FRACTIONAL PERFECT b-MATCHING POLYTOPES I: GENERAL THEORY

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Abstract. The fractional perfect b-matching polytope of an undirected graph G is the polytope of all assignments of nonnegative real numbers to the edges of G such that the sum of the numbers over all edges incident to any vertex v is a prescribed nonnegative number b_v. General theorems which provide conditions for nonemptiness, give a formula for the dimension, and characterize the vertices, edges and face lattices of such polytopes are obtained. Many of these results are expressed in terms of certain spanning subgraphs of G which are associated with subsets or elements of the polytope. For example, it is shown that an element u of the fractional perfect b-matching polytope of G is a vertex of the polytope if and only if each component of the graph of u either is acyclic or else contains exactly one cycle with that cycle having odd length, where the graph of u is defined to be the spanning subgraph of G whose edges are those at which u is positive.

Contents

2010 *Mathematics Subject Classification.* 52B05; 05C50, 05C70, 52B11, 90C27, 90C35.
Key words and phrases. graphs, polytopes, perfect matchings.

1. Introduction

The focus of this paper, and its expected sequels [3, 4], is the fractional perfect b-matching polytope of a graph. For any finite, undirected graph G, which may contain loops and multiple edges, and any assignment b of nonnegative real numbers to the vertices of G, this polytope, denoted $\mathcal{P}(G, b)$, is defined to be the set of all assignments of nonnegative real numbers to the edges of G such that the sum of the numbers over all edges incident to any vertex is the prescribed value of b at that vertex.

Certain fractional perfect b-matching polytopes, or polytopes which are affinely isomorphic to these, have been studied and used in the contexts of combinatorial matrix classes (see, for example, the book by Brualdi [9]), and combinatorial optimization (see, for example, the books by Korte and Vygen [17], or Schrijver [22]). The terminology 'fractional perfect b-matching polytope' is derived mainly from the latter context, and will be discussed further in Section 1.5.

1.1. Main results.

The primary aim of this paper is to present general theorems concerning the nonemptiness, existence of positive elements, dimension, vertices, edges and faces of such polytopes, together with uniform, and in most cases self-contained, proofs.

The main theorems apply to an arbitrary graph G, which may be nonbipartite. However (as is often the case with results related to matchings of graphs), many of these theorems take simpler forms if G is bipartite, so these forms will also be given.

A list of the main results of this paper, including those which are restricted to the case of bipartite G, is as follows.

- *Conditions for $\mathcal{P}(G, b)$ to be nonempty:* Theorem 6.
  Bipartite case: Theorem 3.
- *Conditions for $\mathcal{P}(G, b)$ to contain a positive element (i.e., one whose value at each edge of G is positive):* Theorem 7.
  Bipartite case: Theorem 4.
- *Results for the dimension of $\mathcal{P}(G, b)$:* Corollaries 18 and 19.
- *Results for the vertices of $\mathcal{P}(G, b)$:* Corollary 21, and Theorems 22, 24, 29 and 39.
  Bipartite case: Corollary 23.
- *Results for the edges of $\mathcal{P}(G, b)$:* Theorem 25 and Corollary 31.
  Bipartite case: Corollary 26.
- *Results for the faces of $\mathcal{P}(G, b)$*: Theorems 17, 30, 32, 33, 34, 35, 37 and 38.
  Bipartite case: Corollaries 20 and 36.

Several of these results are expressed in terms of certain spanning subgraphs of G (i.e., subgraphs of G with the same vertex set as that of G) which are associated with subsets or elements of $\mathcal{P}(G, b)$. These graphs are defined in (51)–(53), and certain characterizations of the graphs are given in Theorems 32, 33, 34, 35 and 37, and Corollary 36.

1.2. **Structure of paper.** The structure of this paper, and the interdependence of its sections, is as follows.

In Section 2, alternative forms, involving the incidence matrix or a certain generalized adjacency matrix of G, are given for $\mathcal{P}(G,b)$.

In Section 3, results which give conditions for $\mathcal{P}(G,b)$ to be nonempty, or to contain a positive element, are derived.

In Section 4, some relevant general results for graphs, and their incidence matrices, are obtained.

In Sections 5 and 6, some relevant general results for polytopes, and their face lattices, are obtained or stated.

In Section 7, results for the faces, dimension, vertices and edges of $\mathcal{P}(G,b)$ are derived. The proof of each of the main results of Section 7 involves a relatively simple combination of a general result for graphs from Section 4 with a general result for polytopes from Section 5.

In Section 8, further results for the vertices, edges, faces and graphs of $\mathcal{P}(G,b)$ are derived. The proof of each of the main results of Section 8 involves a relatively simple application of a general result for polytopes from Section 6 to the context of $\mathcal{P}(G,b)$.

In Section 9, some additional results for the graphs of $\mathcal{P}(G,b)$ are obtained, using certain results from Section 3.

Finally, in Section 10, results which are relevant for the case in which G contains multiple edges are obtained. These results identify relationships between the faces and vertices of $\mathcal{P}(G,b)$ and those of $\mathcal{P}(G_{\mathrm{rd}},b)$, where G_{rd} is a graph obtained from G by reducing each set of multiple edges to a single edge

For the sake of completeness, this paper includes some previously-known results. However, many of these results have appeared previously in the literature only under slightly less general conditions (for example, graphs without loops or multiple edges), in terms of slightly different objects (for example, matrices rather than graphs), or with somewhat different proofs (for example, those in which the aspects which depend on graph theory and the aspects which depend on polytope theory are interspersed throughout the proof, rather than being considered separately until the final step).

1.3. **Notation and basic facts.** The main notation and conventions, and some basic facts, which will be used in this paper are as follows.

Throughout the paper, G is a finite, undirected graph which, unless stated otherwise, may be nonbipartite, and may contain loops and multiple edges. Furthermore, V and E are the vertex and edge sets of G, and b is a function from V to the nonnegative real numbers.

It will always be assumed that V is nonempty, but, unless stated otherwise, that E may be empty.

The set of all edges incident with vertex v of G will be denoted as $\delta_G(v)$, with the implication that a loop attached to v appears once rather than twice in $\delta_G(v)$. Each edge of G will be taken to have two endpoints in V, with these being identical if the edge is a loop.

The conventions used for cycles will be as follows. For a positive integer n, a cycle in G of length n corresponds to all cyclic permutations of a sequence $(v_1, e_1, v_2, e_2, \ldots, v_n, e_n)$, in which $v_1, \ldots, v_n$ are distinct vertices of V, $e_1, \ldots, e_n$ are distinct edges of E, any two consecutive terms of the sequence are incident, and the endpoints of e_n are v_n and v_1. (It follows that if $n = 1$ then e_1 is a loop incident to v_1, that if $n = 2$ then e_1 and e_2 belong to a set of multiple edges with the same incident vertices v_1 and v_2, and that if $n \geq 3$ then the distinctness of $e_1, \ldots, e_n$ is implied by the distinctness of $v_1, \ldots, v_n$.) With these conventions, the following standard facts, which are often applied only to graphs without loops or multiple edges, are valid. The graph G is bipartite if and only if G does not contain any odd-length cycles. For a connected graph G, $|E| + 1 = |V|$ if and only if G is acyclic (i.e., a tree), and $|E| = |V|$ if and only if G contains exactly one cycle.

A graph which is obtained from G by reducing each set of multiple edges to a single edge will be referred to as a reduced graph of G. More specifically, a reduced graph of G is a graph G_{rd} which has vertex set V, does not have multiple edges, and such that, for any $u, w \in V$, u and w are adjacent in G_{rd} if and only if u and w are adjacent in G. Thus, for example, G itself is a reduced graph of G if and only if G does not have multiple edges.

Matrices and vectors whose rows and columns are indexed by finite sets will often be used. If such a matrix or vector is written out as an explicit array of entries, then an ordering of the elements of each associated index set needs to be chosen. However, all of the results of this paper are independent of such choices.

The sets of nonnegative real numbers and positive real numbers will be denoted as $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{>0}$, respectively.

For nonempty sets S and N, the set of functions from N to S will be denoted as S^N. For N finite, as will always be the case here, the value of a function $x \in S^N$ at $i \in N$ will be denoted as x_i, so that x is also regarded as a vector whose entries are indexed by N. The case $S^{\emptyset}$, for $S \subset \mathbb{R}$, will also occasionally be needed, and this will denote a set $\{0\}$ containing only the zero vector if $0 \in S$, or $\emptyset$ if $0 \notin S$. A vector $x \in \mathbb{R}^N$ will be referred to as positive if $x_i > 0$ for each $i \in N$, i.e., if $x \in \mathbb{R}^N_{>0}$.

The fractional perfect b-matching polytope of G can now be written, using some of the notation above, as

$$\mathcal{P}(G, b) := \left\{ x \in \mathbb{R}^E_{\geq 0} \;\middle|\; \sum_{e \in \delta_G(v)} x_e = b_v \text{ for each } v \in V \right\}. \tag{1}$$

This set is a polytope in $\mathbb{R}^E$ since it is a polyhedron in $\mathbb{R}^E$ (being the intersection of the closed halfspaces $\{x \in \mathbb{R}^E \mid x_e \geq 0\}$ for each $e \in E$, and the hyperplanes $\{x \in \mathbb{R}^E \mid \sum_{e \in \delta_G(v)} x_e = b_v\}$ for each $v \in V$), and it is bounded (since any $x \in \mathcal{P}(G, b)$ satisfies $0 \leq x_e \leq b_{v_e}$ for each $e \in E$, where v_e is an endpoint of e).

It will be assumed, for some of the results of this paper, that b is nonzero. It can be seen that this is equivalent to the assumption that $\mathcal{P}(G, b) \neq \{0\}$.

Note also that for the case $E=\emptyset$, $\mathcal{P}(G,b)$ is $\{0\}$ if $b=0$, or $\emptyset$ if $b\neq 0$.

The set of positive elements of $\mathcal{P}(G,b)$ will be denoted as $\mathcal{P}(G,b)_{>0}$, i.e.,

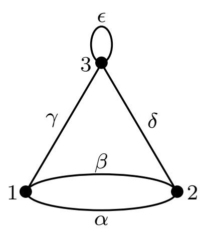
$$\mathcal{P}(G,b)_{>0} := \left\{ x\in\mathbb{R}^E_{>0} \;\middle|\; \sum_{e\in\delta_G(v)} x_e = b_v \text{ for each } v\in V \right\}. \tag{2}$$

For convenience, a brief summary will now be given of some further notation which will be introduced properly later in the paper.

- I_G denotes the incidence matrix of G. First used in Section 2.
- $G[U,W]$, for subsets U and W of V, denotes the set of all edges of G which connect a vertex of U and a vertex of W. First used in Section 3. See also (14).
- $U=W_1\Cup W_2\Cup\ldots\Cup W_n$, for sets $U,W_1,W_2,\ldots,W_n$, means that $U=W_1\cup W_2\cup\ldots\cup W_n$ and that $W_1,\ldots,W_n$ are pairwise disjoint. First used in Section 3.
- $\operatorname{supp}(X)$ and $\operatorname{supp}(x)$, for $X\subset\mathbb{R}^N$ and $x\in\mathbb{R}^N$ with N a finite set, denote the supports of X and x, respectively. First used in Section 5. See also (24)–(25).
- $\operatorname{vert}(P)$, $\operatorname{facets}(P)$ and $\mathcal{F}(P)$, for a polytope P, denote the set of vertices, set of facets and face lattice, respectively, of P. First used in Section 6.
- The appearance of $\subset'$ within a result means that the result is valid if $\subset'$ is taken to be $\subset$, and also valid if $\subset'$ is taken to be $=$. First used in Section 6.
- $\operatorname{gr}(X)$, for $X\subset\mathbb{R}^E$, denotes the spanning subgraph of G with edge set $\{e\in E \mid$ there exists $x\in X$ with $x_e\neq 0\}$, and is referred to as the graph of X. First used in Section 7. See also (51).
- $\operatorname{gr}(x)$, for $x\in\mathbb{R}^E$, denotes the spanning subgraph of G with edge set $\{e\in E\mid x_e\neq 0\}$, and is referred to as the graph of x. First used in Section 7. See also (52).
- $\mathcal{G}(G,b)$ denotes the set of graphs of subsets of $\mathcal{P}(G,b)$. First used in Section 7. See also (53).

1.4. **Example.** A simple example of a fractional perfect b-matching polytope will now be introduced. This example will be used to illustrate various results later in the paper.

For the rest of this section, consider the graph, with $V=\{1,2,3\}$ and $E=\{\alpha,\beta,\gamma,\delta,\epsilon\}$, given by

$G =$ 

. (3)

In subsequent diagrams of spanning subgraphs of G, the edges α and β will always be represented by curves which lie below and above, respectively, the straight line between the vertices 1 and 2, as in (3).

The fractional perfect b-matching polytope of G for this case is

$$\mathcal{P}(G,b)=\Big\{(x_\alpha,x_\beta,x_\gamma,x_\delta,x_\epsilon)\in\mathbb{R}_{\ge 0}^{\{\alpha,\beta,\gamma,\delta,\epsilon\}}\;\Big|\; x_\alpha+x_\beta+x_\gamma=b_1,\ x_\alpha+x_\beta+x_\delta=b_2,\ x_\gamma+x_\delta+x_\epsilon=b_3\Big\}, \quad (4)$$

which gives

$$\mathcal{P}(G,b)=\Big\{\big(x_\alpha,x_\beta,b_1-x_\alpha-x_\beta,b_2-x_\alpha-x_\beta,b_3-b_1-b_2+2(x_\alpha+x_\beta)\big)\in\mathbb{R}^{\{\alpha,\beta,\gamma,\delta,\epsilon\}}\;\Big|\; x_\alpha\ge 0,\ x_\beta\ge 0,\ \tfrac{b_1+b_2-b_3}{2}\le x_\alpha+x_\beta\le\min(b_1,b_2)\Big\}. \quad (5)$$

It follows immediately from (5) that $\mathcal{P}(G,b)\ne\emptyset$ if and only if $\frac{b_1+b_2-b_3}{2}\le\min(b_1,b_2)$, which gives

$$\mathcal{P}(G,b)\ne\emptyset \text{ if and only if } b_3\ge|b_1-b_2|. \quad (6)$$

It can also be deduced that

$$\mathcal{P}(G,b)_{>0}\ne\emptyset \text{ if and only if } b_1>0,\ b_2>0 \text{ and } b_3>|b_1-b_2|. \quad (7)$$

In particular, if there exists $x\in\mathcal{P}(G,b)_{>0}$, then $b_1=x_\alpha+x_\beta+x_\gamma>x_\alpha+x_\beta>0$, $b_2=x_\alpha+x_\beta+x_\delta>x_\alpha+x_\beta>0$ and $x_\epsilon=b_3-b_1-b_2+2(x_\alpha+x_\beta)>0$, which gives $b_1>0$, $b_2>0$ and $b_3>b_1+b_2-2(x_\alpha+x_\beta)>b_1+b_2-2\min(b_1,b_2)=|b_1-b_2|$. Conversely, if $b_1>0$, $b_2>0$ and $b_3>|b_1-b_2|$, then setting $x_\alpha=x_\beta=\big(\max(0,\frac{b_1+b_2-b_3}{2})+\min(b_1,b_2)\big)/4$, $x_\gamma=b_1-x_\alpha-x_\beta$, $x_\delta=b_2-x_\alpha-x_\beta$ and $x_\epsilon=b_3-b_1-b_2+2x_\alpha+2x_\beta$ gives $x\in\mathcal{P}(G,b)_{>0}$.

It can be seen, using (5), that if $b_1>0$, $b_2>0$ and $|b_1-b_2|<b_3<b_1+b_2$, then $\mathcal{P}(G,b)$ is a quadrilateral with vertices $\big(\frac{b_1+b_2-b_3}{2},0,\frac{b_1-b_2+b_3}{2},\frac{-b_1+b_2+b_3}{2},0\big)$, $\big(0,\frac{b_1+b_2-b_3}{2},\frac{b_1-b_2+b_3}{2},\frac{-b_1+b_2+b_3}{2},0\big)$, $\big(\min(b_1,b_2),0,\max(0,b_1-b_2),\max(0,b_2-b_1),b_3-|b_1-b_2|\big)$ and $\big(0,\min(b_1,b_2),\max(0,b_1-b_2),\max(0,b_2-b_1),b_3-|b_1-b_2|\big)$. It can also be seen that $\mathcal{P}(G,b)$ is a triangle for $b_1>0$, $b_2>0$ and $b_3\ge b_1+b_2$, a line segment for $b_1>0$, $b_2>0$ and $b_3=|b_1-b_2|$, and a single point for $b_1=0$ or $b_2=0$ and $b_3\ge|b_1-b_2|=\max(b_1,b_2)$.

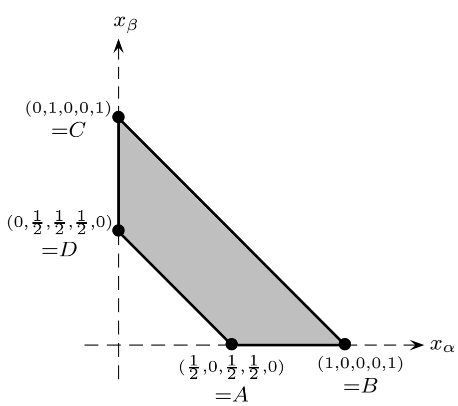


FIGURE 1. $\mathcal{P}(G,b)$, for G given by (3) and $b_1=b_2=b_3=1$.

For the specific case $b_1 = b_2 = b_3 = 1$, the quadrilateral $\mathcal{P}(G, b)$ (or, more precisely, its projection onto the x_α, x_β plane) is shown in Figure 1. In that figure, the coordinates of each vertex are indicated explicitly, and the vertices are also denoted as A, B, C and D, where this notation will be used further in Sections 7–9.

1.5. **Related matching polytopes.** In order to place fractional perfect b-matching polytopes into a wider context, some related matching polytopes will now be defined.

For a graph G and a vector $b \in \mathbb{R}^V_{\geq 0}$, define eight types of vector $x \in \mathbb{R}^E_{\geq 0}$ which satisfy

$$\sum_{e \in \delta_G(v)} x_e \leq b_v \quad \text{for each } v \in V, \tag{8}$$

where the types are obtained by prefixing any of the terms 'fractional', 'perfect' or 'b-' to the term 'matching', and these terms have the following meanings. If 'fractional' is omitted, then all entries of b and x are integers. If 'perfect' is included, then each case of (8) holds as an equality. If 'b-' is omitted, then each entry of b is 1.

fractional perfect b-matching polytope of G, for $b \in \mathbb{R}^V_{\geq 0}$	$\left\{x \in \mathbb{R}^E_{\geq 0} \,\middle|\, \sum_{e \in \delta_G(v)} x_e = b_v \text{ for each } v \in V\right\}$
fractional perfect matching polytope of G	$\left\{x \in \mathbb{R}^E_{\geq 0} \,\middle|\, \sum_{e \in \delta_G(v)} x_e = 1 \text{ for each } v \in V\right\}$
fractional b-matching polytope of G, for $b \in \mathbb{R}^V_{\geq 0}$	$\left\{x \in \mathbb{R}^E_{\geq 0} \,\middle|\, \sum_{e \in \delta_G(v)} x_e \leq b_v \text{ for each } v \in V\right\}$
fractional matching polytope of G	$\left\{x \in \mathbb{R}^E_{\geq 0} \,\middle|\, \sum_{e \in \delta_G(v)} x_e \leq 1 \text{ for each } v \in V\right\}$
perfect b-matching polytope of G, for $b \in \mathbb{Z}^V_{\geq 0}$	$\operatorname{conv}\left\{x \in \mathbb{Z}^E_{\geq 0} \,\middle|\, \sum_{e \in \delta_G(v)} x_e = b_v \text{ for each } v \in V\right\}$
perfect matching polytope of G	$\operatorname{conv}\left\{x \in \{0,1\}^E \,\middle|\, \sum_{e \in \delta_G(v)} x_e = 1 \text{ for each } v \in V\right\}$
b-matching polytope of G, for $b \in \mathbb{Z}^V_{\geq 0}$	$\operatorname{conv}\left\{x \in \mathbb{Z}^E_{\geq 0} \,\middle|\, \sum_{e \in \delta_G(v)} x_e \leq b_v \text{ for each } v \in V\right\}$
matching polytope of G	$\operatorname{conv}\left\{x \in \{0,1\}^E \,\middle|\, \sum_{e \in \delta_G(v)} x_e \leq 1 \text{ for each } v \in V\right\}$

TABLE 1. Types of matching polytope.

For each of these types of matching, define an associated polytope as the set of all such matchings if 'fractional' is included, or as the convex hull in $\mathbb{R}^E$ of all such matchings if 'fractional' is omitted. The definitions of these polytopes are given explicitly for each case in Table 1, where conv denotes the convex hull and $\mathbb{Z}_{\geq 0}$ denotes the set of nonnegative integers. It can be seen that the current definition of the fractional perfect b-matching polytope of G coincides with the previous definition (1). For more information on the other seven cases of such matchings and polytopes (and further related special cases, such as those which are 'capacitated' or 'simple'), see, for example, Korte and Vygen [17], Lovász and Plummer [19], or Schrijver [22]. Some of these cases, and certain relationships among them, will also be considered in [3].

