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Double Implementation in a Market for Indivisible Goods with a Price Constraint

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Abstract

I consider the problem of assigning agents to indivisible objects, in which each agent pays a price for his object and all prices sum to a given constant. The objective is to select an assignment-price pair that is envy-free with respect to the agents' true preferences. I propose a simple mechanism whereby agents announce valuations for all objects and an envy-free allocation is selected with respect to these announced preferences. I prove that the proposed mechanism implements both in Nash and strong Nash equilibrium the set of true envy-free allocations.

Journal of Economic Literature classification numbers: C78, C71, D78

Keywords: Indivisible Goods, Envy-Freeness, Implementation, Strong Nash Equilibrium

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1 Introduction

I study the problem of assigning a set of indivisible objects to a set of agents. Each agent wants exactly one object, and his preferences are quasilinear in money. Each agent must pay a price corresponding to the object he gets, and prices are required to sum to a given number. A standard example is the housemate problem where a group of tenants is sharing an apartment. The objective is to determine who gets which room and how much each must pay subject to the constraint that the sum of their contributions must equal the rent of the apartment. However, neither the valuations of objects, nor the number that the prices must sum up to, need to be positive. An example with a negative price constraint includes heirs sharing inheritance that besides indivisible objects contains a divisible object - money. Negative valuations imply that the objects are not goods but rather bads or burdens. For example, a central government with fixed budget assigning waste disposal sites and other duties/projects to municipalities.

When deciding on the assignment of agents to objects and the corresponding prices, we may want to meet certain criteria. The usual requirements include efficiency and envy-freeness. Efficiency ensures that the welfare of society as a whole is maximized, while envy-freeness guarantees that each agent prefers his own object-price pair to the object-price pair of any other agent. In this sense envy-freeness is a sufficient condition for the stability of the assignment. Moreover, in this model it implies efficiency which makes it an attractive solution concept.

There exists a wide literature that uses the above framework. Shapley and Shubik [13] showed that the set of efficient and envy-free allocations can be found as a solution to a linear programming problem. Subsequent contributions have proposed different algorithms to find a particular envy-free allocation. Examples selecting envy-free allocations when prices are required to sum to a given number include algorithms by Abdulkadiroğlu *et al.* [1] and Haake *et al.* [8]. Since the set of efficient and envy-free allocations is usually not a singleton, these algorithms pick up different solutions, corresponding to different price vectors. In related works, Brams and Kilgour [5] and Chin Sung and Vlach [6] impose the additional constraint that prices must be nonnegative and analyze when the selected allocation is envy-free. The requirement of nonnegative prices is justified when objects are goods and the price constraint is positive, like in the room-sharing problem.

A shortcoming of all of the above studies is that they treat the valuations of objects as known by the social planner. However, in a more realistic setup the social planner lacks such perfect knowledge and instead solicits agents' valuations. Yet agents are interested to maximize their own utility and, in general, have no incentives to reveal their true valuations. Therefore, if we insist on using an algorithm to reach a particular allocation, a question arises on the scope that agents may manipulate the outcome by misrepresenting the valuations.

Motivated by these algorithms, I consider a mechanism that, given the announced valuations, selects an allocation that is efficient and envy-free with respect to these announced preferences. It is a direct revelation mechanism where agents' actions are messages of the valuations they attach to each object. The particular price vector that the mechanism selects among all possible envy-free prices, coincides with the one that would be selected by the algorithm of Abdulkadiroğlu *et al.* [1]. The advantage of this algorithm is that it provides a formula for the selected price vector in terms of the announced preferences, making it easy to establish whether there is a profitable deviation. In addition, since the algorithm by Abdulkadiroğlu *et al.* [1] does not specify which efficient assignment to be selected with respect to the announced preferences, I introduce a tie-breaking rule to ensure that the assignment is efficient with respect to the true preferences.

I prove that the proposed mechanism double implements the set of efficient and envy-free allocations both in Nash and strong Nash equilibrium. That is, I show, first, that all envy-free allocations (with respect to the true valuations) are outcomes of some (strong) Nash equilibrium of the game induced by the mechanism and, second, all (strong) Nash equilibrium outcomes of the game are envy-free. One implication of the result is that when choosing an efficient and envy-free allocation a social planner does not need to worry about strategic issues since the selected allocation will be envy-free not only with respect to the announced preferences but also with respect to the true ones. This provides a justification on strategic grounds for the use of social choice functions selecting envy-free allocations.

Although truth-telling need not be an equilibrium strategy, and in this sense, the allocation rule defined by the mechanism can be said to be manipulable, the scope for agents to manipulate the allocations is limited in equilibrium. In any (strong) Nash equilibrium the reported preferences will induce an allocation such that agents will be envy-free with respect to their true preferences. Thus, it also provides an answer to the question posed by Abdulkadiroğlu *et al.* [1] on the possible equilibria of the preference manipulation game induced by their algorithm.

When there is no requirement that prices must sum to a certain amount, it has been shown (Leonard [10]) that truth-telling is a dominant strategy for the mechanism which selects an efficient assignment of objects and agents pay the so-called agent-optimal prices. However, when prices are required to sum to a given amount, there is a trade off between strategy-proofness and envy-freeness: if the price that an agent pays depends on his valuations, then he would have incentives to misrepresent them, ruling out truth telling as a dominant strategy. On the other hand, if prices are independent of preferences, then envy-freeness is not guaranteed. Therefore, I use (strong) Nash equilibrium as a solution concept.

Another desirable feature of the proposed mechanism is that it is balanced, unlike the implementation in dominant strategies which requires side-payments to the third party. The latter can be justified in certain cases, for instance, when objects are auctioned and a seller receives the revenue such as in Demange *et al.* [7]. However, there are economic examples where such side-payments are ruled out or their amount is fixed in advance in which case a balanced mechanism is the appropriate one.

Tadenuma and Thomson [15] are the first to study the strategic aspects of using envy-freeness as a solution concept to allocate a single indivisible good to one of several agents when monetary compensations are available. They prove that both the set of Nash equilibrium allocations and that of strong Nash equilibrium allocations coincide with the set of envy-free allocations with respect to the true preferences. They showed that the result holds for any selection rule from the set of envy-free allocations and for any preference relation that is continuous and strictly monotone in money. Beviá [4] extends the result to the multiple objects case while restricting the preferences to quasi-linear in money. Beviá's [4] approach is more general in that she works with correspondences while my mechanism selects a single-valued outcome. As a consequence, Beviá [4] must use modified equilibrium concepts appropriate to multi-valued outcomes.

The remaining of the paper is organized as follows. The following section provides the formal model and some results necessary for the proof. Section 3 defines the implementation problem and states the theorem. An example for two-agent two-object case is provided in Section 4 before proving the theorem in Section 5. Final remarks in Section 6 conclude the paper. Some of the proofs are relegated to the Appendix.

