REID'S RECIPE AND DERIVED CATEGORIES

TIMOTHY LOGVINENKO

Abstract. We prove two conjectures from [CL09] which describe the geometrical McKay correspondence for a finite abelian $G \subset \mathrm{SL}_3(\mathbb{C})$ such that $\mathbb{C}^3/G$ has a single isolated singularity. We do it by studying the relation between the derived category mechanics of computing a certain Fourier-Mukai transform and a piece of toric combinatorics known as 'Reid's recipe', effectively providing a categorification of the latter.

1. Introduction

The classical McKay correspondence is a one-to-one correspondence

$$\mathrm{Irr}(G) \setminus \rho_0 \quad \xleftrightarrow{\text{1-to-1}} \quad \mathrm{Exc}(Y)$$

between the non-trivial irreducible representations of a finite subgroup G of $\mathrm{SL}_2(\mathbb{C})$ and the irreducible exceptional divisors on the minimal resolution Y of the singular quotient space $\mathbb{C}^2/G$. It first arose from an observation by McKay in [McK80] which implied a coincidence of the representation graph of G, less the trivial representation ρ_0, and the intersection graph of $\mathrm{Exc}(Y)$. Gonzales-Sprinberg and Verdier in [GSV83] gave a geometric construction where this coincidence was shown to arise naturally from a K-theory isomorphism $\Theta : K^G(\mathbb{C}^3) \rightarrow K(Y)$ between the G-equivariant K-theory of $\mathbb{C}^2$ and the K-theory of Y. In modern language, Θ is defined by identifying Y with G-$\mathrm{Hilb}(\mathbb{C}^2)$, the fine moduli space of G-clusters[1] in $\mathbb{C}^2$, and setting Θ to be the K-theoretic Fourier-Mukai transform defined by the universal G-cluster family $\mathcal{M}$ on $Y \times \mathbb{C}^2$

$$\Theta(-) = \left[\pi_{Y*}\left(\mathcal{M} \otimes \pi^*_{\mathbb{C}^2}(-)\right)\right]^G$$

where π_Y and $\pi_{\mathbb{C}^2}$ are projections from $Y \times \mathbb{C}^2$ to Y and $\mathbb{C}^2$, respectively. The functor $[-]^G : K^G(Y) \rightarrow K(Y)$ is the functor of taking the G-invariant part of a G-sheaf. It was then proved in [GSV83] that for every $\rho \in \mathrm{Irr}(G) \setminus \rho_0$ there exists a unique $E_\rho \in \mathrm{Exc}(Y)$ such that $\Theta(\mathcal{O}_0 \otimes \rho) = \left[\mathcal{O}_{E_\rho}(-1)\right]$, where $\mathcal{O}_0$ is the skyscraper sheaf of the origin $(0,0) \in \mathbb{C}^2$. The group G acts on Y trivially, so every G-sheaf $\mathcal{F}$ on Y splits up as a direct sum $\bigoplus_{\rho \in \mathrm{Irr}(G)} \mathcal{F}_\rho \otimes \rho$ where each $\mathcal{F}_\rho$ is a G-invariant sheaf called *the ρ-eigensheaf* of $\mathcal{F}$. Observe that not only we have $[\mathcal{F}]^G = \mathcal{F}_{\rho_0}$, by definition, but more generally $[\mathcal{F} \otimes \rho]^G = \mathcal{F}_{\rho^\vee}$ for every $\rho \in \mathrm{Irr}(G)$. Thus by looking at $\Theta(\mathcal{O}_0 \otimes \rho)$ we are looking at how does $\pi_{Y*}\left(\mathcal{M}_{|\mathrm{Exc}(Y)\times\{0\}}\right)$ break up into G-eigensheaves. Very roughly, to obtain the correspondence $\mathrm{Irr}(G) \setminus \rho_0 \leftrightarrow \mathrm{Exc}(Y)$ we break up the exceptional set of Y with respect to the G-action on its natural G-cluster scheme structure and observe that for each non-trivial $\rho \in \mathrm{Irr}(G)$ we get a different irreducible curve.

In [CL09] we proposed a program of the geometric McKay correspondence which generalises the ideas of [GSV83] to dimension three. In a celebrated result of [BKR01] it was shown that the K-theoretic equivalence Θ of [GSV83] lifts naturally to the level of derived categories and gives for any finite subgroup $G \subset \mathrm{SL}_n(\mathbb{C})$, where $n = 2$ or 3, the equivalence $\Phi : D(Y) \rightarrow$

[1] A *G-cluster* is a finite-length G-invariant subscheme Z such that $H^0(Z)$ is isomorphic to the regular representation of G. It serves as a scheme-theoretic generalization of a concept of a set-theoretic orbit of G.

$D^G(\mathbb{C}^n)$ between the bounded derived categories of coherent sheaves on the distinguished crepant resolution $Y = G\text{-Hilb}(\mathbb{C}^n)$ and of coherent G-sheaves on $\mathbb{C}^n$. In [CL09] we showed that the inverse $\Psi : D^G(\mathbb{C}^n) \to D(Y)$ of Φ is the Fourier-Mukai transform

$$\Psi(-) = \left[\pi_{Y*} \left(\tilde{\mathcal{M}} \overset{\mathbf{L}}{\otimes} \pi^*_{\mathbb{C}^n}(-) \right) \right]^G$$

defined by the dual family $\tilde{\mathcal{M}}$ of the universal family $\mathcal{M}$ of G-clusters on $Y \times \mathbb{C}^n$. We then showed how to compute the transforms $\Psi(\mathcal{O}_0 \otimes \rho)$ and although apriori each of these transforms is a complex in $D(Y)$ we were able to show that for an abelian G all the cohomologies of this complex vanish except for one, i.e. for every $\rho \in \operatorname{Irr}(G)$ the transform $\Psi(\mathcal{O}_0 \otimes \rho)$ is a shift of a coherent sheaf ([CL09], Theorem 1.1). This is expected to also hold for non-abelian G. We then proposed the geometric McKay correspondence to be $\rho \mapsto \operatorname{Supp}(\Psi(\mathcal{O}_0 \otimes \rho))$, assigning to every ρ a closed subscheme of the exceptional set of Y. In dimension 2 this gives precisely the classical $\operatorname{Irr}(G) \setminus \rho_0 \leftrightarrow \operatorname{Exc}(Y)$ correspondence of [GSV83]. In dimension 3 the correspondence is more complicated - as was expected given that generally G has more irreducible representations then there are irreducible divisors on Y. For abelian $G \subset \mathrm{SL}_3(\mathbb{C})$ we were able to employ the numerical methods of [Log08b] to compute this correspondence very explicitly ([CL09], §6). Based on numerous computational evidence we made a conjecture as to the form that $\Psi(\mathcal{O}_0 \otimes \rho)$ take in dimension three and in the present paper we prove it:

Theorem 1.1. *([CL09], Conj. 1.1) Let $G \subset \mathrm{SL}_3(\mathbb{C})$ be a finite abelian subgroup such that $\mathbb{C}^3/G$ has a single isolated singularity at the origin. Then for any $\chi \in \operatorname{Irr}(G)$ the Fourier-Mukai transform $\Psi(\mathcal{O}_0 \otimes \chi)$ is one of the following:*

(1) $\mathcal{L}_\chi^{-1} \otimes \mathcal{O}_{E_i}$
(2) $\mathcal{L}_\chi^{-1} \otimes \mathcal{O}_{E_i \cap E_j}$
(3) $\mathcal{F}[1]$ *where* $\operatorname{Supp}_Y(\mathcal{F}) = E_{i_1} \cup \dots \cup E_{i_k}$
(4) $\mathcal{O}_Y(\operatorname{Exc}(Y)) \otimes \mathcal{O}_{\operatorname{Exc}(Y)}[2]$

where E_i are irreducible exceptional divisors, $\mathcal{F}$ is a coherent sheaf and $\mathcal{L}_\chi = \Psi(\mathcal{O}_{\mathbb{C}^3} \otimes \chi^{-1}) = (\pi_{Y*}\mathcal{M})_\chi$ are the tautological bundles on Y.

We prove Theorem 1.1 by investigating the relation we uncovered between the geometric McKay correspondence and a piece of toric geometric combinatorics known as 'Reid's recipe'. It was originally developed by Reid in [Rei97] and then employed by Craw in [Cra05] to tackle the problem of finding a basis for $H^*(Y, \mathbb{Z})$ naturally bijective to $\operatorname{Irr}(G)$. This problem also has its roots in [GSV83] where it was shown that $\{c_1(\mathcal{L}_\rho)\}_{\rho \in \operatorname{Irr}(G) \setminus \rho_0}$ is the basis of $H^2(Y, \mathbb{Z})$ dual to the basis $\{[E]\}_{E \in \operatorname{Exc}(Y)}$ of $H_2(Y, \mathbb{Z})$ with $[E_\chi]$ being precisely the vector dual to $c_1(\mathcal{L}_\chi)$. Taking $\mathcal{L}_{\rho_0} = \mathcal{O}_Y$ to base $H^0(Y, \mathbb{Z})$ we obtain a natural basis of $H^*(Y, \mathbb{Z})$ in dimension two. In dimension three $c_1(\mathcal{L}_\chi)$ still span $H^2(Y, \mathbb{Z})$, but there are relations. Reid's recipe singles out $\mathcal{L}_\chi$ whose first Chern classes are redundant and replaces them by abstract elements of $K(Y)$ in such a way that the second Chern classes of these 'virtual' bundles base $H^4(Y, \mathbb{Z})$. The recipe is based around a marking which via some simple toric geometric calculations assigns a character $\chi \in \operatorname{Irr}(G)$ to every exceptional toric curve on Y and then a character or a pair of characters to every exceptional toric divisor $E \in \operatorname{Exc}(Y)$ (see [Cra05], Section 3 or our short summary of it in the Section 2.2 of the present paper). Based on some more computational evidence we conjectured in [CL09], Conj. 1.3, the relation between the marking of Reid's recipe and the transforms $\Psi(\mathcal{O}_0 \otimes \chi)$. A stronger version of this conjecture we prove in the present paper. The "only if" implication of item (1) was proved by Craw and Ishii in [CI04], Prop. 9.3, but the rest are original to this paper and for the first time a complete categorification is obtained which for every possibility for χ in Reid's recipe describes the transform $\Psi(\mathcal{O}_0 \otimes \chi)$ in $D(Y)$:

Theorem 1.2. *Let $G \subset \mathrm{SL}_3(\mathbb{C})$ be a finite abelian subgroup such that $\mathbb{C}^3/G$ has a single isolated singularity at the origin. Let χ be a character of G. Then in Reid's recipe χ marks*

(1) *a divisor $E \in \mathrm{Exc}(Y)$ if and only if $\Psi(\mathcal{O}_0 \otimes \chi) = \mathcal{L}_\chi^{-1} \otimes \mathcal{O}_E$.*
(2) *a single curve C if and only if $\Psi(\mathcal{O}_0 \otimes \chi) = \mathcal{L}_\chi^{-1} \otimes \mathcal{O}_C$.*
(3) *several curves if and only if $\Psi(\mathcal{O}_0 \otimes \chi)$ is supported in degree -1. The support of $H^{-1}(\Psi(\mathcal{O}_0 \otimes \chi))$ is then the union of all $E \in \mathrm{Exc}(Y)$ which contain two or more of the curves marked by χ.*
(4) *nothing (i.e. χ is the trivial character χ_0) if and only if $\Psi(\mathcal{O}_0 \otimes \chi) = \mathcal{O}_Y(\mathrm{Exc}(Y)) \otimes \mathcal{O}_{\mathrm{Exc}(Y)}[2]$.*

In this stronger form Theorem 1.2 easily implies Theorem 1.1. To prove Theorem 1.2 it was necessary to relate the derived category mechanics of computing the transforms $\Psi(\mathcal{O}_0 \otimes \chi)$ to the toric combinatorics of the markings in Reid's recipe. The first step towards this was made with the *sink-source graphs* of [CL09], Section 4. Given an exceptional divisor $E \in \mathrm{Exc}(Y)$ the sink-source graph $SS_{\mathcal{M},E}$ of $\mathcal{M}$ along E is a graph drawn on top of the *McKay quiver* $Q(G)$ whose vertices are certain vertices of $Q(G)$ and whose edges are certain paths in $Q(G)$. Which vertices and which paths is determined by the behaviour of the family $\mathcal{M}$ generically along E (see Section 2.4). Something employed throughout [CL09] but never stated explicitly was that for any $\chi \in \mathrm{Irr}(G)$ the divisor E belongs to the support of $H^i\Psi(\mathcal{O}_0 \otimes \chi)$ for a certain i if and only if χ is a vertex of $SS_{\mathcal{M},E}$ of a certain type (Prps. 2.7 of the present paper).

It was shown in [CL09] that there are only three possible shapes that the sink-source graph $SS_{\mathcal{M},E}$ of any divisor $E \in \mathrm{Exc}(Y)$ can have (see Section 2.4). On the other hand, in Reid's recipe all the divisors in $\mathrm{Exc}(Y)$ are divided into three classes and the marking for a divisor is determined in a different fashion in each class (see §3 of [Cra05] or our account of it in Section 2.2). The crucial step at the heart of the present paper is showing that these three classes of the divisors in Reid's recipe and the three possible shapes of the sink-source graphs of $\mathcal{M}$ are in exact correspondence (Theorem 3.1). Moreover, we can calculate the monomials which define the toric curves contained in E in terms of the lengths of the edges of $SS_{\mathcal{M},E}$[2] and vice versa (see Prps. 3.1-3.3). With this established most of the work needed to prove Theorem 1.2 is done as for any $E \in \mathrm{Exc}(Y)$ and any $\chi \in \mathrm{Irr}(G)$ we can translate the information that $E \in H^i \operatorname{Supp} \Psi(\mathcal{O}_0 \otimes \chi)$ for some i via the sink-source graph of E into the information on the markings of E and of the toric curves E contains. It then remains only to exclude some cases of $\operatorname{Supp} \Psi(\mathcal{O}_0 \otimes \chi)$ containing extra curves where it shouldn't, which we do via straightforward calculations with long exact sequences in sheaf cohomology.

Acknowledgments: The author would like to thank Sabin Cautis, his co-author on [CL09]. The author would also like to thank Alastair King, Alastair Craw and Akira Ishii for useful discussions and suggestions. Most of the paper was written during the author's stay at the University of Liverpool and he would like to thank everyone at the maths department there and to thank Prof. Viacheslav Nikulin in particular. The paper was completed during the COE-COW conference in the University of Tokyo in December 2008 and the author would like to thank the organisers for inviting him to speak there and thus providing him with a good deadline to have this paper finished by.

[2] A length of an edge in $SS_{\mathcal{M},E}$ is the number of arrows which compose the corresponding path in the McKay quiver.

