Properties of the Operator Product Expansion in Quantum Field Theory

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Abstract

We prove that the operator product expansion (OPE), which is usually thought of as an asymptotic short distance expansion, actually *converges* at arbitrary finite distances within perturbative quantum field theory. The result is derived for the massive scalar field with φ^4-interaction on Euclidean spacetime. This constitutes a generalisation of an earlier result by Hollands and Kopper, which states that the OPE of exactly *two* quantum fields converges. We also show that the OPE coefficients satisfy factorisation conditions for certain configurations of the spacetime arguments. Such conditions are known to encode information on the algebraic structure of the underlying quantum field theory.

Both results rely on modified versions of the renormalisation group flow equations, which allow us to derive explicit bounds on the remainder of these expansions. Within this framework, we also derive a new formula for the perturbation of OPE coefficients, i.e. an equation relating coefficients at a given perturbation order to those of lower order. By iteration of this formula, a new constructive method for the computation of OPE coefficients in perturbation theory is obtained, which only requires the coefficients of the free theory as initial data.

Finally, we investigate a strategy to restrict renormalisation ambiguities in quantum field theory via the condition that the OPE coefficients depend analytically on the coupling constant(s) of the respective model. We apply this strategy to the computation of the vacuum expectation value of the stress energy operator in the two dimensional Gross-Neveu model and we obtain a unique prediction for the non-perturbative contribution to this expectation value, which is of the order $\exp(-2\pi/g^2)$ (here g is the coupling constant). We discuss the possibility that a similar effect, if present in the Standard Model of particle physics, could account for the "unnatural" smallness of the cosmological constant.

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Notation and conventions

Conventions used in chapter 3: The convention for the Fourier transform in $\mathbb{R}^4$ used in this thesis is

$$f(x) = \int_p \hat{f}(p)\, \mathrm{e}^{ipx} := \int_{\mathbb{R}^4} \frac{\mathrm{d}^4 p}{(2\pi)^4}\, \mathrm{e}^{ipx} \hat{f}(p)\,. \tag{0.1}$$

We use a standard multi-index notation. Our multi-indices are elements $w = (w_1, \ldots, w_n) \in \mathbb{N}^{4n}$, where each $w_i \in \mathbb{N}^4$ is a four-tuple with components $w_{i,\mu} \in \mathbb{N}$ and $\mu = 1, \ldots, 4$. For $f(p_1, \ldots, p_n)$ a smooth function on $\mathbb{R}^{4n}$, we use the shorthand $f(\vec{p})$ and we set

$$\partial^w f(\vec{p}) = \prod_{i,\mu} \left(\frac{\partial}{\partial p_{i,\mu}} \right)^{w_{i,\mu}} f(\vec{p}) \tag{0.2}$$

and

$$w! = \prod_{i,\mu} w_{i,\mu}!\,, \quad |w| = \sum_{i,\mu} w_{i,\mu}\,. \tag{0.3}$$

If a function $f(\vec{x}; \vec{p})$ depends on two sets of variables, $(\vec{x}, \vec{p}) \in \mathbb{R}^{4n_1} \times \mathbb{R}^{4n_2}$, then we write $\partial^w_{\vec{p}}$ to indicate that the partial derivatives are taken with respect to the variables $(p_1, \ldots, p_{n_2})$ as in (0.2). Derivatives ∂^w of a product of functions $f_1 \cdots f_r$ are distributed over the factors using the Leibniz rule, which results in the sum of all terms of the form $c_{\{v_i\}}\, \partial^{v_1} f_1 \cdots \partial^{v_r} f_r$. Here each v_i is now a $4n$-dimensional multi-index, where $v_1 + \ldots + v_r = w$, and where

$$c_{\{v_i\}} = \frac{(v_1 + \ldots + v_r)!}{v_1! \cdots v_r!} \leq r^{|w|} \tag{0.4}$$

is the associated multi-nomial weight factor. We will denote sets of indices by $I = \{i_1, \ldots, i_k\}$ with $i_j \in \mathbb{N}$ and we denote their cardinality by $|I|$.

Given a set of momenta $(p_1, \ldots, p_n) \in \mathbb{R}^{4n}$, we agree on the shorthand notation

$$\vec{p} := (p_1, \ldots, p_n) \quad , \quad |\vec{p}|_n := \sup_{J \subseteq \{1,\ldots,n\}} \left| \sum_{i \in J} p_i \right| \quad , \quad \vec{p}_{n+2} := (\vec{p}, k, -k) \tag{0.5}$$

Later we will often simply write $|\vec{p}|$ instead of $|\vec{p}|_n$. Further we define $\kappa := \sup(\Lambda, m)$ for later convenience. We also use the notation

$$(c)_+ = \sup(0, c) \tag{0.6}$$

to denote the *positive part* of $c \in \mathbb{R}$. In particular, we often write $\log_+(x) = \sup(0, \log(x)) = \log(\sup(1, x))$.

If $F(\varphi)$ is a differentiable function (in the Frechet space sense) of the Schwartz space function $\varphi \in \mathcal{S}(\mathbb{R}^4)$, we denote its functional derivative as

$$\frac{\mathrm{d}}{\mathrm{d}t} F(\varphi + t\psi)|_{t=0} = \int \mathrm{d}^4 x \, \frac{\delta F(\varphi)}{\delta \varphi(x)} \, \psi(x) \, , \quad \psi \in \mathcal{S}(\mathbb{R}^4) \, , \tag{0.7}$$

where the right side is understood in the sense of distributions in $\mathcal{S}'(\mathbb{R}^4)$. Multiple functional derivatives are denoted in a similar way and define in general distributions on multiple Cartesian copies of $\mathbb{R}^4$.

Conventions used in chapter 4: We use the sign convention

$$(\eta_{\mu\nu}) := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{0.8}$$

for the two dimensional Minkowski-metric. Our Dirac-matrices are defined as

$$\gamma^0 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \gamma^1 := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \, , \tag{0.9}$$

and the Dirac conjugate is given by $\overline{\psi} = \psi^\dagger \gamma^0$. As usual, we also define $\gamma^5 := \gamma^0 \gamma^1$. We use letters from the beginning of the Greek alphabet, such as $\alpha, \beta \in \{1, 2\}$, to denote spinor indices, and letters from the middle of the Greek alphabet, such as $\mu, \nu \in \{0, 1\}$, for spacetime indices.

1 Introduction

Quantum field theory (QFT) is the mathematical framework underlying the Standard Model of particle physics, which provides our current understanding of the electromagnetic-, weak- and strong interactions of elementary particles [1, 2, 3]. In other words, it offers a unified model for all fundamental forces except gravity. The Standard Model has been gradually developed throughout the 1970s as a collaborative effort of many theorists, and it has since then continued to pass experimental tests to an ever growing - and certainly quite astonishing - precision. This process culminated in the recent discovery of the Higgs boson at the Large Hadron Collider (LHC), which added the last missing piece to the experimental verification of Standard Model predictions. It should further be mentioned that quantum field theory has found important applications also in the areas of condensed matter physics and the theory of critical phenomena [4]. Despite this remarkable phenomenological and quantitative success, our understanding of quantum field theory in a mathematically rigorous sense is still somewhat unsatisfying and incomplete.

One of the main difficulties in formulating a theory of quantum fields lies in the characteristic singular nature of *operator products*: Given a quantum field $\phi(x)$ representing some physical observable of interest localised at a spacetime point x, it is a general feature that the product $\phi(x)\phi(y)$ diverges as the spacetime arguments approach each other, i.e. in the limit $y \to x$. Technically speaking, this behaviour is due to the fact that local quantum fields are not regular functions, but (operator valued) distributions, which in general have ill-defined products. In terms of physical intuition, one may view the appearance of divergences in the attempt to define sharply localised quantum observables as a manifestation of Heisenberg's uncertainty relation, which would imply complete de-localisation in momentum space for these objects. The singular nature of the product $\phi(x)\phi(y)$ also complicates the definition of *composite operators*, i.e. monomials in the basic field ϕ. The importance of overcoming this difficulty should be evident in view of the fact that powers of ϕ show up in various quantities of physical interest, such as Lagrangians, field equations or energy momentum tensors.

The most important tool in the analysis of operator products in quantum field theory is Wilson's *operator product expansion* (OPE) [5], which states that any product of local quantum fields can be expanded as

$$\mathcal{O}_{A_1}(x_1)\cdots\mathcal{O}_{A_N}(x_N) \sim \sum_B \mathcal{C}^B_{A_1\ldots A_N}(x_1,\ldots,x_N)\,\mathcal{O}_B(x_N)\,. \tag{1.0.1}$$

Here the symbols $\mathcal{O}_A$ denote the composite fields that appear in the given theory, where the label A also incorporates the tensor or spinor character of the field. The so called OPE coefficients, $\mathcal{C}^B_{A_1\ldots A_N}$, are distributions with singularities on the diagonals $x_i = x_j$ for $1 \leq i < j \leq N$. The relation (1.0.1) is supposed to hold in the weak sense, i.e. it holds when inserted into an arbitrary (suitably well-behaved) quantum state. Further, the symbol "$\sim$" signifies that the OPE is usually understood as an *asymptotic expansion*, which means that the remainder of such an expansion, when truncated at a sufficiently high dimension D for the operators $\mathcal{O}_B$ on the right side, should go to zero as we take the limit $x_1,\ldots,x_{N-1} \to x_N$ at a rate that improves as we increase D.

The operator product expansion is the answer to the problem described above: It is designed to study the singularity structure of operator products, and it allows for a meaningful definition of composite operators. It is therefore no surprise that Wilson's idea caught on quickly in the physics community, and that it is by now a well established tool in most approaches to quantum field theory. The first proof that perturbative quantum field theory admits an operator product expansion is due to Zimmermann [6]. In the context of particle physics, the OPE has found applications for example in the understanding of *deep inelastic scattering* [3, 7]. It has further played a crucial role in the development of *conformal field theories* [8, 9, 10], has been derived within *axiomatic settings* [11], and has also been proven to hold order by order in perturbative quantum field theory on *curved spacetimes* [12].

In this thesis we derive mathematical properties of the operator product expansion within perturbative quantum field theory, which are of both technical as well as conceptual interest. More precisely, we will deal with the topics of convergence, factorisation and deformation of the OPE. A brief explanation of what is meant by these terms as well as a discussion of their relevance can be found in the following sections of this introduction.

The second line of research presented in the present work concerns an application of the

OPE in the presence of non-perturbative effects. We will follow a proposal of Hollands and Wald [13] by which the OPE may be used to reduce renormalisation ambiguities in the computation of vacuum expectation values of composite operators. We will apply this strategy in the two dimensional Gross-Neveu model [14] and discuss possible ramifications for the cosmological constant problem.

1.1 Convergence of the operator product expansion

As mentioned above, the operator product expansion was originally introduced as an asymptotic expansion, which means that the difference between the left and right hand side of (1.0.1) goes to zero as the space time arguments are scaled together, $x_1, \ldots, x_{N-1} \to x_N$, if we sum over all B up to sufficiently high dimension. This interpretation of the symbol $\sim$ in eq.(1.0.1) has generally remained unchanged since then. However, it has been shown recently in [15] that the operator product expansion of two fields in fact *converges* to any order in perturbation theory in the setting of Euclidean $g\varphi^4$-theory. More precisely, it was shown that, for a product of two fields $\mathcal{O}_{A_1}(x_1)\mathcal{O}_{A_2}(x_2)$, the difference between left and right hand side of eq. (1.0.1) goes to zero in the weak sense (i.e. as an insertion into any suitably well-behaved state) as we sum over all B. The result holds for any finite separation $(x_1 - x_2)^2$ of the two spacetime arguments!

One would expect that this result can be generalised to a product of any number of quantum fields, i.e. to products of the form $\mathcal{O}_{A_1}(x_1) \cdots \mathcal{O}_{A_N}(x_N)$ with $N \geq 2$. In the first main result of this thesis we will show that this is indeed the case (see section 3.3), i.e. we will show that the OPE of any number of quantum fields converges up to any perturbation order in (massive) Euclidean $g\varphi^4$-theory. The proof utilises a refinement of the Wilson-Wegner-Polchinski renormalisation group flow method [16, 17, 18, 19], which is due to Kopper et al. (see e.g. [20] for a review). This technique will allow us to derive explicit bounds on the remainder of the OPE, i.e. it will provide us with an estimate for the difference between the left and right hand side of equation (1.0.1) as we sum over all operators $\mathcal{O}_B$ of dimension $\leq D$ on the right hand side. This bound on the remainder will be found to go to zero as we take the limit $D \to \infty$.

While the general idea of the proof will be the same as in the $N = 2$ case analysed in [15], an additional complication arises for products of more than two quantum fields due to the presence of nested subdivergences in the spacetime arguments. Our main technical advance in the mentioned flow equation approach to perturbative quantum field theory therefore lies in the understanding and regularisation of these subdivergences (see in particular sections 3.1.3 and 3.1.4). As a side result of this analysis, we will also be able to show that the OPE coefficients are real analytic functions in the spacetime arguments for non-coinciding points.

We would like to stress that convergence of the operator product expansion is not merely a technicality. It is rather a property that offers conceptual insights into the general structure of quantum field theory. For the sake of the argument, let us switch to a Minkowskian context for the moment. Here the analog of our result would be that the N-point correlation functions $\langle \mathcal{O}_{A_1}(x_1)\cdots\mathcal{O}_{A_N}(x_N)\rangle_\Psi$ in a (well-behaved) state Ψ are entirely determined by the collections of OPE coefficients, which are state-independent, together with 1-point functions $\langle \mathcal{O}_B(x_N)\rangle_\Psi$:

$$\langle \mathcal{O}_{A_1}(x_1)\cdots\mathcal{O}_{A_N}(x_N)\rangle_\Psi = \sum_B \mathcal{C}^B_{A_1\ldots A_N}(x_1,\ldots,x_N)\,\langle \mathcal{O}_B(x_N)\rangle_\Psi \tag{1.1.1}$$

Here it is understood that the infinite sum over B would be convergent, and the distances $(x_i - x_j)^2$ would not necessarily have to be small. It follows that all the state-independent algebraic information of quantum field theory would be encoded in the OPE coefficients, which play a role similar to the structure constants of an ordinary algebra, whereas all the information about the quantum state is contained in the 1-point functions (”form factors”) only.

The convergence result further yields strong support to a novel approach to quantum field theory, due to Hollands [21], which elevates the OPE to a defining object of a quantum field theory. In this axiomatic framework, a QFT is defined in terms of its OPE coefficients and one-point functions. As we have seen, convergence of the OPE would imply that these data indeed contain the same information as the n-point functions.

1.2 Factorisation of the operator product expansion

Consider an operator product $\mathcal{O}_{A_1}(x_1)\cdots\mathcal{O}_{A_N}(x_N)$ of $N \geq 3$ fields. Assume that the points $x_1,\ldots,x_{M-1}$ are closer to x_M than the points $x_{M+1},\ldots,x_N$. We may express this condition in terms of the inequality

$$\frac{\max_{1\leq i\leq M}|x_i - x_M|}{\min_{M+1\leq j\leq N}|x_j - x_M|} < \frac{1}{K} \qquad , K \geq 1\,, \tag{1.2.1}$$

where the parameter K specifies *how much* closer one set of points has to be to x_M compared to the other points. As an example, a configuration satisfying this requirement is depicted in fig. 1.1.

Since the OPE is designed to be a short distance expansion, one would expect that we may perform the OPE of only the product $\mathcal{O}_{A_1}(x_1)\cdots\mathcal{O}_{A_M}(x_M)$ around the point x_M first, since the other fields are farther away from $\mathcal{O}_{A_M}(x_M)$. Such an expansion would have the

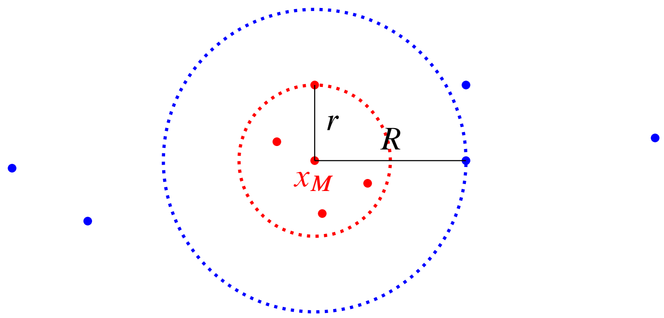


Figure 1.1.: Sketch of a configuration satisfying (1.2.1). Here the points $x_1, \ldots, x_M$ are coloured red, and the points $x_{M+1}, \ldots, x_N$ are blue. Further, the radii of the circles are $r = \max_{1\le i\le M} |x_i - x_M|$ and $R = \min_{M<j\le N} |x_j - x_M|$, respectively. By equation (1.2.1), they are required to satisfy $r/R < 1/K$.

form

$$\mathcal{O}_{A_1}(x_1)\cdots\mathcal{O}_{A_N}(x_N) \sim \sum_B \mathcal{C}^B_{A_1\ldots A_M}(x_1,\ldots,x_M)\,\mathcal{O}_B(x_M)\mathcal{O}_{A_{M+1}}(x_{M+1})\cdots\mathcal{O}_{A_N}(x_N)\,. \tag{1.2.2}$$

We refer to this kind of expansion as a *partial* OPE. What does the symbol $\sim$ stand for in this case? When inserted into a state, one would certainly expect such an expansion to hold in an asymptotic sense, i.e. as the points $x_1, \ldots, x_M$ are scaled together. But what can be said about the convergence properties of the infinite sum over B? In this thesis we will show, again within massive Euclidean $g\varphi^4$-theory, that the partial OPE converges in the weak sense for suitable configurations of the spacetime arguments. The significance of this result becomes apparent if one performs another OPE on the right hand side of equation (1.2.2), which leads to the relation

$$\mathcal{O}_{A_1}(x_1)\cdots\mathcal{O}_{A_N}(x_N)\sim\sum_{B_1,B_2}\mathcal{C}^{B_1}_{A_1\ldots A_M}(x_1,\ldots,x_M)\,\mathcal{C}^{B_2}_{B_1A_{M+1}\ldots A_N}(x_M,\ldots,x_N)\,\mathcal{O}_{B_2}(x_N)\,. \tag{1.2.3}$$

Comparing this expansion to eq.(1.0.1), we can simply read off the following non-trivial algebraic relations between the OPE coefficients[1]:

$$\mathcal{C}^B_{A_1\ldots A_N}(x_1,\ldots,x_N) = \sum_C \mathcal{C}^C_{A_1\ldots A_M}(x_1,\ldots,x_M)\,\mathcal{C}^B_{CA_{M+1}\ldots A_N}(x_M,\ldots,x_N) \tag{1.2.4}$$

We will show that such relations indeed hold in $g\varphi^4$-theory, i.e. the sum over C converges on certain spacetime domains. In the case $(N-M)\le 2$ this domain coincides with the one described in eq.(1.2.1), where K is a (potentially large) constant. In other words, the

[1]Here we have implicitly exchanged the order of the infinite sums over B_1 and B_2. Strictly speaking, this would have to be justified of course. In section 3.5 we will derive eq.(1.2.4) in a slightly different, a little more complicated, but rigorous manner.

points $x_{M+1}, \dots, x_N$ have to be *very far away* from the cluster $x_1, \dots, x_M$ in order for (1.2.4) to hold. For general N, M our result holds on a somewhat smaller, but quite similar domain.

The relations (1.2.4) will be called *factorisation conditions* in the following[2]. Our interest in these conditions is rooted in the fact that they put potentially powerful restrictions on the OPE coefficients. These restrictions are clearly the stronger the smaller the constant K from (1.2.1) is. In the case $K = 1$, for example, one can deduce from the factorisation conditions that all the N-point OPE coefficients, $\mathcal{C}_{A_1 \dots A_N}^B$, can be determined just from the two point coefficients [21]. Further, the factorisation condition in that case suggests a close connection to *vertex operator algebras* [22], which usually appear in the context of conformal field theories [8, 9]. It was also observed in [22] that, as a side result, one might even obtain non-trivial results in the theory of special functions if the factorisation condition holds for $K = 1$.

1.3 Deformation of the operator product expansion

The explicit computation of OPE coefficients in perturbation theory is usually presented in textbooks by expanding certain correlation functions at short distance/large momentum [7, 3]. Fundamentally, however, the OPE coefficients are state-independent objects, so one would expect to be able to compute them without any reference to particular correlation functions.

In this thesis, we derive a rigorous formula that relates the OPE coefficients of Euclidean $g\varphi^4$-theory at any finite perturbation order $r \in \mathbb{N}$ to those at lower order. For the OPE of two fields, this formula reads explicitly (the general case of more than two fields can be found below in theorem 4)

$$
\begin{aligned}
(\mathcal{C}_{r+1})_{A_1 A_2}^B(x_1, x_2) = \frac{-1}{4!\,(r+1)} \int \mathrm{d}^4 y \\
\Bigg[(\mathcal{C}_r)_{A_g A_1 A_2}^B(y, x_1, x_2) - \sum_{s=0}^{r} \Bigg(\sum_{[C] \le [A_1]} (\mathcal{C}_s)_{A_g A_1}^C(y, x_1)\, (\mathcal{C}_{r-s})_{C A_2}^B(x_1, x_2) \\
+ \sum_{[C] \le [A_2]} (\mathcal{C}_s)_{A_g A_2}^C(y, x_2)\, (\mathcal{C}_{r-s})_{A_1 C}^B(x_1, x_2) + \sum_{[C] < [B]} (\mathcal{C}_s)_{A_1 A_2}^C(x_1, x_2)\, (\mathcal{C}_{r-s})_{A_g C}^B(y, x_2) \Bigg) \Bigg].
\end{aligned}
\tag{1.3.1}
$$

Here we used the notation $(\mathcal{C}_n)$ to denote the OPE coefficients at perturbation order n, and $[A]$ stands for the *engineering dimension* of the operator $\mathcal{O}_A$ (the precise notions are given in chapter 3). Further, the index A_g corresponds to the interaction operator, i.e. $\mathcal{O}_{A_g} := \varphi^4$

[2] Similar conditions are sometimes also referred to in the literature [21] as *consistency conditions* or *associativity conditions*.

in our example of $g\varphi^4$-theory.

Our derivation of this formula is again formulated within the flow equation approach to perturbative quantum field theory, based on a definition of the operator product expansion which was first given in [23]. An advantage of this setup is that it is conceptually very clean. In one stroke, it allows for a manifestly state-independent definition of all OPE coefficients to arbitrary order in perturbation theory. We derive formula (1.3.1) from first principles in this framework, i.e. we do not make any assumptions besides BPHZ-renormalisation conditions. The formula is shown to hold in the theory with a finite ultraviolet-cutoff, which, as we show, can be safely removed in the end.

Given the OPE coefficients of the free theory (i.e. zeroth perturbation order) as "initial data", our formula (1.3.1) provides a concrete algorithm for the calculation of any OPE coefficient to arbitrary (finite) order in perturbation theory. To our knowledge, this constitutes the first method for the perturbative computation of OPE coefficients which is entirely "self contained", in the sense that it does not require any additional information besides the zeroth order coefficients, and which at the same time does not make any additional assumptions[3]. The result yields further support to the proposed formulation of quantum field theory in terms of OPE coefficients and 1-point functions.

In the context of ordinary algebra one usually refers to perturbations of the algebra product as *deformation* [24]. In view of the analogy between OPE coefficients on the one side and structure constants of an algebra on the other, we will also refer to (1.3.1) as a "deformation of the operator product expansion".

From a purely computational perspective, we do not expect this new algorithm to yield a significant simplification over the usual techniques for the calculation of OPE coefficients, since we expect to encounter the same types of loop integrals. On the other hand, an advantage of our method is that it provides a very clear prescription for the computation of any OPE coefficient $\mathcal{C}_{A_1\ldots A_N}^{B}$ to any order. The computations in standard textbooks [7, 3] usually cover only particular examples, such as the coefficient $\mathcal{C}_{\varphi\varphi}^{\varphi^2}$ to first order, and it is often not entirely clear, in our opinion, how to generalise these calculations in order to obtain other OPE coefficients.