Note that, using the previous definitions, a matching or perfect matching x is an assignment x_e of 0 or 1 to each edge e of G such that the sum of the numbers over all edges incident to any vertex is at most 1, or exactly 1 in the perfect matching case. However, using standard graph theory terminology, such an x is actually the incidence vector of a matching (i.e., a subset M of E such that each vertex is incident to at most one edge in M), or perfect matching (i.e., a subset M of E such that each vertex is incident to exactly one edge in M). More specifically, x_e is 1 or 0 according to whether or not the edge e is in the matching.

1.6. **Further papers.** This is the first paper in a projected series of three papers on fractional perfect b-matching polytopes.

In the second paper [3], various polytopes which are special cases of fractional perfect b-matching polytopes, or which are affinely isomorphic to such special cases, will be considered, and results (including certain standard theorems) for these cases will be obtained by applying the general theorems of this paper. The cases which will be considered in [3] will include the following.

- Polytopes $\mathcal{P}(G,b)$ in which each entry of b is an integer.
- Polytopes defined by modifying (1) so that $\sum_{e\in\delta_G(v)} x_e = b_v$ is replaced by $\sum_{e\in\delta_G(v)} x_e \le b_v$ for each vertex v of G.
- Polytopes defined by modifying (1) so that additional conditions $x_e \le c_e$ apply to each edge e of G, where c_e is a prescribed nonnegative number.
- Polytopes of b-flows (or b-transshipments) on directed graphs. See, for example, Schrijver [22, Secs. 11.4 & 13.2c], or Korte and Vygen [17, Sec. 9.1].
- Certain other matching polytopes, including some of those discussed in Section 1.5.
- Polytopes of magic labelings of graphs. See, for example, Ahmed [1].
- The symmetric transportation polytope $\mathcal{N}(R)$, and the related polytopes $\mathcal{N}(\le R)$, $\mathcal{N}_{\le Z}(R)$ and $\mathcal{N}_{\le Z}(\le R)$. See, for example, Brualdi [9, Sec. 8.2] for definitions of the notation, and further information. The cases $\mathcal{N}(R)$ and $\mathcal{N}_{\le Z}(R)$ will also be discussed briefly in Section 2.
- The transportation polytope $\mathcal{N}(R,S)$, and the related polytopes $\mathcal{N}(\le R, \le S)$, $\mathcal{N}_{\le Z}(R, S)$ and $\mathcal{N}_{\le Z}(\le R, \le S)$. See, for example, Brualdi [9, Secs. 8.1 & 8.4] for definitions of the notation, and further information. The cases $\mathcal{N}(R,S)$ and $\mathcal{N}_{\le Z}(R,S)$ will also be discussed briefly in Section 2.

- The polytope of doubly stochastic matrices, also known as the Birkhoff or assignment polytope, and various related polytopes, including the polytopes of doubly substochastic matrices, extensions of doubly substochastic matrices, symmetric doubly stochastic matrices, symmetric doubly substochastic matrices, and tridiagonal doubly stochastic matrices. See, for example, Brualdi [9, Ch. 9], and (for the tridiagonal case) Dahl [11].
- The alternating sign matrix polytope. See, for example, Behrend and Knight [5, Sec. 6], or Striker [23].

In the third paper [4], the polytope of all elements of $\mathcal{P}(G,b)$ (for certain cases of G and b) which are invariant under a certain natural action of all elements of a group of automorphisms of G will be considered.

2. Matrix forms of $\mathcal{P}(G,b)$

In this section, some alternative forms involving matrices are given for $\mathcal{P}(G,b)$. The form (9) will be used first in Section 7, while the other forms, (12) and (13), will be used in [3]. Certain matrix classes, and their relationship with certain cases of fractional perfect b-matching polytopes, are also discussed.

The incidence matrix I_G of a graph G is the matrix with rows and columns indexed by V and E respectively, and entries $(I_G)_{ve}$ given by 1 or 0 according to whether or not vertex v is incident with edge e. It follows immediately from this definition, and the definition (1) of the fractional perfect b-matching polytope of G, that

$$\mathcal{P}(G,b) = \bigl\{x \in \mathbb{R}^E_{\geq 0} \bigm| I_G\, x = b\bigr\}. \tag{9}$$

As an example, for G given by (3),

$$\mathcal{P}(G,b) = \left\{(x_\alpha, x_\beta, x_\gamma, x_\delta, x_\epsilon) \in \mathbb{R}^{\{\alpha,\beta,\gamma,\delta,\epsilon\}}_{\geq 0} \;\middle|\; \begin{pmatrix} 1&1&1&0&0\\ 1&1&0&1&0\\ 0&0&1&1&1 \end{pmatrix} \begin{pmatrix} x_\alpha\\ x_\beta\\ x_\gamma\\ x_\delta\\ x_\epsilon \end{pmatrix} = \begin{pmatrix} b_1\\ b_2\\ b_3 \end{pmatrix}\right\}, \tag{10}$$

For $x \in \mathbb{R}^E$, define the generalized adjacency matrix $A_G(x)$ of G to be the matrix with rows and columns indexed by V, and entries given by

$$A_G(x)_{vw} = \sum_{e\in\delta_G(v)\cap\delta_G(w)} x_e \tag{11}$$

for each $v, w \in V$. Note that the sum here is simply over all edges which connect v and w, that $A_G(x)$ is symmetric, and that if $x_e = 1$ for all $e \in E$ then $A_G(x)$ is the standard adjacency matrix of G. It follows that

$$\mathcal{P}(G,b) = \left\{x \in \mathbb{R}^E_{\geq 0} \;\middle|\; \sum_{w\in V} A_G(x)_{vw} = b_v \text{ for each } v \in V\right\}, \tag{12}$$

i.e., $\mathcal{P}(G,b)$ is the polytope of all assignments x of nonnegative real numbers to the edges of G such that the sum of entries in row/column v of $A_G(x)$ is b_v, for each vertex v.

As an example, for G given by (3), $\mathcal{P}(G,b)$ is the set of all $(x_\alpha,x_\beta,x_\gamma,x_\delta,x_\epsilon)\in\mathbb{R}_{\geq 0}^{\{\alpha,\beta,\gamma,\delta,\epsilon\}}$ such that the sum of entries in row/column 1, 2 or 3 of $\left(\begin{smallmatrix} 0 & x_\alpha+x_\beta & x_\gamma \\ x_\alpha+x_\beta & 0 & x_\delta \\ x_\gamma & x_\delta & x_\epsilon \end{smallmatrix}\right)$ is b_1, b_2 or b_3, respectively.

For the case in which G is bipartite with bipartition (U,W), and for $x\in\mathbb{R}^E$, define the generalized (U,W)-biadjacency matrix $A_G^{(U,W)}(x)$ of G to be the submatrix of $A_G(x)$ obtained by restricting the rows to those indexed by U and the columns to those indexed by W. If $x_e=1$ for all $e\in E$, then $A_G^{(U,W)}(x)$ is the standard (U,W)-biadjacency matrix of G. It follows that

$$\mathcal{P}(G,b)=\Bigg\{x\in\mathbb{R}_{\geq 0}^E\;\Bigg|\;\sum_{w\in V}A_G^{(U,W)}(x)_{uw}=b_u\text{ for each }u\in U,\\ \sum_{u\in U}A_G^{(U,W)}(x)_{uw}=b_w\text{ for each }w\in W\Bigg\},\quad(13)$$

i.e., $\mathcal{P}(G,b)$ is the polytope of all assignments x of nonnegative real numbers to the edges of G such that the sum of entries in row u of $A_G^{(U,W)}(x)$ is b_u and the sum of entries in column w of $A_G^{(U,W)}(x)$ is b_w, for each $u\in U$, $w\in W$.

The relationship between certain cases of fractional perfect b-matching polytopes, and certain matrix classes will now be considered briefly. Further details for these cases will be given in [3].

It follows from (12) that if G does not contain multiple edges, then $\mathcal{P}(G,b)$ is affinely isomorphic (using simple and obvious mappings) to the polytope of all $|V|\times|V|$ symmetric matrices with nonnegative real entries, for which certain entries are prescribed to be zero, and the sum of entries in any row/column is a prescribed nonnegative real number for that row/column. An account of such polytopes is given by Brualdi in [9, Sec. 8.2]. The notation used there is that, given a vector $R\in\mathbb{R}^n$ with nonnegative entries, and a symmetric $n\times n$ matrix Z each of whose entries is 0 or 1, $\mathcal{N}_{\leq Z}(R)$ is the polytope, or matrix class, of all $n\times n$ symmetric matrices with nonnegative real entries, for which the i,j entry is zero if Z_{ij} is zero, and the sum of entries in row/column i is R_i. Accordingly, if G does not contain multiple edges, then $\mathcal{P}(G,b)$ and $\mathcal{N}_{\leq Z}(R)$ are affinely isomorphic, where Z is the adjacency matrix of G, and associated entries of b and R are equal. For the case in which G is a complete graph with loops (i.e., a graph in which any two distinct vertices are connected by a single edge, and a single loop is incident to each vertex), Z is a square matrix each of whose entries is 1, and $\mathcal{N}_{\leq Z}(R)$ is a so-called symmetric transportation polytope, denoted in [9, pp. 39 & 348] as $\mathcal{N}(R)$. It follows (using the fact that setting a subset of a polyhedron's defining inequalities to equalities gives a, possibly empty, face of the polyhedron) that $\mathcal{N}_{\leq Z}(R)$ is a face of $\mathcal{N}(R)$. Similarly, for an arbitrary graph G without multiple edges, $\mathcal{P}(G,b)$ is affinely isomorphic to a face of $\mathcal{P}(K_V,b)$, where K_V is a complete graph with loops, and vertex set V.

It follows from (13) that if G is bipartite with bipartition (U,W), and does not contain multiple edges, then $\mathcal{P}(G,b)$ is affinely isomorphic (again, using simple and obvious mappings) to the polytope of all $|U|\times|W|$ matrices with nonnegative real entries, for which certain entries are prescribed to be zero, and the sum of entries in any row or column is a prescribed nonnegative real number for that row or column. An account of such polytopes is given by Brualdi in [9, Secs. 8.1 & 8.4]. The notation used there is that, given vectors $R\in\mathbb{R}^m$ and $S\in\mathbb{R}^n$ with nonnegative entries, and an $m\times n$ matrix Z each of whose entries is 0 or 1, $\mathcal{N}_{\leq Z}(R,S)$ is the polytope, or matrix class, of all $m\times n$ matrices with nonnegative real entries, for which the i,j entry is zero if Z_{ij} is zero, the sum of entries in row i is R_i, and the sum of entries in column j is S_j. Accordingly, if G is bipartite with bipartition (U,W), and does not contain multiple edges, then $\mathcal{P}(G,b)$ and $\mathcal{N}_{\leq Z}(R,S)$ are affinely isomorphic, where Z is the (U,W)-biadjacency matrix of G, the entries b_v with $v\in U$ are equal to associated entries of R, and the entries b_v with $v\in W$ are equal to associated entries of S. For the case in which G is a complete bipartite graph (i.e., a graph in which each vertex of U and each vertex of W are connected by a single edge), Z is a matrix each of whose entries is 1, and $\mathcal{N}_{\leq Z}(R,S)$ is a so-called transportation polytope, denoted in [9, pp. 26 & 337] as $\mathcal{N}(R,S)$. It follows that $\mathcal{N}_{\leq Z}(R,S)$ is a face of $\mathcal{N}(R,S)$. Similarly, for an arbitrary bipartite graph G without multiple edges, and with bipartition (U,W), $\mathcal{P}(G,b)$ is affinely isomorphic to a face of $\mathcal{P}(K_{(U,W)},b)$, where $K_{(U,W)}$ is a complete bipartite graph with bipartition (U,W). For further information regarding transportation polytopes, see, for example, De Loera and Kim [12], Klee and Witzgall [16], Schrijver [22, Sec. 21.6], or Yemelichev, Kovalev and Kravtsov [25, Ch. 6].

3. Conditions for nonemptiness of $\mathcal{P}(G,b)$ and $\mathcal{P}(G,b)_{>0}$

In this section, results which provide necessary and sufficient conditions for $\mathcal{P}(G,b)$ and $\mathcal{P}(G,b)_{>0}$ to be nonempty are obtained (where $\mathcal{P}(G,b)_{>0}$, as defined in (2), is the set of positive elements of $\mathcal{P}(G,b)$). The conditions for $\mathcal{P}(G,b)$ to be nonempty take the form of finitely-many weak inequalities for certain sums of entries of b, while the conditions for $\mathcal{P}(G,b)_{>0}$ to be nonempty take the form of finitely-many strict inequalities and equalities for certain sums of entries of b. Since $\mathcal{P}(G,b)_{>0}$ is contained in $\mathcal{P}(G,b)$, the conditions for the nonemptiness of $\mathcal{P}(G,b)_{>0}$ correspond to a strengthening of the conditions for the nonemptiness of $\mathcal{P}(G,b)$. The conditions for the nonemptiness of $\mathcal{P}(G,b)_{>0}$ will be used in Section 9.

It will be simplest, in this section, to obtain results first for the case of bipartite G (in Theorems 3 and 4), and then to use these to obtain results for the case of arbitrary G (in Theorems 6 and 7). By contrast, in later sections of the paper, results will be obtained first for arbitrary G, with results for bipartite G then following as corollaries.

At the end of this section, Theorems 6 and 7 will be verified for the case of the graph G of (3).

For subsets U and W of V, let $G[U,W]$ denote the set of all edges of G which connect a vertex of U and a vertex of W, i.e.,

$$G[U,W] := \{e \in E \mid \text{the endpoints of } e \text{ are } u \text{ and } w, \text{ for some } u \in U \text{ and } w \in W\}. \tag{14}$$

Some simple properties of such sets are that, for any $U_1, U_2, U_3 \subset V$, $G[U_1, U_2] = G[U_2, U_1]$ and $G[U_1, U_2 \cup U_3] = G[U_1, U_2] \cup G[U_1, U_3]$. Also, a vertex cover of G (i.e., a subset of V which contains an endpoint of every edge of G) is any $C \subset V$ for which $G[V \setminus C, V \setminus C] = \emptyset$

The partitioning of a set into a union of finitely-many pairwise disjoint subsets will be expressed using the notation $\Cup$. More specifically, for sets $U, W_1, W_2, \ldots, W_n$, the statement $U = W_1 \Cup W_2 \Cup \ldots \Cup W_n$ will mean that $U = W_1 \cup W_2 \cup \ldots \cup W_n$ and $W_i \cap W_j = \emptyset$ for each $i \neq j$.

Several of the results of this section will be expressed in terms of vertex covers of G. However, such results could easily be restated in terms of stable (or independent) sets of G (i.e., subsets of V which do not contain any adjacent vertices), since $S \subset V$ is a stable set of G if and only if $V \setminus S$ is a vertex cover of G.

In the first proposition of this section, it is seen that there would not be any loss of generality in the main theorems of this section if G were assumed to contain only single edges.

Proposition 1. *Let G_{rd} be a reduced graph (as defined in Section 1.3) of G. Then $\mathcal{P}(G,b)$ is nonempty if and only if $\mathcal{P}(G_{\mathrm{rd}},b)$ is nonempty, and $\mathcal{P}(G,b)_{>0}$ is nonempty if and only if $\mathcal{P}(G_{\mathrm{rd}},b)_{>0}$ is nonempty.*

This elementary result will now be proved directly. However, its validity will also follow from later theorems which give conditions for $\mathcal{P}(G,b)$ and $\mathcal{P}(G,b)_{>0}$ to be nonempty, since it will be apparent that these conditions depend only on whether or not certain pairs of vertices of G are adjacent, rather than the actual number of edges which connect such vertices.

Proof. First, assume that $\mathcal{P}(G,b) \neq \emptyset$, and choose an $x \in \mathcal{P}(G,b)$. Define $x' \in \mathbb{R}^{E'}$ by $x'_{e'} = \sum_{e \in M(e')} x_e$ for each $e' \in E'$, where E' is the edge set of G_{rd}, and $M(e')$ is the set of edges of G which have the same endpoints as e'. Then $x' \in \mathcal{P}(G_{\mathrm{rd}},b)$. Also, if $\mathcal{P}(G,b)_{>0} \neq \emptyset$ and $x \in \mathcal{P}(G,b)_{>0}$, then $x' \in \mathcal{P}(G_{\mathrm{rd}},b)_{>0}$.

Conversely, assume that $\mathcal{P}(G_{\mathrm{rd}},b) \neq \emptyset$ and choose an $x' \in \mathcal{P}(G_{\mathrm{rd}},b)$. Define $x \in \mathbb{R}^E$ by $x_e = x'_{e'}/m(e)$ for each $e \in E$, where e' is the single edge of G_{rd} which has the same endpoints as e, and $m(e)$ is the number of edges of G which have the same endpoints as e. Then $x \in \mathcal{P}(G,b)$. Also, if $\mathcal{P}(G_{\mathrm{rd}},b)_{>0} \neq \emptyset$ and $x' \in \mathcal{P}(G_{\mathrm{rd}},b)_{>0}$, then $x \in \mathcal{P}(G,b)_{>0}$. ☐

The following elementary result provides necessary conditions for $\mathcal{P}(G,b)$ and $\mathcal{P}(G,b)_{>0}$ to be nonempty.

Lemma 2.

(i) A necessary condition for $\mathcal{P}(G,b)$ to be nonempty is that $\sum_{v \in C} b_v \geq \sum_{v \in V \setminus C} b_v$ for each vertex cover C of G.

(ii) *A necessary condition for $\mathcal{P}(G,b)_{>0}$ to be nonempty is that the condition of (i) is satisfied, with its inequality holding as an equality if and only if $V\setminus C$ is also a vertex cover of G.*

Note that, by adding $\sum_{v\in C} b_v$ to each side, the inequality in (i) of this lemma is equivalent to $\sum_{v\in C} b_v \geq \frac{1}{2}\sum_{v\in V} b_v$.

Note also that, for a subset C of V, C and $V\setminus C$ are both vertex covers of G if and only if $(C, V\setminus C)$ is a bipartition for G. Hence, if G is not bipartite, then a necessary condition for $\mathcal{P}(G,b)_{>0}$ to be nonempty is that $\sum_{v\in C} b_v > \sum_{v\in V\setminus C} b_v$ for each vertex cover C of G.

Proof. Assume that $\mathcal{P}(G,b)$ is nonempty, and choose an $x\in\mathcal{P}(G,b)$. Then $\sum_{e\in\delta_G(v)} x_e = b_v$ for each $v\in V$, which gives $\sum_{v\in C} b_v = \sum_{e\in G[C,C]} \mu_e\, x_e + \sum_{e\in G[C,V\setminus C]} x_e$, for any $C\subset V$, where $\mu_e = 2$ if e is not a loop and $\mu_e = 1$ if e is a loop. Therefore,

$$\sum_{v\in C} b_v - \sum_{v\in V\setminus C} b_v = \sum_{e\in G[C,C]} \mu_e\, x_e - \sum_{e\in G[V\setminus C,V\setminus C]} \mu_e\, x_e, \tag{15}$$

for any $C\subset V$.

Part (i) of the lemma now follows from (15), and the facts that $\mu_e\, x_e \geq 0$ for each $e\in E$, and $G[V\setminus C, V\setminus C] = \emptyset$ if C is a vertex cover. Part (ii) of the lemma follows from (15) by assuming that $\mathcal{P}(G,b)_{>0}$ is nonempty, choosing $x\in\mathcal{P}(G,b)_{>0}$, and using the facts that $G[C,C]\neq\emptyset = G[V\setminus C, V\setminus C]$ if C is a vertex cover and $V\setminus C$ is not a vertex cover, while $G[C,C] = G[V\setminus C, V\setminus C] = \emptyset$ if C and $V\setminus C$ are both vertex covers. $\square$

The next two results, Theorems 3 and 4, state that if G is bipartite, then the conditions of Lemma 2 are also sufficient to ensure that $\mathcal{P}(G,b)$ and (for $E\neq\emptyset$) $\mathcal{P}(G,b)_{>0}$ are nonempty.

Theorem 3. *Let G be bipartite. Then a necessary and sufficient condition for $\mathcal{P}(G,b)$ to be nonempty is that $\sum_{v\in C} b_v \geq \sum_{v\in V\setminus C} b_v$ for each vertex cover C of G.*

It can be seen that, if (U,W) is a bipartition for G, then the condition of the theorem is equivalent to the alternative condition that $\sum_{v\in U_1} b_v + \sum_{v\in W_1} b_v \geq \sum_{v\in U_2} b_v + \sum_{v\in W_2} b_v$ for all sets U_1, U_2, W_1 and W_2 such that $U = U_1 \uplus U_2$, $W = W_1 \uplus W_2$ and $G[U_2, W_2] = \emptyset$. (In particular, for C satisfying the condition of the theorem, set $U_1 = U\cap C$, $U_2 = U\setminus C$, $W_1 = W\cap C$ and $W_2 = W\setminus C$, and conversely, for U_1, U_2, W_1 and W_2 satisfying the alternative condition, set $C = U_1\cup W_1$.) It can also be checked that the alternative condition remains unchanged if its single inequality is replaced by $\sum_{v\in U_1} b_v \geq \sum_{v\in W_2} b_v$ and $\sum_{v\in W_1} b_v \geq \sum_{v\in U_2} b_v$, by $\sum_{v\in U} b_v = \sum_{v\in W} b_v$ and $\sum_{v\in U_1} b_v \geq \sum_{v\in W_2} b_v$, or by $\sum_{v\in U} b_v = \sum_{v\in W} b_v$ and $\sum_{v\in W_1} b_v \geq \sum_{v\in U_2} b_v$. (For example, the condition $\sum_{v\in U} b_v = \sum_{v\in W} b_v$ follows from the condition of the theorem by using the vertex covers $C = U$ and $C = W$.)