2 Preliminaries

The set of agents is $I = \{1, ..., n\}$ and generic elements of I will be denoted by i and k. The set of objects is $J = \{1, ..., n\}$ with generic elements of J denoted by j and l. Throughout it is assumed that the number of agents and objects is the same n.[1] It is assumed that each agent consumes one and only one object. The matrix of true valuations is $A = [a_{ij}]_{i \in I, j \in J}$ where $a_{ij} \in \mathbb{R}$ is the valuation that agent i assigns to object j. The assignment of agents to objects is given by a one-to-one mapping $\mu : I \to J$. I denote a price vector by $p = (p_1, ..., p_n) \in \mathbb{R}^n$. Utilities are quasi-linear in money, namely, the utility of agent i from being assigned to object $\mu(i)$ and paying its price $p_{\mu(i)}$ is $u_i(p_{\mu(i)}) = a_{i\mu(i)} - p_{\mu(i)}$. Let M denote the set of assignments. An allocation is an assignment-price pair $(\mu, p) \in M \times \mathbb{R}^n$.

Definition 1 *An assignment $\mu \in M$ is* **efficient** *if $\sum_{i \in I} a_{i\mu(i)} \geq \sum_{i \in I} a_{i\eta(i)}$ for all assignments $\eta \in M$.*

[1]See the discussion in Section 6 when the number of agents and objects is different. Note, however, that the results stated in this section do not require that $|I| = |J|$.

Definition 2 *An allocation* $(\mu, p) \in M \times \mathbb{R}^n$ *is* **envy-free** *if* $u_i(p_{\mu(i)}) \geq u_i(p_j)$ *for all* $i \in I$ *and* $j \in J$.

Given an envy-free allocation $(\mu, p) \in M \times \mathbb{R}^n$ we will refer to p as an envy-free price. Also denote by M^A the set of efficient assignments relative to the matrix of valuations A. Svensson [14] proves that if the allocation (μ, p) is envy-free then the assignment μ is efficient. One can also think of envy-freeness as a requirement of stability since each individual prefers his object to any other object, given the vector of prices. Therefore, envy-freeness is used as a solution concept in most models dealing with indivisible objects, see for example Alkan *et al.* [2], Aragonés [3], Haake *et al.* [8], and Klijn [9].

The assignment problem of indivisible objects was first addressed by Shapley and Shubik [13] who proved that the problem can be translated into a linear programming problem where the efficient assignments are obtained from the primal problem but envy-free prices and the corresponding utilities come from the dual problem as shadow prices. Given the matrix of valuations A, define with the coalitional function $w(A, T, Q)$ the maximal value that a coalition of agents $T \subseteq I$ can obtain when assigned to a set of objects $Q \subseteq J$. It can be expressed in terms of the following linear programming problem[2]: given the matrix of valuations A and subsets T and Q, choose $(x_{ij})_{i \in T, j \in Q}$ to solve for

$$w(A, T, Q) \equiv \max \sum_{i \in T, j \in Q} a_{ij} x_{ij} \tag{1}$$

subject to

$$\sum_{i \in T} x_{ij} \leq 1 \qquad \text{for any } j \in Q \tag{2}$$

$$\sum_{j \in Q} x_{ij} \leq 1 \qquad \text{for any } i \in T \tag{3}$$

$$x_{ij} \geq 0 \qquad \text{for any } i \in T, j \in Q \tag{4}$$

$$\sum_{i \in T, j \in Q} x_{ij} = \min(|T|, |S|). \tag{5}$$

This primal problem has a corresponding dual problem where the costs of inputs — agents and objects — are minimized. Shadow prices are prices of objects and utilities of agents. Given the matrix of valuations A and subsets T and Q, choose $(u_i)_{i \in T}$ and $(p_j)_{j \in Q}$ to solve for

$$w(A, T, Q) \equiv \min \sum_{i \in T} u_i + \sum_{j \in Q} p_j \tag{6}$$

[2]The last condition is needed to ensure that agents will be assigned to objects even if their valuations are negative.

subject to

$$u_i + p_j \geq a_{ij} \quad \text{for any } i \in T, j \in Q. \tag{7}$$

Then the solution of the primal has the property that x_{ij} takes values 0 or 1 for all $i \in T$ and all $j \in Q$. Assume that $T = I$ and $Q = J$. The primal problem solves for an efficient assignment of objects as follows: given the solution $(x_{ij})_{i\in I, j\in J}$, define the assignment μ by letting $\mu(i) = j$ if and only if $x_{ij} = 1$. The dual problem gives the set of envy-free prices (this follows from constraint (7) since $u_i \geq a_{ij} - p_j$ for all $i \in I$ and all $j \in J$) and the corresponding utilities $(u_i \equiv u_i(p_{\mu(i)}))$.

The set of envy-free prices forms a lattice that possess the following property: if p' and p'' are two envy-free price vectors then so are the price vectors $\underline{p}$ and $\overline{p}$ where $\underline{p}_i = \min(p'_i, p''_i)$ and $\overline{p}_i = \max(p'_i, p''_i)$. This property is proven in Shapley and Shubik [13], see also Roth and Sotomayor [12] (chapter 8). The lattice has an agent-optimal price vector $p_* \geq 0$ such that $p \geq p_* \geq 0$ for all envy-free and non-negative prices p.

Given an efficient assignment μ, an agent-optimal price can be calculated[3] (see Leonard [10] or Roth and Sotomayor [12]) using the coalitional function, defined by equation (1), as

$$p_{*\mu(i)} = w(A, I\backslash\{i\}, J) - w(A, I\backslash\{i\}, J\backslash\{\mu(i)\}), \tag{8}$$

for each $i \in I$. From (8) it follows that $p_{*\mu(i)}$ does not depend on the object valuations of agent i. Using this property Leonard [10] proves that the mechanism that selects the agent-optimal prices p_* is strategy-proof. His result is a consequence of the well-known Clark's pivotal mechanism and is a special case of the results proven by Roberts [11] for quasilinear utility functions.

Here I state some additional results that will be useful later in proving the theorem.[4]

Proposition 1 *Given a matrix of valuations A, the coalitional function $w(A, T, Q)$ is continuous and weakly increasing in a_{ij}.*

It is shown in the proof of the Proposition 1 that when $i \in T$ and $j \in Q$ equation (1) can be written as

$$w(A, T, Q) = \max(const_1 + a_{ij} \cdot 0, const_2 + a_{ij} \cdot 1) \tag{9}$$

where $const_2 = w(A, T\backslash\{i\}, Q\backslash\{j\})$ and $const_1$ is a coalitional worth that is obtained by solving the original assignment problem subject to the additional constraint that agent i is not assigned to object j. Since neither $const_1$ nor $const_2$ is affected by the change in a_{ij} they can be considered constants. Function (9) is obviously continuous and weakly increasing in a_{ij}.