2. Preliminaries

Throughout the paper, we take G to be a finite abelian subgroup of $\mathrm{SL}_3(\mathbb{C})$ such that $\mathbb{C}^3/G$ has a single isolated singularity at the origin.

2.1. **G-Hilb $\mathbb{C}^3$.** As is usual we approach the resolution $Y = G\text{-Hilb}\,\mathbb{C}^3$ via the methods of toric geometry. For detailed explanation of this see [Log03], Section 3.1 or [CR02]. In brief, let $\mathbb{Z}^3$ be the lattice of Laurent monomials, where we identify point $m = (m_1, m_2, m_3)$ with the monomial $x_1^{m_1}x_2^{m_2}x_3^{m_3}$. Let $M \subset \mathbb{Z}^3$ be the sublattice of G-invariant monomials. Dually, we have the overlattice $(\mathbb{Z}^3)^\vee \subset L$, known as the lattice of weights. As G is finite we have $L \subset \mathbb{Q}^3$ and we think of any point $l \in L$ as of a triplet $(l_1, l_2, l_3) \in \mathbb{Q}^3$.

Let σ_+ be the positive octant cone in $L \otimes \mathbb{R} = \mathbb{R}^3$ defined by $\{(e_i) \in \mathbb{R}^3 \mid e_i \geq 0\}$. The original affine space $\mathbb{C}^3$ is defined as a toric variety by a single cone σ_+ and the lattice $\mathbb{Z}^3$. The singular quotient $\mathbb{C}^3/G$ is defined by σ_+ and the lattice L. The crepant resolutions of $\mathbb{C}^3/G$ are defined in the lattice L by the fans which subdivide the cone σ_+ into regular subcones. Let Δ denote the section of σ_+ by the hyperplane $\sum e_i = 1$ in $\mathbb{R}^3$. It is a regular triangle which we call *the junior simplex*. We identify the subdivisions of σ_+ into regular subcones with the corresponding triangulations of the junior simplex Δ.

It is described in [Cra05], Section 2, how to construct the triangulation Σ of Δ whose corresponding fan gives the crepant resolution $Y = G\text{-Hilb}\,\mathbb{C}^3$. Denote this fan by $\mathfrak{F}$. To each k-dimensional cone σ in $\mathfrak{F}$ corresponds a $(3-k)$-dimensional torus orbit S_σ in Y. Denote by E_σ the subscheme of Y given by the closure of S_σ, then E_σ is the union of $S_{\sigma'}$ for all cones σ' which contain σ as a face. Denote by $\mathfrak{E}$ the set $L \cap \Delta$, these are the vertices of the triangles in Σ and, correspondingly, the generators of the one-dimensional cones in $\mathfrak{F}$. Then $\{E_e\}_{e \in \mathfrak{E}}$, where we write E_e for $E_{\langle e \rangle}$, are precisely the exceptional divisors of Y together with strict transforms of the hyperplanes $x^{|G|} = 0$, $y^{|G|} = 0$ and $z^{|G|} = 0$ in $\mathbb{C}^3/G$.

Two-dimensional cones in $\mathfrak{F}$ are the sides of the triangles in Σ. For any $e, f \in \mathfrak{E}$ the cone $\langle e, f \rangle$ lies in $\mathfrak{F}$ if and only if the exceptional divisors E_e and E_f intersect. The orbit closure $E_{e,f}$ is precisely the intersection $E_e \cap E_f$ and it is always a $\mathbb{P}^1$.

Three-dimensional cones in $\mathfrak{F}$ are the triangles in Σ. For any such cone σ we denote by A_σ the toric affine chart which consists of the torus orbits corresponding to all cones in $\mathfrak{F}$ which are faces of σ. We have a natural isomorphism $A_\sigma \simeq \mathbb{C}^3$ which maps the torus fixed point E_σ to the origin $0 \in \mathbb{C}^3$.

2.2. **Reid's recipe.** Let $\mathcal{M}$ be the universal family of G-clusters on $Y \times \mathbb{C}^3$. It is a G-equivariant coherent sheaf on $Y \times \mathbb{C}^3$. The category of G-equivariant quasi-coherent sheaves on $Y \times \mathbb{C}^3$ is equivalent to the category of quasi-coherent $(R \rtimes G) \otimes_{\mathbb{C}} \mathcal{O}_Y$-modules on Y via the pushdown functor. So we shall often abuse the notation by identifying $\mathcal{M}$ with its pushdown to Y considered as an $(R \rtimes G) \otimes_{\mathbb{C}} \mathcal{O}_Y$-module.

The family $\mathcal{M}$ is defined up to an equivalence of families, that is - up to a twist by a (G-invariant) line bundle on Y. We normalize by assuming the line bundle $[\mathcal{M}]^G$ to be trivial and identifying it with $\mathcal{O}_Y$. This uniquely determines $\mathcal{M}$ up to isomorphism. Write the decomposition of $\mathcal{M}$ into G-eigensheaves as $\mathcal{M} = \bigoplus_{\chi \in G^\vee} \mathcal{L}_\chi$ where G acts on $\mathcal{L}_\chi$ by χ. The sheaves $\mathcal{L}_\chi$ are line bundles on Y known in the literature as *tautological* or *Gonzales-Sprinberg and Verdier* sheaves.

Reid's recipe ([Rei97], [Cra05]) is an algorithm to construct the cohomological version of the McKay correspondence. It takes the Chern classes of the tautological sheaves and modifies some of them to turn them into a basis of the cohomology ring $H^\bullet(Y, \mathbb{Z})$. It consists of two parts: in the first each edge and each vertex in the triangulation Σ are marked by a character of G in accordance with the geometry of the toric fan $\mathfrak{F}$ of Y. In the second the data of this marking is used to dictate the way in which the Chern classes of the tautological sheaves are modified to produce a basis of $H^{\bullet}(Y, \mathbb{Z})$. This cohomological construction is not relevant to this paper - it is replaced by the version of the McKay correspondence proposed in [CL09] where we look at the images of $\mathcal{O}_0 \otimes \chi$ under the derived equivalence of [BKR01]. However, the data of the marking constructed in the first half of the Reid's recipe turns out to dictate the way our correspondence goes as well – as demonstrated by our main result, Theorem 1.2. In a sense, our correspondence takes the same source data of the Reid's recipe marking but realises it in the derived category instead of the cohomology ring.

Below we give a brief summary of the construction of this marking. Denote by R the coordinate ring $\mathbb{C}[x, y, z]$ of $\mathbb{C}^3$. First we mark each edge (e, f) in the triangulation Σ with a character of G according to the following rule. The one-dimensional ray in M perpendicular to the hyperplane $\langle e, f \rangle$ in L has two primitive generators: $\frac{m_1}{m_2}$ and $\frac{m_2}{m_1}$, where m_1, m_2 are co-prime regular monomials in R. As M is the lattice of G-invariant Laurent monomials, m_1 and m_2 have to be of the same character χ for some $\chi \in G^{\vee}$. We say that (e, f) is *carved out* by the ratio $m_1 : m_2$ (or $m_2 : m_1$) and *mark it* by χ.

Then we mark the vertices of Σ according to a recipe which is based on the following classification:

Proposition 2.1 ([Cra05], see §2-§3)**.** *For any $e \in \mathfrak{E}$ the corresponding vertex in the triangulation Σ is one of the following:*

1. *A meeting point of three lines emanating from the three vertices of Δ, as depicted on Figure 1.*
2. *An interior point of exactly one line emanating from a vertex of Δ. Other than the two edges coming from this line, it also has 2, 3 or 4 other edges incident to it, as depicted on Figure 2 (up to permutation of x, y and z).*
3. *An intersection point of three straight lines none of which emanate from a vertex of Δ, as depicted on Figure 3.*

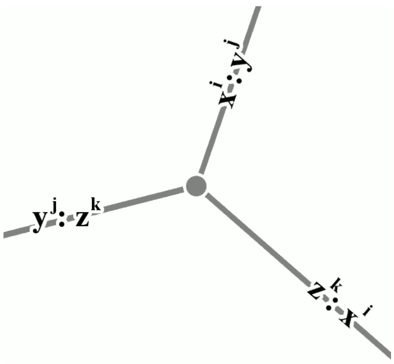
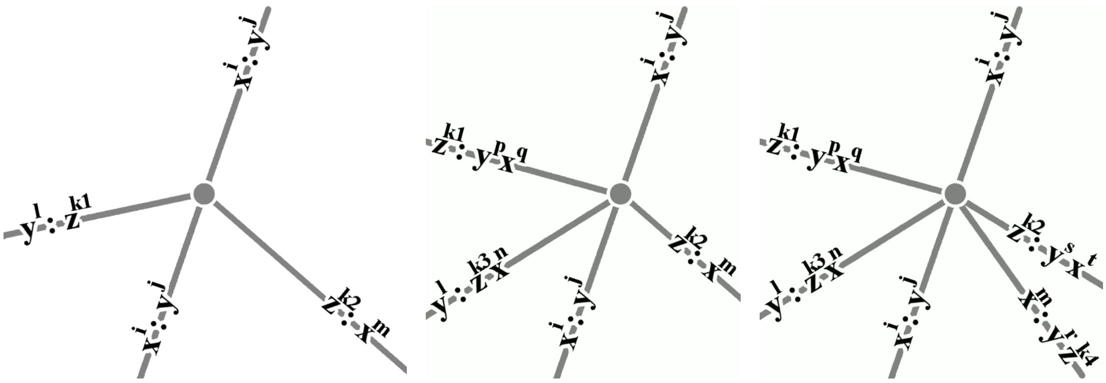


FIGURE 1. Case 1

When vertex $e \in \mathfrak{E}$ belongs to the Case 1 it is clear from Figure 1 that all the three edges incident to e are marked with the same character $\chi \in G^{\vee}$, which is the common character of x^i, y^j and z^k. Reid's recipe prescribes for such e to be marked with the character $\chi \cdot \chi$.

When e belongs to the Case 2 it is proved in [Cra05], Lemmas 3.2-3.3 that $k_1 = k_2$. Reid's recipe prescribes for such e to be marked with the character $\chi \cdot \chi'$, where χ is the common character of y^j and z^k, i.e. the character which marks the unique line which emanates from one of the vertices of Δ and contains e as interior point, and χ' is the character of $x^{k_1} = x^{k_2}$, i.e. the character which marks precisely two of the remaining edges incident to e.



(a) Valency 4 (b) Valency 5 (c) Valency 6

FIGURE 2. Case 2

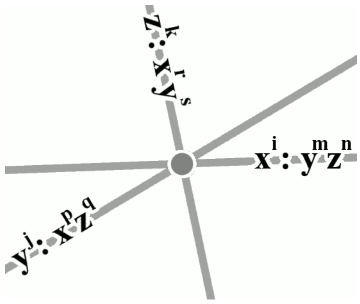


FIGURE 3. Case 3

Finally, when e belongs to the Case 3 it is proved in [Cra05], Lemma 3.4 that the monomials $x^i z^q$, $y^j x^r$ and $z^k y^m$ are all of the same character. Denote it by $\chi \in G^\vee$. It is also proved that the monomials $x^i y^s$, $z^k x^p$ and $y^j z^n$ are also all of the same character. Denote it by $\chi' \in G^\vee$. Reid's recipe prescribes for such e to be marked by two characters - χ and χ'.

2.3. *G-***clusters and** *G-***graphs.** The resolution map $Y \to \mathbb{C}^3/G$ induces an inclusion of $K(Y)$ into $K(\mathbb{C}^3)$ and thus allows to view $K(\mathbb{C}^3)$ as a constant sheaf of $(R \rtimes G) \otimes_{\mathbb{C}} \mathcal{O}_Y$-modules on Y. As shown in [Log08b], §3.1, there is a unique $(R \rtimes G) \otimes_{\mathbb{C}} \mathcal{O}_Y$-module embedding of $\mathcal{M}$, normalized as above, into $K(\mathbb{C}^3)$ which maps $\mathcal{O}_Y \subset \mathcal{M}$ identically to $\mathcal{O}_Y \subset K(\mathbb{C}^3)$. Then for each character $\chi \in G^\vee$ the image of the embedding $\mathcal{L}_\chi \hookrightarrow K(\mathbb{C}^3)$ is $\mathcal{L}(-D_\chi)$ for some uniquely defined fractional G-Weil divisor $D_\chi = \sum_{f \in \mathfrak{E}} q_{\chi,f} E_f$ ([Log08b], §2). Thus $\mathcal{M}$ can be written canonically as $\bigoplus_{\chi \in G^\vee} \mathcal{L}(-D_\chi)$.

Let σ be a three-dimensional cone in $\mathfrak{F}$ generated by some $i, j, k \in \mathfrak{E}$. Then $\mathcal{L}(-D_\chi)$ is generated inside $K(\mathbb{C}^3)$ on the affine chart A_σ by the unique Laurent monomial r for which

$$i(r) = q_{\chi,i}, \quad j(r) = q_{\chi,j}, \quad k(r) = q_{\chi,k}. \tag{2.1}$$

This is natural, considering that for any $i \in \mathfrak{E}$ and any Laurent monomial $m \in \mathbb{Z}^3$ we have $i(m) = \mathrm{val}_{E_i}(m)$ with the RHS being a $\mathbb{Q}$-valuation defined as $\frac{1}{|G|}\mathrm{val}_{E_i} m^{|G|}$, where G-invariant Laurent monomial $m^{|G|}$ is treated as a rational function on Y and is valuated at prime Weil divisor E_i ([Log03], Prps. 3.2).

Let σ be a three-dimensional cone in $\mathfrak{F}$. The set $\Gamma_\sigma = \{r_\chi\}_{\chi \in G^\vee}$, where r_χ is the unique monomial generator of $\mathcal{L}(-D_\chi)$ over the affine chart A_σ, is called the *G-graph* of A_σ. The monomials in Γ_σ are precisely the monomials which do not lie in the ideal $I_\sigma \subset R$ defining the G-cluster parametrised by the torus fixed point E_σ of the chart A_σ.

Let (e,f) be any edge in the triangulation Σ. Let (e,f,g) and (e,f,g') be the two triangles containing it and let σ and σ' be the corresponding three-dimensional cones in $\mathfrak{F}$. Let $\Gamma_\sigma = \{r_\chi\}_{\chi \in G^\vee}$ and $\Gamma_{\sigma'} = \{r'_\chi\}_{\chi \in G^\vee}$ be the G-graphs of affine toric charts A_σ and $A_{\sigma'}$. Suppose that the hyperplane $\langle e, f\rangle$ in L is carved out by the ratio $m : m'$ for some co-prime regular monomials m, m' in R. Suppose, without loss of generality, that $g(\frac{m'}{m}) > 0$. The following simple observation is fundamental in explaining the link between Reid's recipe and the geometrical McKay correspondence:

Lemma 2.2. *If $r_\chi \neq r'_\chi$ for some $\chi \in G^\vee$ then $m | r_\chi$ and $m' | r'_\chi$.*

Proof. Since r_χ and r'_χ generate $\mathcal{O}_Y$-module $\mathcal{L}(-D_\chi)$ on toric affine charts A_σ and $A_{\sigma'}$, respectively, $\frac{r'_\chi}{r_\chi}$ has to be invertible on $A_\sigma \cap A'_\sigma$. Therefore the valuation of $\frac{r'_\chi}{r_\chi}$ is zero on E_e and E_f. In other words $\frac{r'_\chi}{r_\chi} \in \langle e, f\rangle^\perp$. Since $\langle e, f\rangle^\perp$ is a one-dimensional ray in M generated by $\frac{m'}{m}$ we must have $\frac{r'_\chi}{r_\chi} = (\frac{m'}{m})^k$ for some $k \in \mathbb{Z}$.