It can also be seen that, if the condition of this theorem is satisfied, and if C and $V\setminus C$ are both vertex covers of G, then $\sum_{v\in C} b_v = \sum_{v\in V\setminus C} b_v$ (i.e., the inequality then holds as an equality).

It can also be seen that, if the condition of this theorem is satisfied, and if C and $V \setminus C$ are both vertex covers of G, then $\sum_{v\in C} b_v = \sum_{v\in V\setminus C} b_v$ (i.e., the inequality then holds as an equality).

This theorem is a standard result. See, for example, Schrijver [22, Thm. 21.11]. It can be proved using linear programming duality (as done in the proof given by Schrijver [22, Thm. 21.11]), or using standard theorems from network flow theory. (For example, it follows from Schrijver [22, Cor. 11.2h] by using a directed graph which is formed from G by directing each edge from U to W, where (U, W) is a bipartition for G). For completeness, a proof will also be given here. This is a direct and self-contained proof, which uses an approach based on that used by Schrijver [22, Thms. 10.3 and 11.2] for proofs of the max-flow min-cut theorem and Hoffman's circulation theorem.

Proof. The necessity of the condition is given by (i) of Lemma 2.

The sufficiency of the condition will be obtained by using a bipartition (U, W) for G, and showing that if $\mathcal{P}(G, b)$ is empty and $\sum_{v\in U} b_v = \sum_{v\in W} b_v$, then there exist U_1, U_2, W_1 and W_2 such that $U = U_1 \uplus U_2$, $W = W_1 \uplus W_2$, $G[U_2, W_2] = \emptyset$ and $\sum_{v\in W_1} b_v < \sum_{v\in U_2} b_v$.

So, assume that $\mathcal{P}(G, b) = \emptyset$ and $\sum_{v\in U} b_v = \sum_{v\in W} b_v$. For any $x \in \mathbb{R}^E_{\ge 0}$, let $f(x) = \sum_{v\in V} \bigl|\sum_{e\in\delta_G(v)} x_e - b_v\bigr|$. (Note that $f(x) > 0$ for all $x \in \mathbb{R}^E_{\ge 0}$, since $\mathcal{P}(G, b) = \emptyset$.) Now choose an x which minimizes f over $\mathbb{R}^E_{\ge 0}$, where the forms of $\mathbb{R}^E_{\ge 0}$ and f guarantee the existence of such an x. (In particular, the polyhedron $\mathbb{R}^E_{\ge 0}$ can be subdivided into finitely-many nonempty polyhedra, on each of which f is a positive affine function. Specifically, each such polyhedron P has the form $\{x \in \mathbb{R}^E_{\ge 0} \mid \sigma(P)_v \bigl(\sum_{e\in\delta_G(v)} x_e - b_v\bigr) \ge 0$ for each $v \in V\}$, for some assignment $\sigma(P)_v$ of -1 or 1 to each $v \in V$, so that $f(x) = \sum_{v\in V} \sigma(P)_v \bigl(\sum_{e\in\delta_G(v)} x_e - b_v\bigr)$ for all $x \in P$. The standard fact, as given for example in Korte and Vygen [17, Prop. 3.1], that a real affine function which is bounded below on a nonempty polyhedron attains a minimum over the polyhedron then implies that f attains a minimum over $\mathbb{R}^E_{\ge 0}$.)

Define $S = \{u \in U \mid \sum_{e\in\delta_G(u)} x_e < b_u\} \cup \{w \in W \mid \sum_{e\in\delta_G(w)} x_e > b_w\}$ and $T = \{u \in U \mid \sum_{e\in\delta_G(u)} x_e > b_u\} \cup \{w \in W \mid \sum_{e\in\delta_G(w)} x_e < b_w\}$. Since $\mathcal{P}(G, b) = \emptyset$, $S \cup T$ is nonempty. It then follows, using $\sum_{v\in U} b_v = \sum_{v\in W} b_v$, that S and T are each nonempty (since $S \neq \emptyset$ and $T = \emptyset$ would give $\sum_{v\in U} b_v > \sum_{v\in W} b_v$, while $S = \emptyset$ and $T \neq \emptyset$ would give $\sum_{v\in U} b_v < \sum_{v\in W} b_v$). Now define

$$S' = \{v \in V \mid \text{there exists } s \in S \text{ and a path } P \text{ in } G \text{ from } s \text{ to } v \text{ satisfying}$$
$$x_e > 0 \text{ for each edge } e \text{ corresponding to a step of } P \text{ from } W \text{ to } U\}, \quad (16)$$

i.e., S' is the set of vertices of G which are reachable from S by a path P with the property that x_e is positive for each edge e which corresponds to a step of P from W to U. It follows immediately that $S \subset S'$, $G[U \cap S', W \setminus S'] = \emptyset$, and $x_e = 0$ for each $e \in G[W \cap S', U \setminus S']$. Also, $S' \cap T = \emptyset$, where this can be deduced as follows. If $S' \cap T$ were nonempty, then there would exist $s \in S$, $t \in T$ and a path P from s to t satisfying the property of (16). Taking $y \in \mathbb{R}^E$ as $y_e = \epsilon$ for each edge e corresponding to a step of P from U to W, $y_e = -\epsilon$ for each edge e corresponding to a step of P from W to U, and $y_e = 0$ for each edge e not in P, it would follow that, for sufficiently small $\epsilon > 0$, $x + y \in \mathbb{R}^E_{\geq 0}$ and $f(x+y) < f(x)$, but this is impossible since x minimizes f over $\mathbb{R}^E_{\geq 0}$.

Now define $U_1 = U \setminus S'$, $U_2 = U \cap S'$, $W_1 = W \cap S'$ and $W_2 = W \setminus S'$. Then $U = U_1 \uplus U_2$, $W = W_1 \uplus W_2$, $G[U_2, W_2] = \emptyset$, and $x_e = 0$ for each $e \in G[U_1, W_1]$. Also, $\sum_{e\in\delta_G(u)} x_e \leq b_u$ for each $u \in U_2$, and $\sum_{e\in\delta_G(w)} x_e \geq b_w$ for each $w \in W_1$ (since $S \subset S' = U_2 \cup W_1$ and $T \subset V \setminus S'$), with strict inequality holding for at least one $u \in U_2$ or $w \in W_1$ (since $S \neq \emptyset$). Therefore $\sum_{w\in W_1} b_w \leq \sum_{e\in G[U,W_1]} x_e = \sum_{e\in G[U_2,W_1]} x_e = \sum_{e\in G[U_2,W]} x_e \leq \sum_{u\in U_2} b_u$, with at least one of the inequalities holding strictly, so that $\sum_{v\in W_1} b_v < \sum_{v\in U_2} b_v$, as required. □

Theorem 4. *Let G be bipartite, with E nonempty. Then a necessary and sufficient condition for $\mathcal{P}(G,b)_{>0}$ to be nonempty is that $\sum_{v\in C} b_v \geq \sum_{v\in V\setminus C} b_v$ for each vertex cover C of G (i.e., the condition of Theorem 3 is satisfied), with the inequality holding as an equality if and only if $V \setminus C$ is also a vertex cover of G.*

Note that if (U,W) is a bipartition for G, then the condition of the theorem is equivalent to the condition that $\sum_{v\in U_1} b_v + \sum_{v\in W_1} b_v \geq \sum_{v\in U_2} b_v + \sum_{v\in W_2} b_v$ for all sets U_1, U_2, W_1 and W_2 such that $U = U_1 \uplus U_2$, $W = W_1 \uplus W_2$ and $G[U_2, W_2] = \emptyset$, with the inequality holding as an equality if and only if $G[U_1, W_1] = \emptyset$. Furthermore, this condition remains unchanged if its inequality is replaced by $\sum_{v\in U_1} b_v \geq \sum_{v\in W_2} b_v$, or by $\sum_{v\in W_1} b_v \geq \sum_{v\in U_2} b_v$ (since if the condition, in any of these forms, is satisfied, then taking $U_1 = U$, $W_2 = W$ and $U_2 = W_1 = \emptyset$, or $U_2 = U$, $W_1 = W$ and $U_1 = W_2 = \emptyset$, gives $\sum_{v\in U} b_v = \sum_{v\in W} b_v$).

This theorem, stated in terms of matrices, is due to Brualdi. See [7, Thm. 2.1], [8, Thm. 2.7] and [9, Thm. 8.1.7]. The statement given by Brualdi can be translated to that given here using the correspondence, discussed in Section 2, between $\mathcal{P}(G,b)$ and $\mathcal{N}_{\leq Z}(R,S)$ for the case in which G is bipartite and does not contain multiple edges, and Proposition 1.

Proof. The necessity of the condition is given by (ii) of Lemma 2.

Proceeding to the proof of sufficiency, define $d \in \mathbb{R}^V$ by $d_v = |\delta_G(v)|$ for each $v \in V$ (i.e., d_v is the degree of v), and define $y \in \mathbb{R}^E$ (where $E \neq \emptyset$ ensures that $\mathbb{R}^E \neq \{0\}$) by $y_e = 1$ for each $e \in E$. Then $y \in \mathcal{P}(G,d)_{>0}$. Therefore, using (ii) of Lemma 2, $\sum_{v\in C} d_v \geq \sum_{v\in V\setminus C} d_v$ for each vertex cover C of G, with equality holding if and only if $V \setminus C$ is also a vertex cover of G.

Now assume that the condition of the theorem is satisfied. It can then be shown that $b_v > 0$ for each $v \in V$ with $d_v > 0$, i.e., for each nonisolated vertex v. (More specifically, this can be done by considering a nonisolated vertex w, and a bipartition (U,W) for G, with $w \in W$. Then, choosing the vertex cover $C = U$ gives $\sum_{v\in U} b_v = \sum_{v\in W} b_v$, while choosing the vertex cover $C = U \cup \{w\}$ gives $\sum_{v\in U\cup\{w\}} b_v > \sum_{v\in W\setminus\{w\}} b_v$, from which it follows that $b_w > 0$.)

Now choose an $\epsilon > 0$ which satisfies $\epsilon\, d_v \leq b_v$ for each $v \in V$, and $\epsilon\,(\sum_{v\in C} d_v - \sum_{v\in V\setminus C} d_v) \leq \sum_{v\in C} b_v - \sum_{v\in V\setminus C} b_v$ for each vertex cover C of G, where the conditions satisfied by b and d guarantee the existence of such an ϵ. It follows that $b-\epsilon\, d$ has all of its entries nonnegative, and satisfies the condition of Theorem 3 (i.e., $\sum_{v\in C}(b_v-\epsilon\, d_v)\ge\sum_{v\in V\setminus C}(b_v-\epsilon\, d_v)$ for each vertex cover C of G), so that $\mathcal{P}(G,b-\epsilon\, d)\neq\emptyset$. Finally, choose an $x\in\mathcal{P}(G,b-\epsilon\, d)$. Then it can be seen that $x+\epsilon\, y\in\mathcal{P}(G,b)_{>0}$. $\square$

Theorems 3 and 4, which apply to the case of bipartite G, can now be used to give analogous results for the case of arbitrary G. These results will be expressed in preliminary forms in Lemma 5, and then restated in more compact forms in Theorems 6 and 7.

Lemma 5.

(i) A necessary and sufficient condition for $\mathcal{P}(G,b)$ to be nonempty is that $\sum_{v\in U_1}b_v\ge\sum_{v\in W_2}b_v$ for all sets U_1, U_2, W_1 and W_2 such that $V=U_1\uplus U_2=W_1\uplus W_2$ and $G[U_2,W_2]=\emptyset$.

(ii) Let E be nonempty. A necessary and sufficient condition for $\mathcal{P}(G,b)_{>0}$ to be nonempty is that the condition of (i) is satisfied, with its inequality holding as an equality if and only if $G[U_1,W_1]=\emptyset$.

Note that the conditions in the lemma remain unchanged if the inequality is replaced by $\sum_{v\in W_1}b_v\ge\sum_{v\in U_2}b_v$.

Proof. Let G' be a bipartite double graph of G. Specifically, let G' have vertex set $V'=V\times\{1,2\}$ and edge set $E'=E\times\{1,2\}$, where if $e\in E$ connects vertices u and w of V, then one of the edges $(e,1)$ or $(e,2)$ of E' connects vertices $(u,1)$ and $(w,2)$ of V', while the other connects vertices $(w,1)$ and $(u,2)$ of V'. Also define $b'\in\mathbb{R}^{V'}$ by $b'_{(v,1)}=b'_{(v,2)}=b_v$ for each $v\in V$. It can now be checked that $\mathcal{P}(G,b)\neq\emptyset$ if and only if $\mathcal{P}(G',b')\neq\emptyset$. In particular, if there exists $x\in\mathcal{P}(G,b)$, then there exists $x'\in\mathcal{P}(G',b')$ given by $x'_{(e,1)}=x'_{(e,2)}=\mu_e x_e/2$ for each $e\in E$, where $\mu_e=2$ if e is not a loop and $\mu_e=1$ if e is a loop. Conversely, if there exists $x'\in\mathcal{P}(G',b')$, then there exists $x\in\mathcal{P}(G,b)$ given by $x_e=(x'_{(e,1)}+x'_{(e,2)})/\mu_e$ for each $e\in E$. It follows similarly that $\mathcal{P}(G,b)_{>0}\neq\emptyset$ if and only if $\mathcal{P}(G',b')_{>0}\neq\emptyset$.

Since G' is bipartite, with bipartition $(V\times\{1\},V\times\{2\})$, it follows from Theorem 3 (using one of the alternative forms given after the statement of that theorem, and noting that $\sum_{v\in V\times\{1\}}b'_v=\sum_{v\in V\times\{2\}}b'_v$), that a necessary and sufficient condition for $\mathcal{P}(G,b)$ to be nonempty is that $\sum_{v\in U'_1}b'_v\ge\sum_{v\in W'_2}b'_v$ for all sets U'_1, U'_2, W'_1 and W'_2 such that $V\times\{1\}=U'_1\uplus U'_2$, $V\times\{2\}=W'_1\uplus W'_2$ and $G'[U'_2,W'_2]=\emptyset$.

Similarly, it follows from Theorem 4 (using one of the alternative forms given after the statement of that theorem), that a necessary and sufficient condition for $\mathcal{P}(G,b)_{>0}$ to be nonempty is that the previous condition for the nonemptiness of $\mathcal{P}(G,b)$ is satisfied, with its inequality holding as an equality if and only if $G'[U'_1,W'_1]=\emptyset$.

Finally, it can easily be seen that the previous two conditions are equivalent to the corresponding conditions of the lemma. $\square$

Theorem 6. *A necessary and sufficient condition for $\mathcal{P}(G,b)$ to be nonempty is that $\sum_{v\in V_1} b_v \geq \sum_{v\in V_3} b_v$ for all sets V_1, V_2 and V_3 such that $V = V_1 \uplus V_2 \uplus V_3$ and $G[V_2 \cup V_3, V_3] = \emptyset$.*

Note that the appearance of V_2 in this theorem could be removed by rewriting the condition as $\sum_{v\in V_1} b_v \geq \sum_{v\in V_3} b_v$ for all disjoint subsets V_1 and V_3 of V such that $G[V \setminus V_1, V_3] = \emptyset$. Note also that $G[V_2 \cup V_3, V_3] = \emptyset$ is equivalent to $G[V_2, V_3] = G[V_3, V_3] = \emptyset$.

Furthermore, in the condition of the theorem, V_3 can be restricted to being nonempty, since if sets V_1, V_2 and V_3 satisfy $V = V_1 \uplus V_2 \uplus V_3$ and $V_3 = \emptyset$, then $G[V_2 \cup V_3, V_3] = \emptyset$ and $\sum_{v\in V_1} b_v \geq \sum_{v\in V_3} b_v$ $(= 0)$ are automatically satisfied.

It can also be seen that, if the condition of this theorem is satisfied, and if sets V_1, V_2 and V_3 satisfy $V = V_1 \uplus V_2 \uplus V_3$ and $G[V_1, V_1 \cup V_2] = G[V_2 \cup V_3, V_3] = \emptyset$, then $\sum_{v\in V_1} b_v = \sum_{v\in V_3} b_v$ (i.e., the inequality then holds as an equality).

Proof. It will be shown that the condition of the theorem is equivalent to the condition of (i) of Lemma 5. (Alternatively, the necessity of the condition of the theorem could easily be proved directly.)

First, let the condition of (i) of Lemma 5 be satisfied, and consider any sets V_1, V_2 and V_3 for which $V = V_1 \uplus V_2 \uplus V_3$ and $G[V_2 \cup V_3, V_3] = \emptyset$. Now take U_1, U_2, W_1 and W_2 to be $U_1 = V_1$, $U_2 = V_2 \cup V_3$, $W_1 = V_1 \cup V_2$ and $W_2 = V_3$. Then $V = U_1 \uplus U_2 = W_1 \uplus W_2$ and $G[U_2, W_2] = \emptyset$, so that, since the condition of (i) of Lemma 5 is satisfied, $\sum_{v\in U_1} b_v \geq \sum_{v\in W_2} b_v$. Therefore $\sum_{v\in V_1} b_v \geq \sum_{v\in V_3} b_v$, and hence the condition of the theorem is satisfied.

Conversely, let the condition of the theorem be satisfied, and consider any sets U_1, U_2, W_1 and W_2 for which $V = U_1 \uplus U_2 = W_1 \uplus W_2$ and $G[U_2, W_2] = \emptyset$. Now take V_1, V_2', V_2'', V_2 and V_3 to be $V_1 = U_1 \cap W_1$, $V_2' = U_2 \cap W_1$, $V_2'' = U_1 \cap W_2$, $V_2 = V_2' \cup V_2''$ and $V_3 = U_2 \cap W_2$. Then $U_1 = V_1 \cup V_2''$, $U_2 = V_2' \cup V_3$, $W_1 = V_1 \cup V_2'$, $W_2 = V_2'' \cup V_3$, and $V = V_1 \uplus V_2 \uplus V_3$. Also, $\emptyset = G[U_2, W_2] = G[V_2' \cup V_3, V_2'' \cup V_3] = G[V_2' \cup V_3, V_3] \cup G[V_2'' \cup V_3, V_3] \cup G[V_2', V_2''] = G[V_2 \cup V_3, V_3] \cup G[V_2', V_2'']$, and therefore $G[V_2 \cup V_3, V_3] = \emptyset$. So, since the condition of the theorem is satisfied, $\sum_{v\in V_1} b_v \geq \sum_{v\in V_3} b_v$, which gives $\sum_{v\in V_1\cup V_2''} b_v \geq \sum_{v\in V_3\cup V_2''} b_v$, and thus $\sum_{v\in U_1} b_v \geq \sum_{v\in W_2} b_v$. Hence, the condition of (i) of Lemma 5 is satisfied. $\square$

Theorem 7. *Let E be nonempty. A necessary and sufficient condition for $\mathcal{P}(G,b)_{>0}$ to be nonempty is that $\sum_{v\in V_1} b_v \geq \sum_{v\in V_3} b_v$ for all sets V_1, V_2 and V_3 such that $V = V_1 \uplus V_2 \uplus V_3$ and $G[V_2 \cup V_3, V_3] = \emptyset$ (i.e., the condition of Theorem 6 is satisfied), with the inequality holding as an equality if and only if $G[V_1, V_1 \cup V_2] = \emptyset$.*

This theorem, stated in terms of matrices, is due to Brualdi. See [8, Thm. 3.7] and [9, Thm. 8.2.3]. The statement given by Brualdi can be translated to that given here using the correspondence, discussed in Section 2, between $\mathcal{P}(G,b)$ and $\mathcal{N}_{\leq Z}(R)$ for the case in which G does not contain multiple edges, and Proposition 1.

Note that if G does not have any isolated vertices, then the condition of the theorem taken with $V_1 = \{u\}$, $V_2 = V \setminus \{u\}$ and $V_3 = \emptyset$, for each $u \in V$, simply gives the condition that b is positive (since in these cases $G[V_2\cup V_3,V_3]=\emptyset$ and $G[V_1,V_1\cup V_2]\neq\emptyset$, so the condition gives $\sum_{v\in V_1}b_v>\sum_{v\in V_3}b_v$, i.e., $b_u>0$).

Proof. It will be shown, by extending the proof of Theorem 6, that the condition of the theorem is equivalent to the condition of (ii) of Lemma 5. (Again, the necessity of the condition of the theorem could easily be proved directly instead.)

First, let the condition of (ii) of Lemma 5 be satisfied, consider any sets V_1, V_2 and V_3 for which $V=V_1\Cup V_2\Cup V_3$ and $G[V_2\cup V_3,V_3]=\emptyset$, and take U_1, U_2, W_1 and W_2 to be the same as in the first part of the proof of Theorem 6. Then $V=U_1\Cup U_2=W_1\Cup W_2$, $G[U_2,W_2]=G[V_2\cup V_3,V_3]=\emptyset$, $G[U_1,W_1]=G[V_1,V_1\cup V_2]$, and $\sum_{v\in U_1}b_v-\sum_{v\in W_2}b_v=\sum_{v\in V_1}b_v-\sum_{v\in V_3}b_v$. It can now be seen that the condition of the theorem is satisfied, since the condition of (ii) of Lemma 5 is satisfied.