[3]For an example how to calculate agent-optimal prices, see Section 4.

[4]The proofs are provided in the Appendix.

Proposition 2 *Given a matrix of valuations A, the set of envy-free prices is the same for all efficient assignments of objects.*

Proposition 2 allows us to establish immediately the following result.

Corollary 1 *Fix an envy-free price vector p. Under all efficient assignments of objects each agent gets the same utility:*

$$a_{i\mu_1(i)} - p_{\mu_1(i)} = a_{i\mu_2(i)} - p_{\mu_2(i)}$$

for all $i \in I$ and where μ_1 and μ_2 are any two efficient assignments of objects.

3 Implementation Problem

In the implementation problem that I consider I restrict the set of feasible price vectors and require the prices to sum to a given number C: $\sum_{j \in J} p_j = C$. One can think in terms of an economy that consists of the set J of indivisible objects and a quantity C of the divisible object - money - that must be distributed among n agents.

Let $\triangle_C$ denote the set of price vectors that sum to C. Since all feasible price vectors are required to belong to this set, from now on it is understood that an allocation is an assignment-price pair $(\mu, p) \in M \times \triangle_C$. And with an envy-free price vector p I will only refer to price vectors that meet the price constraint: $p \in \triangle_C$. Notice that the set of envy-free prices in $\triangle_C$ is non-empty. For example, if we take the agent-optimal price vector p_* and add a constant c (positive or negative) to each element of p_*, the envy-freeness is preserved. We can always choose c such that $p_* + c\mathbf{1} \in \triangle_C$, where vector $\mathbf{1} \equiv (1, ..., 1) \in \mathbb{R}^n$.

Given C and the matrix of valuations A, denote the set of envy-free allocations in $M \times \triangle_C$ with $G(A)$. If a social planner were to choose an allocation (μ, p), arguably, he would prefer to select one from the set of envy-free allocations $(\mu, p) \in G(A)$ since these allocations meet the desirable normative criteria of envy-freeness and hence efficiency. The algorithms proposed by Abdulkadiroğlu *et al.* [1], Aragonés [3], Brams and Kilgour [5], Haake *et al.* [8], and Klijn [9] were designed to select allocations from the set $G(A)$. However, all of them rely on the knowledge of matrix A. If the social planner does not know the true preferences of agents, he will need to solicit them. A question arises whether agents have strategic incentives to reveal their true valuations. That is, an agent can find it profitable to announce valuations of objects different from his true ones.[5] Given this misrepresentation of preferences there is no guarantee anymore that the selected allocation by any of the algorithms will satisfy envy-freeness with

[5] Truth-telling is a dominant strategy when the assignment μ is efficient and agent i pays the agent-optimal price $p_{*\mu(i)}$ of the object he gets. However, in general the agent-optimal prices will not meet the price constraint, $p_* \notin \triangle_C$, and therefore, $(\mu, p_*) \notin G(A)$.

respect to the true preferences. However, it will be demonstrated that, with the help of an appropriate tie-breaking rule, selecting an allocation that is envy-free with respect to the announced preferences, not necessarily the true ones, achieves in the equilibrium envy-freeness with respect to the true valuations.

I propose a direct revelation mechanism where each agent is required to announce only his own valuations of objects and the mechanism selects an envy-free allocation with respect to the matrix of announced valuations. Formally, a strategy of agent i is a vector of object valuations $b_i = (b_{i1}, ..., b_{in}) \in \mathbb{R}^n$ that he announces. Given the matrix of reported valuations $B = [b_{ij}]_{i \in I, j \in J}$, denote the set of envy-free allocations implied by the matrix B by $G(B)$. A mechanism g is a mapping from the space of valuations into the space of allocations $g : R^{n \times n} \to M \times \triangle_C$. I restrict attention to mechanisms that, for each matrix of valuations B, will select an allocation (μ, p) from the envy-free set $G(B)$. The justification why to consider such mechanisms was provided before - the envy-freeness is the standard solution concept for such type of problems and there exist works that analyze how to select an allocation from the set $G(B)$ although usually ignoring strategic issues.

In general, the set of envy-free prices is not a singleton. From all envy-free prices the price vector that I select corresponds to the one that would be selected according to the algorithm of Abdulkadiroğlu *et al.* [1] when applied to the matrix B. The advantage of this price vector is its explicit linear relationship with the agent-optimal prices, given by the equation:

$$p_j = p_{*j} + \frac{C - \sum\limits_{m \in J} p_{*m}}{n} \text{ for all } j \in J. \tag{10}$$

where p_* is the vector of agent-optimal prices implied by B. According to this formula each agent $i \in I$ pays the agent-optimal price corresponding to the object he gets $p_{*\mu(i)}$ plus the equal share of the difference between the price constraint and the sum of all agent-optimal prices.

The utility of agent i having object $\mu(i)$ and paying price $p_{\mu(i)}$, by applying equation (10), is

$$u_i(p_{\mu(i)}) = a_{i\mu(i)} - p_{\mu(i)} = a_{i\mu(i)} - \frac{C}{n} - \frac{n-1}{n} p_{*\mu(i)} + \frac{1}{n} \sum_{l \neq i} p_{*\mu(l)}. \tag{11}$$

It follows that the utility of agent i is decreasing in its own agent-optimal price but increasing in each of other agent-optimal prices keeping the assignment μ fixed. We know from equation (8) that $p_{*\mu(i)}$ does not depend on the valuations of objects reported by agent i, that is, he cannot affect his own agent-optimal price. However, he can affect the agent-optimal prices of other objects.

If there are several efficient assignments of agents to objects with respect to the reported valuations B then the mechanism g will break ties according to the following rule. Order all objects and all agents, and without loss of generality assume that the order corresponds to the natural one: $\sigma(i) = i$ for all $i \in I$ and $\sigma(j) = j$ for all $j \in J$, and keep these orders fixed. Start with object 1 and proceed iteratively. If all efficient assignments allocate object 1 to the same agent i, then let agent i get it. Otherwise choose among all efficient assignments the one that assigns object 1 to the agent that has announced the smallest valuation for object 1: $\mu(i) = 1$ if $b_{i1} < b_{k1}$ for any k such that there exists an efficient assignment $\nu \in M^B$ under which $\nu(k) = 1$. If $b_{i1} = b_{k1}$ for two or more agents then select the agent from this set who has been assigned the lowest number: $\mu(i) = 1$ if $i < k$ when $b_{i1} = b_{k1}$. In general, assume that objects 1 to $l-1$ are already assigned. If all remaining efficient assignments allocate object l to the same agent i, then let agent i get it. Otherwise choose among all efficient assignments the one that assigns object l to the agent that has announced the smallest valuation of object l: $\mu(i) = l$ if $b_{il} < b_{kl}$ for any k such that there exists an efficient assignment $\nu \in M^B$ such that $\nu(k) = l$ and $\nu^{-1}(j) = \mu^{-1}(j)$ for already assigned objects $j \in \{1, ..., l-1\}$. If $b_{il} = b_{kl}$ for two or more agents select the agent from this set who has been assigned the lowest number: $\mu(i) = l$ if $i < k$ when $b_{il} = b_{kl}$. Thus the tie-breaking rule selects a unique assignment among all efficient assignments with respect to B. Thus, the mechanism g defines a game form, and given A, the pair (A, g) is a game in normal form.

