Type III Subfactors and Planar Algebras

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This thesis is submitted in partial fulfillment of
the requirements for the degree of
Doctor of Philosophy

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Summary

In this thesis we investigate the theory of planar algebras for type III subfactors. We show directly how to associate a planar algebra to a type III subfactor using endomorphisms and intertwiners. We begin by describing how to define a type III version of the Temperley-Lieb planar algebra before giving a general definition of a type III planar algebra. We define a presenting map using endomorphisms and intertwiners and prove that this defines a type III subfactor planar algebra. We show that the definition of a type III subfactor planar algebra may be extended by removing the sphericality condition.

We also investigate the reverse implication, and show that if we start with a type III subfactor planar algebra we can produce a type III subfactor using techniques from Guionnet-Jones-Shlyakhtenko and free probability.

In the final chapter we investigate the type III version of A_2 planar algebras. We extend the results of Chapter 3 to the A_2 setting, defining a type III string algebra for $SU(3)$ $\mathcal{ADE}$ graphs and relating this to planar algebras. We also discuss further work here relating to A_2 planar algebras.

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Chapter 1

Introduction

Interest in the theory of subfactors was initiated by the paper of Jones [37], where he defined the index and proved that it takes values in the set $\{4\cos(\pi/n), n \geq 3\} \cup [4, \infty)$. Besides the index, many other invariants for subfactors have been defined. These include the principal and dual principal graphs and the standard invariant [37]. The standard invariant consists of commuting squares of finite dimensional C^*-algebras. Alternative characterisations of the standard invariant have been given by Popa's λ-lattices [76] and Ocneanu's paragroups [71], [70]. Under certain conditions the standard invariant is a complete invariant. The index, principal graph and standard invariant of a subfactor $N \subset M$ can be described in terms of bimodules. The theory of bimodules is one of the key tools in the study of II_1 subfactors.

The concept of a planar algebra was first introduced by Jones in [36]. A planar algebra is a way of representing the standard invariant of an extremal subfactor using a collection of finite dimensional vector spaces and multilinear maps which are represented graphically by planar tangles. The idea of a planar algebra grew from the graphical representation of the Temperley-Lieb algebra, which first appeared in [44]. The Temperley-Lieb algebra appears as a planar subalgebra of any planar algebra, this is because the Jones projections satisfy the Temperley-Lieb relations. Recently it has been shown [15] that planar algebras may be used to define invariants for non-extremal subfactors also.

Planar algebras have proved to be a useful tool in the study of subfactors. Firstly, as mentioned above a subfactor planar algebra is equivalent to the standard invariant of a subfactor. Another use is in proving the existence of subfactors with particular principal graphs [74], [5]. They have also been used to construct subfactors. In particular the work of Guionnet-Jones-Shlyakhtenko [28] provided the first step in this construction and demonstrated the connections between subfactor theory, free probability and random matrices. The link with free probability is due to the central importance of the Temperley-Lieb algebra in subfactor theory and the non-crossing pair partitions in free probability. A Temperley-Lieb diagram with n boundary points is just a non-crossing pair partition of n elements. This work has been continued by Jones-Shlyakhtenko-Walker [42], Kodiaylam-Sunder [49], [50], [51] and others. They have proved that any subfactor planar algebra P may be used to construct an inclusion of interpolated free group factors with P as its standard invariant. This should be compared with results of Popa and Shlyakhtenko [78], on the universality of $L\mathbb{F}_\infty$ in subfactor theory.

Another area of research is the A_2-planar algebras of Evans-Pugh [23], [24]. These can be used to study $SU(3)$ subfactors.

In parallel to the theory of II_1 subfactors is the theory of type III subfactors. In the II_1 case, the existence of a canonical positive definite trace is crucial in proving many results. The trace defines a conditional expectation, and it is this expectation which is used to define the index. In the type III case this trace no longer exists. Despite this Jones index theory has successfully been extended to the type III case, by Kosaki [52], Longo [58],[59], Hiai [31] and others. In the type III theory, instead of using the index of the trace preserving conditional expectation, the index is defined as the minimum over all possible conditional expectations. The role of bimodules in the type II theory is replaced with the role of endomorphisms here, since every bimodule is determined up to isomorphism by an endomorphism in the type III case. An equivalence relation may be defined on endomorphisms and the equivalence classes are called sectors. The theory of superselection sectors first appeared in the work of Doplicher, Haag and Roberts

[17], in the context of quantum field theory. It was noticed by Longo that this theory was applicable to the theory of type III subfactors and this was exploited in the papers [58], [59]. Sector theory can be used to describe the principal and dual principal graphs and is used in work of Izumi [32] on the classification of subfactors.

Popa [77] shows under certain conditions the standard invariant is a complete invariant for type III subfactors and in this case the type III subfactor $N \subset M$ is isomorphic to $(\mathcal{N} \subset \mathcal{M}) \otimes M$ where $\mathcal{N} \subset \mathcal{M}$ is a type II_1 subfactor.

In this thesis we investigate the theory of planar algebras for type III subfactors. We show directly how to associate a planar algebra to a type III subfactor. We do so using endomorphisms and intertwiners, relying on techniques of Izumi [34], and in particular the characterisation of the spaces of intertwiners between endomorphisms as string algebras. We show that most of the theory for type II subfactors carries over to the type III case with some minor changes. We also investigate the Guionnet-Jones-Shlyakhtenko construction in the type III setting. We prove that, starting with a type III subfactor planar algebra P, we can construct a type III subfactor with P as its planar algebra. We use techniques from free probability to study this tower of algebras. We also show how to define a type III version of the A_2-planar algebras.

1.1 Outline of Thesis

We begin in Chapter 2 with a detailed discussion of the background theory. We focus mainly on the theory of planar algebras for type II subfactors and the general theory of type III subfactors. We collect here all the definitions and results from the literature which we will need in the rest of the thesis.

In Chapter 3 we describe how to extend the theory of planar algebras to describe type III subfactors. We begin with a couple of simple examples. We define a type III analogue of the Temperley-Lieb planar algebra and show that it may be used to define inclusions of hyperfinite type III_λ factors. Using results of Popa mentioned above, we show that this inclusion can be split into a type II inclusion tensored with a type III factor. Next we make the general definition of a type III subfactor planar algebra. We define a presenting map using endomorphisms and intertwiners and prove that this defines a type III subfactor planar algebra. We also give a detailed construction of type III subfactors using string algebras. We show that our definition of a type III planar algebra may be extended further, to define non-spherical planar algebras. This relies on the type II construction of [15] and the 2-categories associated to a subfactor in the work of Longo-Roberts. Non spherical planar algebras correspond to subfactors which are not necessarily extremal. We also discuss how to extend the theory of planar modules to the type III setting.

In Chapter 4 we show how to use the planar algebras defined in Chapter 3 to construct a tower of type III factors. We define a type III Guionnet-Jones-Shlyaktenko construction and show that the subfactor constructed from a planar algebra P has P as its planar algebra. We use a graph construction, similar to [51], to split the factors into an amalgamated free product of simpler factors.

In Chapter 5 we investigate the type III version of A_2-planar algebras. We extend the results of Chapter 3 to the A_2 setting, defining a type III string algebra for $SU(3)$ $\mathcal{ADE}$ graphs and relating this to planar algebras. We also discuss further work here relating to A_2-planar algebras, namely the extension of the results of Chapter 4 to the A_2 setting and skein theory for A_2 algebras.

Chapter 2

Background and Preliminaries

2.1 von Neumann Algebras

A C^*-*algebra* A is a Banach *-algebra with a norm $\|\cdot\|$ satisfying $\|x^*x\| = \|x\|^2$ for all $x \in A$. It can be shown that any C^*-algebra is isomorphic to a norm closed *-subalgebra of $B(H)$, the space of bounded linear operators on a Hilbert space H. A *von Neumann algebra* M is a weakly closed unital *-subalgebra of $B(H)$. In this thesis we will always assume H to be a separable Hilbert space. A theorem of von Neumann proves that if M is a unital *-subalgebra of $B(H)$, then M is a von Neumann algebra if and only if $M = M''$, where $M' = \{x \in B(H) : xm = mx \quad \text{for all} \quad m \in M\}$ is the commutant of M. Given a C^*-algebra A, an element $x \in A$ is said to be *positive* if there exists $y \in A$ with $yy^* = x$. The collection of all positive elements of A is denoted by A_+. A simple unital C^*-algebra A is called *purely infinite* if for every non-zero $x \in A$ there exists $y \in A$ with $yxy^* = 1$.

A weight ϕ on a unital C^*-algebra A is a positive linear functional on A. If a weight ϕ also satisfies $\phi(1) = 1$ then we call ϕ a state. A linear map $\phi : A \to B$ between C^*-algebras is *positive* if $\phi(A_+) \subseteq B_+$. A linear map between von Neumann algebras is called *normal* if it is σ-weakly continuous. A *trace* is a state ϕ with $\phi(xy) = \phi(yx)$ for all $x, y \in A$. A state ϕ on A is said to be *faithful* if $\phi(x) = 0$ implies $x = 0$ for $x \in A_+$.

Let ϕ be a faithful normal semifinite weight on a von Neumann algebra M and let $\beta > 0$. Then there exists a unique one paramater group of automorphisms $\{\sigma_t : t \in \mathbb{R}\}$ on M such that $\phi(xy) = \phi(y\sigma_{i\beta}(x))$ for all $y \in M$ and all x entire for σ. In this case σ is called the *modular automorphism group* and ϕ is said to satisfy the KMS condition for σ at inverse temperature β. Given a von Neumann algebra M with state ϕ the centraliser of ϕ is $\{x \in M : \phi(xy) = \phi(yx) \quad \text{for all} \quad y \in M\}$ and is denoted by M_ϕ. For two von Neumann algebras $N \subset M$ the map $E : M \to N$ is a *conditional expectation* if it is positive and bounded and satisfies $E(1) = 1$ and $E(n_1 m n_2) = n_1 E(m) n_2$ for $n_i \in N$ and $m \in M$. If there is a trace tr on M then for $x \in M$ by Theorem 5.20 of [21] there is a unique $x' \in N$ such that $tr(xy) = tr(x'y)$ for all $y \in N$. We call the map $E : M \to N$ defined by $x' =: E(x)$ the conditional expectation relative to the trace.

Definition 2.1.1. Let B_3 be a von Neumann algebra with a finite faithful normal trace and let B_i be von Neumann subalgebras for $i = 0, 1, 2$ then the four von Neumann algebras

$$\begin{array}{ccc} B_0 & \subset & B_1 \\ \cap & & \cap \\ B_2 & \subset & B_3 \end{array} \tag{2.1}$$

are said to form a *commuting square* if they satisfy one of the following conditions, which are shown to be equivalent in Proposition 9.51 of [21]:

1. $E_{B_1}(B_2) \subset B_0$
2. $E_{B_2}(B_1) \subset B_0$
3. $E_{B_1} E_{B_2} = E_{B_0}$
4. $E_{B_2} E_{B_1} = E_{B_0}$
5. $E_{B_1} E_{B_2} = E_{B_2} E_{B_1}$ and $B_0 = B_1 \cap B_2$
6. $E_{B_0}(x) = E_{B_1}(x)$ for all $x \in B_2$
7. $E_{B_0}(x) = E_{B_2}(x)$ for all $x \in B_1$

where $E_{B_i} : B_3 \to B_i$ is the conditional expectation from $B_3 \to B_i$ relative to the trace for $i = 0, 1, 2$.

A *factor* is a von Neumann algebra M with trivial centre, that is $M' \cap M = \mathbb{C}$. Factors are important as any von Neumann algebra may be written as a direct integral of factors. Factors may be classified as follows.

1. Type I_n factors are matrix algebras $M_n(\mathbb{C})$ for $1 \le n < \infty$, type I_∞ are $B(\ell^2)$
2. Type II_1 factors are factors with a finite trace, type II_∞ are tensor products of a II_1 factor with a I_∞ factor
3. Type III are all other factors.

Type III factors were classified further into type III_λ for $0 \le \lambda \le 1$ in [10], where this classification depends on the Connes spectrum of the factor. Let M be a von Neumann algebra with modular automorphism group σ_t. Then the *Arveson spectrum* $Sp(\sigma_t) := \{s \in \mathbb{R} : \hat{f}(s) = 0, f \in I(\sigma)\}$ where $I(\sigma)$ is the intersection $\cap_{x \in M}\{f \in L^1(\mathbb{R}) : \sigma_f(x) = 0\}$ and $\sigma_f(x) = \int_{\mathbb{R}} f(t)\sigma_t(x)dt$ for $x \in M$. If p is a projection in M^σ we write $\sigma_t^p(x) = \sigma_t(x)$ for $x \in M_p = pMp$. The *Connes spectrum* $\Gamma(\sigma) = \cap_p\{Sp(\sigma_t^p)\}$ where the intersection is taken over all non zero projections p in the fixed point algebra M^σ. In fact, by Lemma XI 2.2 of [87] the intersection may be taken over all projections in the centre of M^σ and thus in the case where M^σ is a factor the Connes spectrum and Arveson spectrum coincide. A factor is of type III_1 if $\Gamma = \mathbb{R}_+$, type III_0 if the Connes spectrum is $\{1\}$ and type III_λ if the Connes spectrum is $\{\lambda^n : n \in \mathbb{Z}\}$. An *approximately finite dimensional* (AFD) or *hyperfinite* factor is a factor M which has an increasing sequence of finite dimensional subalgebras whose union is weakly dense in M. There is a unique AFD factor of type II_1, II_∞ and III_λ for each $\lambda \in (0, 1]$ this was proved for type II_1 in [68], for II_∞ in [11] and for type III in [10].

Suppose N, M are factors with N contained in M then $N \subset M$ is called a *subfactor*. In this thesis we will only consider the case when N and M are either both type II or both type III.

2.2 Bratteli Diagrams and Path Algebras

The material in this section may be found for example in Chapter 2 of [21]. Let $A := \varinjlim A_n$ be an *approximately finite dimensional* (AF) C^*-algebra, that is A is the inductive limit of finite dimensional C^*-algebras A_n with inclusion maps $i_n : A_n \to A_{n+1}$. It can be shown that there is a unique C^*-norm on the algebraic inductive limit. Recall that a finite dimensional C^*-algebra is isomorphic to a direct sum of matrix algebras $M_{n_1} \oplus \cdots \oplus M_{n_r}$. The minimal central projections of such an algebra are of the form $0\oplus\cdots\oplus 0\oplus 1\oplus 0\oplus\cdots\oplus 0$. Suppose A_n has minimal central projections $\Omega[n] := \{p_1^{(n)}, \dots, p_{r(n)}^{(n)}\}$ and let $\lambda_n = (\lambda_{ij}^{(n)})_{\substack{i=1,\dots,r(n)\\ j=1,\dots,r(n+1)}}$ be the multiplicity matrix for the inclusion A_n in A_{n+1}, that is, the simple subalgebra of A_n corresponding to the projection $p_i^{(n)}$ is embedded in the simple subalgebra of A_{n+1} corresponding to the projection $p_j^{(n+1)}$ with multiplicity $\lambda_{ij}^{(n)}$. Then the multiplicity graph G_n for the inclusion is the bipartite graph with $r(n)$ vertices along the top and $r(n+1)$ vertices along the bottom with $\lambda_{ij}^{(n)}$ edges joining vertex i along the top with vertex j along the bottom. The *Bratteli diagram* of A is obtained by concatenating the multiplicity graphs, identifying the vertices along the bottom of each G_n with those along the top of G_{n+1}. An example is shown in Figure 2.1. A Bratteli diagram can be associated to any AF algebra and two AF algebras with the same Bratteli diagrams are isomorphic.

Suppose we have a Bratteli diagram which describes unital embeddings. We now describe the path algebra model for such a Bratteli diagram. Suppose $m < n$ and let $i \in \Omega[m]$ and $j \in \Omega[n]$ be vertices in the Bratteli diagram. Let $Path(i, j)$ denote the collection of paths (of length $n-m$) from i to j in the Bratteli diagram. Given paths $\alpha \in Path(i, j)$ and $\beta \in Path(j, k)$, $\alpha \cdot \beta \in Path(i, k)$ denotes the concatenation of the paths. Denote by $A_{ij} := M_{|(Path(i,j))|}$. Then A_{ij} can be thought of as being generated by matrix units (μ, ν) where μ and ν are paths in $Path(i, j)$. Then let $A[m, n] = \bigoplus_{\substack{i\in\Omega[m]\\ j\in\Omega[n]}} A_{ij}$. For $m' < m < n < n'$ the algebra $A[m, n]$ may be embedded in $A[m', n']$ by

$$(\mu, \nu) \to \sum (\alpha \cdot \mu \cdot \beta, \alpha \cdot \nu \cdot \beta) \tag{2.2}$$

where the sum is over all paths $\alpha \in Path(k,i)$ for $k \in \Omega[m']$ and $\beta \in Path(j,l)$ where $l \in \Omega[n']$. The AF C^*-algebra is given by $A = \varinjlim A[0,n]$ with the inclusions of $A[0,n] \to A[0,n+1]$ given above.

2.3 II_1 subfactors

There are many invariants for II_1 subfactors including the *index*, the *standard invariant*, *planar algebras*, the *principal graph* and λ*-lattices*. Throughout this subsection $N \subset M$ will be an inclusion of II_1 factors. Standard references for the theory of II_1 factors are [21] and [43].

Using the uniquely defined trace on M define an inner product on M by $\langle x, y\rangle = tr(y^*x)$ and denote the completion of M by this inner product by $L^2(M)$. If the action of M on some Hilbert space H is isomorphic to the action of M on $(\bigoplus_{i=1}^n L^2(M))p$ where $p = (p_{ij})$ is a projection in $M_n(\mathbb{C}) \otimes M$ for some $n < \infty$ then we define the coupling constant $dim_M H := \sum_{i=1}^n tr(p_{ii})$. The Hilbert space $L^2(M)$ can be thought of as a left N module, where N acts by left multiplication. The index $[M : N]$ is defined to be $dim_N L^2(M)$. If the action is not of the above form we say $[M : N] = \infty$. Jones proved in [37] that the index of a subfactor may only take the values $\{4\cos^2 \pi/n \,:\, n = 3, 4, 5, \ldots\} \cup [4, \infty]$.

Given a subfactor $N \subset M$ there is a unique trace preserving conditional expectation $E_N : M \to N$. The map E_N may be extended uniquely to a projection $e_1 : L^2(M) \to L^2(N)$ where $L^2(N)$ may be identified with the subspace of $L^2(M)$ generated by the image of the elements of N. This projection is called the *Jones projection*. Then let M_1 be the von Neumann algebra generated by M and e_1 (both acting on $L^2(M)$). It can be shown that M_1 is a II_1 factor if and only if $[M : N]$ is finite. This is called the *basic construction*. In the finite index case the trace tr on M extends uniquely to a trace on M_1 which we denote also by tr and this satisfies $tr(xe_1) = [M : N]^{-1}tr(x)$. We may repeat the process on the subfactor $M \subset M_1$, with Jones projection $e_2 : L^2(M_1) \to L^2(M)$, to get $M_2 = \langle M_1, e_2\rangle$. Continuing iteratively, we get a tower of II_1 factors M_i and a sequence of projections e_i with

$$N \subset M \subset M_1 \subset M_2 \subset \cdots$$

Then it can be shown that when $[M : N] < \infty$ the grid of relative commutants

$$\begin{array}{ccccccccc} N' \cap N & \subset & N' \cap M & \subset & N' \cap M_1 & \subset & N' \cap M_2 & \subset \cdots \\ & & \cup & & \cup & & \cup & \\ & & M' \cap M & \subset & M' \cap M_1 & \subset & M' \cap M_2 & \subset \cdots \end{array}$$

is a grid of finite dimensional C^*-algebras with a consistent trace. This is called the *standard invariant* of the subfactor $N \subset M$.

Recall, e.g. from [21] Section 9.5, that the Temperley-Lieb algebra is the universal *-algebra on n generators $E_1, \dots, E_n$ satisfying the relations

1. $E_i^2 = E_i = E_i^*$

2. $E_i E_j = E_j E_i$ if $|i - j| > 1$

3. $E_i E_{i\pm1} E_i = \delta E_i$ for all i

The Jones projections $\{e_i\}$ satisfy the Temperley-Lieb relations, with $\delta = [M : N]^{-1}$.

Note that we have $\{1, e_1, \dots, e_n\}'' \subset N' \cap M_n$ for all n.

In [76] an alternative formulation of the standard invariant called a λ-lattice is defined. In order to define this we first need to define the index for inclusions of finite dimensional algebras. This definition may be found for example in Chapter 3 of [27]. The index $[B : A]$ for finite dimensional von Neumann algebras $A \subset B$ is defined as follows. If $A \simeq M_n(\mathbb{C})$, $B \simeq M_m(\mathbb{C})$ then $[B : A] = m^2/n^2$. Otherwise let $A = \bigoplus_{i=1}^n Ap_i$ and $B = \bigoplus_{j=1}^m Bq_j$ where p_i, q_j are the collections of minimal central projections of A, B respectively. Define

$$A_{i,j} := q_j p_i A p_i q_j, \qquad B_{i,j} := p_i q_j B q_j p_i.$$

Then define the inclusion matrix Λ_A^B by $\lambda_{ij} = [B_{ij}, A_{ij}]^{\frac{1}{2}}$ if $q_j p_i \neq 0$ and $\lambda_{ij} = 0$ otherwise. Then the index $[B : A]$ is defined as $\|\Lambda_A^B\|^2$.

A λ-*lattice* is a collection of finite dimensional algebras $A_{i,j}, i, j \in \mathbb{N}$ with $A_{i,i} = \mathbb{C}$ for all i, $A_{i,j} \subset A_{k,l}$ for $k \leq i$ and $j \leq l$ with a faithful trace on $\cup_{n\in\mathbb{N}}$ $A_{0,n}$ and such that

1. $E_{A_{i,j}} E_{A_{k,l}} = E_{A_{k,l}} E_{A_{i,j}} = E_{A_{m,n}}$ for $m = \max\{i,k\}$, $n = \min\{j,l\}$ where E_A is the trace preserving conditional expectation onto A

2. there exists a representation of the sequence of Jones projections $\{e_i\}_{i\geq 2}$ in $\cup_n A_{0,n}$ with

   $e_j \in A_{i-2,k}$ for $2 \leq i \leq j \leq k$

   $e_{j+1} x e_{j+1} = E_{A_{i,j+1}}(x) e_{j+1}$ for all $x \in A_{i,j}$, $i \leq j-1$

   $e_{i+1} x e_{i+1} = E_{A_{i+1,j}}(x) e_{i+1}$ for all $x \in A_{i,j}$ $i \leq j-1$

3. $[A_{i,j+1} : A_{i,j}] \leq \lambda^{-1} E_{A_{i,j}}(e_{j+1}) = \lambda I$

   $[A_{i-1,j} : A_{i,j}] \leq \lambda^{-1} E_{A_{i-1,j}}(e_i) = \lambda I$

A λ-lattice is called *standard* if $[A_{ij}, A_{kl}] = 0$ for all $0 \leq i \leq j \leq k \leq l$.

A finite index subfactor is called *extremal* if $E_{N'\cap M}(e_1) \in \mathbb{C}$. In [76] it is proved that the tower of relative commutants of an extremal subfactor forms a standard λ-lattice and that every standard λ-lattice is the tower of relative commutants for some extremal subfactor. However even if we start with a hyperfinite subfactor and associate a λ-lattice to it, the subfactor associated to that λ-lattice in [76] may not be hyperfinite.

Drawing the Bratteli diagram for the tower of relative commutants at each step we get the reflection of the previous step plus a possible new part. The *principal graph* is obtained by deleting all the reflected parts of the Bratteli diagram, this is shown in bold in Figure 2.1. The dual principal graph is obtained by repeating the same procedure for the tower $M' \cap M \subset M' \cap M_1 \subset \cdots$. A subfactor is called *finite depth* if its principal graph is finite. Note that the principal graph is finite if and only if the dual principal graph is finite.

The principal graph may alternatively be described as follows. The collection of even vertices is given by the irreducible $N - N$ bimodules ${}_N X_N$ occuring in

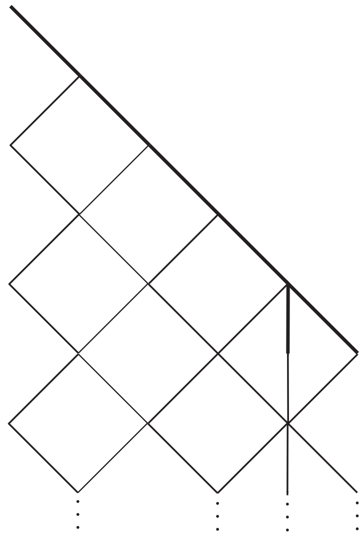


Figure 2.1: Bratteli Diagram

the decomposition of the bimodules ${}_N L^2(M_n)_N$. The odd vertices are bimodules ${}_N X_M$ occuring in the decomposition of the ${}_N L^2(M_n)_M$. A vertex corresponding to ${}_N X_N$ is joined to the vertex labelled ${}_N Y_M$ by n edges if the decomposition of the bimodule ${}_N Y \otimes_M L^2(M)_N$ contains n copies of ${}_N X_N$.

For subfactors of index less than 4 the only possible principal graphs are the Dynkin diagrams, shown in Figure 2.2. The A_n graphs, correspond to index $4\cos^2(\pi/n+1)$, D_{2n} correspond to index $4\cos^2(\pi/4n-2)$, and E_6 and E_8 correspond to indices $4\cos^2(\pi/12)$ and $4\cos^2(\pi/30)$ respectively. For index greater than 4 it is still an open problem, but all possible graphs of index less than $3+\sqrt{2}$ were listed by Haagerup in [30], more recently, in the series of papers [64], [62], [73], [35] it is shown that there are exactly five finite graphs which appear as principal graphs of subfactors with index in the interval $(4,5)$. It has been proved by Izumi that there are exactly five principal graphs subfactors with index 5, all of these are subgroup subfactors.

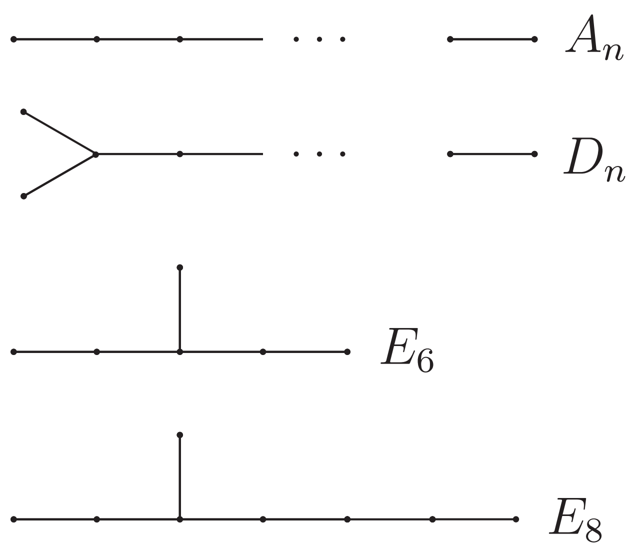


Figure 2.2: ADE graphs

2.4 String Algebra Construction of II_1 Subfactors

Consider the finite graph $\mathcal{G}$ with subgraphs $\mathcal{G}_0$, $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ as shown.

$$\begin{array}{ccc} V_0 & \xrightarrow{\mathcal{G}_0} & V_1 \\ {\scriptstyle \mathcal{G}_3}\downarrow & & \downarrow{\scriptstyle \mathcal{G}_1} \\ V_3 & \xrightarrow[\mathcal{G}_2]{} & V_2 \end{array}$$

We assume that $\mathcal{G}_0$ and $\mathcal{G}_2$ are connected and that $\mathcal{G}_0$ has more than one edge. We denote the common vertices of $\mathcal{G}_0$ and $\mathcal{G}_3$ by V_0, the common vertices of $\mathcal{G}_0$ and $\mathcal{G}_1$ by V_1 and so on, as shown in the diagram above. For each edge ξ denote its source vertex by $s(\xi)$ and range by $r(\xi)$. Denote by $\tilde{\xi}$ the edge ξ with the opposite orientation. Denote the number of edges between vertices v and w by $n_{v,w}$. Suppose we have an assignment of a strictly positive number $\mu(v)$ to each vertex v. For each possible square $\sigma_0, \ldots, \sigma_3$ such that σ_i is an edge in $\mathcal{G}_i$ we assign a complex number denoted $\begin{array}{ccc} A & \xrightarrow{\sigma_0} & B \\ {\scriptstyle \sigma_3}\downarrow & & \downarrow{\scriptstyle \sigma_1} \\ C & \xrightarrow[\sigma_2]{} & D \end{array}$. This assignment should satisfy the following conditions.

1. *Unitarity* Suppose A,D,B, B' are vertices in $\mathcal{G}$ such that B and B' are in

the same V_i and suppose σ_0,σ_0',σ_1,σ_1' are edges as shown. Then we have the identity

$$\sum_{C,\sigma_2,\sigma_3} \begin{array}{ccc} A & \xrightarrow{\sigma_0} & B \\ \sigma_3\downarrow & & \downarrow\sigma_1 \\ C & \xrightarrow[\sigma_2]{} & D \end{array} \; \overline{\begin{array}{ccc} A & \xrightarrow{\sigma_0'} & B' \\ \sigma_3\downarrow & & \downarrow\sigma_1' \\ C & \xrightarrow[\sigma_2]{} & D \end{array}} = \delta_{B,B'}\delta_{\sigma_0,\sigma_0'}\delta_{\sigma_1,\sigma_1'}.$$

2. *Renormalisation*

$$\begin{array}{ccc} A & \xrightarrow{\sigma_0} & B \\ \sigma_3\downarrow & & \downarrow\sigma_1 \\ C & \xrightarrow[\sigma_2]{} & D \end{array} = \sqrt{\frac{\mu(B)\mu(C)}{\mu(D)\mu(A)}} \; \overline{\begin{array}{ccc} B & \xrightarrow{\tilde\sigma_0} & A \\ \sigma_1\downarrow & & \downarrow\sigma_3 \\ D & \xrightarrow[\tilde\sigma_2]{} & C \end{array}} = \sqrt{\frac{\mu(B)\mu(C)}{\mu(D)\mu(A)}} \; \overline{\begin{array}{ccc} C & \xrightarrow{\sigma_2} & D \\ \tilde\sigma_3\downarrow & & \downarrow\tilde\sigma_1 \\ A & \xrightarrow[\sigma_0]{} & B \end{array}}$$

3. *Harmonicity*: There exists $\delta' > 0$ such that

$$\delta'\mu(v) = \sum_{w\in V_k} n_{v,w}\mu(w)$$

for any pair of vertices v and w connected by an edge in $\mathcal{G}_1$ or $\mathcal{G}_3$. There also exists $\delta > 0$ such that

$$\delta\mu(v) = \sum_{w\in V_k} n_{v,w}\mu(w)$$

for any pair of vertices v and w connected by an edge in $\mathcal{G}_0$ or $\mathcal{G}_2$.

Such an system is called a *biunitary connection.* Fix a vertex in V_0 and denote it by $*$. Normalise μ so that $\mu(*) = 1$. A connection is called *flat* if it satisfies

$$\begin{array}{ccccccc} * & \xrightarrow{\sigma_1} & \xrightarrow{\sigma_2} & \cdots & & \xrightarrow{\sigma_{2n}} & * \\ \rho\downarrow & & & & & & \downarrow\rho_1' \\ \rho_2\downarrow & & & & & & \downarrow\rho_2' \\ \vdots & & & & & & \vdots \\ \rho_{2m}\downarrow & & & & & & \downarrow\rho_{2m}' \\ * & \xrightarrow[\sigma_1']{} & \xrightarrow[\sigma_1']{} & \cdots & & \xrightarrow[\sigma_{2n}']{} & * \end{array} = \delta_{\sigma_1,\sigma_1'}\cdots\delta_{\sigma_{2n},\sigma_{2n}'}\delta_{\rho_1,\rho_1'}\cdots\delta_{\rho_{2m},\rho_{2m}'}$$

for all choices of $n, m, \sigma_i, \sigma_i', \rho_i, \rho_i'$. The left hand side of the equation is calculated as follows. Suppose we fill the inside of the rectangle with a grid of squares in $\mathcal{G}$. We call any such choice of edges a configuration. The value of a configuration will be the product of the $4mn$ values of the connection contained in the configuration. The left hand side of the equation is then defined to be the sum over all possible choices of configuration of the values of the configuration.

Starting with the graph $\mathcal{G}_0$ we define a collection of *-algebras $A_{0,k}$ as follows. A path in $\mathcal{G}_0$ is a succession of edges $\xi = \sigma_1 \cdots \sigma_n$ with $r(\sigma_i) = s(\sigma_{i+1})$ for all i. We denote by $|\xi|$ the length of ξ. As a vector space $A_{0,k}$ has basis consisting of pairs of paths (ξ_1, ξ_2) on $\mathcal{G}_0$ with $s(\xi_1) = s(\xi_2) = *$, $r(\xi_1) = r(\xi_2)$ and $|\xi_1| = |\xi_2| = k$. We may define a multiplication and *-operation by

$$(\xi_1, \xi_2)(\zeta_1, \zeta_2) = \delta_{\xi_2, \zeta_1}(\xi_1, \zeta_2)$$

$$(\xi_1, \xi_2)^* = (\xi_2, \xi_1).$$

Under these operations $A_{0,k}$ is a *-algebra. The algebra $A_{0,k}$ may be embedded in the algebra $A_{0,k+1}$ as in Equation 2.2 by

$$(\xi_1, \xi_2) \to \sum_{|\sigma|=1} (\xi_1 \cdot \sigma, \xi_2 \cdot \sigma)$$

where $\xi_1 \cdot \sigma$ means the concatenation of the two paths and the sum is over all edges σ with $s(\sigma) = r(\xi_1)$. We may define a trace tr on $\cup A_{0,k}$ by $tr(\xi_1, \xi_2) = \delta_{\xi_1, \xi_2}(\delta')^{-|\xi|}\mu(r(\xi_1))$. Let $A_{0,\infty}$ denote the von Neumann algebra obtained by taking the completion of $\cup A_{0,k}$ with respect to this trace. It follows from Theorem 10.3 in [21] that the trace on $\cup A_{0,k}$ is unique and hence $A_{0,\infty}$ is a hyperfinite II_1 factor.

We define the string algebras $A_{l,k}$ in a similar manner. As a vector space $A_{l,k}$ has basis given by pairs of paths $(\alpha_1 \cdot \beta_1, \alpha_2 \cdot \beta_2)$, where α_i is a path in $\mathcal{G}_0$ with $s(\alpha_i) = *$ and $|\alpha_i| = k$. The path β_i is a path in $\mathcal{G}_1$ if k is odd and $\mathcal{G}_3$ if k is even. It satisfies $r(\alpha_i) = s(\beta_i)$, $|\beta_i| = l$ and $r(\beta_1) = r(\beta_2)$. The multiplication and *-operation are defined exactly as for $A_{0,k}$. Using the connection it is possible to transform the basis of $A_{l,k}$ into a basis where we first travel l steps along $\mathcal{G}_3$ and then k steps along $\mathcal{G}_0$ or $\mathcal{G}_2$ or any other combination of paths on $\mathcal{G}$ with

k horizontal steps and l vertical steps. Define a trace on $A_{l,k}$ by $tr_{l,k}(\xi_1, \xi_2) = \delta^{-l}(\delta')^{-k}\delta_{\xi_1,\xi_2}\mu(r(\xi))$. Let $A_{n,\infty}$ be the von Neumann algebra completion $\cup_{k\in\mathbb{N}} A_{n,k}$ with respect to the trace. As before this is a hyperfinite II_1 factor.

Define the vertical Jones projection $e_n \in A_{n,0}$ by

$$e_n = \sum_{\xi,\zeta,\eta} \frac{1}{\delta} \frac{\sqrt{\mu(r(\eta))\mu(r(\zeta))}}{\mu(r(\xi))} (\xi \cdot \eta \cdot \tilde{\eta}, \xi \cdot \zeta \cdot \tilde{\zeta})$$

where the sum is over all paths ξ of length $n-2$ starting at $*$ and all edges ζ and η for which the sum makes sense.

Theorem 11.9 of [21] states that the construction above defines a subfactor $A_{0,\infty} \subset A_{1,\infty}$ with $[A_{1,\infty}, A_{0,\infty}] = \delta^2$ and tower

$$A_{0,\infty} \subset A_{1,\infty} \subset A_{2,\infty} \subset \dots$$

where the Jones projection of the inclusion $A_{n,\infty} \subset A_{n+1,\infty}$ is given by e_n. Flatness of the connection is equivalent to the condition that any element of $A_{k,0}$ commutes with $A_{0,\infty}$. Theorem 11.15 of [21] then states that if the connection is flat, the relative commutant $A'_{0,\infty} \cap A_{k,\infty}$ is then $A_{k,0}$.

2.5 Planar Algebras

The idea of planar algebras was first introduced by Jones in [36] as an alternative formulation of the standard invariant of a subfactor.

A planar tangle is a disc D_0 in the plane with a collection of internal discs D_i, $1 \leq i \leq n$. Each disc D_i, $0 \leq i \leq n$ has a $2k_i$ marked points on its boundary. The interior of D also contains a collection of non-overlapping strings. Strings may only intersect the boundaries of the discs at the marked points. Every marked point is the endpoint of exactly one string and each string either forms a closed loop or has exactly two endpoints, both occuring at marked points of some disc. For each disc D_i one boundary region will be marked with a star. Regions of a tangle are shaded black or white and adjacent regions are required to have different colours. A + tangle is one which has the star of D_0 in a white region

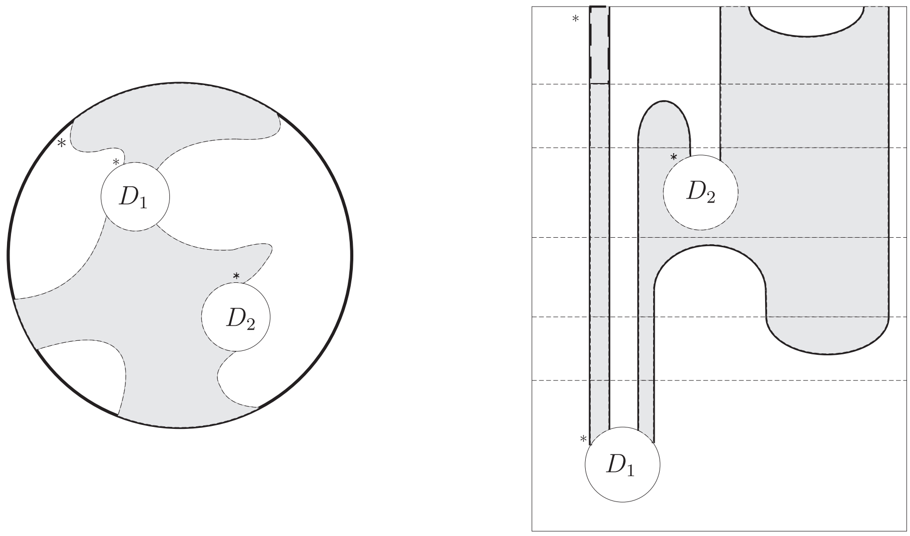


Figure 2.3: Tangle T and T in a standard form

and a $-$ tangle has the star of D_0 in a black region. An example of a $+, 3$ tangle is shown in Figure 2.3. A planar tangle is said to be in *standard form* if all the marked points of the discs are along the top edge and the tangle is drawn in such a way that each horizontal strip contains at most one cup or cap of a string or one internal disc. A planar tangle with one input disc is called an annular tangle.

If the outer pattern of a tangle S is the same as the pattern of some inner disc D_i of a tangle T then we can form the tangle $T \circ_i S$ by gluing S inside D_i, removing the boundary and smoothing the strings.

A *planar algebra* is a collection of vector spaces $P_i^{\pm}$, $i \in \mathbb{N}$, with a collection of multilinear maps Z_T (one for each tangle) that are consistent with composition of tangles and relabelling of internal discs in the obvious way. If T is a tangle with k_0 marked boundary points and n internal discs D_i, each with $2k_i$ marked boundary points then $Z_T : \otimes_{i=1}^n P_{k_i} \to P_{k_0}$.

Given two elements $x, y \in P_k$, $k \in \mathbb{N}$, their product is defined as $Z_M(x, y)$ where M is the multiplication tangle shown in Figure 2.4. This multiplication, along with the inclusion maps gives P the structure of an associative algebra. The *Fourier transform* tangle, shown in Figure 2.5 gives a canonical identification of the P_n^+ with P_n^- and so we usually just work with the P_n^+ and we often write P_n instead of P_n^+. Thus when we draw tangles we omit the shading with the

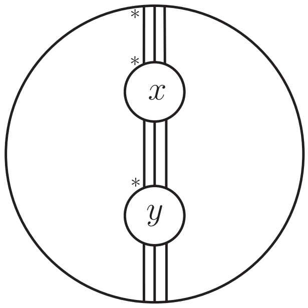


Figure 2.4: Multiplication Tangle

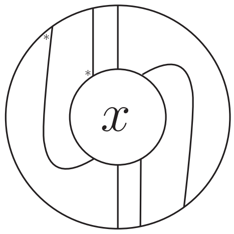


Figure 2.5: Fourier Transform

understanding that it is determined by the fact that the region containing $*$ is unshaded.

A planar algebra has *modulus* δ if, given a tangle T containing a closed loop, we have $Z_T = \delta Z_{T'}$ where T' is the tangle T with the closed loop removed.

The *adjoint* T^* of a tangle T is the tangle obtained by reflecting the tangle through a horizontal line through its centre.

A *planar* **-algebra* is a planar algebra where the vector spaces P_n have a *-structure which is compatible with the adjoint of a tangle, that is $Z_{T^*}(x^*) = (Z_T(x))^*$ for all $x \in P$ and all tangles T.

Given a tangle T we may define two operations tr_l and tr_r by joining corresponding points along the top and bottom of the tangle to the left or right as shown in Figure 2.6. This operation gives a left and right trace on the algebra P.

A planar algebra is called *spherical* if Z_T is invariant under isotopies of the 2 sphere for all 0-tangles T. This is equivalent to the statement that the left and right traces are equal. In this case let $Tr = \delta^{-n} tr_l = \delta^{-n} tr_r$ be the normalised

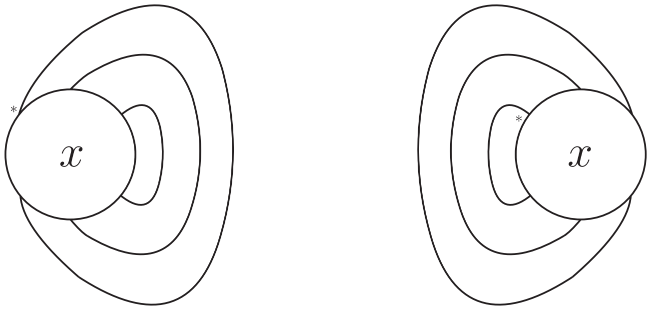


Figure 2.6: The traces $tr_r(x)$ and $tr_l(x)$

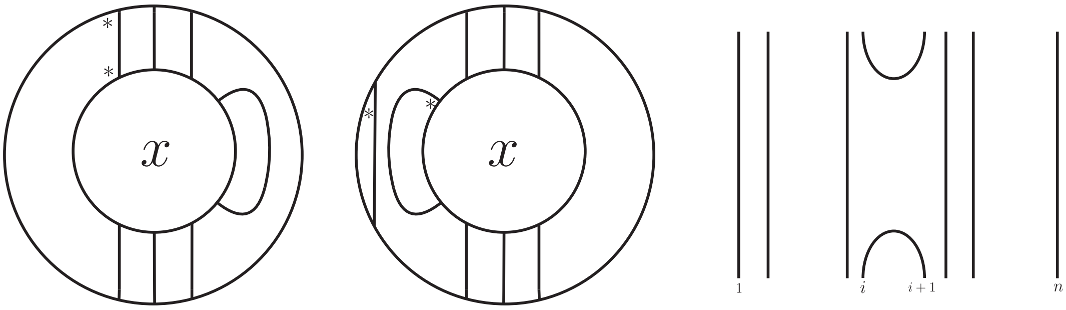


Figure 2.7: Right and Left Conditional expectations and Jones Projection

trace on P_n.

If the trace is positive definite we may define a postive definite inner product on P by $\langle x, y\rangle := Tr(y^*x)$.

A planar algebra is called *connected* if the spaces P_0 and P_1 have dimension one. A planar algebra is said to be *finite dimensional* if $dimP_n < \infty$ for all n.

A planar *-algebra is a C^**-planar algebra* if the trace is non degenerate, it is finite dimensional and Z_T is positive for all 0-tangles T. In this case there is a unique C^*-norm on P.

A *subfactor planar algebra* is a planar *-algebra with some extra structure: sphericality, finite dimensionality, positive modulus, connectedness and a positive definite trace.

In [36] the following important theorem was proved, which justifies the use of the adjective 'subfactor' in the above definition:

Theorem 2.5.1. *Let $N \subset M \subset M_1 \subset \ldots$ be the tower of the basic construction* *of an extremal II_1 subfactor $N \subset M$ with finite index δ. Then there exists a unique subfactor planar algebra P with modulus δ such that the vector spaces $P_k = N' \cap M_{k-1}$ for all $k \geq 1$ and the presenting map Z_T is compatible with the trace, conditional expectations, inclusions and Jones projections.*

The converse to this theorem, i.e. that, given a subfactor planar algebra P, one may find an extremal finite index subfactor with P as its planar algebra, has been proved, for example by Jones using Popa's λ-lattices. Another proof using planar algebra techniques appeared in [28], [42], [49] where starting with a subfactor planar algebra P a subfactor $N \subset M$ is explicitly constructed such that the subfactor planar algebra of $N \subset M$ is isomorphic to P. In [2], [29] it was shown that given any planar algebra it is realised as the standard invariant of a subfactor of an interpolated free group factor. In [15] it was recently shown that it is possible to remove the extremality assumption. A non extremal subfactor corresponds to a non spherical subfactor planar algebra.

Another important construction is the planar algebra of a bipartite graph, first defined in [38]. This is a useful tool which has been used (for example in [5],[74]) to find new subfactors by finding their subfactor planar algebra in the planar algebra of the principal graph. It was recently shown [40] that all finite depth subfactor planar algebras are planar subalgebras of the planar algebra of a bipartite graph. The planar algebra of a bipartite graph may be defined as follows.

Given a bipartite graph $\mathcal{G}$ with edges E and vertices V_+ and V_-, let Δ be its adjacency matrix, δ be its Perron-Frobenius eigenvalue and let $\mu(v)$ be the entry of the Perron Frobenius eigenvector corresponding to the vertex v. The planar algebra $P^{\mathcal{G}}$ is the planar algebra with vector spaces given by bounded functions on loops of the graph of length $2n$, starting in V_+ for P_n^+ and V_- for P_n^-. Given a tangle T in standard form a *spin state* $\sigma : \{\text{strings}\} \cup \{\text{regions}\} \to E \cup V_+ \cup V_-$ is a map taking strings of the tangle to edges of the graph and regions to vertices such that if an string s borders regions r_1 and r_2 then $\sigma(s)$ is an edge from vertex $\sigma(r_1)$ to $\sigma(r_2)$.

A spin state assigns a label l_i to the i^{th} disc of a tangle T, the label is the loop of length n_i on the graph gotten by reading the assignments of vertices and edges around the edge of the disc, starting with the starred region.

Let $S(T)$ denote the collection of singularities (maxima and minima) of the strings of the tangle T in standard form. For a tangle T with a certain spin state σ each $s \in S(T)$ is assigned a coefficient ψ_s. For each cup $\cup_j$ the value of ψ is $\psi_{\cup_j} = \mu(v_1)/\mu(v_2)$ where v_1 is the vertex corresponding to the region under the cup and v_2 is the vertex corresponding to the region above it. Similarly for a cap we put $\psi_{\cap_j} = \mu(v_1)/\mu(v_2)$ where now v_1 is the region above the cap and v_2 is the region below.

The maps Z_T are then defined as follows. Suppose T is a k-tangle, and γ is a loop of length $2k$ in $\mathcal{G}$. Then the coefficient of Z_T corresponding to the basis element γ is

$$\sum \prod_{discs D_i} l_i \prod_{s \in S(T)} \psi_s$$

where the sum is over all states inducing γ on the boundary.

The planar algebra of a bipartite graph was shown in [38] to be a spherical, positive definite planar *-algebra with modulus δ.

An important example of a planar algebra is the Temperley-Lieb planar *-algebra TL(δ) where $\delta > 0$ is the modulus. The vector spaces TL_i^+ (TL_i^-) are the linear span of all planar diagrams with $2i$ marked points on the boundary, no internal discs and the marked point in an unshaded (shaded) region. It can be shown [27] that the vector spaces are generated multiplicatively by the Jones projections e_i, shown in Figure 2.7, which are easily seen to satisfy the Temperley-Lieb relations. The maps Z_T are just given by insertion of the relevant TL diagrams inside the appropriate inner discs of T, removing all closed loops by multiplying the result by δ. The Temperley-Lieb planar *-algebra is a subalgebra of the planar algebra of the bipartite graph A_{n-1}, where the value of n determines the value of δ. TL(δ) is a subfactor planar algebra for $\delta \geq 2$. For $\delta = 2\cos(\frac{\pi}{n})$ ($n \geq 3$) we may form a subfactor planar algebra by taking a quotient of TL(δ) by the ideal $I(\delta)$ of all vectors $x \in$TL(δ) with $tr(x^*x) = 0$.

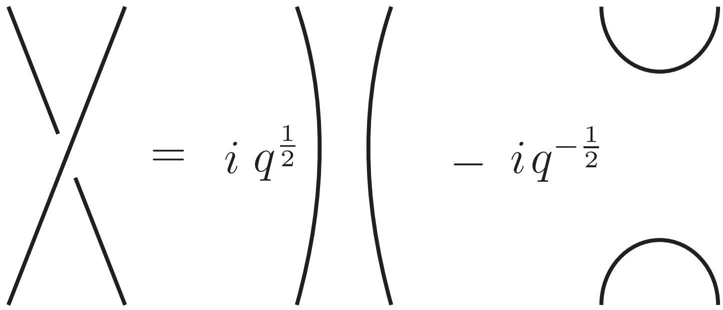


Figure 2.8: Braiding on Temperley-Lieb Tangles

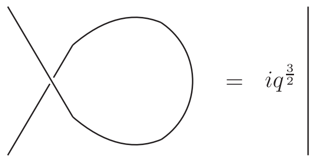


Figure 2.9: Removing a twist

We can define a crossing on tangles using the linear combination of diagrams in Figure 2.8, with $\delta = q + q^{-1}$.

The braid group B_n is the group with $n-1$ generators $\sigma_1, \ldots, \sigma_{n-1}$ satisfying the braid relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if } |i-j| > 1$$

and

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}.$$

It is shown for example in [44] Section 4, [46] Section 2.3 that if σ_k is the crossing with k straight strands to the left and $n-k-2$ straight strands to the right, the σ_k define a representation of the Braid group B_n on the n- strand Temperley-Lieb algebra TL_n.

The crossing defined above also satisfies the second and third Reidemeister moves and the first Reidemeister move up to a constant as shown in Figure 2.9. A proof of this may be found for example in Corollory 3.4 and Theorem 3.5 of [45]

We may use the Temperley-Lieb planar algebra to define a C^*-algebra as follows. Use the trace on TL_n to define an inner product by $\langle x, y\rangle = tr_n(y^*x)$ and take the GNS construction. The C^* algebra $\mathcal{TL}_n$ generated by TL_n is a finite dimensional C^* algebra which may be embedded in $\mathcal{TL}_{n+1}$ using the inclusion of TL_n in TL_{n+1}. Taking the inductive limit $\varinjlim \mathcal{TL}_n$ we get an AF C^*-algebra which we denote by $\mathcal{TL}$.

2.6 A_2-Planar Algebras

Here we introduce the A_2-planar algebras which were first defined in [23].