1.4 Non-perturbative effects and Dark Energy

Composite operators in quantum field theory are generally subject to so called *renormalisation ambiguities*. These ambiguities are a finite remainder of the process of removing the characteristic divergences from a quantum field theory. Unless reasonable restrictions can

[3]A similar algorithm for the computation of OPE coefficients in perturbation theory was given in [21]. However, a crucial requirement for that scheme to work is that factorisation conditions of the type (1.2.4) hold on the domain (1.2.1) with $K = 1$. As discussed in the previous section, it is currently not known whether these conditions are satisfied by interacting quantum fields.

be imposed on such ambiguities, these effects make it impossible to make quantitative predictions for vacuum expectation values of composite operators. While various symmetry conditions can be imposed quite naturally in a given theory, these do not suffice to uniquely define composite operators in general. In the usual applications of quantum field theory this is not actually a problem, since experiments in condensed-matter or particle physics usually do not measure absolute expectation values of observables, but rather different values between these observables in different states (such as e.g. vacuum- and multi-particle states). If one is interested in cosmological applications, however, the absolute expectation value of the energy-momentum tensor is of prime importance, since it appears in the (semi-classical) Einstein field equations as a source term generating spacetime curvature. Some mechanism to further restrict renormalisation ambiguities of composite operators would therefore be highly desirable.

Such a mechanism, based on the OPE, was proposed in [13]. Namely, it was observed that OPE coefficients appear to have a more regular behaviour in the parameters of a theory than the correlation functions. In particular, it was suggested that one may impose the requirement that the OPE coefficients depend *analytically* on the parameters of the model. It was further noticed that, in the presence of non-perturbative (i.e. non-analytic in the coupling constant) effects, such a condition on the OPE coefficients could allow one to *uniquely* single out the non-perturbative contributions to expectation values of composite operators. While non-perturbative effects are well known to exist in the Standard Model of particle physics, it is extremely difficult to make rigorous calculations beyond perturbation theory within any quantum field theory in four spacetime dimensions. In order to analyse the proposed analyticity condition on the OPE coefficients, as well as its consequences for the ambiguities in the definition of composite operators, we therefore focus on a simpler quantum field theory in this thesis. Our model should be sufficiently complex to include non-perturbative effects, but at the same time simple enough to allow for explicit calculations. In this thesis, our toy model of choice, which satisfies these requirements, is the two dimensional *Gross-Neveu model* (GN-model) [14].

Non-perturbative results in the Gross-Neveu model are obtained via a large flavour (or $1/N$-) expansion. In the present work, we compute the OPE coefficients that are needed for the construction of the energy momentum operator of the GN-model to leading order in $1/N$. We will see that, indeed, these OPE coefficients can be chosen to be analytic in the coupling constant. Imposing the analyticity condition on the OPE coefficients in order to restrict renormalisation ambiguities, we obtain a unique prediction for the non-perturbative contribution to the vacuum expectation value of the stress energy operator. More precisely, this non-perturbative contribution is found to be of the form $\exp(-2\pi/g^2)$, where g is the coupling constant of the model. Pursuing the analogy with the Standard Model, it is conceivable that a similar mechanism produces a non-perturbative, exponentially small factor, which could be responsible for the ”unnatural smallness” of the observed cosmological constant [25]. This could possibly be an explanation for the measured value of Dark Energy.

1.5 Outline

The research presented in this thesis was largely motivated by the approach to quantum field theory based on the operator product expansion proposed in [21]. It has been mentioned a few times in this introduction that many of our results are of relevance to the viability of this framework. To put our research into perspective, we will devote chapter 2 to a brief account of the axiomatic setting presented in [21] before we come to the actual derivation of our results. Chapter 3 contains the proofs of the convergence-, factorisation- and deformation results mentioned above in this introduction. All these results are derived within a pertubative setting, and we will start off in section 3.1 by reviewing the *renormalisation flow equation approach* to perturbative quantum field theory, which will be used throughout chapter 3. In section 3.2 we present certain bounds on quantities of interest, such as Green’s functions with regularisation. These bounds are then used in sections 3.3-3.5 to obtain the convergence and factorisation results, followed by the derivation of our deformation formula in 3.6. Finally, our non-perturbative results in the Gross-Neveu model, as well as possible cosmological implications, are outlined in chapter 4, and we draw conclusions from our findings and discuss possible future lines of research in chapter 5.

Let us take a moment to clarify the distinction between literature review and original research presented in this thesis. All of chapter 2 as well as most of section 3.1 provide background material from the existing literature that will help to put our results into context. Our original contributions start in sections 3.1.3 and 3.1.4 with the discussion of the regularisation of Schwinger functions with multiple operator insertions (the case of up to two insertions was first discussed by Kopper and Keller in [23], and the case of three insertions was treated by the present author in collaboration with Hollands in [26]). Similarly, the bounds on Schwinger functions with N operator insertions presented in sections 3.2.2 and 3.2.3 as well as the OPE convergence and factorisation proofs (theorems 1 and 3) constitute new results due to the author, which generalise earlier results for the case $N = 2$ [15] by Hollands and Kopper and $N = 3$ [26] by Hollands and the present author. The deformation formula for OPE coefficients derived in section 3.6 is also an original result of this thesis. Finally, the contents of chapter 4 are based on a collaboration with Hollands [27].

Some of the results published in [28, 29], which deal with properties of black holes in classical general relativity, were also obtained as part of the author’s PhD project. We chose not to include these results in this thesis due to the fact that they are completely unrelated to the main theme of the present work.

2

Quantum field theory based on the operator product expansion

The idea to put special emphasis on the algebraic relations between quantum field observables as a fundamental characteristic of a quantum field theory dates back to the work of Haag and Kastler [30]. Algebraic approaches have also been useful in conformal quantum field theories (cQFT), see e.g. [8, 9], and, in view of the lack of a preferred vacuum state, turned out to be essential in the construction of quantum field theories on curved spacetimes [31, 32, 33, 34, 35, 36].

In the present chapter we will briefly review an axiomatic approach to quantum field theory, due to Hollands [21], which elevates Wilson's operator product expansion (OPE) [5] to a defining feature of the theory, encoding all the algebraic relations between the fields. A particular appeal of this framework is that, with slight adjustments, it can be generalised to give an axiomatic definition of quantum field theories on *curved spacetimes* [36]. But even in the Euclidean context, the shift of perspective provided by this approach has already led to valuable insights (see section 2.2 for a brief account).

It is one aim of this thesis to analyse the relation between this axiomatic setting and the usual perturbative treatment of quantum field theory, based on a Lagrangian and the path integral. At the end of this chapter we will pose more precise questions that will be tackled in the following chapters.

2.1 An axiomatic definition of quantum field theory

A quantum field theory in the sense of [21] is defined as a pair $(V, \mathcal{C})$. Here V is an infinite dimensional vector space, whose basis elements are in one-to-one correspondence with the composite fields $\mathcal{O}_A$ of the theory. It is therefore graded by the fermionic/bosonic character of these fields, by their dimension as well as by their transformation properties under rotations of $\mathbb{R}^4$. Thus, we have

$$V = \bigoplus_{i\in\{0,1\}} \bigoplus_{\Delta\in\mathbb{R}+} \bigoplus_{S\in\text{irrep}} V^{i,\Delta,S} \tag{2.1.1}$$

where i distinguishes the bosonic and fermionic subspace, Δ denotes the scaling dimension and "irrep" stands for all finite dimensional, irreducible unitary representations of Spin(4). The sum over the field dimensions Δ is assumed to be infinite, but countable. On the vector space V we would like to have an anti-linear, involutive operation called $* : V \to V$, which should be thought of as taking the hermitian adjoint of the quantum fields. We would also like to have a linear grading map $\gamma : V \to V$ with the property $\gamma^2 = \mathrm{id}$. The vectors corresponding to eigenvalue $+1$ are to be thought of as "bosonic", while those corresponding to eigenvalue -1 are to be thought of as "fermionic".

The dynamical content of the theory is encoded in the collection of operator product expansion coefficients

$$\mathcal{C} = (\mathcal{C}(x_1,x_2), \mathcal{C}(x_1,x_2,x_3), \mathcal{C}(x_1,x_2,x_3,x_4), \dots) \ , \tag{2.1.2}$$

which is a hierarchy of linear maps

$$\mathcal{C}(x_1,\dots,x_N) : \ V^{\otimes N} \to V \tag{2.1.3}$$

which are (real) analytic in $(x_1,\dots,x_N)$ on

$$M_N = \{(x_1,\dots,x_N) \in \mathbb{R}^{4N} \,|\, x_i \neq x_j \text{ for all } 1 \leq i < j \leq N\} . \tag{2.1.4}$$

For one point we set $\mathcal{C}(x_1) = \mathrm{id}$, where id is the identity map on V. The components of these maps in a basis of V correspond to the actual OPE coefficients. More precisely, if we denote a basis of V by $\{|v_A\rangle\}$ and a basis of the corresponding dual vector space V^* by $\{\langle v_A|\}$, then

$$\mathcal{C}^B_{A_1\dots A_N}(x_1,\dots,x_N) = \langle v_B|\, \mathcal{C}(x_1,\dots,x_N)\, |v_{A_1} \otimes \cdots \otimes v_{A_N}\rangle \tag{2.1.5}$$

where we used the standard physicist bra-ket notation $|v_{A_1} \otimes \cdots \otimes v_{A_N}\rangle := |v_{A_1}\rangle \otimes \cdots \otimes$

$|v_{A_N}\rangle$. The OPE coefficients are required to have the following properties (see [21] for a more detailed account of the motivations behind these axioms):

Hermitian conjugation: Denoting by $\iota : V \to V$ the anti-linear map given by the star operation $*$, we have

$$\overline{\mathcal{C}(x_1, \dots, x_N)} = \iota\, \mathcal{C}(x_1, \dots, x_N)\, \iota^N \tag{2.1.6}$$

where $\iota^N := \iota \otimes \cdots \otimes \iota$ is the N-fold tensor product of the map ι.

Euclidean invariance: Let R be the representation of Spin(4) on V, let $a \in \mathbb{R}^4$ and let $g \in \mathrm{Spin}(4)$. The OPE coefficients satisfy

$$\mathcal{C}(gx_1 + a, \dots, gx_N + a) = R^*(g)\, \mathcal{C}(x_1, \dots, x_N)\, R(g)^N \tag{2.1.7}$$

where $R(g)^N$ is the N-fold tensor product $R(g) \otimes \cdots \otimes R(g)$.

Bosonic nature: The OPE coefficients themselves should be "bosonic" in the sense that

$$\mathcal{C}(x_1, \dots, x_N) = \gamma\, \mathcal{C}(x_1, \dots, x_N)\gamma^N \tag{2.1.8}$$

where γ^N is again the N-fold tensor product $\gamma \otimes \cdots \otimes \gamma$.

Identity element: There exists a unique element $\mathbb{1} \in V^{0,0,e}$, where e is the identity element in our representation of Spin(4). This vector has the properties $\mathbb{1}^* = \mathbb{1}$, $\gamma(\mathbb{1}) = \mathbb{1}$. The OPE coefficients satisfy

$$\mathcal{C}(x_1, \dots, x_N)\, |v_1 \otimes \cdots \mathbb{1} \otimes \cdots v_{N-1}\rangle = \mathcal{C}(x_1, \dots \widehat{x_i}, \dots, x_N)\, |v_1 \otimes \cdots \otimes v_{N-1}\rangle \tag{2.1.9}$$

where $\mathbb{1}$ is in the i-th tensor position, with $i < N$. When $\mathbb{1}$ is in the N-th position, the condition takes a slightly more complicated form. This is due to the fact that, by convention, we expand around the point x_N, which therefore stands on a different footing than the other points. In this case, we require the OPE coefficients to satisfy the equation

$$\mathcal{C}(x_1, \dots, x_N)\, |v_1 \otimes \cdots v_{N-1} \otimes \mathbb{1}\rangle = t(x_{N-1}, x_N)\mathcal{C}(x_1, \dots, x_{N-1})\, |v_1 \otimes \cdots \otimes v_{N-1}\rangle \tag{2.1.10}$$

where the "Taylor expansion map" is a linear map $t(x_1, x_2) : V \to V$ for each $x_1, x_2 \in \mathbb{R}^4$ with the following properties: The map should have the same transformation properties as the OPE coefficients, see the Euclidean invariance axiom.

Further, it satisfies the properties

$$t(x_1, x_2)\, V^{\Delta} \subset \bigoplus_{\widehat{\Delta} \geq \Delta} V^{\widehat{\Delta}} \tag{2.1.11}$$

and

$$t(x_1, x_2) t(x_2, x_3) = t(x_1, x_3)\ . \tag{2.1.12}$$

Finally, the restriction of any vector of $t(x_1, x_2) V^{\Delta}$ to any subspace $V^{\widehat{\Delta}}$ should have a polynomial dependence on $(x_1 - x_2)$.

Scaling: Let $|v_{A_1}\rangle \in V^{\Delta_1}, \ldots, |v_{A_N}\rangle \in V^{\Delta_N}$ and $\langle v_B| \in (V^*)^{\Delta_{N+1}}$. Then the scaling degree[1] of the $\mathbb{C}$-valued distribution (2.1.5) can be estimated by

$$\mathrm{sd}\ \mathcal{C}^{B}_{A_1 \ldots A_N} \leq \Delta_1 + \ldots + \Delta_N - \Delta_{N+1}\ . \tag{2.1.14}$$

If $\langle v_B| \in (V^*)^0$, if $N = 2$ and if $|v_{A_1}\rangle, |v_{A_2}\rangle \neq 0$, then the inequality is required to be saturated.

(Anti-)symmetry: Let $\pi_{i-1,i}$ be the permutation acting on $V \otimes \cdots \otimes V$ by exchanging the $(i-1)$-th and i-th tensor factors. We require that for all $1 < i < N$

$$\mathcal{C}(x_1, \ldots, x_{i-1}, x_i, \ldots, x_N)\, \pi_{i-1,i} = \mathcal{C}(x_1, \ldots, x_i, x_{i-1}, \ldots, x_N)\, (-1)^{F_{i-1} F_i} \tag{2.1.15}$$

where $F_i := 1/2\ \mathrm{id}^{i-1} \otimes (\mathrm{id} - \gamma) \otimes \mathrm{id}^{n-i}$. For $i = N$ we demand that

$$\mathcal{C}(x_1, \ldots, x_{N-1}, x_N)\, \pi_{N-1,N} = t(x_{N-1}, x_N) \mathcal{C}(x_1, \ldots, x_N, x_{N-1})\, (-1)^{F_{N-1} F_N} \tag{2.1.16}$$

where $t(x_1, x_2)$ is the Taylor expansion map defined in identity element axiom above.

Factorisation: [2] For any $M < N \in \mathbb{N}$, let the OPE coefficients satisfy the identity

$$\mathcal{C}^{B}_{A_1 \ldots A_N}(x_1, \ldots, x_N) = \sum_{C} \mathcal{C}^{C}_{A_1 \ldots A_M}(x_1, \ldots, x_M)\, \mathcal{C}^{B}_{C A_{M+1} \ldots A_N}(x_M, \ldots, x_N) \tag{2.1.17}$$

[1] We define the scaling degree as

$$\mathrm{sd}\, \mathcal{C}^{B}_{A_1 \ldots A_N} = \inf_{p \in \mathbb{R}} \lim_{\epsilon \to 0} \epsilon^{p} \mathcal{C}^{B}_{A_1 \ldots A_N}(\epsilon x_1, \ldots, \epsilon x_N) = 0 \quad \text{for all } (x_1, \ldots, x_N) \in M_N \tag{2.1.13}$$

[2] The axiom differs slightly from the original version presented in [21]. By iteration of our factorisation identities (2.1.17) and (2.1.19), one can in fact derive the relation required in [21]. The only difference lies in the domain where this relation is supposed to hold. The factorisation axiom presented here turns out to be slightly weaker than the original one (i.e. the mentioned spacetime domain is smaller). This seems to be necessary in order to guarantee that the axiom is fulfilled by the OPE of a free quantum field.

on the spacetime domain

$$\max_{1\le i\le M} |x_i - x_M| < \min_{M<j\le N} |x_j - x_M| . \tag{2.1.18}$$

We require in particular that the infinite sum over C converges on the indicated domain, which, as we note, coincides with the domain (1.2.1) mentioned in the introduction for the choice $K = 1$. We note also that the identity

$$\mathcal{C}_{A_1\ldots A_N}^B(x_1,\ldots,x_N) = \sum_C \mathcal{C}_{A_{M+1}\ldots A_N}^C(x_{M+1},\ldots,x_N)\, \mathcal{C}_{A_1\ldots A_M C}^B(x_1,\ldots,x_M,x_N) \tag{2.1.19}$$

holds on the spacetime domain

$$\max_{M+1\le i\le N} |x_i - x_N| < \min_{1\le j\le M} |x_j - x_N| , \tag{2.1.20}$$

which follows from eq.(2.1.17) combined with the symmetry axiom above.

Analyticity (optional): [3] The OPE coefficients are analytic functions in the coupling constant(s) in a quantum field theory described by a Lagrangian.

We agree that two theories are equivalent if they differ only by a redefinition of the composite fields $\mathcal{O}_A \to \hat{\mathcal{O}}_A = \sum_B Z_A^B \mathcal{O}_B$, or equivalently $|v_A\rangle \to \sum_B Z_A^B |v_B\rangle$ in terms of the vector space V, where Z_A^B is a matrix of complex numbers. The OPE coefficients transform under such a redefinition by factors of this matrix, i.e.

$$\mathcal{C}(x_1,\ldots,x_N) = Z^{-1}\hat{\mathcal{C}}(x_1,\ldots,x_N)Z^N . \tag{2.1.21}$$

The axioms stated above put various restrictions on the admissible matrices Z_A^B:

- $Z_A^B \cdot \mathcal{O}_B$ should have the same tensor/spinor character as $\mathcal{O}_A$.
- Field redefinitions should be consistent with Euclidean invariance.
- The redefinition should not increase the dimension of the fields, i.e. for a redefinition $V^\Delta \ni |v_A\rangle \to \sum_B Z_A^B |v_B\rangle$ and we require $|v_B\rangle \in V^{\Delta'}$ with $\Delta' \le \Delta$.
- If the theory depends on a coupling constant g, it is reasonable to require that $Z_A^B(g)$ has a smooth dependence on g.
- We require $Z* = *Z$.

[3] This property was not required in [21]. However, it was argued in [13] that a condition of this type could be fruitful in that it leads to a restriction of renormalisation ambiguities. We will discuss potential ramifications of the analyticity condition in chapter 4.

We will re-encounter the ambiguities in the definition of composite fields later in the flow equation framework, see the discussion following eq.(3.1.28), and in the two dimensional Gross-Neveu model, see in particular section 4.1.

Finally, we require the existence of a collection of Schwinger functions, denoted by $\langle \mathcal{O}_{A_1}(x_1)\cdots\mathcal{O}_{A_N}(x_N)\rangle_\Omega$, which are analytic on M_N and satisfy the Osterwalder-Schrader (OS) axioms for the vacuum state Ω [37, 38]. They should also satisfy the OPE in the sense of an asymptotic expansion, i.e.

$$\langle \mathcal{O}_{A_1}(x_1)\cdots\mathcal{O}_{A_N}(x_N)\rangle_\Omega \sim \sum_B \mathcal{C}^B_{A_1\dots A_N}(x_1,\dots,x_N)\langle \mathcal{O}_B(x_N)\rangle_\Omega \,. \tag{2.1.22}$$

Here the symbol $\sim$ means that the difference between the left and right side is a distribution on M_N whose scaling degree is smaller than any given number δ provided the above sum goes over all of the finitely many fields $\mathcal{O}_B$ whose dimension is smaller than some number $\Delta = \Delta(\delta)$.

2.2 Features of the framework

The approach outlined in the previous section has been proposed quite recently, but it has nevertheless already proven to exhibit some interesting features and applications. Here we would like to give a rough overview of these existing results. We refer the reader to the cited literature for more details.

Coherence [21]: One can show that, due to the factorisation axiom, the N-point OPE coefficients, $\mathcal{C}(x_1,\dots,x_N)$, are uniquely determined in terms of the two point coefficients $\mathcal{C}(x_1,x_2)$. Further, the factorisation conditions with $N>3$ pose no additional conditions on the OPE coefficients beyond those already present in the factorisation condition with $N=3$.

Hochschild Cohomology [21]: The problem of finding perturbations of the OPE coefficients satisfying the above axioms (in particular the factorisation axiom) can be expressed quite elegantly in terms of cohomology theory. This formulation of perturbation theory is similar to the theory of *deformations* of an ordinary algebra.

Vertex algebras [22]: One can define *vertex operators* $Y(x,v): V \to V$ as the endomorphism of V whose matrix elements are given by

$$\langle v_C|Y(x,v_A)|v_B\rangle := \mathcal{C}^C_{AB}(x) \tag{2.2.1}$$

for any $x \neq 0$. It is possible to express the framework of the previous section entirely in terms of these vertex operators. In particular, the factorisation axiom then takes the form

$$Y(v_A, x)Y(v_B, y) = Y(Y(v_A, x-y)v_B, y)\,, \tag{2.2.2}$$

where the spacetime arguments are required to satisfy $|x| > |y| > |x-y| > 0$. This quadratic relation actually first appeared in the study of conformal field theories in two dimensions, where it is one of the crucial properties of the *vertex operator algebras* [9]. It should be stressed, however, that in our context, where conformal symmetry is not required, the condition above is a statement on the *convergence* of the infinite sums implicit in eq.(2.2.2), whereas the same equality in the CFT context is understood in terms of formal power series.

Field equations [21]: If a (massless) classical theory is described by a field equation such as $\Box\varphi = \partial^\mu\partial_\mu\varphi = g\varphi^3$, one would expect that this relation should also be realised on the level of OPE coefficients. It was noted in [21] that this relation, combined with the factorisation condition on the OPE coefficients, allows one to construct the OPE coefficients in perturbation theory via an iterative algorithm. The idea is as follows: One requires the identity[4]

$$\Box\, \mathcal{C}_{\varphi A}^{B}(x) = g\, \mathcal{C}_{\varphi^3 A}^{B}(x)\,, \tag{2.2.3}$$

which is just the field equation expressed on the level of OPE coefficients. If we expand the OPE coefficients in this equation as a formal power series in g, i.e. if we write

$$\mathcal{C}_{AB}^{C}(x) =: \sum_{i=0}^{\infty} \frac{g^i}{i!}(\mathcal{C}_i)_{AB}^{C}(x)\,, \tag{2.2.4}$$

then equation (2.2.3) yields a relation between OPE coefficients of different perturbation order due to the additional factor g on the right hand side.

$$\Box\, (\mathcal{C}_i)_{\varphi A}^{B}(x) = (\mathcal{C}_{i-1})_{\varphi^3 A}^{B}(x) \tag{2.2.5}$$

If we can solve this differential equation, i.e. invert the Laplace operator, then we can determine the OPE coefficient $(\mathcal{C}_i)_{\varphi A}^{B}$ in terms of a coefficient of lower order. In order to construct all the other coefficients $(\mathcal{C}_i)_{AB}^{C}$ at this order, one can make use of the factorisation condition. In this way, one could in principle construct the OPE up to any order in perturbation theory (see also [39] for an explicit construction of coefficients up to second perturbation order). An interesting feature of this algorithm is that one can avoid the familiar Feynman integrals in the explicit computations. Instead, one has to perform nested

[4] We slightly abuse notation here by writing for example $\mathcal{C}_{\varphi A}^{B}$ instead of $\mathcal{C}_{CA}^{B}$ in the case $\mathcal{O}_C = \varphi$.

infinite sums.