Assume that the strategy profile B has been announced. When a set of agents T deviates and announces a different vector of valuations $b'_T \in \mathbb{R}^{|T| \times n}$, that leads to another profile $B' = (b'_T, b_{-T})$. When there is only one deviator, $T = \{i\}$, the strategy profile after the deviation is denoted by $B' = (b'_i, b_{-i})$. Denote the allocation induced by the deviation by $g(B') = (\mu', p')$. The solution concept that I use is strong Nash equilibrium.

Definition 3 *A strategy profile $B \in \mathbb{R}^{n \times n}$ is* **a strong Nash equilibrium** *relative to (A, g) if there is no coalition T and strategy profile b'_T such that $u_i(p'_{\mu'(i)}) \geq u_i(p_{\mu(i)})$ for all $i \in T$ and $u_i(p'_{\mu'(i)}) > u_i(p_{\mu(i)})$ for at least one $i \in T$.*

Denote the set of strong Nash equilibrium outcomes relative to (A, g) by $O^{SNE}_{(A,g)}$, that is $O^{SNE}_{(A,g)} = \{(\mu, p) \in M \times \triangle_C | g(B) = (\mu, p)$ for some pure strategy strong Nash equilibrium B relative to $(A, g)\}$. Similarly, we can define Nash equilibrium if we restrict the set of deviators T to a single agent i and denote the set of Nash equilibrium outcomes relative to (A, g) by $O^{NE}_{(A,g)}$.

Now we are ready to state the main result of the paper:

Theorem *The mechanism g double implements the social choice correspondence G both in Nash and strong Nash equilibrium: $O^{NE}_{(A,g)} = O^{SNE}_{(A,g)} = G(A)$ for all $A \in \mathbb{R}^{n \times n}$.*

4 An Example

Before providing the proof of the theorem, consider the following numeric two agent-two object example with the price constraint $C = 20$ and the matrix of valuations

$$A = \begin{pmatrix} 15 & 18 \\ 6 & 22 \end{pmatrix}.$$

The efficient assignment is $\mu(1) = 1$ and $\mu(2) = 2$ since $15 + 22 > 6 + 18$. The set of all envy-free prices is delimited by the equations $p_2 = 3 + p_1$ and $p_2 = 16 + p_1$ and shown in Figure 1 by the shaded area. To obtain the agent-optimal price of object 1, we find that $w(A, I\backslash\{1\}, J) = 22$ and $w(A, I\backslash\{1\}, J\backslash\{\mu(1)\}) = 22$, and by applying equation (8), $p_{*1} = 22 - 22 = 0$. In the same way we can find that $p_{*2} = 18 - 15 = 3$. Thus the agent-optimal prices are $p_* = (0, 3)$.

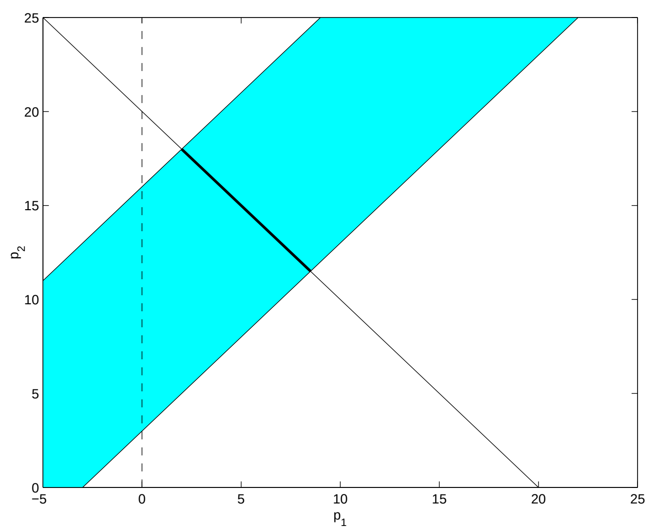

Figure 1: The set of envy-free prices

The prices that sum up to C are represented with the line connecting the points (20,0) and (0,20). The intersection of this line with the shaded region gives the set of envy-free prices that meet the price constraint. In general, there are an infinity of prices that are envy-free and meet the price constraint. The mechanism g that I consider selects, given the announced valuations, envy-free prices obtained from the agent-optimal prices by increasing all of them by the same amount so that the price constraint is met. If the announced valuations are A, then the vector of prices selected by the mechanism is $p = (8.5, 11.5)$ (found by adding 8.5 to the agent-optimal prices p_*).

The algorithm by Abdulkadiroğlu *et al.* [1] would also select the prices $p = (8.5, 11.5)$. Their algorithm finds the first envy-free price when we move from the initial price vector $p^0 = \frac{C}{n}$ along the rent constraint. Thus, the price p obtained by the algorithm is the most 'equal' price among all envy-free prices. In the example we start from $p^0 = (10, 10)$ and reach $p = (8.5, 11.5)$. The proposed mechanism does not ensure neither nonnegative prices nor individual rationality since it depends on the magnitude of C.[6] In the example, if $C < 3$ all envy-free price vectors have at least one negative price and if $C > 37$ then there is no envy-free price that would be individually rational.

In general, agents do not have incentives to announce the true valuations. Agent 1, by announcing the vector $b = (2, 18)$, still gets object 1 according to the tie-breaking rule but pays 2 instead of 8.5. Since, in order to find an efficient assignment, what matters is the relative magnitudes of valuations we can define $\beta_i \equiv b_{i2} - b_{i1}$. Then agent i gets object 2 and agent k gets object 1 if $\beta_i > \beta_k$ or if $\beta_i = \beta_k$ and $b_{i1} > b_{k1}$. One can check that when agent i gets object 2 and agent k gets object 1 the agent-optimal prices are given by $p_{*1} = \max(-\beta_i, 0)$ and $p_{*2} = \max(\beta_k, 0)$. It is easy to check that any strategy profile B where $3 \leq \beta_1 = \beta_2 \leq 16$ and $b_{11} \leq b_{21}$ is a Nash equilibrium. When $\beta_1 = \beta_2$ both assignments are efficient with respect to the announced preferences, but by announcing $b_{11} \leq b_{21}$ the tie-breaking rule ensures that the mechanism will select the assignment that is also efficient with respect to the true preferences.