Conversely, let the condition of the theorem be satisfied, consider any sets U_1, U_2, W_1 and W_2 for which $V=U_1\Cup U_2=W_1\Cup W_2$ and $G[U_2,W_2]=\emptyset$, and take V_1, V_2', V_2'', V_2 and V_3 to be the same as in the second part of the proof of Theorem 6. Then $V=V_1\Cup V_2\Cup V_3$, and $\emptyset=G[U_2,W_2]=G[V_2\cup V_3,V_3]\cup G[V_2',V_2'']$, so that $G[V_2\cup V_3,V_3]=G[V_2',V_2'']=\emptyset$. Also, $G[U_1,W_1]=G[V_1\cup V_2'',V_1\cup V_2']=G[V_1,V_1\cup V_2']\cup G[V_1,V_1\cup V_2'']\cup G[V_2',V_2'']=G[V_1,V_1\cup V_2]$ (using $G[V_2',V_2'']=\emptyset$), and $\sum_{v\in V_1}b_v-\sum_{v\in V_3}b_v=\sum_{v\in V_1\cup V_2''}b_v-\sum_{v\in V_3\cup V_2''}b_v=\sum_{v\in U_1}b_v-\sum_{v\in W_2}b_v$. It can now be seen that the condition of (ii) of Lemma 5 is satisfied, since the condition of the theorem is satisfied □

Theorems 6 and 7 will now be illustrated using the example of G given by (3). For this case, the set triples (V_1,V_2,V_3) which satisfy $V=\{1,2,3\}=V_1\Cup V_2\Cup V_3$ and $G[V_2\cup V_3,V_3]=\emptyset$ are $(\{1,3\},\emptyset,\{2\})$, $(\{2,3\},\emptyset,\{1\})$ and $(U,\{1,2,3\}\backslash U,\emptyset)$, for each $U\subset\{1,2,3\}$. Therefore, using Theorem 6, and the fact (as pointed out after the statement of the theorem) that V_3 can be restricted to being nonempty (since $\sum_{v\in U}b_v\geq 0$ is automatically satisfied for each $U\subset V$), it follows that a necessary and sufficient condition for $\mathcal{P}(G,b)$ to be nonempty is that $b_1+b_3\geq b_2$ and $b_2+b_3\geq b_1$, which coincides with the condition already found in (6).

Among the set triples (V_1,V_2,V_3) which satisfy $V=V_1\Cup V_2\Cup V_3$ and $G[V_2\cup V_3,V_3]=\emptyset$, the only one which satisfies $G[V_1,V_1\cup V_2]=\emptyset$ is $(\emptyset,V,\emptyset)$, for which $\sum_{v\in V_1}b_v=\sum_{v\in V_3}b_v$ $(=0)$ is automatically satisfied. Therefore, using Theorem 7, a necessary and sufficient condition for $\mathcal{P}(G,b)_{>0}$ to be nonempty is that $b_1+b_3>b_2$, $b_2+b_3>b_1$ and $\sum_{v\in U}b_v>0$, for each nonempty $U\subset V$, which can be seen to coincide with the condition already found in (7).

Further examples involving some of the conditions of this section will be considered at the end of Section 9.

4. Relevant results for graphs

In this section, some relevant general results concerning the incidence matrix I_G of an arbitrary graph G (which may contain loops and multiple edges) are obtained. Most of these results involve the nullity of I_G with respect to the field $\mathbb{R}$, i.e., the dimension of the kernel, or nullspace, of I_G with respect to $\mathbb{R}$, where this kernel is explicitly $\{x \in \mathbb{R}^E \mid I_G\,x = 0\} = \{x \in \mathbb{R}^E \mid \sum_{e\in\delta_G(v)} x_e = 0 \text{ for each } v \in V\}$. The results will be applied to $\mathcal{P}(G,b)$ in Section 7.

Note that in the literature, the field $\{0,1\}$ is often used instead of $\mathbb{R}$, and in this case the kernel of I_G is the so-called cycle space of G. Note also that various results which are closely related to those of this section have appeared in the literature. See, for example, Akbari, Ghareghani, Khosrovshahi and Maimani [2, Thm. 2], or Villarreal [24, Cor. 3.2].

The rank and nullity of a real matrix A, with respect to the field $\mathbb{R}$, will be denoted as $\operatorname{rank}(A)$ and $\operatorname{nullity}(A)$, respectively.

Proposition 8. *The nullity of the incidence matrix of G is $|E|-|V|+B$, where B is the number of bipartite components of G.*

Note that the fact that $\operatorname{rank}(A) = \operatorname{rank}(A^T) = n - \operatorname{nullity}(A)$, for any real matrix A with n columns, implies that the result of this proposition is equivalent to

$$\operatorname{rank}(I_G) = |V| - B, \tag{17}$$

and to

$$\operatorname{nullity}({I_G}^T) = B, \tag{18}$$

where B is again the number of bipartite components of G. These results, at least for the case of graphs without loops or multiple edges, are standard. (See, for example, Godsil and Royle [13, Thm. 8.2.1].)

Note also that if G is bipartite and planar, then it follows from this proposition, and Euler's formula for planar graphs (which remains valid for graphs with multiple edges), that the nullity of the incidence matrix of G is the number of bounded faces in a planar embedding of G.

Proof. The validity of the form (18) of the proposition will be confirmed.

The kernel of ${I_G}^T$ is $\{y \in \mathbb{R}^V \mid {I_G}^T y = 0\} = \{y \in \mathbb{R}^V \mid y_u = -y_w \text{ for all pairs } u, w \text{ of adjacent vertices of } G\}$. By considering pairs of adjacent vertices successively along paths through each component of G, forming a bipartition (U_C, W_C) for each bipartite component C, and using the fact that a nonbipartite component contains an odd-length cycle, it can be seen that the general solution of the equations for y is

$$y_v = \begin{cases} \lambda_C, & v \in U_C, \\ -\lambda_C, & v \in W_C, \\ 0, & v \text{ is a vertex of a nonbipartite component,} \end{cases}$$

where $\lambda_C \in \mathbb{R}$ is arbitrary for each C. It now follows that $\operatorname{nullity}({I_G}^T) = B$. $\square$

Proposition 9. *The nullity of the incidence matrix of G is zero if and only if each component of G either is acyclic or else contains exactly one cycle with that cycle having odd length.*

In this proposition, the choice of conditions for the components of G applies independently to each component. An alternative statement of the proposition is that the nullity of the incidence matrix of G is zero if and only if G has no even-length cycles and no component containing more than one odd-length cycle. It can also be seen that $\operatorname{nullity}(I_G)=0$ is equivalent to the condition that $x=0$ is the only $x\in\mathbb{R}^E$ which satisfies $\sum_{e\in\delta_G(v)}x_e=0$ for each $v\in V$.

Note that for the case of bipartite G, it follows from this proposition, and the fact that a bipartite graph does not contain any odd-length cycles, that the nullity of the incidence matrix of G is zero if and only if G is a forest.

Proof. The kernel of I_G is the direct sum of the kernels of the incidence matrices of its components. Therefore, $\operatorname{nullity}(I_G)=0$ if and only if $\operatorname{nullity}(I_C)=0$ for each component C of G. Applying Proposition 8, these equations are $|E_C|+1=|V_C|$ for each bipartite component C of G, and $|E_C|=|V_C|$ for each nonbipartite component C of G, where E_C and V_C are the edge and vertex sets of C. Using the fact that a connected graph C satisfies $|E_C|+1=|V_C|$ if and only if C is acyclic, and satisfies $|E_C|=|V_C|$ if and only if C contains exactly one cycle, it now follows that $\operatorname{nullity}(I_G)=0$ if and only if each bipartite component of G is acyclic, and each nonbipartite component of G contains exactly one cycle. Finally, using the fact that a graph is bipartite if and only if it does not contain any odd-length cycles, it follows that $\operatorname{nullity}(I_G)=0$ if and only if each component of G either is acyclic or else contains exactly one cycle with that cycle having odd length. ☐

Propositions 8 and 9 can also be proved more directly. Such alternative proofs provide further insight into these results, so will now be outlined briefly.

Alternative proof of Proposition 9. First, let each component of G either be acyclic or else contain exactly one cycle with that cycle having odd length, and let $x\in\mathbb{R}^E$ satisfy $I_G\,x=0$, i.e., $\sum_{e\in\delta_G(v)}x_e=0$ for each $v\in V$. It follows immediately that $x_e=0$ for each pendant edge e (i.e., an edge incident to a univalent vertex). By iteratively deleting such edges from E and considering the equation for x at each univalent vertex v in the resulting smaller graph, it then follows that $x_e=0$ for all edges e of E, except possibly those which are part of disjoint cycles, where the length of each such cycle is odd and at least 3. But if $e_1,\ldots,e_n$ are the edges of such a cycle, then the associated entries of x satisfy $x_{e_n}+x_{e_1}=x_{e_1}+x_{e_2}=x_{e_2}+x_{e_3}=\ldots=x_{e_{n-1}}+x_{e_n}=0$, and the fact that n is odd implies that all of these entries are also 0. Therefore, $x=0$ is the only solution of $I_G\,x=0$, and so $\operatorname{nullity}(I_G)=0$.

Now, conversely, let it not be the case that each component of G is acyclic or contains exactly one cycle with that cycle having odd length. Then G contains an even-length cycle or two odd-length cycles connected by a path. (It is assumed here that the two odd-length cycles either share no vertices, or else share only one vertex, in which case the connecting path has length zero. For if G contains two odd-length cycles which share more than one vertex, then G also has an even-length cycle, comprised of certain segments of the odd-length cycles.) If G has an even-length cycle, then there exists $x \in \mathbb{R}^E$ which satisfies $I_G\,x = 0$, where x_e is alternately 1 and -1 for each edge e along the cycle, and $x_e = 0$ for each edge e not in the cycle. If G contains two odd-length cycles connected by a path, then it can be seen that there exists $x \in \mathbb{R}^E$ with $|x_e| = 2$ for e in the path or for e a loop, $|x_e| = 1$ for e in a nonloop cycle, and $x_e = 0$ for e not in the path or either cycle, and where signs are assigned to the nonzero entries of x so that $I_G\,x = 0$. (In the case in which G contains two loops connected by a path, there also exists $x \in \mathbb{R}^E$ with $|x_e| = 1$ if e is in the path or is one of the loops, and $x_e = 0$ otherwise.) Therefore, in each of these cases, there exists a nonzero $x \in \mathbb{R}^E$ which satisfies $I_G\,x = 0$, and so $\text{nullity}(I_G) > 0$. □

Alternative proof of Proposition 8. In this proof, the arguments used in the alternative proof of Proposition 9 will be used to construct an explicit basis for the kernel of I_G. Let H be any spanning subgraph of G with the property that, for each component C of G, the subgraph of H induced by the vertices of C is a tree if C is bipartite, and is connected and contains exactly one cycle with that cycle having odd length if C is nonbipartite. The existence of such a H is guaranteed by the facts that a connected graph has a spanning tree and that a nonbipartite graph has an odd-length cycle. It follows from the formulae relating numbers of edges and vertices in trees and in connected graphs with exactly one cycle that $|E'| = |V| - B$, and so $|E \setminus E'| = |E| - |V| + B$, where E' is the edge set of H and B is the number of bipartite components of G. It can also be seen that, for each $f \in E \setminus E'$, the spanning subgraph of G with edge set $E' \cup \{f\}$ has an even-length cycle containing f, or two odd-length cycles (which share at most one vertex) connected by a path, with one of those cycles containing f. Therefore, using the same argument as in the second part of the alternative proof of Proposition 9, for each $f \in E \setminus E'$, there exists $x(f) \in \mathbb{R}^E$ satisfying the properties that $I_G\,x(f) = 0$, $x(f)_f \neq 0$, and the edges e for which $x(f)_e \neq 0$ are all contained in $E' \cup \{f\}$ and form either a single even-length cycle or two odd-length cycles connected by a path. Choosing a particular such $x(f)$ for each $f \in E \setminus E'$, it follows immediately that these are $|E| - |V| + B$ linearly independent elements of the kernel of I_G.

It will now be shown that these vectors also span the kernel of I_G. First, let y be any vector in the kernel of I_G, and set $y' = \sum_{f \in E \setminus E'} y_f\, x(f)/x(f)_f$ (with $y' = 0$ if $E \setminus E' = \emptyset$). Then $y'_e = y_e$ for each $e \in E \setminus E'$ (since $x(f)_e = 0$ for all $e \in E \setminus (E' \cup \{f\})$). Also, $I_G\,y = I_G\,y' = I_G(y - y') = 0$, and using the same argument as in the first part of the alternative proof of Proposition 9, it then follows that $(y - y')_e = 0$ for each $e \in E'$, so that $y = y'$.

Therefore, the vectors $x(f)$ with $f \in E \setminus E'$ form a basis of the kernel of I_G, and $\text{nullity}(I_G) = |E| - |V| + B$. □

Proposition 10. *Consider an $a \in \mathbb{R}^V$, and for each bipartite component C of G let (U_C, W_C) be a bipartition for C. Then a necessary and sufficient condition for there to* *exist an* $x \in \mathbb{R}^E$ *with* $I_G\, x = a$ *is that*

$$\sum_{v \in U_C} a_v = \sum_{v \in W_C} a_v, \quad \textit{for each bipartite component } C \textit{ of } G. \tag{19}$$

Note that this result provides a necessary and sufficient condition for there to exist an assignment of real numbers to the edges of G such that the sum of the numbers over all edges incident to any vertex v is a prescribed real number a_v.

Proof. Consider any $x \in \mathbb{R}^E$, and any bipartite component C of G. It can be seen that $\sum_{v \in U_C} \sum_{e \in \delta_G(v)} x_e = \sum_{v \in W_C} \sum_{e \in \delta_G(v)} x_e$, since each side is the sum of x_e over all edges e of C. Therefore,

$$\textstyle\sum_{v \in W_C} \Bigl(\sum_{e \in \delta_G(v)} x_e - a_v\Bigr) - \sum_{v \in U_C} \Bigl(\sum_{e \in \delta_G(v)} x_e - a_v\Bigr) = \sum_{v \in U_C} a_v - \sum_{v \in W_C} a_v. \tag{20}$$

If $I_G\, x = a$ then the LHS of (20) immediately vanishes, and so (20) implies that (19) is satisfied. Conversely, if (19) is satisfied then the RHS of (20) immediately vanishes, and so (20) enables an equation $\sum_{e \in \delta_G(v)} x_e = a_v$ for a single vertex v of each bipartite component of G to be eliminated from the $|V|$ constituent equations of $I_G\, x = a$. This leaves $|V| - B$ equations, where B is the number of bipartite components of G. Using (17), these remaining equations are linearly independent, and therefore have a solution. □

Proposition 11. *Consider an* $a \in \mathbb{R}^V$*, and for each bipartite component* C *of* G *let* (U_C, W_C) *be a bipartition for* C*.*

(i) A necessary and sufficient condition for there to exist a unique $x \in \mathbb{R}^E$ *with* $I_G\, x = a$ *is that* (19) *is satisfied, and each component of* G *either is acyclic or else contains exactly one cycle with that cycle having odd length.*

(ii) If the condition of (i) is satisfied, then the unique $x \in \mathbb{R}^E$ *with* $I_G\, x = a$ *is given explicitly by*

$$x_e = k_e \sum_{v \in V_{G\setminus e}(t_e)} (-1)^{d_{G\setminus e}(v,t_e)}\, a_v, \quad \textit{for each } e \in E, \tag{21}$$

where

$$k_e = \begin{cases} \frac{1}{2}, & e \textit{ is an edge of a nonloop cycle of } G, \\ 1, & \textit{otherwise}, \end{cases} \tag{22}$$

$$t_e = \begin{cases} \textit{the endpoint of } e \textit{ furthest from } L, & e \textit{ is an edge of a component of } G \textit{ that contains a single cycle } L, \textit{ but } e \textit{ is not in } L, \\ \textit{an arbitrarily-chosen endpoint of } e, & \textit{otherwise}, \end{cases} \tag{23}$$

$G \setminus e$ *is the spanning subgraph of* G *obtained by deleting edge* e *from* G*,* $V_{G\setminus e}(t_e)$ *is the vertex set of the component of* $G \setminus e$ *which contains* t_e*, and* $d_{G\setminus e}(v, t_e)$ *is the length of the (necessarily unique) path between* v *and* t_e *in* $G \setminus e$*.*

Note that the path between v and t_e in $G \setminus e$ is unique since, for all cases of (21), v and t_e are vertices of a component of $G \setminus e$ which is acyclic.

It can also be checked that if there is choice for t_e in (23), which occurs if e is an an edge of a tree or of a nonloop cycle, then the RHS of (21) is independent of that choice. For example, consider an edge e of a tree with vertex set T, let the endpoints of e be u and w, and denote the length of the path between any two vertices v and v' in T as $d_T(v, v')$. Then the RHS of (21) is $\sum_{v \in V_{G\setminus e}(u)} (-1)^{d_T(v,u)}\, a_v$ for the choice $t_e = u$, and $\sum_{v \in V_{G\setminus e}(w)} (-1)^{d_T(v,w)}\, a_v = \sum_{v \in V_{G\setminus e}(w)} (-1)^{d_T(v,u)-1}\, a_v = -\sum_{v \in V_{G\setminus e}(w)} (-1)^{d_T(v,u)}\, a_v$ for the choice $t_e = w$. Therefore, since $T = V_{G\setminus e}(u) \uplus V_{G\setminus e}(w)$, the difference between the previous expressions is $\sum_{v \in T} (-1)^{d_T(v,u)}\, a_v$, which vanishes due to the condition (19) satisfied by a.

Proof. The validity of (i) follows from Propositions 9 and 10.

Now let the condition of (i) be satisfied. Then it can be verified directly that x, as given by (21), satisfies $\sum_{e \in \delta_G(v)} x_e = a_v$ for each $v \in V$, and hence that (ii) is valid. The nature of the verification process depends on whether v is a vertex of a tree, v is a vertex with a loop attached, v is a vertex of a nonloop cycle, or v is a vertex of a component which contains a cycle but with v not in the cycle. The details for the first of these cases will now be given explicitly, with those for the others being similar. So, let v be a vertex of a tree with vertex set T, for each $e \in \delta_G(v)$ choose t_e to be the endpoint of e other than v, and (as before) denote the length of the path between vertices v' and v'' in T as $d_T(v', v'')$. Then

$$\sum_{e \in \delta_G(v)} x_e = \sum_{e \in \delta_G(v)} \sum_{u \in V_{G\setminus e}(t_e)} (-1)^{d_T(u,t_e)}\, a_u = \sum_{e \in \delta_G(v)} \sum_{u \in V_{G\setminus e}(t_e)} (-1)^{d_T(u,v)-1}\, a_u$$
$$= -\sum_{e \in \delta_G(v)} \sum_{u \in V_{G\setminus e}(t_e)} (-1)^{d_T(u,v)}\, a_u = a_v - \sum_{u \in T} (-1)^{d_T(u,v)}\, a_u = a_v,$$

where the second-last equality follows from the fact that $T \setminus \{v\}$ is the union of the mutually disjoint sets $V_{G\setminus e}(t_e)$ over all $e \in \delta_G(v)$, and the last equality follows from the condition (19) satisfied by a. □

Note that if a and G, as given in Proposition 10, satisfy the condition (19) in that proposition (but not necessarily the condition of (i) in Proposition 11), then a (not necessarily unique) $x \in \mathbb{R}^E$ with $I_G\, x = a$ (whose existence is guaranteed by Proposition 10) can be obtained as follows. First, let H be a spanning subgraph of G, chosen to satisfy the same properties as the H used in the alternative proof of Proposition 8. Then it follows from (ii) of Proposition 11 (using H instead of G) that there exists a unique $x' \in \mathbb{R}^{E'}$ with $I_H\, x' = a$, where E' is the edge set of H. The required $x \in \mathbb{R}^E$ is then given by $x_e = x'_e$ for each $e \in E'$, and $x_e = 0$ for each $e \in E \setminus E'$.

Proposition 12. *The nullity of the incidence matrix of G is 1 if and only if G has a component C such that each component of G other than C either is acyclic or else contains exactly one cycle with that cycle having odd length, while the cycle content of C is one of the following:*

- *exactly one cycle, with that cycle having even length, or*
- *exactly two cycles, with at least one of those cycles having odd length, or*
- *exactly one even-length and exactly two odd-length cycles, with any two of those cycles sharing at least one edge.*

Note that for the case of bipartite G, it follows from this proposition, and the fact that a bipartite graph does not contain any odd-length cycles, that the nullity of the incidence matrix of G is 1 if and only if G contains exactly one cycle.