Extension to curved spacetimes [36, 13]: The framework has been adapted to curved spacetimes by Hollands and Wald. Some of the axioms presented above have to be changed slightly in this context. For example, the factorisation axiom is only required to hold in terms of asymptotic scaling relations, as opposed to convergent power series. Also, the coefficients are required to satisfy a *microlocal spectrum condition*, which is a requirement on their singularity structure.

It has been shown in [12] that the OPE satisfies the adapted version of the axioms within perturbative quantum field theory on curved spacetime.

Spin-statistics and CPT [40, 36]: An OPE satisfying the versions of the axioms adapted to curved spacetimes can be used to prove a version of the spin-statistics theorem as well the CPT-theorem, also on curved spacetimes, Thus, this axiomatic framework captures much of the same content as the Minkowski space Wightman axioms.

2.3 Relation to perturbative quantum field theory

The quantitative predictions of perturbative quantum field theory based on the Feynman path integral have been verified experimentally to extraordinary precision. Therefore it seems natural, as a first test, to verify whether a new approach to quantum field theory is consistent with the customary perturbative treatment.

As mentioned above, the existence of the OPE as an asymptotic expansion in perturbation theory is by now well established [41]. Hence, our question is: Does the operator product expansion in perturbative quantum field theory satisfy the axioms of section 2.1? For most of the axioms this is quite easy to check. The factorisation identity, however, is an exception. While it can be shown to hold for non-interacting quantum fields (i.e. in zeroth order of perturbation theory), it is not known whether the factorisation property carries over to interacting fields. The problem will be addressed in this thesis, and we will be able to prove the somewhat weaker *long distance* factorisation identity discussed in the introduction (see also section 3.5).

Even if the axiomatic approach is consistent with perturbative quantum field theory, one might ask whether the information provided by the collection of OPE coefficients and one-point functions is really equivalent to that provided by the n-point Schwinger functions in customary perturbative quantum field theory. We know that the operator product expansion approximates n-point functions in an asymptotic sense. If these expansions were to even *converge* at finite distances, then we could deduce that, indeed, the OPE coefficients combined with the one-point functions contain the same information as the

n-point functions. Generalising earlier results of Hollands and Kopper [15], we will show below that this is indeed the case.

To summarise, we are interested in the following questions within the setting of perturbative quantum field theory:

1. Does the operator product expansion converge for finite distances of the spacetime arguments?

2. Does the operator product expansion satisfy the factorisation identity?

3

The operator product expansion in perturbation theory

In 1970 Zimmermann gave the first proof that the operator product expansion holds (as an asymptotic expansion) in perturbative quantum field theory [6]. He also used this new tool in order to define *normal products* of quantum fields, which were essential in defining a sensible notion of composite fields within interacting models. In the following decades, the OPE was shown to exist within a large variety of settings, and it has also been found to be an indispensable computational tool in the study of high energy phenomena, such as e.g. *deep inelastic scattering* [7].

In this thesis we will study some fundamental properties of the OPE within massive Euclidean, perturbative $g\varphi^4$-theory. In particular, our main results in this chapter will be (the precise theorems including technical details will be given later in the corresponding sections):

Result 1: The operator product expansion converges (in the weak sense) up to any finite order in perturbation theory and for arbitrary finite separation of the spacetime arguments (see theorem 1 in section 3.3).

Result 2: The OPE coefficients factorise at large spacetime separation, i.e.

$$\mathcal{C}^{B}_{A_1\dots A_N}(x_1,\dots x_N) = \sum_C \mathcal{C}^{C}_{A_1\dots A_M}(x_1,\dots x_M)\mathcal{C}^{B}_{CA_{M+1}\dots A_N}(x_M,,\dots x_N) \tag{3.0.1}$$

holds for $\max_{1\le i\le M}|x_i - x_M| << \min_{M+1\le j\le N}|x_j - x_M|$ (see theorem 3 in section 3.5).

Result 3: We will give an explicit formula for the deformation of the operator product, i.e. we will express the OPE coefficients at perturbation order r in terms of (an integral over) those of order $s < r$ (see theorem 4 in section 3.6.2).

In view of the framework outlined in the previous chapter, it should be clear that these results are of conceptual interest, as they provide a rigorous underpinning for the approach to quantum field theory based on the OPE. The first two results yield direct answers to the questions raised at the end of chapter 2. The third result provides a very direct way of computing OPE coefficients in perturbation theory. Given the OPE coefficients of the free theory, the formula yields an algorithm for the construction of OPE coefficients up to any order. This algorithm is based purely on the OPE coefficients, i.e. no other objects, such as Schwinger functions, appear in the construction. Conceptually, this way of computing OPE coefficients is remarkably simple, compared to standard methods [7]. Practically, one should, of course, expect to encounter the same loop integrals as in the customary computation methods.

Our results are obtained within a framework of quantum field theory based on the renormalisation group flow equations. Early versions of this approach are due to Wilson and Wegner [16, 17, 19]. More recent treatments, which are closer to the methods employed in this thesis, were given by Polchinski, Kopper, Keller [18, 42, 43, 23]. For applications in a non-perturbative setting, see for example the work of Wetterich et.al. [44]. One of the appeals of this framework is that the objects of interest for the purposes of this thesis, such as Schwinger functions, composite operators and OPE coefficients, can be defined in a clear and mathematically rigorous fashion. Starting with the work of Polchinski, the flow equations have been applied to derive explicit bounds on important quantities, such as e.g. Schwinger functions, to all orders in perturbation theory. Refined versions of such bounds will be essential ingredients in the derivation of the results mentioned above.

One of our main technical advances in the flow equation framework developed in this thesis is the analysis of Schwinger functions with insertions of multiple composite operators. We will study the singular behaviour of these functions in the spacetime arguments, and we will define certain "regularised" (sometimes called "oversubtracted") versions of them. The regularisation of composite operators in perturbative quantum field theory was first developed by Zimmermann [6, 41], who introduced the so called *normal products*. Our work builds upon the results of [43, 23, 15], where Schwinger functions with insertion of up to two composite operators have been analysed within the flow equation setting. The generalisation of these results is non-trivial due to the presence of nested sub-divergences in the spacetime arguments in the case of three or more operator insertions, which clearly have no analogue in the case of two operator insertions and therefore pose a qualitatively new problem.

We will continue the present chapter with a review of the flow equation framework, where we will define all objects of relevance for us, in 3.1. Our original work starts with section 3.1.3, where the mentioned regularisation of Schwinger functions with multiple operator insertions is defined. In section 3.2 we present various bounds on these objects, which are derived using an inductive method based on the renormalisation flow equations. In sections 3.3 and 3.5 we will put these bounds to use and prove convergence and factorisation of the operator product expansion. To conclude this chapter, we will derive the mentioned perturbation formula for the OPE coefficients in section 3.6.

3.1 The flow equation framework

Wilson's view of renormalisation as a continuous evolution of effective actions was originally introduced with a focus primarily on non-perturbative theories. It was Polchinski who had the key insight that Wilson's renormalisation group flow equations, when applied to perturbation theory, allow for a closed inductive proof of renormalisability [18]. A great advantage of this proof lies in its remarkable simplicity as compared to the previous results on perturbative renormalisation, which had to deal with the great complexity of Feynman diagram expansions (see for example [45] for a review). The subsequent extensions of the framework [42, 43, 23] revealed that the flow equation method is also particularly well suited for the study of composite operators and of the Wilson operator product expansion. We will give a brief account of this framework in the following.

The model studied in this chapter is the hermitian scalar field theory with self-interaction $g\varphi^4$ and mass $m > 0$ on flat 4-dimensional Euclidean space. The quantities of interest in this (perturbative) quantum field theory will be defined in this section via the flow equation (FE) method [18, 19, 16, 17]. We will give a brief outline of the general formalism with a focus on objects of relevance to our study of the OPE, following closely [15]. The original presentation of the particular method used in this thesis can be found in [42], and for more detailed reviews we refer the reader to [20] and in [46] (in German).

3.1.1. Connected amputated Green's functions (CAG's)

We begin by introducing an infrared cutoff Λ, and an ultraviolet cutoff Λ_0. These cutoffs are implemented into the theory through a modification of the propagator C^{Λ,Λ_0}, which reads in momentum space:

$$C^{\Lambda,\Lambda_0}(p) := \frac{1}{p^2+m^2}\left[\exp\left(-\frac{p^2+m^2}{\Lambda_0^2}\right)-\exp\left(-\frac{p^2+m^2}{\Lambda^2}\right)\right] \tag{3.1.1}$$

Removing the cutoffs corresponds to taking the limits $\Lambda \to 0$ and $\Lambda_0 \to \infty$, which recovers the full propagator $1/(p^2+m^2)$. In the following, we always assume

$$0 < \Lambda\,, \quad \kappa := \sup(\Lambda, m) < \Lambda_0\,. \tag{3.1.2}$$

Other choices of regularisation than (3.1.1) are equally legitimate. The definition (3.1.1) has the advantage of being analytic in p^2 for $\Lambda > 0$. As we are dealing with a massive theory, an infrared cutoff is of course not actually necessary. It is introduced in the flow equation framework as a technical device, which will later allow us to derive the name-giving differential equations.

The propagator (3.1.1) defines a corresponding Gaussian measure μ^{Λ,Λ_0}, whose covariance is $\hbar C^{\Lambda,\Lambda_0}$. Here the factor $\hbar$ is introduced in order to obtain a consistent *loop expansion*[1] in the following. Since we are interested in $g\varphi^4$-theory, the interaction, including renormalisation counter terms, is taken to be (we also require the symmetry $\varphi \to -\varphi$, which causes odd powers of the basic field to vanish)

$$L^{\Lambda_0}(\varphi) = \int \mathrm{d}^4x \left(a^{\Lambda_0}\,\varphi(x)^2 + b^{\Lambda_0}\,\partial\varphi(x)^2 + c^{\Lambda_0}\,\varphi(x)^4\right). \tag{3.1.3}$$

Here the *basic field* $\varphi \in \mathcal{S}(\mathbb{R}^4)$ is any Schwartz space function. The counter terms $a^{\Lambda_0} = O(\hbar)$, $b^{\Lambda_0} = O(\hbar^2)$ and $c^{\Lambda_0} = \frac{g}{4!} + O(\hbar)$ will be adjusted–and actually diverge–when $\Lambda_0 \to \infty$, in order to obtain a well defined limit of the quantities of interest. This has been anticipated by making them "running couplings", i.e. functions of the ultra violet cutoff Λ_0. The correlation (= Schwinger- = Green's- = n-point-) functions of n basic fields with cutoff are defined by the expectation values

$$\begin{aligned}\langle\varphi(x_1)\cdots\varphi(x_n)\rangle &\equiv \mathbb{E}_{\mu^{\Lambda,\Lambda_0}}\left[\exp\left(-\frac{1}{\hbar}L^{\Lambda_0}\right)\varphi(x_1)\cdots\varphi(x_n)\right]\Big/ Z^{\Lambda,\Lambda_0} \\ &= \int \mathrm{d}\mu^{\Lambda,\Lambda_0}\,\exp\left(-\frac{1}{\hbar}L^{\Lambda_0}\right)\varphi(x_1)\cdots\varphi(x_n)\Big/ Z^{\Lambda,\Lambda_0}\,.\end{aligned} \tag{3.1.4}$$

This expression is simply the standard Euclidean path-integral, but with the free part in the Lagrangian absorbed into the Gaussian measure $\mathrm{d}\mu^{\Lambda,\Lambda_0}$. The normalisation factor Z^{Λ,Λ_0} is chosen so that $\langle 1\rangle = 1$. To keep this factor finite one actually has to impose an additional volume cutoff, but the infinite volume limit can be taken without difficulty once we pass to

[1] If one considers the usual Feynman diagram expansion of the quantities of interest defined below, then every closed loop yields a power of $\hbar$.

perturbative connected correlation functions, which we shall do in a moment. For more details on this limit see [47, 20]. In the perturbative approach to quantum field theory, which we will follow in this chapter, the exponentials in the path integral are expanded out and the Gaussian integrals are then performed. As mentioned earlier, the full theory is obtained by removing the cutoffs, $\Lambda_0 \to \infty$ and $\Lambda \to 0$, for a suitable choice of the running couplings. The correct behaviour of these couplings is determined, in the flow equation framework, by deriving first a differential equation in the parameter Λ for the Schwinger functions, and by then defining the running couplings implicitly through the boundary conditions for this equation.

These differential equations, referred to from now on as flow equations, are written more conveniently in terms of the hierarchy of "connected, amputated Green's functions" (CAG's), whose generating functional is given by the following convolution[2] of the Gaussian measure with the exponentiated interaction,

$$-L^{\Lambda,\Lambda_0} := \hbar \, \log \, \mu^{\Lambda,\Lambda_0} \star \exp\left(-\frac{1}{\hbar} L^{\Lambda_0} \right) - \hbar \log Z^{\Lambda,\Lambda_0} \, . \tag{3.1.5}$$

One can expand the functionals L^{Λ,Λ_0} as formal power series in terms of Feynman diagrams with l loops, n external legs and propagator $C^{\Lambda,\Lambda_0}(p)$. One can show that, indeed, only connected diagrams contribute, and the (free) propagators on the external legs are removed. While we will not use diagrammatic decompositions in terms of Feynman diagrams here, we will also analyse the functional (3.1.5) in the sense of formal power series

$$L^{\Lambda,\Lambda_0}(\varphi) := \sum_{n>0}^{\infty} \sum_{l=0}^{\infty} \hbar^l \int \mathrm{d}^4 x_1 \ldots \mathrm{d}^4 x_n \; \mathcal{L}_{n,l}^{\Lambda,\Lambda_0}(x_1, \ldots, x_n) \, \varphi(x_1) \cdots \varphi(x_n) \, . \tag{3.1.6}$$

No statement is made about the convergence of the series in $\hbar$. It turns out that the objects on the right side, the CAG's, are easier to work with than the full Schwinger functions. Thus, we will use these objects as basic quantities in our analysis. Of course, the full Schwinger functions can be recovered from the CAG's in the end.

Translation invariance of the connected amputated functions in position space implies that their Fourier transforms, denoted $\mathcal{L}_{n,l}^{\Lambda,\Lambda_0}(p_1, \ldots, p_n)$, are supported at $p_1 + \ldots + p_n = 0$. Therefore, we can write, by abuse of notation

$$\mathcal{L}_{n,l}^{\Lambda,\Lambda_0}(p_1, \ldots, p_n) = \delta^4 (\sum_{i=1}^{n} p_i) \, \mathcal{L}_{n,l}^{\Lambda,\Lambda_0}(p_1, \ldots, p_{n-1}) \, , \tag{3.1.7}$$

i.e. the momentum variable p_n is determined in terms of the remaining $n-1$ independent

[2]The convolution is defined in general by $(\mu^{\Lambda,\Lambda_0} \star F)(\varphi) = \int \mathrm{d}\mu^{\Lambda,\Lambda_0}(\varphi') \, F(\varphi + \varphi')$.

momenta by momentum conservation. One can show that, as functions of these remaining independent momenta, the connected amputated Green's functions are smooth for $\Lambda_0 < \infty$, $\mathcal{L}^{\Lambda,\Lambda_0}_{n,l}(p_1,\dots,p_{n-1}) \in C^\infty(\mathbb{R}^{4(n-1)})$.

To obtain the flow equations for the CAG's, we take the Λ-derivative of eq.(3.1.5):

$$\partial_\Lambda L^{\Lambda,\Lambda_0} = \frac{\hbar}{2}\langle\frac{\delta}{\delta\varphi}, \dot{C}^\Lambda \star \frac{\delta}{\delta\varphi}\rangle L^{\Lambda,\Lambda_0} - \frac{1}{2}\langle\frac{\delta}{\delta\varphi}L^{\Lambda,\Lambda_0}, \dot{C}^\Lambda \star \frac{\delta}{\delta\varphi}L^{\Lambda,\Lambda_0}\rangle + \hbar\partial_\Lambda \log Z^{\Lambda,\Lambda_0} . \tag{3.1.8}$$

Here we use the following notation: We write $\dot{C}^\Lambda$ for the derivative $\partial_\Lambda C^{\Lambda,\Lambda_0}$, which, as we note, does not depend on Λ_0. Further, by $\langle\ ,\ \rangle$ we denote the standard scalar product in $L^2(\mathbb{R}^4, \mathrm{d}^4x)$, and $\star$ stands for convolution in $\mathbb{R}^4$. As an example,

$$\langle\frac{\delta}{\delta\varphi}, \dot{C}^\Lambda \star \frac{\delta}{\delta\varphi}\rangle = \int \mathrm{d}^4x\, \mathrm{d}^4y\, \dot{C}^\Lambda(x-y)\frac{\delta}{\delta\varphi(x)}\frac{\delta}{\delta\varphi(y)} \tag{3.1.9}$$

is the "functional Laplace operator". Using eq.(3.1.6) to expand the functionals L^{Λ,Λ_0}, we can also write the flow equation (3.1.8) as

$$\begin{aligned}\partial_\Lambda \mathcal{L}^{\Lambda,\Lambda_0}_{2n,l}(p_1,\dots,p_{2n}) &= \binom{2n+2}{2}\int_k \dot{C}^\Lambda(k)\mathcal{L}^{\Lambda,\Lambda_0}_{2n+2,l-1}(k,-k,p_1,\dots,p_{2n})\\ -2\sum_{\substack{l_1+l_2=l\\ n_1+n_2=n+1}} n_1 n_2\, \mathbb{S}&\left[\mathcal{L}^{\Lambda,\Lambda_0}_{2n_1,l_1}(q,p_1,\dots,p_{2n_1-1})\,\dot{C}^\Lambda(q)\,\mathcal{L}^{\Lambda,\Lambda_0}_{2n_2,l_2}(p_{2n_1},\dots,p_{2n})\right],\end{aligned} \tag{3.1.10}$$

with $q = p_{2n_1} + \ldots + p_{2n}$ and where $\mathbb{S}$ is the symmetrisation operator acting on functions of the momenta $(p_1,\dots,p_{2n})$ by taking the mean value over all permutations π of $1,\dots,2n$ satisfying $\pi(1) < \pi(2) < \ldots < \pi(2n_1-1)$ and $\pi(2n_1) < \ldots < \pi(2n)$. The CAG's are defined uniquely as a solution to this differential equation only after we impose suitable boundary conditions. These are[3], using the multi-index convention introduced above in "Notations and Conventions":

$$\partial^w_{\vec{p}}\mathcal{L}^{0,\Lambda_0}_{n,l}(\vec{0}) = \delta_{w,0}\,\delta_{n,4}\,\delta_{l,0}\,\frac{g}{4!} \quad \text{for } n+|w| \le 4, \tag{3.1.11}$$

as well as

$$\partial^w_{\vec{p}}\mathcal{L}^{\Lambda_0,\Lambda_0}_{n,l}(\vec{p}) = 0 \quad \text{for } n+|w| > 4. \tag{3.1.12}$$

Here $\delta_{a,b}$ is the Kronecker-delta. The second boundary condition, (3.1.12), simply follows by noting that $L^{\Lambda_0,\Lambda_0} = L^{\Lambda_0}$, see (3.1.5), and by recalling the definition of the interaction L^{Λ_0}, (3.1.3). The actual renormalisation conditions are encoded in (3.1.11). The CAG's are then determined by integrating the flow equations subject to these boundary conditions,

[3] We restrict to BPHZ renormalisation in this thesis. Other choices are of course possible, and equally legitimate.

see e.g. [42, 20].

3.1.2. Insertions of composite fields

In the previous section we have defined Schwinger functions of products of the basic field. We now turn to the so called *composite operators* (or "composite fields"), which are given by the monomials

$$\mathcal{O}_A = \partial^{w_1}\varphi \cdots \partial^{w_n}\varphi \,, \quad A = \{n, w\}\,. \tag{3.1.13}$$

Here $w = (w_1, \dots, w_n) \in \mathbb{N}^{4n}$ is a multi-index (see also our notation and conventions section), and we denote the canonical dimension of such a field by

$$[A] := n + \sum_i |w_i|\,. \tag{3.1.14}$$

The Schwinger functions with insertions of composite operators are obtained by replacing the action L^{Λ_0} with an action containing additional sources, expressed through smooth functionals. Particular examples of such functionals are *local* ones. Any such local functional can by definition be written as

$$F(\varphi) = \sum_A \int \mathrm{d}^4x \; \mathcal{O}_A(x)\; f^A(x)\;, \quad f^A \in C_0^\infty(\mathbb{R}^4)\,, \tag{3.1.15}$$

where the composite operators $\mathcal{O}_A$ are as in eq. (3.1.13) and where the sum is finite. Recall that we may restrict attention to composite fields (3.1.13) with an even number of factors of φ as a result of our symmetry requirement $\varphi \to -\varphi$. We now modify the action L^{Λ_0} by adding sources f^A as follows:

$$L^{\Lambda_0} \to L^{\Lambda_0}_F := L^{\Lambda_0} - F - \sum_{j=0}^{\infty} B_j^{\Lambda_0}(\underbrace{F \otimes \cdots \otimes F}_{j}) \tag{3.1.16}$$

Here the last term represents the counter terms which are needed to eliminate the additional divergences arising from composite field insertions in the limit $\Lambda_0 \to \infty$. For each j it is a linear functional[4]

$$B_j^{\Lambda_0} : [C^\infty(\mathcal{S}(\mathbb{R}^4))]^{\otimes j} \to C^\infty(\mathcal{S}(\mathbb{R}^4))\;, \tag{3.1.17}$$

that is symmetric, and of order $O(\hbar)$. To obtain the Schwinger functions with insertions of r composite operators we now simply take functional derivatives with respect to the

[4] $C^\infty(\mathcal{S}(\mathbb{R}^4))$ denotes the space of smooth (in the Frechet sense) functionals. All our functionals are actually formal power series in $\hbar$, so we should write more accurately $C^\infty(\mathcal{S}(\mathbb{R}^4))[[\hbar]]$ for the space appearing below.

sources, setting the sources $f^{A_i} = 0$ afterwards:

$$\langle \mathcal{O}_{A_1}(x_1) \cdots \mathcal{O}_{A_r}(x_r) \rangle := \\ \hbar^r \frac{\delta^r}{\delta f^{A_1}(x_1) \ldots \delta f^{A_r}(x_r)} (Z^{\Lambda,\Lambda_0})^{-1} \int \mathrm{d}\mu^{\Lambda,\Lambda_0} \exp\left(-\frac{1}{\hbar} L_F^{\Lambda_0}(\varphi) \right)\Bigg|_{f^{A_i}=0} \tag{3.1.18}$$

Note that the CAG's discussed in the previous section are a special case of this equation; there we take $F = \int \mathrm{d}^4x \; f(x) \; \varphi(x)$, and we have $B_j^{\Lambda_0}(F^{\otimes j}) = 0$, because no extra counter terms are required for this insertion. As above, we can define a corresponding effective action as

$$-L_F^{\Lambda,\Lambda_0} := \hbar \, \log \, \mu^{\Lambda,\Lambda_0} \star \exp\left(-\frac{1}{\hbar}(L^{\Lambda_0} - F - \sum_{j=0}^{\infty} B_j^{\Lambda_0}(F^{\otimes j})) \right) - \log Z^{\Lambda,\Lambda_0} \tag{3.1.19}$$

which now depends on the sources f^{A_i}, as well as on φ. From this modified effective action we determine the generating functionals of the CAG's with r operator insertions:

$$L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_1}(x_1) \otimes \cdots \otimes \mathcal{O}_{A_r}(x_r)) := \frac{\delta^r \; L_F^{\Lambda,\Lambda_0}}{\delta f^{A_1}(x_1) \ldots \delta f^{A_r}(x_r)} \Bigg|_{f^{A_i}=0} . \tag{3.1.20}$$