In the proof of the theorem I consider two types of deviations when an agent feels envy. The first occurs when the agent still gets the same object after the deviation but the other agent now must pay a higher price and thus, according to the price constraint, the deviating agent pays a lower price. For example, consider $\beta_2 > \beta_1$, $\beta_2 > 0$ and $\beta_1 < 3$ and matrix A represents the true preferences. Given the announced preferences, $\mu(1) = 1$ and $\mu(2) = 2$ and the agent-optimal prices are $p_{*1} = \max(-\beta_2, 0) = 0$ and $p_{*2} = \max(\beta_1, 0)$. Therefore $p_1 = p_{*1} + (20 - p_{*1} - p_{*2})/2 = (20 - \max(\beta_1, 0))/2 > 8.5$ and $p_2 = 20 - p_1 < 11.5$. Agent 1 feels envy since $15 - p_1 < 18 - p_2$. Agent 1 can deviate and announce $\beta'_1 = \beta_2$ and $b'_{11} < b_{21}$. Then agent 1 still gets object 1 but pays only $p'_1 = (20 - \beta_2)/2 < p_1$ since $p'_{*1} = \max(-\beta_2, 0) = 0$ and $p'_{*2} = \max(\beta'_1, 0) = \beta_2$. Thus he had a profitable deviation.

The second type of deviation occurs when an agent gets the object he envies at the price that the agent who was originally assigned to it paid. For example, if $0 \geq \beta_2 > \beta_1$ and matrix A represents the true preferences then $\mu(1) = 1$ and $\mu(2) = 2$, $p_{*1} = \max(-\beta_2, 0) = -\beta_2$ and $p_{*2} = \max(\beta_1, 0) = 0$. Agent 1 must pay $p_1 = (20 - \beta_2)/2 > 8.5$ and like in the previous case, feels envy. Agent 1 can profitably deviate by announcing $\beta'_1 = \beta_2$ and $b'_{11} > b_{21}$. After the deviation the efficient assignment is $\mu'(1) = 2$ and $\mu'(2) = 1$ with agent-optimal prices $p'_{*1} = \max(-\beta'_1, 0) = -\beta_2$ and $p'_{*2} = \max(\beta_2, 0) = 0$. Agent 1 pays $p'_2 = p_2 = (20 + \beta_2)/2 < 11.5$. A similar profitable deviation exists when $\beta_2 = \beta_1$ and $b_{11} = b_{21}$. Then agent 1 is assigned to object 1 and will feel envy if $\beta_1 < 3$. Agent 1 is strictly better off by announcing $b'_{11} > b_{21}$ while keeping $\beta'_1 = \beta_2$. If

[6] An allocation $(\mu, p) \in M \times \triangle_C$ is **individually rational** if $u_{i\mu(i)}(p_{\mu(i)}) \geq 0$ for all $i \in I$.

$\beta_1 = 3$ then agent 1 is indifferent between getting object 1 and 2. Observe that the examples discussed cover all the cases when agent 1 could feel envy when he is originally assigned to object 1.

When there are more than two agents, it gets a little bit more complicated to demonstrate the existence of a profitable deviation when an agent feels envy. It may not be anymore possible either to increase the price paid by the agent who is assigned to the object that is envied or to obtain that object at the price that the agent who was originally assigned to it paid. For example, consider the following matrix of announced valuations

$$B = \begin{pmatrix} 5 & 10 & 15 \\ 5 & 10 & 0 \\ 0 & 10 & 20 \end{pmatrix}$$

and $C = 30$. The agent-optimal prices are $p_* = (0, 5, 10)$ and the prices selected that sum to 30 are $p = (5, 10, 15)$. There are two efficient assignments $\mu_1(1) = 1, \mu_1(2) = 2, \mu_1(3) = 3$ and $\mu_2(1) = 2, \mu_2(2) = 1, \mu_2(3) = 3$. The tie-breaking rule selects the first assignment. Suppose that agent 3 envies object 2 at the given prices: $a_{32} - 10 > a_{33} - 15$. Agent 3 can not increase the prices of objects 1 and/or 2 and thus decrease the price of object 3 and still get it. And neither he can obtain object 2 at price $p_2 = 10$. By announcing the vector of valuations $b_3' = (0, 15 + \epsilon, 20)$ where $\epsilon > 0$ ensures that $\mu'(3) = 2$ and the agent-optimal prices will be $p_*' = (0, 5, 10 - \epsilon)$ and the selected prices $p' = (5 + \epsilon/3, 10 + \epsilon/3, 15 - 2\epsilon/3)$. For ϵ sufficiently small agent 3 will find it advantageous to deviate since $a_{32} - 10 - \epsilon/3 > a_{33} - 15$.

5 Proof of The Theorem

Throughout the proof fix a matrix of true valuations A, and assume without loss of generality that the orders of agents and objects needed to define g are both $1, 2, ..., n$.

The set of Nash equilibria contains the set of strong Nash equilibria: $O^{SNE}_{(A,g)} \subseteq O^{NE}_{(A,g)}$. To establish the statement of the theorem, one needs to demonstrate, first, that for every envy-free allocation one can construct a strategy profile B that is a strong Nash equilibrium of the proposed game (A, g) (Lemma 1) implying $G(A) \subseteq O^{SNE}_{(A,g)} \subseteq O^{NE}_{(A,g)}$; second, that a strategy profile B where an agent feels envy at allocation $g(B) = (\mu, p)$ can not be a Nash equilibrium of the game (A, g) (Lemma 2) implying $O^{SNE}_{(A,g)} \subseteq O^{NE}_{(A,g)} \subseteq G(A)$. Combining the results of both Lemmas gives the desired result: $O^{NE}_{(A,g)} = O^{SNE}_{(A,g)} = G(A)$.

Lemma 1 *Let (μ, p) be an envy-free allocation. Then there is a strong Nash equilibrium B of (A, g) such that $g(B) = (\mu, p)$.*

Proof: Take an envy-free allocation $(\mu, p) \in G(A)$. Consider the following strategy profile B: each agent $i \in I$ announces $b_i = p + c_i\mathbf{1} \in \mathbb{R}^n$ where scalars c_i satisfy the following relationship for any two agents i and k: $c_i < c_k$ if and only if $\mu(i) < \mu(k)$. I claim that the given strategy profile constitutes a strong Nash equilibrium.

Observe that any possible assignment of objects is efficient with respect to B. The only envy-free price vector is p. The way how the scalars c_i for $i = \{1, ..., n\}$ were chosen ensures that the unique assignment, selected according to the tie-breaking rule, will be μ: an agent i who announced the smallest b_{i1} among all agents will be assigned to object 1 and by construction it was agent $\mu^{-1}(1)$. Among the remaining $n-1$ agents, agent $\mu^{-1}(2)$ announced the smallest b_{i2} therefore he is assigned to object 2, and so forth.