On the other hand, recall that we have $\mathcal{M}^G = \mathcal{O}_Y$ and therefore 1 is a global section of $\mathcal{M}$. So is any $f \in R$ as R acts on $\mathcal{M} \subset K(\mathbb{C}^3)$ by restriction of the natural action of R on $K(\mathbb{C}^3)$ by multiplication and so $f = f \cdot 1$. In particular, both r_χ and r'_χ are sections of $\mathcal{L}(-D_\chi)$ on A_σ. Since r_χ generates $\mathcal{L}(-D_\chi)$ on U_σ as an $\mathcal{O}_Y$-module we must have $\frac{r'_\chi}{r_\chi} \in \mathcal{O}_Y(U_\sigma)$ and hence $g(\frac{r'_\chi}{r_\chi}) = \mathrm{val}_{E_g}(\frac{r'_\chi}{r_\chi}) \geq 0$. As we assumed that $g(\frac{m'}{m}) > 0$, we conclude that $k \geq 0$. And as by assumption $r_\chi \neq r'_\chi$ we further have $k > 0$. The claim now follows. $\square$

Corollary 2.3. *Let $\chi \in G^\vee$ be the character of G which marks $\langle e, f\rangle$, i.e. χ is the common character of m and m'. Then $r_\chi = m$, whilst $r'_\chi = m'$.*

Proof. There has to exist $\chi' \in G^\vee$ for which $r_{\chi'} \neq r'_{\chi'}$ as otherwise the G-clusters parametrised by the torus fixed points E_σ and E'_σ would be isomorphic. By Lemma 2.2 we must then have $m|r_{\chi'}$ and $m'|r_{\chi'}$. But any regular monomial which divides an element of a G-graph must itself belong to that G-graph, as the compliment of a G-graph in a set of regular monomials is the monomial part of an ideal in R. The claim follows. $\square$

Corollary 2.4. *There doesn't exist a G-invariant Laurent monomial $\frac{m_1}{m'_1} \neq 1$ such that m_1 and m'_1 strictly divide m and m' respectively.*

Proof. Suppose such $\frac{m_1}{m'_1}$ were to exist. By Corollary 2.3 we have $m \in \Gamma_\sigma$ and as $m_1 \mid m$ we must also have $m_1 \in \Gamma_\sigma$. Similarly, we must have $m'_1 \in \Gamma_{\sigma'}$. Since $\frac{m_1}{m'_1}$ is G-invariant the monomials m_1 and m'_1 have to be of the same character and as $\frac{m_1}{m'_1} \neq 1$ we must have $m_1 \neq m'_1$. But then by Lemma 2.2 we must have $m \mid m_1$ and $m' \mid m'_1$, contradicting the assumption that m_1 and m'_1 strictly divide m and m'. $\square$

2.4. **The McKay quiver of G and the sink-source graphs.** A detailed account of this is given in [CL09], Section 4. Below we briefly summarize the essentials.

The action of G on R is obtained from the action of G on $\mathbb{C}^3$ by setting $g \cdot m(\mathbf{v}) = \mathrm{m}(\mathrm{g}^{-1} \cdot \mathbf{v})$ for all $m \in R$ and $\mathbf{v} \in \mathbb{C}^3$. For any regular monomial $m \in R$ denote by $\kappa(m)$ the character with which G acts on m. Quite generally, to any finite subgroup $G \subset \mathrm{GL}_n(C)$ we can associate a quiver $Q(G)$ called *the McKay quiver of* G. In our case of a finite abelian subgroup of $\mathrm{SL}_3(\mathbb{C})$ the quiver $Q(G)$ has as its vertices the characters $\chi \in G^\vee$ of G and from every vertex χ there are three arrows going to $\kappa(x)\chi$, $\kappa(y)\chi$ and $\kappa(z)\chi$. We denote these arrows by (χ, x), (χ, y) and (χ, z) and say that they are x-, y- and z-oriented, respectively.

There exists a standard planar embedding of $Q(G)$ into a real two dimensional torus first constructed by Craw and Ishii in [CI04]. We use the version of it detailed in [CL09], Section 4.1. The torus, which we denote by T_G, is tesselated by the embedded $Q(G)$ into $2|G|$ regular triangles. Locally, this tesselation looks as depicted on Figure 4. When depicting T_G in

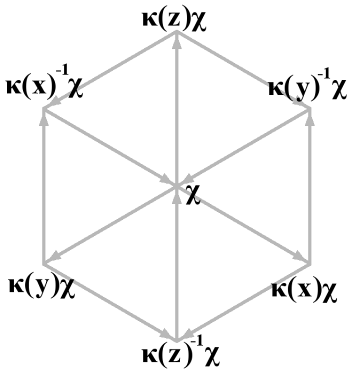


FIGURE 4. The tesselation of T_G by $Q(G)$ (locally)

diagrams we draw its fundamental domain in $\mathbb{R}^2$. For an example see Figure 5 where we give a depiction of the McKay quiver of $G = \frac{1}{13}(1,5,7)$ embedded into T_G. $\frac{1}{13}(1,5,7)$ is a common shorthand for the image in $\mathrm{SL}_3(\mathbb{C})$ of the group of 13th roots of unity under the embedding $\xi \mapsto \begin{pmatrix} \xi^1 & & \\ & \xi^5 & \\ & & \xi^7 \end{pmatrix}$.

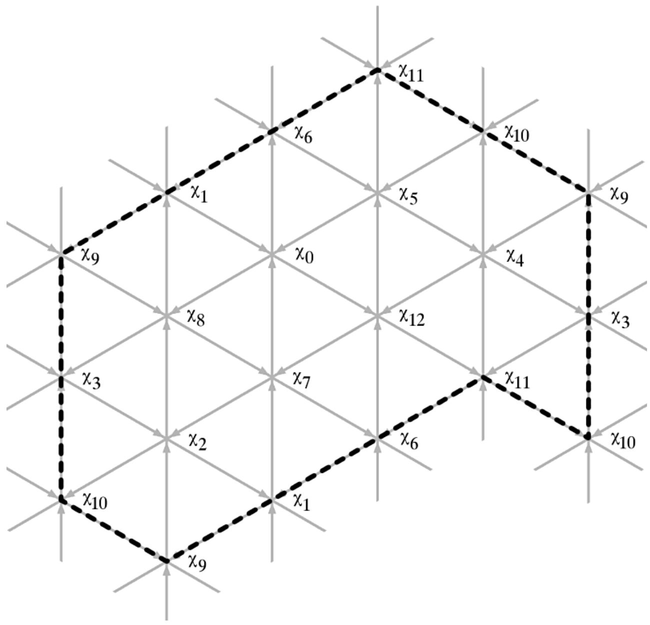


FIGURE 5. The McKay quiver of $G = \frac{1}{13}(1,5,7)$

The importance of the McKay quiver for us is due to the fact that G-clusters are a special case of a more general concept of *G-constellations*, which are coherent G-sheaves on $\mathbb{C}^3$ whose global sections are the regular representation V_{reg}. The category of G-constellations is equivalent to the category of $R \rtimes G$-modules whose underlying G-representation is V_{reg} and the latter category is equivalent to the category of representations of the McKay quiver into the graded vector space $V_{\mathrm{reg}} = \oplus_\chi \mathbb{C}_\chi$ where $\mathbb{C}_\chi$ is a copy of $\mathbb{C}$ on which G acts by χ. This equivalence enables us to define for the universal family $\mathcal{M}$ of G-clusters (or, more generally, for any *gnat*-family[3]) its *associated representation* $Q(G)_\mathcal{M}$ of the McKay quiver over Y(see [CL09], Section 4.2). To any arrow (χ, x) of $Q(G)$ in this representation corresponds a map $\alpha_{\chi,x}$ from $\mathcal{L}(-D_{\chi^{-1}})$, the χ-eigensheaf of $\mathcal{M}$, to $\mathcal{L}(-D_{\kappa(x_k)^{-1}\chi^{-1}})$, the $\kappa(x)\chi$-eigensheaf of $\mathcal{M}$. This map is given by $s \mapsto x \cdot s$. Denote by $B_{\chi,x}$ the locus in Y where map $\alpha_{\chi,x}$ vanishes. It follows from [CL09], Propositions 4.4 and 4.5 that $B_{\chi,x}$ is an effective divisor of form $\sum_{e \in \mathfrak{E}} b_e E_e$ where $b_e \in \{0,1\}$. We say that arrow (χ, x) vanishes along E_e if so does the corresponding map $\alpha_{\chi,x}$ in the associated representation $Q(G)_\mathcal{M}$, i.e. $E_e \subset B_{\chi,x}$. Similarly for the arrows (χ, y) and (χ, z).

Let $E \in \operatorname{Exc} Y$. For every character $\chi \in G^\vee$ we classify the corresponding vertex of $Q(G)$ according to which arrows in the subquiver $\operatorname{Hex}(\chi)$, formed by the six triangles containing χ as per Figure 4, vanish along the divisor E and which do not ([CL09], Prps. 4.7). On Figures 6 - 10 we list all possible cases, drawing in black the arrows which vanish and in grey the arrows which don't. These cases divide into four basic classes : the *charges*, the *sources*, the *sinks* and the *tiles*. The reason for this choice of names is that charge vertices always occur in $Q(G)$ in straight lines propagating from a source vertex to a sink vertex. An x-oriented charge propagates along x-oriented arrows of $Q(G)$ and similarly for y and z. A type $(1,0)$-charge propagates in the direction of the arrows, while a type $(0,1)$-charge propagates against the direction of the arrows. A type (a,b)-source (resp. sink) emits (resp. receives) a charges of type $(1,0)$ and b-charges of type $(0,1)$.

The *sink-source graph* $SS_{\mathcal{M},E}$ is a graph drawn on top of $Q(G)$ whose vertices are the sinks and the sources and whose edges are the charge lines. The torus T_G is then divided up by $SS_{\mathcal{M},E}$ into several regions with all vertices interior to any one region being tiles of same orientation. It turns out ([CL09], Proposition 4.14 and then as per Corollary 4.16) that there are only three possible shapes that $SS_{\mathcal{M},E}$ can take and we depict them on Figures 11(a) - 11(c) indicating by thin grey lines the orientation of tiles in that region. In case of Figure 11(b) the rotations of the depicted shape by $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ are also possible and since one of the tile regions is non-contractible we indicated by a dotted line the fundamental domain of $Q(G)$.

It is worth noting that sink-source graph $SS_{\mathcal{M},E}$ determines completely the divisor E which gave rise to it. This is because it completely determines which arrows of $Q(G)$ do and which do not vanish along E: no arrow which belongs to one of the charge lines vanishes along E and within each of the regions into which $SS_{\mathcal{M},E}$ divides up T_G the arrows that vanish are those that have the same orientation as the tile vertices of that region. Then the following result can be applied to determine the toric coordinates of $e \in \mathfrak{E}$ which corresponds to E:

Lemma 2.5. *Let $e \in \mathfrak{E}$. Suppose the total number of x-oriented arrows of $Q(G)$ which vanish along E_e in $Q(G)_\mathcal{M}$ is a, the total number of y-oriented arrows is b and the total number of z-oriented arrows is c. Then $e = \frac{1}{|G|}(a,b,c) \in L \subset \mathbb{Q}^3$.*

Proof. Consider the sum of the vanishing divisors of all x-oriented arrows of $Q(G)$:

$$\sum_{\chi \in G^\vee} B_{\chi,x} \tag{2.2}$$

[3]A *gnat-family is a flat family of G-constellations satisfying certain geometrical naturality criteria. See* [Log08a], *§3.2.*

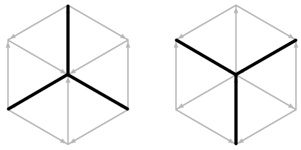


FIGURE 6. The $(3,0)$-sink and the $(0,3)$-sink.

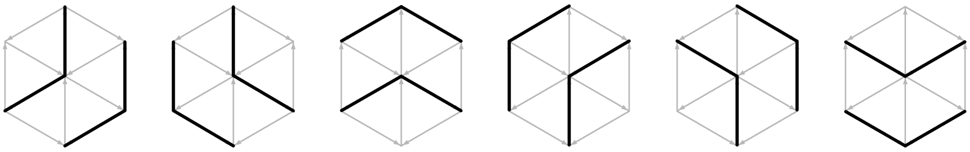


FIGURE 7. The x-, y- and z- $(1,0)$-charges, the x-, y- and z- $(0,1)$-charges

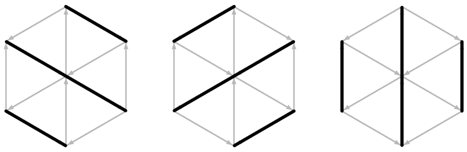


FIGURE 8. The x-tile, the y-tile and the z-tile.

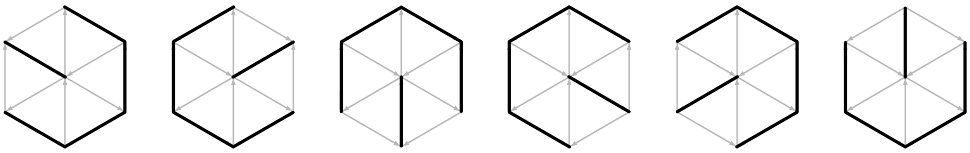


FIGURE 9. The x-, y- and z- $(1,2)$-sources, the x-, y- and z- $(2,1)$-sources.