Let $\sigma = \sigma_1 \cdots \sigma_m$ be a sign string, that is, each σ_i is either $+$ or $-$ and we write $\sigma^* = \sigma_m \cdots \sigma_1$. A planar σ tangle is a disc D_0 in $\mathbb{R}^2$ with m marked boundary points containing a possibly empty collection of internal discs $D_1, \ldots, D_n$ and a collection of oriented strings. Each disc D_k has m_k marked boundary points with orientations given by $\sigma^{(k)} = \sigma_1^{(k)} \cdots \sigma_{m_k}^{(k)}$, and each boundary point is the endpoint for some string. A boundary point is called a *source* if the string is orientated away from it and a *sink* if it is oriented towards it. Boundary points with positive orientation are source vertices and those with negative orientation are sinks. Strings may not intersect the discs at points other than the marked boundary points but they are allowed to meet at incoming and outgoing trivlalent vertices and closed loops are also allowed. The tangle is equipped with a colouring such that each region has colour $\overline{0}, \overline{1}$ or $\overline{2}$ and when crossing a downwards oriented string from left to right the colour increases by 1 mod 3.

The boundary of each disc D_k between the last and first marked points is marked with a $*_{b_k}$, where $\overline{b}_k \in \{\overline{1}, \overline{2}, \overline{3}\}$ is the colour of the region it is adjacent to. For tangles with no marked boundary points, there are three types, depending on the colour of the region adjacent to the outer disc.

Similarly to the tangles defined in Section 2.5, tangles can be composed if the pattern of the outer disc of some tangle S is the same as the pattern of some inner disc D_i of a tangle T (i.e. the two discs have the same number of strings and all orientations and colourings are compatible). Then $T \circ_i S$ is formed by

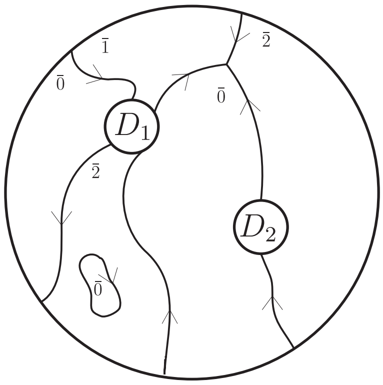


Figure 2.10: A ++++- Tangle

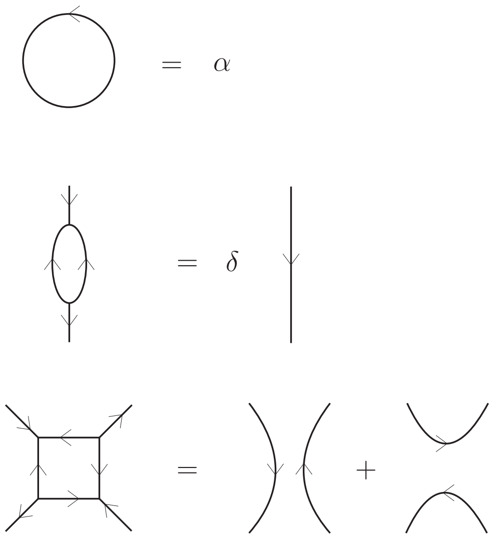


Figure 2.11: Kuperberg Relations

gluing S inside D_i of T, then removing the outer disc of S and smoothing all the strings.

Let $\mathcal{P}_\sigma(L)$ be the free vector space generated by coloured σ tangles, with inner discs labelled by elements of L, quotiented by the Kuperberg relations defined in Figure 2.11, where δ, α are related by $\alpha = \delta^2 - 1$.

A partial braiding may be defined on local parts of tangles by Figure 2.12, with $q \in \mathbb{C}$ such that $[2]_q = \delta$ and $[3]_q = \alpha$. The quantum number $[n]_q$ is $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$ for $n \in \mathbb{Z}$ and $q \in \mathbb{C}$.

The *A_2-planar operad* $\mathcal{P}$ is $\mathcal{P} = \mathcal{P}(L) = \cup_\sigma P_\sigma(L)$.

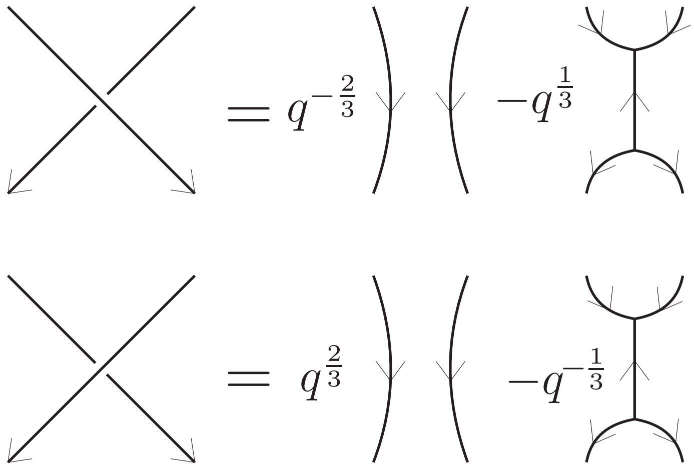


Figure 2.12: Braiding

With composition as defined above this has the structure of a coloured operad.

Definition 2.6.1. A *general A_2-planar algebra* is a family of vector spaces $P = \{P_\sigma^{\overline{a}}, \sigma$ any sign string, $a \in \{0,1,2\}\}$ such that every σ-tangle T has an associated mulitlinear map $Z_T : \otimes_{1\leq m\leq n} P_{\sigma_m}^{\overline{a}_m} \rightarrow P_\sigma^{\overline{a}}$. The maps Z_T are called *presenting maps* and they are required to be compatible with composition of tangles and relabelling of inner discs in a similar manner to the planar algebras defined in the previous section.

In other words a general A_2-planar algebra is an algebra over the A_2-planar operad $\mathcal{P}$. In a general A_2-planar algebra, given two tangles a and b in $\mathcal{P}_{\sigma\sigma^*}$, we can define a multiplication and trace analagously to Section 2.5, i.e. the product $a.b \in \mathcal{P}_{\sigma\sigma^*}$ is the $\sigma\sigma^*$ tangle formed by stacking a on top of b.

Definition 2.6.2. A general A_2-planar algebra is *spherical* if the presenting map is invariant under isotopies of the 2-sphere for any $\emptyset$-tangle.

For a tangle $a \in P_\sigma$ we may define normalised traces $Tr_L(a)$ and $Tr_R(a)$ as the tangles formed by joining corresponding points along the top and bottom of a to the left of a for Tr_L and to the right of a for Tr_R and then dividing by $\delta^{-|\sigma|}$. If P is spherical then these two traces are the same and we write $Tr := Tr_L = Tr_R$.

Each general A_2-planar algebra contains a copy of the A_2-Temperley-Lieb algebra which is just the planar algebra with labelling set $L = \emptyset$.

For a σ-tangle T, there is an involution given by reflecting T in a horizontal line through its centre and reversing the orientations of all strings. The resulting tangle is denoted by T^*. We say that P is a general A_2-planar *-algebra if $(Z_T(x))^* = Z_{T^*}(x^*)$ for all tangles T and $x \in P$.

Definition 2.6.3. A spherical general A_2-planar algebra is said to be *non-degenerate* if the trace defines a non degenerate bilinear form on $P_{\sigma\sigma^*}$ for all σ.

Definition 2.6.4. An *A_2-planar algebra* P is a spherical, non-degenerate general A_2-planar *-algebra which satisfies the following requirements: $\dim(P_{0,0}) = \dim(P_{0,1}) = \dim(P_{0,2}) = 1$ and removal of a closed loop causes the presenting map to be multiplied by δ for some $\delta > 0$.

For a non-degenerate A_2-planar algebra we may define an inner product by $\langle a, b \rangle = Tr(a^*b)$. It can be shown that a non-degenerate spherical A_2-planar algebra P has a unique C^*-norm, and we call such a P an *A_2-planar C^*-algebra*.

An A_2-planar algebra is called *flat* if strings may be passed over discs. That is, given any tangle T and any internal disc D_k, let T' be the liner combination of tangles obtained by pulling i strings of T over the disc D_k. If the planar algebra is flat we require that $Z_T = Z_{T'}$, where if $T' = \sum c_i T_i$ we define $Z_{T'} = \sum c_i Z_{T_i}$.

2.7 A_2-Planar Algebras for Subfactors

In this section we recall the constructions from [20], [23] of subfactors and planar algebras associated to the the SU(3) $\mathcal{ADE}$ graphs shown in Figure 2.13, a more complete list may be found for example in [16] or [22]. The $\mathcal{A}^{(n)}$ graphs are the Weyl alcove of SU(3) at level k. The $\mathcal{D}^{(n)}$ are obtained as $\mathbb{Z}_3$ orbifold of $\mathcal{A}^{(n)}$. We also have the exceptional graphs $\mathcal{E}$ and the $\mathcal{A}^{(n)*}$ and $\mathcal{D}^{(n)*}$ graphs which are conjugations of the $\mathcal{A}^{(n)}$ and $\mathcal{D}^{(n)}$ graphs. Alternatively we can describe $\mathcal{A}^{(n)}$

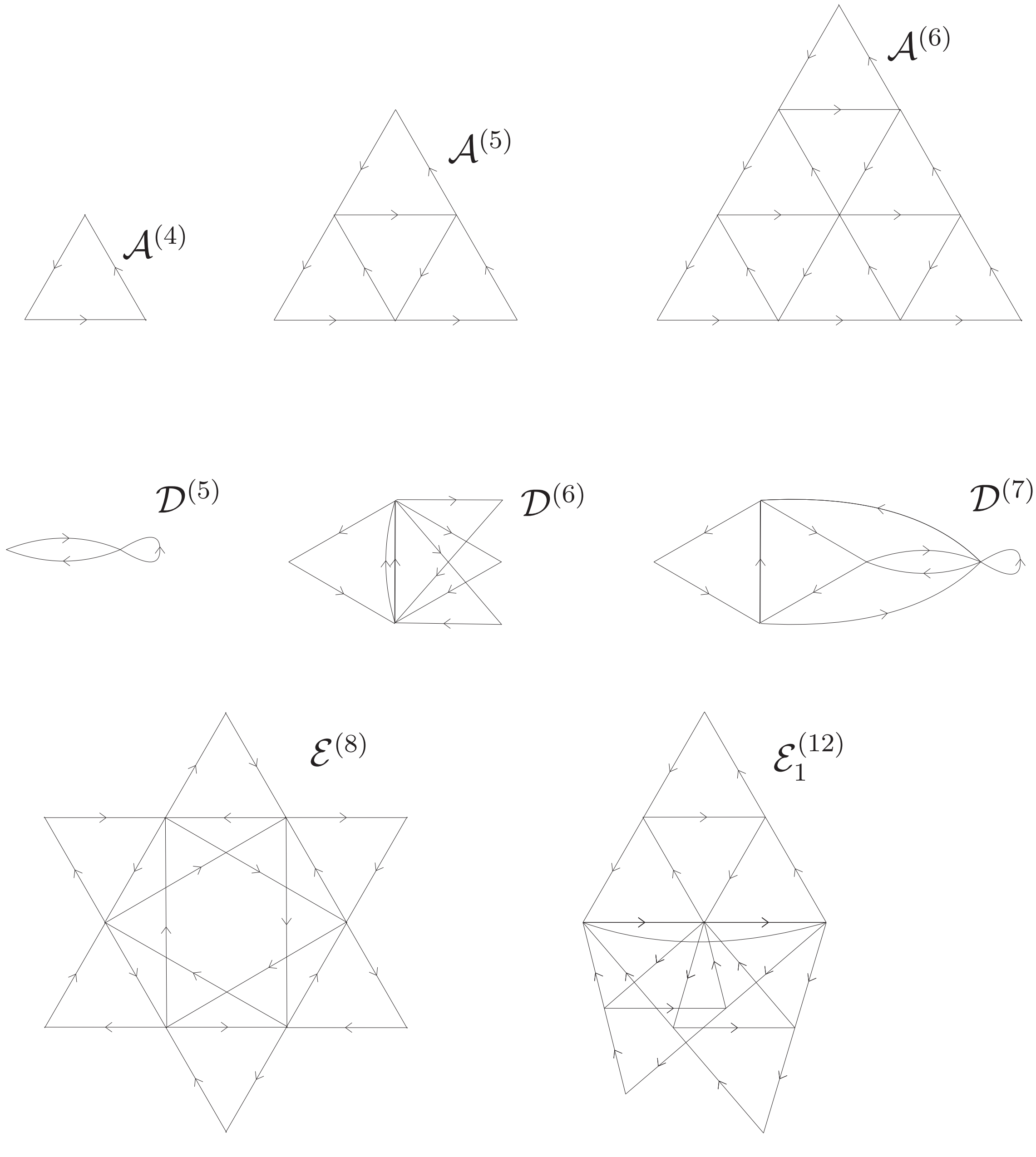


Figure 2.13: $\mathcal{ADE}$ graphs

as the graphs associated to diagonal modular invariants. The graphs $\mathcal{D}^{(n)}$ are orbifold modular invariants. The $\mathcal{A}^{(n)*}$ and $\mathcal{D}^{(n)*}$ graphs are associated to the conjugate of the $\mathcal{A}^{(n)}$, $\mathcal{D}^{(n)}$ modular invariants. Finally the graphs $\mathcal{E}^{(8)}$, $\mathcal{E}^{(8)*}$, $\mathcal{E}_i^{(12)}$, $i = 1, \ldots, 5$ and $\mathcal{E}^{(24)}$ are the graphs associated to exceptional invariants. For $\mathcal{A}^{(n)}$, $\mathcal{A}^{(n)*}$, $\mathcal{D}^{(n)}$, $\mathcal{D}^{(n)*}$ $\mathcal{E}^{(n)}$ we call n the Coxeter number.

For each graph $\mathcal{G}$, let $\{\mu(v)\}_v$ be its Perron Frobenius eigenvector with eigenvalue $[3]_q$, $q = e^{i\pi/n}$ where n is the Coxeter number of $\mathcal{G}$. Denote by $*$ the vertex with lowest Perron Frobenius weight, and normalise the eigenvector so that $\mu(*) = 1$.

Let G be any finite subgroup of $SU(3)$. We may associate to G a graph $\mathcal{G}_G$ called its McKay quiver. Let L be the fundamental representation of G. The vertices of the graph are the set of irreducible representations $\{L_i\}$ of G and the number of edges from vertex L_i to L_j is the dimension of $Hom_G(L_i, L_j \otimes L)$.

A type I frame in a graph $\mathcal{G}$ is a pair of edges e_1, e_2 with $s(e_1) = s(e_2)$ and $r(e_1) = r(e_2)$. A type II frame is 4 edges $e_1, \ldots, e_4$ with $s(e_1) = s(e_4)$, $s(e_2) = s(e_3)$, $r(e_1) = r(e_2)$ and $r(e_3) = r(e_4)$.

Given a graph $\mathcal{G}$ which is either an $\mathcal{ADE}$ graph (but not $\mathcal{E}_4^{(12)}$) or $\mathcal{G}_G$ for some subgroup $G \subseteq SU(3)$ we can associate a complex number $W\triangle_{\alpha,\beta,\gamma}$ to each oriented triangle in the graph edges α, β, γ as follows.

Definition 2.7.1. A cell system on $\mathcal{G}$ is a map that associates a complex number $W\triangle_{\alpha,\beta,\gamma}$ to each oriented triangle in $\mathcal{G}$ such that the W satisfy the following rules.

- For any type I frame

$$\sum_{\substack{k,\beta,\gamma \\ r(\beta)=s(\gamma)=k}} W\triangle_{\alpha,\beta,\gamma}\overline{W\triangle_{\alpha',\beta,\gamma}} = [2]_q \mu(s(\alpha))\mu(r(\alpha))\delta_{\alpha,\alpha'}$$

- For any type II frame

$$\sum_{\substack{k,\beta_1,\beta_2,\beta_3,\beta_4 \\ r(\beta_1)=s(\beta_2)=r(\beta_3)=s(\beta_4)=k}} \mu(k)^{-1} W\triangle_{\alpha_2,\beta_1,\beta_2}\overline{W\triangle_{\alpha_3,\beta_2,\beta_3}}W\triangle_{\alpha_4,\beta_3,\beta_4}\overline{W\triangle_{\alpha_1,\beta_4,\beta_1}} =$$

$$\mu(s(\alpha_1))\mu(s(\alpha_2))\mu(s(\alpha_4))\delta_{\alpha_1,\alpha_4}\delta_{\alpha_2,\alpha_3} + \mu(s(\alpha_1))\mu(s(\alpha_2))\mu(s(\alpha_3))\delta_{\alpha_1,\alpha_2}\delta_{\alpha_3,\alpha_4} \quad (2.3)$$

In [22] the cells were shown to exist and calculated for the SU(3) $\mathcal{ADE}$ graphs, apart from the graphs $\mathcal{E}_3^{(12)}$ and $\mathcal{E}_4^{(12)}$.

Let $\mathcal{G}$ be any finite $SU(3)$ $\mathcal{ADE}$ graph with vertices V and distinguished vertex $* \in V$. Let $\{\mu(v) : v \in V\}$ be the Perron Frobenius eigenvector with eigenvalue $[3]_q$. We construct the double sequence of finite dimensional string algebras in a similar manner to the construction in Section 2.4.

$$\begin{array}{ccccccccc}
B_{0,0} & \subset & B_{0,1} & \subset & B_{0,2} & \subset & \cdots & \rightarrow & B_{0,\infty} \\
\cap & & \cap & & \cap & & & & \cap \\
B_{1,0} & \subset & B_{1,1} & \subset & B_{1,2} & \subset & \cdots & \rightarrow & B_{1,\infty} \\
\cap & & \cap & & \cap & & & & \cap \\
B_{2,0} & \subset & B_{2,1} & \subset & B_{2,2} & \subset & \cdots & \rightarrow & B_{2,\infty} \\
\cap & & \cap & & \cap & & & & \cap \\
\vdots & & \vdots & & \vdots & & & & \vdots
\end{array}$$

Here the horizontal inclusions are given by the full graph $\mathcal{G}$. If $\mathcal{G}$ is not three colourable, the vertical inclusions are given by all of $\mathcal{G}$ but if it is three colourable the inclusion $B_{i,j} \subset B_{i+1,j}$ is given by the $\overline{j} - \overline{j+1}$ part of $\mathcal{G}$. We identify $B_{0,0} = \mathbb{C}$ with the starred vertex of $\mathcal{G}$.

For the square

$$\begin{array}{ccc}
B_{i,j} & \subset & B_{i,j+1} \\
\cap & & \cap \\
B_{i+1,j} & \subset & B_{i+1,j+1}
\end{array} \tag{2.4}$$

if i is even we define a connection on the graph $\mathcal{G}$ by

$$X^{\rho_1,\rho_2}_{\rho_3,\rho_4} = \begin{array}{ccc} i & \xrightarrow{\rho_1} & j \\ {\scriptstyle\rho_3}\big\downarrow & & \big\downarrow{\scriptstyle\rho_2} \\ k & \xrightarrow[\rho_4]{} & l \end{array} = q^{2/3}\delta_{\rho_1,\rho_3}\delta_{\rho_2,\rho_4} - q^{-1/3}\,\mathcal{U}^{\rho_1,\rho_2}_{\rho_3,\rho_4}, \tag{2.5}$$

and

$$\mathcal{U}^{\rho_1,\rho_2}_{\rho_3,\rho_4} = \sum_{\lambda} \mu(s(\rho_1))^{-1}\mu(r(\rho_2))^{-1} W\triangle_{\lambda,\rho_3,\rho_4}\overline{W\triangle_{\lambda,\rho_1,\rho_2}}.$$

If i is odd we define the connection on $\mathcal{G}$ by

$$\begin{array}{ccc} & \xrightarrow{\rho_1} & \\ \tilde{\rho}_3\downarrow & & \downarrow\tilde{\rho}_2 \\ & \xrightarrow[\rho_4]{} & \end{array} = \sqrt{\frac{\mu(s(\rho_3))\mu(r(\rho_2))}{\mu(s(\rho_2))\mu(r(\rho_3))}} \begin{array}{ccc} & \xrightarrow{\rho_1} & \\ \rho_3\downarrow & & \downarrow\rho_2 \\ & \xrightarrow[\rho_4]{} & \end{array} . \tag{2.6}$$

Thus we may transform a path from $B_{i,j} \to B_{i+1,j} \to B_{i+1,j+1}$ to a path of the form $B_{i,j} \to B_{i,j+1} \to B_{i+1,j+1}$.

Denote by $B_{i,\infty}$ the GNS completion of $\bigcup_{n\geq 1} B_{i,n}$ with respect to the Markov trace defined for $(\xi,\zeta) \in B_{i,j}$ by

$$tr(\xi,\zeta) := \delta_{\xi,\eta}[3]^{-(i+j)}\mu(r(\xi)).$$

It is known [20] that for $\mathcal{G} = \mathcal{A}^{(n)}, \mathcal{D}^{(n)}$ the algebra $B_{i,\infty}$ is a II_1 factor and the double sequence satisfies $B'_{0,\infty} \cap B_{i,\infty} = B_{i,0}$ for all i.

Define operators $U_{-k} \in B_{i,j}$ by

$$U_{-k} = \sum_{\substack{|\xi_1|=j-2-k |\rho_i|=1 \\ |\xi_2|=i, |\zeta|=k}} \mathcal{U}^{\rho_3,\rho_4}_{\rho_1,\rho_2}(\xi_1\cdot\rho_1\cdot\rho_2\cdot\xi_2\cdot\zeta, \xi_1\cdot\rho_3\cdot\rho_4\cdot\xi_2\cdot\zeta) \quad 0 \leq k \leq j-2$$

$$U_{-j+1} = \sum_{\substack{|\xi_1|=j-1 |\rho_i|=1 \\ |\zeta|=k}} \mathcal{U}^{\rho_3,\rho_4}_{\rho_1,\rho_2}(\xi_1\cdot\rho_1\cdot\rho_2\cdot\zeta, \xi_1\cdot\rho_3\cdot\rho_4\cdot\zeta)$$

where the sum is over all horizontal paths ξ_i and vertical paths ζ.

In order to relate A_2-planar algebras and subfactors, we restrict our attention to a certain subcollection of tangles, called (i,j) tangles where $i,j \in \mathbb{N}$. An (i,j) tangle T is a planar σ tangle such that the first j boundary points are required to be sources, the following $2i$ alternate between sources and sinks and the final j are all sinks. The $(j+1)$-th marked point is a source for a $+$ tangle and a sink for a $-$ tangle. For a σ of this form, we denote P_σ by $P_{i,j}$.

We now define an A_2-planar algebra whose vector spaces $P_{i,j} \simeq B_{i,j}$ and whose presenting map is defined as follows. First we isotope T into *standard form.* That is, we draw T in such a way that it can be divided into horizontal strips so that each strip contains at most one of a cup, a cap, and incoming or outgoing Y fork, an inverted incoming or outgoing Y fork or a labelled rectangle. For each

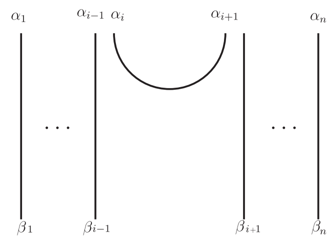


Figure 2.14: Labelling for the strip $\cup^i$

horizontal strip, label the sequence of vertices along the top and bottom edges by all possible paths in the graph $\mathcal{G}$ starting at $*$ and such that downward oriented strings are labelled by edges in $\mathcal{G}$ and upwards oriented strings are labelled by edges in $\tilde{\mathcal{G}}$, the graph $\mathcal{G}$ with all orientations reversed.

For a strip $\cup^i$ containing a cup joining the i-th and $(i+1)$-th vertices as shown in Figure 2.14 the vertices along the top and bottom edge are labelled by paths in $\mathcal{G}$, in order for the labelling to be consistent we must have $\alpha_i = \beta_i$ for all i and $\alpha_{i+1} = \tilde{\alpha}_i$. Thus the contribution to the presenting map for each pair of paths is

$$Z(\cup^i) = \sum_{\alpha} \frac{\sqrt{\mu(r(\alpha_i))}}{\sqrt{\mu(s(\alpha_i))}} \delta_{\alpha_i, \tilde{\alpha}_{i+1}} (\alpha_1 \cdots \alpha_n, \alpha_1 \cdots \alpha_{i-1} \cdot \alpha_{i+2} \cdots \alpha_n)$$

where the sum is over all possible paths $\alpha := \alpha_1 \cdots \alpha_n$

For a cap joining the i-th and $(i+1)$-th vertices along the bottom of the rectangle the contribution to the presenting map is $Z(\cap^i) = Z(\cup^i)^*$.

For a strip $\curlyvee^i$ containing an outgoing Y-fork joining the i-th and $(i+1)$-th vertices on the top to the i-th vertex on the bottom, label the vertices as in Figure 2.15. Then we must have $\alpha_j = \beta_j$ for $1 \leq j \leq i-1$ and $\alpha_j = \beta_{j+1}$ for $i+2 \leq j \leq n$. Thus the contribution to the presenting map is

$$Z(\curlyvee^i) = \sum_{\alpha,\beta} \frac{1}{\sqrt{\mu(s(\beta))\mu(r(\beta))}} \overline{W(\triangle_{\tilde{\beta}.\alpha_i.\alpha_{i+1}})} (\alpha_1 \cdots \alpha_n, \alpha_1 \cdots \alpha_{i-1} \cdot \beta \cdot \alpha_{i+2} \cdots \alpha_n)$$

where once again we sum over all possible paths α and β, the edge $\tilde{\alpha}_i$ is α_i with the orientation reversed and W is the complex number defined by the cell system on the graph.

Similarly for a strip $\overline{\curlyvee}^i$ containing an incoming inverted Y-fork joining the i-th vertex on the top to the i-th and $(i+1)$-th vertex on the bottom, we label

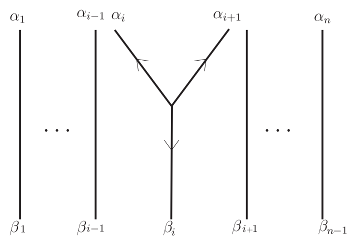


Figure 2.15: Labelling for the strip $\curlyvee^i$

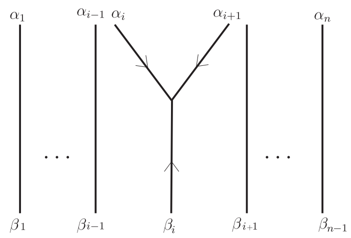


Figure 2.16: Labelling for the strip $\bar{\curlyvee}$

as in Figure 2.16 and the presenting map is

$$Z(\curlywedge^i) = \sum \frac{1}{\sqrt{\mu(s(\beta)\mu(r(\beta))}} W(\triangle_{\beta.\tilde{\alpha}_{i+1}.\tilde{\alpha}_i})(\alpha_1 \cdots \alpha_n, \alpha_1 \cdots \alpha_{i-1}\beta\alpha_{i+1} \cdots \alpha_n).$$

For inverted Y-forks $\curlywedge^i$ and inverted outgoing Y-forks $\bar{\curlywedge}^i$ we have that $Z(\curlywedge^i) = Z(\curlyvee^i)^*$ and $Z(\bar{\curlywedge}^i) = Z(\bar{\curlyvee}^i)^*$.

For a strip x containing a rectangle with label $(\xi, *)$, if there are no through strings to the left or right of the rectangle the presenting map is just $Z(x) = (\xi, *)$. If there are n through strings to the right, the presenting map $Z(x) = \sum_{|\alpha|=n}(\xi \cdot \alpha, \alpha)$. If there are m through strings to the right, we start by adding m through strings to the right as above and then use the connection to transform to a path of the form $Z(x) = \sum_{|\alpha|=m,\zeta} c_{\xi,\zeta}(\alpha \cdot \zeta, \alpha)$ where the constants $c_{\xi,\zeta}$ come from the connection. Then if there are n through strings to the left and m to the right the presenting map is

$$Z(x) = \sum_{|\alpha|=n,|\beta|=m} c_{\xi,\zeta}(\alpha \cdot \zeta \cdot \beta, \alpha \cdot \beta)$$

where the sum is over all possible paths α of length n and β of length m. If the rectangle is labelled by a linear combination of paths $\sum \lambda_i(\xi_i, *)$ then just extend the definition linearly.

Finally suppose T is a tangle in standard form which is divided into n horizontal strips $t_1, \ldots, t_n$ where t_1 is the top strip, t_2 is the next one underneath and so on. Then the presenting map Z_T is the product $Z(t_1) \ldots Z(t_n)$ where the presenting maps of the individual strips are as defined above.

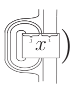
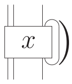
Theorem 2.7.2 (Theorem 6.4, [23]). *Let $\mathcal{G}$ be an $\mathcal{ADE}$ graph with $*$ chosen as the vertex with lowest Perron Frobenius weight and suppose $\mathcal{G}$ has a flat connection. The above definition of Z_T for an A_2-tangle makes the double sequence $B_{i,j}$ into a flat A_2-planar C^*-algebra with parameter α, the Perron-Frobenius eigenvalue of $\mathcal{G}$, and* $\dim P_{0,0} = \dim P_{0,1}^{(0,1)} = \dim P_{0,2}^{(0,2)} = 1$.

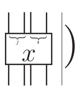
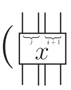
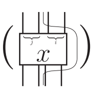
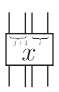
In the above theorem we use the notation $P_{i,j}^{(m,n)} = Z(\mathcal{P}_{i,j}^{(m,n)})$, where $\mathcal{P}_{i,j}^{(m,n)}$ is the subspace of $\mathcal{P}_{i,j}$ spanned by discs where the first n strings are through strings and strings $n+j+1$ to $n+j+m$ are strings which pass over any strings they cross and are such that there are no discs between them and the right edge of the tangle.

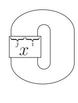
Definition 2.7.3. An A_2-planar algebra is called the *A_2-planar algebra for the subfactor $N \subset M$* if $P_{0,\infty} = N$, $P_{1,\infty} = M$, $P_{i,\infty} = M_{i-1}$ and $P'_{0,\infty} \cap P_{i,\infty} = P_{i,0}$ and Z_T satisfies the following conditions.

(*i*) $Z(W_{-k}) = U_{-k}$, $\quad k \geq 0$,

(*ii*) $Z(f_l) = \alpha e_l$, $\quad l \geq 1$,

(*iii*) $Z($$) = \alpha^{j+1} E_{M' \cap M_{i-1}}(x)$ $\qquad$ $Z($$) = \alpha E_{M_{i-2}}(x)$

(*iv*) $Z($$) = Z($$)$ $\qquad$ $Z($$) = Z($$)$

(*v*) $Z($$) = \alpha^{-i-j} tr(x)$

where $\alpha = [3]_q$ for q such that $[M : N] = [2]_q$.

2.8 Type III Factors and Sectors

Let M and N be type III factors and let $Mor(M, N)$ be the collection of unital normal endomorphisms $M \to N$. If $M = N$ we write $End(M) := Mor(M, M)$. Suppose $N \subset M$ is a subfactor with N and M isomorphic, then there exists $\rho \in End(M)$ with $\rho(M) = N$ ([60], Section 2.8). Suppose there exists a faithful normal conditional expectation $E : M \to N$ then Kosaki describes in [52] a canonical construction of an operator valued weight $E^{-1} : M' \to N'$. If 1 is in the domain of E^{-1} then we put $\mathrm{ind}(E) = E^{-1}(1)$, otherwise we put $\mathrm{ind}(E) = \infty$. It is shown in Theorem 1 of [31] that if the index is finite then there exists a unique conditional expectation with $\mathrm{ind}(E_0) = \inf \mathrm{ind}(E)$ where the infimum is taken over all faithful normal conditional expectations from M to N. If the subfactor is irreducible, that is, if $N' \cap M = \mathbb{C}$ then the conditional expectation is unique. If not, let p_i be the minimal projections of $N' \cap M$. Then there are unique conditional expectations $E_p : M_p \to N_p$ and Proposition 5.4 of [58] states that the minimal expectation $E_0 : M \to N$ is related to E_p by

$$E_p(m_p) = (\mathrm{ind}E_p)E_0(m_p)p$$

for $m_p \in M_p$. The dimension $d(\rho)$ of an endomorphism ρ is defined as $d(\rho) = [M : \rho(M)]_0^{\frac{1}{2}}$, where $[M, N]_0$ is the minimal index of N in M. For $\rho_1, \rho_2 \in Mor(M, N)$ let

$$(\rho_1, \rho_2) := \{n \in N : n\rho_1(x) = \rho_2(x)n \, \text{for all} \, x \in M\}$$

be the intertwiner space. A morphism $\rho \in Mor(M, N)$ is said to be irreducible if the intertwiner space (ρ, ρ) is just $\mathbb{C}$. Define an equivalence relation on endomorphisms by $\rho_1 \sim \rho_2$ if there exists a unitary u such that $\rho_1 = Ad(u)\rho_2$. Denote by $[\rho]$ the equivalence class of ρ in $Mor(M, N)/\sim \; = Sect(M, N)$. In the case $N = M$ we write $Sect(M)$ for $Sect(M, M)$. Sums and products of sectors are defined as follows. Let

$$[\rho_1][\rho_2] = [\rho_1\rho_2] \qquad\qquad [\rho_1] \oplus [\rho_2] = [v_1\rho_1v_1^* + v_2\rho_2v_2^*]$$

where the $v_i \in M$ are isometries satisfying $v_1v_1^* + v_2v_2^* = 1$. The index satisfies $d(\rho_1\rho_2) = d(\rho_1)d(\rho_2)$ and if $[\rho] = [\rho_1] \oplus [\rho_2]$ then $d(\rho) = d(\rho_1) + d(\rho_2)$[54]. The canonical endomorphism γ for a subfactor $\rho(M) \subset M$ is defined as $\gamma(x) = \Gamma x \Gamma^*$, where $\Gamma = J_{\rho(M)} J_M$ and J_M and J_N are the modular conjugations. Then the conjugate sector of $[\rho]$ is the equivalence class of $\bar{\rho}$ denoted by $[\bar{\rho}]$ where $\rho\bar{\rho} = \gamma$. Let $\theta = \bar{\rho}\rho$ be the dual canonical endomorphism.

Given a finite index subfactor $N := \rho(M) \subset M$ it was shown [57] that there exist isometries $r \in (id|_N, \gamma|_N)$ and $\bar{r} \in (id, \gamma)$ which satisfy $r^*\gamma(\bar{r}) = \bar{r}^* r = \frac{1}{d(\rho)}$ and $\bar{r}\bar{r}^* = e_1$ and $rr^* = e_2$, where the e_i are the Jones projections for the basic construction of $M_{i-1} \subset M_i$, defined for type III factors in Section 3 of [52]. Define $r_\rho = \rho^{-1}(r)$ and $\bar{r}_\rho = \bar{r}$.

The downward basic construction for a type III subfactor $N \subset M$ is the tunnel

$$M \supset N \supset \gamma(M) \supset \gamma(N) \supset \dots$$

which gives the tower of relative commutants

$$\mathbb{C} = M' \cap M \subset M \cap N' \subset M \cap \gamma(M)' \subset \dots$$

If we define $\zeta(x) = J_M J_N x J_N J_M$ then we may construct the Jones tower

$$N \subset M \subset \zeta(N) \subset \zeta(M) \subset \dots$$

Note that the subfactor $\gamma(M) \subset N$ is isomorphic to $M \subset \zeta(N)$.

As in the case of II_1 factors, we have the notion of principal and dual principal graph. These may be defined in terms of endomorphisms and intertwiners [34]. To define the principal graph for $\rho(M) \subset M$ first decompose the endomorphisms $1, \bar{\rho}, \bar{\rho}\rho, \bar{\rho}\rho\bar{\rho}, \bar{\rho}\rho\bar{\rho}\rho, \dots$ into irreducibles ρ_i. The vertices of the graph are labelled by the $[\rho_i]$ and an edge joins the two vertices labelled ρ_i and ρ_j if ρ_j appears in the decomposition of $\rho\rho_i$ into irreducibles. The dual graph is defined similarly but this time we decompose the endomorphisms $1, \rho, \rho\bar{\rho}, \rho\bar{\rho}\rho, \rho\bar{\rho}\rho\bar{\rho}, \cdots$. For the graphs A_n, D_{2n}, E_6 and E_8 the fusion rules are given in [32].

Let $\mathcal{G}$ be the dual principal graph of the inclusion $N = \rho(M) \subset M$. The even vertices $\mathcal{G}^{even}$ are labelled by irreducible sectors $\rho_v \in Sect(M, M)$ which occur in the decomposition of $[(\rho\bar{\rho})^n]$ and the odd vertices $\mathcal{G}^{odd}$ are labelled by irreducible sectors $\rho_v \in Sect(N, M)$ which occur in the decomposition of the $[(\rho\bar{\rho})^n\rho]$. The number of edges between $v \in \mathcal{G}^{even}$ and $w \in \mathcal{G}^{odd}$ is the dimension of $(\rho_v, \rho_w \rho)$. Let $\{T(\xi)\}_\xi$ be an orthonormal basis of $(\rho_v, \rho\rho_w)$, indexed by all paths ξ of length one starting at v and ending at w. For a path ξ of length n, $\xi = \xi_1 \cdots \xi_n$ let $T(\xi) = T(\xi_1) \ldots T(\xi_n)$. If $\tilde{\xi}$ is the opposite path then let $T(\tilde{\xi}) = r_\rho T(\xi)$. This is an orthonormal basis for $(\rho_w, \rho_v \bar{\rho})$.

Similarly, let $\mathcal{H}$ be the principal graph of the inclusion $\rho(M) \subset M$. Even vertices $\mathcal{H}^{even}$ are labelled by irreducible sectors $\sigma_v \in Sect(M, M)$ which occur in the decomposition of $[(\bar{\rho}\rho)^n]$ and odd vertices $\mathcal{H}^{odd}$ are labelled by irreducible sectors $\sigma_v \in Sect(N, M)$ which occur in the decomposition of the $[(\overline{\rho}\rho)^n \overline{\rho}]$. Again, we let $\{T(\eta)\}_\eta$ be an orthonormal basis of the intertwiner spaces $(\sigma_x, \sigma_y \bar{\rho})$ for paths η of length one with $s(\eta) = x \in \mathcal{H}^{odd}$ $t(\eta) = y \in \mathcal{H}^{even}$ and extend this to paths of arbitrary length. Note that the conjugation map allows us to identify the odd vertices of $\mathcal{G}$ with those of $\mathcal{H}$.

Next we fix an orthonormal basis $\{S(\xi)\}_\xi$ of the intertwiner spaces $(\sigma_{v'}, \rho\rho_v)$, with $v' \in \mathcal{G}^{odd}$ and $v \in \mathcal{G}^{even}$, and $\{S(\eta)\}$ of $(\sigma_x, \overline{\rho}\rho_y)$ with $x \in \mathcal{H}^{even}$ and $y \in \mathcal{H}^{odd}$. In this way we get bases for the intertwiner spaces $(\rho_v, (\rho\bar{\rho})^n \rho_{v'})$ and $(\rho_v, \bar{\rho}(\rho\bar{\rho})^n \rho_{v'})$ etc. In order to define bases for spaces such as $(\rho_v, (\rho\bar{\rho})^n \rho_w \rho)$ we need to define a connection, since both $\{\bar{\rho}(T(\xi)S(\nu)\}$ and $\{S(\zeta)T(\eta)\}$ are natural choices for this basis.

In Section 2.4 of [34] this connection is defined by the rule

$$\overline{\rho}(T(\xi))S(\nu) = \sum_{\zeta,\eta,y} \begin{array}{ccc} v & \xrightarrow{\xi} & w \\ {}_\zeta\!\downarrow & & \downarrow_\nu \\ y & \xrightarrow[\eta]{} & x \end{array} S(\zeta)T(\eta). \tag{2.7}$$

It was shown that this satisfies the flatness, unitarity and renormalization axioms.

There is a unique (up to equivalence) corespondence between endomorphisms σ and bimodules X_σ for type III factors. For any endomorphism σ the space $X_\sigma = H_\phi$, where H_ϕ is the GNS Hilbert space associated to the von Neumann algebra M with state ϕ, is an M-M bimodule where M acts on the left by ordinary multiplication and on the right by σ as shown

$$x\xi y = x\xi\sigma(y) \qquad \xi \in M, \qquad x, y \in M.$$

For type III factors all bimodules arise in this way up to isomorphism and $\sigma \simeq \sigma'$ if and only if $X_\sigma \simeq X_{\sigma'}$, whereas for type II_1 factors there are bimodules which do not have this form [21]. Tensor products of bimodules corresponds to composition of the corresponding endomorphisms, i.e. $X_\sigma \otimes X_{\sigma'} \simeq X_{\sigma\sigma'}$.

Often it is possible to split an inclusion of type III factors into a type II subfactor tensored with a type III factor. This has been investigated for example in [77], [56], [33], of particular use to us is the following theorem which was proved in [77].

Theorem 2.8.1. *Let $N \subset M$ be a finite index inclusion of type III factors and suppose there exists a conditional expectation $E : M \to N$. Suppose also the following conditions hold:*

1. *$N \simeq N \otimes R$, where R is the hyperfinite type II_1 factor*
2. *$N \subset M$ is approximately inner and centrally free*
3. *E and its extensions define a trace on the relative commutants $N' \cap M_k$*
4. *$\Gamma_{N,M}$ is strongly amenable*

Then $N \subset M \simeq \left(\overline{(\cup_k N_k' \cap N)} \subset \overline{(\cup_k N_k' \cap M)}\right) \otimes M$.

The principal graph $\Gamma_{N,M}$ is called *amenable* if $\|\Gamma_{N,M}\|^2$ is equal to the minimal index. $\Gamma_{N,M}$ is called *ergodic* if the trace on the relative commutants $N_k' \cap M$ defined by $E_{-k} \cdots E$ is factorial, where N_k is a tunnel and $E_k : N_k \to N_{k+1}$ is the conditional expectation. Then $\Gamma_{N,M}$ is called *strongly amenable* if it is both amenable and ergodic.

Theorem 2.9 of [77] gives the following characterisation of approximate innerness for hyperfinite subfactors, a more general definition may be found in [77]. Suppose $N \subset M$ is an inclusion of hyperfinite factors of type III_λ for some $\lambda \in (0, 1]$ then $N \subset M$ is called *approximately inner* if either

1. $\lambda = 1$ and E is the minimal expectation or

2. $\lambda \in (0,1)$ and there exists a common discrete decomposition, that is, there exists a II_∞ subfactor $\tilde{N} \subset \tilde{M}$ and a trace scaling automorphism ϕ of $\tilde{M}$ fixing $\tilde{N}$ with $\tilde{N} \rtimes_\phi \mathbb{Z} \subset \tilde{M} \rtimes_\phi \mathbb{Z} \simeq N \subset M$.

We also define central freeness only for hyperfinite subfactors, our definition is a result of Theorem 3.5 and 4.1 of [77], where a more general definition may be found. The inclusion $N \subset M$ of hyperfinite type III factors is called *centrally free* if either

1. N is of type III_λ for some $\lambda \in (0,1)$, ϕ is a λ trace and $N' \cap M_k \simeq N'_\phi \cap M_k$ for all k or

2. N and M are type III_1 and there exists a faithful normal state ϕ on N such that $N' \cap M_k \simeq N'_\phi \cap M_k$ for all k.

The following theorem, which is Theorem 3.12 of [53] describes the basic construction for type III factors.

Theorem 2.8.2. *Let $N \subset M$ be a type III subfactor with normal conditional expectation $E : M \to N$. Suppose L is a von Neumann algebra containing M with normal conditional expectation $\mathcal{E} : L \to M$. If there exists a projection $e \in L$ and a constant $\lambda > 0$ such that*

1. $\mathcal{E}(e) = \lambda^{-1} I$

2. $\lambda \mathcal{E}(xe)e = xe$ *for all* $x \in L$

3. $exe = E(x)e$ *for all* $x \in M$

then the subfactor $M \subset L$ is isomorphic to the basic construction $M \subset M_1$ and this isomorphism takes e to the Jones projection.

Let α be an automorphism of $N \subset M$. Then α may be extended to M_k inductively by setting $\alpha(e_k) = e_k$, where e_k is the Jones projection for $M_{k-2} \subset M_{k-1}$. Then Φ defined by

$$\Phi(\alpha) := \{\alpha|_{M' \cap M_k}\}_k$$

is called the *Loi invariant*.

2.9 The Cuntz-Krieger Algebras

Given an $N \times N$ $(N < \infty)$ matrix A with entries in $\{0,1\}$ and no zero rows or columns the Cuntz-Krieger algebra $\mathcal{O}_A$ is defined as the universal C*-algebra generated by n non zero partial isometries S_i $1 \leq i \leq N$ satisfying the relations

$$Q_i = \sum_j A_{i,j} P_j \qquad P_i P_j = \delta_{i,j}$$

where $P_i = S_i S_i^*$ and $Q_i = S_i^* S_i$. The algebra $\mathcal{O}_A$ was first defined by Cuntz and Krieger in [12]. In the case where A is an $N \times N$ matrix with $N \geq 2$ and all entries equal to one, $\mathcal{O}_A$ is the Cuntz algebra $\mathcal{O}_N$.

Given a multi index $\mu = (\mu_1, \ldots, \mu_n)$ the operator $S_\mu = S_{\mu_1} \ldots S_{\mu_n}$ is non zero if and only if $A_{\mu_i,\mu_{i+1}} = 1$ for all $1 \leq i \leq N$. The algebra $\mathcal{O}_A$ is the completion of the linear span of operators of the form $S_\mu S_\nu^*$ for multi indices μ, ν. Denote by $\mathcal{F}_A$ the AF algebra which is the inductive limit of the finite dimensional algebras $F_A^n = span\{S_\mu P_i S_\nu^*; |\mu| = |\nu| = n, 1 \leq i \leq N\}$. Each F_A^n may be written as a direct sum of matrix algebras F_n^i which are the closed linear span of the operators $S_\mu P_i S_\nu$, $|\mu| = |\nu| = n$, $1 \leq i \leq N$, and the embedding of F_A^n in F_A^{n+1} is given by A, that is, the algebra F_n^i is contained in the algebra F_{n+1}^j with multiplicity $A_{i,j}$.

For an $N \times N$ matrix A with entries in the positive integers, let $\Sigma = \{(i,k,j)| \; i,j \in \{1,\ldots,N\}, 1 \leq k \leq A_{i,j}\}$ and $A'((i_1,j_1,k_1),(i_2,k_2,j_2)) := \delta_{j_1,i_2}$. If A' is irreducible and not a permutation matrix then $\mathcal{O}_A$ is the algebra on $|\Sigma| = m$ generators S_i subject to the relations $S_i^* S_i = \sum_{j\in\Sigma} A'_{i,j} S_j S_j^*$.

There is an automorphism group of $\mathcal{O}_A$ given by $\lambda_t^A(S_i) = tS_i$ for $t \in \mathbb{T}$. It can be shown [12] that $\mathcal{F}_A$ is the fixed point algebra of $\mathcal{O}_A$ under the action of λ.

As in [72] we define states on $\mathcal{O}_A$ as follows.

Given a Cuntz-Krieger algebra $\mathcal{O}_A$ with A an $N \times N$ matrix, let $\omega = (\omega_1, \ldots, \omega_N) \in \mathbb{R}^N$ and $\sigma_t^\omega(S_i) = e^{it\omega_i} S_i$. Let $\Omega_A = \{(a_i)_{i=1}^\infty | A_{a_i,a_{i+1}} = 1\}$ be the set of one sided infinite admissible words.

Let Φ be the conditional expectation from $\mathcal{O}_A$ onto $\overline{\text{span}}\{S_\xi S_\xi^*\}$. It was shown in [12] that $C(\Omega_A) \simeq \overline{\text{span}}\{S_\xi S_\xi^*\}$. Hence we may define a state on $\mathcal{O}_A$ by first projecting onto $\overline{\text{span}}\{S_\xi S_\xi^*\}$ and then integrating with respect to some measure.

Assume that there exist $x_i > 0$, $1 \leq i \leq N$ and $\beta > 0$ such that $\sum_{i=1}^{N} x_i = 1$ and $x_i = \sum_j \exp[-\beta\omega_i]A_{ij}x_j$. Define a probability measure ν on Ω_A by its value on the cylinder sets $\Omega_A(\xi_1, \ldots \xi_n) := \{(a_i) \in \Omega_A : a_i = \xi_i, 1 \leq i \leq n\}$ by $\nu(\Omega_A(\xi_1, \ldots \xi_n)) = e^{\beta\omega_{\xi_1}} \cdots e^{\beta\omega_{\xi_1}} x_{\xi_n}$. Then the state $\phi^\omega := \nu \circ \Phi$ is the unique KMS state for the modular automorphism group σ^ω at inverse temperature β [72].

It has been shown (eg in [72], Theorem 4.2) that the weak completion of $\mathcal{O}_A$ with respect to this state gives the AFD type III$_\lambda$ factor, with λ dependent on the choice of ω. More specifically, given $\omega = (\omega_1, \ldots, \omega_N)$ then we may define a word length ω_ξ on words ξ in Ω_A by $\omega_\xi = \omega_{\xi_i} + \ldots + \omega_{\xi_n}$ for $\xi = \xi_1 \cdots \xi_n$. If for every two loops ξ and η the ratio $\omega_\xi/\omega_\eta \in \mathbb{Q}$ then $\mathcal{O}_A$ completes to give the AFD type III$_\lambda$ factor. In this case the closed additive subgroup of $\mathbb{R}$ generated by $\beta\omega_\xi$ is $r\mathbb{Z}$ for some $r > 0$ and $\lambda = e^{-r}$. If $\omega_\xi/\omega_\eta \notin \mathbb{Q}$ for some ξ, η then we get the AFD type III$_1$ factor.

2.10 Free probability

Most of the results in this section may be found for example in [69] or [88]. A *noncommutative probability space* is a unital algebra A over $\mathbb{C}$ with a linear functional $\phi : A \to \mathbb{C}$ such that $\phi(1) = 1$. The space (A, ϕ) is called a C^*-*probability space* if A is a C^*-algebra and ϕ is a state and it is called a W^*-*probability space* if A is a von Neumann algebra and ϕ is normal. An element $a \in A$ is called a *noncommutative random variable* and its distribution μ_a is the function $\mathbb{C}[a] \to \mathbb{C}$ defined by $\mu_a(p) = \phi(p(a))$ for any polynomial $p \in \mathbb{C}[a]$. If $(A_i)_{i\in I}$ are disjoint subalgebras of A then A_i are said to be *free* if $\phi(a_1 \cdots a_n) = 0$ whenever $a_j \in A_{i_j}$, $i_j \neq i_{j+1}$ for all j and $\phi(a_j) = 0$ for all j. If (A_i, ϕ_i) $i \in I$ is a family of noncommutative probability spaces let $A := *_{i\in I} A_i$ be the algebraic free product, then by 1.4.1 of [88] there exists a unique linear map ϕ on A such that $\phi|_{A_i} = \phi_i$ and the A_i are free with respect to ϕ.

Suppose (H_i, Ω_i), $i \in I$, is a collection of Hilbert spaces H_i with distinguished unit vector Ω_i. The free product $(H,\Omega) = *_{i\in I}(H_i,\Omega_i)$ is the Hilbert space

$$H = \mathcal{C}\Omega \oplus \bigoplus_{n\in\mathbb{N}} \Big(\bigoplus_{i_1\neq i_2\neq \ldots \neq i_n} H^o_{i_1} \otimes \cdots \otimes H^o_{i_n} \Big)$$

where $H^o_i = H_i \ominus \Omega_i$.