Proof. The structure of this proof is similar to that of the proof of Proposition 9. The fact that the kernel of I_G is the direct sum of the kernels of the incidence matrices of its components now implies that $\operatorname{nullity}(I_G) = 1$ if and only if there exists a component C of G such that $\operatorname{nullity}(I_C) = 1$, and $\operatorname{nullity}(I_{C'}) = 0$ for each component C' of G other than C. Using Proposition 9, the equation $\operatorname{nullity}(I_{C'}) = 0$, for each component C' other than C, is equivalent to the condition that C' either is acyclic or else contains exactly one cycle with that cycle having odd length. Using Proposition 8, the equation $\operatorname{nullity}(I_C) = 1$ is equivalent to $|E_C| = |V_C|$ if C is bipartite, or $|E_C| = |V_C| + 1$ if C is nonbipartite, where E_C and V_C are the edge and vertex sets of C. Using the facts that a connected graph C satisfies $|E_C| = |V_C|$ if and only if C contains exactly one cycle, and satisfies $|E_C| = |V_C| + 1$ if and only if C contains either exactly two cycles or else exactly three cycles, with any two of the three cycles sharing at least one edge (where the latter fact can be derived straightforwardly), it now follows that $\operatorname{nullity}(I_C) = 1$ if and only if C either is bipartite and contains exactly one cycle, or else is nonbipartite and contains exactly two cycles or exactly three cycles, with any two of the three cycles sharing at least one edge. Finally, the conditions on the parities of the lengths of the cycles of C, as given in the statement of the proposition, follow from the fact that a graph is bipartite if and only if it does not contain any odd-length cycles. □

5. Relevant results for polytopes

In this section, definitions are given for supports, and some relevant standard results involving faces, dimensions, vertices and edges of polytopes are obtained. These general results will be applied to $\mathcal{P}(G, b)$ in Section 7.

Let N be a finite set, and denote the support of any $X \subset \mathbb{R}^N$ as

$$\operatorname{supp}(X) := \{i \in N \mid \text{there exists } x \in X \text{ with } x_i \neq 0\}, \tag{24}$$

and the support of any $x \in \mathbb{R}^N$ as

$$\begin{aligned}\operatorname{supp}(x) &:= \operatorname{supp}(\{x\}) \\ &= \{i \in N \mid x_i \neq 0\}.\end{aligned} \tag{25}$$

Some simple but useful properties of supports, which follow immediately from (24), are that, for any $X_1, X_2 \subset \mathbb{R}^N$,

$$X_1 \subset X_2 \text{ implies } \operatorname{supp}(X_1) \subset \operatorname{supp}(X_2), \tag{26}$$

and that, for any set $\mathcal{R}$ of subsets of $\mathbb{R}^N$,

$$\operatorname{supp}\Bigl(\bigcap_{X\in\mathcal{R}} X\Bigr) \subset \bigcap_{X\in\mathcal{R}} \operatorname{supp}(X), \tag{27}$$

$$\operatorname{supp}\Bigl(\bigcup_{X\in\mathcal{R}} X\Bigr) = \bigcup_{X\in\mathcal{R}} \operatorname{supp}(X), \tag{28}$$

with empty intersections in (27) taken as $\bigcap_{X\in\emptyset} X = \mathbb{R}^N$ and $\bigcap_{X\in\emptyset} \operatorname{supp}(X) = N$. It follows that, for any $X \subset \mathbb{R}^N$,

$$\operatorname{supp}(X) = \bigcup_{x\in X} \operatorname{supp}(x). \tag{29}$$

Now let M be a further finite set, A be a real matrix with rows and columns indexed by M and N respectively, a be a vector in $\mathbb{R}^M$, and P be a polytope which can be written as

$$P = \{x \in \mathbb{R}^N_{\geq 0} \mid A\,x = a\}. \tag{30}$$

The following results, Propositions 13, 14 and 16, will provide information regarding the faces, vertices and edges, respectively, of the polytope P given by (30). These results are all closely related to standard results for polyhedra.

Proposition 13. *Let F be a nonempty face of the polytope P given by (30), and define $N' = \operatorname{supp}(F)$, A' to be the submatrix of A obtained by restricting the columns of A to those indexed by N', and*

$$F' := \bigl\{y \in \mathbb{R}^{N'}_{\geq 0} \bigm| A'\,y = a\bigr\}. \tag{31}$$

Then:

(i) F' is a polytope which is affinely isomorphic to F.

(ii) The dimension of F (and also the dimension of F') equals the nullity of A'.

Proof. The fact that any nonempty face of a polyhedron can be obtained by setting a subset of the polyhedron's defining inequalities to equalities (see, for example, Korte and Vygen [17, Prop. 3.4], Schrijver [21, Sec. 8.2, Eq. (11)] or Schrijver [22, Eq. (5.16)]) means that there exists some $N'' \subset N$ for which

$$F = \{x \in \mathbb{R}^N_{\geq 0} \mid x_i = 0 \text{ for each } i \in N \setminus N'',\ A\,x = a\}.$$

It can be seen, using $N' = \operatorname{supp}(F)$, that $N' \subset N''$, and hence that

$$F = \{x \in \mathbb{R}^N_{\geq 0} \mid x_i = 0 \text{ for each } i \in N \setminus N',\ A\,x = a\}.$$

Now consider the affine mapping from any $x \in \mathbb{R}^N$ to $y \in \mathbb{R}^{N'}$ in which $y_i = x_i$ for each $i \in N'$, and the affine mapping from any $y \in \mathbb{R}^{N'}$ to $x \in \mathbb{R}^N$ in which $x_i = y_i$ for each $i \in N'$, and $x_i = 0$ for each $i \in N \setminus N'$. Then these (essentially trivial) mappings, with their domains restricted to F and F' respectively, are mutual inverses, and so F' is a polytope which is affinely isomorphic to F, thus confirming (i).

It follows immediately that $\dim(F) = \dim(F')$. It can also be seen that $N' = \operatorname{supp}(F')$, so that none of the inequalities $y_i \geq 0$ for $i \in N'$ in (31) is an implicit equality. Hence, using the fact that the dimension of a nonempty polyhedron is the nullity of the matrix associated with those of the polyhedron's defining inequalities which are implicit equalities (see, for example, Schrijver [21, Sec. 8.2, Eq. (9)] or Schrijver [22, Thm. 5.6]), it follows that $\dim(F') = \operatorname{nullity}(A')$, thus confirming (ii). $\square$

Proposition 14. *Let u be an element of the polytope P given by (30), and define A' to be the submatrix of A obtained by restricting the columns of A to those indexed by $\operatorname{supp}(u)$. Then u is a vertex of P if and only if $\operatorname{nullity}(A') = 0$.*

Proof. If u is a vertex of P, then $\{u\}$ is a face of P, and so, using (ii) of Proposition 13, $0 = \dim(\{u\}) = \operatorname{nullity}(A')$. Conversely, if $\operatorname{nullity}(A') = 0$, then the equation $A'y = a$ has the unique solution $y \in \mathbb{R}^{N'}$ given by $y_i = u_i$ for each $i \in N'$, where $N' = \operatorname{supp}(u)$. It then follows that $\{x \in P \mid x_i = 0 \text{ for each } i \in N \setminus N'\} = \{u\}$, which implies that $\{u\}$ is face of P, and hence that u is a vertex of P. $\square$

Corollary 15. *Let u be a vertex of the polytope P given by (30). Then $|\operatorname{supp}(u)| \leq \operatorname{rank}(A)$.*

Proof. Using Proposition 14, $\operatorname{nullity}(A') = 0$, where A' is the submatrix of A obtained by restricting the columns of A to those indexed by $\operatorname{supp}(u)$. Therefore, $|\operatorname{supp}(u)| = \operatorname{rank}(A') \leq \operatorname{rank}(A)$. $\square$

Note that, in some contexts in the literature, a vertex u of the polytope P given by (30) is referred to as nondegenerate or degenerate according to whether or not $|\operatorname{supp}(u)| = \operatorname{rank}(A)$.

Proposition 16. *Let u and w be distinct vertices of the polytope P given by (30), and define A' to be the submatrix of A obtained by restricting the columns of A to those indexed by $\operatorname{supp}(\{u, w\})$. Then u and w are the vertices of an edge of P if and only if $\operatorname{nullity}(A') = 1$.*

Note that, using (28), $\operatorname{supp}(\{u, w\}) = \operatorname{supp}(u) \cup \operatorname{supp}(w)$.

Proof. It can be seen that $\operatorname{supp}(\{u, w\}) = \operatorname{supp}([u, w])$, where $[u, w]$ is the closed line segment between u and w. Therefore, if u and w are the vertices of an edge of P, then $[u, w]$ is a face of P, and so, using (ii) of Proposition 13, $1 = \dim([u, w]) = \operatorname{nullity}(A')$. Conversely, if $\operatorname{nullity}(A') = 1$, then the equation $A'y = a$ for $y \in \mathbb{R}^{N'}$ has the general solution $y = \lambda u' + (1 - \lambda) w'$, where $N' = \operatorname{supp}(\{u, w\})$, $u', w' \in \mathbb{R}^{N'}$ are given by $u'_i = u_i$ and $w'_i = w_i$ for each $i \in N'$, and $\lambda \in \mathbb{R}$ is arbitrary. It then follows, since u and w are vertices of P, that $\{x \in P \mid x_i = 0 \text{ for each } i \in N \setminus N'\} = [u, w]$, which implies that $[u, w]$ is a face of P, and hence that u and w are the vertices of an edge of P. $\square$

6. Relevant results for polytope face lattices

In this section, some relevant, essentially standard, results concerning polytope face lattices are outlined. In particular, for a polytope P given by (30) (with $a \neq 0$), three isomorphic lattices (all partially ordered by set inclusion) are considered, namely the face lattice of P, the lattice of vertex sets of the faces of P, and the lattice of supports of the faces of P (where these are denoted as $\mathcal{F}(P)$, $\mathcal{V}(P)$ and $\mathcal{S}(P)$, respectively). Various expressions will be presented for elements of these lattices, for the meet and join of subsets of these lattices, and for isomorphisms among the lattices. The general results of this section will be applied to $\mathcal{P}(G, b)$ in Section 8.

In contrast to the results in other sections of this paper, some details of the proofs of the results of this section will be omitted. However, full proofs can be obtained straightforwardly using standard polyhedron and polytope theory, as given, for example, in the books of Brønsted [6], Grünbaum [14], Korte and Vygen [17, Ch. 3], Schrijver [21, Ch. 8–9], Yemelichev, Kovalev and Kravtsov [25], or Ziegler [26]. The simple properties (26)–(28) of supports are also useful for some of these proofs.

Let N be a finite set. For any subset X of $\mathbb{R}^N$, denote the convex hull of X as $\operatorname{conv}(X)$, and for any polytope P in $\mathbb{R}^N$, denote (as indicated in Section 1.3), the set of vertices of P as $\operatorname{vert}(P)$, the set of facets of P as $\operatorname{facets}(P)$, and the face lattice of P as $\mathcal{F}(P)$.

Some of the statements which follow will involve the notation $\subset'$, where (as also indicated in Section 1.3) this means that such a statement is valid if $\subset'$ is taken to be $\subset$, and also valid if $\subset'$ is taken to be $=$.

Now consider a given polytope P. Then the face lattice $\mathcal{F}(P)$ is the set of all faces of P (including $\emptyset$ and P) partially ordered by set inclusion, where, for any $\mathcal{H} \subset \mathcal{F}(P)$, the infimum or meet of $\mathcal{H}$ is the intersection of all the faces in $\mathcal{H}$, and the supremum or join of $\mathcal{H}$ is the intersection of all those faces, or alternatively facets, of P which contain each face in $\mathcal{H}$, i.e.,

$$\inf(\mathcal{H}) = \bigcap_{F \in \mathcal{H}} F, \tag{32}$$

$$\sup(\mathcal{H}) = \bigcap_{\substack{F \in \mathcal{F}(P) \\ \cup_{F' \in \mathcal{H}} F' \subset F}} F = \bigcap_{\substack{F \in \operatorname{facets}(P) \\ \cup_{F' \in \mathcal{H}} F' \subset F}} F, \tag{33}$$

with an intersection over $\emptyset$ taken to be P.

Define $\mathcal{V}(P)$ to be the set of vertex sets of the faces of P, i.e.,

$$\mathcal{V}(P) := \{\operatorname{vert}(F) \mid F \in \mathcal{F}(P)\}. \tag{34}$$

This can also be written as

$$\begin{aligned} \mathcal{V}(P) &= \left\{\textstyle\bigcap_{F \in \mathcal{H}} \operatorname{vert}(F) \;\middle|\; \mathcal{H} \subset \operatorname{facets}(P)\right\} \\ &= \left\{U \subset \operatorname{vert}(P) \;\middle|\; \textstyle\bigcap_{\substack{F \in \operatorname{facets}(P) \\ U \subset \operatorname{vert}(F)}} \operatorname{vert}(F) \subset' U\right\}, \end{aligned} \tag{35}$$

with intersections over $\emptyset$ taken to be $\operatorname{vert}(P)$.

The mapping from each $F \in \mathcal{F}(P)$ to $\mathrm{vert}(F) = \mathrm{vert}(P) \cap F \in \mathcal{V}(P)$ is bijective, with an inverse which maps each $U \in \mathcal{V}(P)$ to $\mathrm{conv}(U) = \sup(\{\{u\} \mid u \in U\}) = \bigcap_{\substack{F' \in \mathcal{F}(P) \\ U \subset F'}} F' = \bigcap_{\substack{F' \in \mathrm{facets}(P) \\ U \subset F'}} F' \in \mathcal{F}(P)$, and with the property that, for any $F_1, F_2 \in \mathcal{F}(P)$, $F_1 \subset F_2$ if and only if $\mathrm{vert}(F_1) \subset \mathrm{vert}(F_2)$. Thus, $\mathcal{V}(P)$ is a lattice, partially ordered by set inclusion, which is isomorphic to the face lattice $\mathcal{F}(P)$. Also, for any $\mathcal{W} \subset \mathcal{V}(P)$, the meet and join in this lattice are

$$\inf(\mathcal{W}) = \bigcap_{U \in \mathcal{W}} U, \tag{36}$$

$$\sup(\mathcal{W}) = \bigcap_{\substack{U \in \mathcal{V}(P) \\ \cup_{U' \in \mathcal{W}} U' \subset U}} U = \bigcap_{\substack{F \in \mathrm{facets}(P) \\ \cup_{U' \in \mathcal{W}} U' \subset \mathrm{vert}(F)}} \mathrm{vert}(F), \tag{37}$$

with an intersection over $\emptyset$ taken to be $\mathrm{vert}(P)$.

Now let P be a polytope which can be written as (30), for some real matrix A with rows and columns indexed by finite sets M and N respectively, and some real nonzero vector a with entries indexed by M. Note that the condition $a \neq 0$, or equivalently $P \neq \{0\}$, is needed to ensure the validity of some of the results which follow. For example, this condition is needed to guarantee that $\emptyset$ is contained in the RHS of (38).

The face lattice of P can now be written as

$$\mathcal{F}(P) = \bigl\{\{x \in P \mid x_i = 0 \text{ for each } i \in N'\} \bigm| N' \subset N\bigr\}. \tag{38}$$

Some useful properties involving supports, as defined in (24)–(25), and faces of P are that, for any $X \subset P$,

$$\mathrm{supp}(X) = \mathrm{supp}\Biggl(\bigcap_{\substack{F \in \mathcal{F}(P) \\ X \subset F}} F\Biggr) = \mathrm{supp}\Biggl(\bigcap_{\substack{F \in \mathrm{facets}(P) \\ X \subset F}} F\Biggr), \tag{39}$$

and so, for any $x \in P$,

$$\mathrm{supp}(x) = \mathrm{supp}\Biggl(\bigcap_{\substack{F \in \mathcal{F}(P) \\ x \in F}} F\Biggr) = \mathrm{supp}\Biggl(\bigcap_{\substack{F \in \mathrm{facets}(P) \\ x \in F}} F\Biggr), \tag{40}$$

with an intersection over $\emptyset$ taken to be P.

Also, for any $X \subset P$ and $F \in \mathcal{F}(P)$,

$$X \subset F \text{ if and only if } \mathrm{supp}(X) \subset \mathrm{supp}(F), \tag{41}$$

and so, for any $x \in P$ and $F \in \mathcal{F}(P)$,

$$x \in F \text{ if and only if } \mathrm{supp}(x) \subset \mathrm{supp}(F). \tag{42}$$

Note that the 'only if' part of (41) follows immediately from (26).

A condition for the vertices of P in terms of supports is that an element u of P is a vertex of P if and only if there is no other element of P whose support is contained in, or alternatively equal to, the support of u, i.e., for any $u \in P$,

$$\begin{aligned} u \in \mathrm{vert}(P) &\text{ if and only if } \bigl\{x \in P \bigm| \mathrm{supp}(x) = \mathrm{supp}(u)\bigr\} \subset' \{u\} \\ &\text{ if and only if } \bigl\{x \in P \bigm| \mathrm{supp}(x) \subset \mathrm{supp}(u)\bigr\} \subset' \{u\}. \end{aligned} \tag{43}$$

Furthermore, for any $U \subset \mathrm{vert}(P)$,

$$\{u \in \mathrm{vert}(P) \mid \mathrm{supp}(u) \subset \mathrm{supp}(U)\} = \bigcap_{\substack{F \in \mathrm{facets}(P) \\ U \subset \mathrm{vert}(F)}} \mathrm{vert}(F), \tag{44}$$

from which it follows, using (35), that $\mathcal{V}(P)$ can be expressed in terms of supports as

$$\mathcal{V}(P) = \bigl\{U \subset \mathrm{vert}(P) \bigm| \{u \in \mathrm{vert}(P) \mid \mathrm{supp}(u) \subset \mathrm{supp}(U)\} \subset' U\bigr\}. \tag{45}$$

By considering two-element subsets U of $\mathrm{vert}(P)$ in (45), it follows that a condition for the edges of P is that, for any distinct vertices u and w of P,

$$\begin{aligned} &u \text{ and } w \text{ are the vertices of an edge of } P \text{ if and only if} \\ &\qquad\qquad \bigl\{y \in \mathrm{vert}(P) \bigm| \mathrm{supp}(y) \subset \mathrm{supp}(\{u, w\})\bigr\} \subset' \{u, w\}. \end{aligned} \tag{46}$$

Now define $\mathcal{S}(P)$ as the set of supports of the faces of P, i.e.,

$$\mathcal{S}(P) := \{\mathrm{supp}(F) \mid F \in \mathcal{F}(P)\}. \tag{47}$$

This can also be written as

$$\begin{aligned} \mathcal{S}(P) &= \{\mathrm{supp}(X) \mid X \subset P\} \\ &= \{\mathrm{supp}(x) \mid x \in P\} \cup \{\emptyset\} \\ &= \{\mathrm{supp}(U) \mid U \subset \mathrm{vert}(P)\} \\ &= \Bigl\{S \subset N \Bigm| S \subset' \textstyle\bigcup_{\substack{u \in \mathrm{vert}(P) \\ \mathrm{supp}(u) \subset S}} \mathrm{supp}(u)\Bigr\}. \end{aligned} \tag{48}$$

It follows from (42) that the mapping from each $F \in \mathcal{F}(P)$ to $\mathrm{supp}(F) \in \mathcal{S}(P)$ is bijective, with an inverse which maps each $S \in \mathcal{S}(P)$ to $\{x \in P \mid \mathrm{supp}(x) \subset S\} = \{x \in P \mid x_i = 0 \text{ for all } i \in N \setminus S\}$. Furthermore, it follows from (41) that, for any $F_1, F_2 \in \mathcal{F}(P)$, $F_1 \subset F_2$ if and only if $\mathrm{supp}(F_1) \subset \mathrm{supp}(F_2)$. Thus, $\mathcal{S}(P)$ is a lattice, partially ordered by set inclusion, which is isomorphic to the face lattice $\mathcal{F}(P)$. Also, for any $\mathcal{T} \subset \mathcal{S}(P)$, the meet and join in this lattice are

$$\inf(\mathcal{T}) = \bigcup_{\substack{S \in \mathcal{S}(P) \\ S \subset \cap_{S' \in \mathcal{T}} S'}} S \;=\; \bigcup_{\substack{u \in \mathrm{vert}(P) \\ \mathrm{supp}(u) \subset \cap_{S' \in \mathcal{T}} S'}} \mathrm{supp}(u), \tag{49}$$

$$\sup(\mathcal{T}) = \bigcup_{S \in \mathcal{T}} S. \tag{50}$$

In (49), $\bigcap_{S' \in \emptyset} S'$ can be taken as N or as $\mathrm{supp}(P)$, giving $\inf(\emptyset) = \mathrm{supp}(P)$.

Combining the isomorphisms between $\mathcal{F}(P)$ and $\mathcal{V}(P)$, and between $\mathcal{F}(P)$ and $\mathcal{S}(P)$, gives an isomorphism between $\mathcal{V}(P)$ and $\mathcal{S}(P)$ in which each $U \in \mathcal{V}(P)$ is mapped to $\mathrm{supp}(U) \in \mathcal{S}(P)$ and, inversely, in which each $S \in \mathcal{S}(P)$ is mapped to $\{u \in \mathrm{vert}(P) \mid \mathrm{supp}(u) \subset S\} \in \mathcal{V}(P)$.