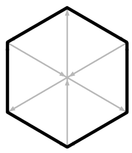


FIGURE 10. The $(3,3)$-source

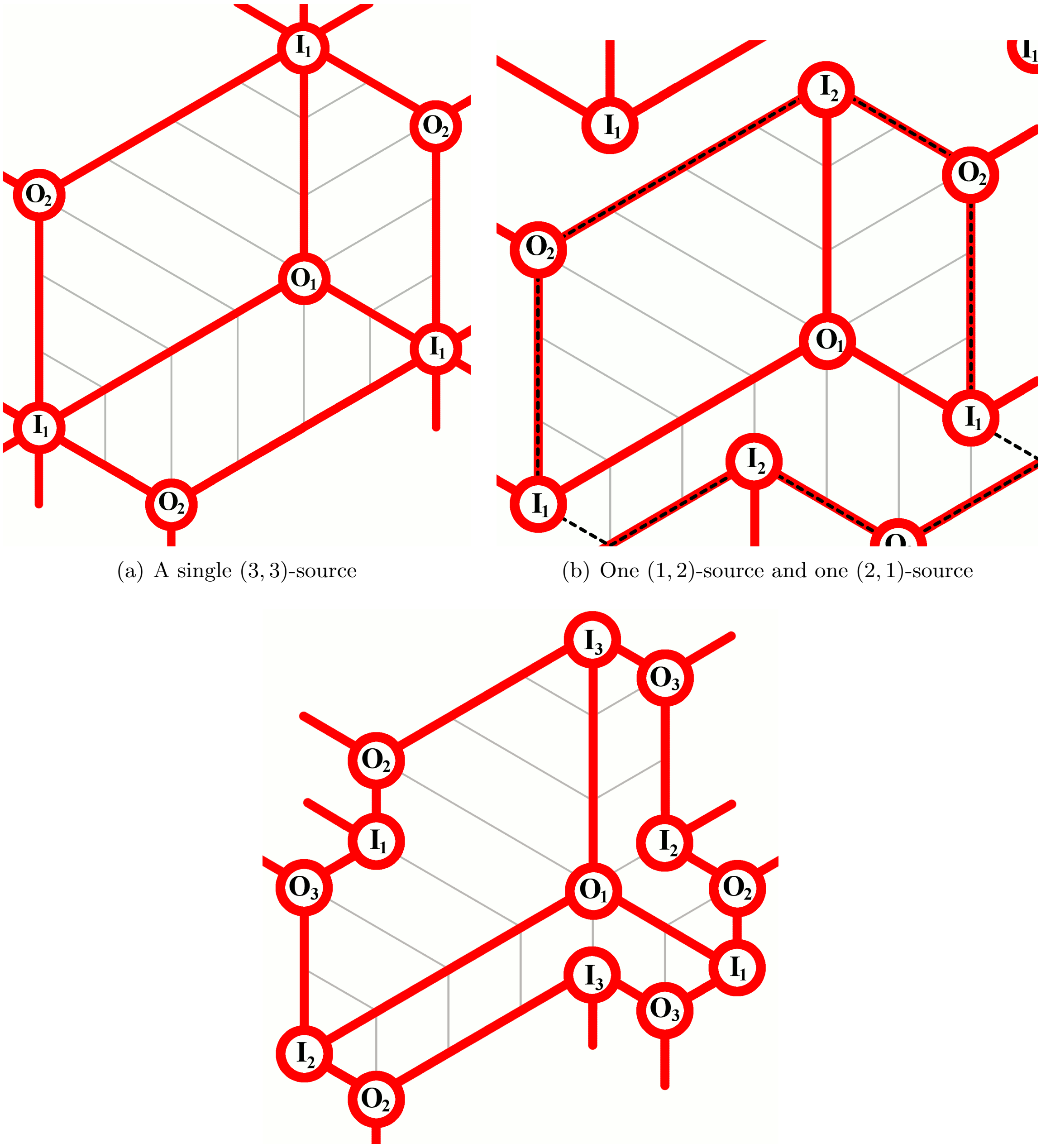
The divisor E_e appears in each $B_{\chi,x}$ with multiplicity 1 if the arrow (χ, x) vanishes along E_e and with multiplicity 0 if it doesn't. We conclude that the multiplicity with which E_e occurs in (2.2) is a, the number of the x-oriented arrows of $Q(G)$ which vanish along E_e.

On the other hand, write $\mathcal{M}$ as $\bigoplus_{\chi\in G^\vee} \mathcal{L}(-D_\chi)$ (see Section 2.3). Then $B_{\chi,x} = D_{\chi^{-1}} + (x) - D_{\kappa(x)^{-1}\chi^{-1}}$, where $(x) = \sum_{f\in\mathfrak{E}} \operatorname{val}_{E_f}(x)$ is the principal G-Weil divisor of x (see [Log08a], Section 4.3). So we re-write (2.2) as

$$\sum_{\chi\in G^\vee} \left(D_{\chi^{-1}} + (x) - D_{\kappa(x)^{-1}\chi^{-1}}\right). \tag{2.3}$$

All D_χ in the sum cancel out and we are left with $|G|(x)$. Therefore the multiplicity of E_e in (2.3) is $|G|\operatorname{val}_{E_e}(x) = \operatorname{val}_{E_e}(x^{|G|}) = e(|G|,0,0) = |G|e(1,0,0)$. We conclude that $e(1,0,0) = \frac{1}{|G|}a$. Similarly $e(0,1,0) = \frac{1}{|G|}b$ and $e(0,0,1) = \frac{1}{|G|}c$. The result follows. $\square$

Most of the results on sink-source graphs in [CL09] are stated for the dual $\tilde{\mathcal{M}}$ of the universal family $\mathcal{M}$ of G-clusters, as it is $\tilde{\mathcal{M}}$ that is used for computing the transforms $\Psi(\mathcal{O}_0 \otimes \chi)$. It is



(a) A single $(3,3)$-source

(b) One $(1,2)$-source and one $(2,1)$-source

(c) Three $(2,1)$-sources

FIGURE 11. Sink-source graphs in the case of a single $(0,3)$-sink

more convenient for us to work with $\mathcal{M}$ in the present paper, so the following general lemma is useful:

Lemma 2.6. *Let $\mathcal{F}$ be any gnat-family on Y, let $E \in \operatorname{Exc}(Y)$ and let $\chi \in G^\vee$.*

(1) *χ is an (a,b)-source (resp. sink) in $SS_{\mathcal{F},E}$ if and only if χ^{-1} is a (b,a)-source (resp. sink) in $SS_{\tilde{\mathcal{F}},E}$.*
(2) *χ is an x-(a,b)-charge in $SS_{\mathcal{F},E}$ if and only if χ^{-1} is an x-(b,a)-charge in $SS_{\tilde{\mathcal{F}},E}$. Similarly for y- and z-oriented charges.*

(3) *χ is an x-tile in $SS_{\mathcal{F},E}$ if and only if χ^{-1} is an x-tile in $SS_{\tilde{\mathcal{F}},E}$. Similarly for y- and z-oriented tiles.*

Proof. From the definition of the dual of a *gnat*-family in [CL09], Section 2.2 it follows that for any $\xi \in G^\vee$ the ξ-eigensheaf $\tilde{\mathcal{F}}_\xi$ of $\tilde{\mathcal{F}}$ is precisely the dual of the ξ^{-1}-eigensheaf $\mathcal{F}_{\xi^{-1}}$ of $\mathcal{F}$. It further folows that the map $\alpha'_{\xi,x}$ which corresponds to the arrow (ξ, x) in the associated representation $Q(G)_{\tilde{\mathcal{F}}}$ is precisely the dual of the map $\alpha_{\xi^{-1}\kappa(x)^{-1},x}$ in $Q(G)_{\mathcal{F}}$. Therefore $\alpha'_{\xi,x}$ vanishes along E if and only if $\alpha_{\xi^{-1}\kappa(x)^{-1},x}$ vanishes along E. Applying this to every arrow in the subquiver $\mathrm{Hex}(\chi)$ surrounding χ for every case depicted on Figures 6 - 10 yields the claim. $\square$

The importance of sink-source graphs for us lies in the fact that the sinks and the sources of $SS_{\mathcal{M},E}$ are precisely the characters $\chi \in G^\vee$ for which $E \subset \mathrm{Supp}(\Psi(\mathcal{O}_0 \otimes \chi))$:

Proposition 2.7. *Let $E \in \mathrm{Exc}(Y)$ and $\chi \in G^\vee$. Then:*

(1) *$E \subset \mathrm{Supp}\, H^0(\Psi(\mathcal{O}_0 \otimes \chi))$ if and only if χ is a $(3,0)$-sink in $SS_{\mathcal{M},E}$.*
(2) *$E \subset \mathrm{Supp}\, H^{-1}(\Psi(\mathcal{O}_0 \otimes \chi))$ if and only if χ is a source in $SS_{\mathcal{M},E}$.*
(3) *$E \subset \mathrm{Supp}\, H^{-2}(\Psi(\mathcal{O}_0 \otimes \chi))$ if and only if χ is a $(0,3)$-sink in $SS_{\mathcal{M},E}$, that is if χ is the trivial character χ_0.*

Proof. By [CL09], Proposition 4.6, the transform $\Psi(\mathcal{O}_0 \otimes \chi)$ is given by the total complex of the skew-commutative cube of line-bundles induced by the subrepresentation $\mathrm{Hex}(\chi^{-1})_{\tilde{\mathcal{M}}}$ of the associated representation $Q(G)_{\tilde{\mathcal{M}}}$. Then [CL09], Lemma 3.1 reduces the question of whether $E \subset \mathrm{Supp}\, H^{-i}(\Psi(\mathcal{O}_0 \otimes \chi))$ for $i = 0$, -1 or -2 to knowing which of the maps in the cube vanish along E, i.e. which of the arrows in $\mathrm{Hex}(\chi^{-1})_{\tilde{\mathcal{M}}}$ vanish along E. We can therefore go through every vertex class depicted on Figures 6-10 and apply [CL09], Lemma 3.1 to see whether $E \subset \mathrm{Supp}\, H^{-i}(\Psi(\mathcal{O}_0 \otimes \chi))$ for some i if χ^{-1} is of that class in $SS_{\tilde{\mathcal{M}},E}$. Finally, we translate the class of χ^{-1} in $SS_{\tilde{\mathcal{M}},E}$ to the class of χ in $SS_{\mathcal{M},E}$ via Lemma 2.6.

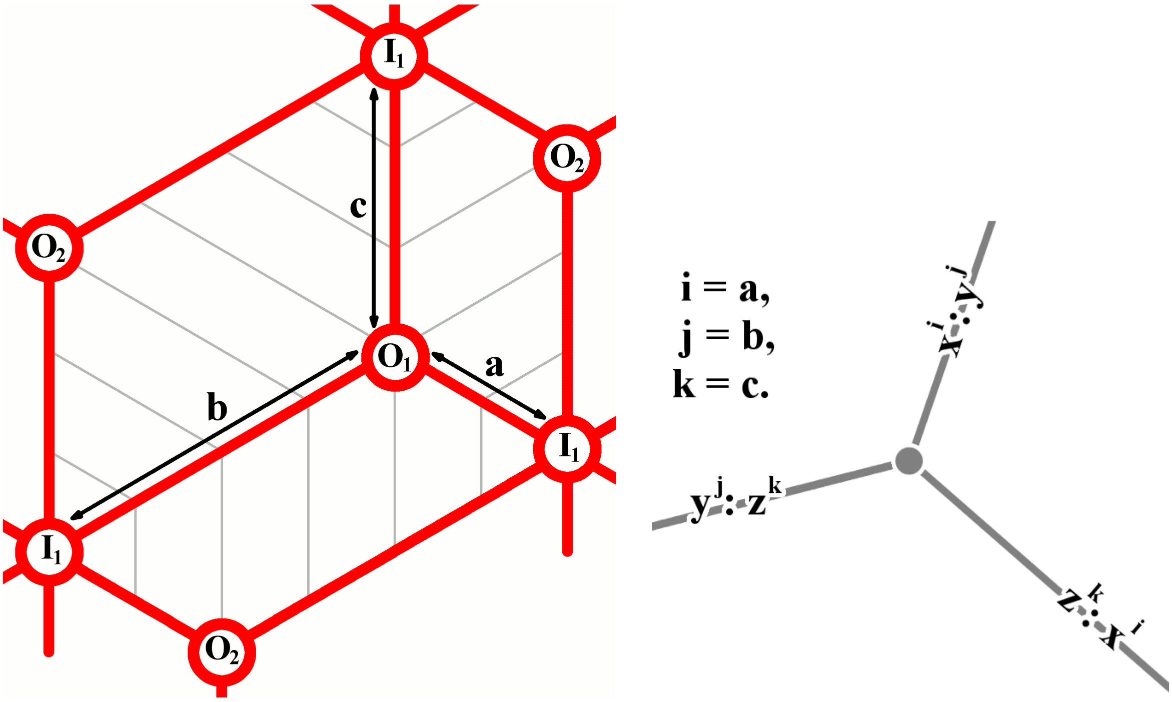
For example, by [CL09], Lemma 3.1 we have $E \subset \mathrm{Supp}\, H^0(\Psi(\mathcal{O}_0 \otimes \chi))$ if and only if E belongs to the vanishing divisors of maps α^1, α^2 and α^3 of the cube, which translates to arrows $(\chi^{-1}\kappa(x)^{-1}, x)$, $(\chi^{-1}\kappa(y)^{-1}, y)$ and $(\chi^{-1}\kappa(z)^{-1}, z)$ in $\mathrm{Hex}(\chi^{-1})_{\tilde{\mathcal{M}}}$. From Figures 6-10 this can only happen when χ^{-1} is a $(0,3)$-sink in $SS_{\tilde{\mathcal{M}},E}$, i.e. χ is a $(3,0)$-sink in $SS_{\mathcal{M},E}$. $\square$

3. Main results

Proposition 3.1. *Let $e \in \mathfrak{E}$ be such that $E_e \subset \mathrm{Exc}(Y)$. If the graph $SS_{\mathcal{M},E_e}$ is as depicted on Figure 12(a) then the vertex e in the triangulation Σ looks locally as depicted on Figure 12(b). The monomial ratios carving out the edges incident to e can be computed in terms of the indicated lengths in $SS_{\mathcal{M},E_e}$ as shown on Figure 12(b). And $e = \frac{1}{|G|}(bc, ac, ab)$ in $L \subset \mathbb{Q}^3$.*

Proof. We proceed by showing how the shape of the sink-source graph $SS_{\mathcal{M},E_e}$ imposes restrictions on which monomial ratios can mark the edges which are incident to the vertex e in the triangulation Σ.