In order to define the free product of W^*-probability spaces, suppose (A_i, ϕ_i) are von Neumann algebras and $\pi_i : A_i \to B(H_i)$ is a representation of A_i on the Hilbert space H_i for all i. Let $H = *H_i$ and let

$$H(i) = \mathcal{C}\Omega \oplus \bigoplus_{n\in\mathbb{N}} \bigoplus_{\substack{i_1\neq i_2\neq \ldots \neq i_n \\ i_1 \neq i}} (H^o_{i_1} \otimes \cdots \otimes H^o_{i_n}).$$

Then the free product $A = *A_i$ is the von Neumann algebra $(\cup\lambda(A_i))''$ where A_i is represented on H by $\lambda_i(a) = V_i(\pi_i(a) \otimes 1_{H(i)})V_i^*$. Here $V_i : H_i \otimes H(i) \to H$ is defined by

$$\Omega_i \otimes \Omega \to \Omega$$

$$H^o_i \otimes \Omega \to H^o_i$$

$$\Omega_i \otimes H^o_{i_1} \otimes \cdots \otimes H^o_{i_n} \to H^o_{i_1} \otimes \cdots \otimes H^o_{i_n}$$

$$H^o_i \otimes H^o_{i_1} \otimes \cdots \otimes H^o_{i_n} \to H^o_i \otimes H^o_{i_1} \otimes \cdots \otimes H^o_{i_n}.$$

2.10.1 Free Group Factors

Let G be a group. Then the left regular representation of G on $\ell^2(G)$ is defined by $(\lambda_g\xi)(h) = \xi(g^{-1}h)$ for $\xi \in \ell^2(G)$, $g, h \in G$. The group von Neumann algebra $L(G)$ is defined as the von Neumann algebra generated by the λ_g with trace given by $tr(\lambda_g) = \delta_{g,e}$ where e is the unit of G. It is well known that in the case that G is an infinite conjugacy class (ICC) group, i.e. a group where the conjugacy class of every non identity element is infinite, $L(G)$ is a factor, a proof may be found for example in [86], Proposition 7.9. In particular the free group factors $L(\mathbb{F}_n)$ are II_1 factors. The problem of whether $L(\mathbb{F}_n)$ is isomorphic to $L(\mathbb{F}_m)$ for $m \neq n$ was one of the motivations for the beginning of the study of free probability, however this problem still remains open. It is a well known result that $L(\mathbb{F}_n) = *_{i=1}^n L(\mathbb{Z})$, this follows from the fact $\mathbb{F}_n \cong *_{i=1}^n \mathbb{Z}$ and $L(G * H) \cong L(G) * L(H)$. The interpolated free group factors $L(\mathbb{F}_r)$ for $1 < r \leq \infty$ were defined by Dykema [18] and Rădulescu [80]. If $r \in \mathbb{N}$, with $n \geq 2$ then the interpolated free group factor $L(\mathbb{F}_r)$ is a free group factor. The interpolated free group factors satisfy $(L(\mathbb{F}_r))_t \simeq L(\mathbb{F}_{1+(r-1)/t})$ for all $0 < t < \infty$ and $1 < r \leq \infty$. For a II_1 factor M with trace tr we use the notation $M_t := pMp$ where p is a projection in M with $tr(p) = t$, it was shown in [68] that M_t is well defined, that is $pMp \simeq qMq$ if p and q are projections, both of trace t.

2.10.2 Free cumulants

Before we define the free cumulants we must introduce non-crossing partitions. Let $n \in \mathbb{N}$ be fixed. Then $\pi = \{U_1, \ldots, U_m\}$ is a partition of $\{1, \ldots, n\}$ if the U_i are disjoint subsets of $\{1, \ldots, n\}$ and their union is $\{1, \ldots, n\}$. We call the U_i the blocks of π. The partition π is called a *non-crossing partition* if whenever $a_1, a_2 \in U_i$ and $b_1, b_2 \in U_j$ with $i \neq j$ the situation $1 \leq a_1 < b_1 < a_2 < b_2 \leq n$ does not occur. The collection of all non-crossing partitions is denoted by $NC(n)$. The number of non-crossing partitions of a set with n elements is $C_n = 1/(n+1)\binom{2n}{n}$ the n^{th} Catalan number. There is a natural partial order on $NC(n)$, for π, $\sigma \in NC(n)$ we say $\pi \leq \sigma$ if each block of π is contained in some block of σ. We write 0_n, 1_n for the smallest and largest element of $NC(n)$ respectively. For any two non-crossing partitions $\pi_1, \pi_2 \in NC(n)$ the join $\pi_1 \vee \pi_2$ is the smallest $\sigma \in NC(n)$ with $\pi_1 \leq \sigma$ and $\pi_2 \leq \sigma$. The partition π is called a *non-crossing pair partition* if each block has exactly two elements. The collection of non-crossing pair partitions of n elements is denoted $NC_2(n)$, where obviously here we must have n even. There are C_n elements of $NC_2(2n)$ and hence there is a bijection between $NC(n)$ and $NC_2(2n)$. The *Kreweras complement* of a partition $\pi \in NC(n)$ is the largest non-crossing partition σ of the set $\{\overline{1}, \ldots, \overline{n}\}$ such that $\pi \cup \sigma$ is a non-crossing partition of the set $\{1, \overline{1}, \ldots, n, \overline{n}\}$.

Let (A, ϕ) be a non commutative probability space. Then we may define functions $\phi_n : \times^n A \to \mathbb{C}$ by $\phi_n(a_1, \ldots, a_n) = \phi(a_1 \cdots a_n)$. For any $\pi \in NC(n)$ we can define the extensions ϕ_π of the ϕ_n by

$$\phi_\pi(a_1,\ldots,a_n) = \begin{cases} \phi_n(a_1,\ldots,a_n) & \text{if } \pi = 1_n \\ \phi_\sigma(a_1,\ldots,a_i,\phi(a_{i+1}\cdots a_{i+j})a_{i+j+1},\ldots a_n) & \text{if } \pi = \sigma\cup 1_{[i+1,i+j]} \end{cases}.$$

The *free cumulants* κ_n are defined by $\kappa_\pi(a_1,\ldots,a_n) = \sum_{\sigma\leq\pi}\phi_\sigma(a_1,\ldots,a_n)\mu(\sigma,\pi)$ where μ is the *Möbius function* defined recursively by $\mu(\pi,\pi)=1$, $\mu(\sigma,\pi) = -\sum_{\sigma\leq\tau<\pi}\mu(\sigma,\tau)$ for $\sigma<\pi$.

It was proved (e.g Theorem 4.2.1 [69]) that, given a non commutative probability space (A,ϕ) with subalgebras A_i for $i\in I$, the algebras A_i are free if and only if the cumulants $\kappa_n(a_1,\ldots,a_n)$ are zero unless all the a_j are in the same subalgebra A_i.

2.10.3 Amalgamated Free Products

Given a collection of free probability spaces (A_i,ϕ_i) for $i=1,\ldots,n$ with a common subalgebra B such that $\phi_i|_B = \phi_j|_B$ for all i,j and conditional expectation maps $\psi_i : A_i \to B$. Let $A = *_B A_i$ be the algebraic amalgamated free product. Then the map $\psi : A\to B$ is the amalgamated free product of the ψ_i if it satisfies $\psi|_{A_i} = \psi_i$ and $\psi(a_1\ldots a_m) = 0$ for $a_k\in A_{i(k)}$ with $\psi_{i(k)}(a_k)=0$ and $i(k)\neq i(k+1)$ for $1\leq k\leq m$. In the case of amalgamated free products we have B valued cumulants, defined by the formula

$$\kappa_n(a_1,\ldots,a_n) := \sum_{\pi\in NC(n)}\psi_\pi(a_1,\ldots,a_n)\mu(\pi,1_n).$$

An alternative definition was given by Speicher in [83]. If an algebra A is the algebraic free product with amalgamation over B of subalgebras A_i then ϕ is the free product state if and only if its B valued cumulants $\kappa_n(a_1,\ldots,a_n)$ are all zero for all n and whenever there exists j,k such that $a_j\in A_{i(j)}$, $a_k\in A_{i(k)}$ and $i(j)\neq i(k)$.

Often it is useful to think of the free product $A_1 *_B A_2$ with amalgamation over B as the algebra generated by $B+\Lambda(A_1^\circ,A_2^\circ)$ where A_i° denotes the kernel of the conditional expectation ϕ_i and $\Lambda(C,D)$ means the collection of alternating products of elements of C and D.

2.11 Crossed product of a C^*-algebra by an endomorphism

Let A be a unital C^*-algebra and Φ be an endomorphism of A. A *covariant representation* of (A, Φ) is a pair (π, S) where π is a non degenerate representation of A on some Hilbert space H and S is an isometry with $\pi(\Phi(x)) = S\pi(x)S^*$ for all $x \in A$. Then the *crossed product* of A by Φ is a triple (B, i_A, t) consisting of a unital C^*-algebra B, a unital homomorphism $i_A : A \to B$ and an isometry t in B such that

- $i_A(\Phi(a)) = t i_A(a) t^*$
- for every covariant representation (π, S) of (A, α) on a Hilbert space H there exists a unital representation σ of B on H with $\sigma \circ i_A = \pi$ and $\sigma(t) = S$
- t and $i_A(A)$ generate B.

Given such a pair (A, Φ) denote by A_∞ the inductive limit of $A \xrightarrow{\Phi} A \xrightarrow{\Phi} A \cdots$. It was shown in Proposition 2.2 of [84] that the crossed product defined above exists if and only if $A_\infty \neq 0$ and in this case it is unique and we usually write $A \rtimes_\Phi \mathbb{N}$ for the crossed product B.

2.12 Bicategories

The theory of bicategories was first introduced in [3], all the material in this section may be found in [55] or [14].

Definition 2.12.1. A bicategory $\mathcal{B}$ is

- a collection of objects $\mathcal{B}_0$ called 0-cells.
- for each $a, b \in \mathcal{B}_0$ there is a category $\mathcal{B}(a, b)$ whose objects $\alpha \in ob(\mathcal{B}(a, b))$ are called 1-cells of $\mathcal{B}$, denoted $a \xrightarrow{\alpha} b$, and whose morphisms $f \in Mor(\mathcal{B}(a, b))$ are called 2-cells of $\mathcal{B}$, denoted $\alpha \xrightarrow{f} \beta$.

- for each $a, b, c \in \mathcal{B}_0$ there is a functor $\otimes : \mathcal{B}(a,b) \times \mathcal{B}(c,a) \to \mathcal{B}(c,b)$

- for each triple $\alpha \in B(a,b)$, $\beta \in \mathcal{B}(c,a)$, $\gamma \in \mathcal{B}(d,c)$ of 1 cells there is an isomorphism $f_{\alpha,\beta,\gamma} : (\alpha \otimes \beta) \otimes \gamma \to \alpha \otimes (\beta \otimes \gamma)$ such that the following diagram commutes

$$\begin{array}{ccccc} ((\alpha \otimes \beta) \otimes \gamma) \otimes \delta & \xrightarrow{f_{\alpha,\beta,\gamma} \otimes 1_\delta} & (\alpha \otimes (\beta \otimes \gamma)) \otimes \delta & \xrightarrow{f_{\alpha,\beta \otimes \gamma,\delta}} & \alpha \otimes ((\beta \otimes \gamma) \otimes \delta) \\ \downarrow{\scriptstyle f_{\alpha \otimes \beta,\gamma,\delta}} & & & & \downarrow{\scriptstyle 1_\alpha \otimes f_{\beta,\gamma,\delta}} \\ (\alpha \otimes \beta) \otimes (\gamma \otimes \delta) & & \xrightarrow[f_{\alpha,\beta,\gamma \otimes \delta}]{} & & \alpha \otimes (\beta \otimes (\gamma \otimes \delta)) \end{array}$$

  for all $\alpha \in B(a,b)$, $\beta \in \mathcal{B}(c,a)$, $\gamma \in \mathcal{B}(d,c)$ and $\delta \in \mathcal{B}(e,d)$

- for each $a \in \mathcal{B}_0$ there is an identity morphism $a \xrightarrow{1_a} a$

- for each 1 cell $a \xrightarrow{\alpha} b$ there exist isomorphisms $1_b \otimes \alpha \xrightarrow{\lambda_a} \alpha$ and $\alpha \otimes 1_a \xrightarrow{\rho_a} \alpha$ in $Mor(B(a,b))$ such that the following diagram commutes

$$\begin{array}{ccc} (\alpha \otimes 1_a) \otimes \beta & \xrightarrow{\alpha_{\alpha,1,\beta}} & \alpha \otimes (1_a \otimes \beta) \\ & {\scriptstyle \rho_\alpha \otimes 1_\beta} \searrow \quad \swarrow {\scriptstyle 1_\alpha \otimes \lambda_\beta} & \\ & \alpha \otimes \beta & \end{array}$$

  for $\alpha \in ob(\mathcal{B}(a,b))$, $\beta \in ob(\mathcal{B}(c,a))$.

A 2-*category* is a strict bicategory, that is, a bicategory where the associativity and unit constraints are just the identity.

Let $\mathcal{B}$, $\mathcal{B}'$ be two bicategories. A *weak morphism* (F, ϕ) from $\mathcal{B}$ to $\mathcal{B}'$ consists of

1. a function $F : \mathcal{B}_0 \to \mathcal{B}'_0$

2. functors $F_{ab} : \mathcal{B}(a,b) \to \mathcal{B}'(F(a), F(b))$

3. natural isomorphism ϕ such that for all $a, b, c \in \mathcal{B}_0$ we have $\phi^{abc} : \otimes' \circ (F^{bc} \otimes F^{ab}) \to F^{ac} \circ \otimes$

4. for all $a \in \mathcal{B}_0$ there is an invertible $\phi_a \in Mor\mathcal{B}(a,a)$ $1_{F(a)} \xrightarrow{\phi_a} F(1_a)$

Suppose (F, ϕ) and (G, ψ) are morphisms between bicategories $\mathcal{B}$, $\mathcal{B}'$. Then a *weak transfomation* between F and G is a 1-cell $\sigma_a \in ob\mathcal{B}'(F(a), G(a))$ for all $a \in \mathcal{B}_0$ a natural transformation $\sigma^{a,b} : \sigma_b \otimes' F^{a,b} \to G^{a,b} \otimes \sigma_a$ satisfying

$$\begin{array}{ccccc}
\sigma_a \otimes' 1_{F(a)} & \xrightarrow{\rho'_{\sigma_a}} & \sigma_a & \xleftarrow[\lambda'_{\sigma_a}]{} & 1_{G(a)} \otimes \sigma_a \\
\downarrow{\scriptstyle 1_{\sigma_a} \otimes' \phi_a} & & & & \downarrow{\scriptstyle \psi_a \otimes 1_{\sigma_a}} \\
\sigma_a \otimes' F(1_a) & & \xrightarrow[\sigma_{1_a}]{} & & G(1_a) \otimes' \sigma_a
\end{array}$$

$$\begin{array}{ccccc}
\sigma_c \otimes' F(\alpha) \otimes' F(\beta) & \xrightarrow{\sigma_f \otimes' 1_{F(\alpha)}} & G(\alpha) \otimes' \sigma_b \otimes' F(\beta) & \xrightarrow{1_{G(\alpha)} \otimes' \sigma_g} & G(\alpha) \otimes' G(\beta) \otimes' \sigma_a \\
\downarrow{\scriptstyle 1 \otimes \phi_{\alpha,\beta}} & & & & \downarrow{\scriptstyle \psi_{\alpha,\beta} \otimes 1_{\sigma_a}} \\
\sigma_c \otimes' F(\alpha \otimes \beta) & & \xrightarrow[\sigma_{\alpha \otimes \beta}]{} & & G(\alpha \otimes \beta) \otimes' \sigma_a
\end{array}$$

for all $\alpha \in ob\mathcal{B}(b, c)$, $\beta \in \mathcal{B}(a, b)$ $a, b, c \in \mathcal{B}_0$.

Definition 2.12.2. A one cell $a \xrightarrow{\alpha} b$ in a bicategory $\mathcal{B}$ is said to have a right dual $\alpha^{\#}$ if there exists a one cell $b \xrightarrow{\alpha^{\#}} a$ such that there exist 2-cells $\alpha^{\#} \otimes \alpha \xrightarrow{e_\alpha} 1_a$ and $1_b \xrightarrow{c_\alpha} \alpha \otimes \alpha^{\#}$ satisfying

$$(1_\alpha \otimes e_\alpha) \circ (c_\alpha \otimes 1_\alpha) = 1_\alpha$$

and

$$(e_\alpha \otimes 1_{\alpha^{\#}}) \circ (1_{\alpha^{\#}} \otimes c_\alpha) = 1_{\alpha^{\#}}$$

A bicategory is said to be *rigid* if every 1 cell has a right dual.

The composition $\# \circ \#$ defines a functor $\#\#$ from $\mathcal{B}$ to $\mathcal{B}$.

A bicategory is said to be *pivotal* if it is right rigid and there exists a weak trasformation $a : id_{\mathcal{B}} \to \#\#$ with $a_\epsilon = 1_\epsilon$ for all $\epsilon \in \mathcal{B}_0$

Chapter 3

Planar Algebra for Type III subfactors

In this chapter we explain how to define a planar algebra for a type III subfactor. The main idea we use is from [34], where it was show that starting from a subfactor $\rho(M) \subset M$ one can construct a C^*-algebra $\mathcal{O}_{\rho\bar{\rho}} = \cup_{n,m}((\rho\bar{\rho})^n, (\rho\bar{\rho})^m)$ which is isomorphic to the Cuntz Krieger algebra $\mathcal{O}_{\Delta\Delta^t}$ where Δ is the adjacency matrix for the principal graph.

Using the characterisation of the principal graph in terms of intertwiners described in Section 2.8 it was shown in [34] that $\mathcal{O}_{\rho\bar{\rho}}$ is characterised by the two conditions

- $\mathcal{O}_{\rho\bar{\rho}}$ is generated by $\bigcup_{n\geq 0}((\rho\bar{\rho})^n, (\rho\bar{\rho})^n)$ and $\overline{r}_\rho$ and
- $\overline{r}_\rho T(\xi_+)T^*(\xi_-)\overline{r}_\rho^* = \frac{1}{d(\rho)}\sum_{\eta,\zeta,|\eta|=|\zeta|=1}\sqrt{\mu(r(\eta))\mu(r(\zeta))}T(\eta\cdot\tilde{\eta}\cdot\xi_+)T^*(\zeta\cdot\tilde{\zeta}\cdot\xi_-)$

for T and r_ρ, $\overline{r}_\rho$ as defined in Section 2.8.

In Section 3.3 we show that using a string algebra construction similar to Section 2.4 we can define hyperfinite type III_λ subfactors and characterise their relative commutants in terms of paths in the graph.

Next, in Section 3.4 we give the general definition for a type III planar algebra and we prove that the algebra $\mathcal{O}_{\rho\bar{\rho}}$ has the structure of a type III subfactor planar algebra.

We then extend the definition slightly, to define the planar algebra of subfactors which are not necessarily extremal, using a method similar to [15].

We conclude the chapter by introducing type III planar modules, this is a tool that has proved very useful in the type II theory.

We now begin by defining some simple extensions of two planar algebras to the type III setting, first the Temperley-Lieb algebra and next a matrix example.

3.1 Type III Temperley-Lieb Planar Algebra

A *planar diagram* is a rectangle in the plane with a collection of marked points along the top and bottom edges. The interior of the diagram contains a collection of non intersecting strings. Each marked point is the endpoint of exactly one string. All strings either form closed loops or have exactly two endpoints, each occuring at a marked point. Planar diagrams are defined up to planar isotopies which leave the boundary fixed. Often when drawing tangles we will denote n parallel strands by a thick strand with the number n adjacent to it. We may equip diagrams with a checkerboard shading. We call a diagram a $+$ diagram if the region adjacent to the first marked point is unshaded and a $-$ diagram if it is shaded.

Let $T^m_{n,\pm}$ denote the collection of all planar $\pm$ diagrams with n points along the bottom and m points along the top where $n+m$ is even.

A diagram $x \in T^m_{n,\pm}$ may be embedded in $T^{m+k}_{n+k,\pm}$ by adding k vertical through strings on the right and thus we have an embedding of $T^m_{n,\pm}$ in $T^{m+k}_{n+k,\pm}$ for any $k \geq 0$. The product xy of a diagram $x \in T^{m_1}_{n_1,\pm}$ with $y \in T^{m_2}_{n_2,\pm}$ is defined as follows. If $n_1 < m_2$ then embed x in $T^{m_1+m_2-n_1}_{m_2,\pm}$, then stack x on top of y, aligning corresponding marked points. Then remove the marked points and if necessary smooth the strings. The product xy is therefore an element of $T^{m_1+m_2-n_1}_{n_2,\pm}$. If $n_1 > m_2$ then we embed y in $T^{n_1}_{n_2+n_1-m_2,\pm}$ and proceed similarly. It is easy to see that this multiplication is associative. Figure 3.1 shows the product of $x \in T^3_{5,+}$ with $y \in T^2_{2,+}$.

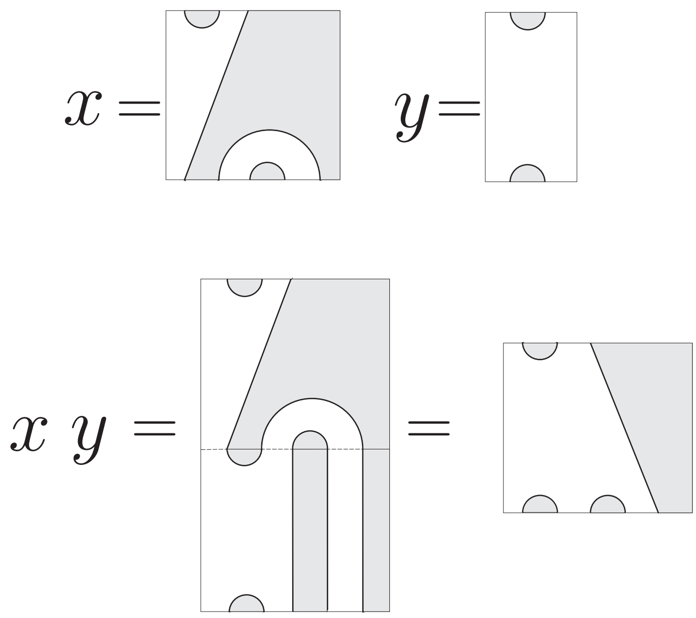


Figure 3.1: Multiplication of planar diagrams

For $x \in T^m_{n,\pm}$ define $x^* \in T^n_{m,\pm}$ to be the diagram obtained by reflecting x in a horizontal line through its centre. Note that this satisfies $x^{**} = x$ and $(xy)^* = y^* x^*$.

Let $\delta > 1$. For each n, m let $\mathcal{T}^m_{n,\pm}$ be the linear span over $\mathbb{C}$ of $T^m_{n,\pm}$. Let $I^m_{n,\pm}$ be the ideal in $\mathcal{T}^m_{n,\pm}$ generated by the relation 'closed loop=δ' (i.e any diagram x containing a closed loop is equivalent to $\delta x'$ where x' is just x with the loop removed). Then let $\mathcal{V}^m_{n,\pm}$ be the quotient of $\mathcal{T}^m_{n,\pm}$ by $I^m_{n,\pm}$. Note that $\mathcal{V}^0_{0,\pm} \simeq \mathbb{C}$, since the only elements of $\mathcal{V}^0_0$ are scalar multiples of the empty diagram. The dimension of $\mathcal{V}^m_{n,\pm}$ is the Catalan number $C_N = \frac{1}{N+1}\binom{2N}{N}$ where $N = \frac{1}{2}(n+m)$, since the dimension of $\mathcal{V}^m_{n,\pm}$ is the number of non crossing pair partitions on $2N$ elements. Hence $\mathcal{V}^m_{n,\pm}$ is finite dimensional for all n, m.

The space $\mathcal{V}^m_{n,\pm}$ can be naturally embedded in the space $\mathcal{V}^{m+1}_{n+1,\pm}$ using the linear extension of the embedding of $T^m_{n,\pm}$ in $T^{m+1}_{n+1,\pm}$. Using this embedding we take the algebraic inductive limit $\varinjlim \mathcal{V}^{j+2k}_{j,\pm} =: \mathcal{V}_{k,\pm}$. Let $O^{\pm}_{TL} := \oplus_{k\in\mathbb{Z}} \mathcal{V}_{k,\pm}$, so elements of $O^{\pm}_{TL}$ are finite direct sums of elements of the $\mathcal{V}_{k,\pm}$. Then we may equip $O^{\pm}_{TL}$ with the multiplication given by the bilinear extension of the multiplication and the *-operation given by the conjugate linear extension of the *-operation defined

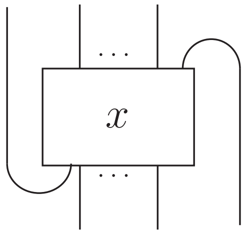


Figure 3.2: Map from O^+_{TL} to O^-_{TL}

above. It is easily seen by drawing the diagrams that the multiplication and *-operations are compatible with the identification of $x \in \mathcal{V}^{n+2k}_{n,\pm}$ with $x' \in \mathcal{V}^{n+2k+N}_{n+N,\pm}$. Under these operations $O^{\pm}_{TL}$ is an associative unital *-algebra. For each n let F^n_{TL} be the subalgebra of $O^{\pm}_{TL}$ spanned by diagrams with n marked points along the top and bottom edge. Using the map shown in Figure 3.2, where x is any O_{TL} tangle, it is easy to see that $O^+_{TL} \simeq O^-_{TL}$. Thus we usually just work with O^+_{TL} and write O_{TL} instead of O^+_{TL}.

Let $\alpha = \ln \delta$. For any $n, m \in \mathbb{N}$ and any diagram $x \in T^m_n$, define $\sigma_t(x) := e^{\alpha i(m-n)t}x$, and extend linearly to all $\mathcal{V}^m_n$. The action σ_t is compatible with the embedding of T^m_n in T^{m+k}_{n+k} and so σ_t can be extended to $\mathcal{V}_k$. Since for $x \in T^{m_1}_{n_1}$, $y \in T^{m_2}_{n_2}$ we have $\sigma_t(xy) = e^{\alpha i(m_1+m_2-n_1-n_2)t}xy = \sigma_t(x)\sigma_t(y)$, σ_t is multiplicative on all of O_{TL}. Also $(\sigma_t(x))^* = \sigma_t(x^*)$ and so σ_t is a *-automorphism. Let $S(x) = \alpha/2\pi \int_0^{\frac{2\pi}{\alpha}} \sigma_t(x)dt$ for $x \in O_{TL}$. Note that the map $t \mapsto \sigma_t(x)$ is continuous and so the integral is well defined. The operator S is positive since the integral is the limit of Riemann sums of the form $\sum_{i=0}^{n-1} \sigma_{t_i}(x)(t_{i+1} - t_i)$ which are obviously positive. Since S satisfies $S^2 = S$, S is a projection from O_{TL} onto $TL := \cup_n F^n_{TL}$.

Let tr_n be the usual trace on F^n_{TL}, that is tr is the trace on the Temperley-Lieb algebra defined in Section 2.5, defined on elements of V^n_n by joining corresponding points along the top and bottom of the diagram and let Tr be the normalised trace, defined by $Tr(x) = \delta^{-n} tr_n(x)$ for $x \in V^n_n$. The trace Tr may be extended linearly to F^n_{TL}, noting that it is zero on the ideal I^n_n since it is zero on the generator by definition. Note that Tr is scalar valued, since the only elements of

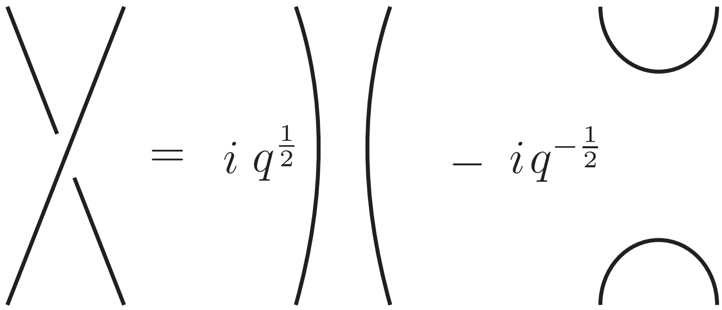


Figure 3.3: Braiding

F^0_{TL} are scalar multiples of the empty diagram.

When $\delta < 2$, let $\mathcal{I}^n_n$ be the ideal in F^n_{TL} generated by $x \in F^n_{TL}$ with $Tr(x^*x) = 0$ and let $\mathcal{F}^n_{TL} = F^n_{TL}/\mathcal{I}^n_n$ and let $\mathcal{O}_{TL}$ be the quotient of O_{TL} by traceless vectors. When $\delta \geq 2$, Tr is positive definite and we put $\mathcal{F}^n_{TL} = F^n_{TL}$ and $\mathcal{O}_{TL} = O_{TL}$. Then $\mathcal{F}^n_{TL}$ is just the ordinary Temperley-Lieb algebra on n generators defined in 2.5.

We define a state on the algebra $\mathcal{O}_{TL}$ by $\phi := Tr \circ S$. The state ϕ is positive since it is the composition of two positive operators, S and Tr.

We define an inner product on $\mathcal{O}_{TL}$ by $(x, y) = \phi(y^*x)$ and let H_{TL} be the Hilbert space completion. Let $\lambda : \mathcal{O}_{TL} \to \mathcal{O}_{TL}$ be the action of $\mathcal{O}_{TL}$ on itself by left multiplication. We wish to show that λ is bounded and hence may be extended uniquely to an element of $B(H_{TL})$. Let $\cup \in V^2_0$ be the diagram with two marked points along the top joined by a single string.

Using the crossing defined in Figure 3.3, with $q + q^{-1} = \delta$, we may define a braiding on O_{TL} in the same way as in Section 2.5.

Lemma 3.1.1. *The planar algebra O_{TL} is generated as a *-algebra by the type II Temperley-Lieb algebra and the element $\cup \in T^2_0$. Every element $x \in \mathcal{O}_{TL}$ may be written as a finite sum*

$$x = \sum \cup^n x_n + x_0 + \sum (\cup^*)^n x_{-n} \tag{3.1}$$

for some x_i in the type II Temperley-Lieb planar algebra

Proof. This is clear since if we have any diagram in T^m_n with $m = n+2k$ for some $k \geq 0$ then there must be at least k cups along the top. At least one of these

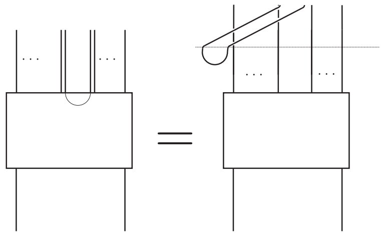


Figure 3.4: Writing tangles in O_{TL} as products of $\cup$ and TL tangles

joins two adjacent vertices. This may be moved to one side using the braiding as in Figure 3.4. This shows that a tangle in $x \in T_n^{n+2k}$ can be written as a product of a linear combination of Temperley-Lieb tangles, a copy of $\cup$ and a tangle in $T_n^{n+2(k-1)}$. The procedure can be repeated k times and thus we get a product of a linear combinationn of T_{n+2k}^{n+2k} tangles with some copies of $\cup$ and a T_n^n tangle. Thus there exists $\tilde{x} \in T_{n+2k}^{n+2k}$ with $x = \tilde{x}\cup^k$ and so every tangle in O_{TL} may be written in the form 3.1. Since we know from Section 2.5 that the ordinary Temperley-Lieb algebra is generated by Jones projections we have that O_{TL} is generated by Jones projections and $\cup$. □

Proposition 3.1.2. *Let $a \in \mathcal{O}_{TL}$ and let λ_a denote the action of a on $\mathcal{O}_{TL}$ by left multiplication. Then λ_a is bounded and may be extended uniquely to an element of $B(H_{TL})$ and so $\mathcal{O}_{TL} \subset B(H_{TL})$.*

Proof. Since, by Lemma 3.1.1, O_{TL} is generated by TL diagrams and $\cup$, all that needs to be proved is that left multiplication by $\cup$ or by $a \in TL$ are bounded. First let $a \in TL$ and suppose a has k strings along its top and bottom edge. Let $x = \sum c_i x_i$ where the $x_i \in T_{n_i}^{m_i}$ are diagrams in O_{TL} and $c_i \in \mathbb{C}$. It is easy to see that $\langle ax_i, ax_j \rangle$ is zero unless $n_i - m_i = n_j - m_j$ and so we just need to prove that there exists $C > 0$ such that $\|ax\| \le C\|x\|$ for $x = \sum c_i x_i$ where all the $x_i \in T_{n_i}^{n_i+N}$. Here we are using $\|\cdot\|$ for the inner product norm on H_{TL}. We may suppose that $x_i \in T_M^{M+N}$ for some M with $M + N > k$. Then

$$\begin{aligned}\|ax\|^2 =& \langle ax, ax \rangle = \sum_{i,j} \langle c_i a x_i, c_j a x_j \rangle \\ =& \sum_{i,j} \langle c_i a' x_i', c_j a' x_j' \rangle = \|a'x'\|^2\end{aligned}$$

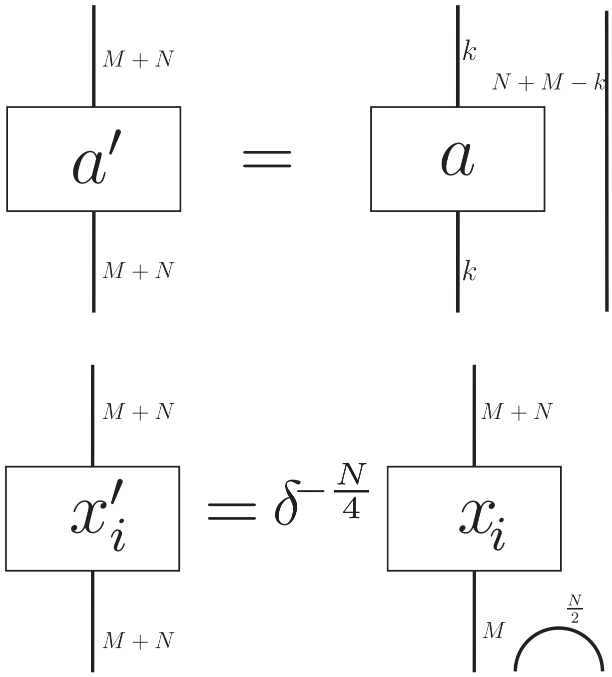


Figure 3.5: The tangles a' and x'

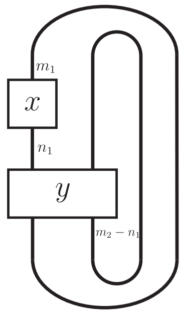
where a' is the image of a in T^{M+N}_{M+N}, x'_i is the T^{M+N}_{M+N} tangle shown in Figure 3.5 and $x' = \sum c_i x'_i$. Since a' and x' are both elements of the Temperley-Lieb algebra and the inner products on TL and O_{TL} agree on TL, we may use boundedness of the multiplication there to get $\|a'x'\| \leq \|a'\|_{op}\|x'\|$ where $\|\cdot\|_{op}$ is the operator norm. Thus, since $\|x'\| = \|x\|$ we have $\|ax\| \leq \|a'\|_{op}\|x\|$ as required.

Next we wish to show that there exists $C > 0$ such that $\langle \cup x, \cup x\rangle \leq C\|x\|^2$ for all $x \in O_{TL}$. To show this, let $x = \sum c_i x_i$ where the $x_i \in V^{m_i}_{n_i}$ are diagrams in O_{TL} and $c_i \in \mathbb{C}$. Then

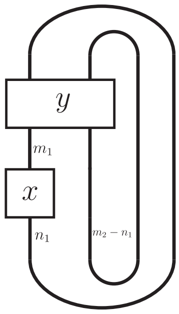
$$\begin{aligned}
\langle \cup x, \cup x\rangle &= \sum_{i,j} c_i c_j^* \langle \cup x_i, \cup x_j\rangle \\
&= \sum_{i,j} \delta_{n_i - m_i, n_j - m_j} \delta^{-(p_{i,j}+2)} c_i c_j^* tr_{p_{i,j}+2}(\cup x_i x_j^* \cup^*) \\
&= \sum \delta_{n_i - m_i, n_j - m_j} \delta^{-(p_{i,j}+1)} c_i c_j^* tr_{p_{i,j}}(x_i x_j^*) = \delta\|x\|^2
\end{aligned}$$

where $p_{i,j} = n_i$ if $m_i > n_j$ and $p_{i,j} = m_j$ if $m_i < n_j$. It is easy to see that the map $a \to \lambda_a$ is multiplicitive since $\lambda_{ab} = \lambda_a\lambda_b$ and the map is injective since, for any $a \neq 0$, $\langle \lambda_a(1), \lambda_a(1)\rangle = \langle a, a\rangle > 0$ and so λ_a is zero if and only if a is zero. $\square$

$\varphi(xy) = \delta^{-n_2}$



$= \delta^{-n_2}$



$$= \quad \delta^{n_1-m_1}\varphi(yx)$$

Figure 3.6: KMS condition

Thus we can define a C^*-algebra $\overline{\mathcal{O}}_{TL}$ by taking the norm completion of $\mathcal{O}_{TL}$ in $B(H_{TL})$ and a von Neumann algebra $\mathcal{M}_{TL}$ by taking the weak completion. Let $\Omega \in H_{TL}$ be the image of $1 \in \mathcal{O}_{TL}$ in H_{TL} and define a state ψ on $B(H_{TL})$ by $\psi(x) = \langle x\Omega, \Omega\rangle$. Then ψ is a faithful normal state on $B(H_{TL})$ which agrees with ϕ on the image of $\mathcal{O}_{TL}$. Denote by φ the restriction of ψ to $\mathcal{M}_{TL}$.

Proposition 3.1.3. *The state φ is the unique KMS state on $\mathcal{M}_{TL}$ for σ_t and the inverse temperature β is also unique and equal to* 1*.*

Proof. To show φ is a KMS state at inverse temperature $\beta = 1$ we calculate for example $\varphi(xy)$ for $x \in T_{n_1}^{m_1}$, $y \in T_{n_2}^{m_2}$. Then $\varphi(xy) = 0$ unless $m_1 - n_1 = n_2 - m_2$. Suppose that with $m_1 < n_2$ and $n_1 < m_2$. In this case we see as in Figure 3.6 that $\varphi(xy) = Tr_{n_2}(xy) = \delta^{m_2-n_2}Tr_{m_2}(yx) = \delta^{n_1-m_1}\varphi(yx) = \varphi(y\sigma_i(x))$. The other cases may be proved similarly. The element x is entire since $\sigma_{it}(x) = e^{\alpha(m_1-n_1)t}$ is an analytic continuation of σ_t. Hence ϕ is a KMS state for σ_t.

Uniqueness can be proved in a similar way to Example 5.3.27 of [7]. Let ψ be any KMS state for σ_t at temperature β. Since $\psi(\sigma_t(x)) = e^{\alpha it(m-n)}\psi(x)$ for $x \in T_n^m$ and by properties of the KMS state ψ is σ_t invariant, we must have that $\psi(x) = 0$ for all $x \in T_n^m$ with $n \neq m$. Hence ψ is the trace on TL, since it is contained in the fixed point algebra $\mathcal{O}_{TL}^{\sigma}$, and zero everywhere else. Since the trace on TL is unique, we must have $\psi = \varphi$. To prove uniqueness of β let $x \in T_{n_1}^{m_1}$ and $y \in T_{n_2}^{m_2}$. By the KMS condition $\varphi(xy) = \varphi(y\sigma_{i\beta}(x)) = e^{-\alpha\beta(n_1-m_1)}\varphi(yx)$. Both sides of this equation are zero unless $m_1 - n_1 = n_2 - m_2$, so suppose this is the case and also suppose $n_1 > m_2$ (a similar calculation shows the result holds for $m_2 > n_1$ also). Then, since $\varphi(xy) = \delta^{-m_2}tr(xy)$ and $\varphi(yx) = \delta^{-n_2}tr(yx)$ it is easy to see that the KMS condition is satisfied if and only if $\beta = 1$. $\square$

Since the KMS state φ is unique, by Theorem 5.3.30 of [7], the von Neumann algebra completion of $\mathcal{O}_{TL}$ is a factor. The centraliser of $\mathcal{M}_{TL}$ is the II_1 factor M_{TL} generated by the Temperley-Lieb algebra. It is easy to see that M_{TL} is contained in the centraliser of $\mathcal{M}_{TL}$, for the reverse inclusion note that the conditional expectation S maps $\mathcal{M}_{TL}$ to $\mathcal{M}_{TL}^{\sigma}$. For any $x \in (\mathcal{M}_{TL})_{\varphi}$ there exists a net x_i in O_{TL} converging weakly to x and so $S(x_i)$ is a net in TL converging weakly to $S(x) = x$ and hence $x \in M_{TL}$. Thus the Connes spectrum and Arveson spectrum coincide. Since σ_t is periodic with period $2\pi/\alpha$, $\mathcal{M}_{TL}$ is a type III_λ factor for $\lambda = e^{-\alpha} = 1/\delta$.

Proposition 3.1.4. *The C^*-algebra $\overline{\mathcal{O}}_{TL}$ is simple and purely infinite.*

Proof. We prove this in a similar way to Theorem 3.1 of [82]. We need to show that for each non-zero $x \in \overline{\mathcal{O}}_{TL}$ there exists $v \in \overline{\mathcal{O}}_{TL}$ such that $v^*xv = 1$. As in [82] we prove this in four steps. Let $\mathcal{TL}$ be the C^*-algebra generated by (type II) Temperley-Lieb tangles and define $\sqcup := \delta^{-\frac{1}{2}}\cup$. Let Φ be the endomorphism on $\mathcal{TL}$ defined by $\Phi(x) = \sqcup x \sqcup^*$ for $x \in \mathcal{TL}$. Let $E : \overline{\mathcal{O}}_{TL} \to \mathcal{TL}$ be the conditional expectation defined by $E(x) = \alpha/2\pi \int_0^{2\pi/\alpha} \sigma_t(x)dt$.

Step 1: We prove that for every non-zero projection $p \in \mathcal{TL}$ there is a partial isometry $u \in \mathcal{TL}$ and $m \in \mathbb{N}$ such that $(\sqcup^*)^m u^* p u \sqcup^m = (\sqcup^*)^m u^* u \sqcup^m = 1$.

Let p be any non-zero projection in $\mathcal{TL}$. Choose $m \in \mathbb{N}$ such that $\delta^{-2m} = Tr(\sqcup^m(\sqcup^*)^m) \leq Tr(p)$. Then we may find $u \in \mathcal{TL}$ with $\sqcup^m(\sqcup^*)^m = u^*u$ and $uu^* \leq p$ and thus $(\sqcup^*)^m u^* p u \sqcup^m = (\sqcup^*)^m u^* u \sqcup^m = 1$.

Step 2: Next we prove that for every non-zero $x \in \mathcal{TL}$ and $m \in \mathbb{N}$ there is a non-zero projection p such that $\|px\Phi^m(p)\| \leq \frac{1}{3}\|x\|$.

Let $\lambda = \inf sp|x^*|$, where we denote by $sp(x)$ the spectrum of x. If $\lambda < \frac{1}{3}\|x\|$ then choose a continuous function $f : [0, \|x\|] \to \mathbb{R}_+$ with $f(t) = 0$ for $t > \frac{1}{3}\|x\|$ and $f(|x^*|) \neq 0$. Then let p be a non-zero projection in the closure of the subalgebra $f(|x^*|)\mathcal{TL}f(|x^*|)$. Then p satisfies $\|px\| \leq \frac{1}{3}\|x\|$. Hence $\|px\Phi^m(p)\| \leq \frac{1}{3}\|x\|$. If $\lambda > \frac{1}{3}\|x\|$ then $|x^*|$ is invertible. Hence $u = |x^*|^{-1}x \in \mathcal{TL}$ is a unitary. If we take $p = 1 - u\sqcup^m(\sqcup^*)^m u^*$ then $pu\Phi^m(p) = 0$ and so $px\Phi^m(p) = p(|x^*| - \frac{2}{3}\|x\|)u\Phi^m(p)$. Hence

$$\|px\Phi^m(p)\| \leq \frac{1}{3}\|x\|.$$

Step 3: For any $x \in \overline{\mathcal{O}}_{TL}$ with $E(x) = 1$ and any $\epsilon > 0$ there is an isometry $v \in \overline{\mathcal{O}}_{TL}$ with $\|v^*xv - 1\| \leq \epsilon$.

By Proposition 3.1.1 any x in O_{TL} may be written as $\sum \sqcup^n x_n + x_0 + \sum(\sqcup^*)^n x_{-n}$. For x of this form, $E(x) = x_0$. Thus any $x \in \overline{\mathcal{O}}_{TL}$ with $E(x) = 1$ may be approximated by elements of the form $\sum_{n=1}^{N}(\sqcup^*)^n x_{-n} + 1 + \sum_{n=1}^{N} x_n \sqcup^n$ with $x_n \in \mathcal{TL}$ and so it is sufficient to show that for any x of the above form there exists an isometry v and $j \neq 0$ such that $v^*xv = \sum(\sqcup^*)^n x'_{-n} + 1 + \sum x'_n \sqcup^n$ with $x'_i \in TL$ such that $\|x'_i\| \leq \|x_i\|$ and $\|x'_j\| \leq \frac{1}{3}\|x_j\|$. By the previous step, there exists a projection $p \in \mathcal{TL}$ with $\|px_j\Phi^j(p)\| \leq \frac{1}{3}\|x_j\|$. Hence, with u,m as in step 1, $v = pu\sqcup^m$ is an isometry with the required property.

Step 4: Finally we show that for any non-zero $x \in (\overline{\mathcal{O}}_{TL})_+$ there exists $z \in \overline{\mathcal{O}}_{TL}$ with $z^*xz = 1$.

It suffices to show that there exists z such that $\|z^*xz - 1\| < 1$, since then z^*xz is invertible. Hence, by the previous step, we just need to find z such that $E(zxz^*) = 1$. Let $x_0 = E(x)$. Then x_0 is a non-zero positive element of $\mathcal{TL}$. Choose ϵ with $0 < \epsilon < \|x_0\|$ and put $f(t) = \max\{t - \epsilon, 0\}$ Then $\overline{f(x_0)TLf(x_0)} \subset$

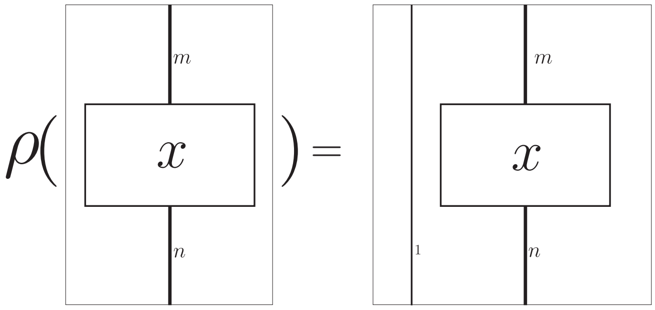


Figure 3.7: The endomorphism ρ

$x_0^{\frac{1}{2}}TLx_0^{\frac{1}{2}}$. Since $\mathcal{TL}$ is AF and so real rank zero, by Theorem 2.6 of [8] there exists a non zero projection $p \in \overline{f(x_0)TLf(x_0)}$. Then $p = x_0^{\frac{1}{2}}yx_0^{\frac{1}{2}}$ for some positive $y \in \mathcal{TL}$. Let $e = y^{\frac{1}{2}}x_0y^{\frac{1}{2}}$ then by step one we may find $u \in \mathcal{TL}$ and $m \in \mathbb{N}$ such that $(\sqcup^*)^m u^* eu\sqcup^m = 1$. Then $z = y^{\frac{1}{2}}u\sqcup^m$ has expectation one since

$$E(z^*xx) = E((\sqcup^*)^m u^* y^{\frac{1}{2}}xy^{\frac{1}{2}}u\sqcup^m) = (\sqcup^*)^m u^* eu\sqcup^m = 1.$$

□

Proposition 3.1.5. *The C^*-algebra $\overline{\mathcal{O}}_{TL}$ is isomorphic to the crossed product $\overline{\mathcal{O}}_{TL} \simeq \mathcal{TL} \rtimes_\Phi \mathbb{N}$ where $\mathcal{TL}$ is the C^*-algebra generated by Temperley-Lieb diagrams and the endomorphism Φ is given by $\Phi(x) = \delta^{-1} \cup x \cup^*$ for $x \in TL$.*

Proof. By Lemma 3.1.1 $\mathcal{O}_{TL}$ is generated by TL and $\delta^{-1/2}\cup \in \mathcal{O}_{TL}$. Since $(\delta^{-1/2}\cup)^*\delta^{-1/2}\cup = 1$ and $\delta^{-1/2} \cup (\delta^{-1/2}\cup)^* = e_0$, $\delta^{-\frac{1}{2}}\cup$ is an isometry. Since $\mathcal{T}L$ is simple and AF it follows from Theorem 2.6 of [8] that it has real rank zero, and from [6], Example 5.1 that it has comparability of projections. Thus it follows from Theorem 3.1 of [82] that the crossed product $\mathcal{TL} \rtimes \mathbb{N}$ is simple. There exists a homomorphism from $\mathcal{TL} \rtimes \mathbb{N} \to \overline{\mathcal{O}}_{TL} = C^*(TL, \cup)$ and since we know $\mathcal{TL} \rtimes \mathbb{N}$ is simple this must be an isomorphism. Hence $\overline{\mathcal{O}}_{TL}$ is the crossed product of $\mathcal{TL}$ with $\mathbb{N}$ by the endomorphism Φ of TL defined by $\Phi(x) = \delta^{-1} \cup (x)\cup^*$ for any $x \in TL$. □

An endomorphism ρ can be defined on $\mathcal{O}_{TL}$ as in Figure 3.7. From Figure 3.7 it is clear that $\rho(xy) = \rho(x)\rho(y)$, $\rho(x^*) = (\rho(x))^*$ and $\rho(1) = 1$ for all $x, y \in \mathcal{O}_{TL}$.

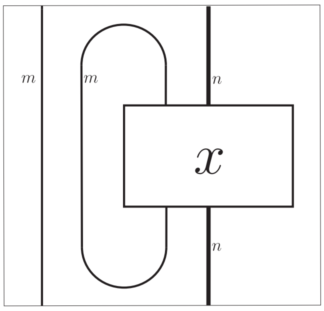


Figure 3.8: Conditional expectation tangle

For the case $\delta < 2$ we will show in Proposition 3.4.8 that $\mathcal{O}_{TL}$ may be realised as a subalgebra of the planar algebra of the graph A_n, where $\delta = 2\cos(\pi/n)$. It follows from this and Proposition 3.3.6 that the commutant $(\rho^m(\mathcal{M}_{TL}))' \cap \mathcal{M}_{TL}$ is just the *-algebra F^m_{TL}. We use this fact in the proof of the following proposition, however we note here that the following proposition is not used to prove any further results in this thesis, in particular it is not needed to prove Propositions 3.3.6 and 3.4.8.

Proposition 3.1.6. *Suppose $\delta = 2\cos\pi/n$ for some $n \geq 3$. Then $\rho(\mathcal{M}_{TL}) \subset \mathcal{M}_{TL}$ is an extremal finite index subfactor which is isomorphic to the subfactor $N \otimes \mathcal{M}_{TL} \subset M \otimes \mathcal{M}_{TL}$ where M is the type II_1 factor which is the weak closure of the algebra generated by Temperley-Lieb diagrams with the same number of marked points on the top and bottom and N is the II_1 factor generated by diagrams with a through strand joining the leftmost marked points on top and bottom.*

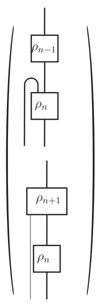
Proof. To see this we need to check the conditions of Theorem 2.8.1. There is a conditional expectation $E : M \to \rho(M)$ defined by Figure 3.8 with $m = 1$. Morevoer, for any m, Figure 3.8 defines a conditional expectation $E_m : M \to \rho^m(M)$. By the comment above the proposition the relative commutants $\rho^k(M)' \cap M \simeq F^k_{TL}$. Thus it is clear that the conditional expectation E_k implements a trace on $\rho^k(M)' \cap M$. For finite graphs it is shown in 1.3.6 of [75] that the condition $\Gamma_{N,M}$ is strongly amenable is always true. To show $\rho(\mathcal{M}_{TL}) \subset \mathcal{M}_{TL}$ is approximately inner we need to prove the existence of a simultaneous discrete decomposition. A theorem of Loi ([56], Theorem 2.8) states that for a type III_λ subfactor $N \subset M$ the simultaneous discrete decomposition always exists if $N' \cap M$ is a factor and the conditional expectation is minimal. Since $\rho(\mathcal{M}_{TL})' \cap \mathcal{M}_{TL}$ is one dimensional, there is only one conditional expectation. Thus $\rho(\mathcal{M}_{TL}) \subset \mathcal{M}_{TL}$ is approximately inner. The only thing left to check is central freeness. Corollary 3.7 of [77] states that for III_λ factors, central freeness of $N \subset M$ is equivalent to the condition that the principal graph of $N \subset M$ is the same as the principal graph for the type II_∞ subfactor appearing in the common discrete decomposition. It is known (e.g. [53] Theorem 5.13) that this is always true for type III subfactors with principal graphs A_n for $n \neq 4m-3$. In the case $n = 4m-3$ the type II principal graph may be A_n or D_{2m}. However, since we know that the higher relative commutants are the Temperley-Lieb algebras which are generated by Jones projections, the Loi invariant must be trivial. Hence, by the discussion preceding Theorem 5.13 of [53], the principal graph must be A_{4m-3}. Hence all the conditions of Theorem 2.8.1 are satisfied, and $\rho(\mathcal{M}_{TL}) \subset \mathcal{M}_{TL}$ is isomorphic to the subfactor $\left(\overline{(\cup_k \rho^k(M)' \cap \rho(\mathcal{M}_{TL})} \subset \overline{(\cup_k \rho^k(\mathcal{M}_{TL})' \cap \mathcal{M}_{TL})}\right) \otimes \mathcal{M}_{TL}$.