The CAG's with insertions defined this way are multi-linear, as indicated by the tensor product notation, and symmetric in the insertions. We can also expand the CAG's with insertions in φ and $\hbar$ again (in momentum space):

$$L^{\Lambda,\Lambda_0}\left(\otimes_{i=1}^r \mathcal{O}_{A_i}(x_i) \right) = \\ \sum_{n,l \geq 0} \hbar^l \int \mathrm{d}^4 p_1 \ldots \mathrm{d}^4 p_n \; \mathcal{L}_{n,l}^{\Lambda,\Lambda_0}\left(\otimes_{i=1}^r \mathcal{O}_{A_i}(x_i); p_1, \ldots, p_n \right) \prod_{j=1}^n \hat{\varphi}(p_j) \tag{3.1.21}$$

Due to the insertions in $\mathcal{L}_{n,l}^{\Lambda,\Lambda_0}(\otimes_j \mathcal{O}_{A_j}(x_j), \vec{p})$, there is no restriction on the momentum set $\vec{p}$ in this case. Translation invariance, however, implies that the CAG's with insertions at a translated set of points $x_j + y$ are obtained from those at $y = 0$ through multiplication by $\mathrm{e}^{iy \sum_{i=1}^n p_i}$, i.e.

$$\mathcal{L}_{n,l}^{\Lambda,\Lambda_0}\left(\otimes_{i=1}^r \mathcal{O}_{A_i}(x_i + y); p_1, \ldots, p_n \right) = \mathrm{e}^{iy \sum_{i=1}^n p_i} \; \mathcal{L}_{n,l}^{\Lambda,\Lambda_0}\left(\otimes_{i=1}^r \mathcal{O}_{A_i}(x_i); p_1, \ldots, p_n \right). \tag{3.1.22}$$

Note also that only moments of CAG's with an even number n are non-vanishing, again by our $\mathbb{Z}_2$-symmetry requirement. The flow equation for the CAG's with insertions reads:

$$\begin{aligned}\partial_\Lambda L^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}) &= \frac{\hbar}{2}\langle \frac{\delta}{\delta\varphi}, \dot{C}^\Lambda \star \frac{\delta}{\delta\varphi}\rangle L^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}) \\ &-\frac{1}{2}\sum_{\substack{I_1\cup I_2=\{1,\dots,N\}\\ I_1\cap I_2=\emptyset}} \langle \frac{\delta}{\delta\varphi} L^{\Lambda,\Lambda_0}(\otimes_{i\in I_1}\mathcal{O}_{A_i}), \dot{C}^\Lambda \star \frac{\delta}{\delta\varphi} L^{\Lambda,\Lambda_0}(\otimes_{j\in I_2}\mathcal{O}_{A_j})\rangle ,\end{aligned} \tag{3.1.23}$$

In the second line it is understood that in the case $I = \emptyset$ we obtain the CAG's without insertions, i.e. $L^{\Lambda,\Lambda_0}(\otimes_{i\in I=\emptyset}\mathcal{O}_{A_i}) := L^{\Lambda,\Lambda_0}$. We also suppressed the coordinate space variables $(x_1,\dots,x_N)$ by writing $\mathcal{O}_{A_i}$ instead of $\mathcal{O}_{A_i}(x_i)$. This convention will also be used regularly in the following for the sake of brevity. For convenience, let us define $\mathfrak{P}(I)$ to be the *set of partitions* of a set I, i.e.

$$\mathfrak{P}(I) = \{(I_1,\dots,I_n) : \cup_{i=1}^n I_i = I, I_i \neq \emptyset, I_i \cap I_j = \emptyset \text{ for } i \neq j\}. \tag{3.1.24}$$

We can alternatively write the flow equation in the expanded version

$$\begin{aligned}&\partial_\Lambda \mathcal{L}^{\Lambda,\Lambda_0}_{2n,l}(\otimes_{i=1}^N \mathcal{O}_{A_i}; p_1,\dots,p_{2n}) = \\ &\binom{2n+2}{2}\int_k \dot{C}^\Lambda(k)\mathcal{L}^{\Lambda,\Lambda_0}_{2n+2,l-1}(\otimes_{i=1}^N \mathcal{O}_{A_i}; k,-k,p_1,\dots,p_{2n}) \\ &-4\sum_{\substack{l_1+l_2=l\\ n_1+n_2=n+1}} n_1 n_2\, \mathbb{S}\Bigg[\mathcal{L}^{\Lambda,\Lambda_0}_{2n_1,l_1}(\otimes_{i=1}^N \mathcal{O}_{A_i}; q,p_1,\dots,p_{2n_1-1})\,\dot{C}^\Lambda(q)\,\mathcal{L}^{\Lambda,\Lambda_0}_{2n_2,l_2}(p_{2n_1},\dots,p_{2n}) \\ &\quad+\frac{1}{2}\sum_{(I_1,I_2)\in\mathfrak{P}\{1,\dots,N\}}\int_k \mathcal{L}^{\Lambda,\Lambda_0}_{2n_1,l_1}(\bigotimes_{i\in I_1}\mathcal{O}_{A_i}; k,p_1,\dots,p_{2n_1-1}) \\ &\qquad\qquad\times \dot{C}^\Lambda(k)\,\mathcal{L}^{\Lambda,\Lambda_0}_{2n_2,l_2}(\bigotimes_{j\in I_2}\mathcal{O}_{A_j}; -k,p_{2n_1},\dots,p_{2n})\Bigg].\end{aligned} \tag{3.1.25}$$

Note that the flow equation for the CAG's with $N \geq 2$ insertions involves inhomogeneities (called *source terms* in the following) in the last line, which are quadratic in the CAG's with less than N insertions. Therefore, we have to ascend in the number of insertions if we want to integrate the flow equations (3.1.25). To complete the definition of the CAG's with insertions, we again have to specify boundary conditions on the corresponding flow equation. The simplest choice in the case of $N \geq 2$ insertions is

$$\partial^w_{\vec{p}}\mathcal{L}^{\Lambda_0,\Lambda_0}_{n,l}(\otimes_{i=1}^N \mathcal{O}_{A_i}(x_i); \vec{p}) = 0 \quad \text{for all } w,n,l. \tag{3.1.26}$$

For CAG's with one insertion we choose ("normal ordering")[5]

$$\partial^w_{\vec{p}} \mathcal{L}^{0,\Lambda_0}_{n,l}(\mathcal{O}_A(0);\vec{0}) = i^{|w|} w! \delta_{w,w'}\delta_{n,n'}\delta_{l,0} \quad \text{for } n+|w| \leq [A] \tag{3.1.27}$$

$$\partial^w_{\vec{p}} \mathcal{L}^{\Lambda_0,\Lambda_0}_{n,l}(\mathcal{O}_A(0);\vec{p}) = 0 \quad \text{for } n+|w| > [A] \quad . \tag{3.1.28}$$

Our freedom to choose boundary conditions different from (3.1.27) can be seen to correspond to field redefinitions of the type discussed at the end of section 2.1 (cf. [43]). Although the connected amputated Green's functions (CAG's) with insertions can be used as the basic building blocks of the correlation functions, it will turn out to be useful to also consider certain non-connected versions of these, called "AG's with insertions" in the following. They are defined as

$$G^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}) := \sum_{\alpha=1}^{N} \sum_{(I_1,\dots,I_\alpha)\in\mathfrak{P}\{1,\dots,N\}} \prod_{i=1}^{\alpha} (-\hbar)^{N-\alpha} L^{\Lambda,\Lambda_0}(\otimes_{j\in I_i}\mathcal{O}_{A_j})\,. \tag{3.1.29}$$

Note that the case $N=1$ just reduces to the CAG's with one insertion, i.e. $G^{\Lambda,\Lambda_0}(\mathcal{O}_A) = L^{\Lambda,\Lambda_0}(\mathcal{O}_A)$. As usual, we also consider the expanded quantities in $\hbar$ and $\hat{\varphi}$; these are denoted in the present case as $\mathcal{G}^{\Lambda,\Lambda_0}_{n,l}(\otimes_{i=1}^N \mathcal{O}_{A_i},\vec{p})$, where as usual, l indicates the power of $\hbar$, and n the power of $\hat{\varphi}$. As the name suggests, these are the amputated versions of the Schwinger (=Green's) functions[6],

$$\begin{aligned}&\Big\langle \prod_{i=1}^N \mathcal{O}_{A_i}(x_i) \prod_{j=1}^n \hat{\varphi}(p_j)\Big\rangle \prod_{k=1}^n (C^{\Lambda,\Lambda_0}(p_k))^{-1} \\ &= \sum_{j=1}^n \sum_{\substack{I_1\cup\dots\cup I_j=\{1,\dots,n\}\\ I_i\cap I_j=\emptyset \\ l_1+\dots+l_j=l\geq 0}} \hbar^{n+l+1-j}\, \mathcal{G}^{\Lambda,\Lambda_0}_{|I_1|,l_1}(\otimes_{i=1}^N \mathcal{O}_{A_i}(x_i),\vec{p}_{I_1})\, \bar{\mathcal{L}}^{\Lambda,\Lambda_0}_{|I_2|,l_2}(\vec{p}_{I_2})\cdots \bar{\mathcal{L}}^{\Lambda,\Lambda_0}_{|I_j|,l_j}(\vec{p}_{I_j})\end{aligned} \tag{3.1.30}$$

where $\bar{\mathcal{L}}^{\Lambda,\Lambda_0}_{n,l}$ are the expansion coefficients of the generating functional $\bar{L}^{\Lambda,\Lambda_0}(\varphi) = -L^{\Lambda,\Lambda_0}(\varphi) + \frac{1}{2}\langle\varphi,\,(C^{\Lambda,\Lambda_0})^{-1}\star\varphi\rangle$ without the momentum conservation delta functions taken out. We will use this relation later.

[5] See [43, 23] for a more detailed motivation of these boundary conditions. It should be mentioned that our definition of the functionals $L^{\Lambda,\Lambda_0}(\mathcal{O}_A)$ differs from the one given in those papers by a minus sign.

[6] Strictly speaking, the functionals $G^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i})$ do not generate *all* the amputated Feynman diagrams with operator insertions, since connected pieces without any operator insertion are excluded, see also eq.(3.1.30). For lack of a better name, we will however continue to refer to these functionals as amputated Green's functions with insertions by a slight abuse of language.

By contrast to the CAG's, the AG's satisfy linear *homogeneous* flow equations,

$$\begin{aligned}\partial_\Lambda G^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}) = &\frac{\hbar}{2}\langle\frac{\delta}{\delta\varphi}, \dot{C}^\Lambda \star \frac{\delta}{\delta\varphi}\rangle\, G^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}) \\ &- \langle\frac{\delta}{\delta\varphi} G^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}), \dot{C}^\Lambda \star \frac{\delta}{\delta\varphi} L^{\Lambda,\Lambda_0}\rangle\,.\end{aligned} \tag{3.1.31}$$

The fact that the AG's satisfy a linear homogeneous flow equation is a welcome simplification, which is unfortunately counterbalanced by the fact that the boundary conditions for the AG's are more complicated. Therefore, as a compromise between simple flow equation and simple boundary conditions, we will not work with the full AG's in the following, but instead define the slightly modified objects

$$\hbar F^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}) := G^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}) - \prod_{i=1}^N L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_i})\,. \tag{3.1.32}$$

Using the definitions of the CAG's given above, these functionals are seen to obey the flow equation

$$\begin{aligned}&\partial_\Lambda F^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}) \\ =&\frac{\hbar}{2}\langle\frac{\delta}{\delta\varphi}, \dot{C}^\Lambda \star \frac{\delta}{\delta\varphi}\rangle\, F^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}) - \langle\frac{\delta}{\delta\varphi} F^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}), \dot{C}^\Lambda \star \frac{\delta}{\delta\varphi} L^{\Lambda,\Lambda_0}\rangle \\ +& \sum_{1\le i<j\le N} \langle\frac{\delta}{\delta\varphi} L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_i}), \dot{C}^\Lambda \star \frac{\delta}{\delta\varphi} L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_j})\rangle \prod_{r\in\{1,\dots,N\}\setminus\{i,j\}} L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_r})\end{aligned} \tag{3.1.33}$$

and the trivial boundary conditions

$$\partial_{\vec{p}}^w \mathcal{F}_{n,l}^{\Lambda_0,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}; \vec{p}) = 0 \quad \text{for all } n, l, w, \tag{3.1.34}$$

with a calligraphic letter $\mathcal{F}_{n,l}^{\Lambda,\Lambda_0}$ denoting as usual the objects appearing in the expansion of F^{Λ,Λ_0} in powers of $\hbar, \varphi$. In terms of Feynman diagrams we may interpret these functionals as follows: As mentioned above, the G-functionals correspond to the (not necessarily connected) amputated Feynman graphs with N extra vertices corresponding to the operator insertions. On the other hand, the F-functionals correspond to the subset of these diagrams where *at least two* of the operator insertions belong to the same connected component of the graph. Like the CAG's with multiple insertions, the F-functionals are divergent on the partial diagonals, i.e. whenever two or more spacetime arguments coincide. Since the CAG's with one insertion are smooth in the spacetime argument [see equation (3.1.22)], the decomposition (3.1.32) separates the contributions to G which are regular in the spacetime arguments from those which are singular at short distances. We also note that translation invariance again implies

$$\mathcal{F}^{\Lambda,\Lambda_0}_{n,l}(\otimes_{i=1}^N \mathcal{O}_{A_i}(x_i); p_1,\dots,p_n) = \mathrm{e}^{iy(p_1+\ldots+p_n)}\mathcal{F}^{\Lambda,\Lambda_0}_{n,l}(\otimes_{i=1}^N \mathcal{O}_{A_i}(x_i - y); p_1,\dots,p_n)\,. \tag{3.1.35}$$

3.1.3. Regularisation of Schwinger functions with insertions

The purpose of introducing a UV-cutoff is that as long as we keep Λ_0 finite, the CAG's with insertions depend smoothly on the points $x_1,\dots,x_N$, as well as on the momenta $p_1,\dots,p_n$ (see also remark 5 below). In the limit $\Lambda_0 \to \infty$, smoothness in the x_i's however is lost, and the CAG's develop singularities for configurations such that some of the points x_i coincide. This is of course not a problem, nor unexpected–the Green's functions in quantum field theory are usually singular for coinciding points–reflecting the singular nature of the operators themselves. In the following we will discuss certain *regularised* (sometimes also called *oversubtracted*) versions of the Green's functions with insertions defined in the previous section, which possess a higher degree of regularity in the spacetime arguments. As we will see later, these regularised Green's functions play a crucial role in the definition and application of the operator product expansion.

A method for improving regularity of Green's functions with two operator insertions was first given, in the context of the present framework, in [23], and the case of three insertions has been analysed by Hollands and the present author in [26]. The procedure described in the present- and the following section constitutes, as far as we are aware, the first discussion of the regularisation of Green's functions with more than three operator insertions within the flow equation framework.

Let us first try to understand the process of regularisation for Green's functions with a low number of operator insertions. For $N = 1$ insertion, the functionals $L^{\Lambda,\Lambda_0}(\mathcal{O}_A(x))$ are already smooth in x, see eq.(3.1.22), so there is no need to improve regularity in this case. For $N = 2$, however, one can show that the CAG's $L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_1}(x) \otimes \mathcal{O}_{A_2}(0))$ are singular at $x = 0$ if we remove the cutoff $\Lambda_0 \to \infty$. As mentioned above, a method of removing this divergence was first given in [23]. Namely, one simply changes the boundary conditions for the flow equation defining these CAG's . These regularised CAG's are then parametrised by a single integer $D \geq -1$ and defined by the same flow equation, eq.(3.1.23), but subject to the boundary conditions

$$\partial^w_{\vec{p}} \mathcal{L}^{0,\Lambda_0}_{D,n,l}(\mathcal{O}_{A_1}(x) \otimes \mathcal{O}_{A_2}(0); \vec{0}) = 0 \quad \text{for } n + |w| \leq D \tag{3.1.36}$$

as well as

$$\partial^w_{\vec{p}} \mathcal{L}^{\Lambda_0,\Lambda_0}_{D,n,l}(\mathcal{O}_{A_1}(x) \otimes \mathcal{O}_{A_2}(0); \vec{p}) = 0 \quad \text{for } n + |w| > D \tag{3.1.37}$$

instead of eq.(3.1.26). It has been shown in [23] that these functions are of differentiability class $C^{D-[A_1]-[A_2]}$ in x[7]. This justifies the interpretation of D as a regularisation parameter.

As we progress to the case of $N = 3$ insertions, however, the situation starts to get more complicated. This is due to the fact that now the source terms in the corresponding flow equation, (3.1.23), contain not only the regular CAG's with one insertion, but also CAG's with two insertions, which themselves are singular when the spacetime arguments of the two insertions coincide. Therefore, in order to remove the singularities from the CAG's with three insertions, simply changing the boundary conditions for the flow equation does not suffice. One also has to alter the flow equation itself so as to regularise these "source terms". This way, one has to specify not only one regularisation parameter, but a whole hierarchy (one for each partial diagonal). This strategy can indeed be carried through, and the procedure has been outlined in [26]. However, as one tries to progress to larger N, i.e. insertions of more operators, the method quickly becomes somewhat cumbersome and heavy on notation due to the need to regularise nested subdivergences. In the following we will present an alternative method for regularising the amputated Green's functions (AG's) instead of the CAG's. While the method based on the AG's will leave us with less control over the exact degree of regularisation for each nested subdivergence, it will be sufficiently versatile for our needs, and it has the benefit of being much more economical and also easier to follow in the case of large N.

Recall from eq.(3.1.32) that we can decompose the AG's $G^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i})$ into a factorised part, where the composite operators $\mathcal{O}_{A_i}$ are inserted into diagrams which are disconnected from each other, and a contribution called $F^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i})$, which corresponds to diagrams where at least one pair of composite operators is inserted into the same connected piece. Since the CAG's with one insertion are smooth, so will be the factorised part. All the divergences on the (partial) diagonals are thus included in the F functionals. Therefore, the question now is how to regularise these functionals.

Note that the source term in the flow equation for the F-functionals, i.e. the last line of eq.(3.1.33), contains only CAG's with one insertion, which are smooth. Thus, there is no need to alter the flow equation if we want to regularise the F-functionals. The regularised versions of the F-functionals, called $F_D^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i})$, are therefore quite easily obtained in an analogous manner to the CAG's $L_D^{\Lambda,\Lambda_0}(\mathcal{O}_{A_1}(x) \otimes \mathcal{O}_{A_2}(0))$ by simply changing the boundary conditions:

[7] Explicit bounds on these functions have been given in [15] (for the case $D = [A_1] + [A_2]$) and [26].

Definition 1 (Regularised AG's): *The* amputated Green's functions (AG's) with insertions and regularisation *are defined for any* $D \geq -1$ *as*

$$G_D^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}) := \hbar F_D^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}) + \prod_{i=1}^N L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_i}) \,, \tag{3.1.38}$$

where the functionals F_D^{Λ,Λ_0} *are required to satisfy the flow equation* (3.1.33) *and the boundary conditions*

$$\partial_{\vec{p}}^w \mathcal{F}_{D,n,l}^{0,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}(x_i); \vec{0})\Big|_{x_N=0} = 0 \quad \text{for } n + |w| \leq D \tag{3.1.39}$$

$$\partial_{\vec{p}}^w \mathcal{F}_{D,n,l}^{\Lambda_0,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}(x_i); \vec{p})\Big|_{x_N=0} = 0 \quad \text{for } n + |w| > D \,. \tag{3.1.40}$$

Evidently, $F_{D=-1}^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}) = F^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i})$ are the functionals without regularisation. We will see below in bound 1 that, indeed, the functionals defined above are of scaling degree (recall eq.(2.1.13) for the definition of this concept)

$$\mathrm{sd}(F_D^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i})) \leq [A_1] + \ldots + [A_N] - D \tag{3.1.41}$$

in the spacetime arguments x_i, which confirms the role of D as a regularisation parameter. Note that in the $N = 2$ case, F_D reduces to the CAG with two insertions, i.e.

$$F_D^{\Lambda,\Lambda_0}(\mathcal{O}_A(x) \otimes \mathcal{O}_B(0)) = -L_D^{\Lambda,\Lambda_0}(\mathcal{O}_A(x) \otimes \mathcal{O}_B(0)) \tag{3.1.42}$$

since both sides of the equation share the same flow equation and boundary conditions. For $N \geq 3$, however, such a simple relation does not seem to exist.

The spacetime derivatives of the AG's with insertions satisfy some properties which will come in handy in later sections. These relations are generalisations of the so called *Lowenstein rules*, which can be found for example in [43, 23].

Proposition 1: *The amputated Green's functions with operator insertions satisfy the relations*

$$\partial_{x_j}^v G_D^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}(x_i))\Big|_{x_N=0} = G_D^{\Lambda,\Lambda_0}\left(\partial_{x_j}^v \otimes_{i=1}^N \mathcal{O}_{A_i}(x_i)\right)\Big|_{x_N=0} , \tag{3.1.43}$$

for any $1 \leq j < N$, *and*

$$(\partial_{x_1} + \ldots + \partial_{x_N})^v G_D^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}(x_i)) = G_{D+|v|}^{\Lambda,\Lambda_0}\left((\partial_{x_1} + \ldots + \partial_{x_N})^v \otimes_{i=1}^N \mathcal{O}_{A_i}(x_i)\right) , \tag{3.1.44}$$

where $v \in \mathbb{N}^4$.

Remark 1: Note that on the left hand side the derivatives act on the functional $G^{\Lambda,\Lambda_0}_{D,n,l}$, whereas they act directly on the composite fields $\mathcal{O}_{A_i}$ on the right hand side. By $\partial^w \mathcal{O}_A$ we mean the linear combination of monomials which are obtained by carrying out the derivatives in the obvious manner. We also note that in the case $N = 1$, these Lowenstein rules imply

$$\partial^v_x L^{\Lambda,\Lambda_0}(\mathcal{O}_A(x)) = L^{\Lambda,\Lambda_0}(\partial^v_x \mathcal{O}_A(x))\,. \tag{3.1.45}$$

Proof. We derive all the claimed relations using the same general strategy: *If two functionals satisfy the same linear flow equations and the same boundary conditions, then these functionals must coincide.* We will, in fact, apply this strategy repeatedly in this thesis.

To begin with, let us verify the Lowenstein rule for the case of one insertion, eq.(3.1.45). Note that both sides of the equation obey the same linear homogeneous flow equation. Further, note that, using the translation properties of the CAG's with insertions, we can write

$$\begin{aligned}\partial^v_x \mathcal{L}^{\Lambda,\Lambda_0}_{n,l}(\mathcal{O}_A(x);\vec{p}) &= \partial^v_x \mathrm{e}^{i(p_1+\ldots+p_n)x}\,\mathcal{L}^{\Lambda,\Lambda_0}_{n,l}(\mathcal{O}_A(0);\vec{p})\\ &= i^{|v|}(p_1+\ldots+p_n)^v\,\mathcal{L}^{\Lambda,\Lambda_0}_{n,l}(\mathcal{O}_A(x);\vec{p})\,.\end{aligned} \tag{3.1.46}$$

Combining this equation with the boundary conditions for the CAG's with one insertion, eqs.(3.1.27) and (3.1.28), it is easy to check that both sides of eq.(3.1.45) are also subject to the same boundary conditions. Thus, both sides of eq.(3.1.45) must coincide.

To continue, let us come to the proof of eq.(3.1.43). Recall first the definition of the AG's with insertions, G^{Λ,Λ_0}_D, from equation (3.1.38). In view of eq.(3.1.45), we immediately find that the factorised contributions to both sides of (3.1.43) coincide, i.e.

$$\partial^v_{x_j}\prod_{i=1}^N L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_i}(x_i)) = L^{\Lambda,\Lambda_0}(\partial^v_{x_j}\mathcal{O}_{A_j}(x_j))\prod_{\substack{i=1\\ i\neq j}}^N L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_i}(x_i))\,. \tag{3.1.47}$$

Concerning the remaining contributions from the F^{Λ,Λ_0}_D functionals, we again simply compare flow equations and boundary conditions to see that both sides of the equation coincide.