Suppose there is an edge incident to e which is carved out by a ratio $x^{i'} : y^{j'}z^{k'}$ for some $i', j', k' \neq 0$. Let σ and σ' be the two triangles containing the edge in question. Then by Corollary 2.3 one of $x^{i'}$ and $y^{j'}z^{k'}$ must belong to the G-graph Γ_σ of the toric fixed point E_σ and the other to $\Gamma_{\sigma'}$. Without loss of generality, assumed $x^{i'} \in \Gamma_\sigma$ and $y^{j'}z^{k'} \in \Gamma_{\sigma'}$. Then, by definition of a G-graph, $x^{i'}$ doesn't belong to the ideal defining the G-cluster $\mathcal{M}_{|E_\sigma}$. Therefore $x^{i'} \cdot 1 \neq 0$ in $\mathcal{M}_{|E_\sigma}$ and, similarly, $y^{j'}z^{k'} \cdot 1 \neq 0$ in $\mathcal{M}_{|E_{\sigma'}}$. Translating this into the language of the associated representation $Q(G)_{\mathcal{M}}$ we see that in the path of i' x-oriented arrows in $Q(G)$



(a) A one $(3,3)$-source graph (b) Case 1 of Prps. 2.1

FIGURE 12. The correspondence of Proposition 3.1

which starts at χ_0 no arrow can vanish at E_σ. Similarly, in any path which starts at χ_0 and consists of j' y-oriented arrows and k' z-oriented arrows no arrow can vanish at $E_{\sigma'}$. But the vanishing locus of any arrow in $Q(G)_{\mathcal{M}}$ is a divisor of form $\sum_{f \in \mathfrak{E}} b_f E_f$ where $b_f = 1$ or 0. Therefore, if an arrow doesn't vanish at E_σ it doesn't vanish at any E_f such that $E_\sigma \in E_f$. In other words, it doesn't vanish at any of the three divisors corresponding to the three vertices of the triangle σ in $\mathfrak{E}$. In particular, it doesn't vanish at E_e. Similarly if an arrow doesn't vanish at E'_σ it also doesn't vanish along E_e.

Denote by X the vertex of $Q(G)$ corresponding to the common character of $x^{i'}$ and $y^{j'}z^{k'}$. Recall that on Figure 12(a) the vertex O_1 corresponds to the trivial character χ_0. So we have a path of i' x-oriented arrows which begins at O_1, terminates at X and none of the arrows vanish along E_e. It must not therefore enter the region tiled with x-oriented tiles as within this region every x-oriented arrow vanishes at E_e. It is evident from Figure 12(a) that the whole path, together with its endpoint X, must therefore be contained in x-$(0,1)$-charge line $O_1 I_1$ and the x-$(1,0)$-charge line $I_1 O_2$ which follows upon it. On the other hand, the paths which start at O_1 terminate at X, and consist of j' y-oriented arrows and k' z-oriented arrows, sweep out a parallelogram betwen O_1 and X with sides of length j' and k'. As by assumption neither $j' = 0$ nor $k' = 0$ this parallelogram is non-degenerate. As no y- or z-oriented arrow within this parallelogram vanishes along E_e the whole parallelogram must be contained within the region tiled with x-oriented tiles. In particular, X itself must lie within this region or on its boundary. From Figure 12(a) it is evident that the only common points of the x-arrow path $O_1 I_1 O_2$ on which X must lie and of the x-tiled region are O_1, I_1 and O_2. We can't have $X = O_1$ or I_1, as then the parallelogram would be degenerate which contradicts $j', k' \neq 0$. We conclude that $X = O_2$, the parallelogram is the whole of the x-tiled region and $(i', j', k') = (2a, b, c)$. The ratio would then be $x^{2a} : y^b z^c$. But this is impossible since $\frac{x^{2a}}{y^b z^c}$ decomposes as $\frac{x^a}{y^b} \frac{x^a}{z^c}$ and so can't carve out an edge of Σ by Corollary 2.4. Repeating the same argument for the ratios of form $y^{j'} : z^{k'}x^{i'}$ and $z^{k'} : x^{i'}y^{j'}$ we see that neither of them can occur either.

Suppose now there is an edge incident to e which is carved out by the ratio $x^{i'} : y^{j'}$ for some $i', j' \neq 0$. Denote by X the vertex which corresponds to the common character of $x^{i'}$ and $y^{j'}$. Arguing as above we see that X must lie both somewhere on the x-arrow path $O_1I_1O_2$ and somewhere on the y-arrow path $O_1I_1O_2$. From Figure 12(a) it is evident that X must then be either O_1, I_1 or O_2. If $X = O_1$, then $i' = j' = 0$ which contradicts our assumption. If $X = O_2$ then $i' = 2a$ and $j' = 2b$, so the ratio marking the incident edge would be $x^{2a} : y^{2b}$. This contradicts the fact that by its definition any ratio marking an edge must come from a primitive Laurent monomial. Therefore we must have $X = I_1$ and then $(i', j') = (a, b)$ and the ratio is $x^a : y^b$. Repeating the same argument for ratios of form $y^{j'} : z^{k'}$ and $z^{k'} : x^{i'}$ we conclude that the only monomial ratios which can mark the edges incident to e in Σ are:

$$x^a : y^b, \quad y^b : z^c, \quad z^c : x^a. \tag{3.1}$$

Consulting the classification of the Proposition 2.1 we see that e must necessarily belong to Case 1 reproduced on Figure 12(b), and that we must have $i = a$, $j = b$ and $k = c$.

Finally, to obtain $e = \frac{1}{|G|}(bc, ac, ab)$ we apply Lemma 2.5. Observe that the number of x-oriented arrows which vanish along E_e is precisely the number of x-arrows in the x-tiled region. We count the latter by breaking up the x-tiled region into little parallelograms whose sides are a single y-arrow and a single z-arrow. Each such parallelogram contains exactly one x-oriented arrow as its diagonal and on Figure 12(a) we see the x-tiled region consists of bc such parallelograms, since the x-tiled region is itself a big parallelogram with two sides consisting one of b y-arrows and the other of c z-arrows. Similarly, we see see that the number of y-arrows which vanish along E_e is ac and the number of z-arrows is ab. Therefore by Lemma 2.5 we have $e = \frac{1}{|G|}(bc, ac, ab)$. ☐

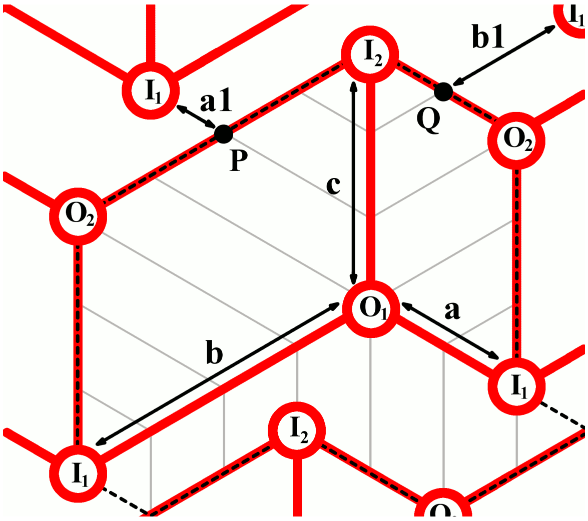
Proposition 3.2. *Let $e \in \mathfrak{E}$ be such that $E_e \in \mathrm{Exc}(Y)$. If the graph $SS_{\mathcal{M},E_e}$ is as depicted on Figure 13(a) then the vertex e in the triangulation Σ looks locally as depicted on:*

1. *Figure 13(b) if $a \mid a1$ and $b \mid b1$.*
2. *Figure 13(c) if $a \mid a1$ and $b \nmid b1$.*
3. *Figure 13(d) if $a \nmid a1$ and $b \mid b1$.*
4. *Figure 13(e) if $a \nmid a1$ and $b \nmid b1$.*

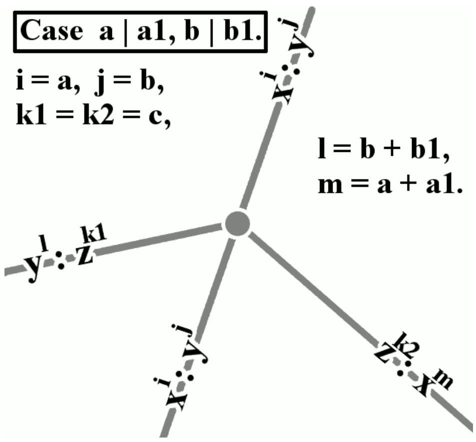
The monomial ratios carving out the edges incident to e can be computed in terms of the indicated lengths in $SS_{\mathcal{M},E_e}$ as shown on Figures 13(b)-13(d). If the shape of $SS_{\mathcal{M},E_e}$ is a rotation of Figure 13(a) by $\frac{2\pi}{3}$ or $\frac{4\pi}{3}$ one permutes x, y and z in Figures 13(b)-13(d) accordingly. The coordinates of e in L are $\frac{1}{|G|}(bc, ac, |G| - bc - ac)$.

Proof. Assume that $SS_{\mathcal{M},E_e}$ is exactly as depicted on Figure 13(a). The cases of rotation of Figure 13(a) by $\frac{2\pi}{3}$ or $\frac{4\pi}{3}$ are dealt with in exactly the same way.

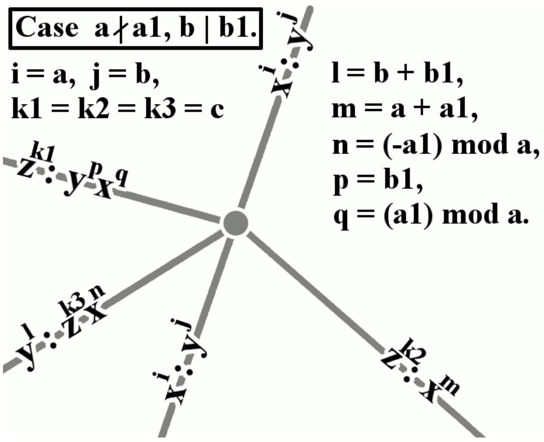
We employ the same method here as for Proposition 3.1. However this time the z-tiled region is non-contractible, wrapping around the torus. We can perfectly well have two straight lines intersecting at more than one point within it. This gives rise to some minor technical difficulties, so we need to establish several auxiliary facts. Consider a path of x-oriented arrows which begins at O_1. It first travels along the x-$(0, 1)$-charge line O_1I_1. Past the vertex I_1 the path enters the z-tiled region and travels within it until it encounters the y-$(1, 0)$-charge line O_2I_2. We define P to be the vertex where it happens and we define a_1 be the length of the x-arrow path I_1P. Similarly, we define Q to be the point where the y-arrow path which starts at O_1 first meets the x-$(1, 0)$-charge line I_2O_2 and b_1 to be the length of the y-arrow path



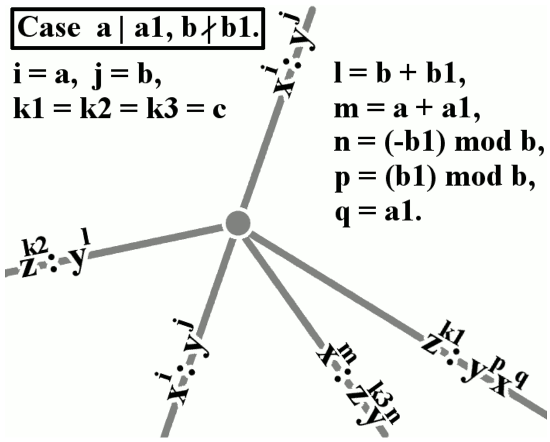
(a) A one $(1,2)$-source and one $(2,1)$-source graph



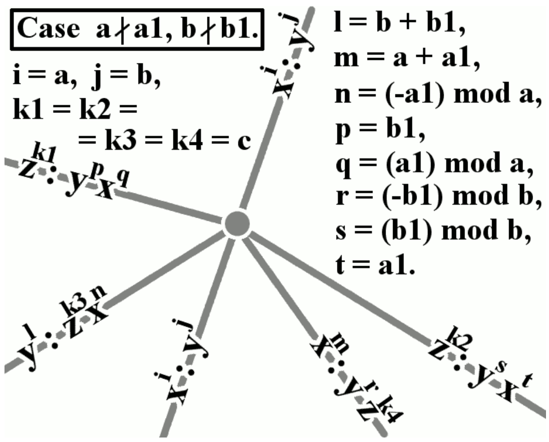
(b) Case 2(a) of Prps. 2.1



(c) Case 2(b) of Prps. 2.1



(d) Case 2(b) of Prps. 2.1, with x and y permuted



(e) Case 2(c) of Prps. 2.1

FIGURE 13. The correspondence of Proposition 3.2

I_1Q. As noted above it is perfectly possible for the paths I_1P and I_1Q to intersect several times within the z-tiled region (even though it doesn't happen on Figure 13(a)). Let C be any vertex where they intersect. Let a_c be the length of the x-arrow path I_1C and b_c be the length of y-arrow path I_1C. We claim that $a_c = n'a$ and $b_c = n'b$ for some integer $n' \in \mathbb{Z}$. This is because there exists a natural isomorphism $M/\mathbb{Z}(1,1,1) \xrightarrow{\sim} H_1(T_G, \mathbb{Z})$ where M is the lattice of G-invariant Laurent monomials (see Section 2.1). This isomorphism sends $[i', j', k']$, the class of the monomial $x^{i'}y^{j'}z^{k'}$, to the class of a loop in T_G consisting of i' x-arrows, j' y-arrows and k' z-arrows ([Log04], Lemma 6.41). Now z-tiled region is clearly contractible to a 1-sphere, so its first homology is $\mathbb{Z}$. On Figure 13(a) we see that the loop $O_1I_1O_1$ formed by x-$(0,1)$- and y-$(0,1)$-charge lines lies within the z-tiled region and wraps around it exactly once. Therefore its class must generate the first homology of the z-tiled region. This class is $[\frac{x^a}{y^b}]$ and therefore the class of every loop contained within the z-tiled region must be a multiple of it. In particular $[\frac{x^{a_c}}{y^{b_c}}]$, which is the class of the loop I_1CI_1. The claim follows.

Similarly, denote by a_2 the length of the x-arrow path I_2Q and by b_2 the length of the y-arrow path I_2P. Consider the loop $I_1PI_2QI_1$. Its homology class is $[\frac{x^{a_1+a_2}}{y^{b_1+b_2}}]$. As this loop lies within the z-tiled region its class must be a multiple of $[\frac{x^a}{y^b}]$. As clearly $0 \leq a_2 < a$ and $0 \leq b_2 < b$ we conclude that $a_2 = (-a_1) \mod a$ and $b_2 = (-b_1) \mod b$.

We can now proceed to the main proof. Suppose there is an edge incident to e which is carved out by a ratio $x^{i'} : y^{j'}z^{k'}$ for some $i', j', k' \neq 0$. Denote by X the vertex of $Q(G)$ corresponding to the common character of $x^{i'}$ and $y^{j'}z^{k'}$. Assume first that $b \nmid b_1$ and therefore $P \neq I_2$. Then travelling along the x-arrow path which starts at O_1 we first encounter an arrow which vanishes at E_e immediately after the vertex P. Arguing as in the proof of Proposition 3.1 we see that on one hand X must lie on the x-arrow path O_1I_1P, while on the other hand it must lie somewhere within the x-tiled region or its boundary. On Figure 13(a) we see that it is only possible if $X = O_1$ or P. We can't have $X = O_1$ as then $i' = j' = k' = 0$, so $X = P$ and $i' = a + a_1$, $j' = b_2 = (-b_1) \mod b$ and $k' = c$. Assume now that $b \mid b_1$. Then $P = I_2$ and one can see on Figure 13(a) that the maximal path of x-arrows which starts at O_1 and in which no x-arrow vanishes at E_e is $O_1I_1I_2O_2$. Arguing again as in the proof of Proposition 3.1 we see that X must belong both to $O_1I_1I_2O_2$ and to the x-tiled region or its boundary. On Figure 13(a) we see that it is only possible if $X = O_1$, I_2 or O_2. We can't have $X = O_1$ or $X = I_2$ as that would contradict $i', j', k' \neq 0$. Therefore $X = O_2$ and the ratio is $x^{a+a_1+a} : y^bz^c$. But this is impossible, since $\frac{x^{a+a_1+a}}{y^bz^c}$ decomposes as $\frac{x^{a+a_1}}{z^c}\ \frac{x^a}{y^b}$ and so can't be carving out an edge of Σ by Corollary 2.4. Similar argument for ratios of form $y^{j'} : x^{i'}z^{k'}$ shows that the only possibility is $y^{b+b_1} : x^{(-a_1) \mod a}z^c$ when $a \nmid a_1$.