□

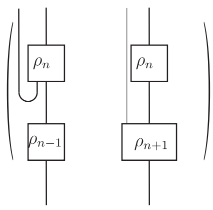
Next we show how to decompose the endomorphism ρ into irreducibles, similar to [63], [25]. In order to do this we need to define a tensor category $\mathcal{C}$ whose objects are elements of $\cup_n \mathcal{T}_n^n$ and for any $x \in \mathcal{T}_n^n$, $y \in \mathcal{T}_m^m$ the morphisms in $Hom(x, y)$ are elements of $\mathcal{T}_n^m$. The trivial object in $\mathcal{C}$ is the empty diagram and the tensor product is given by horizontal juxtaposition. From $\mathcal{C}$ we define a matrix category $Mat(\mathcal{C})$ which has objects given by direct sums of objects in $\mathcal{C}$ and morphisms given by matrices of morphisms in $\mathcal{C}$, that is $Hom(x_1 \oplus \cdots \oplus x_n, y_1 \oplus \cdots \oplus y_m)$ is an $m \times n$ matrix whose i, j entry is an element of $Hom(x_j, y_i)$. The tensor product on $\mathcal{C}$ gives a tensor product on $Mat(\mathcal{C})$, where the tensor product on objects is given by distributing over the direct sum and the tensor product of morphisms is given by the ordinary tensor product of matrices.

Let $\rho \in ob(\mathcal{C})$ be the diagram consisting of a single vertical strand. Clearly ρ is irreducible since $Hom(\rho, \rho)$ is in $\mathcal{V}_1^1$ which is one dimensional. Then $\rho \otimes \rho = \rho^2$

is the diagram with two vertical strands. It is easy to see that $Hom(\rho^2, \rho^2)$ is two dimensional as it is spanned by the identity diagram and the element $\cup\cup^*$. Thus we may decompose ρ^2 into irreducibles. First we can see that (ρ^2, ρ^0) is one dimensional as it is spanned by $\cup$. The space $Hom(\rho^2, \rho^1)$ is empty by definition and thus we must have that $\rho^2 = id \oplus \rho_2$ where ρ_2 is an irreducible endomorphism in $\mathcal{V}_2^2$. The irreducible endomorphisms are represented graphically in the same way as the Jones Wenzl idepotents in the type II case. They satisfy the same relations and we have that $\rho\rho_n \simeq \rho_{n-1} \oplus \rho_{n+1}$ where the isomorphism is represented graphically by



$: \rho\rho_n \to \rho_{n-1} + \rho_{n+1}$



$: \rho_{n-1} + \rho_{n+1} \to \rho_n$

3.2 Planar algebra for infinite tensor product of matrices

Let $\mathcal{N} = 1 \otimes M_p \otimes M_p \otimes \ldots$ and $\mathcal{M} = M_p \otimes M_p \otimes \ldots$ and consider the subfactor $\mathcal{N} \subset \mathcal{M}$ of index p^2. The Jones tower is then $\mathcal{N} \subset \mathcal{M} \subset M_p \otimes \mathcal{M} \subset M_p \otimes M_p \otimes \mathcal{M} \subset \ldots$ and the relative commutants $\mathcal{N}' \cap \mathcal{M}_n = \otimes^n M_p$ where $\mathcal{M}_n$ is the n^{th} step in the basic construction. For this we recall Example 2.6 of [36]: in the setting of type II planar algebras, Jones defines a planar algebra P whose vector spaces are $P_k = (V \otimes V^*)^{\otimes k}$, V is a vector space of dimension p with dual V^* so $V \otimes V^* \simeq M_p \simeq End(V)$. Internal discs of colour k_i of tangles are labelled by elements of $(V \otimes V^*)^{\otimes k_i}$ with $2k_i$ factors in the tensor product. Given a fully labelled tangle T we define the presenting map $Z_T : \otimes P_{k_i} \to P_{k_0}$ as follows.

First let σ be a map (which we will call a state) from the strings of T to the set $\{1, \ldots, p\}$. For any internal disc D the state σ assigns a sequence of indices $j_1, \ldots j_m$ to the strings meeting D along its top edge and $i_1, \ldots i_m$ to the strings meeting D along its bottom edge. Thus if D is labelled by the element $R \in (V \otimes V^*)^{\otimes m}$, following Jones notation we denote the number ${R_{j_1,\ldots j_m}}^{i_1,\ldots i_m}$ by $\sigma(D)$.

The presenting map then gives an element of $(V \otimes V^*)^{\otimes k_0}$, where the entry of Z_T indexed by ${}_{j_1,\ldots,j_k}{}^{i_1,\ldots,i_k}$ is

$$(Z_T)_{j_1,\ldots,j_k}{}^{i_1,\ldots,i_k} := \sum_{\substack{\text{states inducing} \\ \binom{i_1,\ldots,i_k}{j_1,\ldots,j_k} \text{ on boundary}}} \prod_{\text{internal discs} D} \sigma(D)$$

where the sum is taken over all states on the tangle that induce the correct indices on the boundary disc D_0. The value of a closed loop is thus p, the dimension of V, since there will always be a choice of p labels for a loop.

3.2.1 Type III Planar Algebra for Tensors I

We may generalise this example in two ways to give a type III subfactor. For the first generalisation, suppose V has dimension n, the vector spaces P_k are as before as are the maps σ and Z_T. Let $\{c_1, \ldots, c_n\} \in \mathbb{R}^n_+$ and let D be the matrix

$$\begin{pmatrix} e^{-\beta c_1} & 0 & \cdots & 0 \\ 0 & e^{-\beta c_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{-\beta c_n} \end{pmatrix}$$

where β is chosen so that the matrix has trace 1. We modify the trace of the previous example to get a state ϕ defined as follows. Starting with a tangle T we join corresponding points along the top and bottom with a string. Each of these joining strings has a weight, diagramatically we indicate this with a dot on the string. Then when we take the presenting map, if a state assigns the value

k to one of the joining strings we multiply by $e^{\beta c_k}$. Taking the completion with respect to this state gives the hyperfinite type III_λ factor, where $\lambda = \ln \beta$. Thus this is the planar algebra of the subfactor $(N, \phi) \subset (M, \phi)$ of the hyperfinite type III_λ factor since each $P_k = (M_n^{\otimes k}, \phi)$, which is the relative commutant $M_k \cap M$ of this subfactor. Here ϕ is the state defined by $\phi(x) = tr(\otimes^k D \cdot x)$ for $x \in M_n^{\otimes k}$.

3.2.2 Type III Planar Algebra for Tensors II

For our second example we enlarge the vector spaces to get a type III algebra with non-tracial state. Let V be a p dimensional vector space. We define a planar algebra $P = \oplus_k \varinjlim P_i^{i+2k}$ where $P_i^j = (V)^{\otimes i} \otimes (V^*)^{\otimes j}$ and presenting maps Z_T defined below.

A (n, m)-tangle is a disc in the plane which we shall often draw as a box for convenience (i.e. in order to keep track of what we mean by the top and bottom edge), with n marked points along the top edge and m along the bottom edge. The interior of the disc contains a possibly empty set of internal discs D_i $1 \le i \le k(T)$ each with n_i marked points along the top and m_i along the bottom, along with a collection of non-intersecting strings, each of which has both endpoints on marked points of the discs or form closed loops.

The adjoint of a labelled tangle is gotten by reflecting it in a horizontal line through its centre and replacing the labels by their adjoints. As in the above type II example an input disc D_i is labelled by an element of $(V)^{\otimes n_i} \otimes (V^*)^{\otimes m_i}$, so a box with n marked points on the top and bottom is an element of $\otimes^n M_p$. For a tangle T we define a state $\sigma : \{\text{strings of T}\} \to \{1, 2, \dots, p\}$. If the state σ assigns the indices $j_1, \dots, j_{m_k}$ and $i_1, \dots, i_{n_k}$ to the top and bottom edges of a disc D_k (labelled by some $R \in (V)^{\otimes n_k} \otimes (V^*)^{\otimes m_k}$) then we say $\sigma(D_k)$ is the number ${R_{j_1, \dots j_{m_k}}}^{i_1, \dots i_{n_k}}$. As in the type II case, the presenting map of a fully labelled (n, m) tangle T defines an element of $(V)^{\otimes n} \otimes (V^*)^{\otimes m}$, its entry indexed by ${}_{i_1, \dots, i_n}{}^{j_1, \dots, j_m}$ is the number:

$$(Z_T)_{i_1, \dots, i_n}{}^{j_1, \dots, j_m} := \sum_{\substack{\text{compatible} \\ \text{states } \sigma}} \prod_{\text{internal discs } D} \sigma(D).$$

Once again, by compatible state we mean a state on the tangle which induces the correct indices on the boundary of D_0 and here closed loops can be removed by multiplying the resulting tangle by p. This presenting map is compatible with the composition of tangles and the star operation as required. Note that with this presenting map we recover ordinary matrix multiplication for two discs x, y labelled by elements of M_p (or $\otimes^n M_p$) using the Z_T where T is the usual multiplication tangle which stacks x on top of y and joins corresponding strings. The multiplicitive identity here is just the tangle with all vertical through strings.

Define an action of $\mathbb{R}$ on P by $\sigma_t(x) = e^{2\pi i k t}x$ for $x \in P_n^{n+k}$. We define a state in a similar way to the state on O_{TL}, by $\phi := Tr \circ S$, where $S : P \to \bigcup_n P_n^n$ is the projection defined by $S(x) = \int_0^1 \sigma_t(x)dt$ for $x \in P_n^m$ and Tr is defined on (n, n)-tangles by joining corresponding marked points along the top and bottom. Note that ϕ gives the usual trace on elements of $\otimes^n M_p$.

This construction gives the subfactor $\mathcal{N} \otimes Q \subset \mathcal{M} \otimes Q$, with $\mathcal{N}$, $\mathcal{M}$ the II_1 factors above and Q is the type III factor which is the weak completion of P with respect to the trace defined by ϕ. The tower of relative commutants

$$(\mathcal{N} \otimes Q)' \cap (\mathcal{N} \otimes Q) \subset (\mathcal{N} \otimes Q)' \cap (\mathcal{M} \otimes Q) \subset (\mathcal{N} \otimes Q)' \cap (\mathcal{M}_1 \otimes Q) \subset \dots$$

is in this case given by

$$\mathbb{C} \subset M_p \subset M_p \otimes M_p \subset \dots$$

As in the algebra $\mathcal{O}_{TL}$, in the planar picture the commutants are the diagrams with the same number of marked points on the top and bottom. Checking dimensions we see the spaces have the correct dimension, the $(0, 0)$ space has dimension one, and the dimensions of the (n, n) space is p^{2n}, since we have $2n$ marked points and p choices of label for each.

3.2.3 Fixed Point Algebras

Let $\mathcal{N} \subset \mathcal{M}$ be as in the section 3.2.2, but now we require $dimV = 2$. Let G be a finite subgroup of $SU(2)$. Then G acts on $\mathcal{M}$ as follows. Let e_1, e_2 be a basis

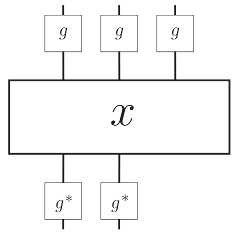


Figure 3.9: Action of $g \in G$ on a tangle $x \in P_2^3$

for V and let $(g_{ij}) \in G$ then $\alpha_g(e_1) = g_{11}e_1 + g_{21}e_2$ and $\alpha_g(e_2) = g_{12}e_1 + g_{22}e_2$. Then the inclusion $\mathcal{N}^G \subset \mathcal{M}^G$ is a subfactor. It was shown in [27] that the tower of relative commutants is just the tower of fixed point algebras of the tower of the original subfactor.

The action of G on the planar algebra is as shown in the Figure 3.9 multiply along the top by g and along the bottom by g^*.

It is well known that $(\otimes M_2)^{SU(2)}$ is isomorphic to the Temperley-Lieb algebra and hence $M^{SU(2)} \simeq \mathcal{M}_{TL}$.

Finite subgroups of $SU(2)$ are classified by the affine Dynkin diagrams. The A_n diagrams correspond to the cyclic subgroups. The cylcic subgroup corresponding to A_n is generated by the diagonal matrix $\{e^{2\pi i/n}, e^{-2\pi i/n}\}$. If the state σ assigns the value 1 to a string, multiply the diagram by $e^{2\pi i/n}$, if it assigns the value 2, multiply by $e^{-2\pi i/n}$. Then $x \in P_n^m$ is in the fixed point algebra if whenever σ assigns the value 1 to k strings and 2 to l strings then $e^{2\pi ki/n}e^{-2\pi li/n} = 1$ where $k + l = n + m$.

3.3 String Algebra construction of type III factors

In this section we generalise the string algebra construction of Section 2.4 to define hyperfinite type III subfactors.

Let $\mathcal{G}$ be the graph:

$$\begin{array}{ccc} V_0 & \xrightarrow{\mathcal{G}_0} & V_1 \\ {\scriptstyle \mathcal{G}_3}\downarrow & & \downarrow{\scriptstyle \mathcal{G}_1} \\ V_3 & \underset{\mathcal{G}_2}{\longrightarrow} & V_2 \end{array}$$

with finite subgraphs $\mathcal{G}_i$ and vertices $\cup_{0\leq i\leq 3} V_i$ such that $\mathcal{G}_0$ and $\mathcal{G}_3$ share common vertices V_0, $\mathcal{G}_0$ and $\mathcal{G}_1$ share vertices V_1, $\mathcal{G}_2$ and $\mathcal{G}_1$ share vertices V_2 and finally $\mathcal{G}_2$ and $\mathcal{G}_3$ share vertices V_3. The graph $\mathcal{G}$ has a distinguished vertex marked $*$ in V_0. We assume that the graphs $\mathcal{G}_0$ and $\mathcal{G}_2$ are connected. Suppose that there exists an assignment of positive numbers $\mu(v)$ to each vertex v and a pair of positive numbers δ, δ' such that there exists a connection on $\mathcal{G}$ satisfying the unitarity, renormalisation and harmonicity conditions described in Section 2.4. A string is a pair of paths (ξ_1, ξ_2) in $\mathcal{G}$ with the same start and end point but possibly different lengths. We can define a multiplication and *-operation on strings as follows. Let

$$(\xi_1,\xi_2).(\zeta_1,\zeta_2) = \begin{cases} (\xi_1, \zeta_2 \cdot \xi_2') & \text{if } \xi_2 = \zeta_1 \cdot \xi_2' \\ (\xi_1 \cdot \zeta_1', \zeta_2) & \text{if } \zeta_1 = \xi_2 \cdot \zeta_1' \\ 0 & \text{otherwise} \end{cases}$$

and

$$(\xi_1, \xi_2)^* = (\xi_2, \xi_1),$$

where $\xi \cdot \zeta$ is the concatenation of ξ and ζ. In other words, if $r(\xi) = s(\zeta)$, ξ is a path of length m and ζ is a path of length m then $\xi \cdot \zeta$ is the path of length $n+m$ obtained by first travelling along ξ and then travelling along ζ, in the case where $r(\xi) \neq s(\zeta)$ we define $\xi \cdot \zeta = 0$. Note that $((\xi_1,\xi_2).(\zeta_1,\zeta_2))^* = (\zeta_1,\zeta_2)^*(\xi_1,\xi_2)^*$.

We may define a collection of vector spaces $A_{(m,n),p}$ as follows. For m, n or p less than 0 the space $A_{(m,n),p}$ is empty by definition. For $m, n \in \mathbb{N}$ then $A_{(m,n),0}$ is the vector space with basis given by the collection of paths (ξ_1, ξ_2) where ξ_1, ξ_2 are paths in $\mathcal{G}_0$ with $s(\xi_1) = s(\xi_2) = *$, $r(\xi_1) = r(\xi_2)$ and $|\xi_1| = m$, $|\xi_2| = n$. Note that in this case $A_{(0,0),0} = \mathbb{C}$ can be identified with the distinguished vertex

$*$. The space $A_{(m,n),0}$ can be naturally embedded in $A_{(m+1,n+1),0}$ using the map

$$(\xi_1, \xi_2) \mapsto \sum_{\sigma} (\xi_1 \cdot \sigma, \xi_2 \cdot \sigma)$$

where the sum is over all edges $\sigma \in \mathcal{G}_0$ for which the composition makes sense. This embedding is compatible with the above multiplication and *-operation. Let $A_{k,0} = \varinjlim A_{(m,m+2k),0}$ and $A_{\infty,0} = \bigoplus_{k\in\mathbb{Z}} A_{k,0}$. The multiplication and *-operation above make $A_{\infty,0}$ into an associative *-algebra with unit $(*, *)$.

We can define a state ψ on $A_{0,0}$ as follows. First define an action of $\mathbb{R}$ on strings (ξ, ζ) by $\sigma_t((\xi,\zeta)) = e^{\ln\delta it(|\xi|-|\zeta|)}(\xi,\zeta)$. Then we define a state by $\psi(\xi,\zeta) = \delta_{\xi,\zeta}\delta^{-|\xi|}\mu(r(\xi))$.

Let $S : A_{\infty,0} \to A_{0,0}$ be the map defined by

$$S(\xi_1, \xi_2) := \frac{\ln\delta}{2\pi}\int_0^{2\pi/\ln\delta} \sigma_t(\xi_1,\xi_2)dt = \delta_{|\xi_1|,|\xi_2|}(\xi_1,\xi_2).$$

Then we may prove in a similar way to Section 3.1 that S is a faithful conditional expectation of $A_{\infty,0}$ into $A_{0,0}$.

We may then define a state ϕ on the algebra $A_{\infty,0}$ by $\phi = \psi \circ S$. Using ϕ we may define an inner product on $A_{\infty,0}$ by $\langle x, y\rangle = \phi(y^*x)$. Let $H_{\infty,0}$ be the Hilbert space completion of $A_{\infty,0}$ in this inner product.

Proposition 3.3.1. *Let $x \in A_{\infty,0}$. Denote by λ_x the action of x on $A_{\infty,0}$ by left multiplication. Then λ_x is a bounded operator for the inner product norm on $A_{\infty,0}$ and so it may be extended uniquely to an element of $B(H_{\infty,0})$.*

Proof. Let $\cup \in A_{(2,0),0}$ denote the element $\sum_e (e\tilde{e}, *)$, where the sum is over all edges $e \in \mathcal{G}_0$. The algebra $A_{\infty,0}$ is generated by $A_{0,0}$ and $\cup$, since if $(\xi,\zeta) \in A_{(n,n-2k),0}$, then $(\xi,\zeta) = (\xi, (e\tilde{e})^k \cdot \zeta) \cdot \sum_{|\sigma|=n+k}((e\tilde{e})^k\sigma, \sigma)$. Let $x = \sum c_i(\alpha_i, \beta_i) \in A_{\infty,0}$ where $c_i \in \mathbb{C}$ and $(\alpha_i,\beta_i) \in A_{(m_i,n_i),0}$. Then

$$\begin{aligned} \| \cup x\|^2 =& \langle \cup x, \cup x\rangle = \phi(x^* \cup^* \cup x) \\ =& \phi(\sum c_i c_j^*(\beta_i,\alpha_i)(*, e\tilde{e})(f\tilde{f}, *)(\alpha_j,\beta_j)) \\ =& \sum_e \phi(x^*x) = n\|x\|^2, \end{aligned}$$

where n is the number of edges adjacent to $*$, so multiplication by $\cup$ is bounded. Next, let $a = (\xi, \zeta) \in A_{0,0}$. Then

$$\begin{aligned}\|ax\|^2 =& \langle ax, ax\rangle = \phi(x^* a^* ax) \\ =& \phi(\sum_{i,j} c_i c_j^* (\beta_i, \alpha_i)(\zeta, \xi)(\xi, \zeta)(\alpha_j, \beta_j)) \\ =& \phi(\sum_{i,j} c_i c_j^* (\beta_i, \alpha_i)(\zeta, \zeta)(\alpha_j, \beta_j))\end{aligned}$$

For each i, j let $(\beta_i, \alpha_i)(\alpha_j, \beta_j) = (\gamma_{ij}, \eta_{ij})$. Suppose $|\alpha_i| \geq |\zeta| = n$. Then $(\beta_i, \alpha_i)(\zeta, \zeta)(\alpha_j, \beta_j) = (\gamma_{ij}, \eta_{ij})\delta_{(\alpha_i)_{[1,n]},\zeta}$ and so

$$\begin{aligned}\phi((\beta_i, \alpha_i)(\zeta, \zeta)(\alpha_j, \beta_j)) =& \delta_{(\alpha_i)_{[1,n]},\zeta}\phi((\gamma_{ij}, \eta_{ij})) \\ =& \delta_{(\alpha_i)_{[1,n]},\zeta}\delta^{-|\gamma_{ij}|}\mu(r(\gamma_{ij}))\end{aligned}$$

where we use the notation $(\alpha)_{[1,n]}$ to mean the path made up of the first n edges of α. Similarly if $|\alpha_j| \geq |\zeta|$ we get

$$\phi((\beta_i, \alpha_i)(\zeta, \zeta)(\alpha_j, \beta_j)) = \delta_{(\alpha_j)_{[1,n]},\zeta}\phi((\gamma_{ij}, \eta_{ij})).$$

If both $|\alpha_i|$ and $|\alpha_j|$ are less than $|\zeta|$ then $(\beta_i, \alpha_i)(\zeta, \zeta)(\alpha_j, \beta_j) = (\gamma_{ij}\zeta_i, \eta_{ij}\zeta_j)$, where ζ_k are paths made up of the last $|\zeta| - |\alpha_k|$ edges of ζ, and so

$$\phi((\beta_i, \alpha_i)(\zeta, \zeta)(\alpha_j, \beta_j)) = \delta^{-(|\gamma_{ij}|+|\zeta_1|)}\mu(r(\zeta)) \leq \delta^{-|\gamma_{ij}|}\mu(r(\gamma_{ij})),$$

where the inequality follows from the eigenvalue condition.

Hence in all cases we have $\phi((\beta_i, \alpha_i)(\zeta, \zeta)(\alpha_j, \beta_j)) \leq \phi((\beta_i, \alpha_i)(\alpha_j, \beta_j))$ and so $\|ax\| \leq \|x\|$ Thus the action of $A_{\infty,0}$ on $A_{\infty,0}$ by left multiplication is bounded and so it may be extended to an operator in $B(H_{\infty,0})$. □

Denote by $\mathcal{A}_{\infty,0}$ the C^*-algebra generated by $A_{\infty,0}$ in the GNS representation with with respect to ϕ and let $\mathcal{M}_{\infty,0}$ denote the weak completion of $A_{\infty,0}$ in the GNS representation with respect to this state. Let $(\mathcal{M}_{0,\infty})_\phi$ be the centraliser of ϕ.

Proposition 3.3.2. *The state ϕ is the unique KMS state on $\mathcal{M}_{\infty,0}$ for the modular automorphism group σ_t at the inverse temperature $\beta = 1$.*

Proof. That ϕ is a KMS state at inverse temperature β is easily checked, we use the same argument to prove uniqueness as in [19]. Suppose ψ is any KMS state for the modular automorphism group σ at inverse temperature β. Then by the KMS condition $\psi(x) = 0$ unless x is in the fixed point algebra of σ. Then for any $(\xi_1, \xi_2) \in A_{(n,m),0}$ with $\sigma_t(\xi_1, \xi_2) = (\xi_1, \xi_2)$ we have

$$\psi((\xi_1, \xi_2)) = \psi((\xi_1, \xi_1)(\xi_1, \xi_2)) = \psi((\xi_1, \xi_2)\sigma_t(\xi_1, \xi_1)) = \psi((\xi_1, \xi_2)(\xi_1, \xi_1)).$$

The product $(\xi_1, \xi_2)(\xi_1, \xi_1)$ is zero unless either $\xi_1 = \xi_2 \cdot \zeta_1$ or $\xi_2 = \xi_1 \cdot \zeta_2$, but in both cases, since (ξ_1, ξ_2) is in the fixed point of σ we must have $\zeta_i = 0$. Hence $\psi(\xi_1, \xi_2)$ is zero unless $\xi_1 = \xi_2$. Hence ψ is zero outside $A_{0,0}$ and is the trace on $A_{0,0}$ and so $\psi = \phi$. □

Let $\mathcal{A}_{0,0}$ be the C^*-algebra generated by $A_{0,0}$ in the norm $\|x\|^2 = \phi(x^*x)$. Let

$$U = \delta^{-1} \sum_{e \in E; s(e) = *} \sqrt{\mu(r(e))}(e \cdot \tilde{e}, *)$$

and let Ψ be the endomorphism of $\mathcal{A}_{0,0}$ defined by $\Psi(x) = UxU^*$.

Proposition 3.3.3. *The C^*-algebra $\mathcal{A}_{\infty,0}$ is simple and purely infinite.*

Proof. This can be proved in exactly the same way as the analogous result for $\mathcal{O}_{TL}$ in Proposition 3.1.4. Using the fact that any pair of paths $(\xi, \eta) \in A_{(n,n+k),0}$ may be written as $(\xi', \eta') \cdot \cup$ for some $(\xi', \eta') \in A_{(n,n+k-2),0}$ we can prove that any $x \in A_{\infty,0}$ may be written as a finite sum

$$x = \sum U^n x_n + x_0 + \sum (U^*)^n x_{-n}$$

for some x_i in the type II string algebra. Since $\mathcal{A}_{0,0}$ is simple and AF and thus has the comparibility property of projections the proof of 3.1.4 carries over. □

Proposition 3.3.4. *Then $\mathcal{A}_{\infty,0} = \mathcal{A}_{0,0} \rtimes_\Psi \mathbb{N}$.*

Proof. This is proved in exactly the same way as Proposition 3.1.5. The algebra $\mathcal{A}_{0,0}$ is simple and AF, hence it satisfies the conditions of Theorem 3.1 of [82] and so the crossed product $\mathcal{A}_{0,0} \rtimes_\Psi \mathbb{N}$ is simple. It is easy to check that U is an isometry and $A_{\infty,0}$ is generated by $A_{0,0}$ and U. By Proposition 3.3.3 $\mathcal{A}_{\infty,0}$ is simple and hence it is isomorphic to the crossed product $\mathcal{A}_{0,0} \rtimes_\Psi \mathbb{N}$. □

Proposition 3.3.5. *The von Neumann algebra $\mathcal{M}_{\infty,0}$ is the type III_λ factor for $\lambda = \delta'$.*

Proof. It was proved in Theorem 3 of [9] that the von Neumann algebra completion of a nuclear algebra is hyperfinite and it was proved in Theorem 3.1 of [67] that the crossed product of a nuclear algebra by an endomorphism is nuclear. The C^*-algebra $\mathcal{A}_{\infty,0}$ is nuclear since by Proposition 3.3.4 it is the crossed product of a nuclear algebra by an endomorphism. Hence the von Neumann algebra $\mathcal{M}_{\infty,0}$ is hyperfinite. It is a factor, since by Proposition 3.3.2 ϕ is the unique KMS state, and by Theorem 5.3.30 of [7] it is a factor state. Using the same method as the discussion preceding Proposition 3.1.4 we may prove that the centraliser $(\mathcal{M}_{\infty,0})_\phi$ is isomorphic to the von Neumann algebra generated by the type II string algebra and so it is a factor. Hence the Connes spectrum and Arveson spectrum are equal. Then the modular automorphism group has period $\frac{2\pi}{\ln\delta}$. Hence $\mathcal{M}_{\infty,0}$ is type III$_\lambda$ with $\lambda = e^{-\ln\delta} = 1/\delta$. □

To define the vector spaces $A_{(m,n),p}$ we start at the vertex $*$ and first travel horizontally along $\mathcal{G}_0$ using a pair of paths in $A_{(m,n),0}$ ending at a vertex v in V_0 or V_1 and then travel vertically along $\mathcal{G}_3$ or $\mathcal{G}_1$ using a pair of paths (ζ_1, ζ_2) with $|\zeta_i| = p$ for $i = 1, 2$. The vector space $A_{(m,n),p}$ is the linear span over $\mathbb{C}$ of all possible pairs of paths of this form. In order to make these vector spaces well defined require the existence of a connection. We could equally have chosen to use a basis where we travel along $\mathcal{G}_3$ first and then out along $\mathcal{G}_0$ or $\mathcal{G}_2$ or any other combination of paths in $\mathcal{G}$ with the required lengths and endpoints. The connection allows us to change basis between these different paths. As in the II$_1$ case the connection must satisfy the unitarity, harmonicity and renormalisation conditions defined in Section 2.4. As before we let $A_{k,p} = \varinjlim A_{(m,m+2k),p}$ and $A_{\infty,p} = \bigoplus_{k\in\mathbb{Z}} A_{k,p}$. The $A_{\infty,p}$ can be given the structure of a unital associative *-algebra, with multiplication and *-operation similar to $A_{\infty,0}$. We can define states ϕ_p on $A_{\infty,p}$ in a similar way to the states defined on $A_{\infty,0}$ above. Let S_p be the projection onto the subalgebra $A_{0,p}$ defined by $S_p(\xi,\zeta) = \delta_{|\xi|,|\zeta|}(\xi,\zeta)$. Suppose ξ,ξ' are paths of length p in the vertical graph $\mathcal{G}_1$ both starting at $*$

and ending at $r(\xi) = r(\xi')$. Suppose also ζ, ζ' are paths in the horizontal graphs $\mathcal{G}_0$ or $\mathcal{G}_2$ (depending on the parity of $r(\xi)$) of length m, n respectively with $s(\zeta) = s(\zeta') = r(\xi)$ and $r(\zeta) = r(\zeta')$. Then we define the state ψ_p on $A_{(m,n),p}$ by

$$\psi_n(\xi \cdot \zeta, \xi' \cdot \zeta') := \delta_{\xi,\xi'}\delta_{\zeta,\zeta'}\delta^{-n}(\delta')^{-p}\delta_{n,m}\mu(r(\xi)).$$

The state ϕ_p is then the composition $\psi_p \circ S_p$. Using this state we can form an inner product in the usual way by $\langle x, y\rangle = \phi_p(y^*x)$ and taking the weak completion with respect to this inner product we have an increasing sequence of von Neumann algebras

$$\mathcal{M}_{\infty,0} \subset \mathcal{M}_{\infty,1} \subset \mathcal{M}_{\infty,2} \subset \mathcal{M}_{\infty,3} \subset \dots$$

A similar proof to that of Proposition 3.3.5 shows that each $\mathcal{M}_{\infty,n}$ is a hyperfinite type III$_\lambda$ factor. Next we will show that the basic extension of $\mathcal{M}_{\infty,n} \subset \mathcal{M}_{\infty,n+1}$ is $\mathcal{M}_{\infty,n+2}$. We also show that the relative commutant $\mathcal{M}'_{\infty,0} \cap \mathcal{M}_{\infty,k}$ is $A_{(k,k),0}$ for all k. In order to do this we first define certain projections and conditional expectations and check the conditions of Theorem 2.8.2 are satisfied.

Define the vertical Jones projection $e_n \in A_{(0,0),n+1}$ by

$$e_n = \sum_{\xi,\zeta,\eta} \frac{1}{\delta}\frac{\sqrt{\mu(r(\eta))\mu(r(\zeta))}}{\mu(r(\xi))}(\xi \cdot \eta \cdot \tilde{\eta}, \xi \cdot \zeta \cdot \tilde{\zeta})$$

where ξ, ζ, η are paths in $\mathcal{G}_3$ with $|\xi| = n-1$, $|\zeta| = |\eta| = 1$, $s(\xi) = *$ and the reverse edge of ξ is indicated by $\tilde{\xi}$. As in the II$_1$ case we can show the e_n satisfy $e_n = e_n^2 = e_n^*$ and $e_n e_{n\pm 1} e_n = \delta^{-2} e_n$. If we use the connection to transform e_n into an element of $A_{(t,t),s}$, $t+s = n+1$, it still has the same form. We will show below that e_n is the Jones projection for the basic construction of $\mathcal{M}_{\infty,n-1} \subset \mathcal{M}_{\infty,n}$.

For m sufficiently large, the vector space $A_{(m,m+k),n}$ is generated by $A_{(m-1,m-1+k),n}$ and the Jones projection e_{m-1}. This is because the graph is finite and so eventually the Bratteli diagram at each step is a reflection of the previous step.

Define the map $E_k : A_{(k,k),0} \to A_{(k-1,k-1),0}$ by

$$E_k(\xi_1 \cdot \sigma_1, \xi_2 \cdot \sigma_2) = \delta_{\sigma_1,\sigma_2}\delta^{-1}\frac{\mu(r(\sigma_1))}{\mu(s(\sigma_1))}(\xi_1, \xi_2)$$

where $|\xi_1| = |\xi_2| = k-1$ and $|\sigma_1| = |\sigma_2| = 1$. Then it is easy to check that E_k is a conditional expectation and it satisfies $\phi \circ E_k = \phi$.

Consider the following inclusion of vector spaces

$$\begin{array}{ccc} A_{(m,n),p} & \subset & A_{(m+1,n+1),p} \\ \cap & & \cap \\ A_{(m,n),p+1} & \subset & A_{(m+1,n+1),p+1} \end{array} \tag{3.2}$$

For the horizontal inclusions we may define maps $E^{(m,n),p+1}_{(m+1,n+1),p+1} : A_{(m+1,n+1),p+1} \to A_{(m,n),p+1}$ as follows. Let $(\xi_1, \xi_2) \in A_{(m,n),p+1}$ and let ζ_i be an edge in the horizontal graph starting at $r(\xi_i)$. Then, similarly to the definition of E_k above, let

$$E^{(m,n),p+1}_{(m+1,n+1),p+1}(\xi_1 \cdot \zeta_1, \xi_2 \cdot \zeta_2) = \delta_{\zeta_1,\zeta_2}\delta^{-1}\frac{\mu(r(\zeta))}{\mu(s(\zeta))}(\xi_1, \xi_2).$$

For the vertical inclusions we may similarly define maps $E^{(m+1,n+1),p}_{(m+1,n+1),p+1} : A_{(m+1,n+1),p+1} \to A_{(m+1,n+1),p}$ as follows. Let $(\xi_1, \xi_2) \in A_{(m+1,n+1),p}$ and let ζ_i be an edge in the vertical graph starting at $r(\xi_i)$. Then let

$$E^{(m+1,n+1),p}_{(m+1,n+1),p+1}(\xi_1 \cdot \zeta_1, \xi_2 \cdot \zeta_2) = \delta_{\zeta_1,\zeta_2}\delta^{-1}\frac{\mu(r(\zeta))}{\mu(s(\zeta))}(\xi_1, \xi_2).$$

The renormalisation condition of the connection gives us the equality

$$\sum_{\sigma_2,\sigma_4} \frac{\mu(r(\sigma_2))\sqrt{\mu(r(\sigma_3))\mu(r(\sigma_3'))}}{\mu(s(\sigma_2))\mu(s(\sigma_3))} \begin{array}{ccc} & \xrightarrow{\sigma_1} & \\ \sigma_3\big\downarrow & & \big\downarrow\sigma_2 \\ & \xrightarrow[\sigma_4]{} & \end{array} \overline{\begin{array}{ccc} & \xrightarrow{\sigma_1'} & \\ \sigma_3'\big\downarrow & & \big\downarrow\sigma_2 \\ & \xrightarrow[\sigma_4]{} & \end{array}}$$

This implies that the maps E defined above satisfy $E^{(m,n),p}_{(m+1,n+1),p}E^{(m+1,n+1),p}_{(m+1,n+1),p+1} = E^{(m,n),p}_{(m,n),p+1}E^{(m,n),p+1}_{(m+1,n+1),p+1}$.

Denote by $\mathcal{E}_{p-1}$ the conditional expectation $\mathcal{E}_{p-1} : \mathcal{M}_{\infty,p} \to \mathcal{M}_{\infty,p-1}$. Then for $x \in A_{(m,n),p}$, we have $\mathcal{E}_{p-1}(x)$ is given by the conditional expectation of x onto $A_{(m,n),p-1}$.

We check the conditions of Theorem 2.8.2 for $N = \mathcal{M}_{\infty,0}$, $M = \mathcal{M}_{\infty,1}$, $L = \mathcal{M}_{\infty,2}$ and the conditional expectations and e_1 as defined above. For the first condition:

$$\begin{aligned}\mathcal{E}_1(e_1) =&\mathcal{E}_1(\sum_{\xi,\zeta,\eta}\frac{1}{\delta}\frac{\sqrt{\mu(r(\eta))\mu(r(\zeta))}}{\mu(r(\xi))}(\xi\cdot\eta\cdot\tilde{\eta},\xi\cdot\zeta\cdot\tilde{\zeta}))\\ =&\sum_{\xi,\eta}\frac{1}{\delta^2}(\xi\cdot\eta,\xi\cdot\eta)=\frac{1}{\delta^2}Id.\end{aligned}\tag{3.3}$$

For condition 2, let $x = (\alpha\cdot\beta_1\cdot\beta_2,\alpha'\cdot\beta_1'\cdot\beta_2') \in A_{(m,n),2}$ where $|\alpha| = m$, $|\alpha'| = n$ and $|\beta_i| = |\beta_i'| = 1$. Then

$$\begin{aligned}xe_1 =& \sum_{|\xi|=n,|\zeta|=|\eta|=1}\frac{1}{\delta}\frac{\sqrt{\mu(r(\eta))\mu(r(\zeta))}}{\mu(r(\xi))}(\alpha\cdot\beta_1\cdot\beta_2,\alpha'\cdot\beta_1'\cdot\beta_2')(\xi\cdot\eta\cdot\tilde{\eta},\xi\cdot\zeta\cdot\tilde{\zeta})\\ =&\sum_{\zeta}\frac{1}{\delta}\frac{\sqrt{\mu(r(\beta_2'))\mu(r(\zeta))}}{\mu(r(\alpha'))}\delta_{\beta_1',\tilde{\beta}_2'}(\alpha\cdot\beta_1\cdot\beta_2,\alpha'\cdot\zeta\cdot\tilde{\zeta}).\end{aligned}\tag{3.4}$$

Then

$$\begin{aligned}\mathcal{E}_1(xe_1) =&\mathcal{E}_1(\sum_{\zeta}\frac{1}{\delta}\frac{\sqrt{\mu(r(\beta_2'))\mu(r(\zeta))}}{\mu(r(\alpha'))}\delta_{\beta_1',\tilde{\beta}_2'}(\alpha\cdot\beta_1\cdot\beta_2,\alpha'\cdot\zeta\cdot\tilde{\zeta}))\\ =&\frac{\mu(r(\beta_2))}{\mu(s(\beta_2))}\frac{1}{\delta^2}\frac{\sqrt{\mu(r(\beta_2'))\mu(r(\beta_2))}}{\mu(r(\alpha'))}\delta_{\beta_1',\tilde{\beta}_2'}(\alpha\cdot\beta_1,\alpha'\cdot\tilde{\beta}_2)\\ =&\frac{1}{\delta^2}\frac{\sqrt{\mu(r(\beta_2'))\mu(r(\beta_2))}}{\mu(s(\beta_2))}\delta_{\beta_1',\tilde{\beta}_2'}(\alpha\cdot\beta_1,\alpha'\cdot\tilde{\beta}_2)\end{aligned}\tag{3.5}$$

and so we have

$$\begin{aligned}\delta^2\mathcal{E}_1(xe_1)e_1 =&\delta^2\sum_{\substack{|\xi|=n-1\\|\zeta|=|\eta|=1}}\frac{1}{\delta^3}\frac{\sqrt{\mu(r(\eta))\mu(r(\zeta))}}{\mu(r(\xi))}\frac{\sqrt{\mu(r(\beta_2'))\mu(r(\beta_2))}}{\mu(s(\beta_2))}\delta_{\beta_1',\tilde{\beta}_2'}\\ &(\alpha\cdot\beta_1,\alpha'\cdot\tilde{\beta}_2)(\xi\cdot\eta\cdot\tilde{\eta},\xi\cdot\zeta\cdot\tilde{\zeta})\\ =&\sum_{\zeta}\frac{1}{\delta}\frac{\sqrt{\mu(r(\beta_2'))\mu(r(\zeta))}}{\mu(r(\alpha'))}\delta_{\beta_1',\tilde{\beta}_2'}(\alpha\cdot\beta_1\cdot\beta_2,\alpha'\cdot\zeta\cdot\tilde{\zeta})\\ =&xe_1\end{aligned}\tag{3.6}$$

as required.

For condition 3, let $x \in A_{(m,n),1}$. Then x is of the form $x = (\alpha_1\cdot\beta_1,\alpha_2\cdot\beta_2)$ with $|\alpha_1| = m$, $|\alpha_2| = n$ and $|\beta_i| = 1$. We embed x in $A_{(m,n),2}$ as $\sum_{|\sigma|=1}(\alpha_1\cdot\beta_1\cdot$

$\sigma, \alpha_2 \cdot \beta_2 \cdot \sigma)$

$$e_1 x e_1 = \sum_{\substack{|\xi|=|\zeta|=n-1,\\ |\nu_i|=|\gamma_i|=|\sigma|=1}} \frac{1}{\delta^2} \frac{\sqrt{\mu(r(\nu_1))\mu(r(\nu_2))}}{\mu(r(\xi))} \frac{\sqrt{\mu(r(\gamma_1))\mu(r(\gamma_2))}}{\mu(r(\zeta))} (\xi \cdot \nu_1 \cdot \tilde{\nu}_1, \xi, \cdot \nu_2 \cdot \tilde{\nu}_2)$$

$$(\alpha_1 \cdot \beta_1 \cdot \sigma, \alpha_2 \cdot \beta_2 \cdot \sigma)(\zeta \cdot \gamma_1 \cdot \tilde{\gamma}_1, \zeta \cdot \gamma_2 \cdot \tilde{\gamma}_2)$$

$$= \sum \delta_{\beta_1,\beta_2} \delta_{\nu_2,\gamma_1} \delta_{\gamma_1,\beta_1} \frac{1}{\delta^2} \frac{\sqrt{\mu(r(\nu_1))\mu(r(\nu_2))}}{\mu(r(\xi))}$$

$$\frac{\sqrt{\mu(r(\gamma_1))\mu(r(\gamma_2))}}{\mu(r(\zeta))} (\alpha_1 \cdot \nu_1 \cdot \tilde{\nu}_1, \alpha_2 \cdot \gamma_2 \cdot \tilde{\gamma}_2)$$

$$= \sum_{\nu_1,\gamma_2} \delta_{\beta_1,\beta_2} \frac{1}{\delta^2} \frac{\mu(r(\beta_1))}{\mu(s(\beta_1))} \frac{\sqrt{\mu(r(\nu_1))\mu(r(\gamma_2))}}{\mu(r(\alpha_1))} (\alpha_1 \cdot \nu_1 \cdot \tilde{\nu}_1, \alpha_2 \cdot \gamma_2 \cdot \tilde{\gamma}_2). \tag{3.7}$$

On the other hand we have

$$\mathcal{E}_1(x) = \frac{1}{\delta} \frac{\mu(r(\beta_1))}{\mu(s(\beta_1))} \delta_{\beta_1,\beta_2} (\alpha_1, \alpha_2)$$

and so

$$\mathcal{E}_1(x) e_1 = \sum_{\xi,\zeta,\eta,\nu} \delta_{\beta_1,\beta_2} \frac{1}{\delta^2} \frac{\mu(r(\beta_1))}{\mu(s(\beta_1))} \frac{1}{\delta} \frac{\sqrt{\mu(r(\eta))\mu(r(\zeta))}}{\mu(r(\xi))} (\alpha_2 \cdot \nu_1 \cdot \nu_2, \alpha_2 \cdot \nu_1 \cdot \nu_2)(\xi \cdot \eta \cdot \tilde{\eta}, \xi \cdot \zeta \cdot \tilde{\zeta})$$

$$= \sum_{\zeta,\eta} \delta_{\beta_1,\beta_2} \frac{\mu(r(\beta_1))}{\mu(s(\beta_1))} \frac{1}{\delta^2} \frac{\sqrt{\mu(r(\eta))\mu(r(\zeta))}}{\mu(r(\alpha_1))} (\alpha_2 \cdot \eta \cdot \tilde{\eta}, \alpha_2 \cdot \zeta \cdot \tilde{\zeta}) \tag{3.8}$$

and so condition 3 holds.

Thus we may apply Theorem 2.8.2 to conclude that $\mathcal{M}_{\infty,0} \subset \mathcal{M}_{\infty,1} \subset \mathcal{M}_{\infty,2}$ is the basic construction with Jones projection e_1. We may repeat this calculation to show that $\mathcal{M}_{\infty,k} \subset \mathcal{M}_{\infty,k+1} \subset \mathcal{M}_{\infty,k+2}$ is the basic construction with Jones projection e_{k+1} for all $k > 0$.

Proposition 3.3.6. *For the algebras defined above, if the connection is flat we have* $\mathcal{M}'_{\infty,0} \cap \mathcal{M}_{\infty,k} = \mathcal{M}_{(k,k),0}$.

Proof. To prove this we use Ocneanu's compactness argument [71]. Let $x \in \mathcal{M}'_{\infty,0} \cap \mathcal{M}_{\infty,k}$ and denote by $x_{l,n}$ its conditional expectation in $A_{(l,l+2n),k}$. Let d be the smallest integer such that the Bratteli diagram for the inclusion $A_{(d,d+2n),k} \subset$

$A_{(d+1,d+1+2n),k}$ is a reflection of the Bratteli diagram for the previous step. Then for all $2t > d$ the algebras $A'_{(2t,2t+2n),0} \cap A_{(2t,2t+2n),k}$ are isomorphic since they are string algebras with basis consisting of pairs of paths of length k on the graph $\mathcal{G}_3$ which are allowed to start at any vertex in V_0. Denote by B a finite dimensional C^*-algebra isomorphic to these algebras and let $\phi_{2t,n} : A'_{(2t,2(t+n)),0} \cap A_{(2t,2(t+n)),k} \to B$ denote the isomorphism. Note also that the algebras $A'_{(2t,2(t+n)),0} \cap A_{(2t+2,2(t+n)+2),k}$ are all isomorphic, denote by $\tilde{B}$ an algebra isomorphic to these algebras with isomorphism $\tilde{\phi}_{2t,n} : A'_{(2t,2(t+n)),0} \cap A_{(2t+2,2(t+n)+2),k} \to \tilde{B}$. Then $\{\phi_{2t,n}(x_{2t,n})\}_t$ is a bounded sequence in B since $\|\phi_{2t,n}(x_{2t,n})\| \le \|x_{2t,n}\|$ and by compactness we may find a sequence $\{t_k\}_k$ such that the subsequences $\{\phi_{2t_k,n}(x_{2t_k,n})\}_k$ and $\{\phi_{2t_k+2,n}(x_{2t_k+2,n})\}_k$ both converge. Let z and z' be their respective limits. We may embed any string in B into $\tilde{B}$ by adding the horizontal identity string of length 2 at the beginning or at the end. Thus, using the connection we can identify $z \cdot id_2$ with $id_2 \cdot z'$, where id_2 is the string $\sum_{|\sigma|=2}(\sigma, \sigma)$, since we know that $\lim_{t\to\infty} \|x_{2t,n} - x_{2t+2,n}\| = 0$. Now we wish to show that we have in fact $z \cdot id_2 = id_2 \cdot z'$ as strings. It is easily shown that $z'.e = e.z'$ and $e.id = id.e$ where e is the horizontal Jones' projection. Hence we have

$$\begin{aligned}(z.id) \times (id.e) =& (id \cdot z') \times (e \cdot id) = e \cdot z' \\ =& z' \cdot e = (z'.id) \times (id.e).\end{aligned}$$

Taking conditional expectations we have that $z = z'$ in B. Let $z(v)$ denote the component of z with initial vertex v. Then we have that $\lim_{t\to\infty} \|x_{2t,n} - z(*)\| = 0$ if the connection is flat and hence $x = z(*) \in A_{(k,k),0}$ □

The algebras $A_{0,k}$ are the type II string algebras, and the state ϕ_k^ω is the ordinary trace here. Denote by $H_{0,k}$ the Hilbert space completion with respect to the trace. Let $\mathcal{A}_{0,k}$ be the C^*-algebra generated by $A_{0,k}$, and $\mathcal{M}_{0,k}$ the von Neumann algebra completion in $B(H_{0,k})$. Similarly to Proposition 3.3.4, it can be proved that $\mathcal{A}_{\infty,k}$ is the crossed product of $\mathcal{A}_{0,k}$ by an endomorphism.

Proposition 3.3.7. *Let $k \ge 0$ then $\mathcal{A}_{\infty,k} \simeq \mathcal{A}_{0,k} \rtimes_\Psi \mathbb{N}$ where $\Psi(x) = U_k x U_k^*$ for $x \in A_{0,0}$ and U_k is the image of the isometry $U = \delta^{-\frac{1}{2}} \sum_e (e\tilde{e}, *)$ in $A_{0,k}$.*

Finally we want to show that the subfactor $\mathcal{M}_{\infty,0} \subset \mathcal{M}_{\infty,1}$ is isomorphic to $(N \subset M) \otimes \mathcal{M}_{\infty,1}$ for some II_1 subfactor $N \subset M$. In order to do this we again use Theorem 2.8.1. In the case when we have a tower of type III_1 factors, by [77] the conditions of the theorem are automatically satisfied. In the III_λ case we need to check central freeness and approximate innerness. It is known that a common discrete decomposition of a subfactor $N \subset M$ of type III_λ exists if $N' \cap M$ is a factor so this is automatically true for $\mathcal{M}_{\infty,0} \subset \mathcal{M}_{\infty,1}$ if the graph $\mathcal{G}$ has only one edge adjacent to the vertex $*$. Central freeness is equivalent to requiring the type II and III graphs to coincide, so we know this is true for the ADE graphs (apart from A_{4n-3}).

3.4 Definition of Type III planar algebra

An (n_0, m_0) *tangle* T is a disc D_0 in the plane with a possibly empty collection of internal discs D_i $1 \leq i \leq k(T)$. Each disc D_i $0 \leq i \leq k(T)$ has n_i marked points along the top and m_i along the bottom (usually discs will be drawn as rectangles so 'top' and 'bottom' are clear) such that for each i the sum $m_i + n_i \in 2\mathbb{N}$. The interior of $D_0 / \bigcup_{1 \leq i \leq k(T)} D_i$ contains a collection of non intersecting strings, each string either has exactly two endpoints, which occur at marked points of the D_i, or forms a closed loop. Each marked point is the endpoint of exactly one string. The interior of the tangle is shaded black and white in such a way that adjacent regions always have different shadings. A disc with pattern (n, m) will be called a $+(n, m)$ disc if the region adjacent to the first marked point (which we usually indicate with a $*$) is unshaded and a $-(n, m)$ disc if the region is shaded. The inner discs may be labelled using some labelling set $L = \cup_{n,m} L_{(n,m)}$. Tangles are defined up to planar isotopy. A tangle is said to be in *standard form* if all the discs are converted to rectangles with all marked points along the top edge and the tangle is drawn in such a way that it may be split horizontally into strips with each strip containing at most one cup, one cap or one inner disc. When drawing tangles we will often denote n parallel strands by a thick strand with the number

n written beside it.