The isomorphisms among the lattices $\mathcal{F}(P)$, $\mathcal{V}(P)$ and $\mathcal{S}(P)$ are summarized in Figure 2.

$$\begin{tikzcd}
& \mathcal{F}(P) \arrow[dl, shift left, "F\mapsto\mathrm{vert}(F)"] \arrow[dr, "F\mapsto\mathrm{supp}(F)"] & \\
\mathcal{V}(P) \arrow[ur, shift left, "U\mapsto\mathrm{conv}(U)"] \arrow[rr, shift left, "U\mapsto\mathrm{supp}(U)"] & & \mathcal{S}(P) \arrow[ul, shift right, "S\mapsto\{x\in P|\mathrm{supp}(x)\subset S\}"'] \arrow[ll, shift left, "S\mapsto\{u\in\mathrm{vert}(P)|\mathrm{supp}(u)\subset S\}"]
\end{tikzcd}$$

FIGURE 2. Isomorphisms among the lattices $\mathcal{F}(P)$, $\mathcal{V}(P)$ and $\mathcal{S}(P)$.

Finally, note that the results of this section can easily be modified so as to be valid for the more general case in which the polytope P has the form $P = \bigcap_{i\in M} K_i$ instead of (30), and the supports of any $X \subset \mathbb{R}^N$ and any $x \in \mathbb{R}^N$ are defined to be $\mathrm{supp}(X) = \{i \in M \mid X \not\subset H_i\}$ and $\mathrm{supp}(x) = \mathrm{supp}(\{x\}) = \{i \in M \mid x \notin H_i\}$ instead of (24)–(25), where M is a finite set and, for each $i \in M$, K_i is a given closed halfspace in $\mathbb{R}^N$ whose bounding hyperplane is H_i, with it being assumed that $\bigcap_{i\in M} H_i = \emptyset$. In particular, the modifications are that (38) becomes $\mathcal{F}(P) = \big\{\big(\bigcap_{i\in M'} K_i\big) \bigcap \big(\bigcap_{i\in M\setminus M'} H_i\big) \mid M' \subset M\big\}$ (with an intersection over $\emptyset$ taken to be $\mathbb{R}^N$), the isomorphism from $S \in \mathcal{S}(P)$ to $F \in \mathcal{F}(P)$ becomes $F = \{x \in P \mid \mathrm{supp}(x) \subset S\} = \big(\bigcap_{i\in S} K_i\big) \bigcap \big(\bigcap_{i\in M\setminus S} H_i\big)$, and N in (48) is replaced by M, with all other statements in this section remaining unchanged.

7. Results for the faces, dimension, vertices and edges of $\mathcal{P}(G, b)$

In this section, graphs associated with subsets and elements of $\mathcal{P}(G, b)$ are defined, and some of the main results of this paper, concerning the faces, dimension, vertices and edges of $\mathcal{P}(G, b)$, are obtained by combining general results for graphs from Section 4 with general results for polytopes from Section 5. At the end of the section, aspects of many of these results are illustrated using the example from Section 1.4.

Using the definitions (24)–(25) of supports (with N taken to be E), define the graph of any $X \subset \mathbb{R}^E$ as

$$\begin{aligned}\operatorname{gr}(X) &:= \text{the spanning subgraph of } G \text{ with edge set } \operatorname{supp}(X)\\ &= \text{the spanning subgraph of } G \text{ with edge set}\\ &\qquad\qquad \{e \in E \mid \text{there exists } x \in X \text{ with } x_e \neq 0\},\end{aligned} \tag{51}$$

and define the graph of any $x \in \mathbb{R}^E$ as

$$\begin{aligned}\operatorname{gr}(x) &:= \operatorname{gr}(\{x\})\\ &= \text{the spanning subgraph of } G \text{ with edge set } \operatorname{supp}(x)\\ &= \text{the spanning subgraph of } G \text{ with edge set } \{e \in E \mid x_e \neq 0\}.\end{aligned} \tag{52}$$

Note that the graphs of X and x depend on the graph G, even though this is not indicated explicitly by the notation.

It will also be useful, for Section 8, to denote the set of graphs of subsets of $\mathcal{P}(G,b)$ as $\mathcal{G}(G,b)$, i.e.,

$$\mathcal{G}(G,b) := \{\operatorname{gr}(X) \mid X \subset \mathcal{P}(G,b)\}, \tag{53}$$

and to refer to the graphs in this set as the graphs of $\mathcal{P}(G,b)$.

Note that any element of $\mathcal{G}(G,b)$ is *a* graph of $\mathcal{P}(G,b)$, whereas the particular element $\operatorname{gr}(\mathcal{P}(G,b))$ is *the* graph of $\mathcal{P}(G,b)$. However, $\operatorname{gr}(\mathcal{P}(G,b))$ should not be confused with the polytope graph of $\mathcal{P}(G,b)$, i.e., $\operatorname{gr}(\mathcal{P}(G,b))$ is not the graph whose vertices and edges correspond to the vertices and edges of the polytope $\mathcal{P}(G,b)$. Indeed, the graphs of $\mathcal{P}(G,b)$ might be described more completely as the support graphs of $\mathcal{P}(G,b)$.

The standard definitions of graph union, intersection and containment will be applied to spanning subgraphs of G in Section 8, and, to some extent, in this section. In particular, for any set $\mathcal{H}$ of spanning subgraphs of G, the union and intersection of the graphs in $\mathcal{H}$ are given by

$$\textstyle\bigcap_{H\in\mathcal{H}} H = \text{the spanning subgraph of } G \text{ with edge set } \bigcap_{H\in\mathcal{H}} E_H, \tag{54}$$

$$\textstyle\bigcup_{H\in\mathcal{H}} H = \text{the spanning subgraph of } G \text{ with edge set } \bigcup_{H\in\mathcal{H}} E_H, \tag{55}$$

where E_H denotes the edge set of H, and $\bigcap_{H\in\emptyset} E_H$ is taken to be E (so that $\bigcap_{H\in\emptyset} H = G$). Similarly, for spanning subgraphs H_1 and H_2 of G with edge sets E_{H_1} and E_{H_2}, graph containment is given by

$$H_1 \subset H_2 \text{ if and only if } E_{H_1} \subset E_{H_2}. \tag{56}$$

For example, it follows from (29) and (55) that, for any $X \subset \mathbb{R}^E$,

$$\operatorname{gr}(X) = \textstyle\bigcup_{x\in X} \operatorname{gr}(x). \tag{57}$$

The main results of this section will now be obtained. These include results which provide formulae for the dimensions of $\mathcal{P}(G,b)$ and its faces (Theorem 17, and Corollaries 18, 19 and 20), characterize the elements of $\mathcal{P}(G,b)$ which are vertices of $\mathcal{P}(G,b)$ (Theorem 22 and Corollary 23), give an explicit formula for a vertex of $\mathcal{P}(G,b)$ in terms of its graph (Theorem 24), and characterize the pairs of distinct vertices of $\mathcal{P}(G,b)$ which form edges of $\mathcal{P}(G,b)$ (Theorem 25 and Corollary 26).

Certain cases of some of these results correspond to previously-known results for the matrix classes $\mathcal{N}_{\le Z}(R)$, $\mathcal{N}(R)$, $\mathcal{N}_{\le Z}(R,S)$ or $\mathcal{N}(R,S)$ (using the notation discussed in Section 2).

Theorem 17. *Let F be a nonempty face of $\mathcal{P}(G,b)$. Then:*

(i) F is affinely isomorphic to $\mathcal{P}(\mathrm{gr}(F),b)$.

(ii) The dimension of F is $|\mathrm{supp}(F)|-|V|+B$, where B is the number of bipartite components of $\mathrm{gr}(F)$.

Proof. Using (9), (51), and both parts of Proposition 13 (taking P, M, N, A and a to be $\mathcal{P}(G,b)$, V, E, I_G and b respectively, so that $A'=I_{\mathrm{gr}(F)}$), it follows that F is affinely isomorphic to $\mathcal{P}(\mathrm{gr}(F),b)$, and that $\dim(F)=\mathrm{nullity}(I_{\mathrm{gr}(F)})$. The expression for $\dim(F)$ in (ii) is then given by Proposition 8 (taking the graph in that proposition to be $\mathrm{gr}(F)$). □

Corollary 18. *If $\mathcal{P}(G,b)$ is nonempty, then its dimension is $|\mathrm{supp}(\mathcal{P}(G,b))|-|V|+B$, where B is the number of bipartite components of $\mathrm{gr}(\mathcal{P}(G,b))$.*

A case of this result applied to $\mathcal{N}_{\le Z}(R,S)$ is given by Brualdi [9, Lem. 8.4.3(ii)], and a case applied to $\mathcal{N}(R,S)$ is given by Brualdi [9, Thm. 8.1.1], Klee and Witzgall [16, Thm. 1], Schrijver [22, Thm. 21.16], and Yemelichev, Kovalev and Kravtsov [25, Ch. 6, Prop. 1.1].

Proof. This result follows from (ii) of Theorem 17, by taking F to be $\mathcal{P}(G,b)$. □

Note that, assuming the validity of (i) of Theorem 17, Corollary 18 and (ii) of Theorem 17 are equivalent, since (ii) of Theorem 17 could be obtained from Corollary 18 by taking G in that corollary to be $\mathrm{gr}(F)$.

Corollary 19. *If $\mathcal{P}(G,b)_{>0}$ is nonempty, then the dimension of $\mathcal{P}(G,b)$ is $|E|-|V|+B$, where B is the number of bipartite components of G.*

Proof. This result follows from Corollary 18, and the fact that if $\mathcal{P}(G,b)_{>0}$ is nonempty, then $\mathrm{gr}(\mathcal{P}(G,b))=G$. □

Corollary 20. *Let G be bipartite and planar. Then the dimension of a nonempty face F of $\mathcal{P}(G,b)$ is the number of bounded faces in a planar embedding of $\mathrm{gr}(F)$.*

Proof. This result follows from (ii) of Theorem 17, and Euler's formula for planar graphs (which remains valid for graphs with multiple edges). □

Corollary 21. *Let u be a vertex of $\mathcal{P}(G,b)$. Then $|\mathrm{supp}(u)|\le|V|-B$, where B is the number of bipartite components of G.*

A case of this result applied to $\mathcal{N}(R)$ is given by Brualdi [9, Cor. 8.2.2], Converse and Katz [10, Lem.], and Lewin [18, Cor. 2], and a case applied to $\mathcal{N}(R,S)$ is given by Brualdi [9, Thm. 8.1.3], Klee and Witzgall [16, Cor. 3], and Yemelichev, Kovalev and Kravtsov [25, p. 264].

Proof. Using (ii) of Theorem 17, and the fact that $\{u\}$ is a face $\mathcal{P}(G,b)$ with dimension 0, it follows that $0 = |\operatorname{supp}(u)| - |V| + B'$, where B' is the number of bipartite components of $\operatorname{gr}(u)$. The result now follows from the fact that $B' \geq B$ (since $\operatorname{gr}(u)$ is a spanning subgraph of G).

Alternatively, this result follows from (17) and Corollary 15 (again taking P, M, N, A and a to be $\mathcal{P}(G,b)$, V, E, I_G and b respectively). ☐

Theorem 22. *An element u of $\mathcal{P}(G,b)$ is a vertex of $\mathcal{P}(G,b)$ if and only if each component of the graph of u either is acyclic or else contains exactly one cycle with that cycle having odd length.*

Note that in this theorem, as in Proposition 9, the choice of conditions for the components of G applies independently to each component. As also indicated after Proposition 9, the condition of this theorem can be restated as the condition that G has no even-length cycles and no component containing more than one odd-length cycle.

A case of this theorem applied to $\mathcal{N}_{\leq Z}(R)$ is given by Brualdi [9, Thm. 8.2.6], and a case applied to $\mathcal{N}(R)$ is given by Brualdi [8, Thm. 3.1], [9, Thm. 8.2.1], and Lewin [18, Thm. 2].

Proof. Using (9), (52) and Proposition 14 (again taking P, M, N, A and a to be $\mathcal{P}(G,b)$, V, E, I_G and b respectively, so that $A' = I_{\operatorname{gr}(u)}$), it follows that u is a vertex of $\mathcal{P}(G,b)$ if and only if $\operatorname{nullity}(I_{\operatorname{gr}(u)}) = 0$. The condition on $\operatorname{gr}(u)$ is then given by Proposition 9 (taking the graph in that proposition to be $\operatorname{gr}(u)$). ☐

Theorem 22 can also be proved more directly, using arguments from the alternative proof of Proposition 9. This will now be outlined briefly.

Alternative proof of Theorem 22. In this proof, the fact is used that an element u of a polytope $P \subset \mathbb{R}^N$ (for a finite set N) is a vertex of P if and only if there does not exist any nonzero $x \in \mathbb{R}^N$ such that $u - x \in P$ and $u + x \in P$.

First, consider $u \in \mathcal{P}(G,b)$ and $x \in \mathbb{R}^E$ such that $u \pm x \in \mathcal{P}(G,b)$, and let each component of $\operatorname{gr}(u)$ either be acyclic or else contain exactly one cycle with that cycle having odd length. Then $I_G\, u = I_G(u \pm x) = b$, and so $I_G\, x = 0$. Also, $u_e \pm x_e \geq 0$ for each $e \in E$, and so if $u_e = 0$ then $x_e = 0$, i.e., $\operatorname{supp}(x) \subset \operatorname{supp}(u)$. Therefore, $I_{\operatorname{gr}(u)}\, y = 0$, where $y \in \mathbb{R}^{\operatorname{supp}(u)}$ is given by $y_e = x_e$ for each $e \in \operatorname{supp}(u)$. Using the same argument as in the first part of the alternative proof of Proposition 9, it follows that $y = 0$. Hence, $x = 0$ and u is a vertex of $\mathcal{P}(G,b)$.

Now, conversely, consider $u \in \mathcal{P}(G,b)$ and let it not be the case that each component of $\operatorname{gr}(u)$ is acyclic or contains exactly one cycle with that cycle having odd length. Then using the same argument as in the second part of the alternative proof of Proposition 9, there exists a nonzero $y \in \mathbb{R}^{\mathrm{supp}(u)}$ which satisfies $I_{\mathrm{gr}(u)}\, y = 0$. Now define $x \in \mathbb{R}^E$ by $x_e = y_e$ for each $e \in \mathrm{supp}(u)$, and $x_e = 0$ for each $e \in E \setminus \mathrm{supp}(u)$. Then $x \neq 0$, $I_G\, x = 0$ and $\mathrm{supp}(x) \subset \mathrm{supp}(u)$. Therefore, an $\epsilon > 0$ can be chosen such that $u_e \pm \epsilon x_e \geq 0$ for all $e \in E$. (In particular, $0 < \epsilon \leq \min(\{u_e/x_e \mid e \in E,\ x_e > 0\} \cup \{-u_e/x_e \mid e \in E,\ x_e < 0\})$.) Hence, $u \pm \epsilon x \in \mathcal{P}(G, b)$, and u is not a vertex of $\mathcal{P}(G, b)$. □

Corollary 23. *Let G be bipartite. Then an element u of $\mathcal{P}(G, b)$ is a vertex of $\mathcal{P}(G, b)$ if and only if the graph of u is a forest.*

A case of this result applied to $\mathcal{N}_{\leq Z}(R, S)$ is given by Brualdi [9, Thm. 8.1.10], and a case applied to $\mathcal{N}(R, S)$ is given by Brualdi [9, Thm. 8.1.2], Klee and Witzgall [16, Thm. 4], and Schrijver [22, Thm. 21.15].

Proof. This result follows from Theorem 22, using the fact that a bipartite graph does not contain any odd-length cycles. □

Theorem 24. *Let H be the graph of a vertex u of $\mathcal{P}(G, b)$. Then u is the only element of $\mathcal{P}(G, b)$ whose graph is H, and it is given explicitly by*

$$u_e = \begin{cases} k_e \displaystyle\sum_{v \in V_{H\setminus e}(t_e)} (-1)^{d_{H\setminus e}(v, t_e)}\, b_v, & e \text{ is an edge of } H, \\ 0, & \text{otherwise}, \end{cases} \tag{58}$$

for each $e \in E$, where

$$k_e = \begin{cases} \frac{1}{2}, & e \text{ is an edge of a nonloop cycle of } H, \\ 1, & \text{otherwise}, \end{cases} \tag{59}$$

$$t_e = \begin{cases} \text{the endpoint of } e \text{ furthest from } L, & e \text{ is an edge of a component of } H \text{ that contains a single cycle } L, \text{ but } e \text{ is not in } L, \\ \text{an arbitrarily-chosen endpoint of } e, & \text{otherwise}, \end{cases} \tag{60}$$

$H \setminus e$ is the spanning subgraph of H obtained by deleting edge e from H, $V_{H\setminus e}(t_e)$ is the vertex set of the component of $H \setminus e$ which contains t_e, and $d_{H\setminus e}(v, t_e)$ is the length of the (necessarily unique) path between v and t_e in $H \setminus e$.

The reasons for the uniqueness of the path between v and t_e in $H \setminus e$, and for the independence of the RHS of (58) on any choices of t_e in (60), will be given in the following proof.

Note that the fact that if H is the graph of a vertex u of $\mathcal{P}(G, b)$, then u is the only element of $\mathcal{P}(G, b)$ with graph H will also be given as part of Theorem 29.

Proof. Denote the edge set of H as E' (i.e., $E' = \mathrm{supp}(u)$), and define $u' \in \mathbb{R}^{E'}$ by $u'_e = u_e$ for each $e \in E'$. It can be seen that $I_H\, u' = b$, so it follows from Proposition 10 (taking G and a in that proposition to be H and b) that $\sum_{v \in U_C} b_v = \sum_{v \in W_C} b_v$ for each bipartite component C of H, where (U_C, W_C) is a bipartition for C. Also, since u is a vertex of $\mathcal{P}(G,b)$, it follows from Theorem 22 that each component of $H = \mathrm{gr}(u)$ either is acyclic or else contains exactly one cycle with that cycle having odd length. It now follows from both parts of Proposition 11 (taking G and a in that proposition also to be H and b), that u' is the only vector in $\mathbb{R}^{E'}$ with $I_H\, u' = b$, and that it is given explicitly by the RHS of (21) (which matches the first case on the RHS of (58)). Furthermore, as discussed after the statement of Proposition 11, the path between v and t_e in $H \setminus e$ is unique, and the expression for u' is independent of any choices of t_e in (60). Therefore, u is given by (58), since $u_e = u'_e$ for each $e \in E'$ and $u_e = 0$ for each $e \in E \setminus E'$. It can also be seen that the uniqueness of u' as a vector in $\mathbb{R}^{E'}$ with $I_H\, u' = b$ implies that u is the only vector in $\mathbb{R}^E$ with $I_G\, u = b$ and support E', and hence that u is the only element of $\mathcal{P}(G,b)$ with graph H. □

Theorem 25. *Let u and w be distinct vertices of $\mathcal{P}(G,b)$. Then u and w are the vertices of an edge of $\mathcal{P}(G,b)$ if and only if $\mathrm{gr}(u) \cup \mathrm{gr}(w)$ has a component C such that each component of $\mathrm{gr}(u) \cup \mathrm{gr}(w)$ other than C either is acyclic or else contains exactly one cycle with that cycle having odd length, while the cycle content of C is one of the following:*

- *exactly one cycle, with that cycle having even length, or*
- *exactly two cycles, with at least one of those cycles having odd length, or*
- *exactly one even-length and exactly two odd-length cycles, with any two of those cycles sharing at least one edge.*

Note that $\mathrm{gr}(u) \cup \mathrm{gr}(w)$ is the graph union (55) of $\mathrm{gr}(u)$ and $\mathrm{gr}(w)$, and that, using (57), $\mathrm{gr}(u) \cup \mathrm{gr}(w) = \mathrm{gr}(\{u,w\})$.

It can also be seen that the graph of the set of vertices of an edge equals the graph of that edge. Hence, the graph of each edge of $\mathcal{P}(G,b)$ satisfies the condition of Theorem 25.

Proof. Using (9) and Proposition 16 (again taking P, M, N, A and a to be $\mathcal{P}(G,b)$, V, E, I_G and b respectively, so that $A' = I_{\mathrm{gr}(u)\cup \mathrm{gr}(w)}$), it follows that u and w are the vertices of an edge of $\mathcal{P}(G,b)$ if and only if $\mathrm{nullity}(I_{\mathrm{gr}(u)\cup \mathrm{gr}(w)}) = 1$. The condition on $\mathrm{gr}(u) \cup \mathrm{gr}(w)$ is then given by Proposition 12 (taking the graph in that proposition to be $\mathrm{gr}(u) \cup \mathrm{gr}(w)$). □

Corollary 26. *Let G be bipartite, and let u and w be distinct vertices of $\mathcal{P}(G,b)$. Then u and w are the vertices of an edge of P if and only if $\mathrm{gr}(u) \cup \mathrm{gr}(w)$ contains exactly one cycle.*

A case of this theorem applied to $\mathcal{N}(R,S)$ is given by Brualdi [9, Thm. 8.4.6], Oviedo [20, Cor. 1], and Yemelichev, Kovalev and Kravtsov [25, Ch. 6, Lem. 4.1].

Proof. This result follows from Theorem 25, using the fact that a bipartite graph does not contain any odd-length cycles. □

Aspects of many of the results of this section will now be illustrated using the case of G given by (3) and $b_1 = b_2 = b_3 = 1$, as already considered in Section 1.4. The nonempty faces of $\mathcal{P}(G,b)$ for this case can easily be identified using Figure 1. The graphs of these faces can then be obtained using (51), and are shown in Figure 3. The vertices of $\mathcal{P}(G,b)$ will again be denoted as A, B, C and D, and the edge between vertices X and Y will be denoted as XY.

