation. *J. Math. Physics*, 3:882–890, 1962.
- [49] B. W. Roos and W. C. Sangren. Three Spectral Theorems for a Pair of Singular First Order Differential Equations. *Pacific J. Math.*, 12:1047–1055, 1962.

- [50] M. Rosenblum. Perturbations of Continuous Spectrum and Unitary Equivalence. *Pacific J. Math.*, 7:997–1010, 1957.
- [51] W. Rudin. *Real and Complex Analysis*. New York-Toronto-London McGraw-Hill Book Company, 1966.
- [52] W. Rudin. *Principles of Mathematical Analysis*. McGraw Hill, Third edition, 1976.
- [53] K. M. Schmidt. Dense Point Spectrum and Absolutely Continuous Spectrum in Spherically Symmetric Dirac Operators. *Forum Mathematicum*, 7:459–475, 1995.
- [54] K. M. Schmidt. Dense Point Spectrum for the One-Dimensional Dirac Operator with an Electrostatic Potential. *Proc. Roy. Soc. Edinburgh Sect. A*, 126:1087–1096, 1996.
- [55] K. M. Schmidt. Absolutely Continuous Spectrum of Dirac Systems with Potentials Infinite at Infinity. *Math. Proc. Camb. Phil. Soc.*, 122, 1997.
- [56] K. M. Schmidt. Spectral Properties of Rotationally Symmetric Massless Dirac Operators. *Letters in Mathematical Physics*, 92(3):231–241, 2000.
- [57] K. M. Schmidt. A short proof for Bargmann-type inequalities. *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.*, 458(2027):2829–2832, 2002.
- [58] B. Simon. Some Schrödinger Operators with Dense Point Spectrum. *Proc. Amer. Math. Soc.*, 125(1):203–208, 1997.
- [59] B. Simon and G. Stolz. Operators with Singular Continuous Spectrum: Sparse Potentials. *Proceedings of the American Mathematical Society*, 124(7):2073–2080, 1996.
- [60] M. H. Stone. *Linear Transformations in Hilbert Space*. American Mathematical Society: Providence, 1964.
- [61] B. Thaller. *The Dirac Equation*. Springer, Berlin, 1992.
- [62] C. R. Thurlow. The Point-Continuous Spectrum of Second-Order Ordinary Differential Operators. *Proc. Roy. Soc. Edinburgh Sect. A*, 84:197–211, 1979.
- [63] S. Timischl. A Trace Formula for One-Dimensional Dirac Operators. Master’s thesis, Karl-Franzens-Universität, Graz, 1995.
- [64] E. C. Titchmarsh. On the Nature of the Spectrum in Problems of Relativistic Quantum Mechanics. *Quart. J. Math. Oxford*, 12(2):227–240, 1961.
- [65] J. Von Neumann. *Mathematical Foundations of Quantum Mechanics*. Princeton University Press: Princeton, 1955.
- [66] J. Von Neumann and E. P. Wigner. Über Merkwürdige Diskrete Eigenwerte. *Phys. Z.*, 30:465–467, 1929.
- [67] J. Weidmann. Oszillationsmethoden für Systeme gewöhnlicher Differentialgleichungen. *Math. Z.*, 119:349–373, 1971.
- [68] J. Weidmann. *Linear Operators in Hilbert Spaces*. Springer, 1980.
- [69] J. Weidmann. *Spectral Theory of Ordinary Differential Operators (Lecture Notes in Mathematics)*. Springer-Verlag, 1987.

- [70] H. Weyl. Über Gewöhnliche Differentialgleichungen mit Singularitäten und die Zugehörigen Entwicklungen Willkürlicher. *Funktionen. Math. Annalen.*, 68:220–269, 1910.
- [71] D. A. W. White. Schrödinger Operators with Rapidly Oscillating Central Potentials. *Trans. Am. Math. Soc.*, 275:641–677, 1983.
- [72] J. Wissel. One-Dimensional Schrödinger and Dirac Operators with Absolutely Continuous Spectrum. Master's thesis, Department of Mathematics, Ludwig-Maximilians-Universität München, 2003.
- [73] O. Yamada. On the Spectrum of Dirac Operators with the Unbounded Potential at Infinity. *Hokkaido Math. J.*, 26:439–449, 1997.