A tangle T is fully labelled if there is an assignment of $l \in L_{(n,m)}$ to each internal (n, m)-disc D_k. Denote the collection of all planar tangles with labels in L by $\mathcal{P}(L)$. We now define a composition in $\mathcal{P}(L)$.

If the pattern of the outer disc of some tangle $S \in \mathcal{P}(L)$ is the same as the pattern of some inner disc D_i of a tangle $T \in \mathcal{P}(L)$ then we may form the tangle $T \circ_i S$ by gluing S inside D_i, removing the outer boundary and smoothing the strings.

Definition 3.4.1. A *type III planar algebra* is a collection of vector spaces $P^m_{n,\pm}$, $n, m \in \mathbb{N}$ with inclusion maps $i : P^m_{n,\pm} \to P^{m+1}_{n+1,\pm}$ and a collection of linear maps Z_T (one for each tangle $T \in \mathcal{P}(L)$) such that, for T as above $Z_T : \otimes_{i=1}^{k(T)} P^{m_i}_{n_i,\pm} \to P^{m_0}_{n_0,\pm}$ such that the Z_T are compatible with composition and relabelling of internal discs, more precisely, we require that the following diagram commutes, for all tangles T with $k(T)$ inner discs and tangles S with $k(S)$ inner discs for which the composition $T \circ_l S$ makes sense:

$$\begin{array}{ccc} (\otimes_{i=1}^{l-1} P^{m_i}_{n_i,\pm}) \otimes (\otimes_{j=1}^{k(S)} P^{t_j}_{s_j,\pm}) \otimes (\otimes_{i=l+1}^{k(T)} P^{m_i}_{n_i,\pm}) & \xrightarrow{Z_{T\circ S}} & P^{m_0}_{n_0,\pm} \\ \downarrow{\scriptstyle 1\otimes Z_S\otimes 1} & \nearrow{\scriptstyle Z_T} & \\ \otimes_{i=1}^{k(T)} P^{m_i}_{n_i,\pm} & & \end{array}$$

where the empty tensor product is defined to be $\mathbb{C}$. Let $P_{k,\pm} = \varinjlim P^{n+2k}_{n,\pm}$ and let $P_\pm = \oplus_{k\in\mathbb{Z}} P_{k,\pm}$. The map Z_T is called the *presenting map* of the tangle T.

Using the Fourier transform tangle shown in Figure 2.5 we can show that $P^m_{n,+}$ is anti-isomorphic to $P^m_{n,-}$ if at least one of n, m are non-zero, and so in general we will work only with the + part and write P^m_n to mean $P^m_{n,+}$, and for $n = m = 0$ we write P_+ or P_- depending on the shading of the region adjacent to the outer boundary. Thus when drawing tangles we usually omit the shading since the region containing $*$ is always assumed to be unshaded.

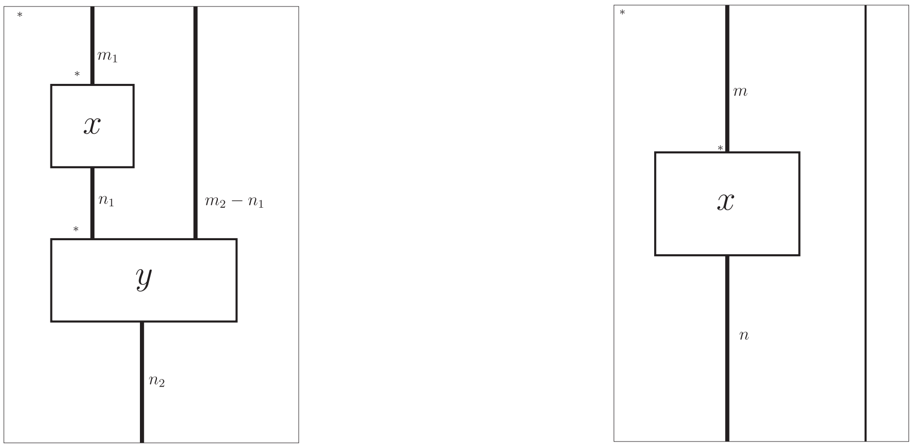


Figure 3.10: The tangles $M(x,y)$ and $I_{(n,m)}^{(n+1,m+1)}(x)$

We say a planar algebra P has modulus (δ_+, δ_-) if there exist constants $\delta_\pm$ with $Z_T = \delta_\epsilon$ where T is the $\epsilon(0,0)$ tangle with a single closed shaded (for $\epsilon = -$) or unshaded (for $\epsilon = +$) loop in its interior. If $\delta_+ = \delta_- = \delta$ we call P unimodular. Each planar algebra $P_\pm$ contains a copy of the Temperley-Lieb planar algebra $O_{TL}^{\pm}$ as the subalgebra with labelling set $L = \emptyset$ where the presenting map is just the identity map.

There are inclusion tangles, shown in Figure 3.10 which embed a $\pm(n,m)$ tangle in a $\pm(n+k, m+k)$ tangle by adding k through strings to the right.

An *annular* $(m,n;m',n')$ tangle is a tangle T for which the outer disc has pattern (m,n) and T exactly one inner disc, with pattern (m',n').

In order to define the multiplication in the planar algebra we have a class of multiplication tangles, shown in Figure 3.10, which map $P_{n_i}^{m_1} \otimes P_{n_2}^{m_2} \to P_{n_1}^{m_2+m_1-n_2}$ (or $P_{n_1+n_2-m_1}^{m_2}$ depending on whether $n_2 - m_1$ is positive or negative). The product of $x \in P_{n_1}^{m_1}$ and $y \in P_{n_2}^{m_2}$ is then defined as $Z_M(x,y)$ where M is the appropriate multiplication tangle. The element $Z_1(1) \in P$ is the identity for this multiplication, where 1 is the tangle where all the strands are through strands.

The adjoint T^* of a tangle T is defined as the tangle obtained by reflecting T in a horizontal line through its centre. If T is labelled (by some labelling set with a *-operation), the labels must be replaced by their adjoints. A *type III planar *-algebra* is a planar algebra with an involution on P satisfying $Z_{T^*}(x^*) = (Z_T(x))^*$

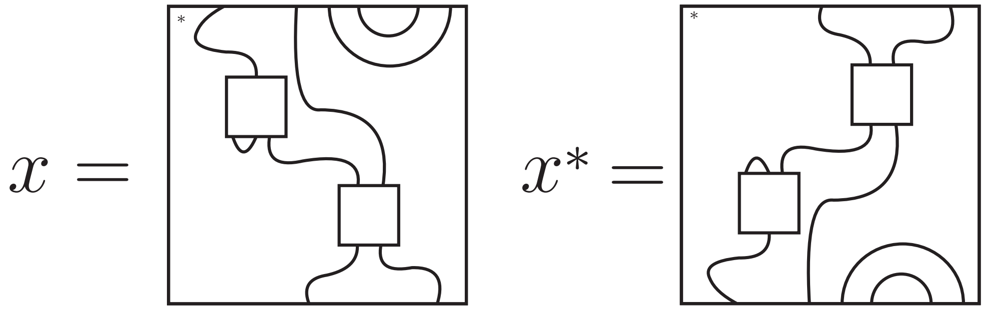


Figure 3.11: The adjoint of a tangle

for all $x \in P$ and all tangles T.

It is easy to see that the collection of all planar tangles is generated by $\mathcal{O}_{TL}$, multiplication tangles and annular tangles. To show this, just draw any tangle in standard form. Each horizontal strip is either an annular tangle or an element of $\mathcal{O}_{TL}$ and the multiplication of the horizontal strips is given by multiplication tangles.

We now define a partial braiding on P. We use the same crossing as in Section 2.5, replacing local parts of planar tangles by the linear combination shown in Figure 3.3. Recall from Section 2.5 that this satisfies the second and third Reidemeister moves, but removing a twist results in multiplication by a scalar as shown in Figure 2.9. However in general we cannot pass strings over or under discs. Given a planar tangle T, let T' be the linear combination of tangles obtained by passing j strings of T over some inner disc D_i of T as shown in Figure 3.12. A planar algebra P is called *flat* if $Z_T = Z_{T'}$ for all T and all possible choices of i and j, where as before if $T' = \sum_{i \in I} c_i T_i$, we denote by $Z_{T'}$ the linear combination $\sum_{i \in I} c_i Z_{T_i}$. Note that in this case it is still not possible in general to pull strings under discs.

A type III planar algebra P is *spherical* if Z_T is invariant under isotopies of the 2-sphere for all $0, 0$ tangles T.

Let σ_t be the action of $\mathbb{R}$ on P defined by $\sigma_t(x) = e^{\alpha i(n-m)t}x$ for $\alpha = \ln \delta$ and $x \in P_n^m$ and extend linearly to all of P. As in O_{TL}, we define a projection S from P onto P_0 by the linear extension of the map $S(x) = \alpha/2\pi \int_0^{2\pi/\alpha} \sigma_t(x)dt$ for

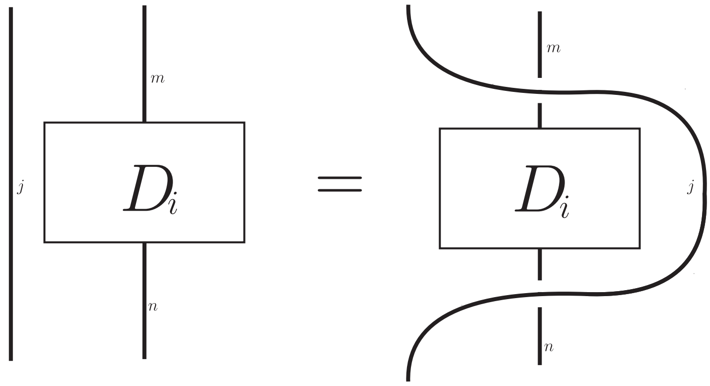


Figure 3.12: Flatness

$x \in P_n^m$. We may define states on the P_0 in a similar way to the trace on the II_1 planar algebra. In terms of planar tangles the states Tr_R and Tr_L are gotten by joining corresponding marked points along the top and bottom of the tangle, where the strings pass either to the right (for Tr_R) or to the left of the tangle (for Tr_L), and possibly multiplying by a scalar factor. Then the states ϕ_R and ϕ_L are the composition of Tr_R or Tr_L with the projection S.

Proposition 3.4.2. *Let P be a type III planar algebra. Then P is spherical if and only if $\phi_R = \phi_L$.*

Proof. This is proved exactly as in [36]. Suppose P is spherical. Then for any $x \in P_n^m$, $\phi_R(x) = 0 = \phi_L(x)$ if $n \neq m$. For any $x \in P_n^n$ $\phi_R(x) = \phi_L(x)$ since if we imagine $\phi_R(x)$ on the surface of a sphere, we can loop the n strings on the right of $\phi_R(x)$ around the sphere to get $\phi_L(x)$. If $\phi_R = \phi_L$ then P is spherical since all spherical isotopies are generated by planar isotopies plus the isotopy taking $\phi_R(x)$ to $\phi_L(x)$[1]. □

Proposition 3.4.3. *A flat planar algebra P is spherical.*

Proof. We just need to check that ϕ_L and ϕ_R agree. This is exactly the same as in the type II case in [36], since ϕ_L and ϕ_R are zero for any $x \in P_n^m$ with $n \neq m$. For $x \in P_n^n$, $\phi_R(x) = \phi_L(x)$ since we may pass the strings on the right of $\phi_R(x)$ over the rest of the tangle as shown in Figure 3.13 and remove the twists using

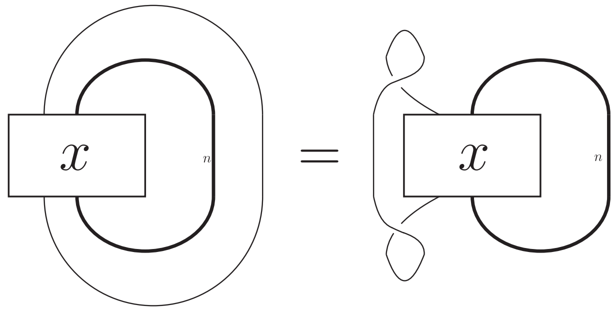


Figure 3.13: Passing a string over a tangle

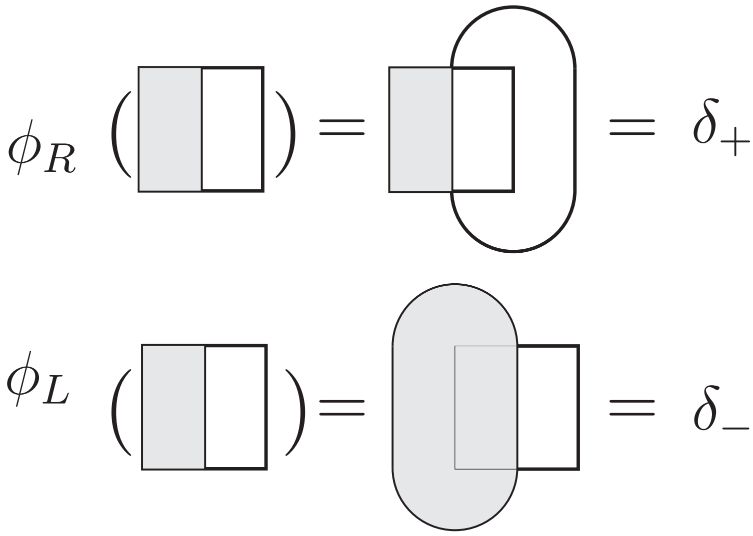


Figure 3.14: Spherical planar algebras are unimodular

the partial braiding. Since there is one positive and one negative crossing, the scalar factors involved in removing the twists cancel. □

For a spherical planar algebra we write ϕ instead of $\phi_R = \phi_L$. A spherical planar algebra must be unimodular since by Figure 3.14 $\phi_L(1) = \delta_+$ and $\phi_R(1) = \delta_-$ where $1 \in P_1^1$ is the one-string identity.

A planar algebra is *non-degenerate* if $\phi(x^*x) > 0$ for all $x \neq 0$ and in this case we may define a non degenerate inner product on P by $\langle x, y\rangle := \phi(y^*x)$.

Proposition 3.4.4. *A planar algebra P is non-degenerate if and only if $Z_A(x) = 0$ for all $(0,0;m,n)$-annular tangles A implies $x = 0$.*

Proof. We can prove this in exactly the same way as the proof of Lemma 1.29 in [36]. First suppose that P is non-degenerate and $x \neq 0$. Then $\phi(x^*x) \neq 0$ and $\phi(x^*x)$ is the result of applying an annular tangle to x. For the converse, given any $(0,0;m,n)$-annular tangle A we may find $y \in P$ such that $A(x) = \phi(xy)$. Thus there exists y such that $\phi(xy) \neq 0$ and by Cauchy-Schwarz we have that $\phi(x^*x) \neq 0$. □

A planar algebra is *finite dimensional* if $dimP_n^m < \infty$ for all n, m, *connected* if $P_\pm$ has dimension 1 and *irreducible* if $dimP_1^1 = 1$.

Proposition 3.4.5. *Let P be a non-degenerate finite dimensional type III planar *-algebra. Then there is a unique C^*-norm on P.*

Proof. It is proved in [36] that a type II planar algebra has a unique C^*-norm. Since the subalgebra $P_0 \subset P$ is a type II planar algebra it has a unique C^*-norm $\|\cdot\|_{C^*}$. We can use this to define a norm $\|\cdot\|_P$ on all of P by $\|x\|_P = \|x^*x\|_{C^*}^{\frac{1}{2}}$ for $x \in P_k$. To see this is a norm, $\|ax\|_P = a\|x\|_P$ and $\|x\|_P = 0$ implies $x = 0$ both follow from the corresponding properties of $\|\cdot\|_{C^*}$, for subadditivity

$$\begin{aligned}\|x+y\|_P^2 =& \|(x+y)^*(x+y)\|_{C^*} \\ \leq& \|x^*x\|_{C^*} + 2\|x^*y\|_{C^*} + \|y^*y\|_{C^*} \\ \leq& \|x\|_P^2 + 2\|x\|_P\|y\|_P + \|y\|_P^2 \\ =& (\|x\|_P + \|y\|_P)^2.\end{aligned}$$

It is easy to see that this is a C^*-norm and it agrees with $\|\cdot\|_{C^*}$ on P_0. To prove uniqueness, suppose $\|\cdot\|$ is any other C^*-norm on P. It must agree with $\|\cdot\|_{C^*}$ on P_0 and so $\|x\|^2 = \|x^*x\| = \|x^*x\|_{C^*} = \|x\|_P^2$. □

Therefore we will call a non-degenerate finite dimensional type III planar *-algebra a *type III C^*-planar algebra*.

Definition 3.4.6. A type III *subfactor planar algebra* is a spherical, finite dimensional, connected planar *-algebra with modulus $\delta > 0$ and a positive definite state ϕ.

We define an endomorphism ρ on (n, m)-tangles as in Figure 3.15.

The endomorphism ρ maps P_+ to P_-. We may similarly define $\bar{\rho}$ as the endomorphism which maps P_- to P_+ by adding a single through string to the left. We also define the *conditional expectation* tangles E_L and E_R as in Figure 3.16.

The map $\delta^{-1}Z_{E_L}$ is the conditional expectation of P onto the subalgebra $P^{(1)}$ which is the subalgebra where the first and last endpoints of each tangle are joined by a through string. Drawing diagrams, it is easy to see that E_L satisfies $E_L(axb) = aE_L(x)b$ for $a, b \in P^{(1)}$ and $x \in P$. To show positivity of E_L, let $x = \sum x_i \in P$ then $E_L(xx^*) = x'x'^*$ where $x' = \sum x'_i$ and x'_i is defined in the Figure 3.17.

The map $\delta^{-1}Z_{E_R}$ defines a positive conditional expectation from $P_n^n \to P_{n-1}^{n-1}$.

3.4.1 Type III Planar Algebra Associated to a Subfactor

Given a finite bipartite graph $\mathcal{G}$ with distinguished vertex $*$, one may associate a type III planar algebra as follows. Suppose the Perron Frobenius eigenvalue of $\mathcal{G}$ is δ with eigenvector entry $\mu(v)$ corresponding to the vertex v. The vector spaces P_n^m are spaces of pairs of paths (ξ, ζ) where $|\xi| = n$, $|\zeta| = m$, $s(\xi) = s(\zeta) = *$ and $r(\xi) = r(\zeta)$. The multiplication and *-operation defined in Section 3.3 makes P into a *-algebra. The labelling set is just P. The presenting map Z_T of a tangle T is as follows. First isotope the tangle to standard form and then presenting maps for individual rectangles are as in [23] which were described in Section 2.7. That is, suppose we have a strip with a cup joining the i^{th} vertex along the top with the $(i+1)^{th}$ vertex along the top and the rest of the strings through strings. We may label the diagram by a pair of paths from $\mathcal{G}$. We require that a through string must have the same label along its top and bottom edges. Thus we have that the n^{th} edge along the bottom must agree with the n^{th} edge along the top for the first $i-1$ edges, and the n^{th} edge along the bottom must agree with the $n+2$ edge for the remaining edges along the bottom. Also the edge i on the top

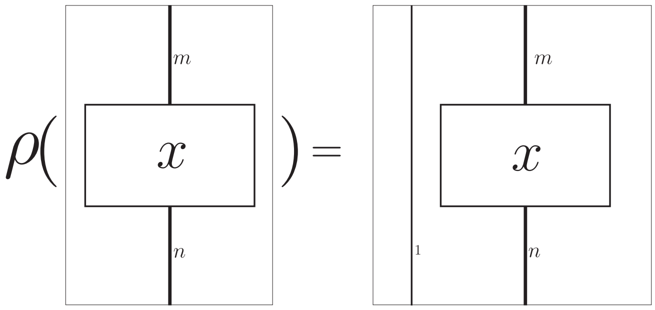


Figure 3.15: The endomorphism $\rho(x)$

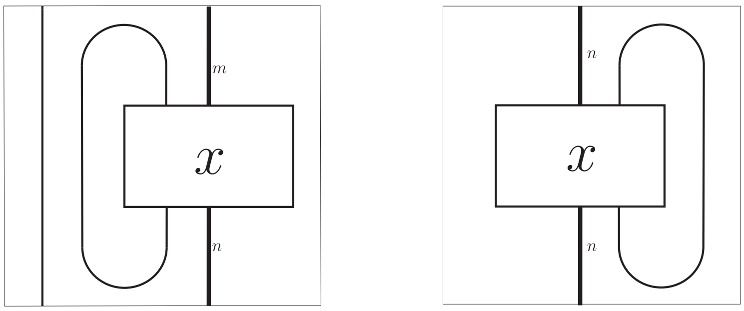


Figure 3.16: Conditional expectations $E_L(x)$ and $E_R(x)$

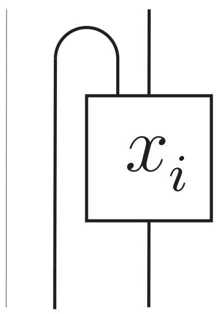


Figure 3.17: The tangle x_i'

must be the opposite edge as $i+1$. Thus the map $Z_{\cup}$ is defined as

$$Z_{\cup} = \sum_{\xi,\eta} \sqrt{\frac{\mu(s(\eta))}{\mu(r(\eta))}} (\xi_1 \cdots \xi_i \eta \cdot \tilde{\eta} \cdot \xi_{i+1} \cdots \xi_n, \xi_1 \cdots \xi_n)$$

where the sum is over all possible paths ξ and η. Similarly we have that

$$Z_{\cap} = \sum_{\xi,\eta} \sqrt{\frac{\mu(s(\eta))}{\mu(r(\eta))}} (\xi_1 \cdots \xi_n, \xi_1 \cdots \xi_i \eta \cdot \tilde{\eta} \cdot \xi_{i+1} \cdots \xi_n).$$

For a strip containing a rectangle with label $(\xi, *)$ and no vertical strings to the left or right then Z_T is just $(\xi, *)$. If the rectangle has n vertical through strings to the left first attach the n string identity $\sum_{|\mu|=n} \mu$ to the right to get $\sum_{\mu} (\xi \cdot \mu, \mu)$. Using the connection transform this to the an element of the string algebra with the identity on the left, i.e. $\sum_{\mu,\zeta} c_{\xi} (\nu \cdot \zeta, \nu)$, where

$$c_{\xi} = \sum_{\zeta,\mu,\nu} \begin{array}{ccc} v & \xrightarrow{\xi} & w \\ \zeta\downarrow & & \downarrow\nu \\ y & \underset{\eta}{\longrightarrow} & x \end{array}$$

are given by the connection defined in Section 2.8. Note that if the connection is flat this presenting map defines a flat planar algebra.Finally, if there are m through strings on the right, label them by the m string identity $\sum_{|\nu|=m} \nu$. Hence the presenting map of a strip b containing a labelled box with n through strings to the left and m to the right is the element

$$Z_b := \sum_{\mu,\zeta,\nu} c_{\zeta} (\mu \cdot \zeta \cdot \nu, \mu \cdot \nu)$$

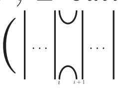
The presenting map of T is then $Z_T = Z(t_1) \dots Z(t_n)$ where t_1 is the top rectangle, t_2 is the one directly below it and so on.

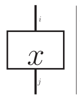
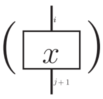
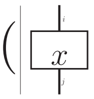
Now we show that for a type III subfactor $\rho(M) \subset M$ using the identification of string algebras with spaces of intertwiners in [34] we can use the definition of Z above to define a type III subfactor planar algebra.

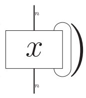
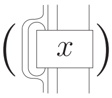
Proposition 3.4.7. *Let $\rho(M) \subset M$ be a extremal type III subfactor with $d(\rho)^{\frac{1}{2}} = \delta < \infty$ and for each n, m let*

$$\begin{aligned} P_{2n}^{2m} &= ((\rho\overline{\rho})^n, (\rho\overline{\rho})^m) \\ P_{2n+1}^{2m+1} &= ((\rho\overline{\rho})^n \rho, (\rho\overline{\rho})^m \rho) \end{aligned}$$

Let $P_k = \cup_n P_n^{n+k}$ and $P = \oplus_k P_k$ and let $\phi = tr \circ S$ be the state defined in Section 3.4. Then the above definition of Z makes P into a type III subfactor planar algebra with modulus δ and $Z(x) = x$ for each $x \in P$ (where by $Z(x)$ we mean the presenting map of a tangle with a single input box labelled by x). For all $n, m \in \mathbb{N}$ and $x \in P$, Z satisfies

(*i*) $Z\Big($  $\Big) = \delta^{-1} e_i \quad i \geq 1$

(*ii*) $Z\Big($  $\Big) = Z\Big($  $\Big)$ $\qquad Z\Big($  $\Big) = \rho(x)$

(*iii*) $Z\Big($  $\Big) = \delta\, E_{N' \cap M_n}(x)$ $\qquad Z\Big($  $\Big) = \delta\, \mathcal{E}(x)$

Here $E_{N' \cap M_n}$ is the conditional expectation $N' \cap M_{n+1} \to N' \cap M_n$ and $\mathcal{E}$ is the minimal expectation $M \to N$.

If Z' is another planar structure on P satisfying the 4 conditions above and such that $Z'(x) = Z(x)$ for all labelled discs x and also $Z'(\cup) = Z(\cup)$ then $Z \simeq Z'$.

Proof. As in [34] (and Section 2.8) we identify the vector spaces $((\rho\overline{\rho})^n, (\rho\overline{\rho})^m)$ with vector spaces with basis given by pairs of paths (ξ_1, ξ_2) in the principal/dual principal graph of $\rho(M) \subset M$ with $|\xi_1| = n$, $|\xi_2| = m$, $s(\xi_1) = s(\xi_2) = *$ and $r(\xi_1) = r(\xi_2)$.

Let Z be as defined above. We begin by showing that the definition of Z is independent of isotopies of the tangle. By the discussion at the beginning of the proof of Theorem 4.2.1 of [36] we just need to check the isotopies of Figures 3.18,3.19, 3.21 and 3.22. This is almost exactly the same as the proof in [23].

Firstly, straightening a string with a cup and cap does not change Z since for a strip t with a string with a cup and cap then we can split the strip into two strips, as shown in Figure 3.18, t_1 has a cap joining points $i+1$ and $i+2$ along the bottom and t_2 has a cap joining points i and $i+1$ on the top.

Then $Z(t) = Z(t_1)Z(t_2)$ with

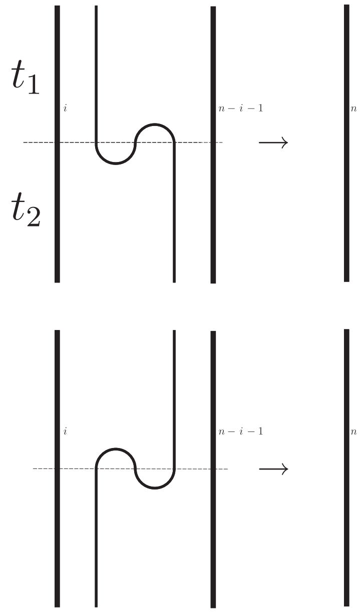


Figure 3.18: Straightening a cup and cap

$$Z(t_1) = \sum_{\xi,\eta} \sqrt{\frac{\mu(s(\eta))}{\mu(r(\eta))}} (\xi_1 \cdots \xi_n, \xi_1 \cdots \xi_{i+1} \cdot \eta \cdot \tilde{\eta} \cdot \xi_{i+2} \cdots \xi_n)$$

$$Z(t_2) = \sum_{\zeta,\nu} \sqrt{\frac{\mu(s(\nu))}{\mu(r(\nu))}} (\zeta_1 \cdots \zeta_i \cdot \nu \cdot \tilde{\nu} \cdot \zeta_{i+1} \cdots \zeta_n, \zeta_1 \cdots \zeta_n)$$

and so $Z(t) = \sum_{\xi,\nu,\eta} \delta_{\mu,\tilde{\nu}} \sqrt{\frac{\mu(s(\nu))}{\mu(r(\nu))}} \sqrt{\frac{\mu(s(\eta))}{\mu(r(\eta))}} (\xi_1 \cdots \xi_n, \xi_1 \cdots \xi_n) = id_n$. A similar computation shows that flattening a string with a cap and then a cup does not change Z either.

Next, we show that isotopies involving labelled discs do not change Z. For the first isotopy of Figure 3.19 we show that interchanging the vertical coordinates of the rectangle and cap does not change Z.

Suppose t_2 is the horizontal strip containing the rectangle x with label $(\xi, *)$ on the left hand side of the equation. Then

$$Z(t_2) = \sum_{\alpha_i,\zeta} c_{\xi,\zeta} (\alpha_1 \cdot \zeta \cdot \alpha_2 \cdot \alpha_3, \alpha_1 \cdot \alpha_2 \cdot \alpha_3)$$

where the sum is over all α_i with $|\alpha_1| = i$, $|\alpha_2| = k$, $|\alpha_3| = l$ and ζ with $|\zeta| = j$ and the constants $c_{\xi,\zeta}$ come from the connection. Then, if t_1 is the strip containing the cup,

$$Z(t_1) = \sum_{\beta_i,\eta} \sqrt{\frac{\mu(s(\eta))}{\mu(r(\eta))}} (\beta_1 \cdot \beta_2 \cdot \beta_3 \cdot \eta \cdot \tilde{\eta} \cdot \beta_4, \beta_1 \cdot \beta_2 \cdot \beta_3 \cdot \beta_4)$$

where now we sum over all β_i with $|\beta_1| = i$, $|\beta_2| = j$, $|\beta_3| = k$, $|\beta_4| = l$ and $|\eta| = 1$. Hence

$$Z(t) = Z(t_1)Z(t_2) = \sum_{\eta,\zeta,\alpha_i} \delta_{\beta_1,\alpha_1}\delta_{\beta_2,\zeta}\delta_{\beta_3,\alpha_2}\delta_{\beta_4,\alpha_3}c_{\xi,\zeta}\sqrt{\frac{\mu(s(\eta))}{\mu(r(\eta))}}(\alpha_1{\cdot}\zeta{\cdot}\alpha_2{\cdot}\eta{\cdot}\tilde{\eta}{\cdot}\alpha_3, \alpha_1{\cdot}\alpha_2{\cdot}\alpha_3).$$

Then if we label the upper and lower strip of the right hand side by s_1, s_2 respectively we have

$$Z(s_1) = \sum_{\substack{|\alpha_1|=i,|\alpha_2|=j,\\ |\alpha_3|=k,|\alpha_4=l,|\eta|=1}} \sqrt{\frac{\mu(s(\eta))}{\mu(r(\eta))}} (\alpha_1 \cdot \alpha_2 \cdot \eta \cdot \tilde{\eta} \cdot \alpha_3, \alpha_1 \cdot \alpha_2 \cdot \alpha_3)$$

and

$$Z(s_2) = \sum_{\substack{|\beta_1|=i,|\beta_2|=k,\\ |\beta_3|=2,|\beta_4|=l,|\zeta|=j}} c_{\xi,\zeta}(\beta_1 \cdot \zeta \cdot \beta_2 \cdot \beta_3 \cdot \beta_4, \beta_1 \cdot \beta_2 \cdot \beta_3 \cdot \beta_4).$$

Hence

$$\begin{aligned} Z(s) =& Z(s_2)Z(s_1) \\ =& \sum_{\eta,\zeta,\alpha_i} \delta_{\beta_1,\alpha_1}\delta_{\alpha_2,\zeta}\delta_{\beta_2,\alpha_3}\delta_{\beta_4,\alpha_3}\delta_{\beta 3,\eta\tilde{\eta}}c_{\xi,\zeta}\sqrt{\frac{\mu(s(\eta))}{\mu(r(\eta))}}(\alpha_1 \cdot \zeta \cdot \alpha_2 \cdot \eta \cdot \tilde{\eta} \cdot \alpha_3, \alpha_1 \cdot \alpha_2 \cdot \alpha_3) \end{aligned}$$

which is equal to the left hand side. To show that we may interchange the vertical coordinates of a labelled rectangle to the right of a cap, as in the second equation of 3.19 we can use the partial braiding as shown in Figure 3.20 to transform this to the above situation.

For the isotopies described in Figure 3.21 we show that pulling a rectangle down to the left of another rectangle does not change Z. Given two rectangles x and y, labelled $(\xi_1, *)$ and $(\xi_2, *)$ then the horizontal strip containing the first rectangle will be

$$Z(s_1) = \sum_{\substack{|\zeta|=k,|\alpha_1|=j,\\ |\beta_1|=l,|\gamma|=m,|\delta_1|=n}} c_{\xi_1,\zeta_1}(\alpha_1 \cdot \zeta_1 \cdot \beta_1 \cdot \gamma \cdot \delta_1, \alpha_1 \cdot \beta_1 \cdot \gamma \cdot \delta_1)$$

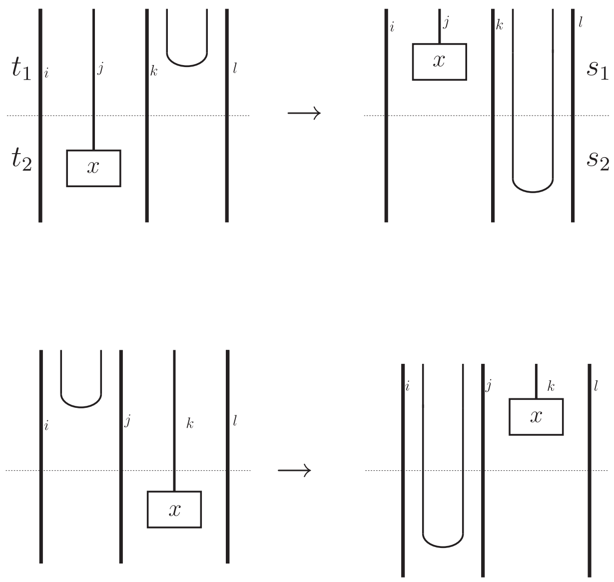


Figure 3.19: Isotopies involving a cup and labelled rectangle

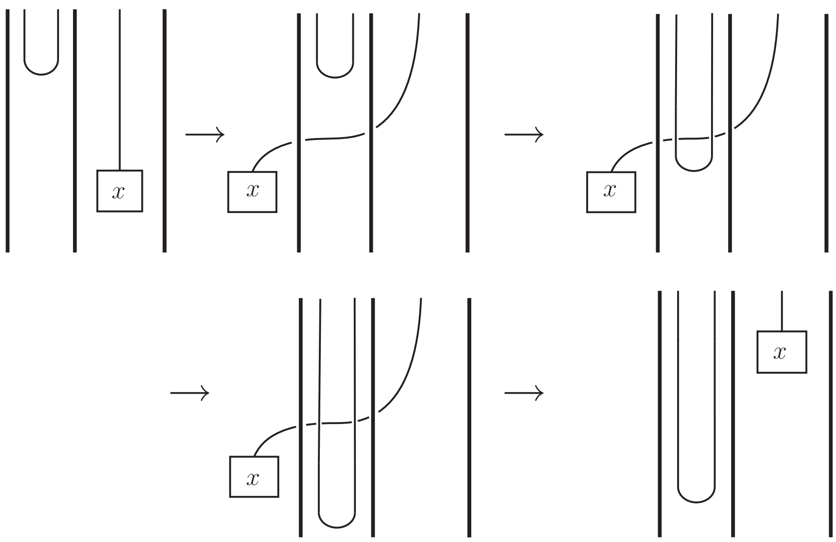


Figure 3.20: Isotopies involving a cup and a labelled rectangle II

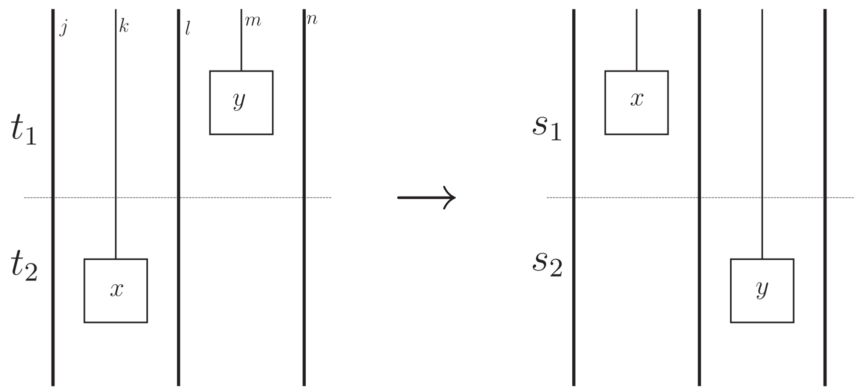


Figure 3.21: Isotopies involving two labelled rectangles

and the strip containing the second rectangle will be

$$Z(s_2) = \sum_{\substack{|\alpha_2|=j,|\beta_2|=l,\\ |\zeta_2|=m,|\delta_2|=n}} c_{\xi_2,\zeta_2}(\alpha_2 \cdot \beta_2 \cdot \zeta_2 \cdot \delta_2, \alpha_2 \cdot \beta_2 \cdot \delta_2).$$

Multiplying these we see

$$Z(s_1)Z(s_2) = \sum_{\substack{|\zeta|=k,|\alpha_1|=j,\\ |\beta_1|=l,|\zeta_2|=m,|\delta_1|=n}} c_{\xi_1,\zeta_1} c_{\xi_2,\zeta_2}(\alpha_1 \cdot \zeta_1 \cdot \beta_1 \cdot \zeta_2 \cdot \delta_1, \alpha_1 \cdot \beta_1 \cdot \delta_1).$$

On the other hand, letting t_1, t_2 be the horizontal strips on the left hand side we get

$$Z(t_2) = \sum_{\substack{|\zeta_1'|=k,|\alpha_4|=j,\\ |\beta_4|=l,|\delta_4|=n}} c_{\xi_1,\zeta_1'}(\alpha_4 \cdot \zeta_1' \cdot \beta_4 \cdot \delta_4, \alpha_4 \cdot \beta_4 \cdot \delta_4)$$

and

$$Z(t_1) = \sum_{\substack{|\zeta_2'|=m,|\alpha_3|=j,|\gamma|=k\\ |\beta_3|=l,|\delta_3|=n}} c_{\xi_2,\zeta_2}(\alpha_3 \cdot \gamma \cdot \beta_3 \cdot \zeta_2' \cdot \delta_3, \alpha_3 \cdot \gamma' \cdot \beta_3 \cdot \delta_3)$$

and so $Z(t_1)Z(t_2) = Z(s_1)Z(s_2)$ as required.

The final planar isotopy that needs to be checked is that rotating a box by 2π does not change Z. We demonstrate this for a box with two marked points, the method carries over to a box with an arbitrary number of marked points. Let the box be labelled by $(\xi, *)$, with $|\xi| = 2$. Then splitting the diagram in Figure

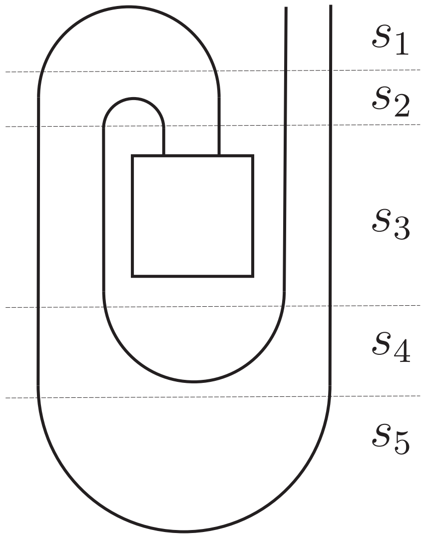


Figure 3.22: Rotation of a rectangle

3.22 into five horizontal strips as shown we get

$$Z(s_1) = \sum_{|\alpha_i|=1} (\alpha_1 \cdot \alpha_2, \alpha_3 \cdot \tilde{\alpha}_3 \cdot \alpha_1 \cdot \alpha_1)$$

$$Z(s_2) = \sum_{|\beta_i|=1} (\beta_1 \cdots \beta_4, \beta_1 \cdot \beta_5 \cdot \tilde{\beta}_5 \cdot \beta_2 \cdots \beta_4)$$

$$Z(s_3) = \sum_{|\gamma_i|=1, |\zeta|=2} c_{\xi,\zeta}(\gamma_1 \cdot \gamma_2 \cdot \zeta \cdot \gamma_3 \cdot \gamma_4, \gamma_1 \cdots \gamma_4)$$

$$Z(s_4) = \sum_{|\delta_i|=1} (\delta_1 \cdot \delta_2 \cdot \tilde{\delta}_2 \cdot \delta_3 \cdot \delta_4, \delta_1 \cdot \cdot \delta_3 \cdot \delta_4)$$

$$Z(s_5) = \sum_{|\epsilon|=1} (\epsilon \cdot \tilde{\epsilon}, *).$$

Hence $Z = \sum c_{\xi,\zeta}(\zeta, *)$ which, by the definition of the connection, is equal to $(\xi, *)$.

Hence our definition is independent of all planar isotopies. To show that P is a subfactor planar algebra, we must show that is is spherical, connected, finite dimensional, has modulus $\delta > 0$ and the state ϕ is positive definite. The finite dimensionality of P_n^m is true by definition, as is the fact that $P_0^0 = (id, id)$ has dimension one. The planar algebra P is spherical since the connection defined in Section 2.8 is flat. To show that the planar algebra has modulus δ, the Perron Frobenius eigenvalue of the graph, let t be a horizontal strip with a closed loop

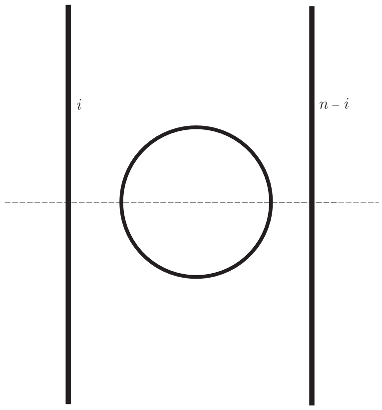


Figure 3.23: Calculating δ

with i through strings to the left and j through strings to the right. Then $Z(t) = Z(t_1)Z(t_2)$ where t_1 is the strip with a cap joining strings i and $i+1$ on the bottom and t_2 is the strip with a cap in the corresponding position on the top edge as shown in Figure 3.23. Then

$$Z(t_1) = \sum_{\xi,\eta} \frac{\sqrt{\mu(r(\eta))}}{\sqrt{\mu(s(\eta))}}(\xi_1 \cdots \xi_n, \xi_1 \cdots \xi_i \cdot \eta \cdot \tilde{\eta} \cdot \xi_{i+1} \cdots \xi_n)$$

$$Z(t_2) = \sum_{\zeta,\nu} \frac{\sqrt{\mu(r(\nu))}}{\sqrt{\mu(s(\nu))}}(\zeta_1 \cdots \zeta_i \cdot \nu \cdot \tilde{\nu} \cdot \zeta_{i+1} \cdots \zeta_n, \zeta_1 \cdots \zeta_n)$$

and

$$Z(t) = \sum_{\zeta,\nu: s(\nu)=r(\zeta_i)} \frac{\mu(r(\nu))}{\mu(s(\nu))}(\zeta,\zeta) = \delta id_n.$$

We now show that the state on the planar algebra is positive. Let x be a tangle with label (ξ_1, ξ_2). Then $Z(\phi(\xi_1,\xi_2)) = \delta_{\xi_1,\xi_2}\delta^{-|\xi_1|}\mu(t(\xi_1))$. This is exactly the state we defined on the string algebra, which is positive since it is the composition of a projection and the unique positive definite trace on the string algebra.

Next we must show that P satisfies the three conditions in the statement of the proposition. For the first one, using the definition of Z we calculate the left hand side to be

$$\sum_{\xi,\alpha,\beta} \frac{\sqrt{\mu(r(\alpha))\mu(r(\beta))}}{\mu(r(\xi_{i-1}))}(\xi_1 \ldots \xi_{i-1} \cdot \alpha \cdot \tilde{\alpha} \cdot \xi_i \cdots \xi_n, \xi_1 \ldots \xi_{i-1} \cdot \beta \cdot \tilde{\beta} \cdot \xi_i \ldots \xi_n)$$

which is which is δ times the Jones projection in the string algebra. The second condition is satisfied by definition.

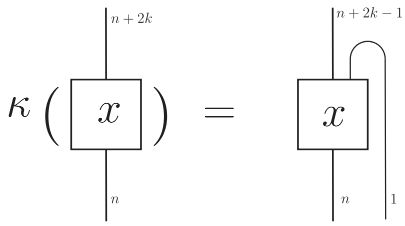
Now we must show that we have the correct conditional expectations. The minimal expectation $E : M \to \rho(M)$ is given by $E(x) = \rho(r^*\rho(x)r)$ for $r \in (id, \bar{\rho}\rho)$. The presenting map of the left conditional expectation tangle gives exactly this map with $r = Z(\cup)$. The right conditional expectation is the expectation from $N' \cap M_n$ onto $N' \cap M_{n-1}$, which is exactly the same as the type II case. This follows from the fact that $P_{2n}^{2n} = ((\rho\bar{\rho})^n, (\rho\bar{\rho})^n) \simeq N' \cap M_{2n}$ and $P_{2n+1}^{2n+1} = ((\rho\bar{\rho})^n\rho, (\rho\bar{\rho})^n\rho) \simeq N' \cap M_{2n+1}$

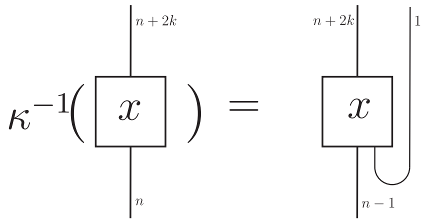
The uniqueness can be proved exactly as in Jones' paper [36]. Suppose Z and Z' are two presenting maps on P satisfying $Z(x) = x = Z'(x)$ for all $x \in M$ and $Z(\cup) = Z'(\cup)$. Then we want to show that for any planar tangle T we have $Z(T) = Z'(T)$. Let T be a type III planar tangle. Putting T in standard form we see it is a product of O_{TL} tangles and tangles containing a single labelled disc, possibly with some strings to the left and right. Since O_{TL} is generated by $\cup$ and Jones projections, we know that $Z = Z'$ on O_{TL}. We also know that $Z = Z'$ for any strip containing a labelled disc or a labelled disc with through strings to the right. All that is left is to show that Z and Z' agree on strips containing a labelled disc with strings to the left. Since we are assuming the connection is flat, this is trivial. □

Next we show that for $\delta < 2$ the planar algebra O_{TL} may be realised as the planar algebra defined using the string algebra construction for the graph A_n where $\delta = 2\cos(\pi/n)$.

Proposition 3.4.8. *Let P be the planar algebra with presenting map defined using the string algebra construction for the Dynkin diagram A_n. Then $P \simeq O_{TL}$.*

Proof. The ordinary Temperley-Lieb algebra may be described as the (type II) string algebra of the graph A_n. In order to prove $\mathcal{O}_{TL}$ may be described as the type III string algebra of A_n, first we define linear maps $\kappa : \mathcal{V}_k \to \mathcal{V}_{k+1}$ for $k \in \mathbb{Z}$ by





Then κ is invertible with inverse $\kappa^{-1} : \mathcal{V}_{k+1} \to \mathcal{V}_k$. We can similarly define maps $\tilde{\kappa} : A_{(n,n+2m),0} \to A_{(n-1,n+2m+1),0}$ by $\tilde{\kappa}(\xi_1 \cdots \xi_n, \zeta_1 \cdots \zeta_{n+2m})$ $= (\xi_1 \cdots \xi_{n-1}, \zeta_1 \cdots \zeta_{n+2m} \cdot \tilde{\xi}_n)$.

If $x \in \mathcal{V}_k$ with $k > 0$ then $(\kappa^{-1})^k(x)$ is in $\mathcal{V}_0$ which is the type II Temperley-Lieb planar algebra. Let χ be the isomorphism from $\mathcal{V}_0$ to the type II string algebra of A_n defined by mapping the diagramatic Jones projection e_i in $\mathcal{V}_0$ to the corresponding Jones projections in A_n. We wish to show that Z defines a surjective map from $V_n^{n+2m} \to A_{(n,n+2m),0}$ and then, since $dim(V_n^{n+2m}) = dim(V_{n+m}^{n+m}) = dimA_{(n+m,n+m),0} = dimA_{(n,n+2m),0}$ we have that Z is a bijection and hence $Z : O_{TL} \to A_{\infty,0}$ is an isomorphism. Let $x \in A_{(n,n+2m),0}$. Then $\tilde{\kappa}^m(x) \in A_{(n+m,n+m),0}$ and then $\chi(\tilde{\kappa}^m(x))$ defines a unique element of $\mathcal{V}_0$. Thus, letting $\tilde{x} = \kappa^{-m}\chi(\tilde{\kappa}^m(x))$ we have that for any $x \in V$ there exists $\tilde{x} \in V$ with $Z(\tilde{x}) = x$ and so Z is surjective. $\square$

3.5 Perturbations of Planar Algebras

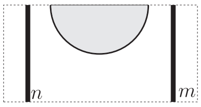
In [15] it was shown that for type II planar algebras it is possible to remove the condition of extremality by defining planar algebras which are not necessarily spherical. We now show that this is also possible for type III planar algebras. First we define an analogue of the bimodule planar algebra of [15], [26]. In the type II setting the appropriate bicategory is the bicategory of bifinite bimodules.

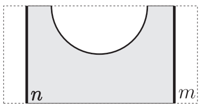
However in the type III setting we use instead the following bicategory $\mathcal{B}$, details of which may be found for example in [65], [61].

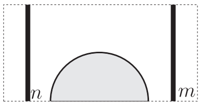
Let $N = \rho(M) \subset M$ be a type III subfactor. Let $\mathcal{B}$ be the 2-category whose objects are $\{\rho(M), M\}$, whose 1-morphisms generated by ρ and $\bar{\rho}$ and whose 2-morphisms are intertwiners. The tensor product of 1-morphisms is given by composition, i.e $\rho_1 \otimes \rho_2 = \rho_1\rho_2$ and the tensor product of intertwiners $u \in (\rho_1, \rho_2)$, $v \in (\sigma_1, \sigma_2)$ is given by $u \otimes v = u\rho_1(v)$.

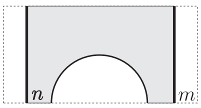
Recall from Section 2.8 that there exist isometries r and $\bar{r}$ with $r \in (id_M, \bar{\rho}\rho)$, $\bar{r} \in (id_N, \rho\bar{\rho})$ and $\bar{r}^* \otimes 1_\rho \circ 1_\rho \otimes r = 1_\rho$ and $r^* \otimes 1_{\bar{\rho}} \circ 1_{\bar{\rho}} \otimes \bar{r} = 1$. Such an r is said to be a solution of the conjugate equations. Suppose $\rho = \sum \rho_i$ is a sum of irreducibles. Then there exist partial isometries w_i with $w_i \rho w_i^* = \rho_i$ and $\sum_i w_i w_i^* = 1$. There is a unique (up to unitary equivalence) $r_i \in (1, \rho_i\bar{\rho}_i)$ and the sum $r = \sum_i w_i^* \otimes w_i \circ r_i$ is said to be the *standard* solution of the conjugate equations. The *standard left inverse* for ρ is a collection of mappings $\phi_{\sigma,\tau} : (\rho\sigma, \rho\tau) \rightarrow (\sigma, \tau)$ defined by $\phi_{\sigma,\tau}(x) = r^* \otimes 1_\tau \circ 1_{\bar{\rho}} \otimes x \circ r \otimes 1_\sigma$. Similarly one can define the standard right inverse $\psi_{\sigma,\tau} : (\sigma\rho, \tau\rho) \rightarrow (\sigma, \tau)$ by $\psi_{\sigma,\tau}(x) = 1_\tau \otimes r^* \circ x \otimes 1_{\bar{\rho}} \circ 1_\sigma \otimes r$. It is proved in Lemma 3.9 of [61] that ψ and ϕ are faithful positive maps and $\psi_{1,1} = \phi_{1,1}$ if and only if r is standard. For any choice of $r' \in (1, \rho\bar{\rho})$, $\bar{r}' \in (1, \bar{\rho}\rho)$ there exists an invertible element $y \in (\rho, \rho)$ such that $r' = (1_\rho \otimes y) \circ r$ and $\bar{r}' = (y^* \otimes 1_\rho) \circ \bar{r}$. The minimal conditional expectation $E : M \rightarrow \rho(M)$ may be defined in terms of standard r by $E(x) = 1_{\bar{\rho}} \otimes \bar{r}^* \circ x \otimes 1_{\bar{\rho}} \circ 1_{\bar{\rho}} \otimes \bar{r}$. The bicategory $\mathcal{B}$ is right rigid if $\rho(M) \subseteq M$ is a finite index subfactor, since in this case a unique $\bar{\rho}$ always exists by Proposition 6.25 of [65]. Thus any choice of r defines a pivotal structure on $\mathcal{B}$.