Suppose there is an edge incident to e which is carved out by a ratio $z^{k'} : x^{i'}y^{j'}$ for some $i', j', k' \neq 0$. Denote by X the vertex corresponding to the common character of $z^{k'}$ and $x^{i'}y^{j'}$. As before, we see that on one hand X must lie somewhere on z-$(0,1)$-charge line O_1I_2 and on the other hand it must lie somewhere within the z-tiled region or its boundary. This is clearly only possible when $X = I_2$. Then $i' = c$, but due to non-contractibility of the z-tiled region we can no longer uniquely determine j' and k'.

Suppose there is an edge incident to e which is carved out by a ratio $x^{i'} : z^{k'}$ with $i', k' \neq 0$. Denote by X the vertice of $Q(G)$ corresponding to the common character of $x^{i'}$ and $z^{k'}$. Then X has to lie on both the x-arrow path O_1I_1P and the z-arrow path O_1I_2. From Figure 13(a) we see that this is only possible when $X = P = I_2$, i.e. when $b \mid b_1$ and so P coincides with I_2. The ratio would then be $x^{a+a_1} : z^c$. A similar argument for ratios of form $y^{j'} : z^{k'}$ yields that we'd have to have $a \mid a_1$ and the ratio would have to be $y^{b+b1} : z^c$.

Finally, suppose there is an edge incident to e which is carved out by a ratio $x^{i'} : y^{j'}$. Denote by X the vertice of $Q(G)$ corresponding to the common character of $x^{i'}$ and $y^{j'}$. As before, we see X would have to lie both on the x-arrow path $O_1 I_1 P$ and on the y-arrow path $O_1 I_1 Q$. One possibility is always $X = I_1$, which yields the ratio $x^a : y^b$. As established above, any other intersection point of $O_1 I_1 P$ and $O_1 I_1 Q$ would give rise to ratios of form $x^{n'a} : y^{n'b}$ for some $n' \geq 2$. As the ratio marking an edge in Σ has to be primitive we conclude that the only possibility is $x^a : y^b$.

Suppose $a \mid a_1$ and $b \mid a_2$. Then from the above we see that the only ratios which could mark an edge incident to e would be:

$$x^a : y^b, \quad y^{b+b_1} : z^c, \quad z^c : x^{a+a_1}, \quad z^c : x^{i'} y^{j'}.$$

for some $i', j' > 0$. Consulting the classification of the Proposition 2.1 we see that e must necessarily belong to Case 2(a), reproduced on Figure 13(b), and we must have $i = a$, $j = b$, $k_1 = k_2 = c$, $l = b + b_1$ and $m = a + a_1$.

Suppose $a \nmid a_1$ and $b \mid a_2$. Then the only ratios which could mark an edge incident to e would be:

$$x^a : y^b, \quad y^{b+b_1} : z^c x^{(-a_1) \mod a}, \quad x^{a+a_1} : z^c, \quad z^c : x^{i'} y^{j'}$$

for some $i', j' > 0$. Consulting the classification of the Proposition 2.1 we see that e must necessarily belong to Case 2(b), reproduced on Figure 13(c) and we must have $i = a$, $j = b$, $k_1 = k_2 = k_3 = c$, $l = b + b_1$, $m = a + a_1$, $n = (-a_1) \mod a$. To compute p and q we use the following method. By construction of Σ the lines carved out by the ratios $x^i : y^j$, $y^l : z^{k_3} x^n$ and $z^{k_1} : y^p x^q$ are parallel to sides of some regular triangle and therefore themselves form a (degenerate) regular triangle (see [Cra05], Section 2 and [CR02], Section 1.2). Three lines in Σ are said to form a regular triangle if the product of ratios carving them out (for some choice of one of the two mutually inverse Laurent monomials corresponding to each ratio) is $(xyz)^{r'}$ for some $r' \geq 0$. Such three lines must intersect at a triangle which has $r' + 1$ lattice points in each side, with the degenerate case $r' = 0$ corresponding to the intersection being a point. Therefore in our case we must have

$$\frac{x^i}{y^j} \frac{y^l}{z^{k3} x^n} \frac{z^{k1}}{y^p x^q} = 1.$$

It follows that $p = l - j = b_1$ and $q = i - n = (a_1) \mod a$.

The case $a \mid a_1$ and $b \nmid a_2$ is entirely analogus to that of $a \nmid a_1$ and $b \mid a_2$.

Suppose that $a \nmid a_1$ and $b \nmid a_2$. Then the only ratios which could mark an edge incident to e would be:

$$x^a : y^b, \quad y^{b+b_1} : z^c x^{(-a_1) \mod a}, \quad x^{a+a_1} : z^c y^{(-b_1) \mod b}, \quad z^c : x^{i'} y^{j'}$$

for some $i', j' > 0$. Consulting all the possibilities for e in the classification of the Proposition 2.1 we see that e could belong to either Case 2(c) or Case 3. But were e to belong to Case 3 it would have to be the intersection point of three straight lines carved out by ratios $x^{a+a_1} : z^c y^{(-b_1) \mod b}$, $y^{b+b_1} : z^c x^{(-a_1) \mod a}$ and $z^c : x^r y^s$ (see Figure 3). These lines, by construction of Σ, form a regular triangle of side 0 and therefore we would have to have:

$$\frac{x^{a+a_1}}{z^c y^{(-b_1) \mod b}} \frac{y^{b+b_1}}{z^c x^{(-a_1) \mod a}} \frac{z^c}{x^r y^s} = 1.$$

This is impossible as the power of z in the expression on LHS is clearly $-c$. We conclude that e has to belong to Case 2(b) of the classification of the Proposition 2.1 and we must have

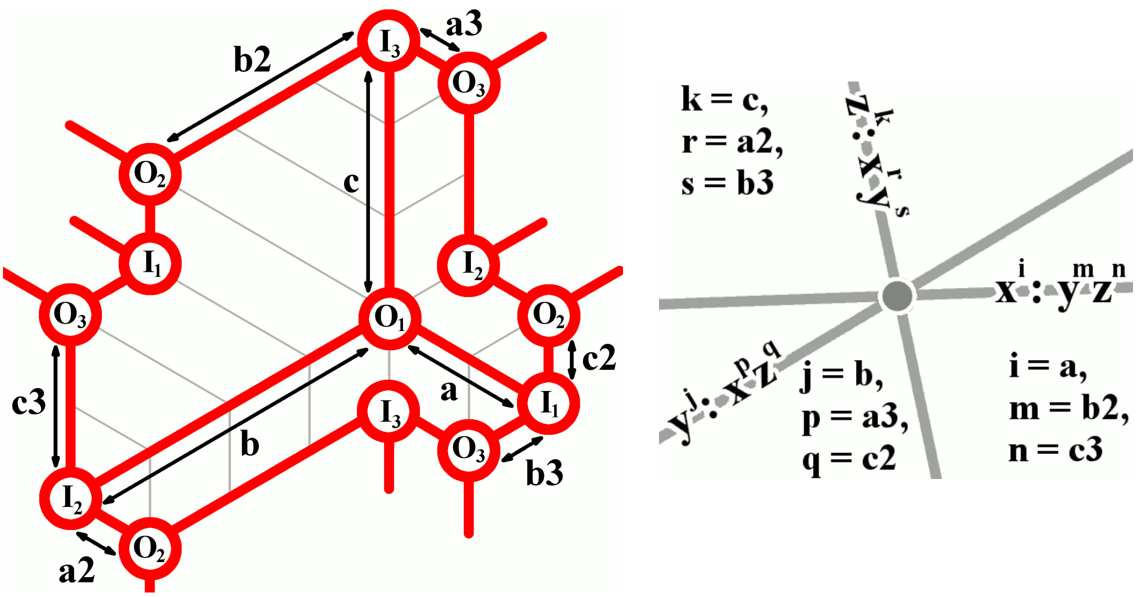
$$i = a, \quad j = b, \quad k_1 = k_2 = k_3 = k4 = c,$$
$$l = b + b_1, \quad m = a + a_1, \quad n = (-a_1) \mod a, \quad r = (-b_1) \mod b.$$

Computing p, q, s and t in the same way p and q were computed in the case $a \nmid a_1$ and $b \mid a_2$, we obtain:

$$p = b_1, \quad q = (a_1) \mod a, \quad s = (b_1) \mod b, \quad t = a_1.$$

Finally, to obtain $e = \frac{1}{|G|}(bc, ac, |G| - bc - ac)$ we apply Lemma 2.5. $\square$

Proposition 3.3. *Let $e \in \mathfrak{E}$ be such that $E_e \subset \operatorname{Exc}(Y)$. If the graph $SS_{\mathcal{M},E_e}$ is as depicted on Figure 14(a) then the vertex e in the triangulation Σ looks locally as depicted on Figure 14(b). The monomial ratios carving out the edges incident to e can be computed in terms of the indicated lengths in $SS_{\mathcal{M},E_e}$ as shown on Figure 14(b). The coordinates of e in L are $\frac{1}{|G|}(bc_3 + b_2c - b_2c_3, ac_2 + a_3c - a_3c_2, ab_3 + a_2b - a_2b_3)$.*



(a) Three $(2,1)$-source graph (b) Case 3 of Prps. 2.1

FIGURE 14. The correspondence of Proposition 3.3

Proof. Suppose there is an edge incident to e which is carved out by ratio $x^{i'} : y^{j'}z^{k'}$ for some $i', j', k' \neq 0$. Denote by X the vertex of $Q(G)$ corresponding to the common character of $x^{i'}$ and $y^{j'}z^{k'}$. Arguing as in the proof of Proposition 3.1 we see that on one hand X must lie on the x-$(0,1)$-charge line O_1I_1, while on the other hand it must lie somewhere within the x-tiled region or its boundary. On Figure 14(a) we see that it is only possible if $X = I_1$ and that the ratio would then have to be $x^a : y^{b_2}z^{c_3}$. Arguing similarly for ratios of form $y^{j'} : x^{i'}z^{k'}$ and $z^{k'} : x^{i'}y^{j'}$ we see that the only possibilities there are $y^b : x^{a_3}z^{c_2}$ and $z^c : x^{a_2}y^{b_3}$.

Suppose there is an edge incident to e which is carved out by ratio $x^{i'} : y^{j'}$ for some $i', j' \neq 0$. Denote by X the vertex corresponding to the common character of $x^{i'}$ and $y^{j'}$. Then X must lie both on the x-$(0,1)$-charge line O_1I_1 and the y-$(0,1)$-charge line O_1I_2. On Figure 14(a) we see that it is impossible unless $X = O_1$, but that would contradict $i', j' \neq 0$. Arguing similarly for ratios of form $y^{j'} : z^{k'}$ and $z^{k'} : x^{i'}$ we see that they are impossible also.

We conclude that the only ratios which could mark an edge incident to e would be:

$$x^a : y^{b_2} z^{c_3}, \quad y^b : x^{a_3} z^{c_2}, z^c : x^{a_2} y^{b_3}$$

Consulting the classification of the Proposition 2.1 we see that e must necessarily belong to Case 3, reproduced on Figure 14(b), and that we must have

$$i = a, m = b_2, n = c_3, j = b, p = a_3, q = c_2, k = c, r = a_2, s = b_3$$

For $e = \frac{1}{|G|}(bc_3 + b_2c - b_2c_3, ac_2 + a_3c - a_3c_2, ab_3 + a_2b - a_2b_3)$ we apply Lemma 2.5. □

Since the sink source graph $SS_{\mathcal{M},E}$ of every exceptional divisor $E \subset \mathrm{Exc}(Y)$ is as depicted on either Figure 12(a), Figure 13(a) or Figure 14(a), Propositions 3.1, 3.2 and 3.3 add together to give a following theorem:

Theorem 3.1. *Let $e \in \mathfrak{E}$ be such that $E_e \subset \mathrm{Exc}(Y)$. Then:*

1. *The graph $SS_{\mathcal{M},E_e}$ looks as on 12(a) if and only if the vertex e in Σ belongs to Case 1 of the classification in Proposition 2.1.*
2. *The graph $SS_{\mathcal{M},E_e}$ looks as on 13(a) if and only if the vertex e in Σ belongs to Case 2 of the classification in Proposition 2.1.*
3. *The graph $SS_{\mathcal{M},E_e}$ looks as on 14(a) if and only if the vertex e in Σ belongs to Case 3 of the classification in Proposition 2.1.*

We are now ready to prove Theorem 1.2. In the course of the proof we repeatedly use the fact that for every $\chi \in G^\vee$ the transform $\Psi(\mathcal{O}_0 \otimes \chi)$ is a shift of a coherent sheaf, that is - a complex all of whose cohomology sheaves are zero except for one ([CL09], Theorem 1.1). If it is k-th cohomology sheaf of $\Psi(\mathcal{O}_0 \otimes \chi)$ that doesn't vanish, we say that $\Psi(\mathcal{O}_0 \otimes \chi)$ is supported in degree k.