Assume on the contrary that there exists a profitable deviation by a group of agents T. Given the strategy profile after deviation $B' = (b'_T, b_{-T})$, the mechanism g selects an allocation (ν, p'). Since before deviation all agents $i \in T$ preferred their object-price pair to any other object-price pair and for a deviation to be profitable it must be that

$$a_{i\nu(i)} - p'_{\nu(i)} \geq a_{i\mu(i)} - p_{\mu(i)} \geq a_{i\nu(i)} - p_{\nu(i)} \tag{12}$$

with the first inequality strict for at least one agent $i \in T$. It follows that for all $i \in T$

$$p'_{\nu(i)} \leq p_{\nu(i)} \tag{13}$$

with at least one inequality strict. Thus there exits an object j whose price has strictly decreased: $p'_j < p_j$. Observe that if $T = I$ it follows immediately that the new price vector does not sum to C, a contradiction.

If $T \subsetneq I$ choose one of the objects j whose price has decreased the most. Since after the deviation the selected allocation $g(B') = (\nu, p')$ is envy-free with respect to the matrix B' then for each non-deviating agent $i \in I \backslash T$ we have an inequality

$$b_{i\nu(i)} - p'_{\nu(i)} \geq b_{ij} - p'_j. \tag{14}$$

Using the fact that before the deviation $b_{i\nu(i)} - p_{\nu(i)} = b_{ij} - p_j$ since $b_i = p + c_i\mathbf{1}$ we obtain for each agent $i \in I \backslash T$ that

$$0 > p'_j - p_j \geq p'_{\nu(i)} - p_{\nu(i)} \tag{15}$$

for the assignment ν. Thus it follows that $p'_{\nu(i)} < p_{\nu(i)}$ for all $i \in I \backslash T$. Combining it with (13) and summing up over all objects gives

$$\sum_{j=1}^{n} p'_j < \sum_{j=1}^{n} p_j,$$

a contradiction since both price vectors must sum to C. ■

Lemma 1 says that an envy-free allocation can be supported as a strong Nash equilibrium of (A, g). Therefore, if $(\mu, p) \in G(A)$ then $(\mu, p) \in O^{SNE}_{(A,g)}$. Note also that the proof does not depend on any particular way the prices are determined as long as they are envy-free with respect to the announced matrix B.

Lemma 2 *Let B be a strategy profile such that $g(B) = (\mu, p) \notin G(A)$. Then B is not a Nash equilibrium of (A, g).*

Proof: Let $g(B) = (\mu, p)$ be given and assume that agent i envies object j:

$$u_i(p_j) > u_i(p_{\mu(i)}). \tag{16}$$

Let $p_* \in \mathbb{R}^n$ be the agent-optimal price for B. I will construct a profitable deviation in two steps. In the first step I increase the announced valuation of object j by agent i to b'_{ij} which is just enough to ensure that there exists an efficient assignment ν such that $\nu(i) = j$.[7] As a result of the first step deviation the price of object j either strictly increases or remains the same. Depending on the case I construct the second step deviation. If the price has increased then the agent lowers valuation b'_{ij} by an ϵ. I show that he is still assigned to object $\mu(i)$ but now pays less. If the price of object j did not change after the first step deviation then the agent now increases valuation b'_{ij} by an ϵ to secure the assignment to object j. It is possible that the price of object j increases as a result of this second step deviation. However, for ϵ sufficiently small the after-deviation utility of agent i is still higher than before the deviation.

Thus, in the first step consider a possible deviation b'_i where agent i announces

$$b'_{ij} = w(B, I, J) - w(B, I\backslash\{i\}, J\backslash\{j\}) \geq b_{ij}$$

and $b'_{ik} = b_{ik}$ for all $k \neq j$. According to (9) we can distinguish between two cases before the deviation. First, there was an efficient assignment $\nu \in M^B$ such that $\nu(i) = j$. Then we have $const_1 \leq w(B, I\backslash\{i\}, J\backslash\{j\}) + b_{ij} = w(B, I, J)$. Then by the construction of the deviation $b'_{ij} = b_{ij}$ and $w(B', I, J) = w(B, I, J)$. Second, there was no efficient assignment $\nu \in M^B$ such that $\nu(i) = j$. It implies that $w(B, I\backslash\{i\}, J\backslash\{j\}) + b_{ij} < const_1 = w(B, I, J)$. By substituting this result for $const_1$ in equation (9) but applied to calculate $w(B', I, J)$, it again follows that $w(B', I, J) = w(B, I, J)$. Since after the deviation every assignment that achieves the coalitional worth equal to $w(B', I, J)$ is efficient, it follows that all assignments that were efficient before the deviation remain efficient after. That is, the deviation b'_i was constructed in such a way that no assignment that was efficient is destroyed by the deviation and if the deviation adds an additional efficient assignment, it must assign agent i to object j: if $\nu \in M^B$ then $\nu \in M^{B'}$,

[7] It could be that there already existed an efficient assignment ν such that $\nu(i) = j$, but according to the tie-breaking rule agent i was assigned to object $\mu(i)$. In that case we do not change the announced valuation b_{ij} in the first step at all.

and if $\nu \in M^{B'}$ but $\nu \notin M^B$ then $\nu(i) = j$. It follows that $\mu \in M^{B'}$. Therefore we can take the assignment μ to find the agent-optimal price of any object $l \in J$ after the deviation according to (8):

$$p'_{*l} = w(B', I\backslash\{\mu^{-1}(l)\}, J) - w(B', I\backslash\{\mu^{-1}(l)\}, J\backslash\{l\}). \tag{17}$$

Since $w(B', I, J) = w(B, I, J)$ and $b'_{\mu^{-1}(l)l} = b_{\mu^{-1}(l)l}$ for all $l \in J$ because the only valuation to change was b_{ij} but $\mu(i) \neq j$, therefore the second term of (17) does not change:

$$w(B', I\backslash\{\mu^{-1}(l)\}, J\backslash\{l\}) = w(B', I, J) - b'_{\mu^{-1}(l)l} = w(B, I, J) - b_{\mu^{-1}(l)l}.$$

By Proposition 1 the first term is weakly increasing in b_{ij}:

$$w(B', I\backslash\{\mu^{-1}(l)\}, J) \geq w(B, I\backslash\{\mu^{-1}(l)\}, J).$$

Therefore none of the agent-optimal prices can decrease as a result of the deviation.