Finally, the relation (3.1.44) can be derived in a very similar way. Again it is not hard to check that both sides of the equation satisfy the same flow equation. To see that also the boundary conditions coincide, we need the translation property for the AG's

$$\mathcal{G}^{\Lambda,\Lambda_0}_{D,n,l}(\otimes_{i=1}^N \mathcal{O}_{A_i}(x_i);\vec{p}) = \exp(i(p_1+\ldots+p_n)y)\,\mathcal{G}^{\Lambda,\Lambda_0}_{D,n,l}(\otimes_{i=1}^N \mathcal{O}_{A_i}(x_i-y);\vec{p})\,, \tag{3.1.48}$$

where we made use of eq.(3.1.35) and (3.1.22). Choosing $y = (x_1+\ldots+x_N)/N$ and taking the corresponding derivatives of this equation then yields

$$(\partial_{x_1} + \ldots + \partial_{x_N})^v \ \mathcal{G}^{\Lambda,\Lambda_0}_{D,n,l}(\otimes_{i=1}^N \mathcal{O}_{A_i}(x_i); \vec{p}) = (p_1+\ldots+p_n)^v \ \mathcal{G}^{\Lambda,\Lambda_0}_{D,n,l}(\otimes_{i=1}^N \mathcal{O}_{A_i}(x_i); \vec{p})\,. \tag{3.1.49}$$

The powers of $\vec{p}$ imply that the right hand side of this equation is subject to the same boundary conditions as the right hand side of (3.1.44), which finishes the proof of the proposition. □

3.1.4. Regularisation of subdivergences

In the previous section we have outlined a procedure that allows us to improve the total scaling degree of the amputated Green's functions $G^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i})$. In other words, we are able to control the singular behaviour of these functionals with respect to the total diagonal $x_1 = \ldots = x_N$. In some applications, however, one might want to remove only divergences associated to the partial diagonals of a subset of the spacetime arguments $x_1, \ldots, x_N$. In the present section we will define this regularisation of subdivergences, which is one of the main technical advances to the flow equation framework provided by this thesis.

It is a priori far from clear how to generalise the strategy of the previous section to subdivergences. The following lemma provides a decomposition of the AG's that will be helpful for this purpose:

Lemma 1: *For any $N \geq 2$ and $M < N$ the following decomposition holds:*

$$\begin{aligned} G^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}) &= G^{\Lambda,\Lambda_0}(\otimes_{i=1}^M \mathcal{O}_{A_i})\, G^{\Lambda,\Lambda_0}(\otimes_{i=M+1}^N \mathcal{O}_{A_i}) \\ &\quad + \hbar H^{\Lambda,\Lambda_0}(\otimes_{i=1}^M \mathcal{O}_{A_i}; \otimes_{i=M+1}^N \mathcal{O}_{A_i}) \end{aligned} \tag{3.1.50}$$

Here the functionals H^{Λ,Λ_0} are defined through the flow equation

$$\begin{aligned} \partial_\Lambda H^{\Lambda,\Lambda_0}(\otimes_{i=1}^M \mathcal{O}_{A_i}; \otimes_{i=M+1}^N \mathcal{O}_{A_i}) &= \frac{\hbar}{2}\langle \frac{\delta}{\delta\varphi}, \dot{C}^\Lambda \star \frac{\delta}{\delta\varphi}\rangle\, H^{\Lambda,\Lambda_0}(\otimes_{i=1}^M \mathcal{O}_{A_i}; \otimes_{i=M+1}^N \mathcal{O}_{A_i}) \\ &\quad - \langle \frac{\delta}{\delta\varphi} H^{\Lambda,\Lambda_0}(\otimes_{i=1}^M \mathcal{O}_{A_i}; \otimes_{i=M+1}^N \mathcal{O}_{A_i}), \dot{C}^\Lambda \star \frac{\delta}{\delta\varphi} L^{\Lambda,\Lambda_0}\rangle \\ &\quad + \langle \frac{\delta}{\delta\varphi} G^{\Lambda,\Lambda_0}(\otimes_{i=1}^M \mathcal{O}_{A_i}), \dot{C}^\Lambda \star \frac{\delta}{\delta\varphi} G^{\Lambda,\Lambda_0}(\otimes_{i=M+1}^N \mathcal{O}_{A_i})\rangle \end{aligned} \tag{3.1.51}$$

and the boundary conditions

$$\partial^w_{\vec{p}}\, \mathcal{H}^{\Lambda_0,\Lambda_0}_{n,l}(\otimes_{i=1}^M \mathcal{O}_{A_i}; \otimes_{i=M+1}^N \mathcal{O}_{A_i}; \vec{p}) = 0 \quad \textit{for all } n, l, w. \tag{3.1.52}$$

Remark 2: The lemma can also be understood diagrammatically. On the l.h.s. of equation

(3.1.50) we have all (amputated) diagrams with insertion of extra vertices corresponding to the composite operators $\mathcal{O}_{A_1}, \dots, \mathcal{O}_{A_N}$. The first term on the r.h.s. stands for the factorised contributions, where the diagrams containing the $\mathcal{O}_{A_1}, \dots, \mathcal{O}_{A_M}$ vertices are disconnected from the diagrams containing the $\mathcal{O}_{A_{M+1}}, \dots, \mathcal{O}_{A_N}$ vertices. The second term on the r.h.s. then contains all contributions where at least one pair of vertices $\mathcal{O}_{A_i}, \mathcal{O}_{A_j}$ with $1 \le i \le M < j \le N$ belong to the same connected diagram.

Proof. We will show that both sides of the equation satisfy the same flow equation and boundary conditions. First, we note that the fully factorised term $\prod_{i=1}^N L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_i})$ appears on both sides of the equation, so we can just subtract it and arrive at the equivalent claim

$$\begin{aligned}
&\hbar F^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i})\\
=&\hbar^2 F^{\Lambda,\Lambda_0}(\otimes_{i=1}^M \mathcal{O}_{A_i})\ F^{\Lambda,\Lambda_0}(\otimes_{i=M+1}^N \mathcal{O}_{A_i}) + \hbar F^{\Lambda,\Lambda_0}(\otimes_{i=1}^M \mathcal{O}_{A_i}) \prod_{j=M+1}^N L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_j})\\
+&\prod_{i=1}^M L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_j})\, \hbar\, F^{\Lambda,\Lambda_0}(\otimes_{j=M+1}^N \mathcal{O}_{A_j}) + \hbar H^{\Lambda,\Lambda_0}(\otimes_{i=1}^M \mathcal{O}_{A_i}; \otimes_{i=M+1}^N \mathcal{O}_{A_i})
\end{aligned} \tag{3.1.53}$$

Note that in view of eqs. (3.1.34) and (3.1.52), all the terms in this expression satisfy the same trivial boundary conditions. Let us now determine the flow equation satisfied by the r.h.s. of the equation. Using eq.(3.1.33) we find for the first term

$$\begin{aligned}
&\partial_\Lambda \left(F^{\Lambda,\Lambda_0}(\otimes_{i=1}^M \mathcal{O}_{A_i})\ F^{\Lambda,\Lambda_0}(\otimes_{i=M+1}^N \mathcal{O}_{A_i}) \right) =\\
&\frac{\hbar}{2}\langle \frac{\delta}{\delta\varphi}, \dot{C}^\Lambda \star \frac{\delta}{\delta\varphi}\rangle\ F^{\Lambda,\Lambda_0}(\otimes_{i=1}^M \mathcal{O}_{A_i})\ F^{\Lambda,\Lambda_0}(\otimes_{i=M+1}^N \mathcal{O}_{A_i})\\
&-\langle \frac{\delta}{\delta\varphi} F^{\Lambda,\Lambda_0}(\otimes_{i=1}^M \mathcal{O}_{A_i})\ F^{\Lambda,\Lambda_0}(\otimes_{i=M+1}^N \mathcal{O}_{A_i}), \dot{C}^\Lambda \star \frac{\delta}{\delta\varphi} L^{\Lambda,\Lambda_0}\rangle\\
&-\hbar\langle \frac{\delta}{\delta\varphi} F^{\Lambda,\Lambda_0}(\otimes_{i=1}^M \mathcal{O}_{A_i}), \dot{C}^\Lambda \star \frac{\delta}{\delta\varphi} F^{\Lambda,\Lambda_0}(\otimes_{i=M+1}^N \mathcal{O}_{A_i})\rangle\\
&+\sum_{1\le i<j\le M} \langle \frac{\delta}{\delta\varphi} L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_i}), \dot{C}^\Lambda \star \frac{\delta}{\delta\varphi} L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_j})\rangle \prod_{r=\{1,\dots,M\}\setminus\{i,j\}} L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_r}) F^{\Lambda,\Lambda_0}(\otimes_{k=M+1}^N \mathcal{O}_{A_k})\\
&+\sum_{M<i<j\le N} \langle \frac{\delta}{\delta\varphi} L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_i}), \dot{C}^\Lambda \star \frac{\delta}{\delta\varphi} L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_j})\rangle \prod_{r=\{M+1,\dots,N\}\setminus\{i,j\}} L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_r}) F^{\Lambda,\Lambda_0}(\otimes_{k=1}^M \mathcal{O}_{A_k})
\end{aligned} \tag{3.1.54}$$

For the second term we obtain

$$
\begin{aligned}
&\partial_\Lambda \left(F^{\Lambda,\Lambda_0}(\otimes_{i=1}^M \mathcal{O}_{A_i}) \prod_{j=M+1}^N L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_j}) \right) = \\
&\frac{\hbar}{2}\langle \frac{\delta}{\delta\varphi}, \dot{C}^\Lambda \star \frac{\delta}{\delta\varphi}\rangle\, F^{\Lambda,\Lambda_0}(\otimes_{i=1}^M \mathcal{O}_{A_i}) \prod_{j=M+1}^N L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_j}) \\
&-\langle \frac{\delta}{\delta\varphi} F^{\Lambda,\Lambda_0}(\otimes_{i=1}^M \mathcal{O}_{A_i}) \prod_{j=M+1}^N L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_j}), \dot{C}^\Lambda \star \frac{\delta}{\delta\varphi} L^{\Lambda,\Lambda_0}\rangle \\
&-\hbar\langle \frac{\delta}{\delta\varphi} F^{\Lambda,\Lambda_0}(\otimes_{i=1}^M \mathcal{O}_{A_i}), \dot{C}^\Lambda \star \frac{\delta}{\delta\varphi} \prod_{j=M+1}^N L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_j})\rangle \\
&-\hbar \sum_{M<i<j\leq N} \langle \frac{\delta}{\delta\varphi} L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_i}), \dot{C}^\Lambda \star \frac{\delta}{\delta\varphi} L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_j})\rangle \prod_{r=\{M+1,\ldots,N\}\setminus\{i,j\}} L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_r}) F^{\Lambda,\Lambda_0}(\otimes_{k=1}^M \mathcal{O}_{A_k}) \\
&+ \sum_{1\leq i<j\leq M} \langle \frac{\delta}{\delta\varphi} L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_i}), \dot{C}^\Lambda \star \frac{\delta}{\delta\varphi} L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_j})\rangle \prod_{r\in\{1,\ldots,N\}\setminus\{i,j\}} L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_r})
\end{aligned}
\tag{3.1.55}
$$

The flow equation for the third term is analogous to the one above, with the roles of the indices $(1,\ldots,M) \leftrightarrow (M+1,\ldots,N)$ exchanged. With these flow equations at hand, it is a straightforward exercise to check that the right hand side of eq.(3.1.53) satisfies a flow equation of the form

$$
\begin{aligned}
&\partial_\Lambda[\text{r.h.s. of (3.1.53)}] \\
&= \frac{\hbar}{2}\langle \frac{\delta}{\delta\varphi}, \dot{C}^\Lambda \star \frac{\delta}{\delta\varphi}\rangle\, [\text{r.h.s. of (3.1.53)}] - \langle \frac{\delta}{\delta\varphi}[\text{r.h.s. of (3.1.53)}], \dot{C}^\Lambda \star \frac{\delta}{\delta\varphi} L^{\Lambda,\Lambda_0}\rangle \\
&+\hbar \sum_{1\leq i<j\leq N} \langle \frac{\delta}{\delta\varphi} L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_i}), \dot{C}^\Lambda \star \frac{\delta}{\delta\varphi} L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_j})\rangle \prod_{r\in\{1,\ldots,N\}\setminus\{i,j\}} L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_r})
\end{aligned}
\tag{3.1.56}
$$

which coincides with the flow equation for $\hbar F^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i})$, see eq.(3.1.33). To summarise, we have shown that both sides of eq.(3.1.53) satisfy the same flow equation and boundary conditions, which establishes their equality. □

The decomposition provided by lemma 1 suggests the following definition:

Definition 2 (Partially regularised AG's): *Let $M < N \geq 2$. We denote the amputated Green's functions with operator insertions $\mathcal{O}_{A_1}(x_1), \ldots, \mathcal{O}_{A_N}(x_N)$, regularised to degree $D \leq [A_1] + \ldots + [A_M]$ in the coordinates $x_1, \ldots, x_M$, by $G^{\Lambda,\Lambda_0}([\otimes_{i=1}^M \mathcal{O}_{A_i}]_D \otimes_{M+1}^N \mathcal{O}_{A_i})$. These functionals are defined as*

$$G^{\Lambda,\Lambda_0}\left([\otimes_{i=1}^M \mathcal{O}_{A_i}]_D \otimes_{M+1}^N \mathcal{O}_{A_i}\right) := G_D^{\Lambda,\Lambda_0}(\otimes_{i=1}^M \mathcal{O}_{A_i})\, G^{\Lambda,\Lambda_0}(\otimes_{i=M+1}^N \mathcal{O}_{A_i}) + \hbar H^{\Lambda,\Lambda_0}\left([\otimes_{i=1}^M \mathcal{O}_{A_i}]_D ; \otimes_{i=M+1}^N \mathcal{O}_{A_i}\right) \tag{3.1.57}$$

where $H^{\Lambda,\Lambda_0}\left([\otimes_{i=1}^M \mathcal{O}_{A_i}]_D ; \otimes_{i=M+1}^N \mathcal{O}_{A_i}\right)$ is defined through the flow equation (3.1.51) with $G^{\Lambda,\Lambda_0}(\otimes_{i=1}^M \mathcal{O}_{A_i})$ replaced by $G_D^{\Lambda,\Lambda_0}(\otimes_{i=1}^M \mathcal{O}_{A_i})$, subject to the boundary conditions (3.1.52).

The bounds on these functionals derived below in section 3.2.3 show that the parameter D does indeed allow us to improve regularity on the partial diagonal $x_1 = \ldots = x_M$, while the behaviour on the other diagonals in $\mathrm{Diag}_{\{1,\ldots,N\}}$ remains unaffected. We also note that the spacetime derivatives of the partially regularised AG's satisfy the Lowenstein rule

$$(\partial_{x_1} + \ldots + \partial_{x_M})^v\, G^{\Lambda,\Lambda_0}\left([\otimes_{i=1}^M \mathcal{O}_{A_i}]_D \otimes_{M+1}^N \mathcal{O}_{A_i}\right) = G^{\Lambda,\Lambda_0}\left([(\partial_{x_1} + \ldots + \partial_{x_M})^v \otimes_{i=1}^M \mathcal{O}_{A_i}]_{D+|v|} \otimes_{M+1}^N \mathcal{O}_{A_i}\right) \tag{3.1.58}$$

which follows straightforwardly from proposition 1, and that they satisfy the translation identity

$$\mathcal{G}_{n,l}^{\Lambda,\Lambda_0}\left([\otimes_{i=1}^M \mathcal{O}_{A_i}(x_i)]_D \otimes_{M+1}^N \mathcal{O}_{A_i}(x_i); \vec{p}\right) = \mathrm{e}^{i(p_1+\ldots+p_n)y}\, \mathcal{G}_{n,l}^{\Lambda,\Lambda_0}\left([\otimes_{i=1}^M \mathcal{O}_{A_i}(x_i - y)]_D \otimes_{M+1}^N \mathcal{O}_{A_i}(x_i - y); \vec{p}\right), \tag{3.1.59}$$

which follows with the help of eq.(3.1.48) by comparing the defining flow equation and boundary conditions for both sides of the equation.

3.1.5. OPE coefficients

We next give the definition of the OPE coefficients. To have a more compact notation, let us define the operator $\mathcal{D}^A$ acting on differentiable functionals $F(\varphi)$ of Schwartz space functions $\varphi \in \mathcal{S}(\mathbb{R}^4)$ by

$$\mathcal{D}^A F(\varphi) = \frac{(-i)^{|w|}}{n!\, w!}\, \partial_{\vec{p}}^w \frac{\delta^n}{\delta\hat{\varphi}(p_1)\cdots\delta\hat{\varphi}(p_n)} F(\varphi)\Big|_{\hat{\varphi}=0,\vec{p}=0}, \tag{3.1.60}$$

where $A = \{n, w\}$. Further, let us also define the multivariate Taylor expansion operator through

$$\mathbb{T}^{j}_{\vec{x}\to\vec{y}}\, f(\vec{x}) = \mathbb{T}^{j}_{(x_1,\ldots,x_N)\to(y_1,\ldots,y_N)}\, f(x_1,\ldots,x_N) = \sum_{|w|=j} \frac{(\vec{x}-\vec{y})^w}{w!}\, \partial^w f(\vec{y}) \tag{3.1.61}$$

where $\vec{x} = (x_1,\ldots,x_N)$ and where f is a sufficiently smooth function on $\mathbb{R}^{4N}$. For expansions around zero we will use the shorthand $\mathbb{T}^{j}_{\vec{x}\to\vec{0}} =: \mathbb{T}^{j}_{\vec{x}}$. Then the OPE coefficients are defined as follows [26]:

Definition 3 (OPE coefficients): *Let $\Delta := [B] - ([A_1] + \ldots + [A_N])$. The OPE coefficients are defined in terms of the regularised AG's with insertions as*

$$\mathcal{C}^{B}_{A_1\ldots A_N}(x_1,\ldots,x_{N-1},0) := \mathcal{D}^B \left\{ G^{0,\Lambda_0}_{[B]-1}\left(\left(1 - \sum_{j<\Delta} \mathbb{T}^{j}_{\vec{x}}\right) \otimes_{i=1}^{N} \mathcal{O}_{A_i}(x_i)\right)\right\}, \tag{3.1.62}$$

where it is understood that $x_N = 0$.

Remark 3: In the case $N = 2$ this definition is equivalent to the one given in [15]. Note also that the OPE coefficients are translation invariant, so we may e.g. put the last point to zero by a translation, as we have done above to get a simpler formula.

3.2 Bounds on Green's functions with insertions

In the previous section we have defined all quantities of interest for the purpose of the present work, such as (regularised) Schwinger functions and OPE-coefficients. As mentioned earlier, the reason we have cast perturbative $g\varphi^4$-theory in this form is that the flow equation approach to the theory allows us to derive bounds on the quantities of interest via an inductive scheme, which is based on the renormalisation group flow equations. These bounds will be presented in the present section. The corresponding proofs, which are somewhat lengthy and technical in nature, can be found in appendix A.

Before we come to the statement of the various bounds, let us first give a brief account of the general idea behind the induction scheme, which is used repeatedly in the derivation of these bounds. First, consider as an example the CAG's without operator insertion. Recall from eq.(3.1.10) that these objects satisfy a flow equation, which is roughly of the form

$$\partial_\Lambda \mathcal{L}^{\Lambda,\Lambda_0}_{2n,l} = \sim \int_k \dot{C}^\Lambda(k)\, \mathcal{L}^{\Lambda,\Lambda_0}_{2n+2,l-1}(k,-k,\ldots) + \sim \sum_{\substack{l_1+l_2=l\\ n_1+n_2=n+1}} \mathcal{L}^{\Lambda,\Lambda_0}_{2n_1,l_1}\, \dot{C}^\Lambda(q)\, \mathcal{L}^{\Lambda,\Lambda_0}_{2n_2,l_2}, \tag{3.2.1}$$

where we used a shortened notation for the sake of simplicity of this schematic outline. In order to show that the CAG's satisfy a certain bound, let us call it $|\mathcal{L}^{\Lambda,\Lambda_0}_{2n,l}| \leq B^{\Lambda,\Lambda_0}_{2n,l}$, we can use the following induction procedure (see also fig. 3.1):

- Assume that the estimate is true for all $\mathcal{L}^{\Lambda,\Lambda_0}_{2n,l}$ satisfying $2n + 2l < 2r$ for some constant $r \in \mathbb{N}$.
- Note that the right hand side of the flow equation for $\mathcal{L}^{\Lambda,\Lambda_0}_{2r,0}$ contains only CAG's with $2n + 2l < 2r$. This can be seen as follows: The first term on the r.h.s. of the flow equation vanishes trivially in the case $l = 0$. For the second term we note that[8] $\mathcal{L}^{\Lambda,\Lambda_0}_{2,0} = 0$, which implies that each of the two factors $\mathcal{L}^{\Lambda,\Lambda_0}_{2n_1,l_1}$ and $\mathcal{L}^{\Lambda,\Lambda_0}_{2n_2,l_2}$ satisfies $2n_i + 2l_i < 2r$.
- Thus, we can substitute our inductive bound, $|\mathcal{L}^{\Lambda,\Lambda_0}_{2n,l}| \leq B^{\Lambda,\Lambda_0}_{2n,l}$, for the CAG's with $2n + 2l < 2r$ on the r.h.s. of the flow equation for $\mathcal{L}^{\Lambda,\Lambda_0}_{2r,0}$ and integrate over Λ in order to derive a bound for this CAG as well. In order for the induction to close, this bound should imply $|\mathcal{L}^{\Lambda,\Lambda_0}_{2r,0}| \leq B^{\Lambda,\Lambda_0}_{2r,0}$.
- Keeping $2n + 2l = 2r$ fixed, we then ascend in l, i.e. we next verify our bound for $\mathcal{L}^{\Lambda,\Lambda_0}_{2(r-1),1}$. The r.h.s. of the flow equation again contains CAG's with $2n + 2l < 2r$, as well as additionally the CAG $\mathcal{L}^{\Lambda,\Lambda_0}_{2r,0}$. All of these have been bounded already in the inductive procedure. By iteration, we thereby verify our bound for all CAG's $\mathcal{L}^{\Lambda,\Lambda_0}_{2n,l}$ with $2n + 2l = 2r$.
- Repeat the procedure to determine the CAG's with $2n + 2l = 2r + 2$, and so on.

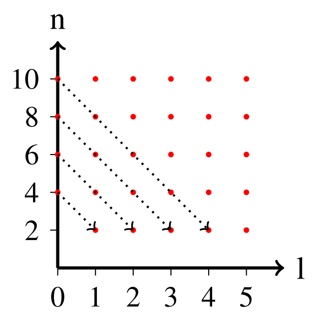


Figure 3.1.: Schematic visualisation of the induction procedure used to derive various bounds on the Schwinger functions

Some further explanations are in order. First, we have mentioned in the outline above that we have to *integrate the flow equation*. However, we have not specified the limits of integration. This is where the boundary conditions (3.1.11) and (3.1.12) come into play.

[8] The equation $\mathcal{L}^{\Lambda,\Lambda_0}_{2,0} = 0$ can be seen either directly from the definition (3.1.5) or alternatively from the flow equation and boundary conditions for these CAG's.

- For $2n \leq 4$ the boundary conditions are given at $\Lambda' = 0$. Hence, we integrate over Λ' from 0 to Λ in this case. At this stage, our choice for the coupling constant enters the scheme via the boundary condition $\mathcal{L}^{0,\Lambda_0}_{4,0}(\vec{0}) = g/4!$. The contributions with $2n \leq 4$ are known as *relevant* terms in the literature.

- For $2n \geq 5$ the boundary conditions are given at $\Lambda' = \Lambda_0$. Hence, we integrate over Λ' from Λ to Λ_0 in this case. These contributions are known as *irrelevant* terms in the literature.