We shall also need the following auxiliary results:

Lemma 3.4. *For any $\chi \in G^\vee$ the support of $\Psi(\mathcal{O}_0 \otimes \chi)$ is connected and each of its irreducible components is either a toric divisor E_i or a toric curve $E_{i,j}$ with $i, j \in \mathfrak{E}$.*

Proof. By [CL09], Prop. 4.6 we have it that $\Psi(\mathcal{O}_0 \otimes \chi)$ is the total complex of the skew-commutative cube of line bundles corressponding to the subprepresentation $\mathrm{Hex}(\chi^{-1})_{\tilde{\mathcal{M}}}$ of the associated representation $Q(G)_{\tilde{\mathcal{M}}}$. The subquiver $\mathrm{Hex}(\chi^{-1})$ is just the hexagonal subquiver consisting of the six triangles in $Q(G)$ which contain χ^{-1}. The support of $\Psi(\mathcal{O}_0 \otimes \chi)$ can then be computed with Lemma 3.1 of [CL09], which expresses it in terms of the vanishing divisors of the arrows of $\mathrm{Hex}(\chi^{-1})_{\tilde{\mathcal{M}}}$. Recall now that for any arrow $q \in Q(G)$ its vanishing divisor B_q in $\tilde{M}$ is of form $\sum_{f \in \mathfrak{E}} b_f E_f$ with $b_f = 0, 1$. With this in mind it follows from [CL09], Lemma 3.1, that every irreducible component of $\mathrm{Supp}\,(\Psi(\mathcal{O}_0 \otimes \chi))$ is an intersection of form $E_{f_1} \cap \cdots \cap E_{f_k}$ for $k \in \{1, 2, 3\}$. Such intersection is non-empty if and only if the cone $\sigma = \langle f_1, \ldots, f_k \rangle$ is in the fan $\mathfrak{F}$ of Y. In which case it is precisely the toric orbit closure E_σ. We conclude that each irreducible component of $\mathrm{Supp}\,(\Psi(\mathcal{O}_0 \otimes \chi))$ is either a toric divisor E_i, a toric curve $E_{i,j}$ or a toric fixed point $E_{i,j,k}$ with $i, j, k \in \mathfrak{E}$.

On the other hand, since Ψ is an equivalence of derived categories

$$\mathrm{End}_{D(Y)}\, \Psi(\mathcal{O}_0 \otimes \chi) = \mathrm{End}_{D^G(\mathbb{C}^3)}\, \mathcal{O}_0 \otimes \chi = \mathbb{C}.$$

Hence $\mathrm{Supp}\,(\Psi(\mathcal{O}_0 \otimes \chi))$ is connected. It can't therefore have a toric fixed point as an irreducible component unless it is the only component. Which is impossible, as then $\Psi(\mathcal{O}_0 \otimes \chi)$ would be a shift of a point sheaf, but we know that [BKR01] equivalence Φ, of which Ψ is the inverse, sends every point sheaf on Y to a G-cluster on $\mathbb{C}^3$. The claim now follows. □

Lemma 3.5. *Let $e \in \mathfrak{E}$ be such that $E_e \subset \operatorname{Exc}(Y)$. For any $\chi \in G^\vee$ the divisor E_e belongs to the support of $H^{-1}(\Psi(\mathcal{O}_0 \otimes \chi))$ if and only if E_e contains two or more curves marked by χ.*

Proof. By Proposition 2.7 the divisor E_e belongs to $\operatorname{Supp} H^{-1}(\Psi(\mathcal{O}_0 \otimes \chi))$ if and only if χ is a source vertex in SS_{M,E_e}. Figures 12-14 list all possible shapes of $SS_{\mathcal{M},E_e}$ together with the corresponding toric fans of e. By inspection of this data we see that χ is a source vertex in $SS_{\mathcal{M},E_e}$ if and only χ marks two or more edges incident to e in the toric fan. Each of these edges corresponds to a toric curve contained in E_e and so the claim follows. □

Lemma 3.6. *Each of the irreducible components of* $\operatorname{Supp} H^{-1}(\Psi(\mathcal{O}_0 \otimes \chi))$ *is a toric divisor E_e for some $e \in \mathfrak{E}$.*

Proof. As in the proof of Lemma 3.4 we use Lemma 3.1(2) of [CL09] to compute the support of $H^{-1}(\Psi(\mathcal{O}_0 \otimes \chi))$ in terms of the vanishing divisors of the arrows in $\operatorname{Hex}(\chi^{-1})_{\tilde{\mathcal{M}}}$. Translating from the language of $Q(G)_{\tilde{\mathcal{M}}}$ into that of $Q(G)_{\mathcal{M}}$ as seen in the proof of Lemma 2.6 we see that the vanishing divisors of $\operatorname{Hex}(\chi^{-1})_{\tilde{\mathcal{M}}}$ are precisely the vanishing divisors of $\operatorname{Hex}(\chi)_{\mathcal{M}}$. On Figure 15 we marked for each arrow of $\operatorname{Hex}(\chi)$ the name for its vanishing divisor in $\operatorname{Hex}(\chi)_{\mathcal{M}}$ as translated into the language of Lemma 3.1 of [CL09].

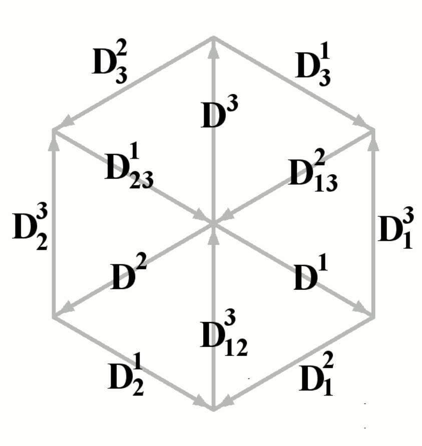


FIGURE 15
The vanishing divisors of $\operatorname{Hex}(\chi)_{\mathcal{M}}$ in the language of Lemma 3.1 of [CL09].

By [CL09], Lemma 3.1, $\operatorname{Supp} H^{-1}(\Psi(\mathcal{O}_0 \otimes \chi))$ has a filtration with successive quotients

- $\mathcal{O}_Z \otimes \mathcal{L}_{12}(\gcd(D_1^2, D_2^1))$ where Z is the scheme theoretic intersection of $\gcd(D_1^2, D_2^1)$ and the effective part of $D^3 + \operatorname{lcm}(\tilde{D}_3^1, \tilde{D}_3^2) - \tilde{D}_1^2 - D^1$
- $\mathcal{O}_Z \otimes \mathcal{L}_{13}(\gcd(D_1^3, D_3^1))$ where Z is the scheme theoretic intersection of $\gcd(D_1^3, D_3^1)$ and the effective part of $D^2 + \operatorname{lcm}(D_2^1, \tilde{D}_2^3) - \tilde{D}_3^1 - D^3$
- $\mathcal{O}_Z \otimes \mathcal{L}_{23}(\gcd(D_2^3, D_3^2))$ where Z is the scheme theoretic intersection of $\gcd(D_2^3, D_3^2)$ and the effective part of $D^1 + \operatorname{lcm}(D_1^2, D_1^3) - \tilde{D}_2^3 - D^2$

where $\tilde{D}_j^i = D_j^i - \gcd(D_j^i, D_i^j)$.

Suppose $\operatorname{Supp} H^{-1}(\Psi(\mathcal{O}_0 \otimes \chi))$ has an irreducible component which is less than a divisor and let us assume that it belongs to the third of the quotients in the filtration above. The other two cases are similar and simpler. As each of D_j^i is a sum $\sum_{e \in \mathfrak{E}} b_e E_e$ with $b_e = 0$ or 1 we see that this component must be of form $E_{e,f} = E_e \cap E_f$ with E_e belonging to $\gcd(D_3^2, D_2^3)$ and E_f to $D^1 + \operatorname{lcm}(D_2^1, D_3^1) - \tilde{D}_3^2 - D^2$. Every possible arrangement of which arrows of $\operatorname{Hex}(\chi)$ vanish along E_e and which don't is listed in Figures 6 - 10. Observe that χ can not be a source for E_e as then $\operatorname{Supp} H^{-1}(\Psi(\mathcal{O}_0 \otimes \chi))$ would contain all of E_e (Proposition 2.7) and $E_{e,f}$ wouldn't be an irreducible component. Similarly χ can not be a sink as then E_e would belong to either the support of $\operatorname{Supp} H^0(\Psi(\mathcal{O}_0 \otimes \chi))$ or $\operatorname{Supp} H^{-2}(\Psi(\mathcal{O}_0 \otimes \chi))$ and $\Psi(\mathcal{O}_0 \otimes \chi)$ wouldn't be supported in a single degree. By inspection we see that the only remaining possibility is that of χ being an x-$(0,1)$-charge for E_e. Arguing similarly for E_f we see that χ must be either an x-tile, or an y-$(1,0)$-charge or a z-$(1,0)$-charge. But observe that in each of these cases D^2_{13} and D^3_{12} contain E_e, while D^1_{23} contains E_f, so $D^1_{23} \cap D^2_{13} \cap D^3_{12}$ contains $E_{e,f}$. Then by Lemma 3.1(3) of [CL09] $E_{e,f}$ would belong $\operatorname{Supp} H^{-2}(\Psi(\mathcal{O}_0 \otimes \chi))$, and again $\Psi(\mathcal{O}_0 \otimes \chi)$ wouldn't be supported in a single degree. Therefore all irreducible components of $\operatorname{Supp} H^{-1}(\Psi(\mathcal{O}_0 \otimes \chi))$ are divisors. The claim now follows from Lemma 3.4. □

Proof of Theorem 1.2. We proceed case by case:

Proof of (4):

A character χ of G marks nothing in Reid's recipe if and only if it is the trivial character χ_0. Using Lemma 3.1(3) of [CL09] as detailed in Lemma 3.6 to compute $H^{-2}(\Psi(\mathcal{O}_0 \otimes \chi))$ we see that for any $\chi \in G^\vee$

$$H^{-2}(\Psi(\mathcal{O}_0 \otimes \chi)) = \mathcal{O}_Y(D) \otimes \mathcal{O}_D$$

where D is the union of all $E \in \operatorname{Exc}(Y)$ such that χ is a $(0,3)$-sink in $SS_{\mathcal{M},E}$. By Proposition 4.14 of [CL09] the sink-source graph $SS_{\mathcal{M},E}$ of any $E \in \operatorname{Exc}(Y)$ has only one $(0,3)$-sink - the vertex χ_0. Thus $H^{-2}(\Psi(\mathcal{O}_0 \otimes \chi))$ is $\mathcal{O}_Y(\operatorname{Exc}(Y)) \otimes \mathcal{O}_{\operatorname{Exc}(Y)}$ if $\chi = \chi_0$ and zero if $\chi \neq \chi_0$. The claim now follows.

Proof of (1):

The 'If' direction: Let $e \in \mathfrak{E}$ be such that $E_e \subset \operatorname{Exc}(Y)$. We claim that it follows from Propositions 3.1, 3.2 and 3.3 that χ marks E_e if and only if the vertex χ is a $(3,0)$-sink. Indeed, suppose the sink-source graph $SS_{\mathcal{M},E_e}$ is as depicted on Figure 12(*a*). Then $SS_{\mathcal{M},E_e}$ has a single $(3,0)$-sink O_2. Since O_1, the $(0,3)$-sink, is the trivial character χ_0 we can see that O_2 is the character $\kappa(x^a y^b)$. By Proposition 3.1 the triangulation Σ around e looks as on Figure 12(*b*). Reid's recipe prescribes then for E_e to be marked by ξ^2 where $\xi = \kappa(x^i) = \kappa(y^j) = \kappa(z^k)$. Since by Proposition 3.1 we have $i = a$, $j = b$ and $k = c$ we see that $O_2 = \kappa(x^a y^b) = \xi^2$ and the claim follows. The cases of $SS_{\mathcal{M},E_e}$ being as on Figure 13(*a*) or on Figure 14(*a*) are treated similarly. After we express the powers of x,y and z in the ratios marking the edges incident to e in terms of the lengths of the marked edges in $SS_{\mathcal{M},E_e}$ the corresponding formula of Reid's recipe becomes the natural formula for calculating $(3,0)$-sinks.

On the other hand, by Proposition 2.7 the vertex χ is a $(3,0)$-sink for E_e if and only if $E_e \subset \operatorname{Supp} H^0(\Psi(\mathcal{O}_0 \otimes \chi))$. We conclude that χ marks E_e if and only if $E_e \subset \operatorname{Supp} H^0(\Psi(\mathcal{O}_0 \otimes \chi))$. In particular, if $\operatorname{Supp} H^0(\Psi(\mathcal{O}_0 \otimes \chi)) = E_e$ then χ marks E_e.

The 'Only if' direction: This was proved in Proposition 9.3 of [CI04]. There it was done by showing that if χ marks some divisor E then χ defines a wall of the G-Hilb$(\mathbb{C}^3)$ chamber in the space of stability conditions for G-constellations and E is the unstable locus corresponding to crossing that wall. It then follows that $\Psi(\mathcal{O}_0 \otimes \chi) = \mathcal{L}^{-1}_\chi \otimes \mathcal{O}_E$.

Proof of (2):

By Lemma 3.1(1) of [CL09] we have:

$$H^0(\Psi(\mathcal{O}_0 \otimes \chi)) = \mathcal{L}^{-1}_\chi \otimes \mathcal{O}_D.$$

where D is the support of $H^0(\Psi(\mathcal{O}_0 \otimes \chi))$. It suffices therefore to prove that χ marks a single toric curve in Reid's recipe if and only if this curve is the whole of $\operatorname{Supp} H^0(\Psi(\mathcal{O}_0 \otimes \chi))$.

Let $\langle e, f \rangle$ be any two-dimensional cone in $\mathfrak{F}$ and let $E_{e,f}$ be the corresponding toric curve.

Claim A: If χ marks $E_{e,f}$ in Reid's recipe, and $\operatorname{Supp} H^0(\Psi(\mathcal{O}_0 \otimes \chi))$ is non-empty then $E_{e,f}$ is an irreducible component of $\operatorname{Supp} H^0(\Psi(\mathcal{O}_0 \otimes \chi))$.

Claim B: If $E_{e,f}$ is an irreducible component of $\operatorname{Supp} H^0(\Psi(\mathcal{O}_0 \otimes \chi))$ then $E_{e,f}$ is the whole of $\operatorname{Supp} H^0(\Psi(\mathcal{O}_0 \otimes \chi))$ and χ marks $E_{e,f}$ in Reid's recipe.

Suppose these claims were true and suppose in Reid's recipe χ marks $E_{e,f}$ and no other curve. Then χ is not the trivial character χ_0 and so, as seen in the proof of (4), $\Psi(\mathcal{O}_0 \otimes \chi)$ is not supported in degree -2. Neither it is supported in degree -1, as we have it from Lemmas 3.5 and 3.6 that the support of $H^{-1}(\Psi(\mathcal{O}_0 \otimes \chi))$ consists of divisors which each contain two or more curves marked by χ. Therefore $\Psi(\mathcal{O}_0 \otimes \chi)$ is supported in degree 0 and we have it from Claim A that $E_{e,f}$ is an irreducible component of $\operatorname{Supp} H^0(\Psi(\mathcal{O}_0 \otimes \chi))$ and then from Claim B that $E_{e,f}$ is the whole of $\operatorname{Supp} H^0(\Psi(\mathcal{O}_0 \otimes \chi))$. Conversely suppose $E_{e,f} = \operatorname{Supp} H^0(\Psi(\mathcal{O}_0 \otimes \chi))$. Then by claim B the character χ marks $E_{e,f}$ and by claim A no other curves can be marked by χ. Thus to prove (2) it suffices to prove these two claims.