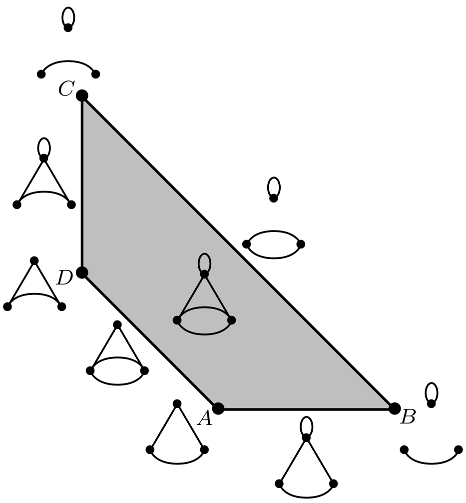


FIGURE 3. $\mathcal{P}(G,b)$, and the graphs of its nonempty faces, for G given by (3) and $b_1 = b_2 = b_3 = 1$.

Some aspects of the results of this section which can be verified for this example are as follows.

- The dimension of each nonempty face F of $\mathcal{P}(G,b)$ can be obtained directly from its graph using the formula in (ii) of Theorem 17, noting that $|\operatorname{supp}(F)|$ is simply the number of edges in $\operatorname{gr}(F)$, and that $|V| = 3$. In particular, the graphs of vertices A and D each have 3 edges and no bipartite components, so $\dim(\{A\}) = \dim(\{D\}) = 3 - 3 + 0 = 0$, the graphs of vertices B and C each have 2 edges and 1 bipartite component, so $\dim(\{B\}) = \dim(\{C\}) = 2 - 3 + 1 = 0$, the graphs of edges AB, AD and CD each have 4 edges and no bipartite components, so $\dim(AB) = \dim(AD) = \dim(CD) = 4 - 3 + 0 = 1$, and the graph of edge BC has 3 edges and 1 bipartite component, so $\dim(BC) = 3 - 3 + 1 = 1$.
- The dimension of $\mathcal{P}(G,b)$ can also be obtained from (ii) of Theorem 17, or Corollary 18, as has just been done for the other faces. Alternatively, since $\mathcal{P}(G,b)_{>0}$ is nonempty and G has no bipartite components, Corollary 19 gives $\dim(\mathcal{P}(G,b)) = |E| - |V| + 0 = 5 - 3 + 0 = 2$.
- The inequality of Corollary 21 can be verified for each vertex of $\mathcal{P}(G,b)$. In particular, for vertices A and D, the inequality is $3 \le 3 - 0 = 3$, while for vertices B and C, the inequality is $2 \le 3 - 0 = 3$.
- It can be observed that the graph of each vertex of $\mathcal{P}(G,b)$ satisfies the condition of Theorem 22, i.e., each component either is acyclic or else contains exactly one cycle

with that cycle having odd length. In particular, the graphs of vertices A and D each consist of a single component, comprised of a cycle of length 3, while the graphs of vertices B and C each consist of two components, one acyclic, and the other comprised of a cycle of length 1.

- The explicit coordinates of each vertex of $\mathcal{P}(G,b)$, as given in Figure 1, can be obtained from the graph of the vertex using the formula in Theorem 24.

  Consider first the vertex A, and let $(u_\alpha, u_\beta, u_\gamma, u_\delta, u_\epsilon)$ be the coordinates of A (using the edge labels from (3)), and H be the graph of A. The second line of (58) gives $u_\beta = u_\epsilon = 0$, since β and ϵ are not edges of H. Using (59) to give $k_\alpha = \frac{1}{2}$, and the second line of (60) to choose t_α as the vertex 1, the first line of (58) then gives $u_\alpha = k_\alpha \sum_{v \in V_{H\setminus\alpha}(t_\alpha)} (-1)^{d_{H\setminus\alpha}(v,t_\alpha)} b_v = \frac{1}{2}\sum_{v\in\{1,2,3\}}(-1)^{d_{H\setminus\alpha}(v,1)} b_v = \frac{1}{2}\big((-1)^0 b_1 + (-1)^2 b_2 + (-1)^1 b_3\big) = \frac{1}{2}(1+1-1) = \frac{1}{2}$. It can be checked that the alternative choice $t_\alpha = 2$ gives $u_\alpha = \frac{1}{2}\big((-1)^2 b_1 + (-1)^0 b_2 + (-1)^1 b_3\big) = \frac{1}{2}$, and that similar calculations give $u_\gamma = \frac{1}{2}(b_1 - b_2 + b_3) = \frac{1}{2}$ and $u_\delta = \frac{1}{2}(-b_1 + b_2 + b_3) = \frac{1}{2}$. Hence, $A = (\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}, 0)$.

  Proceeding to vertex B, let $(u_\alpha, u_\beta, u_\gamma, u_\delta, u_\epsilon)$ and H now be the coordinates and graph of B. The second line of (58) gives $u_\beta = u_\gamma = u_\delta = 0$. Using (59) to give $k_\alpha = 1$, and the second line of (60) to choose t_α as the vertex 1, the first line of (58) then gives $u_\alpha = k_\alpha \sum_{v\in V_{H\setminus\alpha}(t_\alpha)} (-1)^{d_{H\setminus\alpha}(v,t_\alpha)} b_v = \sum_{v\in\{1\}} (-1)^{d_{H\setminus\alpha}(v,1)} b_v = (-1)^0 b_1 = 1$. Alternatively, the choice $t_\alpha = 2$ gives $u_\alpha = \sum_{v\in\{2\}} (-1)^{d_{H\setminus\alpha}(v,2)} b_v = (-1)^0 b_2 = 1$. Now using (59) to give $k_\epsilon = 1$, and the second line of (60) to give t_ϵ as the vertex 3 (where this is a unique choice, since ϵ is a loop), the first line of (58) then gives $u_\epsilon = k_\epsilon \sum_{v\in V_{H\setminus\epsilon}(t_\epsilon)} (-1)^{d_{H\setminus\epsilon}(v,t_\epsilon)} b_v = \sum_{v\in\{3\}} (-1)^{d_{H\setminus\epsilon}(v,3)} b_v = (-1)^0 b_3 = 1$. Hence, $B = (1,0,0,0,1)$.

  Finally, the coordinates of vertices C and D can be obtained similarly to those of vertices B and A, respectively.

- Theorem 25 can be verified for each pair of distinct vertices of $\mathcal{P}(G,b)$, as follows. The union of the graphs of vertices B and C is the graph of edge BC, which consists of two components, one comprised of a cycle of length 1, and the other comprised of a cycle of length 2, so the first case listed in Theorem 25 applies to the second of those components. The unions of the graphs of vertices A and B, or C and D, are the graphs of edges AB, or CD, each of which consists of a single component comprised of a cycle of length 1 and a cycle of length 3, so the second case listed in Theorem 25 applies to that component. The union of the graph of vertices A and D is the graph of edge AD, which consists of a single component containing three cycles, one of length 2 and the other two of length 3, with any two of those cycles sharing at least one edge, so the third case listed in Theorem 25 applies to that component. The unions of the graphs of vertices A and C, or B and D, are the graph G, which consists of a single component containing four cycles (one of length 1, one of length 2 and two of length 3), so the condition of Theorem 25 does not hold.

8. Further results for the vertices, edges, faces and graphs of $\mathcal{P}(G,b)$

In this section, further results concerning the vertices, edges, faces and graphs of $\mathcal{P}(G,b)$ are obtained. The previous such results, in Section 7, combined both general results for graphs, from Section 4, and general results for polytopes, from Section 5. By contrast, the results of this section depend only on general results for polytopes, from Section 6, together with the simple correspondences, as given in (51)–(52), between supports of subsets or elements of $\mathcal{P}(G,b)$, and graphs of $\mathcal{P}(G,b)$.

In particular, this section consists of results from Section 6, in which the polytope P of (30) is now taken to be $\mathcal{P}(G,b)$, in the form (9), with N, M, A and a in (30) taken to be E, V, I_G and b respectively, and with unions, intersections or containments of supports of subsets or elements of $\mathcal{P}(G,b)$ now expressed as unions, intersections or containments (as given in (54)–(56)) of graphs of $\mathcal{P}(G,b)$.

It will be assumed throughout this section that b is nonzero, so that the application of the results of Section 6 to $\mathcal{P}(G,b)$ is valid.

Included among the results of this section are further characterizations of the elements of $\mathcal{P}(G,b)$ which are vertices of $\mathcal{P}(G,b)$ (in Theorem 29), and of the pairs of distinct vertices of $\mathcal{P}(G,b)$ which form edges of $\mathcal{P}(G,b)$ (in Corollary 31), several equivalent conditions for a spanning subgraph of G to be a graph of $\mathcal{P}(G,b)$ (in Theorem 32), and a statement that the set of graphs of $\mathcal{P}(G,b)$ forms a lattice which is isomorphic to the face lattice of $\mathcal{P}(G,b)$ (in Theorem 33).

At the end of this section, the face lattice of $\mathcal{P}(G,b)$ and the lattice of graphs of $\mathcal{P}(G,b)$ are considered for the example of Section 1.4.

Proposition 27. *For any subset X of $\mathcal{P}(G,b)$,*

$$\mathrm{gr}(X)=\mathrm{gr}\Biggl(\bigcap_{\substack{F\in\mathcal{F}(\mathcal{P}(G,b))\\ X\subset F}}F\Biggr)=\mathrm{gr}\Biggl(\bigcap_{\substack{F\in\mathrm{facets}(\mathcal{P}(G,b))\\ X\subset F}}F\Biggr). \tag{61}$$

Note that $\mathcal{F}(\mathcal{P}(G,b))$ and $\mathrm{facets}(\mathcal{P}(G,b))$ are the face lattice and set of facets, respectively, of $\mathcal{P}(G,b)$, using the notation of Section 6.

Proof. This result follows by applying (39) to $\mathcal{P}(G,b)$. $\square$

Proposition 28. *For any subset X of $\mathcal{P}(G,b)$, and any face F of $\mathcal{P}(G,b)$,*

$$X\subset F \text{ if and only if } \mathrm{gr}(X)\subset\mathrm{gr}(F). \tag{62}$$

Proof. This result follows by applying (41) to $\mathcal{P}(G,b)$. $\square$

Note that (40) and (42) could also easily be applied to $\mathcal{P}(G,b)$, giving special cases of (61) and (62), respectively, in which X contains a single element.

Theorem 29. *Let u be an element of $\mathcal{P}(G,b)$, and H be the graph of u. Then the following are equivalent.*

(i) u is a vertex of $\mathcal{P}(G,b)$.

(ii) u is the only element of $\mathcal{P}(G,b)$ whose graph is H.

(iii) u is the only element of $\mathcal{P}(G,b)$ whose graph is contained in H.

Note that the implication of (ii) by (i) in this theorem is also given as the first statement in Theorem 24.

Cases of this theorem applied to $\mathcal{N}_{\leq Z}(R)$, $\mathcal{N}(R)$, $\mathcal{N}_{\leq Z}(R,S)$ and $\mathcal{N}(R,S)$ are given by Brualdi [8, Thm. 3.1], [9, Thms. 8.1.2, 8.1.10, 8.2.1 & 8.2.6], Jurkat and Ryser [15, p. 348], and Klee and Witzgall [16, Cor. 2].

Proof. This result follows by applying (43) to $\mathcal{P}(G,b)$. □

Theorem 30. *Let U be a subset of vertices of $\mathcal{P}(G,b)$. Then U is the set of vertices of a face of $\mathcal{P}(G,b)$ if and only if the elements of U are the only vertices of $\mathcal{P}(G,b)$ whose graphs are contained in the graph of U.*

Proof. This result follows by applying (45) to $\mathcal{P}(G,b)$. □

Corollary 31. *Let u and w be distinct vertices of $\mathcal{P}(G,b)$. Then u and w are the vertices of an edge of $\mathcal{P}(G,b)$ if and only if u and w are the only vertices of $\mathcal{P}(G,b)$ whose graphs are contained in the union of the graphs of u and w.*

Proof. This result follows from Theorem 30 by taking U to be $\{u,w\}$. (It also follows by applying (46) to $\mathcal{P}(G,b)$.) □

Theorem 32. *Let H be a spanning subgraph of G. Then each of the following is a necessary and sufficient condition for H to be a graph of $\mathcal{P}(G,b)$, i.e., for H to be the graph of a subset of $\mathcal{P}(G,b)$.*

(i) H is the graph of an element of $\mathcal{P}(G,b)$, or H has no edges.

(ii) H is the graph of a face of $\mathcal{P}(G,b)$.

(iii) H is a union of graphs of vertices of $\mathcal{P}(G,b)$.

(iv) H is the union of the graphs of all vertices of $\mathcal{P}(G,b)$ whose graph is contained in H.

Note that further equivalent conditions will be added to this list in Section 9.

Proof. The necessity and sufficiency of condition (ii) for H to be a graph of $\mathcal{P}(G,b)$, and its equivalence to conditions (i), (iii) and (iv), follows by applying the equality between the first set, $\mathcal{S}(P)$, in (48) (as defined in (47)) and each of the other four sets in (48) (using (29) in the third of these), respectively, to $\mathcal{P}(G,b)$. □

It follows from (53) and Theorem 32 that the set of graphs of $\mathcal{P}(G,b)$ can now be written as

$$\begin{aligned}\mathcal{G}(G,b) &= \{\mathrm{gr}(X) \mid X \subset \mathcal{P}(G,b)\}\\ &= \{\mathrm{gr}(x) \mid x \in \mathcal{P}(G,b)\} \cup \{\text{the spanning subgraph of } G \text{ with no edges}\}\\ &= \{\mathrm{gr}(F) \mid F \in \mathcal{F}(\mathcal{P}(G,b))\}\\ &= \{\textstyle\bigcup_{u\in U} \mathrm{gr}(u) \mid U \subset \mathrm{vert}(\mathcal{P}(G,b))\}\\ &= \Big\{\text{spanning subgraphs } H \text{ of } G \;\Big|\; H = \textstyle\bigcup_{\substack{u\in\mathrm{vert}(\mathcal{P}(G,b))\\ \mathrm{gr}(u)\subset H}} \mathrm{gr}(u)\Big\}.\end{aligned} \tag{63}$$

A further equality will be added to those of (63) in (66).

Theorem 33. *The face lattice $\mathcal{F}(\mathcal{P}(G,b))$ of $\mathcal{P}(G,b)$ is isomorphic to the set $\mathcal{G}(G,b)$ of graphs of $\mathcal{P}(G,b)$ partially ordered by inclusion.*

The natural isomorphism between these lattices maps each face $F \in \mathcal{F}(\mathcal{P}(G,b))$ to its graph $\mathrm{gr}(F)$, and inversely maps each graph $H \in \mathcal{G}(G,b)$ to the face $\{x \in \mathcal{P}(G,b) \mid \mathrm{gr}(x) \subset H\} = \{x \in \mathcal{P}(G,b) \mid x_e = 0$ for each $e \in E$ which is not an edge of $H\}$. In terms of vertices of faces, $F \in \mathcal{F}(\mathcal{P}(G,b))$ is mapped to the union of the graphs of the vertices of F, i.e., $\mathrm{gr}(F) = \bigcup_{u\in\mathrm{vert}(F)} \mathrm{gr}(u) = \mathrm{gr}(\mathrm{vert}(F))$, and $H \in \mathcal{G}(G,b)$ is mapped to the face whose vertices are $\{u \in \mathrm{vert}(P) \mid \mathrm{gr}(u) \subset H\}$.

For any $\mathcal{H} \subset \mathcal{G}(G,b)$, the meet of $\mathcal{H}$ is the union of all those graphs of $\mathcal{G}(G,b)$ which are contained in each graph in $\mathcal{H}$, or alternatively the union of the graphs of all those vertices of $\mathcal{P}(G,b)$ whose graphs are contained in each graph in $\mathcal{H}$, and the join of $\mathcal{H}$ is the union of all the graphs in $\mathcal{H}$, i.e.,

$$\inf(\mathcal{H}) = \bigcup_{\substack{H\in\mathcal{G}(G,b)\\ H\subset\bigcap_{H'\in\mathcal{H}}H'}} H \;=\; \bigcup_{\substack{u\in\mathrm{vert}(\mathcal{P}(G,b))\\ \mathrm{gr}(u)\subset\bigcap_{H'\in\mathcal{H}}H'}} \mathrm{gr}(u), \tag{64}$$

$$\sup(\mathcal{H}) = \bigcup_{H\in\mathcal{H}} H. \tag{65}$$

Note that, for the case $\mathcal{H} = \emptyset$ in (64), $\bigcap_{H'\in\emptyset} H'$ can be taken as G or as $\mathrm{gr}(\mathcal{P}(G,b))$, giving $\inf(\emptyset) = \mathrm{gr}(\mathcal{P}(G,b))$.

Note also that the dimension of a nonempty face F of $\mathcal{P}(G,b)$ is given by (ii) of Theorem 17, with $|\mathrm{supp}(F)|$ in that theorem being simply the number of edges in $\mathrm{gr}(F)$.

Proof. All of these results follow from the discussion, in Section 6, between (48) and Figure 2, as applied to $\mathcal{P}(G,b)$. $\square$

As an example, consider again G given by (3) and $b_1 = b_2 = b_3 = 1$. The Hasse diagrams of the face lattice $\mathcal{F}(\mathcal{P}(G,b))$ of $\mathcal{P}(G,b)$, and the lattice $\mathcal{G}(G,b)$ of graphs of $\mathcal{P}(G,b)$ for this case, are shown in Figure 4. Various aspects of Theorem 33, and of certain other results of this section, such as Theorem 30 and Corollary 31, can be verified for this case using this figure.

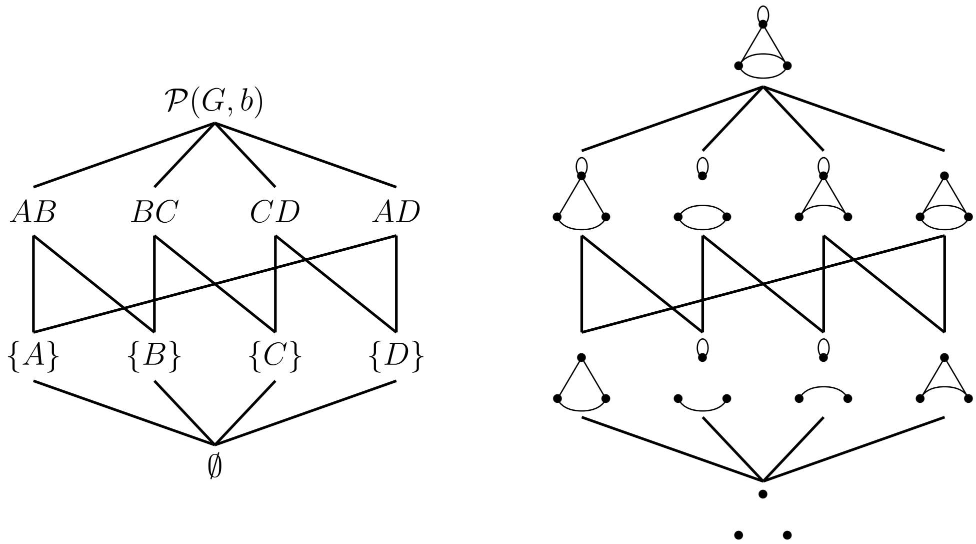


FIGURE 4. Hasse diagrams of the face lattice $\mathcal{F}(\mathcal{P}(G,b))$ of $\mathcal{P}(G,b)$ (left), and the lattice $\mathcal{G}(G,b)$ of graphs of $\mathcal{P}(G,b)$ (right), for G given by (3) and $b_1 = b_2 = b_3 = 1$.

9. FURTHER CONDITIONS FOR THE GRAPHS OF $\mathcal{P}(G,b)$

In this section, further necessary and sufficient conditions for a spanning subgraph H of G to be a graph of $\mathcal{P}(G,b)$ are obtained using Theorems 4 and 7 from Section 3. In contrast to the conditions of Theorem 32, these conditions depend only on H and b, without any reference to $\mathcal{P}(G,b)$, and take the form of finitely-many strict inequalities and equalities for certain sums of entries of b. At the end of the section, some of the results are illustrated using certain spanning subgraphs of the graph G of (3).

It is assumed in this section that b is again nonzero.

Theorem 34. *Let H be a spanning subgraph of G. Then a necessary and sufficient condition for H to be a graph of $\mathcal{P}(G,b)$ is that H has no edges, or $\sum_{v\in V_1} b_v \geq \sum_{v\in V_3} b_v$ for all sets V_1, V_2 and V_3 such that $V = V_1 \uplus V_2 \uplus V_3$ and $H[V_2 \cup V_3, V_3] = \emptyset$, with the inequality holding as an equality if and only if $H[V_1, V_1 \cup V_2] = \emptyset$.*

A case of this theorem applied to $\mathcal{N}(R)$ is given by Brualdi [9, p. 353].

Proof. It will be shown that the condition of this theorem is equivalent to condition (i) of Theorem 32.

Denote the edge set of H as E_H. If $E_H = \emptyset$, then the condition of this theorem and (i) of Theorem 32 are both automatically satisfied.