Note also that the boundary conditions for the relevant terms are given at vanishing momentum. This complicates the induction scheme somewhat. In order to obtain a bound for the relevant terms also at non-vanishing momentum one has to perform a Taylor expansion with remainder in the momenta. Thus, one is forced to also include momentum derivatives of the CAG's in the induction scheme. These subtleties of the induction will be discussed in more detail in the actual proofs in appendix A.

In the following we will not derive new bounds on the CAG's themselves, but we are mainly interested in the AG's with operator insertions and regularisation, defined in sections 3.1.3 and 3.1.4. More precisely, we will derive estimates on the functionals F_D^{Λ,Λ_0} and H^{Λ,Λ_0} defined in those sections. The flow equation for the F-functionals is of the general form

$$\partial_\Lambda \mathcal{F}^{\Lambda,\Lambda_0}_{D,2n,l} = \sim\!\int_k \dot{C}^\Lambda(k)\mathcal{F}^{\Lambda,\Lambda_0}_{D,2n+2,l-1}(k,-k,\ldots) + \sim \sum_{\substack{l_1+l_2=l\\ n_1+n_2=n+1}} \mathcal{F}^{\Lambda,\Lambda_0}_{D,2n_1,l_1}\, \dot{C}^\Lambda(q)\, \mathcal{L}^{\Lambda,\Lambda_0}_{2n_2,l_2}$$
$$+ \text{" Source Terms"} \tag{3.2.2}$$

where *source terms* stands for the momentum integral over a product of CAG's with one operator insertion. We see that the first two terms on the r.h.s. of this flow equation are of a similar form as in the case of the CAG's above. It should therefore not come as a surprise that we can use the same induction scheme for these terms. In contrast to the CAG case, however, we have an additional term on the r.h.s. of the flow equation. Hence, we have to make sure that this source term satisfies a bound that is consistent with our inductive bound. This is in fact the main complication in the derivation of the bounds for the AG's with insertions. The corresponding bounds on the source terms can be found in the appendix, see lemma 6. In order to arrive at this estimate, we make use of the known bounds on the CAG's with one insertion (see [15] and section 3.2.1). For the H-functionals one can proceed in a similar fashion. Here the source terms are the AG's with operator insertions. Since we derive bounds on these quantities first, we can use those bounds in order to estimate the source terms for the H-functionals, see lemma 8 and 9 in the appendix.

Remark 4: The loop expanded (inserted or non-inserted) CAG's depend on the coupling constant g in an obvious manner; the non-inserted functions $\mathcal{L}^{\Lambda,\Lambda_0}_{2n,l}$ carry a power of $g^{(2n-2)/2+l}$ for example. For the sake of simplicity, we will set $g=1$ in the following.

3.2.1. CAG's with up to one insertion

Bounds on CAG's without insertions and on those with one insertion were derived in [48, 15]. It can be seen from the decomposition (3.1.38) that these bounds are a crucial input for the subsequent bounds on AG's with multiple insertions, as both the factorised contribution $\prod_{i=1}^{N} L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_i})$ as well as the F^{Λ,Λ_0}-functionals depend on the CAG's with one insertion [the latter via the flow equation (3.1.33)].

Let us recall the bound for the CAG's without insertions first [48, 15]. There exists a constant $K>0$ such that for $2n+|w|\geq 5$ (recall also the definitions of $|\vec{p}|$ and κ from our notations and conventions section above)

$$\begin{aligned}&|\partial^w_{\vec{p}}\,\mathcal{L}^{\Lambda,\Lambda_0}_{2n,l}(p_1,\dots,p_{n-1})|\\ &\leq \sqrt{|w|!}\,\Lambda^{4-2n-|w|}\,K^{(2n+4l-4)(|w|+1)}\,(n+l-2)!\sum_{\lambda=0}^{\ell(n,l)}\frac{\log^\lambda(\sup(\frac{|\vec{p}|}{\kappa},\frac{\kappa}{m}))}{2^\lambda\,\lambda!}\,,\end{aligned}\tag{3.2.3}$$

where $\ell(n,l)=l$ if $n\geq 2$ and $\ell(n,l)=l-1$ if $n=1$. For $2n+|w|\leq 4$ one has the estimates

$$|\mathcal{L}^{\Lambda,\Lambda_0}_{4,l}(\vec{p})|\;\leq\;\frac{K^{2l}}{(l+1)^2\,2^4}\,(1+l)!\sum_{\lambda=0}^{l}\frac{\log^\lambda\big(\sup(\frac{|\vec{p}|}{\kappa},\frac{\kappa}{m})\big)}{2^\lambda\,\lambda!}\,,\tag{3.2.4}$$

$$|\partial^w_{\vec{p}}\,\mathcal{L}^{\Lambda,\Lambda_0}_{2,l}(p)|\leq\sup(|p|,\kappa)^{2-|w|}\,\frac{K^{2l-1}}{(l+1)^2}\,l!\sum_{\lambda=0}^{l-1}\frac{\log^\lambda\big(\sup(\frac{|p|}{\kappa},\frac{\kappa}{m})\big)}{2^\lambda\,\lambda!}\,.\tag{3.2.5}$$

We also recall the following bound for the CAG's with one insertion [15]. Fix any $A=\{r,v\}$. Then

$$\begin{aligned}|\partial^w_{\vec{p}}\,\mathcal{L}^{\Lambda,\Lambda_0}_{2n,l}(\mathcal{O}_A(0);\vec{p})|\leq{}&\Lambda^{[A]-2n-|w|}\,K^{(4n+8l-4)|w|}\,K^{[A](n+2l)^3}\,\sqrt{|w|!\,|v|!}\\ &\times\sum_{\mu=0}^{d(N=1,n,l,w,[A])}\frac{1}{\sqrt{\mu!}}\left(\frac{|\vec{p}|}{\Lambda}\right)^{\mu}\sum_{\lambda=0}^{2l+n-1}\frac{\log^\lambda(\sup(\frac{|\vec{p}|}{\kappa},\frac{\kappa}{m}))}{2^\lambda\lambda!}\,,\end{aligned}\tag{3.2.6}$$

where $K>0$ is a constant, and where we defined

$$d(N,n,l,w,D'):=2D'(n+l+2(N-1))+\sup(D'+1-2n-|w|,0)\,.\tag{3.2.7}$$

For $n = l = 0$ the CAG's with one insertion vanish, $\mathcal{L}_{0,0}^{\Lambda,\Lambda_0}(\mathcal{O}_A(0)) = 0$. Eventually we would of course like to remove the cutoffs, i.e. take the limits $\Lambda \to 0$ and $\Lambda_0 \to \infty$. In this respect, the following bounds, which hold for $\Lambda \leq m$, will be useful [15]

$$|\partial_{\vec{p}}^w \mathcal{L}_{2n,l}^{\Lambda,\Lambda_0}(\vec{p})| \leq m^{4-2n-|w|} \frac{K^{(2n+4l-4)(|w|+1)}}{n!} (n+l-1)! \times \sqrt{|w|!\,(|w|+2n-4)!} \sum_{\lambda=0}^{l} \frac{\log_+^\lambda(\frac{|\vec{p}|}{m})}{2^\lambda\,\lambda!} \quad \text{for } 2n+|w| \geq 5 \tag{3.2.8}$$

$$|\partial_{\vec{p}}^w \mathcal{L}_{2n,l}^{\Lambda,\Lambda_0}(\mathcal{O}_A(0);\vec{p})| \leq m^{[A]-2n-|w|}\, K^{(4n+8l-4)|w|}\, K^{[A](n+2l)^3} \sqrt{|w|!\,|v|!} \times \sqrt{[2n+|w|-[A]]_+!} \sum_{\mu=0}^{d(N=1,n,l,w,[A])} \left(\frac{|\vec{p}|}{m}\right)^\mu \sum_{\lambda=0}^{2l+n-1} \frac{\log_+^\lambda(\frac{|\vec{p}|}{m})}{2^\lambda\,\lambda!}\,, \tag{3.2.9}$$

where by $[\cdot]_+$ we mean the positive part of the respective expression (see also our notations and conventions section).

3.2.2. Amputated Green's functions with N insertions

The amputated Green's functions (AG's) with regularisation on the total diagonal were defined in section 3.1.3. These functionals are of major interest to us not only since they appear in the definition of the OPE-coefficients, but also because they are closely related to the remainder of the OPE (see lemma 2 below).

We will first derive a bound on the moments of the F_D^{Λ,Λ_0}-functionals and then combine this with the bounds on the CAG's with one insertion, eqs.(3.2.6) and (3.2.9), in order to estimate the AG's, making use of the decomposition (3.1.38). These bounds will also confirm the nature of D as a *regularisation parameter* for the singularity on the total diagonal, and they will further allow us to prove that the AG's with insertions are real analytic functions in the spacetime arguments for non-coinciding points.

Bound 1: *Let $x_N = 0$ and $D \leq D' = [A_1] + \ldots + [A_N]$. There exists a constant $K > 0$ such that*

$$|\partial_{\vec{p}}^w \mathcal{F}_{D,2n,l}^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}(x_i);\vec{p})| \leq m^{D-2n-|w|}\, K^{(4n+8l-3)|w|+D'(n+2l)^3} \sqrt{D'!(D'-D)!} \times \frac{|w|!\, \max_{1\leq i\leq N} |x_i|^{\max(|w|,D+1)}}{\min_{1\leq i<j\leq N} |x_i-x_j|^{D'-D+\max(|w|,D+1)}} \sup\left(1, \frac{|\vec{p}|}{m}\right)^{d(N,n,l,w,D')} \sum_{\lambda=0}^{2l+n} \frac{\log_+^\lambda(\frac{|\vec{p}|}{m})}{2^\lambda \lambda!} \tag{3.2.10}$$

with $d(N,n,l,w,D') := 2D'(n+l+2(N-1)) + \sup(D'+1-2n-|w|,0)$.

Remark 5: The bound leads us to the following observations:

1. If we scale the spacetime arguments by a multiplicative factor $\varepsilon > 0$, i.e.

$$x_1, \dots, x_{N-1} \to \varepsilon x_1, \dots, \varepsilon x_{N-1} \,, \tag{3.2.11}$$

the bound behaves as $\varepsilon^{D-D'}$. In other words, the bound implies

$$\mathrm{sd}\, F_D^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}) \leq D' - D \,, \tag{3.2.12}$$

as indicated in section 3.1.3. Thus, we can see explicitly that the parameter D improves regularity on the total diagonal $x_1 = \dots = x_N = 0$. Note, however, that our bounds do not imply improved regularity on the partial diagonals, i.e. in the case where only a subset of the spacetime arguments are scaled together.

2. The conventional opinion, based on dimensional arguments, is that the short distance behaviour of the quantity in question should in fact be of the form $\varepsilon^{D-D'+1}\log(\varepsilon)^r$ for some $r \in \mathbb{N}$ depending on the loop order. In fact, such a behaviour was indeed shown to hold in the Lorentzian setting using a different approach to quantum field theory [12]. At present, we are only able to give formal arguments that support this expectation in the flow equation framework. Namely, as first observed in [23], one could improve our bound (3.2.10) to yield the scaling behaviour $\varepsilon^{D-D'+1-\delta}$ for any $0 < \delta \in \mathbb{R}$ if we were allowed to calculate with non-integer powers of the partial derivative $(\partial_k)^\delta$ as we do with integer powers. One could then simply follow the proof of bound 1, but replacing $D \to D + 1 - \delta$ throughout the proof. The crucial step where $(\partial_k)^\delta$ would have to be used appears in the proof of lemma 6, where partial integrations in the momentum variables have to be performed to obtain the correct Λ behaviour. It might be possible to put these formal manipulations on solid footing, using the techniques of *fractional calculus* [49, 50], but this is beyond the scope of the present work.

3. One can derive a version of the bound (3.2.10) where the factor $\min |x_i - x_j|^{D-D'}$ is replaced by a factor $\Lambda_0^{D'-D}/\sqrt{(D'-D)!}$. We conclude, also taking into account the Lowenstein rule (3.1.43), that the F-functionals are smooth in the spacetime arguments as long as we keep Λ_0 finite. We will briefly explain in appendix A.1.1 how this version of the bound is obtained.

4. Concerning the behaviour in the infrared (i.e. $|x_i| \to \infty$ for all $1 \leq i < N$), we will also show (see the discussion in appendix A.1.2) that it is possible to introduce any number $r \in \mathbb{N}$ of additional inverse powers of $(m \cdot \min_{1\leq i<j\leq N} |x_i - x_j|)$ on the right hand side of (3.2.10) at the cost of a factor $r!$. This property is not

surprising in view of the characteristic exponential decay of Schwinger functions at large distances in massive quantum field theories.

5. As mentioned previously, in the case $N = 2$ the F-functionals coincide (up to a sign) with the CAG's with two insertions. These have been estimated by Hollands and Kopper in [15] for the particular choice $D = [A_1] + [A_2]$ (full regularisation) and $|w| \leq D + 1$, and by Hollands and the author in [26] for any $D \leq [A_1] + [A_2]$ and any $w \in \mathbb{N}^{8n}$.

The somewhat lengthy proof of the inequality (3.2.10), which is based on the inductive scheme sketched in the previous section, can be found in appendix A.1. It is now an easy exercise to derive a bound on the AG's with insertions, G_D^{Λ,Λ_0}, which is the upshot of this section.

Corollary 1: *Let $D \leq D' = [A_1] + \ldots + [A_N]$. For $\Lambda \leq m$ there exists a constant $K > 0$ such that*

$$|\partial_{\vec{p}}^{w} \mathcal{G}_{D,2n,l}^{\Lambda,\Lambda_0}(\otimes_{i=1}^{N} \mathcal{O}_{A_i}(x_i);\vec{p})| \leq m^{D-2n-|w|}\, K^{(4n+8l-3)|w|}\, K^{D'(n+2l)^3}$$
$$\times \sqrt{D'!(D'-D)!}|w|! \sup\left(1, \frac{|\vec{p}|}{m}\right)^{d(N,n,l,w,D')} \sum_{\lambda=0}^{2l+n} \frac{\log_+^{\lambda}(\frac{|\vec{p}|}{m})}{2^{\lambda}\lambda!} \max_{\mu \leq |w|}\left(\frac{(m|\vec{x}|)^{\mu}}{\sqrt{\mu!}}\right)$$
$$\times \left(\frac{\max_{1\leq i\leq N}|x_i - x_N|}{\min_{1\leq i<j\leq N}|x_i - x_j|}\right)^{\max(|w|,D+1)} \max\left(\frac{1}{\min_{1\leq i<j\leq N}|x_i - x_j|^{D'-D}}, \frac{m^{D'-D}}{\sqrt{(D'-D)!}}\right) \tag{3.2.13}$$

with $d(N,n,l,w,D') := 2D'(n+l+2(N-1)) + \sup(D'+1-2n-|w|,0)$.

Remark 6: This bound implies convergence of the Taylor expansion of the AG's with operator insertions, $\partial_{\vec{p}}^{w} \mathcal{G}_{D,2n,l}^{\Lambda,\Lambda_0}(\otimes_{i=1}^{N} \mathcal{O}_{A_i}(x_i))$, with respect to the spacetime variables in a neighborhood of $\vec{x} \in \mathbb{R}^{4N} \setminus M_N$, for any degree of regularisation $D \leq [A_1] + \ldots + [A_N]$, and uniformly in Λ, Λ_0. This can be seen from the fact that the bound grows like

$$j!\Big(\sup(|\vec{p}|/m, 1)^{2(n+l+2N)} \tilde{K}/|x|\Big)^{j} \tag{3.2.14}$$

if we take j derivatives with respect to $\vec{x}$ and write $\tilde{K} = K^{(n+2l)^3}$. Hence the AG's with insertions are real analytic in $\vec{x}$ away from the partial diagonals, and the same holds for the OPE coefficients, related to them by defn. 3.

Proof of corollary 1. Recall form (3.1.38) that G_D^{Λ,Λ_0} is defined as the sum of F_D^{Λ,Λ_0} and a product of CAG's with one insertion. It follows from bound 1 and from the translation properties (3.1.35) that the F-functionals satisfy the claimed inequality, (3.2.13). For the factorised contributions we may write, using the translation property (3.1.22) (recall from the notation and conventions section that by $c_{\{w_j\}}$ we denote multi-nomial factors),

$$\Big|\, \partial^{w}_{\vec p} \prod_{i=1}^{N} \mathcal{L}^{\Lambda,\Lambda_0}_{2n_i,l_i}(\mathcal{O}_{A_i}(x_i);\vec p_i)\,\Big| \le \sum_{w_1+w_2=w} c_{\{w_j\}}\, |\vec x|^{|w_1|} \Big|\, \partial^{w_2}_{\vec p} \prod_{i=1}^{N} \mathcal{L}^{\Lambda,\Lambda_0}_{2n_i,l_i}(\mathcal{O}_{A_i}(0);\vec p_i)\,\Big| \tag{3.2.15}$$

Combining this inequality with the bound (3.2.9), it is straightforward to check that also the product CAG's with one insertion satisfies the inequality (3.2.13). $\square$

We will see below that the convergence and factorisation property of the operator product expansion are related to coordinate space Taylor expansions of the AG's with insertions. In this context, the following bound will prove to be useful:

Bound 2: *Let $D = D' + \Delta$, where $D' = [A_1] + \ldots + [A_N]$ and $\Delta > 0$. For $\Lambda \le m$ there exists a constant $K > 0$ such that*

$$\begin{aligned}
&|\partial^{w}_{\vec p}(1-\sum_{j\le\Delta}\mathbb{T}^{j}_{(x_1,\ldots,x_N)\to(x_N,\ldots,x_N)})\mathcal{G}^{\Lambda,\Lambda_0}_{D,2n,l}(\otimes_{i=1}^{N}\mathcal{O}_{A_i}(x_i);\vec p)|\\
&\le m^{D-2n-|w|}\, K^{(4n+8l-3)|w|}\, K^{D(n+2l)^3}\sqrt{D'!}\,|w|!\,\sup\left(1,\frac{|\vec p|}{m}\right)^{d(N,n,l,w,D)} \sum_{\lambda=0}^{2l+n}\frac{\log_+^{\lambda}(\frac{|\vec p|}{m})}{2^{\lambda}\lambda!}\\
&\times\left(\frac{\max_{1\le i\le N}|x_i-x_N|}{\min_{1\le i<j\le N}|x_i-x_j|}\right)^{\max(|w|,D+1)-\Delta} \sup_{\mu\le|w|}\left(\frac{(m|\vec x|)^{\mu}}{\sqrt{\mu!}}\right)\frac{\max_{1\le i\le N}|x_i-x_N|^{\Delta}}{\sqrt{\Delta!}}
\end{aligned} \tag{3.2.16}$$

with $d(N,n,l,w,D') := 2D'(n+l+2(N-1)) + \sup(D'+1-2n-|w|,0)$.

Remark 7: The bound is a slight improvement over the estimate one would obtain through a combination of corollary 1 and the remainder formula for the Taylor expansion,

$$\begin{aligned}
&(1-\sum_{j\le\Delta}\mathbb{T}^{j}_{\vec x\to\vec y})f(x_1,\ldots,x_N) =\\
&\sum_{|v|=\Delta+1}\frac{\Delta+1}{v!}(\vec x-\vec y)^{v}\int_0^1 \mathrm{d}\tau\,(1-\tau)^{\Delta}\,\partial^{v}\Big[f\big(y_1+\tau(x_1-y_1),\ldots,y_N+\tau(x_N-y_N)\big)\Big].
\end{aligned} \tag{3.2.17}$$

Namely, this method would yield a bound which is essentially of the same form as (3.2.16), but with the replacement

$$\left(\frac{\max_{1\le i\le N}|x_i-x_N|}{\min_{1\le i<j\le N}|x_i-x_j|}\right)^{\max(|w|,D+1)-\Delta} \to \left(\frac{\max_{1\le i\le N}|x_i-x_N|}{\min_{1\le i<j\le N}|x_i-x_j|}\right)^{\max(|w|,D+1)} \tag{3.2.18}$$

in the last line. While the difference between the two results is irrelevant for our study of convergence of the OPE, the stronger result (3.2.16) will be useful in our proof of the factorisation result, theorem 3. The proof of bound 2 can be found in appendix A.2.

3.2.3. Amputated Green's functions with partial regularisation

The AG's with insertions and regularisation with respect to a partial diagonal have been defined in section 3.1.4. As we will see later in section 3.4, these objects appear when one performs the OPE of a product $\mathcal{O}_{A_1}\ldots\mathcal{O}_{A_N}$ in only a subset of these operators, say $\mathcal{O}_{A_1},\ldots,\mathcal{O}_{A_M}$ with $M<N$, while leaving the others as *spectators*. We are interested in this type of *partial OPE*, because it will allow us to derive non trivial algebraic relations between the OPE coefficients.

Again, we will see that our bounds will justify our terminology, in the sense that the functionals $G^{\Lambda,\Lambda_0}([\otimes_{i=1}^{M}\mathcal{O}_{A_i}]_D\otimes_{j=M+1}^{N}\mathcal{O}_{A_j})$ will indeed be shown to become more regular on the partial diagonal $x_1=\ldots=x_M$ as we increase D.

Bound 3: *Let $D\leq[A_1]+\ldots+[A_M]$, $x_M=0$ and $\Lambda\leq m$. There exists a constant $K>0$ such that*

$$
\begin{aligned}
&|\partial_{\vec p}^{w}\mathcal{H}_{2n,l}^{\Lambda,\Lambda_0}\left([\otimes_{i=1}^{M}\mathcal{O}_{A_i}(x_i)]_D;\otimes_{i=M+1}^{N}\mathcal{O}_{A_i}(x_i);\vec p\right)|\leq\\
&\times\frac{|w|!D'!\,m^{-1-2n-|w|}\,K^{(4n+8l-3)|w|}\,K^{D'(n+2l)^3}}{\min\limits_{\substack{1\leq i\leq N,i\neq j\\1\leq j\leq M}}|x_i-x_j|^{[A_1]+\ldots+[A_M]-D}\min\limits_{\substack{1\leq i\leq N,i\neq j\\M+1\leq j\leq N}}|x_i-x_j|^{[A_{M+1}]+\ldots+[A_N]+1}}\\
&\times\left[\frac{\max\limits_{1\leq i\leq N}|x_i|}{\min\limits_{1\leq i\leq M<j\leq N}|x_i-x_j|}\right]^{|w|}\left(\frac{\max\limits_{1\leq i\leq M}|x_i|}{\min\limits_{1\leq i<j\leq M}|x_i-x_j|}\cdot\frac{\max\limits_{M+1\leq i\leq N}|x_i-x_N|}{\min\limits_{M+1\leq i<j\leq N}|x_i-x_j|}\right)^{D'+|w|+1}\\
&\times\frac{1}{\min\limits_{1\leq i\leq M<j\leq N}|x_i-x_j|^{D}}\sup\left(1,\frac{|\vec p|}{m}\right)^{d(N,n,l,w,D')}\sum_{\lambda=0}^{2l+n}\frac{\log_+^{\lambda}(\frac{|\vec p|}{m})}{2^{\lambda}\lambda!}
\end{aligned}
\tag{3.2.19}
$$

with $D'=[A_1]+\ldots+[A_N]$ and $d(N,n,l,w,D'):=2D'(n+l+2(N-1))+\sup(D'+1-2n-|w|,0)$.

Remark 8: We are led to the following observations:

1. If we scale the spacetime arguments $x_1,\ldots,x_N$ by a multiplicative factor $\varepsilon>0$, the bound behaves as $\varepsilon^{-(D'+1)}$, which suggests that the parameter D does not influence the singular behaviour on the total diagonal.

2. If we only scale the spacetime arguments $x_1, \ldots, x_{M-1}$ by a factor ε, i.e.

$$(x_1, \ldots, x_N) \to (\varepsilon x_1, \ldots, \varepsilon x_{M-1}, x_M = 0, x_{M+1}, \ldots, x_N)\,, \tag{3.2.20}$$

the bound behaves as $\varepsilon^{-([A_1]+\ldots+[A_M]-D)}$ in the limit $\varepsilon \to 0$. Thus, we can see that, as advertised, the parameter D improves regularity on the partial diagonal $x_1 = \ldots = x_M$.