As in [15] we can associate a planar algebra P to the bicategory $\mathcal{B}$ by letting the vector spaces P_n^m be spaces of intertwiners from $(\rho\bar{\rho}\rho\ldots) \rightarrow (\rho\bar{\rho}\rho\ldots)$ or $(\bar{\rho}\rho\ldots) \rightarrow (\bar{\rho}\rho\ldots)$ where there are n terms on the left and m on the right. Fix some $r \in (1, \bar{\rho}\rho)$ and $\bar{r} \in (1, \rho\bar{\rho})$. Then, given a tangle T we can define the presenting map via the assignments shown in Figure 3.24 where σ_n is $(\rho\bar{\rho}\ldots)$ with n terms if the left hand side is shaded and $(\bar{\rho}\rho\ldots)$ with n terms if the left

$Z($  $)\quad = 1_{\sigma_n} \otimes r \otimes 1_{\sigma'_m}$

$Z($  $)\quad = 1_{\sigma_n} \otimes \bar{r} \otimes 1_{\sigma'_m}$

$Z($  $)\quad = 1_{\sigma_n} \otimes \bar{r}^* \otimes 1_{\sigma'_m}$

$Z($  $)\quad = 1_{\sigma_n} \otimes r^* \otimes 1_{\sigma'_m}$

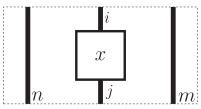
$Z($  $)\quad = 1_{\sigma_n} \otimes x \otimes 1_{\sigma'_m}$

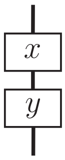
Figure 3.24: Presenting map for horizontal strips

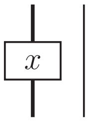
hand side is unshaded.

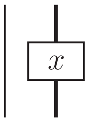
Proposition 3.5.1. *Given a type III factor M and a finite index endomorphism $\rho \in End(M)$ there is a unique finite dimensional, connected, positive definite type III C^*-planar algebra structure with states ϕ_L and ϕ_R and*

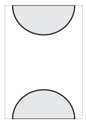
$$P_{2n}^{2m} = \begin{cases} ((\rho\bar{\rho})^n, (\rho\bar{\rho})^m) \\ ((\bar{\rho}\rho)^n, (\bar{\rho}\rho)^m) \end{cases}$$
$$P_{2n+1}^{2m+1} = \begin{cases} (\bar{\rho}(\rho\bar{\rho})^n, \bar{\rho}(\rho\bar{\rho})^m) \\ (\rho(\bar{\rho}\rho)^n, \rho(\bar{\rho}\rho)^m) \end{cases} \tag{3.9}$$

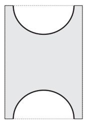
and such that for all $x, y \in P$ we have the following identities

(i) $Z($  $) = xy$

(ii) $Z($  $) = x \otimes 1_\rho \simeq x$

(iii) $Z($  $) = 1_\rho \otimes x \simeq \rho(x)$

(iv) $Z($  $) = e_1^+$

(v) $Z($  $) = e_1^-$

where $e_1^\pm$ *are the Jones projections for the inclusions* $\rho(M) \subset M$ *and* $\bar{\rho}\rho(M) \subset \rho(M)$.

Proof. The first thing that needs to be checked is that the definition of Z above is invariant under the three planar isotopies in Figure 3.25 for all $x, y \in P$ and all $n_k, i_k, j_k \in \mathbb{N}$.

Smoothing a string as in (i) leaves Z unchanged since $Z(\cap \otimes 1_\rho)Z(1\rho \otimes \cup) = r^* \circ \rho(r) = 1 = Z(1)$ by the conjugate equations.

Interchanging the vertical coordinates of discs labelled x, y as shown in (ii) of Figure 3.25 doesn't change Z, since, if we assume n_k, j_k, i_k are all even and the extreme left hand side of the diagram is unshaded, the left hand side is equal to $(\rho\overline{\rho})^{n_1/2}(x)(\rho\overline{\rho})^{(n_1+j_1+n_2)/2}(y)$. Since $\rho^{n_1/2}(x) \in ((\rho\overline{\rho})^{n_1+i_1/2}, (\rho\overline{\rho})^{n_1+j_1/2})$ this equals $(\rho\overline{\rho})^{(n_1+i_1+n_2)/2}(y)(\rho\overline{\rho})^{n_1/2}(x)$ which is equal to the right hand side. Invariance of Z can similarly be shown for other values of n_k, j_k, i_k and either shading.

The final planar isotopy we need to consider is rotation. For simplicity we give a proof here for a disc $x \in (\bar{\rho}, \bar{\rho})$ and the general case may be proved similarly. The presenting map is $r^* \otimes 1_{\bar{\rho}} \circ 1_{\bar{\rho}} \otimes \bar{r}^* \otimes 1_{\rho\bar{\rho}} \circ 1_{\bar{\rho}\rho} \otimes x \otimes 1_{\rho\bar{\rho}} \circ 1_\rho \otimes \bar{r} \otimes 1_{\bar{\rho}} \circ 1_\rho \otimes r = r^* \otimes 1_{\bar{\rho}} \circ 1_{\bar{\rho}} \otimes \bar{r} \otimes 1_\rho \circ 1_{\bar{\rho}\rho} \otimes x \otimes 1_{\rho\bar{\rho}} \circ 1 \otimes r^* \circ 1_\rho \otimes r = x$, where the first equality follows from pivotality and the second follows from the conjugate equations.

The planar algebra is finite dimensional by definition, it is connected because $P_0^0 = (id, id)$ is one dimensional by definition, positivity follows because the state ϕ is the composition of a projection with a trace, the trace is positive definite because $Z(tr(x)) = r^* \circ x \otimes 1 \circ r$ for $x \in P_1^1$, which is positive, since if $x = y^*y$ then $Z(tr(x)) = (y \circ r)^*(y \circ r)$. The four conditions listed follow directly from the definitions of Z and $\mathcal{B}$. Uniqueness is proved in exactly the same way as in the proof of Proposition 3.4.7. □

Proposition 3.5.2. *Let P be the planar algebra defined in the previous proposition. Sphericality of P corresponds to extremality of the subfactor $\rho(M) \subset M$.*

Proof. From the previous section we know that sphericality is equivalent to the condition that the states ϕ_R and ϕ_L agree. From [61] we know that the left and right inverses for ρ agree if and only if the isometry $r \in (id, \rho\bar{\rho})$ is standard, and in this case the conditional expectation is minimal and so the subfactor $\rho(M) \subset M$ is extremal. □

For finite depth subfactors, the above construction is exactly the same as the planar algebra defined in the previous section. To see this just use the identification of the intertwiner spaces $((\rho\bar{\rho})^n, (\rho\bar{\rho})^m)$ etc with spaces of pairs of paths in the graph and dual graph of the subfactor described in Section 2.8.

Given a type III planar algebra we can define a bicategory $\mathcal{B}$ as follows. Let $\mathcal{B}_0$ have exactly two elements denoted $+$ and $-$; for each choice of $\eta, \epsilon \in \{+, -\}$ the category $\mathcal{B}(\epsilon, \eta)$ is the category whose objects are natural numbers and a morphism in $\mathcal{B}(\epsilon, \eta)$ from n to m is an element of $P_{n\epsilon}^m$. Composition of morphisms is by multiplication in the planar algebra, the identity morphism is the empty tangle. The tensor product of objects is just addition in $\mathbb{N}$ and the tensor product of morphisms is by placing the tangles side by side as shown in Figure 3.26.

A rigid structure may be defined on $\mathcal{B}$ using the adjoint in the planar algebra. For a 0-cell ϵ define $\epsilon^{\#} := \epsilon$. For a 1-cell $k \in ob(\mathcal{B}(\epsilon, \eta))$ define $k^{\#} := k \in ob(\mathcal{B}(\eta, \epsilon))$. The evaluation map is given by $Z(\cup)$ and coevaluation by $Z(\cap)$.

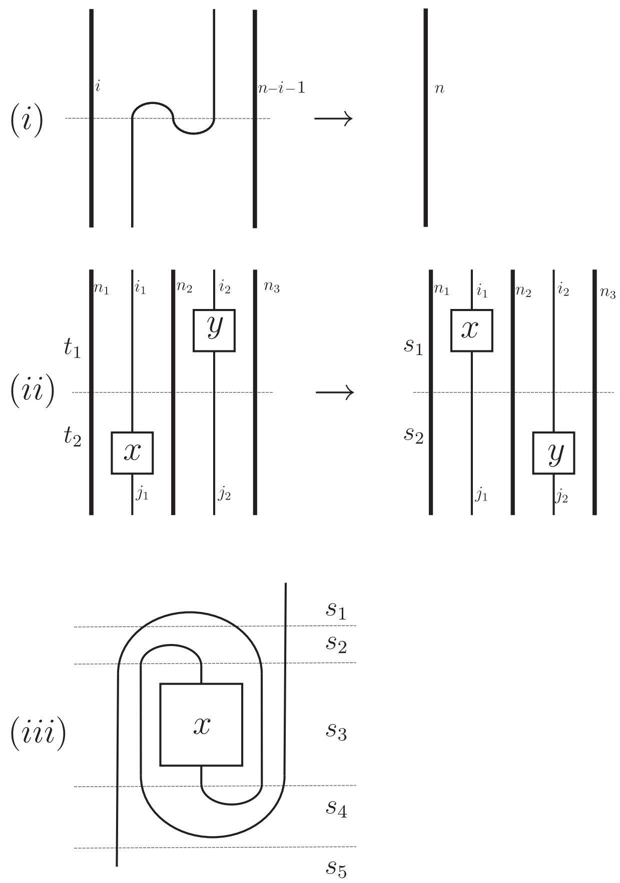


Figure 3.25: Planar isotopies

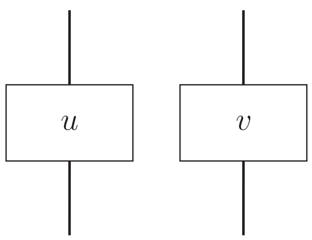


Figure 3.26: Tensor product of morphisms $u \otimes v$

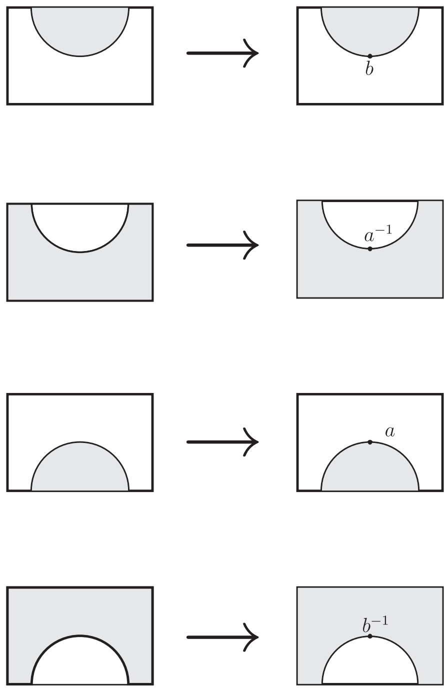


Figure 3.27: Perturbation of Cups and Caps

A 2-cell $x \in Mor(k,l)$, for $k \in ob(\mathcal{B}(\epsilon,\eta))$, $l \in ob(\mathcal{B}(\eta,\gamma))$, corresponds to an element of P_k^l and so to define $x^\# \in Mor(l,k)$ we take the adjoint in the planar algebra.

Similarly to [15] we define a *weight* of a planar algebra to be an invertible element $z \in P_1^1$ such that the element $z_k := (z \otimes z^{-1})^{\otimes k}$ is in the centre of P_k^k for all $k \geq 0$. Suppose that there exist $a, b \in P$ with $z = ab$ then we define a *perturbation* $P^{(a,b)}$ of P to be the planar algebra with the same vector spaces P_n^m as P but where the presenting map is altered by replacing cups and caps as shown in the Figure 3.27.

It is clear that this presenting map is well defined, i.e. it is invariant under planar isotopies. If $z \in \mathbb{C}$ then $P^{(a,b)}$ is called a *scalar perturbation.* Note that for irreducible planar algebras scalar perturbations are the only possible perturbations, since $P_1^1 \simeq \mathbb{C}$. If a planar algebra has modulus (δ_+, δ_-) then its perturbation by a scalar $z = \lambda$ has modulus $(\lambda^{-1}\delta_+, \lambda\delta_-)$ and so it is possible to perturb P to a unimodular planar algebra by perturbing by the weight $\lambda = \sqrt{\delta_+/\delta_-}$.

Proposition 3.5.3. *There is a one to one correspondence between the weights* *of a planar algebra P and pivotal structures on the strict 2-category associated to P.*

Proof. Let z be a weight of P. Then, for $k \in \mathcal{B}(\epsilon, \eta)$, the two cell $k \xrightarrow{\tau_k} k$ defined by $\tau_k = z_k$ is a pivotal structure. To check this we need to show that $\tau^{\epsilon,\eta}$ is a natural transformation, i.e. that $\tau f \tau^{-1} = f$ and that $\tau_{k\otimes l} = \tau_k \otimes \tau_l$. The first equality follows from the fact that z is central. The second equality is easy to verify, since by drawing diagrams it is clear that both sides of the equation are equal to z_{k+l}.

Conversely if τ is a pivotal structure on P then $z = \tau \in Mor(1,1) = P_1^1$ is a weight of P, we just need to check that it is central, but this follows from the fact that τ is a natural transformation. □

For the rest of this section we assume P is a type III finite dimensional connected C^*-planar algebra which has positive definite states ϕ_R and ϕ_L.

Proposition 3.5.4. *The planar algebra P has a spherical planar algebra in its perturbation class.*

Proof. The states ϕ_R and ϕ_L are related by $\phi_R(x) = \phi_L(z^{\otimes k}x)$ for some invertible $z \in P_0^0$. Thus perturbing P by $z^{\frac{1}{2}}$ gives a spherical planar algebra. □

Proposition 3.5.5. *The perturbation class of P contains a unique spherical planar algebra $\mathcal{P}$. $\mathcal{P}$ is the unique unimodular planar algebra attaining the minimal index in its perturbation class.*

Proof. This is proved in exactly the same way as in [15]. If P is spherical and Q is a perturbation of P by the weight $z \in P_1^1$. Then let p_i be the collection of minimal central projections of P_1^1 and let $c_i = trp_i$, so $\delta_P = \sum_i c_i$. Then there exists $\lambda_i > 0$ such that $(\delta_{Q+}, \delta_{Q-}) = (\sum \lambda_i c_i, \sum \lambda_i^{-1} c_i)$. Then the index of Q is $\sum \lambda_i \lambda_j^{-1} c_i c_j \geq (\sum c_i)^2 = \delta_P$. Hence P has the minimal index in its perturbation class if it is spherical.

Conversely, suppose P is unimodular and has minimal index δ. Let Q be a spherical planar algebra in the perturbation class of P. Then Q must have the same index as P and the weight z must be a scalar. Hence $z = 1$ and $Q = P$. □

If P is the planar algebra associated to a subfactor as in Proposition 3.5.1, a choice of perturbation z corresponds to a choice of $r' \in (id, \bar{\rho}\rho)$. Any r' can be written in terms of the standard r as $r' = x \otimes id \circ r$ for some invertible $x \in (\rho, \rho)$.

3.6 Planar Modules

In this section we define modules over a type III planar algebra P. This was first investigated by Jones in [39], and the Temperlely-Lieb modules were studied further in [81], [41]. This was then used, for example in [74] construction of the planar algebra of the Haagerup subfactor and more recently in [14], [13] to investigate the Drinfeld centre of planar algebras. A similar construction for A_2-planar algebras appeared in [24].

It would be interesting in the type III case see if planar modules could be used to investigate the Longo-Rehren subfactors, given the connection between Longo-Rehren subfactors and the Drinfeld centre described for example in [66] and the above mentioned connection between type II Temperley-Lieb modules and the Drinfeld centre.

We now define planar modules for type III planar algebras in exactly the same way as the type II theory developed in [39].

Definition 3.6.1. An *annular tangle* is a tangle T with a distinguished internal disc. If the outer disc has pattern (n_0, m_0) and the inner disc has pattern (n_1, m_1) then it is called an annular $(n_0, m_0) - (n_1, m_1)$ tangle.

Definition 3.6.2. A *planar module* V is a collection of vector spaces V_n^m for $n, m \in \mathbb{N}$ with an action of P. For each annular $(n_0, m_0) - (n_1, m_1)$ tangle with internal discs D_i of pattern (n_i, m_i) there is a linear map $Z_T : V_{n_1}^{m_1} \otimes (\otimes_{i=2}^{N} P_{n_i}^{m_i}) \rightarrow V_{n_0}^{m_0}$ which satisfies all the usual compatibility requirements for presenting maps of planar algebras.

A planar algebra P is always a module over itself, where the action of P on itself is by composition. We will call this the *trivial module*. Another way to define planar modules is in terms of the annular category $AnnP$ as follows. Let $AnnP$ be the category whose objects are pairs of natural numbers (n, m) and whose morphisms are given by annular tangles with labelling set P. Composition of morphisms is given by gluing of tangles. Let FAP be the category with the same objects as $AnnP$ but whose morphisms are linear combinations of morphisms, and composition of morphisms is the linear extension of composition in $AnnP$.

Definition 3.6.3. The *annular algebra* AP is the quotient of FAP by the relation $\delta = \bigcirc$.

Denote by $AP(n, m)$ the subalgebra of annular tangles with pattern (n, m) on both the internal and external discs.

A module V is said to be *irreducible* if it has no non zero proper submodule and it is said to be *indecomposable* if it cannot be written as $V = U \oplus W$ for some submodules U and W. The following proposition may be proved similarly to Lemma 2.11 of [39].

Proposition 3.6.4. *A P-module V is indecomposable if and only if V_n^m is an indecomposable $AP(n, m)$ module for all n, m.*

Proof. If V_n^m is indecomposable for all n, m then it is clear that V is indecomposable. To prove the converse assume that V is indecomposable but that there exists a proper $AP(n, m)$ submodule $W \subset V_n^m$. Then $AP(W)$ is a submodule of V and $AP(W)_n^m \subset V_n^m$ so $AP(W)$ is a proper submodule of V. □

The *rank* of an annular tangle T is the minimum number of strings crossed by any closed loop in the interior of T enclosing the distinguished internal disc. The *weight* of a module V is the smallest n such that V_n^n is non-zero (equivalently, the weight is the smallest n such that V_i^j with $i + j = n$ is non zero). For the rest of the section we assume P is a type III C^*-planar algebra. The *-structure on P induces a *-structure on AP where the *-operation on an annular tangle T is given by reflecting T about a circle halfway between the inner and outer discs. A *Hilbert P-module* is a P module H such that each H_n^m is a finite dimensional

Hilbert space with compatible inner products satisfying $\langle ax, y\rangle = \langle x, a^*y\rangle$ for all $x, y \in H_n^m$ and $a \in AP$. Any spherical C^*-planar algebra P is a Hilbert P-module with inner product $\langle x, y\rangle = \phi(y^*x)$.

Proposition 3.6.5. *Let V be a Hilbert P-module and $W \subset V_{n_0}^{m_0}$ is an irreducible $AP_{n_0}^{m_0}$-submodule of $V_{n_0}^{m_0}$ for some n_0, m_0. Then $AP(W)$ is an irreducible P-submodule of V.*

Proof. This may be proved in a similar way to Lemma 3.4 of [39]. For Hilbert modules irreducibility is the same as indecomposability, since if U is a proper submodule of a Hilbert module V then $U \oplus U^\perp = V$, so by Proposition 3.6.4 we just need to show that $AP(W)_n^m$ is an irreducible AP_n^m submodule for all n, m. Suppose that $AP(W)_n^m$ is not irreducible. Then there exist $x, y \in AP(W)_n^m$ with $AP(x)_n^m \perp AP(y)_n^m$. We may write $x = ax_0$ and $y = by_0$ with $x_0, y_0 \in W$ and $a, b \in AP(W)((n_0, m_0), (n, m))$. Then $AP(a^*ax_0)_{n_0}^{m_0}$ and $AP(b^*by_0)_{n_0}^{m_0}$ are orthogonal submodules of $AP(W)_{n_0}^{m_0}$. □

We may use this lemma, along with the fact that if U, W are AP_n^m invariant subspaces of V_n^m for some module V then $U \perp V$ implies that $AP(U) \perp AP(W)$, to decompose a module V into a countable orthogonal direct sum. First, suppose $k = weight(V)$ and decompose V_k^k into irreducible AP_k^k modules U_i. Then $AP(U_i) \perp AP(U_j)$ for all $i \neq j$. Taking the orthogonal complement we get a module with higher weight that k and we may continue the process, decomposing this module into irreducibles and so on. To decompose the modules V_n^m for $n \neq m$ we just use the decomposition of the V_n^n and the fact that as vector spaces $V_n^m \simeq V_k^k$ for $k = (n+m)/2$. Let $\widehat{AP}_n^m$ be the ideal in $\widehat{AP}_n^m$ generated by elements with rank less than $n + m$.

Proposition 3.6.6. *Let V be a Hilbert P-module and let W_k be the AP_n^m submodule of V_n^m spanned by the (n, m) graded pieces of all P submodules of weight less than $n + m$. Then*

$$(W_n^m)^\perp = \cap_{a \in \widehat{AP}_n^m} ker(a)$$

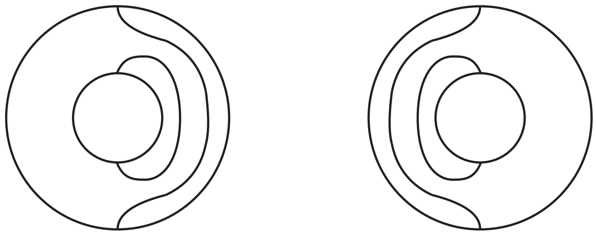


Figure 3.28: The tangles ϵ_1 and ϵ_2

Proof. First, let $w \in W_n^m$. We will show that $w \perp \cap\, ker a$. We can write $w = \sum a_i w_i$ for some annular tangles a_i. We can find tangles t_i with $t_i^* t_i a_i = a_i$ and $t_i a \in \widehat{A}P_n^m$. Let $v \in \cap ker a$ then $\langle v, a_i w_i \rangle = \langle t^* t a_i^* v, w_i \rangle = 0$ for all i.

For the opposite inclusion, suppose $w \in (W_n^m)^{\perp}$. Then we want to show $w \in \cap\, ker(a)$. For any $v \in W_n^m$ we have that $\langle aw, v \rangle = \langle w, a^* v \rangle = 0$. □

3.6.1 Temperley-Lieb modules

As in [39] we may define the Temperley-Lieb modules. The type III annular Temperley-Lieb algebra ATL is the algebra of annular tangles with no internal discs besides the distinguished one. It is easy to see that the quotient $ATL_n^m / \widehat{ATL_n^m}$ is generated by the rotation tangle, since any tangle in TL_n^m with $n+m$ through strings must be a rotation of the identity. Define the map $\sigma_{\pm}$ on annular $(0,0)$ tangles as the map which puts a single non-contractible circle of shading $\pm$ into the interior of the tangle. The module $ATL_{\pm}^{\pm}$ is generated by the map $\sigma_{\mp}\sigma_{\pm}$ since the only possible strings in the interior of the tangle are contractible and non contractible circles, and the contractible ones may be removed by multiplying by δ. Suppose V is an irreducible Hilbert TL module of lowest weight zero. Then the maps σ act by a scalar λ with $0 \leq \lambda \leq \delta$. We prove $\lambda \leq \delta$ using the tangles ϵ_1, ϵ_2 in Figure 3.28. It is easy to see that $\delta^{-1}\epsilon_i$ is a projection and $\epsilon_1 \epsilon_2 \epsilon_1 = \lambda^2 \epsilon_1$. Taking the norm of both sides gives the required inequality. Thus the 0-weight modules are determined by the numbers λ, δ and the dimension of ATL_+^+ and ATL_-^-.

As vector spaces we know $V_n^m \simeq \mathcal{V}_k$ where $k = n+m$ and $\mathcal{V}_k$ is the type II

Temperley-Lieb modules defined in [39], [41].

The main use of Temperley-Lieb modules so far is in decomposing the bipartite graph planar algebras into irreducible Temperley-Lieb modules. In [81], [41] it is shown how to decompose the bipartite graph planar algebra for the ADE graphs into Temperley-Lieb modules and this was also used e.g. in [74] to find the subfactor planar algebra of the Haagerup subfactor inside the bipartite graph planar algebra. The type III planar algebra of a bipartite graph may be defined in a similar manner to the type II case. Basically the idea is to use the same construction as in Section 3.4.1 apart from now we do not require the existence of a connection and we do not constrain the paths on the graph to start only at $*$, now we allow them to start at any vertex. Thus the presenting map may be defined on horizontal strips by

$$Z(\cup) = \sum (\xi \cdot \alpha \cdot \tilde{\alpha} \cdot \zeta, \xi \cdot \zeta)$$

where the sum is over all paths with $|\xi| = i$, $|\alpha| = 1$, $|\zeta| = j$ and $r(\xi) = s(\alpha) = s(\zeta)$. For a strip containing a labelled rectangle with label γ the presenting map is

$$Z(x) = \sum (\xi \cdot \gamma \cdot \zeta, \xi \cdot \zeta)$$

Using the methods of the proof of Proposition 3.4.7 that this presenting map is invariant under planar isotopies. However the bipartite graph planar algebra is not in general a subfactor planar algebra, since the space P_0^0 is usually not isomorphic to $\mathbb{C}$.

The decomposition of type II planar algebras into modules gives a decomposition of P_0 into planar modules, for a type III planar algebra P. Then the P_k may be decomposed by applying annular tangles to P_0.

Chapter 4

Constructing Subfactors from Planar Algebras

In Chapter 3 we showed how to associate a planar algebra to a type III subfactor. In this chapter we would like to perform the opposite construction, that is, starting with a planar algebra P we would like to find a type III subfactor such that the planar algebra P is its subfactor planar algebra. Our construction is similar to the constructions in [28], [49], [42] for II_1 subfactors. In these papers they define alternative algebra structures on P and then take the GNS representation. The difficult thing here is proving boundedness of the representation, but this can be done using planar algebra methods. Then it is proved that the von Neumann algebras obtained form the Jones tower of a subfactor II_1 that has P as its planar algebra. In our construction we use a larger class of planar tangles and obtain a tower of type III factors. Then in Section 4.1.1 we use techniques from free probability to investigate these subfactors. We use methods similar to [2], [51], [29].

4.1 Subfactors Associated to a type III Planar Algebra

Let P be a type III subfactor planar algebra. Let $P_{n+t}^{m+t} =: P_{n,m}^{t}$, where we shall

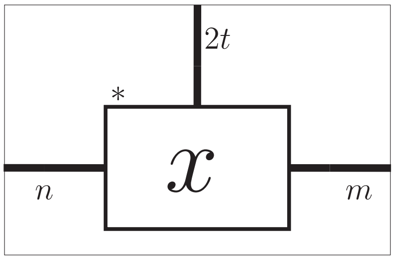


Figure 4.1: A tangle in $P^t_{n,m}$

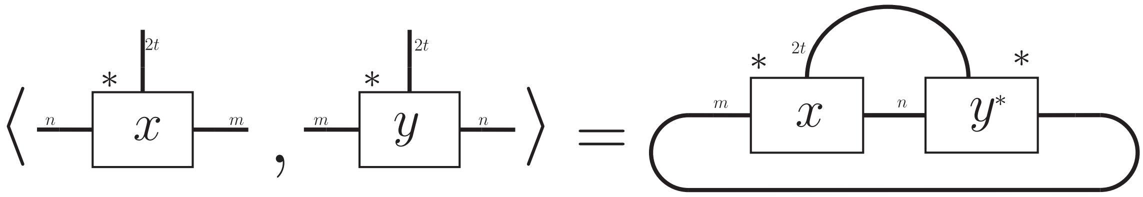


Figure 4.2: Inner product on $P^t_{n,m}$

draw elements of $P^t_{n,m}$ as rectangular boxes with n strings to the left, m strings to the right and $2t$ strings on top with the marked boundary point always at the top left corner. An example is shown in Figure 4.1. Often when drawing tangles we omit the outer boundary. Define an inclusion map $P^t_{n,m} \to P^t_{n+1,m+1}$ by adding a horizontal string underneath the tangle. Let $Gr_k(P^t_m) = \varinjlim P^t_{k+n,k+n+2m}$ and let $Gr_k(P)$ be the vector space direct sum $\oplus_{m,t} Gr_k(P^t_m)$. An element $x \in Gr_k(P)$ is a finite sum $\sum_{n,t} x_{n,t}$ where $x_{n,t} \in Gr_k(P^t_m)$. Denote by $x_t = \sum_n x_{n,t}$. Define an inner product on the $P^t_{n,m}$ by Figure 4.2.

We define an algebraic structure on the $Gr_k(P)$ as follows. For $x \in P^{t_1}_{n_1,m_1}$, $y \in P^{t_2}_{n_2,m_2}$ the multiplication $x \star_k y$ (which we usually just write as $x \star y$) is defined as in Figure 4.3. More precisely, if $m_1 = n_2$ join the vertices along the right edge of x with the corresponding vertices on the left edge of y sum over all diagrams with the last i vertices on the top edge of x joined to the first i vertices from the top of y. If $m_1 \neq n_2$, first apply the inclusion tangle $|m_1 - n_2|$ times to x if $m_1 < n_2$ or to y if $m_1 > n_2$ and then multiply as above. This multiplication is then extended bilinearly to all of $Gr_k(P)$.

This multiplication is associative because $x \star (y \star z)$ and $(x \star y) \star z$ are both

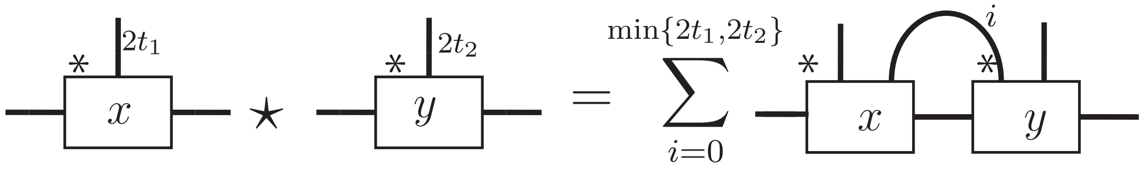


Figure 4.3: Multiplication in $Gr_k(P)$

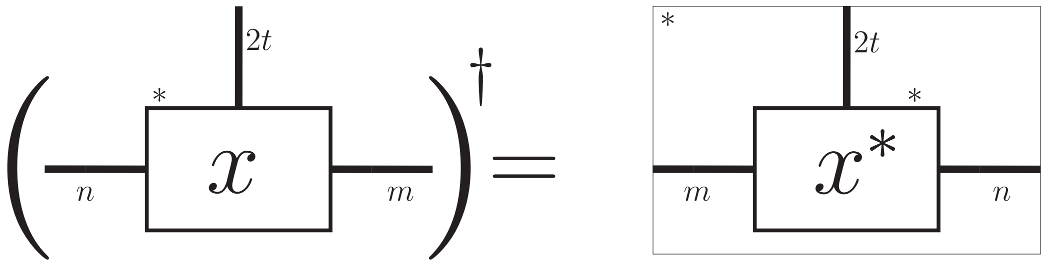


Figure 4.4: Adjoint in $Gr_k(P)$

sums over all ways of joining the top vertices of x, y and z. The multiplicitive unit I_k in Gr_k is just the usual unit in P, that is, I_k is the diagram with k horizontal strings and no other strings.

The adjoint $x^\dagger$ is obtained by reflecting x about a vertical line through its centre and moving the marked boundary point to the top left of the resulting tangle. Drawing the diagrams, it is easy to see that $(x \star y)^\dagger = y^\dagger \star x^\dagger$. Note that for $x \in P^t_{n,m}$ this adjoint operation is the composition of the adjoint in P with a rotation of the marked point $*$ clockwise by $2t$ strings.

Thus with the multiplication and *-operation defined above $Gr_k(P)$ is an associative unital *-algebra. Note that for $t = 0$ these are the ordinary adjoint and multiplication in P defined in Section 3.4.

The algebra $Gr_k(P)$ is included in $Gr_{k+1}(P)$ using the inclusion map which takes a diagram in $Gr_k(P)$ and adds and extra horizontal string below. Thus we have an increasing sequence $Gr_0(P) \subset Gr_1(P) \subset \ldots$ of *-algebras.

There is a state φ_k on $Gr_k(P)$ defined by $\varphi_k(x) := \phi(x_0)$ for $x \in Gr_k(P)$, where ϕ is the state on the planar algebra P defined by composition of the projection onto P_0 with the trace. Note also the tower of algebras constructed in [49] and [42] is contained in the tower of algebras defined above, by restricting to tangles

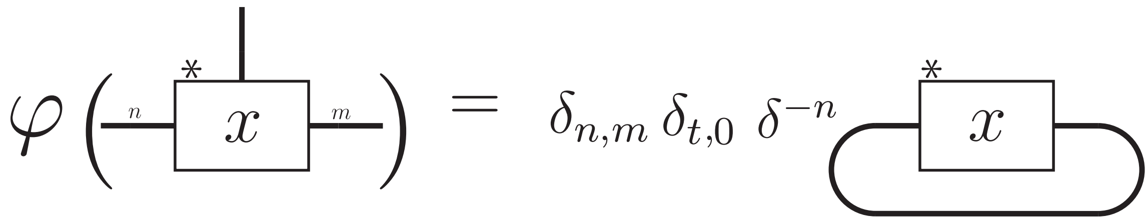


Figure 4.5: The state φ

with exactly k strings to the right and left and our state φ_k restricts to their trace. There is a conditional expectation map $E_k : Gr_{k+1} \to Gr_k$ defined as the tangle with the bottom string on the left and right side joined. It can be seen diagramatically that this satisfies $E_k(x_1 y x_2) = x_1 E_k(y) x_2$ for all $x_i \in Gr_k(P)$ and $y \in Gr_{k+1}(P)$. Note that this also satisfies $\varphi_k \circ E_k = \varphi_{k+1}$. The inner product on $Gr_k(P)$ is defined as usual as $\langle x, y\rangle_k = \varphi_k(y^\dagger \star x)$. Note that in this inner product $Gr_k(P_n^t) \perp Gr_k(P_{n'}^{t'})$ for any $(n,t) \neq (n',t')$. Let H_k denote the Hilbert space completion of $Gr_k(P)$ with respect to this inner product. A vector $x \in H_k$ is a sum $\sum_{n,t} x_{n,t}$ with $x_{n,t} \in Gr_k(P_n^t)$ and $\sum_{n,t} \|x_{n,t}\|^2 < \infty$.

We wish to show that $Gr_k(P)$ acts on the Hilbert space H_k by left multiplication and that $Gr_k(P) \subset B(H_k)$. We prove this in a similar way to [49].

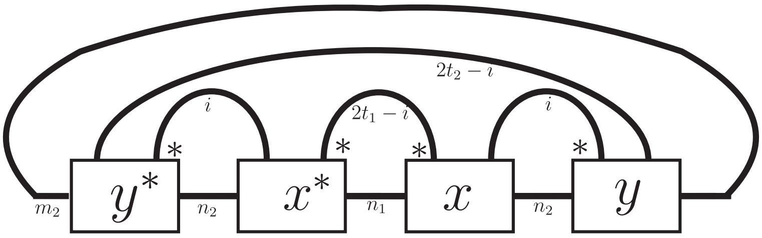
Proposition 4.1.1. *Let $k \in \mathbb{N}$ and let $x \in P_{n_1,m_1}^{t_1} \subset Gr_k(P)$. Then there exists a constant $C > 0$ such that $\|x \star y\|_{H_k} \leq C\|y\|_{H_k}$ for all $y \in P_{n_2,m_2}^{t_2} \subset Gr_k(P)$,*

Proof. We prove this as follows. Firstly we redraw the tangle $\|x \star y\|^2$ as shown in figure 4.6. Then we may replace the left and right hand sides of the tangle with positive elements u^*u and v^*v of the planar algebra. We do this in such a way that we can calculate the norm of u in terms of x and v in terms of y. Then the tangle may be written as the inner product $\langle u^*u, v^*v\rangle$. Using the Cauchy-Schwarz inequality we can then prove the boundedness of the left multiplication.

We may suppose that $m_1 = n_2$, otherwise we may add horizontal strings to x or y without changing its norm in H_k. Then $\|x \star y\|_{H_k}^2$ is shown in Figure 4.6.

Assume that $n_2 \geq m_1$ (otherwise just add $n_2 - m_1$ closed discs around the outside of the tangle and multiply by $\delta^{-(n_2-m_1)}$). Then, for each i we can replace

$$\|x \star y\|_{H_k}^2 = \sum_{i=0}^{\min\{2t_1, 2t_2\}} \delta^{-m_2}$$



$$= \sum_{i=0}^{\min\{2t_1, 2t_2\}} \delta^{-m_2}$$

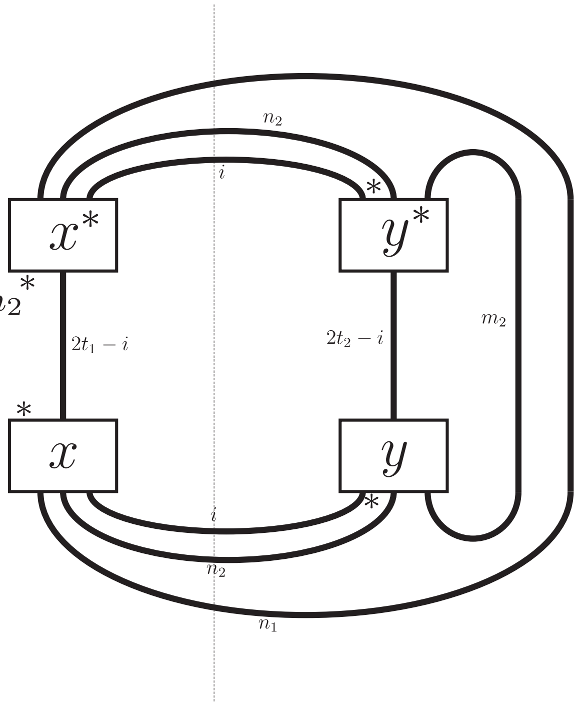


Figure 4.6: $\|x \star y\|_{H_k}^2$

the left and right hand sides of the tangle in Figure 4.6 by positive elements $u_i^* u_i$ and $v_i^* v_i$ in $P_{m_1+n_1+i}^{m_1+n_1+i}$ as described below.

First, we can redraw the left hand side of Figure 4.6 as in Figure 4.7.

In case (i), where $m_1 + n_1 + i = 2t_1 + i + 2d$ for $d \geq 0$, we can take u_i to be the tangle shown inside the dotted line. In the case $m_1 + n_1 + i = 2t_1 + i - 2d$ for some $d > 0$, the tangle in (ii) of Figure 4.7 is $E_d(x'x'^*)$, for x' as defined in Figure 4.8. This is positive, since the tangle $x'x'^*$ is a positive element of the type II planar algebra and $E_d : P_{2t_1-i-d}^{2t_1-i-d} \to P_{2t_1-i-2d}^{2t_1-i-2d}$ is the (right) conditional expectation in the type II planar algebra which is known to be positive. Hence

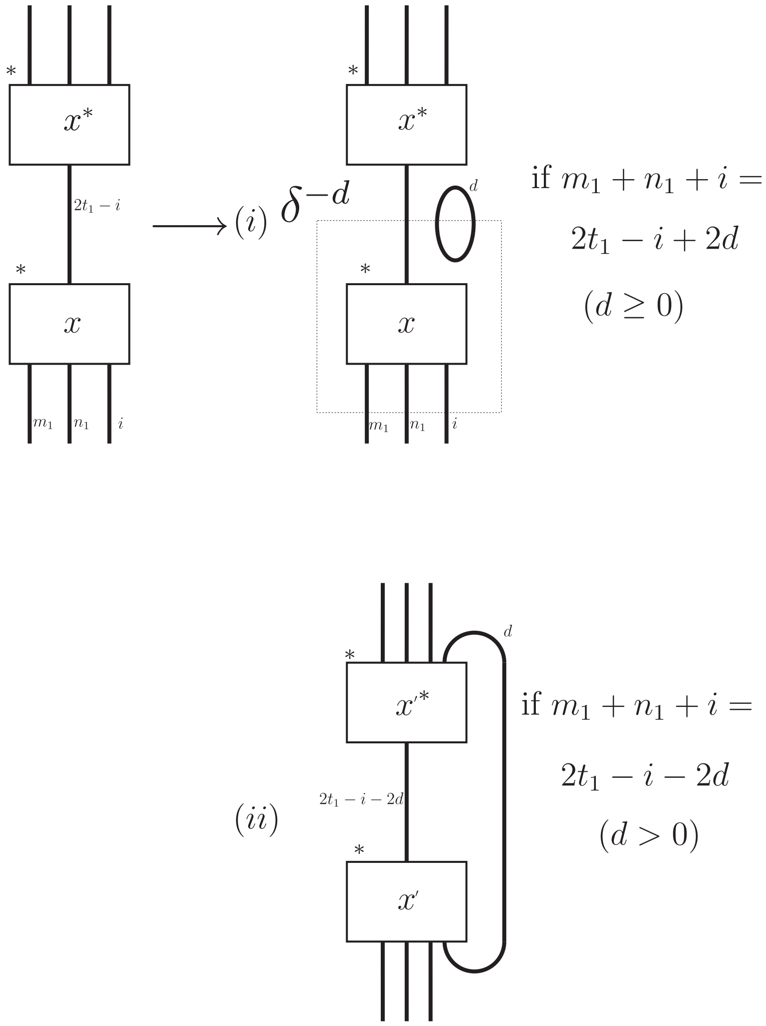


Figure 4.7: Replacing LHS of $\|(x \star y)_i\|^2_{H_k}$ with a positive tangle

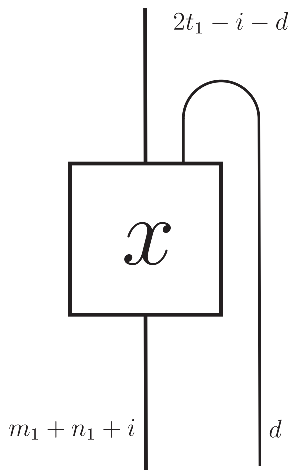


Figure 4.8: The tangle x'

we may choose u_i to be its unique positive square root in the type II subfactor planar algebra P_0.

Next we show how we may replace the right hand side by a positive element of P_0. First flip y about a vertical axis and rotate the marked point of y as shown in Figure 4.9. Denote by $\tilde{y}$ the flipped and rotated tangle obtained from y. The tangle $\tilde{y}_i^*\tilde{y}_i$ can be replaced by a positive $\tilde{v}_i^*\tilde{v}_i$ for $v_i \in P_0$ using a similar procedure to the replacement of x^*x by $u_i^*u_i$ described in the previous paragraph.

Then, since $\tilde{v}_i^*\tilde{v}_i$ is positive, $E_L(\tilde{v}_i^*\tilde{v}_i)$ must be positive, where E_L is the left conditional expectation. Therefore $E_L(\tilde{v}_i^*\tilde{v}_i)$ has a unique positive square root $\overline{v}_i \in P_0$.

Flip $\overline{v}_i$ about a vertical axis and add $n_1 - m_2$ strings to the left. Call this v_i. Then $v_i^*v_i$ is equal to the left hand side of the last tangle in Figure 4.6.

Returning to the expression for $\|x \star y\|_{H_k}^2$ in Figure 4.6 we use u_i, v_i defined above to get the tangle shown in Figure 4.11. This is $\langle u_i^*u_i, v_i^*v_i\rangle$ where we are using the inner product in P_{n+m+i}^{n+m+i}.

Denote by $\|\cdot\|_n$ the inner product norm in P_n^n. It is clear from the diagrams that $\|u\|_{n_1+m_1+i} = \delta^{-(n_1+m_1+i)}tr(x^*x)$, where tr is the non-normalised trace on $P_{n_1+m_1+i}^{n_1+m_1+i}$, and $\|v_i\|_{n_1+m_1+i} = \delta^{-(n_1+m_1+i)}\delta^{m_2}\delta^{n_1}\|y\|_{H_k}$. Thus we have that

$$\begin{aligned}\|x \star y\|_{H_k}^2 =& \| \sum_{i=0}^{\min\{2t_1,2t_2\}} (x \star y)_i\|_{H_k}^2 \\ \leq& M\sum_i \|(x \star y)_i\|_{H_k}^2 \\ =& M\sum_i \delta^{-m_2}\delta^{n_1+m_1+i}\langle u_iu_i^*, v_i^*v_i\rangle \\ \leq& M\sum_i \delta^{-m_2}\delta^{n_1+m_1+i}\|u_i\|^2\|v_i\|^2 \\ \leq& M\sum_i \delta^{n_1}\delta^{-(n_1+m_1+i)}tr(x^*x)\|y\|_{H_k}^2\end{aligned} \tag{4.1}$$

where $M = (1 + \min\{2t_1, 2t_2\})$. Hence the result follows letting

$$C = \sum_{i=0}^{2t_1} \delta^{-(m_1+i)}(1 + 2t_1)tr(x^*x), \tag{4.2}$$

which does not depend on y.

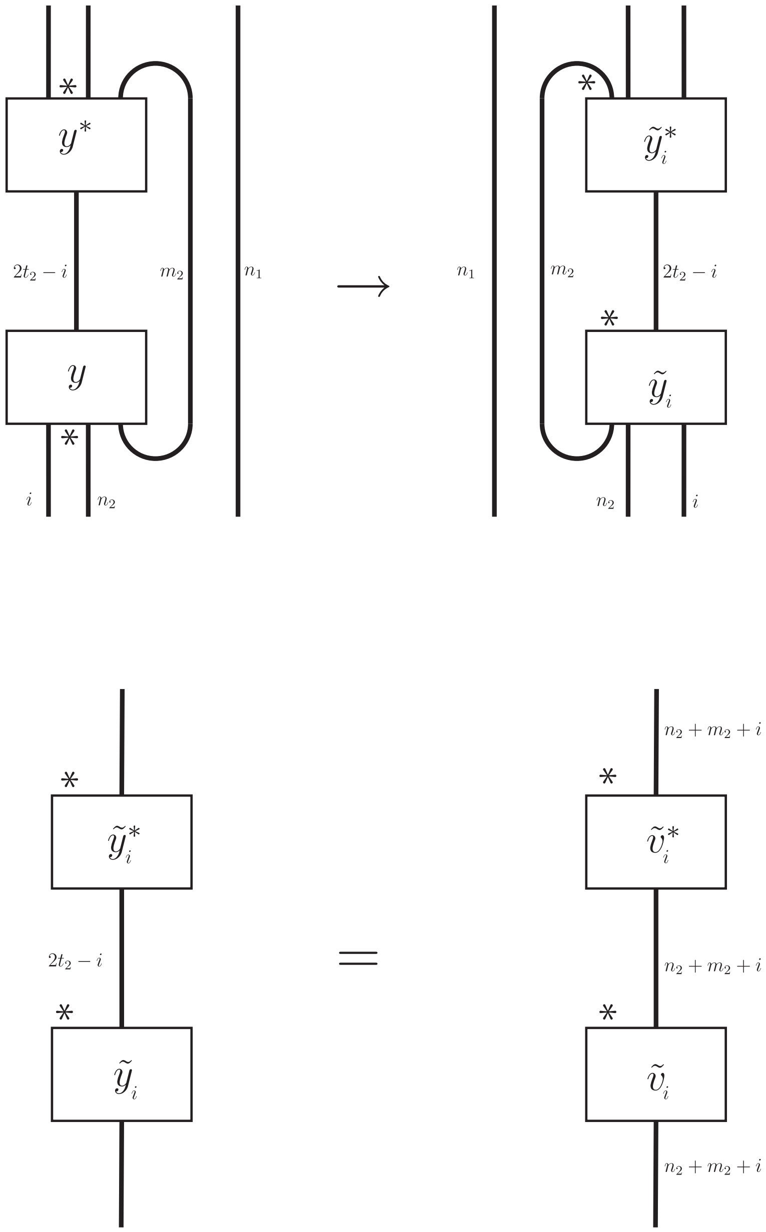


Figure 4.9: Replacing RHS of $\|(x \star y)_i\|^2_{H_k}$ with a positive tangle I

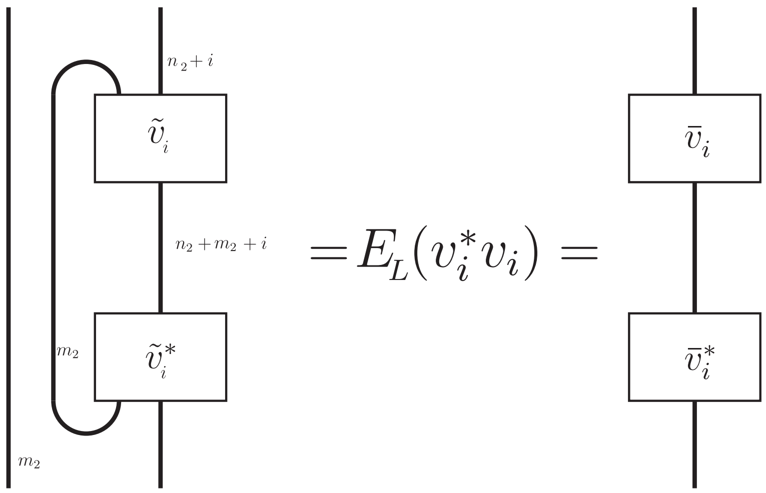


Figure 4.10: Replacing RHS of $\|(x \star y)_i\|^2_{H_k}$ with a positive tangle II

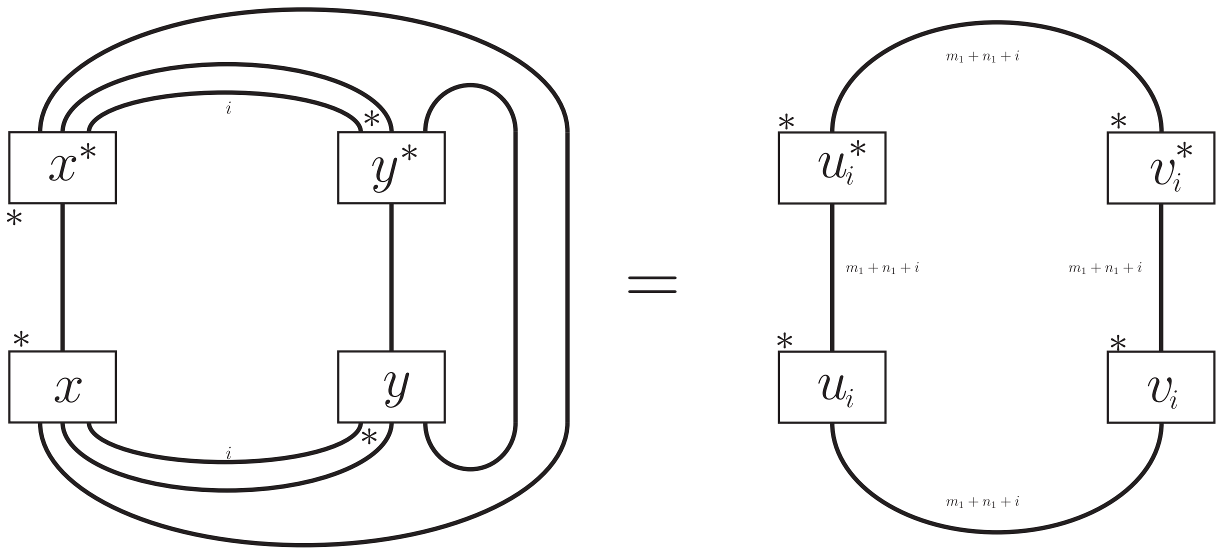


Figure 4.11: $\|(x \star y)_i\|^2_{H_k} = \|u_i v_i\|^2_{P^{n+m+i}_{m+n+1}}$

□

Proposition 4.1.2. *Let $x \in P^t_{n,m} \subset Gr_k(P)$, then there exists a constant $D < \infty$ such that $\|x \star y\|_{H_k} \leq D\|y\|_{H_k}$ for all $y \in Gr_k(P)$.*

Proof. Let $k \in \mathbb{N}$ and let $y \in Gr_k(P)$. Then y is a finite sum of elements $\sum y_{n,t}$ with $y_{n,t} \in Gr_k(P^t_n)$ and

$$\begin{aligned}
\|x \star y\|^2_{H_k} =& \| \sum_{n,t} x \star y_{n,t}\|^2_{H_k} \\
=& \sum_n \| \sum_t x \star y_{n,t}\|^2_{H_k} \\
\leq& \sum_n (2t_1 + 1) \sum_t \|x \star y_{n,t}\|^2_{H_k} \\
\leq& (2t_1 + 1)C \sum \|y_{n,t}\|^2_{H_k} = (2t_1 + 1)C\|y\|^2_{H_k}
\end{aligned} \tag{4.3}$$

with C as in Equation 4.2. □

Then propositions 4.1.1, 4.1.2 show that left multiplication by an element of $Gr_k(P)$ defines a bounded operator on a dense subspace of H_k and hence may be extended to an element of $B(H_k)$. Consequently, $Gr_k(P)$ may be thought of as a *-subalgebra of $B(H_k)$. Let $\mathcal{M}_k(P)$ be the weak completion of $Gr_k(P)$ in $B(H_k)$.