In the continuation I analyze the following two cases:

Case 1 *The agent-optimal price of object j strictly increases: $p'_{*j} > p_{*j}$.*

It means that

$$w(B', I\backslash\{\mu^{-1}(j)\}, J) > w(B, I\backslash\{\mu^{-1}(j)\}, J).$$

This can happen only if $b'_{ij} > b_{ij}$ and there exists an efficient assignment $\eta : I\backslash\mu^{-1}(j) \to J$ for B' such that $\eta(i) = j$. Applying (9) we obtain that[8]

$$\begin{aligned} w(B', I\backslash\{\mu^{-1}(j)\}, J) &= const_2 + b'_{ij} > \\ w(B, I\backslash\{\mu^{-1}(j)\}, J) &= \max(const_1, const_2 + b_{ij}). \end{aligned} \tag{18}$$

Choose $\epsilon > 0$ such that

$$const_2 + b'_{ij} > const_2 + b'_{ij} - \epsilon > \max(const_1, const_2 + b_{ij}).$$

Now consider a deviation where agent i announces $b''_{ij} = b'_{ij} - \epsilon$ and $b''_{ik} = b_{ik}$ for all $k \neq j$. In what follows I compare the strategy profile after the deviation $B'' = (b''_i, b_{-i})$ with the initial strategy profile $B = (b_i, b_{-i})$. First, after the deviation the set of efficient assignments does not change $M^B = M^{B''}$, and hence $\mu \in M^{B''}$ is again selected. Second, using the same argument as when discussing the deviation $B' = (b'_i, b_{-i})$, none of the agent-optimal prices can decrease as a result of the deviation. Third, from (18) it follows that

$$w(B', I\backslash\{\mu^{-1}(j)\}, J) > w(B'', I\backslash\{\mu^{-1}(j)\}, J) > w(B, I\backslash\{\mu^{-1}(j)\}, J),$$

[8]Observe that the values of constants $const_1$ and $const_2$ change depending on the sets $T \subseteq I$ and $Q \subseteq J$ but not on the value of b_{ij} while keeping the rest b_{kl}, for $k \neq i$ and $l \neq j$, fixed.

and, as a result, $p''_{*j} = p'_{*j} - \epsilon > p_{*j}$. Fourth, the agent-optimal price of object $\mu(i)$ does not change $p_{\mu(i)} = p''_{\mu(i)}$ since by (8) it does not depend on the valuations of agent i. Then, according to (11), agent i is strictly better off after the deviation b''_i, thus strategy profile B was not an equilibrium.

Case 2 *The agent-optimal price of object j remains the same: $p'_{*j} = p_{*j}$.*

First I argue that this case implies that none of the agent-optimal prices will change due to the deviation b'_i, namely, $p'_{*k} = p_{*k}$ for all $k \in J$. From Proposition 2, in order to check whether a price vector is envy-free, one may consider any efficient assignment. Choose $\mu \in M^B$ since by the construction of B' the assignment $\mu \in M^{B'}$. Clearly, the only agent who could feel envy under the price vector p_* and the assignment μ, given the matrix of valuations B', is agent i and only with respect to object j. Suppose that agent i envies object j. Then we obtain that

$$b'_{i\mu(i)} - p_{*\mu(i)} < b'_{ij} - p_{*j} = b'_{ij} - p'_{*j} = b'_{i\mu(i)} - p'_{*\mu(i)},$$

where the first equality comes from the assumption that $p'_{*j} = p_{*j}$ and the second equality comes from Corollary 1, since the deviation was constructed to ensure that there exists an assignment $\nu \in M^{B'}$ such that $\nu(i) = j$. However, by (8) $p'_{*\mu(i)} = p_{*\mu(i)}$, a contradiction. Thus nobody feels envy relative to B' under price vector p_*. And it was argued before that as a result of the deviation b'_i, the agent-optimal prices cannot decrease, therefore p_* must be the vector of agent-optimal prices after the deviation.

Now consider a deviation b''_i where agent i announces, for sufficiently small ϵ,

$$b''_{ij} = b'_{ij} + \epsilon$$

and $b''_{ik} = b_{ik}$ for all $k \neq j$. After the deviation all efficient assignments will allocate object j to agent i: $\nu(i) = j$ for all $\nu \in M^{B''}$ and $B'' = (b''_i, b_{-i})$. In what follows I compare the situation when the strategy profile $B' = (b'_i, b_{-i})$ is used with the situation when the strategy profile $B'' = (b''_i, b_{-i})$ is used. Take any efficient assignment after the deviation b''_i: $\nu \in M^{B''}$. This assignment was efficient before the deviation: $\nu \in M^{B'}$. Again, agent i cannot affect his own agent-optimal price, here, the price of object j. According to (8), before the deviation the price of any object $l \neq j$ is equal to

$$p_{*l} = w(B', I\backslash\{\nu^{-1}(l)\}, J) - w(B', I\backslash\{\nu^{-1}(l)\}, J\backslash\{l\}), \tag{19}$$

where

$$w(B', I\backslash\{\nu^{-1}(l)\}, J\backslash\{l\}) = w(B', I\backslash\{\nu^{-1}(l), i\}, J\backslash\{l, j\}) + b'_{ij}$$

since $\nu(i) = j$. After the deviation b''_{ij} the second term of (19) has increased by ϵ, that is,

$$w(B'', I\backslash\{\nu^{-1}(l)\}, J\backslash\{l\}) = w(B', I\backslash\{\nu^{-1}(l)\}, J\backslash\{l\}) + \epsilon$$

for all $l \neq j$. The first term of (19) before the deviation is

$$w(B', I\backslash\{\nu^{-1}(l)\}, J) = \max(const_1, const_2 + b'_{ij}).$$

Therefore, after the deviation b''_{ij}, it belongs to the interval:

$$w(B', I\backslash\{\nu^{-1}(l)\}, J) + \epsilon \geq w(B'', I\backslash\{\nu^{-1}(l)\}, J) \geq w(B', I\backslash\{\nu^{-1}(l)\}, J).$$

It follows that the agent-optimal prices of objects other than j cannot increase and each of them can decrease at most by ϵ: $p_{*l} - \epsilon \leq p''_{*l} \leq p_{*l}$ for all $l \neq j$. Since, according to (11), the utility of agent i is increasing in the agent-optimal prices paid by other agents, consider the worst case: $p''_{*l} = p_{*l} - \epsilon$ for all $l \neq j$. Then the utility of the agent i after the deviation is $u_i(p''_j) = u_i(p_j) - \frac{n-1}{n}\epsilon$. By (16), for sufficiently small ϵ,

$$u_i(p''_j) = u_i(p_j) - \frac{n-1}{n}\epsilon > u_i(p_{\mu(i)}).$$

Thus, for sufficiently small ϵ, announcing

$$b''_{ij} = w(B, I, J) - w(B, I\backslash\{i\}, J\backslash\{j\}) + \epsilon$$

is a profitable deviation for agent i and the matrix B could not form a profile of Nash equilibrium strategies. ■

From Lemma 2 it follows that if B is a Nash equilibrium of (A, g) it must be envy-free, that is, if $g(B) \in O^{NE}_{(A,g)}$ then $g(B) \in G(A)$. Lemma 1 already established the converse inclusion $G(A) \subseteq O^{SNE}_{(A,g)} \subseteq O^{NE}_{(A,g)}$. Therefore $G(A) = O^{SNE}_{(A,g)} = O^{NE}_{(A,g)}$. Thus I have proven that the mechanism doubly implements the *no-envy correspondence*, which associates with each preference profile the set of envy-free allocations, both in Nash and in strong Nash equilibrium.