3. It was mentioned in remark 5 that one can derive bounds on the AG's with insertions with an improved infrared behaviour (i.e. at large separation of the spacetime arguments) without much extra effort. The same is true here for the H-functionals. Using a strategy analogous to the one outlined in appendix A.1.2, we can introduce any number $r \in \mathbb{N}$ of inverse powers of $(m \cdot \min |x_i - x_j|)$ where $1 \le i \le M < j \le N$ at the cost of a factor $r!$.

The proof of bound 3 is given in appendix A.3. As we will see below in section 3.4, the remainder of the *partial OPE* is related to a Taylor expansion of the partially regularised AG's with insertions. The following bound will allow us to estimate this remainder:

Bound 4: *Let $D = [A_1] + \ldots + [A_M] + \Delta$, where and $\Delta > 0$, and assume $|x_i - x_M| \le |x_j - x_M|$ for all $1 \le i \le M$ and $M+1 \le j \le N$. For $\Lambda \le m$ there exists a constant $K > 0$ such that*

$$\begin{aligned}
&\Big|\partial^w_{\vec{p}} (1 - \sum_{j \le \Delta} \mathbb{T}^j_{(x_1,\ldots,x_M)\to(x_M,\ldots,x_M)}) \mathcal{G}^{\Lambda,\Lambda_0}_{2n,l} \left([\otimes_{i=1}^M \mathcal{O}_{A_i}(x_i)]_D ; \otimes_{i=M+1}^N \mathcal{O}_{A_i}(x_i); \vec{p}\right)\Big| \\
&\le m^{-2n-|w|-1}\, |w|! \prod_{i=1}^N [A_i]! \sum_{\lambda=0}^{2l+n} \frac{\log_+^\lambda(\frac{|\vec{p}|}{m})}{2^\lambda \lambda!}\, \tilde{K}^{|w|}\, \sup(1, m|\vec{x}|)^{|w|} \\
&\times \left(\frac{\tilde{K}\, \sup(\frac{|\vec{p}|}{m},1)^{2n+2l+4N}}{\min_{1\le i<j\le N} \left[|x_i - x_j|, 1/m\right]}\right)^{D'+1} \cdot \left(\frac{\max_{1\le i<j\le N} |x_i - x_j|}{\min_{1\le i<j\le N} |x_i - x_j|}\right)^{2D'+3|w|+2} \\
&\times \frac{\left(\tilde{K}\, \sup(\frac{|\vec{p}|}{m},1)^{2n+2l+4N} \max_{1\le i\le M} |x_i - x_M|\right)^\Delta}{\min\left[\left(\min_{M<j\le N} |x_j - x_M|^{\Delta-1} \cdot \min_{1\le i<j\le N} |x_i - x_j|\right), \frac{\sqrt{\Delta!}}{m^\Delta}\right]} \cdot \left(\frac{\max_{M<i\le N} |x_i - x_N|}{\min_{M<i<j\le N} |x_i - x_j|}\right)^\Delta
\end{aligned} \tag{3.2.21}$$

where $D' = [A_1] + \ldots + [A_N]$.

The proof of this bound can be found in appendix A.4.

3.3 Proof of OPE convergence

The bounds presented in the previous section will now be put to use. Our aim is to prove two of the main results of this thesis, namely convergence and (long distance) factorisation of the OPE. We begin with the former. More precisely, we would like insert equation (1.0.1) into a correlation function with suitable *spectator fields* and estimate the difference between left- and right hand side. These spectator fields play the role of a quantum state in the Euclidean context. Our quite simple and natural choice is as follows:

Let $f_p(x)$ be any smooth function on $x \in \mathbb{R}^4$ such that the support of the Fourier transform $\hat{f}_p(q)$ is contained in a ball $|p-q| \leq 1$. Define the *smeared spectator fields* by

$$\varphi(f_p) \equiv \int \mathrm{d}^4 x\, \varphi(x)\; f_p(x)\,. \tag{3.3.1}$$

Using this convention, we are ready to state our result.

Theorem 1 (OPE convergence): *The remainder of the operator product expansion, carried out up to operators of dimension $D = \sum_{i=1}^N [A_i] + \Delta$, at l-loops, is bounded by*

$$\begin{aligned}
&\Big|\Big\langle \mathcal{O}_{A_1}(x_1)\cdots\mathcal{O}_{A_N}(x_N)\,\varphi(f_{p_1})\cdots\varphi(f_{p_n})\Big\rangle \\
&\qquad - \sum_{C:[C]\leq D} \mathcal{C}_{A_1\ldots A_N}^{C}(x_1,\ldots,x_N)\Big\langle \mathcal{O}_C(x_N)\,\varphi(f_{p_1})\cdots\varphi(f_{p_n})\Big\rangle\Big|_{l-loops} \\
\leq\; & m^{n-1}\prod_{i=1}^{N}\sqrt{[A_i]!}\,\prod_j \sup|\hat{f}_{p_j}|\,\sup(1,\frac{|\vec{p}|_n}{m})^{(2\sum[A_i]+2\Delta)(n+l+2N)+3n} \\
&\times \sum_{\lambda=0}^{2l+n/2}\frac{\log^\lambda \sup(1,\frac{|\vec{p}|_n}{m})}{2^\lambda \lambda!}\,\frac{1}{\sqrt{\Delta!}}\,\frac{(\tilde{K}\, m\, \max_{1\leq i\leq N}|x_i-x_N|)^{\sum[A_i]+1+\Delta}}{\min_{1\leq i<j\leq N}|x_i-x_j|^{\sum[A_i]+1}}
\end{aligned} \tag{3.3.2}$$

in $g\varphi^4$-theory. Here $[A]$ denotes the canonical dimension of a composite field $\mathcal{O}_A$ as in eq. (3.1.13). $\tilde{K}$ is a constant depending on n and l.

Remark 9: A version of the bound for $N = 2$ was first given in [15] by Hollands and Kopper, and a version for $N = 3$ due to Hollands and the author can be found in [26]. Evidently, the bound implies convergence of the operator product expansion for any finite distance of the spacetime arguments and to any order in perturbation theory. This can be seen from the fact that $\lim_{\Delta\to\infty} c^\Delta/\sqrt{\Delta!} = 0$ for any $c \in \mathbb{R}$.

Proof. We define the remainder functional

$$R_D^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}(x_i)) := G^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}(x_i)) - \sum_{C:[C]\le D} \mathcal{C}^C_{A_1\dots A_N}(x_1,\dots,x_N)\, L^{\Lambda,\Lambda_0}(\mathcal{O}_C(x_N))\,, \tag{3.3.3}$$

which, in view of equation (3.1.30), allows us to write (for the theory with UV and IR cutoffs Λ_0 and Λ),

$$\begin{aligned}&\Big|\Big\langle \mathcal{O}_{A_1}(x_1)\cdots\mathcal{O}_{A_N}(x_N)\,\varphi(f_{p_1})\cdots\varphi(f_{p_n})\Big\rangle \\ &\qquad - \sum_{C:[C]-D'\le\Delta} \mathcal{C}^C_{A_1\dots A_N}(x_1,\dots,x_N)\Big\langle \mathcal{O}_C(x_N)\,\varphi(f_{p_1})\cdots\varphi(f_{p_n})\Big\rangle\Big| \\ &= \sum_{j=1}^n \sum_{\substack{I_1\cup\ldots\cup I_j=\{1,\dots,n\}\\ I_i\cap I_j=\emptyset\\ l_1+\ldots+l_j=l}} \hbar^{n+l+1-j} \\ &\times \int_{\vec q} \mathcal{R}^{\Lambda,\Lambda_0}_{D,|I_1|,l_1}(\otimes_{i=1}^N\mathcal{O}_{A_i};\vec q_{I_1})\bar{\mathcal{L}}^{\Lambda,\Lambda_0}_{|I_2|,l_2}(\vec q_{I_2})\cdots\bar{\mathcal{L}}^{\Lambda,\Lambda_0}_{|I_j|,l_j}(\vec q_{I_j})\prod_{i=1}^n\Big[\hat f_{p_i}(q_i)\,C^{\Lambda,\Lambda_0}(q_i)\Big]\end{aligned} \tag{3.3.4}$$

where $\bar{\mathcal{L}}^{\Lambda,\Lambda_0}_{n,l}$ are the moments of the generating functional $\bar L^{\Lambda,\Lambda_0}(\varphi) = -L^{\Lambda,\Lambda_0}(\varphi) + \frac{1}{2}\langle\varphi,\,(C^{\Lambda,\Lambda_0})^{-1}\star\varphi\rangle$ without the momentum conservation delta functions taken out. We wish to find a bound for the above expression. Since we already have bounds on $\mathcal{L}^{\Lambda,\Lambda_0}_{n,l}$ from (3.2.3), and since C^{Λ,Λ_0} can be estimated trivially as $C^{\Lambda,\Lambda_0}(p)\le[\sup(m,|p|)]^{-2}$, we will be concerned with $\mathcal{R}^{\Lambda,\Lambda_0}_{D,n,l}$ in the following. The following lemma will allow us to express $\mathcal{R}^{\Lambda,\Lambda_0}_{D,n,l}$ in terms of quantities with known bounds as given in the previous sections:

Lemma 2: *The remainder functionals satisfy*

$$R_D^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}(x_i)) = (1-\sum_{j\le\Delta}\mathbb{T}^j_{\vec x\to(x_N,\dots,x_N)})\,G_D^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}(x_i)) \tag{3.3.5}$$

with $\Delta = D-\sum_{i=1}^N[A_i]$.

This lemma was first given for $N=2$ in [15] and for general N in [26].

Proof. The proof follows the same strategy as the proof of lemma 4.1 in [15]. Let us assume $x_N=0$ for the moment. To begin with, consider the telescopic sum

$$G^{\Lambda,\Lambda_0}(\otimes_{i=1}^N\mathcal{O}_{A_i}) = G_D^{\Lambda,\Lambda_0}(\otimes_{i=1}^N\mathcal{O}_{A_i}) + \sum_{j=0}^D[G_{j-1}^{\Lambda,\Lambda_0}(\otimes_{i=1}^N\mathcal{O}_{A_i}) - G_j^{\Lambda,\Lambda_0}(\otimes_{i=1}^N\mathcal{O}_{A_i})]\,. \tag{3.3.6}$$

Note that for any $j \in \mathbb{N}$ we have

$$G^{\Lambda,\Lambda_0}_{j-1}(\otimes_{i=1}^N \mathcal{O}_{A_i}) - G^{\Lambda,\Lambda_0}_{j}(\otimes_{i=1}^N \mathcal{O}_{A_i}) = \sum_{C:[C]=j} \mathcal{D}^C\{\hbar F^{0,\Lambda_0}_{j-1}(\otimes_{i=1}^N \mathcal{O}_{A_i})\}\, L^{\Lambda,\Lambda_0}(\mathcal{O}_C(0))\,, \tag{3.3.7}$$

which can be seen by checking that both sides of the equation satisfy the same linear homogeneous flow equation and the same boundary conditions, which are

$$\begin{aligned}
&\partial^w_{\vec p}\left(\mathcal{G}^{0,\Lambda_0}_{n,l,j-1}(\otimes_{i=1}^N \mathcal{O}_{A_i};\vec 0) - \mathcal{G}^{0,\Lambda_0}_{n,l,j}(\otimes_{i=1}^N \mathcal{O}_{A_i};\vec 0)\right) = 0 && \text{for } n+|w|<j\\
&\partial^w_{\vec p}\left(\mathcal{G}^{0,\Lambda_0}_{n,l,j-1}(\otimes_{i=1}^N \mathcal{O}_{A_i};\vec 0) - \mathcal{G}^{0,\Lambda_0}_{n,l,j}(\otimes_{i=1}^N \mathcal{O}_{A_i};\vec 0)\right) = \\
&\qquad\qquad\qquad\qquad \hbar\partial^w_{\vec p}\mathcal{F}^{0,\Lambda_0}_{n,l,j-1}(\otimes_{i=1}^N \mathcal{O}_{A_i};\vec 0) && \text{for } n+|w|=j\\
&\partial^w_{\vec p}\left(\mathcal{G}^{\Lambda_0,\Lambda_0}_{n,l,j-1}(\otimes_{i=1}^N \mathcal{O}_{A_i};\vec p) - \mathcal{G}^{\Lambda_0,\Lambda_0}_{n,l,j}(\otimes_{i=1}^N \mathcal{O}_{A_i};\vec p)\right) = 0 && \text{for } n+|w|>j
\end{aligned}$$

in both cases. Further, we will need the identity

$$\begin{aligned}
G^{\Lambda,\Lambda_0}_{D'}(\otimes_{i=1}^N \mathcal{O}_{A_i}(0)) &= \sum_{[C]=D'} \mathcal{D}^C \left\{\prod_{i=1}^N L^{0,\Lambda_0}(\mathcal{O}_{A_i})\right\} L^{\Lambda,\Lambda_0}(\mathcal{O}_C(0))\\
&= L^{\Lambda,\Lambda_0}\left(\left[\mathcal{O}_{A_1}\cdots\mathcal{O}_{A_N}\right](0)\right)
\end{aligned} \tag{3.3.8}$$

where $D' = [A_1] + \ldots + [A_N]$ and where the expression in the second line is a CAG with a single insertion of the composite operator $\mathcal{O}_{A_1}\cdots\mathcal{O}_{A_N} = \mathcal{O}_B$, with

$$\{n_B, v_B\} = \{n_{A_1} + \ldots + n_{A_N}, (v_{A_1},\ldots,v_{A_N})\}\,. \tag{3.3.9}$$

Equation (3.3.8) is again verified by noting that all the terms in that equation satisfy a homogeneous flow equation and the same boundary conditions. As a consequence of eq.(3.3.8), we may write

$$\begin{aligned}
\mathbb{T}^{\Delta+1}_{\vec x\to\vec 0}\, G^{\Lambda,\Lambda_0}_{D+1}(\otimes_{i=1}^N \mathcal{O}_{A_i}) &= \sum_{[C]=D+1} \mathcal{D}^C \left\{\mathbb{T}^{\Delta+1}_{\vec x\to\vec 0}\prod_{i=1}^N L^{0,\Lambda_0}(\mathcal{O}_{A_i})\right\} L^{\Lambda,\Lambda_0}(\mathcal{O}_C(0))\\
&= \sum_{[C]=D+1} \mathcal{D}^C \left\{(1-\sum_{j\le\Delta}\mathbb{T}^{j}_{\vec x\to\vec 0})\prod_{i=1}^N L^{0,\Lambda_0}(\mathcal{O}_{A_i})\right\} L^{\Lambda,\Lambda_0}(\mathcal{O}_C(0))\,,
\end{aligned} \tag{3.3.10}$$

where $\Delta = D - D' \geq 0$. In the last line we applied the formula for the Taylor expansion with remainder, eq.(3.2.17). The Taylor expansion terms of degree $j > \Delta + 1$ vanish due to the boundary conditions of the CAG's with one insertion.

Note also that the boundary conditions for the CAG's with one insertion, eq.(3.1.27), imply

$$\mathcal{D}^C\{G^{0,\Lambda_0}_{[C]-1}(\otimes_{i=1}^N \mathcal{O}_{A_i})\} = \mathcal{D}^C\{\hbar F^{0,\Lambda_0}_{[C]-1}(\otimes_{i=1}^N \mathcal{O}_{A_i})\} \quad \text{for } [C] < D' . \tag{3.3.11}$$

We now prove lemma 2 by induction in D:

1. Induction start: For $D = -1$ the sum in eq.(3.3.3) vanishes and we obtain the lemma for $D = -1$, $R^{\Lambda,\Lambda_0}_{D=-1}(\otimes_{i=1}^N \mathcal{O}_{A_i}) = G^{\Lambda,\Lambda_0}_{D=-1}(\otimes_{i=1}^N \mathcal{O}_{A_i})$, trivially.

2. Induction step: Assume the lemma holds up to degree D, i.e. assume

$$R^{\Lambda,\Lambda_0}_{\tilde{D}}(\otimes_{i=1}^N \mathcal{O}_{A_i}) = (1 - \sum_{j \le \tilde{D}-D'} \mathbb{T}^j_{\vec{x}\to\vec{0}})\, G^{\Lambda,\Lambda_0}_{\tilde{D}}(\otimes_{i=1}^N \mathcal{O}_{A_i}(x_i)) \tag{3.3.12}$$

for all $\tilde{D} \le D$. Using again eq.(3.3.3), we then get

$$\begin{aligned}
R^{\Lambda,\Lambda_0}_{D+1}(\otimes_{i=1}^N \mathcal{O}_{A_i}) &= R^{\Lambda,\Lambda_0}_{D}(\otimes_{i=1}^N \mathcal{O}_{A_i}) - \sum_{[C]=D+1} \mathcal{C}^C_{A_1\dots A_N}\, L^{\Lambda,\Lambda_0}(\mathcal{O}_C)\\
=&(1 - \sum_{j\le\Delta} \mathbb{T}^j_{\vec{x}\to\vec{0}})\, G^{\Lambda,\Lambda_0}_{D}(\otimes_{i=1}^N \mathcal{O}_{A_i}(x_i)) - \sum_{[C]=D+1} \mathcal{C}^C_{A_1\dots A_N}\, L^{\Lambda,\Lambda_0}(\mathcal{O}_C)\\
=&(1 - \sum_{j\le\Delta+1} \mathbb{T}^j_{\vec{x}\to\vec{0}})\, G^{\Lambda,\Lambda_0}_{D+1}(\otimes_{i=1}^N \mathcal{O}_{A_i}(x_i))\\
+&(1 - \sum_{j\le\Delta} \mathbb{T}^j_{\vec{x}\to\vec{0}})\, \Big\{G^{\Lambda,\Lambda_0}_{D}(\otimes_{i=1}^N \mathcal{O}_{A_i}(x_i)) - G^{\Lambda,\Lambda_0}_{D+1}(\otimes_{i=1}^N \mathcal{O}_{A_i}(x_i))\Big\}\\
+& \sum_{j=\Delta+1} \mathbb{T}^j_{\vec{x}\to\vec{0}}\, G^{\Lambda,\Lambda_0}_{D+1}(\otimes_{i=1}^N \mathcal{O}_{A_i}) - \sum_{[C]=D+1} \mathcal{C}^C_{A_1\dots A_N}\, L^{\Lambda,\Lambda_0}(\mathcal{O}_C)
\end{aligned} \tag{3.3.13}$$

where $\Delta = D - D'$. Using eqs.(3.3.7) and (3.3.10) to replace the corresponding terms in the last two lines and also recalling the definition of the OPE coefficients $\mathcal{C}^C_{A_1\dots A_N}$, eq.(3.1.62), we find that the last three terms cancel out (in the case $\Delta < 0$ one also has to take into account eq.(3.3.11) to see this), leaving the claim of the lemma at order $D + 1$ in the case $x_N = 0$. The case $x_N \neq 0$ then follows by translation covariance. □

Lemma 2 combined with bound 2 allows us to estimate the remainder functionals. Substituting this bound along with the estimate (3.2.8) on the CAG's without operator insertions into eq.(3.3.4), and also using $C^{\Lambda,\Lambda_0}(p) \le [\sup(m, |p|)]^{-2}$, we obtain the statement of the theorem (note that the resulting bound is independent of $\Lambda \le m$ and Λ_0, so the cutoffs can be removed safely). □

3.4 Partial OPE

In the previous section we have shown that the remainder

$$G^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}) - \sum_{[C]\leq D} \mathcal{C}^C_{A_1\dots A_N}\, L^{\Lambda,\Lambda_0}(\mathcal{O}_C) \tag{3.4.1}$$

goes to zero as $D \to \infty$. Instead of expanding the complete operator product $\mathcal{O}_{A_1}\cdots\mathcal{O}_{A_N}$, we now consider a similar expansion in just a subset $\mathcal{O}_{A_1}\cdots\mathcal{O}_{A_{M<N}}$ of these operators, while leaving the remaining operators untouched. In other words, we are now interested in the expansion

$$G^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}(x_i)) - \sum_{[C]\leq D} \mathcal{C}^C_{A_1\dots A_M}(x_1,\dots,x_M)\, G^{\Lambda,\Lambda_0}(\mathcal{O}_C(x_M) \otimes_{j=M+1}^N \mathcal{O}_{A_j}(x_j))\,. \tag{3.4.2}$$

We will see in section 3.5 below that convergence of this partial OPE would imply non-trivial algebraic relations between the OPE coefficients. The following lemma will allow us to estimate the remainder of the partial OPE and to thereby analyse under what conditions this expansion converges.

Lemma 3: *Let $\Delta = D - ([A_1] + \ldots + [A_M])$ and $M < N$. The remainder of the partial OPE can be expressed as*

$$\begin{aligned} &G^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}(x_i)) - \sum_{[C]\leq D} \mathcal{C}^C_{A_1\dots A_M}(x_1,\dots,x_M)\, G^{\Lambda,\Lambda_0}(\mathcal{O}_C(x_M) \otimes_{j=M+1}^N \mathcal{O}_{A_j}(x_j)) \\ &= (1 - \sum_{j=0}^{\Delta} \mathbb{T}^j_{(x_1,\dots,x_M)\to(x_M,\dots,x_M)})\, G^{\Lambda,\Lambda_0}\left([\otimes_{i=1}^M \mathcal{O}_{A_i}(x_i)]_D; \otimes_{j=M+1}^N \mathcal{O}_{A_j}(x_j)\right)\,, \end{aligned} \tag{3.4.3}$$

where $G^{\Lambda,\Lambda_0}([\otimes_{i=1}^M \mathcal{O}_{A_i}]_D; \otimes_{j=M+1}^N \mathcal{O}_{A_j})$, defined in section 3.1.4, are the AG's with regularisation on the partial diagonal $x_1 = \ldots = x_M$.

Proof. With the help of lemma 1 we may write the l.h.s. of equation (3.4.3) as

$$\begin{aligned} &G^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}) - \sum_{[C]\leq D} \mathcal{C}^C_{A_1\dots A_M}\, G^{\Lambda,\Lambda_0}(\mathcal{O}_C \otimes_{j=M+1}^N \mathcal{O}_{A_j}) = \\ &G^{\Lambda,\Lambda_0}(\otimes_{i=1}^M \mathcal{O}_{A_i})\, G^{\Lambda,\Lambda_0}(\otimes_{i=M+1}^N \mathcal{O}_{A_i}) + \hbar H^{\Lambda,\Lambda_0}(\otimes_{i=1}^M \mathcal{O}_{A_i}; \otimes_{i=M+1}^N \mathcal{O}_{A_i}) \\ &- \sum_{[C]\leq D} \mathcal{C}^C_{A_1\dots A_M}\left[L^{\Lambda,\Lambda_0}(\mathcal{O}_C)\, G^{\Lambda,\Lambda_0}(\otimes_{j=M+1}^N \mathcal{O}_{A_j}) + \hbar H^{\Lambda,\Lambda_0}(\mathcal{O}_C; \otimes_{j=M+1}^N \mathcal{O}_{A_j})\right] \end{aligned} \tag{3.4.4}$$

We have shown in lemma 2 that

$$
\begin{aligned}
&\Big[G^{\Lambda,\Lambda_0}(\otimes_{i=1}^{M}\mathcal{O}_{A_i}) - \sum_{[C]\le D} \mathcal{C}^{C}_{A_1\ldots A_M} L^{\Lambda,\Lambda_0}(\mathcal{O}_C) \Big] G^{\Lambda,\Lambda_0}(\otimes_{i=M+1}^{N}\mathcal{O}_{A_i}) \\
=&(1-\sum_{j=0}^{\Delta} \mathbb{T}^{j}_{(x_1,\ldots,x_M)\to(x_M,\ldots,x_M)})\, G^{\Lambda,\Lambda_0}_{D}(\otimes_{i=1}^{M}\mathcal{O}_{A_i})\, G^{\Lambda,\Lambda_0}(\otimes_{i=M+1}^{N}\mathcal{O}_{A_i}) \,.
\end{aligned}
\tag{3.4.5}
$$

Further, the equality

$$
\begin{aligned}
&H^{\Lambda,\Lambda_0}(\otimes_{i=1}^{M}\mathcal{O}_{A_i};\otimes_{i=M+1}^{N}\mathcal{O}_{A_i}) - \sum_{[C]\le D} \mathcal{C}^{C}_{A_1\ldots A_M} H^{\Lambda,\Lambda_0}(\mathcal{O}_C;\otimes_{j=M+1}^{N}\mathcal{O}_{A_j}) \\
&= (1-\sum_{j=0}^{\Delta} \mathbb{T}^{j}_{(x_1,\ldots,x_M)\to(x_M,\ldots,x_M)})\, H^{\Lambda,\Lambda_0}\left([\otimes_{i=1}^{M}\mathcal{O}_{A_i}]_D;\otimes_{i=M+1}^{N}\mathcal{O}_{A_i}\right)
\end{aligned}
\tag{3.4.6}
$$

can be seen to hold by comparing the flow equations and boundary conditions for both sides of the equation. One also has to make use of lemma 2 again to verify this equation. Substituting eqs.(3.4.5) and (3.4.6) on the r.h.s. of eq. (3.4.4), we obtain eq.(3.4.3). □

Lemma 3 combined with our bounds on the Taylor expansion of the AG's with partial regularisation, bound 4, directly allows us to estimate the remainder of the partial OPE.