So, assume that $E_H \neq \emptyset$ and that (i) of Theorem 32 is satisfied. Then there exists $x \in \mathcal{P}(G,b)$ with $H = \mathrm{gr}(x)$. Now define $x' \in \mathbb{R}^{E_H}$ by $x'_e = x_e$ for each $e \in E_H$. Then $x' \in \mathcal{P}(H,b)_{>0}$. Therefore, using Theorem 7 (with G in that theorem taken to be H), the condition of this theorem is satisfied.

Conversely, assume that $E_H \neq \emptyset$ and that the condition of this theorem is satisfied. Then, using Theorem 7 (with G in that theorem again taken to be H), there exists $x' \in P(H,b)_{>0}$. Now define $x \in \mathbb{R}^E$ by $x_e = x'_e$ for each $e \in E_H$, and $x_e = 0$ for each $e \in E \setminus E_H$. Then $x \in \mathcal{P}(G,b)$ and $H = \mathrm{gr}(x)$. Therefore, (i) of Theorem 32 is satisfied. $\square$

It follows that the equalities of (63), for the set of graphs of $\mathcal{P}(G,b)$, can now be supplemented by

$$\mathcal{G}(G,b) = \big\{\text{spanning subgraphs } H \text{ of } G \;\big|\; H \text{ satisfies the condition of Theorem 34}\big\}. \tag{66}$$

Theorem 35. *Let H be a spanning subgraph of G. If H is bipartite, then a necessary and sufficient condition for H to be a graph of $\mathcal{P}(G,b)$ is that H has no edges, or $\sum_{v\in C} b_v \geq \sum_{v\in V\setminus C} b_v$ for each vertex cover C of H, with the inequality holding as an equality if and only if $V \setminus C$ is also a vertex cover of H.*

Note that if (U,W) is a bipartition for H, then the condition of this theorem is equivalent to the condition that H has no edges, or $\sum_{v\in U_1} b_v + \sum_{v\in W_1} b_v \geq \sum_{v\in U_2} b_v + \sum_{v\in W_2} b_v$ for all sets U_1, U_2, W_1 and W_2 such that $U = U_1 \uplus U_2$, $W = W_1 \uplus W_2$ and $H[U_2,W_2] = \emptyset$, with the inequality holding as an equality if and only if $H[U_1,W_1] = \emptyset$. Also, this condition remains unchanged if its inequality is replaced by $\sum_{v\in U_1} b_v \geq \sum_{v\in W_2} b_v$, or by $\sum_{v\in W_1} b_v \geq \sum_{v\in U_2} b_v$. The reasons for these equivalences are discussed briefly after the statements of Theorems 3 and 4 (where the graph in those remarks should now be taken to be H).

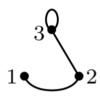
Proof. It follows from Theorems 4 and 7 (taking the graph in each theorem to be H) that if H is bipartite, then the condition of this theorem is equivalent to the condition of Theorem 34. The result of this theorem is then given by Theorem 34. $\square$

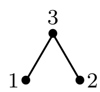
Corollary 36. *Let G be bipartite, and H be a spanning subgraph of G. Then the condition of Theorem 35 is necessary and sufficient for H to be a graph of $\mathcal{P}(G,b)$.*

A case of this theorem applied to $\mathcal{N}(R,S)$ is given by Brualdi [9, p. 343].

Proof. This result follows immediately from Theorem 35, since any spanning subgraph of a bipartite graph is bipartite. $\square$

As simple examples, consider now the spanning subgraphs H_1 and H_2, of the graph G of (3), given by

$H_1 =$ 

and $H_2 =$ 

. (67)

It can be seen directly that H_1 is a graph of $\mathcal{P}(G,b)$ if and only if $(x_\alpha, x_\beta, x_\gamma, x_\delta, x_\epsilon) = (b_1, 0, 0, b_2 - b_1, b_1 - b_2 + b_3)$ is an element of $\mathcal{P}(G,b)_{>0}$, and hence if and only if $b_1 + b_3 > b_2 > b_1 > 0$.

Alternatively, the set triples (V_1, V_2, V_3) which satisfy $V = \{1, 2, 3\} = V_1 \uplus V_2 \uplus V_3$ and $H_1[V_2 \cup V_3, V_3] = \emptyset$ are $(\{2\}, \{3\}, \{1\})$, $(\{1, 3\}, \emptyset, \{2\})$, $(\{2, 3\}, \emptyset, \{1\})$, and $(U, \{1, 2, 3\} \setminus U, \emptyset)$, for each $U \subset \{1, 2, 3\}$. Among these cases, the only one which satisfies $H_1[V_1, V_1 \cup V_2] = \emptyset$ is $(\emptyset, \{1, 2, 3\}, \emptyset)$, for which $\sum_{v \in V_1} b_v = \sum_{v \in V_3} b_v$ $(= 0)$ is automatically satisfied. Therefore, using Theorem 34, H_1 is a graph of $\mathcal{P}(G, b)$ if and only if $b_2 > b_1$, $b_1 + b_3 > b_2$, $b_2 + b_3 > b_1$ and $\sum_{v \in U} b_v > 0$, for each nonempty $U \subset V$, which can be seen to coincide with the condition obtained directly.

Proceeding to H_2, it can be seen directly that H_2 is a graph of $\mathcal{P}(G, b)$ if and only if $(x_\alpha, x_\beta, x_\gamma, x_\delta, x_\epsilon) = (0, 0, b_1, b_2, 0)$ is an element of $\mathcal{P}(G, b)_{>0}$, and hence if and only if $b_3 = b_1 + b_2$, $b_1 > 0$ and $b_2 > 0$.

Alternatively, the vertex covers C of H_2 are $\{3\}$, $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$ and $\{1, 2, 3\}$. Among these vertex covers, those for which $\{1, 2, 3\} \setminus C$ is also a vertex cover of H_2 are $\{3\}$ and $\{1, 2\}$. Therefore, using the fact that H_2 is bipartite and Theorem 35, H_2 is a graph of $\mathcal{P}(G, b)$ if and only if $b_3 = b_1 + b_2$, $b_1 + b_3 > b_2$, $b_2 + b_3 > b_1$ and $b_1 + b_2 + b_3 > 0$, which can be seen to coincide with the condition obtained directly.

It follows that neither H_1 nor H_2 is a graph of $\mathcal{P}(G, b)$ for the previously-considered case of $b_1 = b_2 = b_3$, since neither of the associated conditions is satisfied in that case.

10. Results for the case in which G contains multiple edges

In this section, results which are relevant for the case in which G contains multiple edges (but trivial if G does not contain multiple edges) are obtained. In particular, for a reduced graph (as defined in Section 1.3) G_{rd} of G, the results of this section identify relationships between the graphs of $\mathcal{P}(G, b)$ and $\mathcal{P}(G_{\mathrm{rd}}, b)$ (in Theorem 37), the dimensions of faces of $\mathcal{P}(G, b)$ and $\mathcal{P}(G_{\mathrm{rd}}, b)$ (in Theorem 38), and the vertices of $\mathcal{P}(G, b)$ and $\mathcal{P}(G_{\mathrm{rd}}, b)$ (in Theorem 39). At the end of the section, Theorems 37 and 39 are considered in the context of the example from Section 1.4.

It is assumed in this section that b is again nonzero.

Theorem 37. *Let G_{rd} be a reduced graph of G, H be a spanning subgraph of G, and H_{rd} be both a reduced graph of H and a spanning subgraph of G_{rd}. Then H is a graph of $\mathcal{P}(G, b)$ if and only if H_{rd} is a graph of $\mathcal{P}(G_{\mathrm{rd}}, b)$.*

Proof. It will be shown that condition (i) of Theorem 32 is satisfied by H and G if and only if condition (i) of Theorem 32 is satisfied by H_{rd} and G_{rd}.

It can be seen that H has no edges if and only if H_{rd} has no edges. For the case in which H and H_{rd} both have no edges, (i) of Theorem 32 is automatically satisfied by H and G, and by H_{rd} and G_{rd}. So, for the remainder of this proof, consider the case in which the edge sets of H and H_{rd} are both nonempty.

First, assume that H and G satisfy (i) of Theorem 32. Then there exists $x \in \mathcal{P}(G, b)$ with $H = \mathrm{gr}(x)$. Now define (as in the first definition in the proof of Proposition 1) $x' \in \mathbb{R}^{E'}$ by $x'_{e'} = \sum_{e \in M(e')} x_e$ for each $e' \in E'$, where E' is the edge set of G_{rd}, and $M(e')$ is the set of edges of G which have the same endpoints as e'. (Equivalently, $x'_{e'} = \sum_{e \in M_H(e')} x_e$

for each $e' \in E'$, where $M_H(e')$ is the set of edges of H which have the same endpoints as e'.) Then $x' \in \mathcal{P}(G_{\mathrm{rd}}, b)$ and $H_{\mathrm{rd}} = \mathrm{gr}(x')$, and so H_{rd} and G_{rd} satisfy (i) of Theorem 32.

Conversely, assume that H_{rd} and G_{rd} satisfy (i) of Theorem 32. Then there exists $x' \in \mathcal{P}(G_{\mathrm{rd}}, b)$ with $H_{\mathrm{rd}} = \mathrm{gr}(x')$. Now define $x \in \mathbb{R}^E$ by $x_e = x'_{e'}/m(e)$ for each $e \in E_H$, and $x_e = 0$ for each $e \in E \setminus E_H$, where E_H is the edge set of H, e' is the single edge of G_{rd} (or H_{rd}) which has the same endpoints as e, and $m(e)$ is the number of edges of H which have the same endpoints as e. Then $x \in \mathcal{P}(G, b)$ and $H = \mathrm{gr}(x)$, and so H and G satisfy (i) of Theorem 32. □

It can be seen that if G contains multiple edges, G_{rd} is a reduced graph of G, and the graphs of $\mathcal{P}(G_{\mathrm{rd}}, b)$ are known, then the graphs of $\mathcal{P}(G, b)$ can easily be obtained as follows. First, choose a graph H_{rd} of $\mathcal{P}(G_{\mathrm{rd}}, b)$, and, for each edge e of H_{rd}, choose a nonempty subset E_e of the set of edges of G which have the same endpoints as e. Then let H be the spanning subgraph of G whose edge set is the union of the sets E_e, over all edges e of H_{rd}. It follows, using Theorem 37, that H is a graph of $\mathcal{P}(G, b)$, and that repeating these steps for all possible graphs H_{rd}, and all possible associated sets E_e, produces each graph of $\mathcal{P}(G, b)$ exactly once.

It also follows that the number of graphs of $\mathcal{P}(G, b)$, and hence (using Theorem 33) the number of faces of $\mathcal{P}(G, b)$, is given by

$$|\mathcal{F}(\mathcal{P}(G, b))| = \sum_{F \in \mathcal{F}(\mathcal{P}(G_{\mathrm{rd}}, b))} \prod_{e \in \mathrm{supp}(F)} \big(2^{m(e)} - 1\big), \tag{68}$$

where $m(e)$ is the number of edges of G which have the same endpoints as edge e of G_{rd}.

Theorem 38. *Let G_{rd} be a reduced graph of G, F be a face of $\mathcal{P}(G, b)$, and F' be a face of $\mathcal{P}(G_{\mathrm{rd}}, b)$. If the graph of F' is a reduced graph of the graph of F, then*

$$\dim(F) - \dim(F') = |\,\mathrm{supp}(F)| - |\,\mathrm{supp}(F')| \geq 0. \tag{69}$$

Note that $\mathrm{supp}(F)$ and $\mathrm{supp}(F')$ are simply the edge sets of the graphs of F and F', respectively.

Proof. It can be checked that F is empty if and only if F' is empty. For the case in which F and F' are both empty, (69) is immediately satisfied.

So, assume now that F and F' are both nonempty. Then the equality in (69) follows from (ii) of Theorem 17, and the observations that G and G_{rd} each have vertex set V, and that $\mathrm{gr}(F)$ and $\mathrm{gr}(F')$ each have the same number of bipartite components, while the inequality in (69) follows from the observation that $\mathrm{gr}(F)$ has at least as many edges as $\mathrm{gr}(F')$. □

Theorem 39. *Let G_{rd} be a reduced graph of G, H be a spanning subgraph of G, and H_{rd} be both a reduced graph of H and a spanning subgraph of G_{rd}. Then H is the graph of a vertex u of $\mathcal{P}(G, b)$ if and only if H does not have multiple edges and H_{rd} is the graph of a vertex u' of $\mathcal{P}(G_{\mathrm{rd}}, b)$. In such cases, $u_e = u'_{e'}$ for each pair of edges e of H and e' of H_{rd} which have the same endpoints, while all other entries of u and u' are zero.*

Note that if H does not have multiple edges, then H and H_{rd} are in fact the same, up to the labeling of their edges.

Proof. First, assume that $H=\mathrm{gr}(u)$, for a vertex u of $\mathcal{P}(G,b)$. Then, using Theorem 22, each component of H either is acyclic or else contains exactly one cycle with that cycle having odd length. Therefore, H does not have multiple edges (since if H had multiple edges, then H would contain a cycle of length 2), and so each component of H_{rd} either is acyclic or else contains exactly one cycle with that cycle having odd length. Now define $u'\in\mathbb{R}^{E_{G_{\mathrm{rd}}}}$ (where $E_{G_{\mathrm{rd}}}$ is the edge set of G_{rd}) by $u'_{e'}=u_e$ for each pair of edges e' of H_{rd} and e of H which have the same endpoints, with all other entries of u' (and u) being zero. Then $u'\in\mathcal{P}(G_{\mathrm{rd}},b)$ and $H_{\mathrm{rd}}=\mathrm{gr}(u')$, and so, using Theorem 22, u' is a vertex of $\mathcal{P}(G_{\mathrm{rd}},b)$. Also, using the first statement in Theorem 24 or the implication of (ii) by (i) in Theorem 29, u' is the only element of $\mathcal{P}(G_{\mathrm{rd}},b)$ whose graph is H_{rd}.

Conversely, assume that H does not have multiple edges and that $H_{\mathrm{rd}}=\mathrm{gr}(u')$, for a vertex u' of $\mathcal{P}(G_{\mathrm{rd}},b)$. Then, using Theorem 22, each component of H_{rd} either is acyclic or else contains exactly one cycle with that cycle having odd length. Therefore, each component of H either is acyclic or else contains exactly one cycle with that cycle having odd length. Now define $u\in\mathbb{R}^E$ by $u_e=u'_{e'}$ for each pair of edges e of H and e' of H_{rd} which have the same endpoints, with all other entries of u (and u') being zero. Then $u\in\mathcal{P}(G,b)$ and $H=\mathrm{gr}(u)$, and so, using Theorem 22, u is a vertex of $\mathcal{P}(G,b)$. Also, using the first statement in Theorem 24 or the implication of (ii) by (i) in Theorem 29, u is the only element of $\mathcal{P}(G,b)$ whose graph is H. □

Note that an alternative approach to proving Theorem 39 would involve using Theorems 37 and 38, rather than Theorem 22.

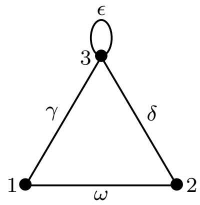
It can be seen that if G contains multiple edges, G_{rd} is a reduced graph of G, and the vertices of $\mathcal{P}(G_{\mathrm{rd}},b)$ (or just their graphs) are known, then the vertices of $\mathcal{P}(G,b)$ (or just their graphs) can easily be obtained as follows. First, choose a vertex u of $\mathcal{P}(G_{\mathrm{rd}},b)$ with graph H_{rd}, and, for each edge e of H_{rd}, choose a single edge f_e from among those edges of G which have the same endpoints as e. Then let H be the spanning subgraph of G with edge set $\{f_e \mid e \text{ is an edge of } H_{\mathrm{rd}}\}$. It follows, using Theorem 39, that H is the graph of a vertex w of $\mathcal{P}(G,b)$, where $w_{f_e}=u_e$ for each edge e of H_{rd}, with all other entries of w being zero (i.e., $w_e=0$ if e is an edge of G but not H), and that repeating these steps for all possible vertices u, and all possible associated edges f_e, produces each vertex of $\mathcal{P}(G,b)$ exactly once.

It also follows that the number of vertices of $\mathcal{P}(G,b)$ is given by

$$|\mathrm{vert}(\mathcal{P}(G,b))|=\sum_{u\in\mathrm{vert}(\mathcal{P}(G_{\mathrm{rd}},b))}\ \prod_{e\in\mathrm{supp}(u)} m(e), \tag{70}$$

where $m(e)$ is the number of edges of G which have the same endpoints as edge e of G_{rd}.

As an example, consider again G given by (3) and $b_1 = b_2 = b_3 = 1$. Consider also a reduced graph of G, with edge set $\{\omega, \gamma, \delta, \epsilon\}$, given by

$G_{\mathrm{rd}} =$  . (71)

In subsequent diagrams of spanning subgraphs of G_{rd}, the edge ω will always be represented by a straight line between the vertices 1 and 2, as in (71).

It can be seen that

$$\begin{aligned}\mathcal{P}(G_{\mathrm{rd}}, b) &= \left\{(x_\omega, x_\gamma, x_\delta, x_\epsilon) \in \mathbb{R}_{\geq 0}^{\{\omega,\gamma,\delta,\epsilon\}} \;\middle|\; x_\omega + x_\gamma = x_\omega + x_\delta = x_\gamma + x_\delta + x_\epsilon = 1\right\}\\ &= \left\{(x_\omega, 1 - x_\omega, 1 - x_\omega, 2x_\omega - 1) \in \mathbb{R}^{\{\omega,\gamma,\delta,\epsilon\}} \;\middle|\; \tfrac{1}{2} \leq x_\omega \leq 1\right\}.\end{aligned} \tag{72}$$

Therefore, $\mathcal{P}(G_{\mathrm{rd}}, b)$ is the line segment between vertices $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0)$ and $(1, 0, 0, 1)$.

The nonempty graphs of $\mathcal{P}(G_{\mathrm{rd}}, b)$ are shown in Figure 5.

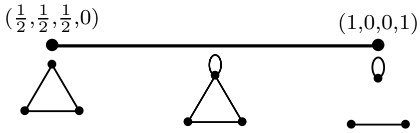


FIGURE 5. $\mathcal{P}(G_{\mathrm{rd}}, b)$, and the graphs of its nonempty faces, for G_{rd} given by (71) and $b_1 = b_2 = b_3 = 1$.

The pairs of spanning subgraphs H of G and H_{rd} of G_{rd}, for which H_{rd} is a reduced graph of H and the conditions of Theorem 37 apply, i.e., such that H is a graph of $\mathcal{P}(G, b)$ and H_{rd} is a graph of $\mathcal{P}(G_{\mathrm{rd}}, b)$, are shown in Table 2.

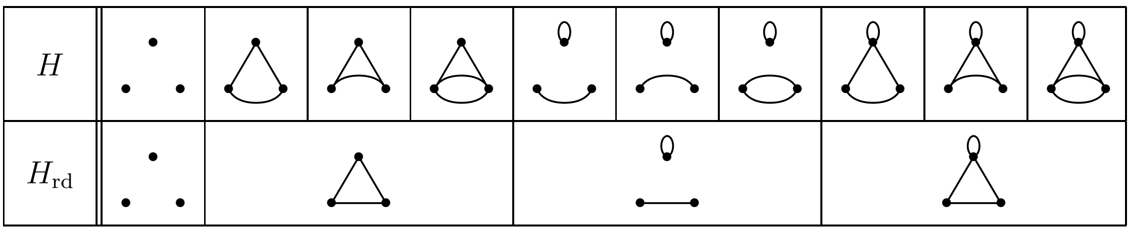


TABLE 2. Graphs H of $\mathcal{P}(G, b)$ and H_{rd} of $\mathcal{P}(G_{\mathrm{rd}}, b)$, such that H_{rd} is a reduced graph of H, for G given by (3), G_{rd} given by (71) and $b_1 = b_2 = b_3 = 1$.

The pairs of spanning subgraphs H of G and H_{rd} of G_{rd}, for which H_{rd} is a reduced graph of H and the conditions of Theorem 39 apply, i.e., such that H is the graph of a vertex u of $\mathcal{P}(G, b)$, H does not have multiple edges and H_{rd} is the graph of a vertex u' of $\mathcal{P}(G_{\mathrm{rd}}, b)$, are shown, together with the vertices u and u', in Table 3.

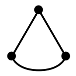
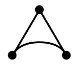
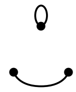
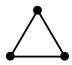
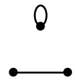
H				
$(u_\alpha, u_\beta, u_\gamma, u_\delta, u_\epsilon)$	$(\frac{1}{2},0,\frac{1}{2},\frac{1}{2},0)$	$(0,\frac{1}{2},\frac{1}{2},\frac{1}{2},0)$	(1,0,0,0,1)	(0,1,0,0,1)
H_{rd}				
$(u'_\omega, u'_\gamma, u'_\delta, u'_\epsilon)$	$(\frac{1}{2},\frac{1}{2},\frac{1}{2},0)$		(1,0,0,1)	

TABLE 3. Graphs H of vertices u of $\mathcal{P}(G,b)$ and H_{rd} of vertices u' of $\mathcal{P}(G_{\mathrm{rd}},b)$, such that H_{rd} is a reduced graph of H, for G given by (3), G_{rd} given by (71) and $b_1 = b_2 = b_3 = 1$.

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