Proof of Claim A: Suppose χ marks $E_{e,f}$ and $\operatorname{Supp} H^0(\Psi(\mathcal{O}_0 \otimes \chi))$ is non-empty. Then $\operatorname{Supp} H^1(\Psi(\mathcal{O}_0 \otimes \chi))$ is empty and so by Proposition 2.7 the vertex corresponding to χ can not be a source in $SS_{\mathcal{M},E}$ for any $E \in \operatorname{Exc}(Y)$. In particular, χ is not a source in $SS_{\mathcal{M},E_e}$. Let us consider all the possibilities listed on Figures 12-14 for $SS_{\mathcal{M},E_e}$ and the corresponding toric fans of e and let us check when is it possible for χ in $SS_{\mathcal{M},E_e}$ not to be a source while marking one of the edges incident to e. We see that it is only possible if the toric fan of e is as depicted on Figure 13(c), 13(d) or 13(e) up to a permutation of x, y and z. Assume without loss of generality that this permutation is such that the unique straight line passing through e is of form $x^\bullet : y^\bullet$, i.e. exactly as on Figures 13(c)-13(e). We also see that χ must be one of the vertices denoted on Figure 13(a) as P or Q. Assume without loss of generality that χ is the vertex P. Then χ is a y-$(1,0)$-charge in $SS_{\mathcal{M},E_e}$ and (e,f) is carved out by a ratio of form $x^\bullet : z^\bullet y^\bullet$. Observe further that one of the two triangles containing (e,f) must also contain the edge carved out by a ratio of form $z^\bullet : y^\bullet x^\bullet$. But if one edge of any triangle in Σ is carved out by a ratio of form $x^\bullet : y^\bullet z^\bullet$ and another edge by a ratio of form $z^\bullet : x^\bullet y^\bullet$, then its third edge must be carved out by a ratio of form $y^\bullet : x^\bullet z^\bullet$ (see [Cra05], §2). We conclude that one of the edges incident to f is carved out by a ratio of form $y^\bullet : x^\bullet z^\bullet$. But now apply the same argument to E_f. We see that f must also be as depicted on Figure 13(c), 13(d) or 13(e) up to a permutation of x, y and z. The permutation can not be such that the straight line which passes through f is carved out by a ratio of form $z^\bullet : y^\bullet$. This is because (e,f) is carved out by a ratio of form $x^\bullet : y^\bullet z^\bullet$ which under such permutation would correspond to $z^\bullet : x^\bullet y^\bullet$ on Figures 13(c)-13(e). Which is impossible as then χ, the character marking (e,f), would be the vertex I_2 on Figure 13(a) which is a source in $SS_{\mathcal{M},E_f}$. But neither can the straight line passing through f be carved out by a ratio of form $x^\bullet : y^\bullet$ as then f has only one edge carved out by the ratio of form $y^\bullet : x^\bullet z^\bullet$ and this edge clearly can not be contained in the same triangle as the edge (e,f) which is marked by a ratio of form $x^\bullet : y^\bullet z^\bullet$ (see Figure 13(e)). We conclude that the straight line passing through f must be carved out by a ratio of form $x^\bullet : z^\bullet$. Then, since (e,f) is carved out by $x^\bullet : y^\bullet z^\bullet$, the vertex χ which marks (e,f) has to be a z-$(1,0)$-charge in $SS_{\mathcal{M},E_f}$. Since χ is also y-$(1,0)$-charge in $SS_{\mathcal{M},E_f}$ we conclude by consulting Figure 7) that each of the three arrows in $Q(G)$ whose tail is χ vanishes either along E_e or along E_f but all three of them vanish neither along E_e nor along E_f. This by Proposition 4.6 and Lemma 3.1(1) of [CL09], and translating from the language of $Q(G)_{\tilde{M}}$ into that of $Q(G)_{\mathcal{M}}$ as seen in the proof of Lemma 2.6, implies that $E_{e,f}$ belongs to the support of $H^0(\Psi(\mathcal{O}_0 \otimes \chi))$, but neither E_e nor E_f do. Therefore $E_{e,f}$ is an irreducible component of $\operatorname{Supp} H^0(\Psi(\mathcal{O}_0 \otimes \chi))$.

Proof of Claim B: Denote by D the support of $H^0\left(\Psi\left(\mathcal{O}_0 \otimes \chi\right)\right)$ and suppose $E_{e,f}$ is an irreducible component of D. By Lemma 3.4 each irreducible component of D is either a toric divisor or a toric curve. But as seen in the proof of the *'if'* direction of (1) D contains a toric divisor E if and only if χ marks E. And by the *'only if'* direction (1) if χ marks E then E is the whole of D. We conclude that every irreducible component of D is a toric curve. We wish to show that $E_{e,f}$ is the only such component and to do that we have to roll up our sleeves and calculate some sheaf cohomology.

Recall that $\Psi : D^G(\mathbb{C}^3) \to D(Y)$ was defined as the inverse of the Fourier-Mukai equivalence $\Phi : D(Y) \to D^G(\mathbb{C}^3)$ of [BKR01]. For any $F \in D(Y)$ we can compute the global sections of χ-eigenparts of $\Phi(F)$ by taking a derived pushdown in two different ways. By definition

$$\Phi(F) = \mathbf{R}\,\pi_{\mathbb{C}^3 *}\left(\mathcal{M} \overset{\mathbf{L}}{\otimes} \pi_Y^*(F \otimes \chi_0)\right)$$

where π_Y and $\pi_{\mathbb{C}^3}$ are the projections from $Y \times \mathbb{C}^3$ onto Y and $\pi_{\mathbb{C}^3}$. Let $\gamma_{\mathbb{C}^3} : \mathbb{C}^3 \to \mathbb{C}^3/G$ and $\gamma_Y : Y \to \mathbb{C}^3/G$ be the quotient map and the resolution morphism. Making use of the projection formula, we have

$$\begin{aligned}
\mathbf{R}\,\gamma_{\mathbb{C}^3 *}\Phi(F) = \mathbf{R}\left(\pi_{\mathbb{C}^3} \circ \gamma_{\mathbb{C}^3}\right)_*\left(\mathcal{M} \overset{\mathbf{L}}{\otimes} \pi_Y^*(F \otimes \chi_0)\right) = \\
= \mathbf{R}\left(\pi_Y \circ \gamma_Y\right)_*\left(\mathcal{M} \overset{\mathbf{L}}{\otimes} \pi_Y^*(F \otimes \chi_0)\right) = \\
= \mathbf{R}\,\gamma_{Y*}\left(\bigoplus_{\chi \in G^\vee} \mathcal{L}_\chi \otimes (F \otimes \chi_0)\right)
\end{aligned}$$

Let F be a sheaf in $\mathrm{Coh}(Y)$. Taking global sections and making use of G-equivariance, we see that for any $\xi \in G^\vee$

$$H^i\,\Gamma(\Phi(F)_\xi) = H^i(\mathcal{L}_\xi \otimes F) \tag{3.2}$$

where on the RHS we take the i-th sheaf cohomology and on the LHS we take the i-th cohomology of the vector space complex $\Gamma(\Phi(F)_\xi)$ where the complex $\Phi(F)_\xi$ is the ξ-eigenpart of the complex $\Phi(F)$.

Let now χ' be any character of G. As $\Phi(\mathcal{L}_\chi^{-1} \otimes \mathcal{O}_D) = \mathcal{O}_0 \otimes \chi$ setting $F = \mathcal{L}_\chi^{-1} \otimes \mathcal{O}_D$ and $\xi = \chi'$ in (3.2) yields that

$$\chi\left(\mathcal{L}_{\chi'} \otimes \mathcal{L}_\chi^{-1} \otimes \mathcal{O}_D\right) = \begin{cases} 0 \text{ for } \chi' \neq \chi \\ 1 \text{ for } \chi' = \chi \end{cases}.$$

By $\chi(-)$ we denote the Euler characteristic $\sum_{i \in \mathbb{Z}} (-1)^i \dim H^i(-)$. Then for $\chi' \neq \chi$ we have

$$\begin{aligned}
0 = \chi\left(\mathcal{L}_{\chi'} \otimes \mathcal{L}_\chi^{-1} \otimes \mathcal{O}_D\right) = \sum_{\sigma \in D} \deg_{E_\sigma}\left(\mathcal{L}_{\chi'} \otimes \mathcal{L}_\chi^{-1}\right) + \chi(\mathcal{O}_D) = \\
= \left(\sum_{\sigma \in D} \deg_{E_\sigma} \mathcal{L}_{\chi'}\right) - \left(\sum_{\sigma \in D} \deg_{E_\sigma} \mathcal{L}_\chi\right) + \chi(\mathcal{O}_D).
\end{aligned} \tag{3.3}$$

where we abuse the notation by writing $\sigma \in D$ to mean that σ is a two-dimensional cone in $\mathfrak{F}$ such that $E_\sigma \subset D$. Observe that the sum $\sum_{\sigma \in D} \deg_{E_\sigma} \mathcal{L}_{\chi'}$ doesn't depend on the choice of $\chi' \neq \chi$. But $\chi_0 \neq \chi$ as otherwise $\Psi(\mathcal{O}_0 \otimes \chi)$ would have to be supported in degree -2 by part (4) of this theorem. Evaluating $\sum_{\sigma \in D} \deg_{E_\sigma} \mathcal{L}_{\chi'}$ for $\chi' = \chi_0$ we obtain zero since $\mathcal{L}_{\chi_0} = \mathcal{O}_Y$. Therefore $\sum_{\sigma \in D} \deg_{E_\sigma} \mathcal{L}_{\chi'} = 0$ for any $\chi' \neq \chi$. On the other hand by Lemma 2.2 the degree of $\mathcal{L}_{\chi'}$ is non-negative on any toric curve E_σ. And by Corollary 2.3 the degree of $\mathcal{L}_{\chi'}$ is 1 on any curve marked by χ'. Therefore for any χ' which marks any of the curves in D we have $\sum_{\sigma \in D} \deg_{E_\sigma} \mathcal{L}_{\chi'} \geq 1$. We conclude that χ marks all the curves in D. Assume that D contains some curve $E_{e',f'}$ other then $E_{e,f}$. As D is connected $E_{e',f'}$ must intersect $E_{e,f}$. Then in Σ the edges (e,f) and (e',f') must be two sides of some regular triangle and therefore have a common vertex. Without loss of generality assume $e = e'$. Then E_e contains two curves marked by χ and by Lemma 3.5 it must belong to $H^{-1} \operatorname{Supp}(\Psi(\mathcal{O}_0 \otimes \chi))$, which contradicts $\Psi(\mathcal{O}_0 \otimes \chi)$ being supported in a single degree. We conclude that $E_{e,f}$ is the whole of D and the claim follows.

Proof of (3):

The 'If' direction: Suppose $\Psi(\mathcal{O}_0 \otimes \chi)$ is supported in degree -1. Then χ can not mark a divisor or mark a single curve or mark nothing in Reid's recipe since then $\Psi(\mathcal{O}_0 \otimes \chi)$ would be necessarily supported in degree 0 or degree -2 by by parts (1), (2), (4) of this theorem which we already proved. Hence χ must mark several curves in Reid's recipe.

The 'Only if' direction:

Assume that χ marks several curves in Reid's recipe. Then $\chi \neq \chi_0$ and by the proof of the part (4) of this theorem $\Psi(\mathcal{O}_0 \otimes \chi)$ is not supported in degree -2. Nor can $\Psi(\mathcal{O}_0 \otimes \chi)$ be supported in degree 0. For by Lemma 3.4 the irreducible components of the support of $\Psi(\mathcal{O}_0 \otimes \chi)$ are toric divisors and toric curves. And as seen in the proof of the 'if' direction of the part (1) of this theorem a divisor belongs to $\operatorname{Supp} H^0 (\Psi(\mathcal{O}_0 \otimes \chi))$ if and only χ marks this divisor in Reid's recipe. Similarly, as we seen in the proof of part (2), a toric curve is an irreducible component of $\operatorname{Supp} H^0 (\Psi(\mathcal{O}_0 \otimes \chi))$ if and only if χ marks just this curve in Reid's recipe. We conclude that $\Psi(\mathcal{O}_0 \otimes \chi)$ is supported in degree -1.

Finally, the fact that $\operatorname{Supp} H^{-1} (\Psi(\mathcal{O}_0 \otimes \chi))$ is precisely the union of the divisors containing two or more curves marked by χ is a consequence of Lemmas 3.5 and 3.6.

This concludes our proof of Theorem 1.2. □

Proof of Theorem 1.1. We get this as a free consequence of Theorem 1.2 by observing that any non-trivial $\chi \in G^\vee$ must either mark a divisor, a single curve or several curves (Corollary 4.6 of [Cra05]). Thus any $\chi \in G^\vee$ must belong to one of the four cases in Theorem 1.2. □

References

[BKR01] Tom Bridgeland, Alastair King, and Miles Reid, *The McKay correspondence as an equivalence of derived categories*, J. Amer. Math. Soc. **14** (2001), 535–554, math.AG/9908027.

[CI04] A. Craw and A. Ishii, *Flops of G-Hilb and equivalences of derived category by variation of GIT quotient*, Duke Math J. **124** (2004), no. 2, 259–307, arXiv:math.AG/0211360.

[CL09] Sabin Cautis and Timothy Logvinenko, *A derived approach to geometric McKay correspondence in dimension three*, J. Reine Angew. Math. **636** (2009), 193–236, arXiv:0803.2990.

[CR02] Alastair Craw and Miles Reid, *How to calculate A-Hilb* $\mathbb{C}^3$, Seminaires et Congres **6** (2002), 129–154, math.AG/9909085.

[Cra05] Alastair Craw, *An explicit construction of the McKay correspondence for A-Hilb* $\mathbb{C}^3$, J. Algebra **285** (2005), no. 2, 682–705.

[GSV83] G. Gonzales-Sprinberg and J.-L. Verdier, *Construction géométrique de la correspondance de McKay*, Ann. sci. ENS **16** (1983), 409–449.

[Log03] Timothy Logvinenko, *Families of G-constellations over resolutions of quotient singularities*, preprint math.AG/0305194, (2003).

[Log04] ———, *Families of G-Constellations parametrised by resolutions of quotient singularities*, Ph.D. thesis, University of Bath, 2004.

[Log08a] ———, *Derived McKay correspondence via pure-sheaf transforms*, Math. Ann. **341** (2008), no. 1, 137–167, arXiv: math/0606791.

[Log08b] ———, *Natural G-constellation families*, Documenta Math. **13** (2008), 803–823, arXiv: math/0601014.

[McK80] J. McKay, *Graphs, singularities and finite groups*, Proc. Symp. Pure Math. **37** (1980), 183–186.

[Rei97] Miles Reid, *McKay correspondence*, preprint math.AG/9702016, (1997).

E-mail address: `T.Logvinenko@liv.ac.uk`

Department of Mathematical Sciences, University of Liverpool, Peach Street, Liverpool, L69 7ZL, UK