As in the type II case we now prove that the $\mathcal{M}_k$'s form an increasing sequence of von Neumann algebras and that

1. $\mathcal{M}_k$ is a type III factor
2. $\mathcal{M}_0 \subset \mathcal{M}_1$ is an extremal subfactor with index δ
3. $\mathcal{M}_0' \cap \mathcal{M}_k = P_{k,k}$ for all k
4. $\mathcal{M}_0 \subset \mathcal{M}_1 \subset \mathcal{M}_2 \subset \ldots$ is the basic construction
5. $\mathcal{M}_i' \cap \mathcal{M}_k$ is the subalgebra of $P_{k,k}$ generated by tangles with i vertical strings to the left for $0 \leq i \leq k$.

Proposition 4.1.3. *The inclusion of $Gr_k(P) \subset Gr_{k+1}(P)$ extends to an inclusion of their weak completions $\mathcal{M}_k(P) \subset \mathcal{M}_{k+1}(P)$.*

The proof of this Proposition follows from Lemma 4.1 of [33] which we state here.

Lemma 4.1.4. *Let A be a C^*-algebra with a state ϕ and let $(\pi_\phi, H_\phi, \Omega_\phi)$ be the GNS triple. Suppose also that Ω_ϕ is separating for $\pi_\phi(A)''$. Let B be a unital *-subalgebra of A such that ψ is the restriction of ϕ to B with GNS triple $(\pi_\psi, H_\psi, \Omega_\psi)$. Then there exists a *-isomorphism $\Phi : \pi_\phi(B)'' \to \pi_\psi(B)''$ with $\Phi(\pi_\phi(x)) = \pi_\psi(x)$ for all $x \in B$.*

To show that $\mathcal{M}_k$ is a type III von Neumann algebra we calculate the Connes spectrum. We do this in exactly the same way as the analagous calculation for $\mathcal{M}_{TL}$ in Section 3.1. Define an action σ_s^k of $\mathbb{R}$ on $\mathcal{M}_k$ by the linear extension of the map $\sigma_s(x) = e^{i(n-m)s\ln\delta}x$, $x \in P_{n,m}^t$. The state ϕ_k is the KMS state for the modular automorphism group σ_s^k at inverse temperature $\beta = 1$. The fixed point subalgebra of σ_s is the von Neumann algebra generated by $Gr_k(P_0)$. It was shown in [49] that this is a type II_1 factor and so the Connes spectrum and Arveson spectrum coincide and equal $\{\lambda^{\mathbb{Z}}\}$ for $\lambda = 1/\delta$. Hence $\mathcal{M}_k$ is type III_λ.

To prove that $\mathcal{M}_0' \cap \mathcal{M}_k = P_{k,k}$, we use a similar method to the one used by Kodiyalam and Sunder in [49]. Their proof shows that $x \in \mathcal{M}_0' \cap \mathcal{M}_k$ implies that $x_{0,t} = 0$ for all $t > 0$, we extend this to show that $x_{n,t} = 0$ unless $n = t = 0$.

First we need some notation. Let $T((m,n), \{a_1, \ldots, a_{2t}\}, \{b_1, \ldots, b_{2t}\})_{s'}^s$, $m, n, a_i, b_i, s, s', t \in \mathbb{N}$, $m+n \in 2\mathbb{N}$, $a_1, b_1 \in 2\mathbb{N}+1$, be the annular tangle shown in Figure 4.12, with $2s+m+n$ marked points along the outer boundary and $2s'+m+n$ along the inner boundary. The last $m+n$ points along the inner boundary are joined by a string to the corresponding point on the inner boundary as shown. The points $2a_1+1, \ldots, 2a_t$ are joined to the points $2b_1+1, \ldots, 2b_t$. The remaining marked points are capped off in pairs with no nested caps.

Let $C_{n,m}^t \subset P_{n,m}^t$ be the subspace generated by tangles with t cups along the top edge. Let $C_k \subset \mathcal{M}_k(P)$ be the subalgebra generated by the (weak) closure of $\oplus_{\substack{m\in\mathbb{Z}\\ t\in\mathbb{N}}} C_{k,m,t}$, where $C_{k,m,t} := \varinjlim C_{k+n,k+n+m}^t$. Let $C^\perp$ be the orthogonal complement of C in $Gr_k(P)$ with respect to the inner product $\langle\cdot,\cdot\rangle_k$. Now we give analogues of Lemma 5.6-Lemma 5.8 in [49].

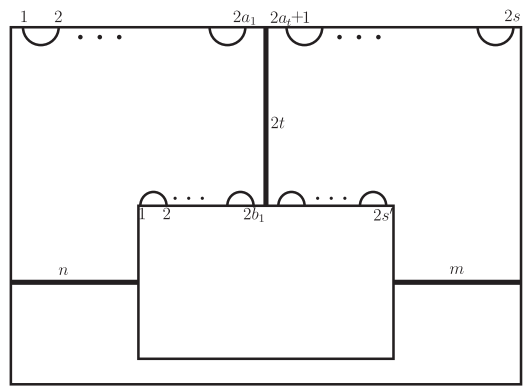


Figure 4.12: $T((m,n),\{a_1,\dots,a_{2t}\},\{b_1,\dots,b_{2t}\})^s_{s'}$

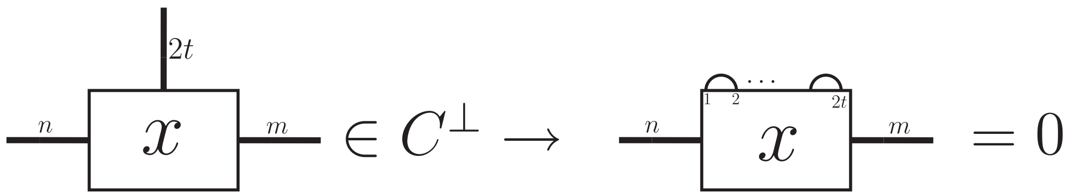


Figure 4.13: $x \in C^{\perp}$

Lemma 4.1.5. *Let $x \in P^t_{m,m+n}$ and suppose that $x \in C^{\perp}$. Then the map $x \to z = ([x_{n,t}, \cup])_{t+1}$ is injective with inverse given by*

$$x = \sum_{s=}^{t} \delta^{-s} Z_{T((m,n),\{1,\dots,2(t-s+1)\},\{2s+1,\dots,2(t+1)\})^t_{t+1}}(z) \tag{4.4}$$

Proof. The condition $x \in C^{\perp}$ is equivalent to the condition in Figure 4.13 for all n, t.

If $x \in C^{\perp}$, let $z = ([x_{n,t}, \cup])_{t+1} \in P^{t+1}_{m,m+n}$, as shown in Figure 4.14.

It is easy to see that the second equation of Figure 4.14 must hold, by applying the tangle shown in Figure 4.15 to the equation above it. Then, summing over all k, most terms on the right hand side cancel and we get the equation of Figure 4.16.

Now, cap off points i and $i+1$ for $i \in \{1, 3, \dots, 2t-1\}$ along the top. The second term on the right is zero and the other terms give Equation 4.4 □

Lemma 4.1.6. *Let $x = \sum x_{n,t} \in C^{\perp}$ with $x_{n,t} \in Gr_k(P^t_n)$ for all n, t and suppose*

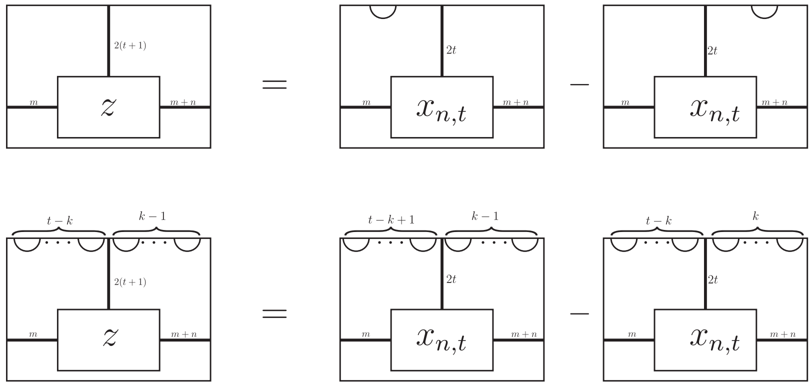


Figure 4.14: $z = ([x_{n,t}, \cup])_{t+1}$

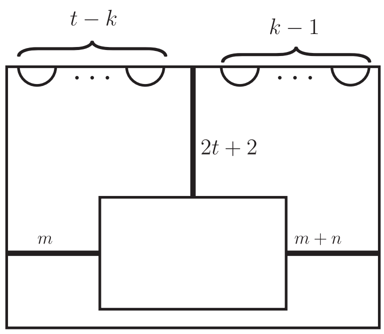


Figure 4.15: Annular tangle

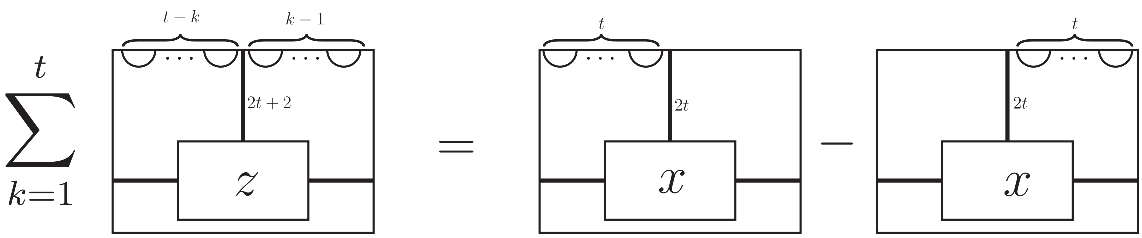


Figure 4.16: Summing over k

$[x, \cup] = 0$. *Then*

$$x_{n,t} = \sum_{s=1}^{t} \delta^{-s+d-1} \Big(Z_{T((n,m),\{1,\ldots,(t-s+1)\},\{2(s+d)-1,\ldots,2(t+d)\})^t_{t'}}(x_{n,s}) - Z_{T((n,m),\{1,\ldots,2(t-s+1)\},\{2(s+d)+1,\ldots,2(t+d+1)\})^t_{t'}}(x_{n,s}) \Big)$$

where $d = t' - t$ *and for each* n*, we choose* m *so that* $x_{n,t} \in P^t_{m,m+n} \subset Gr_k(P)$*.*

Proof. We prove this in exactly the same way as the proof of Corollary 5.7 of

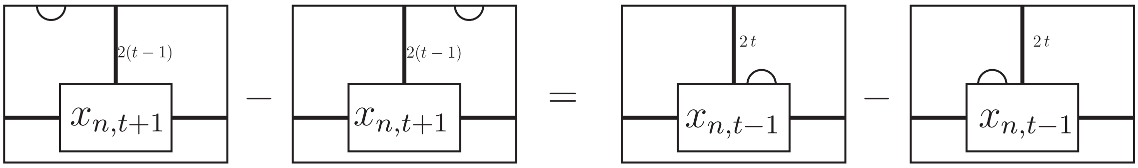


Figure 4.17: $([\sum_{n,t} x_{n,t}, \cup])_s$

[49]. We begin by proving the $d = 1$ case and then prove the general case by induction on d. Since $[x, \cup] = 0$ we must have $([\sum_{n,t} x_{n,t}, \cup])_s = \oplus_n(([x_{n,s-1}, \cup])_s + ([x_{n,s+1}, \cup])_s) = 0$ for all s. Thus we have the equation of Figure 4.17.

The right hand side of Figure 4.17 is equal to

$$Z_{T((m,n),\{1,\dots,2t\},\{1,\dots,2t\})^t_{t+1}}(x_{n,t+1}) - Z_{T(n+m,\{1,\dots,2t\},\{3,\dots,2(t+1)\})^t_{t+1}}(x_{n,t+1})$$

The left hand side is equal to $([x_{n,t-1}, \cup])_t$ and so by Lemma 4.1.5 we have

$$\begin{aligned} x_{n,t-1} =& \sum_{s=0}^{t} \delta^{-s} Z_{T((m,n),\{1,\dots,2(t-s)\},\{2s+1,\dots,2t\})^{t-1}_t}(Z_{T((m,n),\{1,\dots,2t\},\{1,\dots 2t\})^t_{t+1}}(x_{n,t+1}) \\ & - Z_{T((m,n),\{1,\dots,2t\},\{3,\dots,2(t+1)\})^t_{t+1}}(x_{n,t+1})) \\ =& \sum_{s=1}^{t-1} \delta^{-s}(Z_{T((m,n),\{1,\dots,2(t-s)\},\{2s+1,\dots,2t\})^{t-1}_{t+1}}(x_{n,t+1}) \\ & - Z_{T((m,n),\{1,\dots,2(t-s)\},\{2s+3,\dots,2(t+1)\})^{t-1}_{t+1}}(x_{n,t+1})) \end{aligned} \tag{4.5}$$

Thus the lemma is true for $d = 1$. Next assume inductively that

$$\begin{aligned} x_{n,t+1} = \sum_{r=1}^{t+1} \delta^{-r+d-1} & Z_{T((m,n),\{1,\dots,2(t+2-r)\},\{2(r+d)-1,\dots,2(t+d+1)\})^{t+1}_{t'}}(x_{n,t'}) \\ & - Z_{T((m,n),\{1,\dots,2(t+2-r)\},\{2(r+d)+1,\dots,2(t+d+2)\})^{t+2}_{t'}}(x_{n,t'}) \end{aligned}$$

Substituting this into equation 4.5 we get

$$x_{n,t-1} = \sum_{s=1}^{t-1}\sum_{r=1}^{t+1} \delta^{-r-s+d-1}(Z_{T((m,n),\{1,\ldots,2(t-s)\},\{2s+1,\ldots,2t\})_{t+1}^{t-1}}$$

$$(Z_{T((m,n),\{1,\ldots,2(t+2-r)\},\{2(r+d)-1,\ldots,2(t+d+1)\})_{t'}^{t+1}}(x_{n,t'})$$

$$- Z_{T((m,n),\{1,\ldots,2(t+2-r)\},\{2(r+d)+1,\ldots,2(t+d+2)\})_{t'}^{t+1}}(x_{n,t'}))$$

$$- Z_{T((m,n),\{1,\ldots,2(t-s)\},\{2s+3,\ldots,2(t+1)\})_{t+1}^{t-1}}$$

$$(Z_{T((m,n),\{1,\ldots,2(t+2-r)\},\{2(r+d)-1,\ldots,2(t+d+1)\})_{t'}^{t+1}}(x_{n,t'})$$

$$- Z_{T((m,n),\{1,\ldots,2(t+2-r)\},\{2(r+d)+1,\ldots,2(t+d+2)\})_{t'}^{t+2}}(x_{n,t'})))$$

$$= \sum_{s=1}^{t-1} \delta^{-s+d-1}(Z_{T((m,n),\{1,\ldots,(t-s)\},\{2(s+d)-1,\ldots,2(t-1+d)\})_{t'}^{t-1}}(x_{n,t'})$$

$$- Z_{T((m,n),\{1,\ldots,2(t-s)\},\{2(s+d)+1,\ldots,2(t+d)\})_{t'}^{t-1}}(x_{n,t'}))$$

$$+ \sum_{s,r,s+r<t+2} \delta^{-r+d-s}(Z_{T((m,n),\{1,\ldots,2(t+2-r-s)\},\{2(r+d+s)-1,\ldots,2(t+d+1)\})_{t'}^{t-1}}(x_{n,t'})$$

$$- Z_{T((m,n),\{1,\ldots,2(t+2-r-s)\},\{2(r+d+s)+1,\ldots,2(t+d+2)\})_{t'}^{t-1}}(x_{n,t'}))$$

$$+ \sum_{r,s,r+s<t+1} \delta^{-r-s+d-1}(-Z_{T((m,n),\{1,\ldots,2(t+1-r-s)\},\{2(r+d+s)+1,\ldots,2(t+d+1)\})_{t'}^{t-1}}(x_{n,t'})$$

$$+ Z_{T((m,n),\{1,\ldots,2(t+1-r-s)\},\{2(r+d+s)+3,\ldots,2(t+d+2)\})_{t'}^{t-1}}(x_{n,t'}))$$

The first sum gives the required expression for $x_{n,t-1}$ while the r, s term of the second sum cancels with the $r-1, s$ term of the third sum. □

Lemma 4.1.7. *Let $x \in P_{n,m}^t \subset Gr_k(P)$ and let*

$$y = Z_{T((m,n),\{a,\ldots,a+2k\},\{b,\ldots,b+2k\})_t^{t'}}(x) \in P_{n,m}^{t'} \subset Gr_k(P).$$

Then $\|y\|_{H_k}^2 \leq \delta^{t+t'-2k}\|x\|_{H_k}^2$.

Proof. This may be proved similarly to Lemma 5.8 in [49] The norm $\|y\|_{H_k}^2$ is $\delta^{-n}Z_T$ where T is the tangle in Figure 4.18.

We can remove the $t'-k$ closed loops by multiplying by $\delta^{t'-k}$ and the remaining tangle is $\|x'\|_P$ where x' is the tangle obtained from $x \in P$ by applying the appropriate rotations and $t-k$ conditional expectation operators and $\|\cdot\|_P$ is the

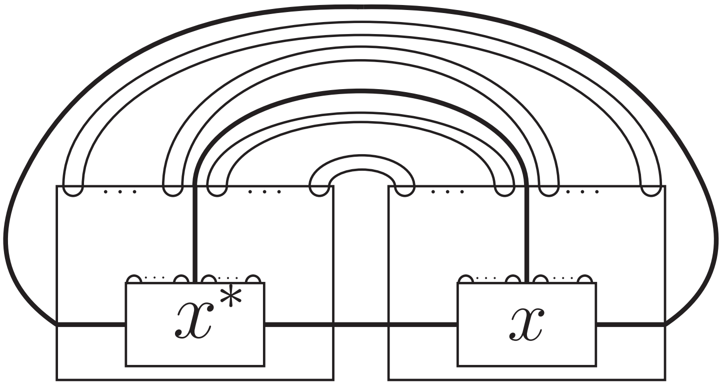


Figure 4.18: $\|y\|^2_{H_k}$

norm in P defined using the state ϕ. Since the rotation operator has norm one and the conditional expectation has norm δ we get $\|y\|^2_{H_k} \leq \delta^{t'-k}\delta^{t-k}\|x\|^2_{H_k}$. $\square$

The following two propositions are analogues of Proposition 5.4 and 5.5 in [49].

Proposition 4.1.8. *Let $x = \sum_{t,n} x_{n,t} \in \mathcal{M}_k$ with $x_{n,t} \in Gr_k(P^t_n)$. Suppose x commutes with the image of $\cup \in O_{TL}$ in $Gr_k(P)$. Then $x \in C$.*

Proof. Suppose $x = \sum x_{n,t} \in C^\perp$ satisfies $[x, \cup] = 0$. Then, by Lemma 4.1.6, we have

$$x_{n,t} = \sum_{s=1}^{t} \delta^{-s+d-1}(Z_{T(n+m,\{1,\ldots,(t-s+1)\},\{2(s+d)-1,\ldots,2(t+d)\})^t_{t'}}(x_{n,s})$$
$$- Z_{T(n+m,\{1,\ldots,2(t-s+1)\},\{2(s+d)+1,\ldots,2(t+d+1)\})^t_{t'}}(x_{n,s}))$$

Hence by Lemma 4.1.7 we have that $\|x_{n,t}\|^2 \leq \sum_{s=1}^{t-1} \delta^{-s+d-1}\delta^{t+t'-(t-s)}\|x_{n,s}\|^2$ for all $s > t$ and hence, since $\|x_{n,s}\| \to 0$ as $s \to \infty$ we must have $x_{n,t} = 0$ for all n and t.

$\square$

Let $\Cup$ be the O_{TL} tangle with exactly two nested cups along its top edge and no other strings.

Proposition 4.1.9. *Suppose $x \in Gr_k(P)$ commutes with the image in $Gr_k(P)$ of both $\cup$ and $\Cup$. Then $x \in P \subset Gr_k(P)$.*

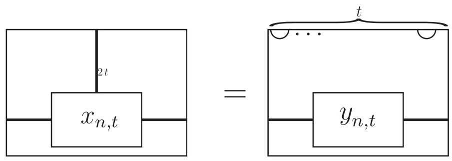


Figure 4.19: Relation between $x_{n,t}$ and $y_{n,t}$

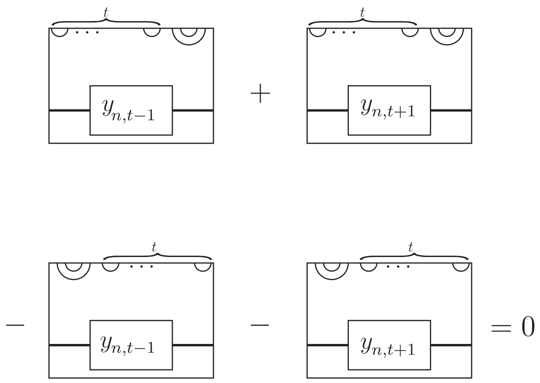


Figure 4.20: $\sum_{s=t-1}^{t+1}([x_{n,s}, \Cup])_t = 0$

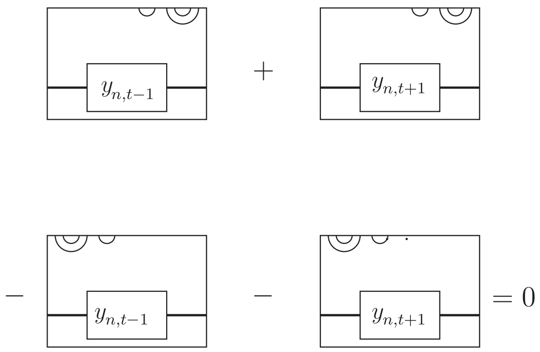


Figure 4.21: Capping off the equation $\sum_{s=t-1}^{t+1}([x_{n,s}, \Cup])_t = 0$

Proof. We know from above that if $x \in Gr_k(P)$ commutes with $\cup$ then $x \in C$. For any $x = \sum_{n,t} x_{n,t} \in C$, then each $x_{n,t}$ can be written as $y_{n,t}$ with t cups along the top as shown in Figure 4.19.

If the commutant $[x, \Cup] = 0$ then for each n and t we must have $\sum_{s=t-1}^{t+1}([x_{n,s}, \Cup])_t = 0$ which is equivalent to the equation in Figure 4.20

If $t > 0$ we may repeatedly cap off the 4^{th} and 5^{th} marked points along the top to get the equation of Figure 4.21

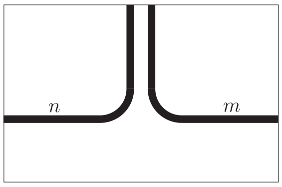


Figure 4.22: The tangle $\theta_{n,m}$

Then capping off points 1 and 2, 3 and 6 and 4 and 5 gives the equation $(\delta^3 - \delta)(y_{n,t} + y_{n,t-1}) = 0$ and since $\delta > 1$ this means that $y_{n,2t} = -y_{n,2t+1} =: y_n$ for all t. Hence

$$\|x\|^2 = \sum \|x_{n,t}\|^2 = \sum \|x_{n,0}\|^2 + \sum_n \|y_n\|^2(\delta + \delta^2 + \delta^3 + \dots)$$

and since we know that $\|x\|$ is finite and $\delta > 1$ we must have $y_n = 0$ for all n. Thus $x = \sum x_{n,0}$ i.e. $x \in P \subset Gr_k(P)$. $\square$

Proposition 4.1.10. *Let $x \in \mathcal{M}_0' \cap \mathcal{M}_k$. Then $x \in P_{k,k}$.*

Proof. By Propositions 4.1.8 and 4.1.9 we have that $\mathcal{M}_0' \cap \mathcal{M}_k \subset P \subset Gr_k(P)$. Let $y \in P \subset Gr_k(P)$, i.e. $y \in Gr_k(P_n^0)$. Then, if y commutes with the element $\theta_{n,m}$ defined in Figure 4.22 for all $n, m \in \mathbb{N}$, then y must be in P_k^k. Hence $\mathcal{M}_0' \cap \mathcal{M}_k \subset P_k^k$. When we embed $\mathcal{M}_0 \subset \mathcal{M}_k$ we do so by adding k strings along the bottom, hence any tangle living on the bottom k strings will commute with everything in $\mathcal{M}_0$. Hence $\mathcal{M}_0' \cap \mathcal{M}_k = P_{k,k}$. $\square$

We now show that $\mathcal{M}_0 \subset \mathcal{M}_1 \subset \mathcal{M}_2 \subset \dots$ is the basic construction. The conditional expectation $E_{\mathcal{M}_i} : \mathcal{M}_{i+1} \to \mathcal{M}_i$ is the restriction to $\mathcal{M}_{i+1}$ of the extension to $H_{i+1} \to H_i$ of $E_i : Gr_{i+1}(P) \to Gr_i$. This follows since we can identify H_i with the GNS Hilbert space associated to $(\mathcal{M}_i, \phi_i)$ and, since H_i is included in H_{i+1} by the extension of the inclusion map from $Gr_i(P) \to Gr_{i+1}(P)$ the conditional expectation must be the extension of the diagramatic one.

Proposition 4.1.11. *For the algebras $\mathcal{M}_i$ defined above we have the following identities*

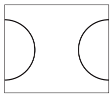
1. $E_{\mathcal{M}_1}(e) = \delta^{-2} I$

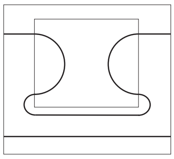
2. $\delta^2 E_{\mathcal{M}_1}(xe)e = xe$ *for all* $x \in \mathcal{M}_2$

3. $exe = E_{\mathcal{M}_0}(x)e$ *for all* $x \in \mathcal{M}_1$

where $E_{M_i} : M_i \to M_{i-1}$ *is the conditional expectation and* $e \in P^0_{2,2}$ *is the tangle defined by* $u^\dagger u$*, where* u *is as in Figure 4.24.*

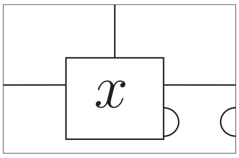
Proof. For the first equation we have

$$E_{\mathcal{M}_0}(e) = E_{\mathcal{M}_0}(\delta^{-1}$$  $$)$$

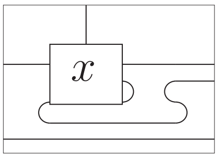
$$= \delta^{-2}$$ 

$$= \delta^{-2} I$$

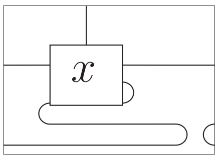
For the second equation, suppose $x \in P^t_{n,m} \subset Gr_2(P)$. Then

$$xe = \delta^{-1}$$ 

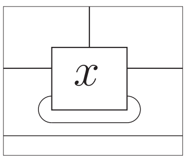
Thus

$$E_{\mathcal{M}_1}(xe) = \delta^{-2}$$ 

and

$$E_{\mathcal{M}_1}(xe)e = \delta^{-3}$$ 

Hence $\delta^2 E_{\mathcal{M}_1}(xe)e = xe$. For the third identity, suppose $x \in P^2_{n,m} \subset Gr_1(P)$. Then

$$E_{\mathcal{M}_0}(x) = \delta^{-1}$$ 

To calculate $E_{\mathcal{M}_0}(x)e$, first embed $E_{\mathcal{M}_0}(x)$ in $Gr_2(P)$ and multiply as shown.

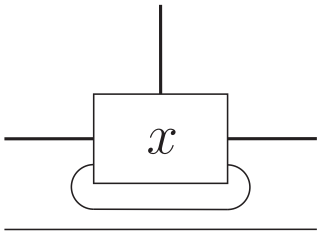
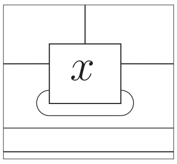
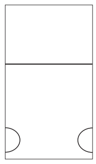
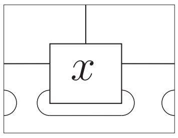
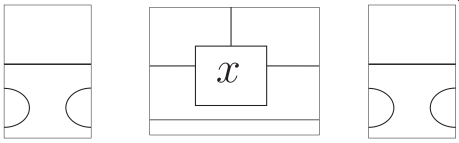


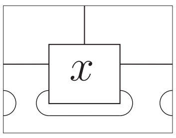
Figure 4.23: The conditional expectation E

$E_{\mathcal{M}_0}(x)e = \delta^{-2}$  

$= \delta^{-2}$ 

Similarly, to calculate exe, first embed x in $Gr_2(P)$ and multiply as follows.

$exe = \delta^{-2}$ 

$= \delta^{-2}$ 

□

Thus by Theorem 2.8.2 $\mathcal{M}_1 \subset \mathcal{M}_2$ is isomorphic to the basic extension of the subfactor $\mathcal{M}_0 \subset \mathcal{M}_1$ and this isomorphism takes e to the Jones projection. We can repeat this argument to show that $\mathcal{M}_0 \subset \mathcal{M}_1 \subset \mathcal{M}_2 \subset \ldots$ is the tower for this subfactor.

Proposition 4.1.12. *The conditional expectation $E : \mathcal{M}_1 \to \mathcal{M}_0$ is minimal.*

Proof. By Proposition 2 of [47] the minimal expectation is the unique conditional expectation satisfying $cE(x) = \sum_i u_i x u_i^*$ for $x \in \mathcal{M}_0' \cap \mathcal{M}_1$, $c > 0$ and a basis $\{u_i\}$ of the conditional expectation. The element u shown in Figure 4.24 is a basis for the conditional expectation. This is because $x = uE(u^*x)$, which is easy

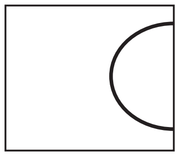


Figure 4.24: The tangle $\delta^{\frac{1}{2}}u$

to verify by drawing diagrams. For any $x \in \mathcal{M}_0' \cap \mathcal{M}_1$, then x has a horizontal strand joining the bottom endpoints on each side. In this case we have that $\delta E(x) = uxu^*$ and so E is minimal. □

Proposition 4.1.13. *Let P be a type III subfactor planar algebra and let $\mathcal{M}_0 \subset \mathcal{M}_1$ be the subfactor defined above using P. Then P is the subfactor planar algebra of $\mathcal{M}_0 \subset \mathcal{M}_1$*

Proof. Let $\mathcal{P}$ be the subfactor planar algebra associated to the subfactor $\mathcal{M}_0 \subset \mathcal{M}_1$. We need to show that $\mathcal{P}_{2n}^{2m} = ((\rho\bar{\rho})^n, (\rho\bar{\rho})^m)$ and $\mathcal{P}_{2n+1}^{2m+1} = ((\rho\bar{\rho})^n\rho, (\rho\bar{\rho})^m\rho)$. If $m = n$ the spaces are isomorphic to the relative commutant $M_0' \cap M_n$. Therefore $P_n^n \simeq \mathcal{P}_n^n$. For $m \neq n$, we know the all the intertwiner spaces are generated by the spaces with $m = n$ and a single element in $(id, \rho\bar{\rho})$. Then, by uniqueness of the planar algebra satisfying the conditions of Proposition 3.4.7 we have $\mathcal{P} \simeq P$. □

4.1.1 Graph Construction

Next we would like to identify the subfactor $\mathcal{M}_0 \subset \mathcal{M}_1$. In [29], [50], [51], [2] the authors use techniques from free probability to prove that for II_1 factors the above construction gives subfactors of interpolated free group factors. In this section we use similar methods to investigate the type III analogue of this construction.

We begin, as in [2] by associating a probability space to a finite graph G with vertices V, edges E and Perron Frobenius eigenvector $(\mu(v))_{v\in V}$ with eigenvalue δ. Let (ξ, η, ζ) be a triple of paths in the graph G such that $r(\xi) = s(\eta)$, $r(\eta) = s(\zeta)$ and $r(\zeta) = s(\xi)$. Let $V_{n,m}^t$ be the linear span of all paths (ξ, η, ζ) in G with $|\xi| = n$, $|\eta| = 2t$ and $|\zeta| = m$.

We define an inclusion map $i : V^t_{n,m} \to V^t_{n+1,m+1}$ by

$$i(\xi,\eta,\zeta) = \sum_{e\in E} (e\cdot\xi, \eta, \zeta\cdot e)$$

where as usual $\xi\cdot\eta$ means the concatenation of ξ and η and is zero if $r(\xi)\neq s(\eta)$ for any paths ξ, η. For each $k\in\mathbb{N}$ let $(V_k)^t_m = \varinjlim V^t_{k+n,k+n+m}$ and let $V_k = \bigoplus_{m\in\mathbb{Z},t\in\mathbb{N}}(V_k)^t_m$. We now define a multiplication $\#_k$ on V_k, let

$$(\xi_1,\eta_1,\zeta_1)\#_k(\xi_2,\eta_2,\zeta_2) = \begin{cases} (\xi_1, \sum_k \sigma_k, \zeta_2) & \text{if } \xi_2 = \zeta_1 \\ (\xi_1, \sum_k \sigma_k, \zeta_2\zeta_1') & \text{if } \xi_2\zeta_1' = \zeta_1 \\ (\xi_2'\xi_1, \sum_k \sigma_k, \zeta_2) & \text{if } \xi_2 = \zeta_1\xi_2' \\ 0 & \text{otherwise} \end{cases}$$

where $\xi_i = (\xi_i)_1\cdots(\xi_i)_{t_i}$ and $\sigma_i = \prod_{k=2t_1-i}^{2t_1}\frac{\mu(s((\eta_1)_k))}{\mu(r((\eta_1)_k))}(\eta_1)_1\ldots(\eta_1)_{2t_1-i}(\eta_2)_i\ldots(\eta_2)_{2t_2}$ if $(\eta_1)_{2t_1-i+j} = (\tilde{\eta}_2)_j$ for $1\leq j\leq i$ and $\sigma_i = 0$ otherwise.

Denote by $Gr_k(G)$ the algebra V_k with multiplication $\#_k$.

Define a state ψ_k on $Gr_k(G)$ by

$$\psi_k(\xi,\eta,\zeta) = \delta_{\xi,\tilde{\zeta}}\delta_{\eta,t(\xi)}\delta^{-|\xi|}\mu(r(\xi)).$$

The state ψ_k is the composition of the projection $P : V_k \to \oplus_m(V_k)^0_m$ with the trace from the type II algebra and so it is positive definite. Similarly to the calculation for $Gr_k(P)$ in the previous section it can be shown that ψ is the KMS state at inverse temperature 1 for the modular automorphism group $\sigma_t(\xi,\eta,\zeta) = e^{\alpha i\beta(|\xi|-|\zeta|)}(\xi,\zeta,\eta)$ where $\alpha = \ln\delta$.

If we think of a triple (ξ,η,ζ) as a planar diagram in the previous section with the strings on the left representing the path ξ, the strings on top representing η and those on the right representing ζ then $Gr_k(G)$ with the multiplication and state defined above is the same as the multiplication and state for $Gr_k(P)$ defined in the previous section.

We may define another algebraic structure on V_k as follows. Let $\mathcal{G}r_k(G)$ be V

with multiplication $\star_k$ and state ϕ_k defined below. Let

$$(\xi_1,\eta_1,\zeta_1)\star_k(\xi_2,\eta_2,\zeta_2)=\begin{cases}(\xi_1,\eta_1\cdot\eta_2,\zeta_2) & \text{if } \xi_2=\zeta_1\\ (\xi_1,\eta_1\cdot\eta_2,\zeta_2\zeta_1') & \text{if } \xi_2\zeta_1'=\zeta_1\\ (\xi_2'\xi_1,\eta_1\cdot\eta_2,\zeta_2) & \text{if } \xi_2=\zeta_1\xi_2'\\ 0 & \text{otherwise}\end{cases}$$

Define a state on $\mathcal{G}r_k(G)$ by

$$\phi_k(\xi,\eta,\zeta)=\delta_{\xi,\tilde{\zeta}}\sum_{\pi\in NC(2t)}\prod_{\{i,j\}\in\pi,i<j}\delta_{\eta_i,\tilde{\eta}_j}\prod_{C\in K(\pi)}(\mu(v_C^{\eta}))^{2-|C|}\delta_{\xi,\tilde{\zeta}}\delta^{-|\xi|}$$

Where $K(\pi)$ denotes the Kreweras complement of π and v_C^{η} is the vertex corresponding to $C\in K(\pi)$ (i.e. if $\bar{i}\in C$ then v_C is the vertex corresponding to $r(\eta_i)=s(\eta_{i+1})$). There is a ϕ_k-preserving conditional expectation E from $\mathcal{G}r_k(G)$ to the subalgebra generated by elements of the form (ξ,v,η) for $v\in V$ and ξ, η paths in G of length at least k. This is defined for any $(\xi,\eta,\zeta)\in\mathcal{G}r_k(G)$ by

$$E(\xi,\eta,\zeta)=\sum_{\pi\in TL(|\eta|)}\prod_{\{(i,j)\in\pi,i<j\}}\delta_{\eta_i,\tilde{\eta}_j}\frac{\mu(t(\eta_i))}{\mu(s(\eta_i))}(\xi,s(\eta),\zeta).$$

In [42] it was shown that for the type II case the algebras Gr and $\mathcal{G}r$ are isomorphic. We now show that their proof carries over to the type III case. The proof involves the natural action of certain classes of Temperley-Lieb diagrams on $\mathcal{G}r_k(G)$ and $Gr_k(G)$. An *epi Temperley-Lieb diagram* is an element of TL_m^n where $n<m$ and all the n marked points along the top are joined to marked points along the bottom. Denote by ETL the collection of epi TL diagrams. Let NNETL be the collection of non nested ETL diagrams i.e. ETL diagrams where no cap along the bottom edge encloses another cap. A TL diagram $T\in T_n^m$ acts on (ξ,η,ζ) as follows. First, split the vertices of T into three collections, V_{cup}, the vertices along the top connected by a cup to another vertex along the top, V_{cap} the vertices along the bottom connected to another vertex by a cap, $V_{through}$ the vertices which are the end points of some through string. Then $T(\xi,\eta,\zeta)$ is zero unless $\eta_i=\tilde{\eta}_j$ if the vertices i, $j\in V_{cap}$ are joined by a cap. In this case the resulting triple $T(\xi,\eta,\zeta):=(\xi,\sigma,\zeta)$ satisfies $\sigma_i=\eta_j$ if vertex i along the top is

connected by a string to vertex j along the bottom and if vertex k along the top is connected to vertex l along the top then $\sigma_k = \sum_{e\in E} e$ and we have $\sigma_l = \tilde{\sigma}_k$.

Let $\Xi : Gr_k(G) \to \mathcal{G}r_k(G)$ be the map defined by $\Xi(\xi, \eta, \zeta) = \sum_{T\in ETL} T(\xi, \eta, \zeta)$ and let $\Psi : \mathcal{G}r(G) \to Gr(G)$ be the map defined by $\Psi(\xi, \eta, \zeta) = \sum_{T\in NNETL} T(\xi, \eta, \zeta)(-1)^{n_T}$, where n_T is the number of caps of T. Similarly to [42] we may prove the following two propositions.

Proposition 4.1.14. *The maps* Ξ *and* Ψ *defined above satisfy* $\Xi\Psi = \Psi\Xi = Id$.

Proof. Let $(\xi, \eta, \zeta) \in Gr(G)$ then $\Xi\Psi(\xi, \eta, \zeta)$ will be a sum of the form $\sum_\sigma c_\sigma(\xi, \sigma, \zeta)$ where σ is a 'subpath' of η, that is, if $\sigma = \sigma_1 \cdots \sigma_n$ and $\eta = \eta_1 \cdots \eta_m$ then $\sigma_i = \eta_{j(i)}$ and for $i < k$ then $j(i) < j(k)$. Then each $\sigma \neq \eta$ with n outermost caps will have coefficient $c_\sigma = \sum_{t=1}^{n} (-1)^t \binom{n}{t} = 0$. Hence $\Xi\Psi(\xi, \eta, \zeta) = (\xi, \eta, \zeta)$. A similar proof shows $\Psi\Xi$ is the identity. □

Proposition 4.1.15. $\psi(\Phi(x)) = \phi(x)$ *for all* $x \in \mathcal{G}r_k(G)$.

Proof. Clearly $\psi(T(\xi, \eta, \zeta)) = 0$ unless $T \in TL_n^0$ with $|\eta| = n$. Hence

$$\begin{aligned}\psi(\Xi(\xi, \eta, \zeta) =& \psi\Big(\sum_{T\in TL} \prod_{(i,j)\in T} \mu(s(\eta))^2 \delta_{\eta_i,\tilde{\eta}_j} \delta_{s(\eta),r(\eta)} \frac{\mu(r(\eta_i))}{\mu(s(\eta_i))}(\xi, \eta, \zeta)\Big)\\ =& \delta_{\zeta,\tilde{\xi}} \sum_{T\in TL} \prod_{(i,j)\in T} \mu(s(\eta))^2 \delta_{\eta_i,\tilde{\eta}_j} \delta_{s(\eta),r(\eta)} \frac{\mu(r(\eta_i))}{\mu(s(\eta_i))} \delta^{-|\xi|}.\end{aligned}$$

Since by definition

$$\phi(\xi, \eta, \zeta) = \delta_{\xi,\zeta} \sum_{\pi\in NC(2t)} \prod_{\{i,j\}\in\pi, i<j} \delta_{\eta_i,\tilde{\eta}_j} \prod_{C\in K(\pi)} \mu(v_C^\eta)^{2-|C|} \delta_{\xi,\tilde{\zeta}} \delta^{-|\xi|},$$

we need to show that

$$\prod_{(i,j)\in T} \mu(s(\eta))^2 \delta_{\eta_i,\tilde{\eta}_j} \delta_{s(\eta),r(\eta)} \frac{\mu(r(\eta_i))}{\mu(s(\eta_i))} = \prod_{C\in K(\pi)} (\mu(v_C^\eta))^{2-|C|}.$$

For any $C \in K(\pi)$, we need to show that the weight $\mu(v_C^\eta)$ appears in the product on the left hand side $2 - |C|$ times. This is true since if C does not contain the vertex $\overline{2n}$ then there must be one pair $(i, j) \in T$ such that $v_C = r(\eta_i)$ and $|C| - 1$ pairs $(i, j) \in T$ with $v_C = s(\eta_i)$ and so $\mu(v_C^\eta)$ must appear to the power $2 - |C|$. If $\overline{2n} \in C$ then $v_C = \mu(s(\eta))$ must appear as the vertex $s(\eta_i)$ for $|C|$ pairs $(i, j) \in T$ and hence appears to the power $2 - |C|$ in the left hand side. □

Similarly to the previous section we may use the algebras $Gr_k(G)$ to define a type III factor. We use the state ψ to define an inner product on $Gr_k(G)$ in the usual way, and let H_k be the Hilbert space completion of Gr_k. Then let Gr_k act on itself by left multiplication.

Proposition 4.1.16. *Let $x \in Gr_k(P)$ and let $\lambda_x : Gr_k(P) \to Gr_k(P)$ be the action of x on $Gr_k(P)$ by left multiplication. Then there exists a constant $C > 0$ such that $\|\lambda_x(y)\| \leq C\|y\|$ for all $y \in Gr_k(P)$ and hence λ_x extends uniquely to an element of $B(H_k)$.*

Proof. Let $x = (\xi, \eta, \zeta) \in Gr_k(P)$. As a first step we show that there exists $K > 0$ such that $\|(x\#y)_t\| \leq K\|y\|$ for all $y = (\xi', \eta', \zeta') \in Gr_k(G)$, where $(x\#y)_t$ is the element of $x\#y$ in $P^t_{n,m}$. To prove this, first suppose that $|\xi'| > |\zeta|$. Then we may suppose $\xi' = \zeta\xi'$ and $\eta_{t_1-i} = \tilde{\eta}'_i$ for all $1 \leq i \leq t$, since if this is not the case the product $(x\#y)_t$ is zero and the proposition is trivially true. Then we have

$$\begin{aligned}\|(x\#y)_t\| =&\|(\xi \cdot \xi'_0, \eta_{[1,t_1-t]}\eta'_{[t,t_2]}, \zeta')\frac{\mu(r(\eta))}{\mu(s(\eta_{t_1-t}))}\| \\ =&\delta^{-|\xi\xi'_0|}\mu(s(\eta))\frac{\mu(r(\eta))}{\mu(s(\eta_{t_1-t}))}\frac{\mu(r(\eta'))}{\mu(s(\eta))} \\ =&\delta^{-|\xi|+|\zeta|}\frac{\mu(r(\eta))}{\mu(s(\eta_{t_1-t}))}\|y\|\end{aligned}$$

and so the claim is proved. Now suppose that $|\xi'| \leq |\zeta|$. Again we may assume $\eta_{t_2-i} = \tilde{\eta}'_i$ for all $1 \leq i \leq t$ and also that $\zeta = \zeta_0\xi'$. Then we have

$$\begin{aligned}\|(x\#y)_t\| =&\delta^{-|\xi|}\mu(s(\eta))\frac{\mu(r(\eta))}{\mu(s(\eta_{t_1-t}))}\frac{\mu(r(\eta'))}{\mu(s(\eta))} \\ \leq&\delta^{|\zeta|-|\xi'|}\delta^{-|\xi|}\mu(s(\eta))\frac{\mu(r(\eta))}{\mu(s(\eta_{t_1-t}))}\frac{\mu(r(\eta'))}{\mu(s(\eta))} \\ =&\delta^{|\zeta|-|\xi|}\frac{\mu(r(\eta))}{\mu(s(\eta))}\|y\|\end{aligned}$$

where the inequality follows from the fact that $\delta^{|\zeta|-|\xi'|} \geq 1$.

Next we must show that for a finite sum $y = \sum_{i=1}^N c_i(\xi_i, \eta_i, \zeta_i)$ there exists a constant $C > 0$ with $\|x\#y\| \leq C\|y\|$.

$$\begin{aligned}\|x\#y\| &= \|\sum_i c_i(\xi,\eta,\zeta)\#(\xi_i,\eta_i,\zeta_i)\| \\ &\leq \sum_{i,t}\|c_i((\xi,\eta,\zeta)\#(\xi_i,\eta_i,\zeta_i))_t\| \\ &\leq \sum_{i,t} K_i c_i \mu(r(\eta_i)) \leq C\|y\|\end{aligned}$$

where $\|(x\#(\xi_i,\eta_i,\zeta_i))_t\| \leq K_i\|(\xi_i,\eta_i,\zeta_i)\|$ □

The above proposition shows that $Gr_k(G) \subset B(H_k)$ and we may take the weak completion to get a von Neumann algebra $\mathcal{M}_k$. Since ϕ_k is the unique KMS state, $\mathcal{M}_k$ is a factor. By calculating the Connes spectrum we can see it is type III. Lemma 4.1.4 proves that the inclusion $Gr_k \subset Gr_{k+1}$ extends to an inclusion of the associated von Neumann algebras.

The basic idea for the rest of the section is that for algebra $Gr_k(P)$ generated by triples (ξ,η,ζ) is made up of the part generated by pairs (ξ,ζ), which is isomorphic to a Cuntz-Krieger algebra, and the part generated by paths η, which is isomorphic to the algebra defined by Kodiyalam and Sunder in [2]. Let A be the adjacency matrix of G and let $\mathcal{O}_A$ be the corresponding Cuntz-Krieger algebra. Let $v \in V$ be a vertex of G and let p_v be the projection onto the subspace of $Gr_k(G)$ generated by triples (ξ,η,ζ) with $r(\xi) = v$. Let $\lambda : Gr_k(P) \to Gr_k(P)$ be the left regular representation. Then for a vertex v, λ_v is the projection of $Gr_k(P)$ onto the subalgebra of tangles with vertex v in the top left corner. Then p_v corresponds to the projection $P_v = \sum_{\alpha\in G, t(\alpha)=v} S_\alpha S_\alpha^*$ in $\mathcal{O}_A$ and the projection λ_v of $Gr_k(G)$ defined in [51]. Then $p_v Gr_k(G) p_w$ is the tensor product of $P_v\mathcal{O}_A P_w$ with the algebra $\lambda([v])Gr(G)\lambda([w])$.

We need the following theorem which is Theorem 4.1.2 of [69]:

Theorem 4.1.17. *Let (A,ϕ) be a non commutative probability space and let $m,n \in \mathbb{N}$. Suppose that there exist $1 \leq i(1) < i(2) < \ldots < i(m) = n$. Then for any $\sigma \in NC(m)$ and $a_1,\ldots,a_n \in A$*

$$\kappa_\sigma(a_1\cdots a_{i(1)},\ldots,a_{i(m-1)+1}\cdots a_{i(m)}) = \sum_{\pi\in NC(n);\pi\vee\hat{0}_m=\hat{\sigma}} \kappa_\pi(a_1,\ldots,a_n)$$

where if σ is a non-crossing partition of the set $(A_1, \ldots, A_m)$ and each $A_j = a_{i(j-1)+1} \cdots a_{i(j)}$ then $\hat{\sigma}$ is defined as the non-crossing partition on the set $(a_1, \ldots, a_{i(m)})$ where for any $a_j \in A_j$ and $a_k \in A_k$ then $a_j \sim_{\hat{\sigma}} a_k$ if and only if $A_j \sim_\sigma A_k$.

Let $B \subset A$ be von Neumann algebras with a conditional expectation $\phi : A \to B$. Suppose A_i, $i \in I$ are subalgebras of A with $B \subset A_i$ for all i, and suppose each A_i is generated by a subset G_i. Then, combining the above theorem with Speichers definition of the amalgamated free product as in Section 2.10.3 we see that the algebra A is the free product of the subalegbras A_i, $i \in I$ with amalgamation over B if and only if for $g_k \in G_{i_k}$ then $\kappa(g_1, \ldots, g_n) = 0$ unless all the g_k are in the same G_i.