Remark 1 *Another definition of strong Nash equilibrium is obtained by requiring that a deviating coalition should have a strategy which makes every member of the coalition strictly better off. Let us denote the set of strong Nash equilibrium outcomes relative to (A, g) for this definition by $O^{SNE^*}_{(A,g)}$. In general, $O^{SNE}_{(A,g)} \subseteq O^{SNE^*}_{(A,g)} \subseteq O^{NE}_{(A,g)}$ holds true. Since it is shown that $O^{SNE}_{(A,g)} = O^{NE}_{(A,g)}$, $O^{SNE}_{(A,g)} = O^{SNE^*}_{(A,g)} = O^{NE}_{(A,g)}$ holds as well.*

6 Concluding Remarks

Given the announced preference profile, the mechanism selects a particular envy-free price vector although there may exist other envy-free prices. Beviá [4] demonstrates that the result extends to other selection rules, both single- and multi-valued, from the set of envy-free allocations when preferences are restricted to be quasilinear in money. However, the use of the given price selection rule entails no loss of generality since with the same rule one can achieve any envy-free price with respect to the true preferences as an equilibrium outcome of the game. Besides, the given selection rule possesses other desirable characteristics that makes it a natural candidate to use. In the absence of the price constraint, an envy-free allocation can be implemented in dominant strategies where envy-free price vector corresponds to the agent-optimal prices. Once there is an additional price constraint, the agent-optimal prices will not in general sum to it. The price vector selected by the mechanism, given the announced preference profile, is simply obtained by increasing or decreasing all agent-optimal prices by the same amount. Abdulkadiroğlu *et al.* [1] propose an algorithm how to find the given price vector and argue that its functioning resembles market process when the prices of over-demanded objects increase while the prices of under-demanded objects decrease in each iteration of the algorithm. They also show that if there exists an envy-free price vector that is nonnegative (with respect to the announced preferences), then the price selected by their algorithm must also be nonnegative.

Tadenuma and Thomson [15] established the result for any preference relation that is continuous and strictly monotone in money. Whether it holds true for multiple object case is an open question. While the construction of an equilibrium strategy profile would proceed along the lines of Lemma 1, the reasoning behind Lemma 2 can not be generalized. Nevertheless, I conjecture that any strategy profile when an agent feels envy with respect to the true preferences can be eliminated as an equilibrium outcome due to the continuity and strong monotonicity of preferences in money.

A feature of the mechanism is that an equilibrium strategy profile B will usually imply multiple efficient assignments with respect to the announced valuations. Therefore the mechanism always needs to rely on the tie-breaking rule to select the right assignment. However, the set of possible equilibria is not affected by the particular tie-breaking rule. One could substitute the present tie-breaking rule with any other rule that selects correctly the efficient assignment with respect to the true valuations. For example, a valid tie-breaking rule could be that additionally to their valuations agents announce the object they prefer. In equilibrium, each agent would announce the object that would be assigned to him if the true valuations were known. An advantage of the proposed tie-breaking rule however is that it requires that agents announce only their own valuations.

Notice that the mechanism does not ensure individual rationality, that is, agents may get lower utility by playing the game than by choosing not to participate. Thus it implicitly assumes that agents are forced to participate in the game. The reason is that the model imposes an exogenous price constraint which if big enough, rules out the existence of individually rational and envy-free allocations. On the other hand, if the constraint is variable so that it accommodates individual rationality, one could simply apply the results of Leonard [10] and Demange *et al.* [7] to implement in dominant strategies.

The model explicitly assumes that the number of agents and objects is the same. If the number of agents exceeded the number of objects one could introduce fictitious objects and the previous analysis would still apply. However, when the number of objects exceeds the number of agents, the introduction of fictitious agents does not work since it implies that some fictitious agent would need to pay a price or receive a transfer of the object he is assigned to. As a result the actually paid prices would not meet the price constraint.

7 Appendix: Proofs of Propositions 1 and 2

Proposition 1 *Given a matrix of valuations A, the coalitional function $w(A,T,Q)$ is continuous and weakly increasing in a_{ij}.*

Proof: If either $i \notin T$ or $j \notin Q$ then $w(A,T,Q)$ does not depend on a_{ij} and can be treated as constant - obviously continuous and weakly increasing in a_{ij}. Assume that $i \in T$ and $j \in Q$. Given a solution $(x_{kl})_{(k,l)\in T\times Q}$ to the primal problem (1)-(5), either $x_{ij} = 0$ or $x_{ij} = 1$. It follows that the maximal value $w(A,T,Q)$ is either the maximal value of the linear programming problem (1)-(5) subject to the additional constraint that $x_{ij} = 0$, or it is the maximal value of the problem (1)-(5) subject to the additional constraint that $x_{ij} = 1$, which ever is greater. Given this result, we can write equation (1) as

$$w(A,T,Q) = \max(const_1 + a_{ij} \cdot 0, const_2 + a_{ij} \cdot 1), \tag{20}$$

where

$$const_r = \sum_{(k,l)\in T\times Q\backslash\{(i,j)\}} a_{kl}x_{kl}$$

given the solution $(x_{kl})_{(k,l)\in T\times Q}$ to (1)-(5) subject to the additional constraint $x_{ij} = 0$ $(r = 1)$ or subject to the additional constraint $x_{ij} = 0$ $(r = 2)$. The function in (20) is obviously continuous and weakly increasing in a_{ij}. Note that $const_2 = w(A, T\backslash\{i\}, Q\backslash\{j\})$ since agent i has been assigned to object j and each agent can be assigned to at most one object and vice versa. ■

Proposition 2 *Given a matrix of valuations A, the set of envy-free prices is the same for all efficient assignments of objects.*

Proof: Take any two efficient assignments μ_1 and μ_2. Assume, on the contrary, that the price vector p is envy-free for the assignment μ_1 but it is not envy-free for the assignment μ_2. Envy-freeness of μ_1 implies that

$$a_{i\mu_1(i)} - p_{\mu_1(i)} \geq a_{i\mu_2(i)} - p_{\mu_2(i)} \tag{21}$$

for all $i \in I$. Assume without loss of generality that agent 1 envies object j under assignment μ_2:

$$a_{1\mu_1(1)} - p_{\mu_1(1)} \geq a_{1j} - p_j > a_{1\mu_2(1)} - p_{\mu_2(1)}. \tag{22}$$

Summing up equation (21) across all agents and using equation (22) we obtain

$$\sum_{i\in I} a_{i\mu_1(i)} - \sum_{i\in I} p_{\mu_1(i)} > \sum_{i\in I} a_{i\mu_2(i)} - \sum_{i\in I} p_{\mu_2(i)},$$

contradicting the assumption that μ_2 was an efficient assignment. ∎

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