The algebra $Gr_k(G)$ is generated by subalgebras $Gr_k(G_e)$ where $Gr_k(G_e)$ is the algebra generated by triples (ξ, η_e, ζ) where η_e is either a path containing only e and $\tilde{e}$ or an empty path. Each $Gr_k(G_e)$ is multiplicatively generated by the set $\Gamma_e := \{(\xi, e, \zeta) : \xi, \zeta \,\text{paths in G}\} \cup \{(\xi, \tilde{e}, \zeta) : \xi, \zeta \,\text{paths in G}\}$. Using the conditional expectation defined above we may calculate the $\mathcal{O}_A$ valued cumulants κ, similarly to Proposition 3.1 of [2].

Proposition 4.1.18. *For the algerbras $Gr_k(G)$ defined above, the $\mathcal{O}_A$-valued cumulants $\kappa_n(x_{i_1}, \ldots, x_{i_n})$ $x_{i_n} \in \Gamma_{e_{i_n}}$ are all zero unless $n = 2$ and x_{i_1} and x_{i_2} are from the same subalgebra $Gr_k(G_e)$.*

Proof. The proof is similar to the proof of Proposition 3.1 in [2]. Define the cumulants κ_n by the rule that $\kappa_n = 0$ for all $n \neq 2$ and $\kappa_2((\xi_i, \eta_1, \zeta_1), (\xi_2, \eta_2, \zeta_2)) = \delta_{\eta_1,e}\delta_{\eta_1,\tilde{\eta}_2}\frac{\mu(r(e))}{\mu(s(e))}(\xi_1, \zeta_1) \cdot (\xi_2, \zeta_2)$. Let κ_π be the multiplicative extensions of the κ_n. Then we wish to show that

$$E(x_1, \ldots, x_n) = \sum_{\pi \in NC(n)} \kappa_\pi(x_1, \ldots, x_n)$$

Note that κ_π must be zero unless π is a pair partition. In that case, there must exist k with $\{k, k+1\} \in \pi$ and it is easy to see that $\kappa_\pi = \kappa_\sigma(x_1, \ldots, x'_{k-1}, x_{k+2}, \ldots, x_n)$

where x'_{k-1} is the product of x_{k-1} with $\kappa(x_k, x_{k+1}) = \mu(r(x_k))$ and then by induction on we see that $\kappa_\pi = \prod_{\substack{\{i,j\}\in\pi \\ i<j}} \frac{\mu(r(e_i))}{\mu(s(e_i))}(\xi, \zeta)$ where $(\xi, \zeta) = \prod(\xi_i, \zeta_i)$. Hence

$$\sum_{\pi\in NC(n)} \kappa_\pi(x_1, \dots, x_n) = \sum_{\pi\in NC(n)} \prod_{\substack{\{i,j\}\in\pi \\ i<j}} \frac{\mu(t(e_i))}{\mu(s(e_i))}(\xi, \zeta) = E(x_1 \cdots x_n)$$

□

Hence we may split the algebra $Gr_k(G)$ into a free product, as in Corollary 3.2 of [2].

Proposition 4.1.19. *$Gr_k(G) = *_{\otimes_v P_v \mathcal{O}_A P_v}\{Gr_k(G_e) : e \in E\}$ and also, taking completions we get $M_k(G) = *_{\otimes_v P_v \mathcal{O}_A P_v}\{M_k(G_e) : e \in E\}$ where $M_k(G_e)$ is the von Neumann algebra completion of $Gr_k(G_e)$ in $B(H_k)$.*

Thus the algebra $Gr_k(G_e)$ can be written as

$$\bigoplus_{v,w\in V} P_v \mathcal{O}_V P_w \otimes p_v Gr(G_e) p_w$$

where p_v is the projection $(v, v, v) \in Gr(G_e)$. In [2] the algebra $Gr(G_e)$ was shown to be equal to $M_2(L\mathbb{Z}) \oplus \mathbb{C} \oplus H_e$ where H_e is a Hilbert space with basis given by $V/\{s(e), r(e)\}$ where the projecions $p_{s(e)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \oplus 1 \oplus 0$, $p_{t(e)} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \oplus 0 \oplus 0$ and the p_v are orthogonal one dimensional projections in H for $v \in V/\{s(e), t(e)\}$.

In the case where there is only one vertex the above algebra has a simple form. By [2], if the graph has two edges e and $\tilde{e}$ the type II part is $L\mathbb{F}_2$ and A is the matrix (2). Hence $M_k(G)$ is a tensor product of $L\mathbb{F}_2$ with a hyperfinite type III factor, since the von Neumann algebra completion of $\mathcal{O}_2$ is a hyperfinite type III factor.

For more complicated graphs, the free product in Proposition 4.1.19 is difficult to understand. Most of the literature on amalgamated free products of type III factors focuses on free products with amalgamation over finite subalgebras, and the results seem difficult to generalise to the infinite case.

Chapter 5

A_2-Planar Algebras

In this chapter we return to the setting of A_2-planar algebras, defined by Evans and Pugh in [23] and described in Section 2.6 of this thesis. Section 5.1 of the chapter describes how to use the methods of Chapter 3 to extend the theory of A_2-planar algebras to describe type III subfactors. The rest of the chapter is concerned with further work. Section 5.2.1 describes how to extend some of the skein theory results of [74], [5], [63] to describe the skein theory of subfactors with graph $\mathcal{D}^{(2n)}$. Finally, in Section 5.2.2 we discuss extensions of the Guionnet Jones Shylaktenko [28], [42], [49] construction to A_2-planar algebras.

5.1 Type III A_2- Planar algebras

5.1.1 Type III A_2- Temperley-Lieb

We now define an A_2 analogue of type III planar algebra $\mathcal{O}_{TL}$. An A_2-$\mathcal{O}_{TL}$ *tangle* is a rectangle with n marked points along the top and $n+3k$ marked points along the bottom for some $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ with $n+3k \geq 0$. The marked points along the top are source vertices for oriented strings which have as their endpoint either a sink vertex along the bottom or an incoming trivalent vertex. Let T_n^{n+3k} be the vector space spanned by such diagrams, quotiented by the Kuperberg relations shown in Figure 2.11. If $k = 2m$ the vector space T_n^{n+3k} is finite dimensional since it is either a type II A_2 Temperley-Lieb diagram or the rotation of a type

II A_2 Temperley-Lieb diagram. Thus finite dimensionality follows from finite dimensionality of the type II A_2-Temperley-Lieb algebra proved in Lemma 3.11 [23]. If k is odd, T_n^{n+3k} is finite dimensional since a basis may be written in terms of a basis of $T_n^{n+3(k+1)}$ multiplied by the diagram with three vertices joined at a trivalent vertex. Thus each T_n^{n+3k} is a finite dimensional vector space. There is an embedding i of T_n^{n+3k} into T_{n+1}^{n+1+3k} by adding a vertical through string to the right. In this way we may take the inductive limit $T_k := \varinjlim T_n^{n+3k}$ and put $T = \bigoplus_{k \in \mathbb{Z}} T_k$. Note that T_0 is the A_2 Temperley-Lieb algebra defined in [23]. We may define a braiding on T in exactly the same way as in Figure 2.12.

Let $x \in T_n^{n+3k}$ and $y \in T_m^{m+3j}$ be single diagrams and suppose $m+3j < n$. To form the product $x \cdot y$ first embed y in T_{n-3j}^n and then stack x on top of y. Join the corresponding strings on the bottom edge of x and the top edge of y, remove the vertices, smooth the strings and if necessary use the Kuperberg relations to remove any closed loops or embedded digons or squares. If $n > m + 3j$ then we embed x in $T_{m+3j}^{m+3j+3k}$ and take the product in a similar way. An example is shown in Figure 5.1. The multiplication on T is the bilinear extension of the multiplication defined above. There is also a *-operation taking T_n^{n+3k} to T_{n+3k}^n, where, given a diagram x the adjoint x^* is formed by flipping x around a horizontal axis.

Let $W \in T_0^3$ be $(\delta \cdot (\delta^2 - 1))^{-\frac{1}{2}} = (\alpha\delta)^{-\frac{1}{2}}$ times the tangle with 3 vertices along the top, joined by a single trivalent vertex as shown in Figure 5.2. Let Φ be the endomorphism of T defined by $\Phi(x) = WxW^*$ for $x \in T$.

Proposition 5.1.1. *The algebra T is generated by T_0 and W. Every $x \in T$ may be written as a finite sum*

$$x = \sum W^n x_n + x_0 + \sum (W^*)^n x_{-n}$$

for some $x_i \in T_0$.

Proof. Suppose $x \in T_n^{n+3k}$. We can write x as a product of T_0 tangles and W's as follows. Let x' be the tangle x with k copies of W^* added along the bottom,

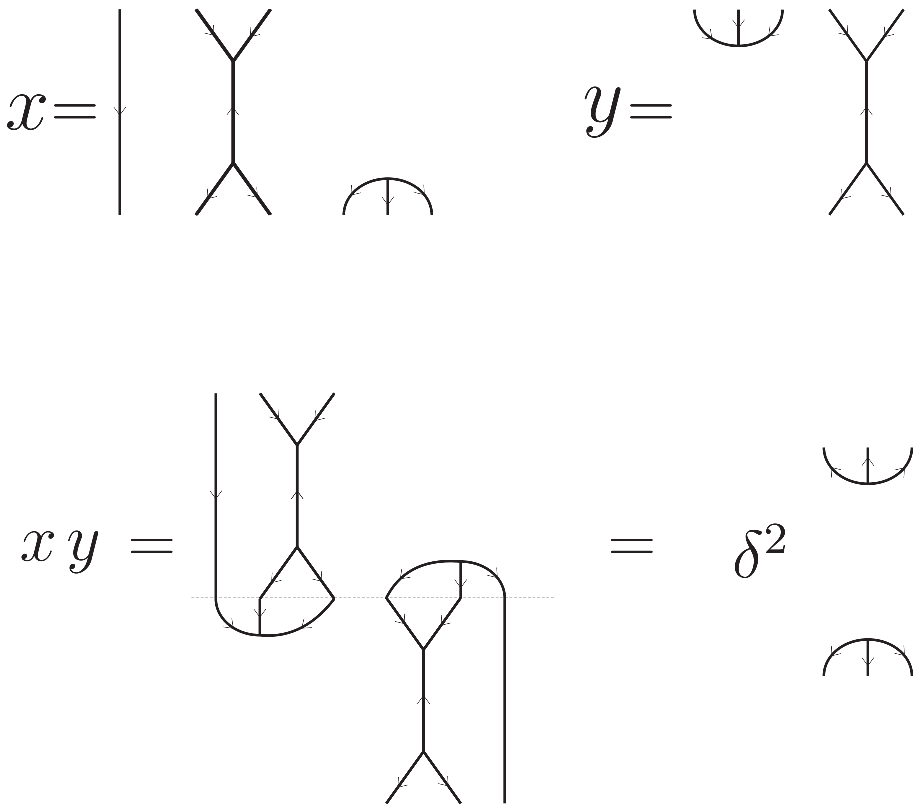


Figure 5.1: Multiplication in A_2-O_{TL}

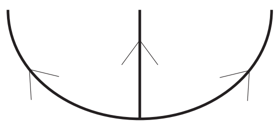


Figure 5.2: The tangle $(\alpha\delta)^{\frac{1}{2}}W$

as shown in Figure 5.3. Let $W_{n,k}$ be the image of the tangle with k copies of W along the top in T_n^{n+3k} showin in Figure 5.4. Then $x = x'W_{n,k}$.

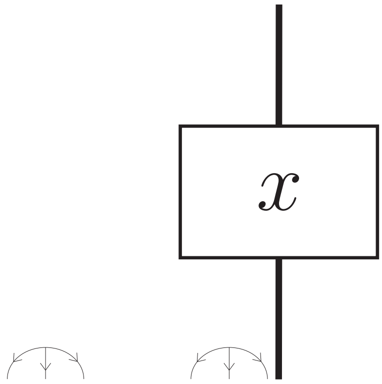


Figure 5.3: The tangle x'

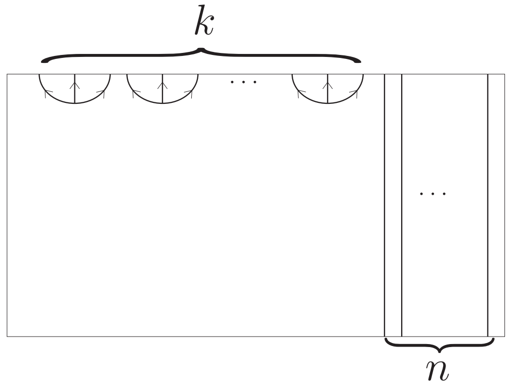


Figure 5.4: The tangle $W_{n,k}$

□

The following proposition may be proved in exactly the same way as Proposition 3.1.4.

Proposition 5.1.2. *The algebra* A_2-$\mathcal{O}_{TL}$ *is simple and purely infinite.*

Let σ_t be the action of $\mathbb{R}$ on T defined by $\sigma_t(x) = e^{\gamma i(m-n)t}x$ for $\gamma = \ln \alpha$ and $x \in T_n^m$. Then let $S : T \to T_0$ be the conditional expectation defined by the linear extension of the map $S(x) = \gamma/2\pi \int_0^{\frac{2\pi}{\gamma}} \sigma_t(x)dt$ for a diagram $x \in T_n^m$. Let Tr be the normalised trace on T_0, defined on $x \in T_n^n$ by joining corresponding points along the top and bottom and multiplying by α^{-n}. Then define a state ϕ on T by $\phi := Tr \circ S$.

Let H be the Hilbert space completion of T with respect to the inner product defined by ϕ and let T act on H by left multiplication.

Proposition 5.1.3. *Let $\lambda_x : T \to T$, $x \in T$ be the action of T on itself by left multiplication. Then λ_x is a bounded operator for all $x \in T$ and hence may be extended uniquely to an element of $B(H)$.*

Proof. Since we know that T is generated by W and T_0 and multiplication by T_0 is bounded, we need to show multiplication by W and by elements of T_0 is bounded. Let $x \in H$ then we wish to show there exists $C > 0$ with $\|Wx\| \leq C\|x\|$. This follows from the computation $\|Wx\|^2 = \langle Wx, Wx\rangle = \alpha^{-3}\langle W^*Wx, x\rangle = \alpha^{-3}\langle x, x\rangle$ and so we obtain the desired inequality with $C = \alpha^{-3}$. To show that multiplication by $a \in A_2 - TL$ is bounded we can use the same proof as for Proposition 3.1.2, except here we must replace $x = \sum_i c_i x_i$ by $x' = \sum_i c_i x'_i$ as shown in Figure 5.3. □

Let $\mathcal{T}$ be the C^*-algebra generated by T in $B(H)$ and let $\mathcal{T}_0$ be the C^*-algebra generated by T_0. The algebra $\mathcal{T}$ is a simple AF C^*-algebra and so we can prove the following proposition in a similar way to 3.1.5

Proposition 5.1.4. *The algebra $\mathcal{T}$ is the crossed product of $\mathcal{T}_0$ by the endomorphism Φ.*

Proof. From above we know that T is generated by the A_2-Temperley-Lieb algebra and the isometry W. Since the C^*-algebra $\mathcal{T}_0$ is simple, its crossed product by Φ is also simple and hence $\mathcal{T} \simeq \mathcal{T}_0 \rtimes_\Phi \mathbb{N}$. □

Let M be the weak completion of T in the GNS representation associated to the state ϕ. Then ϕ is a KMS state at temperature $\beta = 1$ for the modular automorphism group σ_t. The following proposition may be proved in exactly the same way as Proposition 3.1.3

Proposition 5.1.5. *The state ϕ defined above is the unique KMS state on M for the modular automorphism group σ_t and the inverse temperature β is also unique.*

Proposition 5.1.6. *The algebra M is the AFD type III_λ factor, where $\lambda = \frac{1}{\alpha}$.*

Proof. This is proved in a similar way to Proposition 3.3.5. We know that M is a factor since the KMS state is unique and hence a factor state. That M is hyperfinite follows from the nuclearity of the A_2 Temperley-Lieb C^*-algebra which is preserved after taking the crossed product. The modular automorphism group σ_t is periodic with period $2\pi/\beta$ and so M is type III_λ with $\lambda = e^{-\gamma} = 1/\alpha$. $\square$

5.1.2 Type III $\mathcal{ADE}$ string algebra

Now we describe how the string algebra construction for type III factors from Section 3.3 may be extended to construct subfactors from finite $SU(3)$-$\mathcal{ADE}$ graphs.

Let $\mathcal{G}$ be a finite SU(3)-$\mathcal{ADE}$ graph such that there exists a connection on $\mathcal{G}$ and a cell system as in Section 2.7. Let n be the Coxeter number and let $q = e^{i\pi/n}$. Put $\alpha = [2]_q$ and $\delta = [3]_q$. For $m \in \mathbb{N}$ and $k \in \mathbb{Z}$ with $m + 3k \geq 0$, let $B_{(m,m+3k),0}$ be the vector space over $\mathbb{C}$ with basis given by pairs of paths (ξ_1, ξ_2) on $\mathcal{G}$ with $s(\xi_1) = s(\xi_2) = *$, $r(\xi_1) = r(\xi_2)$, $|\xi_1| = m$ and $|\xi_2| = m + 3k$. We may define embeddings $i : B_{(m,m+3k),0} \to B_{(m+n,m+n+3k),0}$ by $i(\xi_+, \xi_-) := \sum_\sigma (\xi_+ \cdot \sigma, \xi_- \cdot \sigma)$ where the sum is over all paths σ of length n starting at $t(\xi_1)$. Thus we define $B_{k,0} := \varinjlim B_{(m,m+3k),0}$ and $B_{\infty,0} := \bigoplus_{k\in\mathbb{Z}} B_{k,0}$. The vector space $B_{\infty,0}$ can be given the structure of an associative *-algebra with the same operations as in section 3.3. That is, there is a multiplication defined by

$$(\xi_1, \xi_2).(\zeta_1, \zeta_2) = \begin{cases} (\xi_1, \zeta_2 \cdot \xi_2') & \text{if } \xi_2 = \zeta_1 \cdot \xi_2' \\ (\xi_1 \cdot \zeta_1', \zeta_2) & \text{if } \zeta_1 = \xi_2 \cdot \zeta_1' \\ 0 & \text{otherwise} \end{cases}$$

and *-operation defined by $(\xi_1, \xi_2)^* = (\xi_2, \xi_1)$.

Suppose $\mathcal{G}$ is three colourable. Then we define the vector space $B_{(m,m+3k),n}$ to be the spaces with basis pairs of paths $(\sigma_1 \cdot \xi_1, \sigma_2 \cdot \xi_2)$ where σ_i is a path of length n starting at $*$ in the $\overline{0}-\overline{1}$ part of $\mathcal{G}$ with $r(\sigma_1)=r(\sigma_2)$ and (ξ_1,ξ_2) is a pair of paths of length $(m,m+3)$ starting at $r(\sigma_1)$ with $r(\xi_1)=r(\xi_2)$. As before we can define inclusion maps $B_{(m,m+3k),n}\to B_{(m+1,m+1+3k),n}$ and then let $B_{k,n}=\varinjlim B_{(m,m+3k),n}$ and $B_{\infty,n}:=\bigoplus_{k\in\mathbb{Z}}B_{k,n}$. Using the same connection defined in 2.5 we can transform paths of the above form to paths where, for example, we travel first along a path $(\xi_1,\xi_2)\in B_{(m,m+3k),0}$ ending at some vertex v of colour b_v and then travel along a path of length n the $\overline{b}_v$-$\overline{b_v+1}$ part of $\mathcal{G}$. Thus we may embed $B_{(m,m+3k),n}$ into $B_{(m,m+3k),n+1}$ by $(\xi_1\cdot\zeta_1,\xi_2\cdot\zeta_2)\mapsto\sum(\xi_1\cdot\zeta_1\cdot\sigma,\xi_2\cdot\zeta_2\cdot\sigma)$ where the sum is over all vertical paths σ in $\mathcal{G}_{\overline{b}_v-\overline{b_v+1}}$. If $\mathcal{G}$ is not three colourable, we use the whole graph for the vertical inclusions.

For $x\in B_{(m,m+3k),n}$ and $y\in B_{(m',m'+3k'),n'}$ we define the product as follows. First put both x and y in the basis where we first travel vertically along the $\bar{0}-\bar{1}$ part of $\mathcal{G}$ and then horizontally along $\mathcal{G}$. Suppose $x=(\xi_1\cdot\zeta_1,\xi_2\cdot\zeta_2)$ and $y=(\xi_1'\cdot\zeta_1',\xi_2'\cdot\zeta_2')$ in this basis. Then if $m+3k=m'$, $n=n'$ we define the product $xy=\delta_{\xi_2,\xi_1'}\delta_{\zeta_2,\zeta_1'}(\xi_1\cdot\zeta_1,\xi_2'\cdot\zeta_2')$ if $m+3k\neq m'$ or $n\neq n'$ we first apply the inclusion maps above to x or y and then multiply.

Define an action of $\mathbb{R}$ on $B_{(m,m+3k),n}$ by $\sigma_t(\xi_1,\xi_2)=e^{\gamma it(3k)}(\xi_1,\xi_2)$. Using this we define a state ψ_n on $B_{0,n}$ by $\psi_n(\xi_1,\xi_2)=\delta_{\xi_1,\xi_2}\alpha^{-3k}\mu(r(\xi_1))$. The map $\Phi_n:B_{\infty,n}\to B_{0,n}$, $\Phi_n(\xi_1,\xi_2)=\gamma/2\pi\int_0^{2\pi/\gamma}\sigma_t(\xi_1,\xi_2)dt$ is a faithful conditional expectation. We can then define a state on $B_{\infty,n}$ by $\phi_n:=\psi_n\circ\Phi_n$. Let $H_{\infty,n}$ be the Hilbert space completion of $B_{\infty,n}$ with respect to the inner product defined by $\langle x,y\rangle:=\phi_n(y^*x)$. It can be shown as in Proposition 3.3.1 that the action of $B_{\infty,n}$ on itself by left multiplication is bounded and hence $B_{\infty,n}$ may be thought of as a subalgebra of $B(H_{\infty,n})$.

Proposition 5.1.7. *Let $\mathcal{B}_{\infty,n}$ be the C^*-algebra obtained by taking the norm completion of the algebra $B_{\infty,n}$ with respect to the inner product norm. Let $\mathcal{W}=(\delta(\delta^2-1))^{-\frac{1}{2}}\sum_{|\sigma|=3}W\triangle_{\sigma_1,\sigma_2,\sigma_3}(\sigma,*)$ be an isometry of $B_{\infty,0}$ and let $\mathcal{W}_n$ be its image in $B_{\infty,n}$. Define the endomorphism Ψ of $B_{\infty,n}$ by $\Psi(x)=\mathcal{W}_n(x)\mathcal{W}_n^*$. Then $B_{\infty,n}=B_{0,n}\rtimes_\Psi\mathbb{N}$*

Let $\mathcal{M}_{\infty,n}$ be the completion of $B_{\infty,n}$ with respect to the inner product defined by ϕ_n. As in Proposition 3.3.2 it can be shown that ϕ is the unique KMS state for the modular automorphism group σ and hence $\mathcal{M}_{\infty,n}$ is a factor. Since $\mathcal{B}_{\infty,n}$ is the crossed product of a nuclear algebra with $\mathbb{N}$, it is nuclear and hence $\mathcal{M}_{\infty,n}$ is hyperfinite. As in Proposition 3.3.5 we can show that $\mathcal{M}_{\infty,n}$ is the hyperfinite III_λ factor for $\lambda = 1/\alpha$. Define the Jones projections by

$$e_n = \sum_{|\xi|=n-2, |\zeta|=|\eta|=1} \frac{1}{\alpha} \frac{\sqrt{\mu(r(\eta))\mu(r(\zeta))}}{\mu(r(\xi))} (\xi \cdot \eta \cdot \tilde{\eta}, \xi \cdot \zeta \cdot \tilde{\zeta})$$

Thus we have an increasing sequence of type III_λ factors $\mathcal{M}_{\infty,0} \subset \mathcal{M}_{\infty,1} \subset \mathcal{M}_{\infty,2} \ldots$.

Proposition 5.1.8. *The sequence $\mathcal{M}_{\infty,0} \subset \mathcal{M}_{\infty,1} \subset \mathcal{M}_{\infty,2} \ldots$ is the basic construction for the subfactor*

$$\mathcal{M}_{\infty,0} \subset \mathcal{M}_{\infty,1}.$$

Proof. Using Theorem 2.8.2 and the Jones projections defined above, the proposition may be proved in exactly the same way as for the bipartite graph case. □

Proposition 5.1.9. *Let $\mathcal{M}_{\infty,i}$ be as above and suppose the connection is flat. Then $\mathcal{M}'_{\infty,0} \cap \mathcal{M}_{\infty,n} = B_{(n,n),0}$ for all $n \in \mathbb{N}$.*

Proof. This may be proved using Ocneanu's compactness argument in exactly the same way as the proof of Proposition 3.3.6 apart from instead of $B_{n,(l+2k,l+2k)}$ we use $B_{n,(l+3k,l+3k)}$. □

5.1.3 A_2-Planar Algebra for type III subfactors

Let P be a general A_2-Planar algebra, as in Definition 2.6.1. Define $\sigma(i,j,k)$ to be the sign string $-^i\tilde{\sigma}_j+^{i+3k}$ where $\tilde{\sigma}_j$ is the alternating sign string of length $2j$ which starts with a $-$. Write $P_{(i,j,k)}$ for $P_{\sigma(i,j,k)}$. Define inclusion maps $I_{(i,j,k)}^{(i,j+1,k)} : P_{(i,j,k)} \to P_{(i,j+1,k)}$ and $I_{(i,j,k)}^{(i+1,j,k)} : P_{(i,j,k)} \to P_{(i+1,j,k)}$ as in Figure 5.5, recall the crossing on A_2-tangles is defined as in Figure 2.12 .

Let $P_{(j,k)} = \varinjlim P_{(i,j,k)}$ be the algebraic direct limit and let $\mathcal{P} := \bigoplus_{j\in\mathbb{N}, k\in\mathbb{Z}} P_{(j,k)}$ be the vector space direct sum, where $P_{(i,j,k)}$ is empty when $i+3k<0$. We can

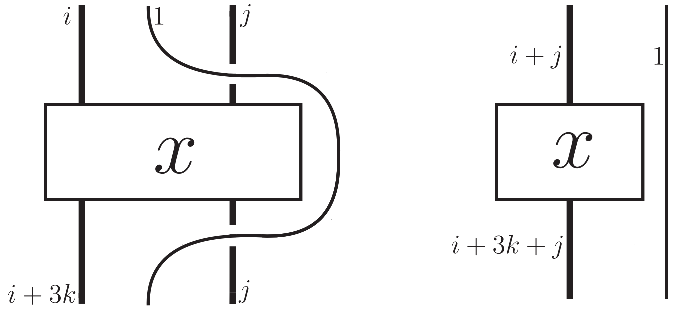


Figure 5.5: Inclusion tangles $I_{(i,j,k)}^{(i+1,j,k)}(x)$ and $I_{(i,j,k)}^{(i,j+1,k)}(x)$

define a multiplication on tangles as follows. If the pattern of the bottom edge of x is the same as the pattern of the top edge of y then the product $x \cdot y$ is just x stacked on top of y. If the patterns do not match, apply the inclusion maps I_R and I_L to x and/or y until the patterns do match and then stack. The bilinear extension of this map defines an associative multiplication on $\mathcal{P}$ via the identification $x \cdot y = Z(M(x,y))$ where M is the appropriate multiplication tangle and $x, y \in \mathcal{P}$.

As usual, we define an involution $*$ on tangles by flipping about a horizontal axis. Then we may define an involution on $\mathcal{P}$ by $Z_T(x)^* = Z_{T^*}(x^*)$. Under these operations $\mathcal{P}$ is an associative *-algebra. We call such a $\mathcal{P}$ a *type III A_2-planar algebra*. We define a state ϕ on $\mathcal{P}$ by $\phi = Tr \circ S$ where Tr is the trace on the type II A_2-planar algebra and S is the projection from $\mathcal{P}$ onto $\bigoplus_{j \in \mathbb{N}} P_{j,0}$. As usual, we may use ϕ to define an inner product $\langle \cdot, \cdot \rangle$ by $\langle x, y \rangle := \phi(y^* x)$.

Definition 5.1.10. Let $\mathcal{P}$ be a general A_2-planar algebra. A *type III A_2-planar algebra* is $\mathcal{P} = \bigoplus_{\substack{j \in \mathbb{N} \\ k \in \mathbb{Z}}} P_{(j,k)}$ where the $P_{(j,k)}$ are as defined above, $P_{(0,0,0)}$ has dimension 1 and $P_{i,j,k}$ is finite dimensional for all i, j, k.

The following proposition may be proved in exactly the same way as Proposition 3.4.5.

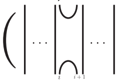
Proposition 5.1.11. *Suppose P is a non degenerate finite dimensional type III A_2-planar algebra such that Z is a positive map on $P_{0,0,0}$. Then there exists a unique C^*-norm on P.*

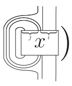
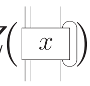
We call such a planar algebra a *type III A_2 C^*-planar algebra.*

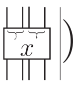
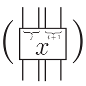
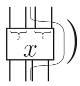
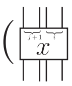
Next we show how to use the string algebra construction of the previous section to define a type III A_2-planar algebra.

Proposition 5.1.12. *Let $\mathcal{G}$ be an $\mathcal{ADE}$ graph with flat connection. Let $[3]_q = \delta$ be its Perron Frobenius eigenvalue and let $[2]_q = \alpha$. Let Z be the presenting map defined in section 2.7 and let $\mathcal{P}$ be the the planar algebra defined above with $P_{i,j,k} \simeq B_{(i,i+3k),j}$. Then $\mathcal{P}$ is a flat type III A_2-C^*-planar algebra such that*

(i) $Z(W_{-k}) = U_{-k}, \qquad k \geq 0,$

(ii) $Z($  $) = \alpha\, e_i \quad i \geq 1$

(iii) $Z($  $) = \alpha^{j+1} E_{M' \cap M_{i-1}}(x)$ $\qquad Z($  $) = \alpha E_{M_{i-2}}(x)$

(iv) $Z($  $) = Z($  $)$ $\qquad Z($  $) = Z($  $)$

Proof. Most of the proof carries over from the proof of Proposition 3.4.7, in terms of invariance of Z under planar isotopies, the only ones that need to be checked are shown in Figures 5.6, 5.7, 5.8. For these invariance is proved exactly as in [23]. We show the first isotopy of Figure 5.6. The presenting map of the top strip on the left hand side is

$$Z(t_1) = \sum_{\substack{|\alpha|=n,|\xi|,|\eta|=1 \\ |\beta|=m}} \sqrt{\frac{\mu(r(\eta))}{\mu(s(\eta))}} (\alpha \cdot \xi \cdot \beta, \alpha \cdot \xi \cdot \eta \cdot \tilde{\eta} \cdot \beta)$$

and the presenting map for the bottom strip is

$$Z(t_2) = \sum_{\substack{|\gamma|=n,|\nu_i|=1 \\ |\delta|=m}} \frac{1}{\sqrt{\mu(s(\nu_1))\mu(r(\nu_1))}} W(\triangle_{\nu_3,\nu_1,\nu_2})(\gamma \cdot \nu_1 \cdot \nu_2 \cdot \nu_4 \cdot \delta, \gamma \cdot \nu_3 \nu_4 \cdot \delta)$$

Multiplying we get

$$Z(t) = \sum_{\substack{|\alpha|=n,|\beta=m| \\ |\nu_i|=1}} \frac{1}{\sqrt{\mu(s(\nu_1))\mu(r(\nu_1))}} W(\triangle_{\nu_3,\nu_1,\nu_4})(\alpha \cdot \nu_1 \cdot \beta, \alpha \cdot \nu_3 \cdot \nu_4 \cdot \beta)$$

which is equal to the right hand side. All the other isotopies in Figure 5.6 may be proved similarly.

For the isotopy in Figure 5.7, the strips on the left hand side give

$$Z(t_1) = \sum_{\substack{|\alpha|=n,|\beta|=1,\\|\gamma|=m,|\nu_i|=1}} \frac{1}{\sqrt{\mu(s(\nu_3))\mu(r(\nu_3))}} W(\triangle_{\nu_1,\nu_3,\nu_2})(\alpha\cdot\beta\cdot\nu_1\cdot\gamma, \alpha\cdot\beta\cdot\nu_2\cdot\nu_3\cdot\gamma)$$

$$Z(t_2) = \sum_{\substack{|\alpha'|=n,|\beta'|=1,\\|\gamma'|=m,|\nu'_i|=1}} \frac{1}{\sqrt{\mu(s(\nu'_3))\mu(r(\nu'_3))}} W(\triangle_{\nu'_3,\nu'_1,\nu'_2})(\alpha'\cdot\nu'_1\cdot\nu'_2\cdot\beta'\cdot\gamma', \alpha'\cdot\nu'_3\cdot\beta'\cdot\gamma').$$

The strips on the right hand side give

$$Z(s_1) = \sum_{\substack{|\alpha|=n,|\beta|=1,\\|\gamma|=m,|\nu_i|=1}} \frac{1}{\sqrt{\mu(s(\nu_1))\mu(r(\nu_1))}} \overline{W(\triangle_{\nu_1,\nu_3,\nu_2})}(\alpha\cdot\nu_1\cdot\beta\cdot\gamma, \alpha\cdot\nu_2\cdot\nu_3\cdot\beta\cdot\gamma)$$

$$Z(s_2) = \sum_{\substack{|\alpha|=n,|\beta|=1,\\|\gamma|=m,|\nu_i|=1}} \frac{1}{\sqrt{\mu(s(\nu'_3))\mu(r(\nu'_3))}} \overline{W(\triangle_{\nu'_3,\nu'_1,\nu'_2})}(\alpha'\cdot\beta'\cdot\nu'_1\cdot\nu'_2\cdot\gamma', \alpha'\cdot\beta'\cdot\nu'_3\cdot\gamma')$$

and so both sides equal

$$\sum_{\substack{|\alpha|=n,|\gamma|=m,\\|\nu'_i|=|\nu_i|=1}} \frac{1}{\sqrt{\mu(s(\nu_3))\mu(r(\nu_3))}} \frac{1}{\sqrt{\mu(s(\nu'_3))\mu(r(\nu'_3))}} W(\triangle_{\nu_1,\nu_3,\nu_2}) W(\triangle_{\nu'_3,\nu'_1,\nu_2})$$
$$(\alpha\cdot\nu_1\cdot\nu'_1\cdot\gamma, \alpha\cdot\nu_3\cdot\nu'_3\cdot\gamma)$$

For isotopies involving rectangles, for the first one, if x has label $(\xi, *)$ the left hand side is

$$Z(t_1) = \sum_{\substack{|\alpha|=n,|\beta|=m,|\gamma|=p\\|\delta|=q,|\nu_i|=1}} \frac{1}{\sqrt{\mu(s(\nu_3))\mu(r(\nu_3))}} W(\triangle_{\nu_3,\nu_1,\nu_2})(\alpha\cdot\beta\cdot\gamma\cdot\nu_1\cdot\nu_2\cdot\delta, \alpha\cdot\beta\cdot\gamma\cdot\nu_3\cdot\delta)$$

$$Z(t_2) = \sum_{\substack{|\alpha'|=n,|\zeta|=m,|\gamma|=p\\|\delta|=q,|\nu'|=1}} c_{\xi,\zeta}(\alpha\cdot\zeta\cdot\gamma\cdot\nu'\cdot\gamma, \alpha\cdot\zeta\cdot\gamma\cdot\nu'\cdot\gamma)$$

and so

$$Z(t) = \sum_{\substack{|\alpha|=n,|\zeta|=m,|\gamma|=p\\|\delta|=q,|\nu_i|=1}} \frac{1}{\sqrt{\mu(s(\nu_3))\mu(r(\nu_3))}} W(\triangle_{\nu_3,\nu_1,\nu_2}) c_{\xi\zeta}(\alpha\cdot\zeta\cdot\gamma\cdot\nu_1\cdot\nu_2\cdot\delta, \alpha\cdot\gamma\cdot\nu_3\cdot\delta)$$

Similarly on the left hand side we have

$$Z(s_1) = \sum_{\substack{|\alpha'|=n,|\zeta|=m,|\gamma|=p\\|\delta|=q,|\nu'|=1}} c_{\xi,\zeta}(\alpha\cdot\zeta\cdot\gamma\cdot\nu'\cdot\gamma, \alpha\cdot\zeta\cdot\gamma\cdot\nu'\cdot\gamma)$$

$$Z(s_2) = \sum_{\substack{|\alpha|=n,|\beta|=m,|\gamma|=p\\|\delta|=q,|\nu_i|=1}} \frac{1}{\sqrt{\mu(s(\nu_3))\mu(r(\nu_3))}} W(\triangle_{\nu_3,\nu_1,\nu_2})(\alpha\cdot\beta\cdot\gamma\cdot\nu_1\cdot\nu_2\cdot\delta, \alpha\cdot\beta\cdot\gamma\cdot\nu_3\cdot\delta)$$

and so $Z(t_1)Z(t_2) = Z(s_1)Z(s_2)$

For the second isotopy in Figure 5.8 we use the same trick with the braiding as for the isotopies involving a cup and rectangle in the proof of Proposition 3.4.7 to transform it into the situation of the first equation in 5.8.

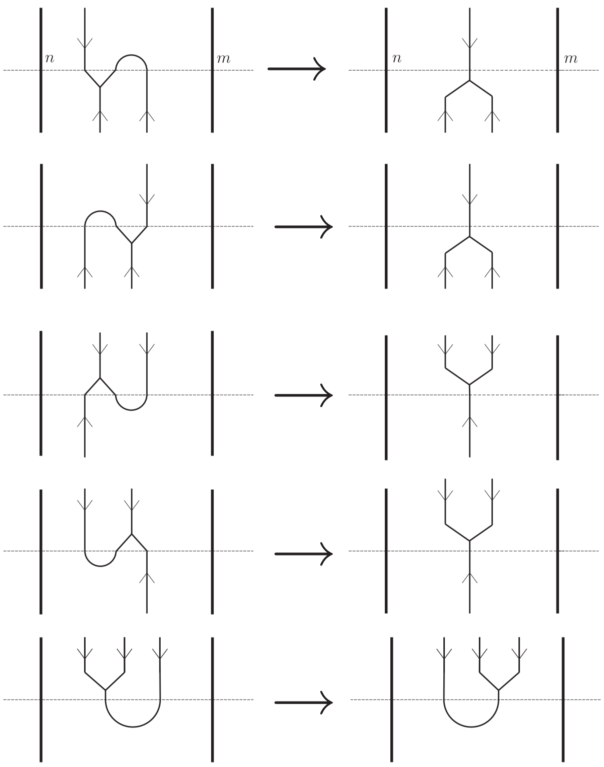


Figure 5.6: Isotopies involving an incoming trivalent vertex and a cup or cap

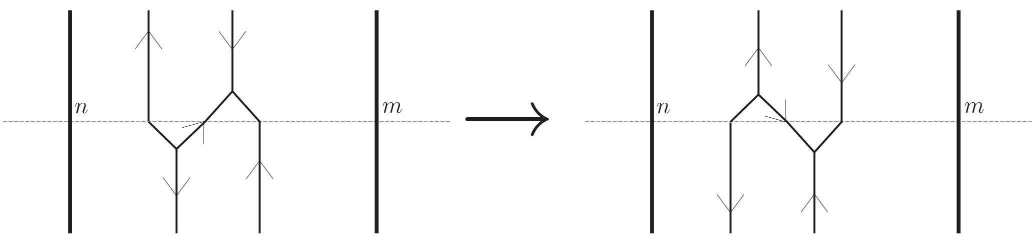


Figure 5.7: Isotopies involving two trivalent vertices

Next we verify that the definition gives a C^*-planar algebra. The spaces $P_{(i,j,k)}$ are finite dimensional by definition, with $P_{(0,0,0)} \simeq \mathbb{C}$ by definition. The state is positive definite, since it is the composition of the projection onto $P_{(j,0)}$ with the positive definite trace on $P_{(j,0)}$ from the type II string algebra. The last thing we

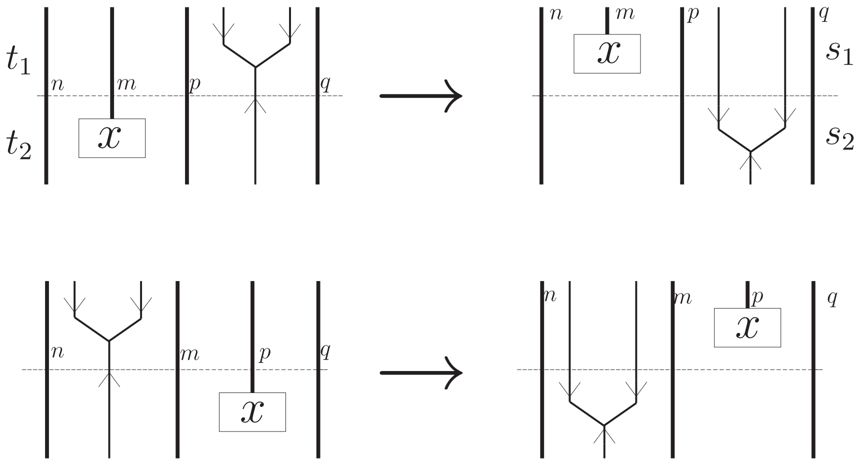


Figure 5.8: Isotopies involving a trivalent vertex and marked point

need to check are the four conditions in the statement of the theorem. The first one is exactly the same as the first condition in Proposition 3.4.7. The second one may be proved as in [23].

$$
\begin{aligned}
Z(W_{-k}) =& (\sum W(\triangle_{\eta_3,\eta_1,\eta_2}) \frac{1}{\sqrt{\mu(s(\eta_1))\mu(r(\eta_1))}} (\xi \cdot \eta_1 \cdot \eta_2 \cdot \zeta, \xi \cdot \eta_3 \cdot \zeta)) \cdot \\
& (\sum W(\triangle_{\eta_4,\eta_5,\eta_6}) \frac{1}{\sqrt{\mu(s(\eta_4))\mu(r(\eta_4))}} (\xi' \cdot \eta_4 \cdot \zeta', \xi \cdot \eta_5 \cdot \eta_5 \cdot \zeta')) \cdot \\
=& \mathcal{U}^{\eta_1\eta_2}_{\eta_5\eta_6} (\xi \cdot \eta_1 \cdot \eta_2 \cdot \zeta, \xi \cdot \eta_5 \cdot \eta_6 \cdot \zeta)
\end{aligned}
$$

The two condtions involving the inclusions follow from the definition of the inclusions in the string algebra. For the conditional expectations, the right expectation is exactly as in the type II case, the left expectation is the minimal expectation defined by $E(x) = \rho(r\rho(x)r^*)$ □

5.2 Further Work

5.2.1 Skein Theory for $\mathcal{D}^{(n)}$ Planar Algebra

Further work to be completed on A_2-planar algebras is the extension of the skein theory results of [63], [5], [74], [4] to the A_2 setting. Here we begin to describe the analogue of the description of the D_{2n} planar algebra using generators and relations in [63]. The D_{2n} planar algebra may be described as the unique planar

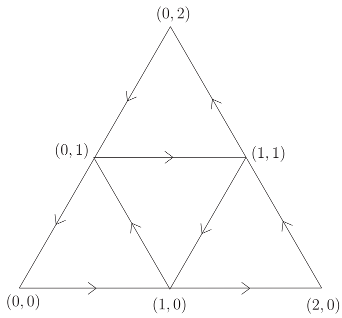


Figure 5.9: The graph $\mathcal{A}^{(6)}$

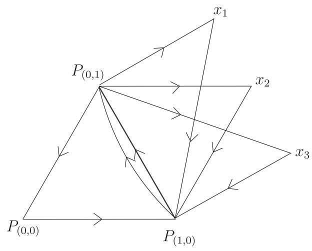
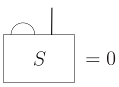
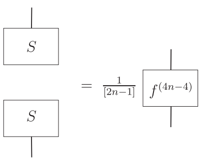


Figure 5.10: The graph $\mathcal{D}^{(6)}$

algebra with a single generator S in the P_{4n-4} modulo the relations

1. the modulus is $\delta = 2\cos(\pi/4n-2)$
2. rotating the marked point of S is equivalent to multiplying by i
3. 

4. 

The $SU(3)$ analogue of this is the planar algebra of the $\mathcal{D}^{2n}$ graphs. Here for simplicity we will look only at the graph $\mathcal{D}^{(6)}$ shown in Figure 5.10. The graph $\mathcal{D}^6$ is an orbifold of the graph $\mathcal{A}^{(6)}$ shown in Figure 5.9.

The $SU(3)$ analogue of the Jones Wenzl idempotents are the Jones Wenzl projectors, defined in [85], [48]. They satisfy the relations $P_{(m,n)} \otimes \downarrow \cong P_{(m+1,n)} \oplus P_{(m-1,n+1)} \oplus P_{(m,n-1)}$ and $P_{(m,n)} = 0$ for all m, n with $m+n \geq 4$ and which may be

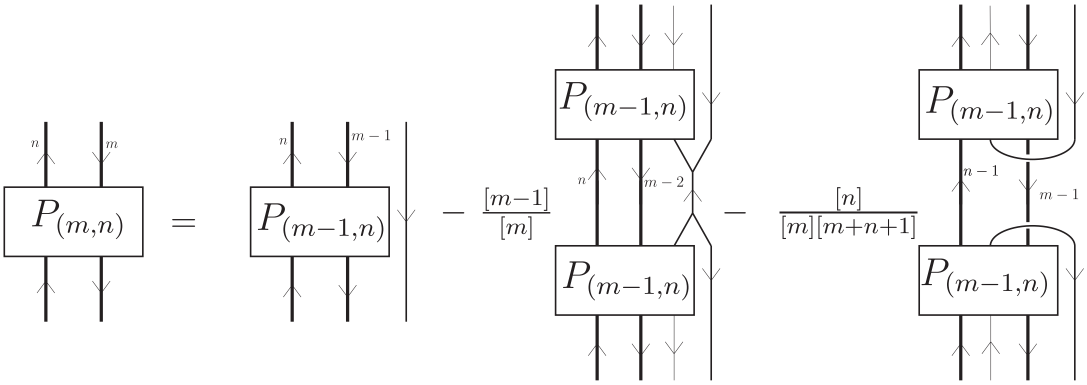


Figure 5.11: Relations between Jones Wenzl Projectors

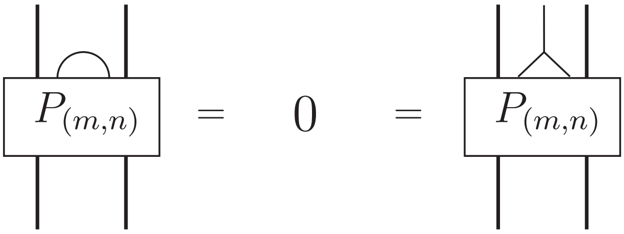


Figure 5.12: Capping off Projectors gives zero

represented graphically by Figure 5.11. They also satisfy the relations in Figure 5.12 for any position of the cap or Y-fork.

The $\mathcal{A}^{(n)}$ planar algebra is generated by the Jones Wenzl projectors with the relation $P_{r,s} = 0$ whenever $r + s = n$.

The vertices of $\mathcal{D}^{(6)}$ correspond to the projectors $P_{(0,0)}, P_{(0,1)}$ and $P_{(1,0)}$ as shown and the projector $P_{(1,1)}$ splits into 3 projections x_1, x_2, x_3. From the graph we can see the projectors satisfy the following relations:

$$x_1 \oplus x_2 \oplus x_3 \cong P_{(1,1)}$$

$$x_i \otimes \downarrow \cong P_{(1,0)}$$

$$x_i \otimes \uparrow \cong P_{(0,1)}$$

$$P_{(0,1)} \otimes \downarrow \cong x_1 + x_2 + x_3 + P_{(0,0)}$$

By drawing the Bratteli diagrams for $\mathcal{D}^{(6)}$ and $\mathcal{A}^{(6)}$ and using Lemma 6.3.1 from [79] which tells us that $\dim B_{(i,j)} = \dim B_{(i+k,j-k)}$ we see that the $(1,1)$ box space of the $\mathcal{D}^{(6)}$ planar algebra has dimension two higher than the corresponding space

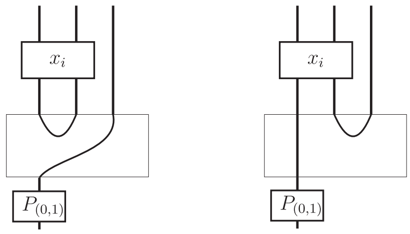


Figure 5.13: $\mathrm{Hom}(x_i \otimes \downarrow, \downarrow)$

for $\mathcal{A}^{(6)}$, this tells us that we must have two linear independent generators in this space. These generators satisfy the relations

$$P_{(1,1)} \oplus S \oplus S^2 = x_1$$

$$P_{(1,1)} \oplus \omega S \oplus \omega^2 S^2 = x_2$$

$$P_{(1,1)} \oplus \omega^2 S \oplus \omega S^2 = x_3$$

where $\omega = e^{\frac{2\pi}{3}}$. This implies

$$x_1 + x_2 + x_3 = P_{(1,1)}$$

$$x_1 + \omega^2 x_2 + \omega x_3 = S$$

$$x_1 + \omega x_2 + \omega^2 x_3 = S^2.$$

Hence S satisfies the property $S^3 = P$ and so multiplying an x_i by S results in a scalar multiple of x_i. The number of edges between any two vertices of the graph $\mathcal{D}^{(6)}$ give us the dimensions of the spaces of homomorphisms between the corresponding projections. Looking at this graphically gives us Figure 5.13. Thus, we see in this case that there are two morphisms between $P_{0,1}$ and $P_{1,0}$. Therefore there is some diagram in P_{++-} that is not a Temperley-Lieb diagram, and so it must contain a copy of S or S^2. We also see that the space of homomorphisms between the $x_i \otimes \downarrow$ and $P_{0,1}$ is one. Graphically we have the picture of Figure 5.13.

Hence one of these diagrams must give zero. Possibly this indicates that we have the uncapability condition on the x_i, similar to condition 3 in the presentation of the D_{2n} planar algebra.

In order to develop a skein theory we must find ways of simplifying diagrams with multiple copies of the generator in them, to do this we must for example find ways of passing strings over or under the generators and relations to simplify diagrams when the generators are joined with a certain amount of strings. From the fact that the projector $P_{(2,2)} = 0$ we would guess that the diagram $S \otimes S$(where we just place two copies of S side by side) should be zero. Therefore we must look at other ways of combining the S, to do so we can for example look at the analogue of Wenzl's relation and use this and the fact that we wish $P_{m,n}$ to be zero when $m + n \geq 4$ to find relations on S.

5.2.2 Constructing Subfactors from A_2-Planar Algebras

Another problem which could be investigated further is using A_2-planar algebras to construct subfactors, both of type II and type III, using some generalisation of the Guionnet-Jones-Shlyakhtenko construction in [28] or the orthogonal version from [49], [42].

As in the Guionnet-Jones-Shlyakhtenko construction we need to define some kind of filtered or graded multiplication to allow multiplication between different P_σ and a tracial state on the planar algebra with this multiplication. In the A_2 case, there is extra complication due to the orientations of the strings.

In the original paper, a Fock space model was used, where the von Neumann algebras were generated as subspaces of $B(H)$ where

$$H = \ell(\Gamma) \oplus \bigoplus H^{\otimes k}$$

where Γ is some bipartite graph and H is a Hilbert space whose orthonormal basis consisting of the edges of Γ. The commutants of the basic operators $\cup$ and $\Cup$ were calculated and this was used to show the von Neumann algebras were factors. In the A_2 case, it may be possible to use the work of Evans and Pugh [22] on spectral measures of the $\mathcal{ADE}$ graphs to study this. In the Fock space model, $\ell(\Gamma)$ should be replaced by $\ell(\Gamma) \otimes \ell(\Gamma)$ and H should be replaced by $H \otimes \bar{H}$.

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