Theorem 2 (Partial OPE convergence): *Assume $|x_i - x_M| \le |x_j - x_M|$ for all $1 \le i \le M$ and $M+1 \le j \le N$ and let $\Lambda \le m$. There exists a constant $\tilde K > 0$ depending on n and l, such that for all $D - [A_1] - \ldots - [A_N] = \Delta$*

$$
\begin{aligned}
&\Big|\partial^{w}_{\vec p}\Big(\mathcal{G}^{\Lambda,\Lambda_0}_{2n,l}(\otimes_{i=1}^{N}\mathcal{O}_{A_i}(x_i);\vec p) \\
&-\sum_{[C]\le D}\mathcal{C}^{C}_{A_1\ldots A_M}(x_1,\ldots,x_M)\,\mathcal{G}^{\Lambda,\Lambda_0}_{2n,l}(\mathcal{O}_C(x_M)\otimes\mathcal{O}_{A_{M+1}}(x_{M+1})\cdots\otimes\mathcal{O}_{A_N}(x_N);\vec p)\Big)\Big| \\
&\le m^{-2n-|w|-1}\,|w|!\,\prod_{i=1}^{N}[A_i]!\,\sum_{\lambda=0}^{2l+n}\frac{\log_+^{\lambda}(\frac{|\vec p|}{m})}{2^{\lambda}\lambda!}\,\tilde K^{|w|}\,\sup(1,m|\vec x|)^{|w|} \\
&\times\left(\frac{\tilde K\ \sup(\frac{|\vec p|}{m},1)^{2n+2l+4N}}{\min_{1\le i<j\le N}[|x_i-x_j|,1/m]}\right)^{D'+1}\cdot\left(\frac{\max_{1\le i<j\le N}|x_i-x_j|}{\min_{1\le i<j\le N}|x_i-x_j|}\right)^{2D'+3|w|+2} \\
&\times\frac{\left(\tilde K\ \sup(\frac{|\vec p|}{m},1)^{2n+2l+4N}\max_{1\le i\le M}|x_i-x_M|\right)^{\Delta}}{\min\left[\left(\min_{M<j\le N}|x_j-x_M|^{\Delta-1}\cdot\min_{1\le i<j\le N}|x_i-x_j|\right),\frac{\sqrt{\Delta!}}{m^{\Delta}}\right]}\cdot\left(\frac{\max_{M<i\le N}|x_i-x_N|}{\min_{M<i<j\le N}|x_i-x_j|}\right)^{\Delta}
\end{aligned}
\tag{3.4.7}
$$

where $D' = [A_1]+\ldots+[A_N]$.

Remark 10: The r.h.s. of (3.4.7) vanishes as we take the limit $\Delta \to \infty$ provided that

$$\left(\tilde{K}\ \sup(\frac{|\vec{p}|}{m}, 1)^{2n+2l+4N} \frac{\max_{1\leq i\leq M} |x_i - x_M|}{\min_{M<j\leq N} |x_j - x_M|} \cdot \frac{\max_{M<i\leq N} |x_i - x_N|}{\min_{M<i<j\leq N} |x_i - x_j|}\right) < 1\,. \tag{3.4.8}$$

This condition defines an open spacetime region $\mathcal{S}(n, l, \vec{p}, m) \in \mathbb{R}^4$ for any finite values of n, l, m and $\vec{p}$. Hence, the partial OPE converges in that region, which is the upshot of the present section.

3.5 Proof of OPE factorisation

Our interest in the partial OPE, analysed in the previous section, is mostly rooted in its implications on the properties of the OPE coefficients. In the present section we will show that convergence of the partial OPE yields non-trivial relations between the OPE coefficients. More precisely, we will show:

Theorem 3 (OPE factorisation): *Let $2 \leq M < N$. Up to any arbitrary but fixed loop order l in $g\varphi^4$-theory, the identity*

$$\mathcal{C}^{B}_{A_1\ldots A_N}(x_1,\ldots,x_N) = \sum_C \mathcal{C}^{C}_{A_1\ldots A_M}(x_1,\ldots,x_M)\ \mathcal{C}^{B}_{CA_{M+1}\ldots A_N}(x_M, x_{M+1},\ldots,x_N) \tag{3.5.1}$$

holds for all configurations $(x_1,\ldots,x_N) \in \mathbb{R}^{4N}$ satisfying

$$\frac{\max\limits_{1\leq i\leq M} |x_i - x_M|}{\min\limits_{M+1\leq j\leq N} |x_j - x_M|} \cdot \frac{\max\limits_{M<i\leq N} |x_i - x_N|}{\min\limits_{M<i<j\leq N} |x_i - x_j|} < \frac{1}{\tilde{K}} \tag{3.5.2}$$

for some (sufficiently large) constant $\tilde{K} > 0$ (depending on l, B).

Remark 11: In the case $(N - M) \leq 2$ the second factor on the l.h.s. of (3.5.2) clearly is equal to one, i.e. the condition reduces in that case to the simpler form

$$\frac{\max\limits_{1\leq i\leq M} |x_i - x_M|}{\min\limits_{M+1\leq j\leq N} |x_j - x_M|} < \frac{1}{\tilde{K}}\,. \tag{3.5.3}$$

Note that this condition would coincide with the spacetime domain specified in the statement of the factorisation axiom, see (2.1.18), if we had $\tilde{K} = 1$. The appearance of the, potentially large, constant $\tilde{K}$ in (3.5.3) means that theorem 3 is a somewhat weaker property than that required by the factorisation axiom in section 2.1. The points $x_1, \ldots, x_{M-1}$ have to be *much* closer to x_M than the points $x_{M+1}, \ldots, x_N$, instead of just closer. The situation is sketched in fig. 3.2. In order for the condition (3.5.3) to be fulfilled, the ratio of the two dashed lines must be smaller then $1/\tilde{K}$.

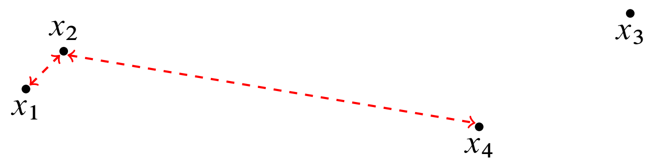


Figure 3.2.: A configuration satisfying the condition (3.5.3) with $M = 2, N = 4$.

For general M, N, the condition (3.5.2) is even weaker. However, the spacetime region (3.5.2) is certainly non-empty, so the OPE coefficients do factorise on a certain domain. This is the main message from theorem 3 in this general case. The condition (3.5.2) seems somewhat awkward from an intuitive standpoint, as one would not expect the mutual distances between the points $x_{M+1}, \ldots, x_N$ to be of any relevance here. We believe that, in principle, an improved result, where (3.5.2) is replaced by (3.5.3), should be achievable.

Proof. Let us first recall our definition of the OPE coefficients (here $\Delta = [B] - [A_1] - \ldots - [A_N]$):

$$\begin{aligned}\mathcal{C}^{B}_{A_1\ldots A_N}(x_1,\ldots,x_N) :=& \mathcal{D}^B\left\{(1-\sum_{j<\Delta}\mathbb{T}^j)\; G^{0,\Lambda_0}_{[B]-1}(\otimes_{i=1}^N \mathcal{O}_{A_i})\right\}\\ =& \mathcal{D}^B\left\{G^{0,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}) - \sum_{[\tilde{C}]<[B]} \mathcal{C}^{\tilde{C}}_{A_1\ldots A_N}(x_1,\ldots,x_N) L^{0,\Lambda_0}(\mathcal{O}_{\tilde{C}})\right\}\end{aligned} \tag{3.5.4}$$

In the second line we used lemma 2. In view of this equation we rewrite the r.h.s. of eq.(3.5.1) as

$$\begin{aligned}&\sum_C \mathcal{C}^{C}_{A_1\ldots A_M}\, \mathcal{C}^{B}_{CA_{M+1}\ldots A_N} =\\ &\mathcal{D}^B\left\{\sum_C \mathcal{C}^{C}_{A_1\ldots A_M}\left(G^{0,\Lambda_0}(\mathcal{O}_C \otimes_{i=M+1}^N \mathcal{O}_{A_i}) - \sum_{[\tilde{C}]<[B]} \mathcal{C}^{\tilde{C}}_{CA_{M+1}\ldots A_N} L^{0,\Lambda_0}(\mathcal{O}_{\tilde{C}})\right)\right\}\end{aligned} \tag{3.5.5}$$

We recognise the expression $\sum_C \mathcal{C}^{C}_{A_1\ldots A_M} G^{0,\Lambda_0}(\mathcal{O}_C \otimes_{i=M+1}^N \mathcal{O}_{A_i})$ as the partial OPE in the fields $\mathcal{O}_{A_1}, \ldots, \mathcal{O}_{A_M}$, which was discussed in the previous section. By assumption, the configuration $(x_1, \ldots, x_N)$ satisfies the inequality (3.5.2), which guarantees convergence of this partial OPE according to theorem 2. Thus, eq.(3.5.5) may equivalently be written as

$$
\begin{aligned}
&\sum_C \mathcal{C}^C_{A_1\dots A_M}\, \mathcal{C}^B_{CA_{M+1}\dots A_N} \\
&= \mathcal{D}^B \left\{ G^{0,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}) - \sum_C \mathcal{C}^C_{A_1\dots A_M} \sum_{[\tilde{C}]<[B]} \mathcal{C}^{\tilde{C}}_{CA_{M+1}\dots A_N} L^{0,\Lambda_0}(\mathcal{O}_{\tilde{C}}) \right\} \\
&= \mathcal{C}^B_{A_1\dots A_N} + \sum_{[\tilde{C}]<[B]} \left(\mathcal{C}^{\tilde{C}}_{A_1\dots A_N} - \sum_C \mathcal{C}^C_{A_1\dots A_M} \mathcal{C}^{\tilde{C}}_{CA_{M+1}\dots A_N} \right) \mathcal{D}^B L^{0,\Lambda_0}(\mathcal{O}_{\tilde{C}})\,.
\end{aligned}
\tag{3.5.6}
$$

To obtain the last equality, we made use of the relation

$$
\mathcal{D}^B \left\{ G^{0,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}) \right\} = \mathcal{D}^B \left\{ \sum_{[\tilde{C}]\leq[B]} \mathcal{C}^{\tilde{C}}_{A_1\dots A_N} L^{0,\Lambda_0}(\mathcal{O}_{\tilde{C}}) \right\}\,. \tag{3.5.7}
$$

which, in turn, can be seen to hold by combining lemma 2 with the boundary conditions for the regularised AG's, see eqs.(3.1.39), (3.1.40), (3.1.27) and (3.1.28). Theorem 3 now follows from equation (3.5.6) by induction:

Induction start ($[B]=0$): In this case the sum over $\tilde{C}$ in the second line of eq.(3.5.6) vanishes, and we are left with the claim of the theorem.

Induction step: Assume that $\mathcal{C}^{\tilde{C}}_{A_1\dots A_N} - \sum_C \mathcal{C}^C_{A_1\dots A_M} \mathcal{C}^{\tilde{C}}_{CA_{M+1}\dots A_N}$ holds for all $\tilde{C}$ with $[\tilde{C}] < [B]$. Then the equation in brackets in the last line of eq.(3.5.6) vanishes, and we are again left with the claim of the theorem. □

3.6 Deformation of the OPE algebra

While the definition of the perturbative OPE coefficients used throughout this chapter, see def. 3, is very clear from a conceptual standpoint, it is somewhat dissatisfying that we have to rely on secondary objects (i.e. regularised AG's with insertions) in order to determine the OPE coefficients in perturbation theory. It would be desirable to be able to construct the perturbed OPE coefficients just in terms of the zeroth perturbation order ones, without reference to any other quantities. This would yield support to the viewpoint that no data other than the OPE coefficients and one point functions are needed to define a quantum field theory [cf. chapter 2].

In the following we are going to show that such a construction is indeed possible. Starting from our definition of the OPE coefficients in terms of amputated Green's functions with insertions, see def.3, we will derive a formula that allows us to express the coefficients at a given order r in terms of (an integral over) lower order ones. Our derivation of this formula is from first principles, i.e. we do not have to make any additional assumptions. It should be mentioned that our formula is derived for the theory with a *finite* UV-cutoff Λ_0. Our bounds in section 3.2 guarantee that the limit $\Lambda_0 \to \infty$ can be taken safely in the end, i.e. after the formula is applied.

To obtain the mentioned perturbation formula, we will first study the effect of taking a derivative with respect to the coupling constant[9] g of Green's functions with and without insertions, see section 3.6.1. In section 3.6.2 we will put these results to use and come to the actual derivation of the perturbation formula for the OPE coefficients, see theorem 4. As mentioned above, this formula requires the OPE coefficients of the free theory as initial data. Therefore, we will derive explicit formulae for these zeroth order coefficients in section 3.6.3. Finally, we will perform a few exemplary calculations of first order coefficients in section 3.6.4 in order to showcase the application of our perturbation formula.

3.6.1. Variation of Green's functions with respect to the coupling constant

In the familiar diagrammatic framework of quantum field theory, increasing the perturbation order is represented by additional insertions of interaction vertices, corresponding in our case to φ^4 insertions, into the Feynman diagrams. This relation between insertions of the interaction operator and the order of perturbation theory takes on a very simple form in our framework. Namely, one can show:

Proposition 2: (Müller [20]) *The derivative with respect to the coupling constant of the CAG's without insertion, which were defined in section 3.1.1, can be expressed as*

$$\partial_g L^{\Lambda,\Lambda_0} = \frac{1}{4!} \int \mathrm{d}^4 y \, L^{\Lambda,\Lambda_0}(\varphi^4(y)) \,, \tag{3.6.1}$$

where we have the CAG's with insertion of the composite operator φ^4 on the right hand side (see section 3.1.2 for the definition of operator insertions).

Proof. We give a slightly different version of the proof compared to the one presented in [20], which is more in the spirit of this thesis. Namely, we note that L^{Λ,Λ_0} is defined through the following conditions:

1. Flow Equation (3.1.8)

2. Boundary conditions (3.1.11), (3.1.12)

[9] Recall that we fixed $g = 1$ in the previous sections (see remark 4). In the present discussion we are interested in variations of the coupling constant, so we leave $g \in \mathbb{R}$ arbitrary for the remainder of section 3.6.

3. Translation invariance

Taking a Λ-derivative[10] on both sides of equation (3.6.1) and substituting the flow equations (3.1.8) and (3.1.23), we find that both expressions indeed obey the same linear homogeneous flow equation. Concerning the boundary conditions, we apply the g-derivative to eqs.(3.1.11) and (3.1.12), which yields the conditions (3.1.27) and (3.1.28) with $A = \{w' = 0, n' = 4\}$. Finally, both sides of equation (3.6.1) are evidently translation invariant. □

The proposition can be generalised to the g-derivative of CAG's with insertions.

Proposition 3: *The CAG's with one insertion satisfy the identity*

$$\partial_g L^{\Lambda,\Lambda_0}(\mathcal{O}_A(x)) = \frac{1}{4!}\int \mathrm{d}^4 y\, L^{\Lambda,\Lambda_0}_{D=[A]}(\mathcal{O}_A(x)\otimes\varphi^4(y))\,. \tag{3.6.2}$$

Proof. It is again our strategy to prove that both sides of the equation satisfy the same flow equations and boundary conditions. Taking the g-derivative of the flow equation for $L^{\Lambda,\Lambda_0}(\mathcal{O}_A)$, (3.1.23), we obtain for the left hand side

$$\begin{aligned}
&\partial_\Lambda \partial_g L^{\Lambda,\Lambda_0}(\mathcal{O}_A)\\
&= \frac{\hbar}{2}\langle\frac{\delta}{\delta\varphi}\,,\,\dot{C}^\Lambda \star \frac{\delta}{\delta\varphi}\rangle\,\partial_g L^{\Lambda,\Lambda_0}(\mathcal{O}_A) - \langle\frac{\delta}{\delta\varphi}\partial_g L^{\Lambda,\Lambda_0}(\mathcal{O}_A)\,,\,\dot{C}^\Lambda \star \frac{\delta}{\delta\varphi}L^{\Lambda,\Lambda_0}(\varphi)\rangle\\
&- \langle\frac{\delta}{\delta\varphi}L^{\Lambda,\Lambda_0}(\mathcal{O}_A)\,,\,\dot{C}^\Lambda \star \frac{\delta}{\delta\varphi}\frac{1}{4!}\int \mathrm{d}^4 y\, L^{\Lambda,\Lambda_0}(\varphi^4(y))\rangle \quad ,
\end{aligned} \tag{3.6.3}$$

where we used proposition 2 in the last line. Further, $\partial_g L^{\Lambda,\Lambda_0}(\mathcal{O}_A)$ is subject to the following boundary conditions:

$$\partial^w_{\vec{p}}\partial_g \mathcal{L}^{0,\Lambda_0}_{n,l}(\mathcal{O}_A(0);\vec{0}) = 0 \quad \text{for } n+|w| \le [A] \tag{3.6.4}$$

$$\partial^w_{\vec{p}}\partial_g \mathcal{L}^{\Lambda_0,\Lambda_0}_{n,l}(\mathcal{O}_A(0);\vec{p}) = 0 \quad \text{for } n+|w| > [A] \quad . \tag{3.6.5}$$

By definition, the right hand side of equation (3.6.2) satisfies the same boundary conditions. Concerning the flow equation, it is easy to check that the Λ-derivative of the right hand side would coincide with equation (3.6.3), if we were allowed to exchange the order of the Λ-derivative with the integral over y. This *differentiation under the integral sign* has to be justified of course, which is the tricky part in the proof. We have to show that the integrand satisfies the following two properties:

1. $L^{\Lambda,\Lambda_0}_{D=[A]}(\mathcal{O}_A(x)\otimes\varphi^4(y))$ is integrable (over y) for all $\Lambda \in [0,\Lambda_0]$.

[10] Note that we can use the translation properties of the CAG's with one insertion in order to write the Λ derivative of the r.h.s. as $\partial_\Lambda \mathcal{L}^{\Lambda,\Lambda_0}_{2n,l}(\varphi^4(0);\vec{p})\frac{1}{4!}\int \mathrm{d}^4 y\, \exp[iy(p_1+\ldots+p_{2n})]$, so we do not have to exchange the order of integration and differentiation here.

2. For some integrable function $g(y)$ we have

$$|\partial_\Lambda L^{\Lambda,\Lambda_0}_{D=[A]}(\mathcal{O}_A(x)\otimes\varphi^4(y))| \leq g(y) \tag{3.6.6}$$

for all $y \in \mathbb{R}^4$ and all $\Lambda \in [0,\Lambda_0]$.

Naturally, we would like to derive these properties with the help of our inductive estimates presented in bound 1 (recall that in the case $N = 2$, the F-functionals coincide with the CAG's with two insertions). Unfortunately, however, we see that these bounds depend on the spacetime variables through a factor $|x-y|^{-4}$ for the CAG with two insertions at hand. The y-integral over this bound therefore diverges logarithmically in both the ultraviolet ($y \to x$) and the infrared ($|y| \to \infty$). Thus, we will need to use different bounds here.

As mentioned in remark 5, the F-functionals decay as $|x-y|^{-r}$ for arbitrary $r \in \mathbb{N}$ at large separation of the spacetime arguments, so $L^{\Lambda,\Lambda_0}_{D=[A]}(\mathcal{O}_A(x)\otimes\varphi^4(y))$ is certainly integrable away from the singularity at $y = x$. It was also mentioned in remark 5 that we can derive an alternative bound on the F-functionals where the singular dependence on the spacetime arguments is replaced by factors of Λ_0. In the case at hand, this version of the bound would be uniform in x and y, but it depends on the UV-cutoff via Λ_0^4. Therefore, as long as we keep Λ_0 finite, the y-integral over $L^{\Lambda,\Lambda_0}_{D=[A]}(\mathcal{O}_A(x)\otimes\varphi^4(y))$ in fact converges absolutely.

Using these alternative bounds, it also easy to find a function $g(y)$ satisfying the requirement (3.6.6). We can pick, for example[11],

$$g(y) = \min\left(\frac{1}{m^r|x-y|^{4+r}},\Lambda_0^4\right)\,\tilde{K}\,\left(\frac{|\vec{p}|}{m}\right)^a\,\log_+^b\!\left(\frac{|\vec{p}|}{m}\right)\,\max(\Lambda_0^{[A]-2n-1},m^{[A]-2n-1}) \tag{3.6.7}$$

for some constant $\tilde{K} > 0$ and some $a,b,r \in \mathbb{N}$. We see that this function is constant for y close to x, that it decays rapidly for large $|y|$ and that it is integrable over y.

To summarise, we have shown that equation (3.6.2) holds for finite Λ_0. To see that both expressions in that equation stay finite as we remove the UV-cutoff, $\Lambda_0 \to \infty$, we recall the bounds on the CAG's with one insertion, (3.2.9), which clearly imply that the left hand side of equation (3.6.2) is finite in that limit. The same must therefore be true for the right hand side, as long as we agree to remove the UV-cutoff *after* performing the y-integral.□

Remark 12: If we could derive a bound on $L^{\Lambda,\Lambda_0}_{D=[A]}(\mathcal{O}_A(x)\otimes\varphi^4(y))$ which scales as $|x-y|^{-3-\delta}$ for any $\delta > 0$ instead of $|x-y|^{-4}$, and which at the same time does not depend on the UV-cutoff Λ_0, then it would be possible to show that equation (3.6.2) holds also if we remove the cutoffs under the integral. As mentioned in remark 5, such a scaling

[11] In the case $\Lambda \leq m$ we can remove the dependence on the IR-cutoff as in the proof of bound 1, see the discussion around (A.1.52).

behaviour is generally expected to hold, but we have no rigorous proof of this property within the flow equation framework at present.

Below we will also be interested in the g-derivative of the amputated Green's functions with insertions, $G^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i})$.

Proposition 4: *The g-derivative of the amputated Green's functions with insertions can be expressed as*

$$
\begin{aligned}
&\hbar\, \partial_g G^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}(x_i)) = \\
&\frac{-1}{4!}\int \mathrm{d}^4 y \Big[G^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}(x_i) \otimes \varphi^4(y)) - G^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}(x_i)) L^{\Lambda,\Lambda_0}(\varphi^4(y)) \\
&-\sum_{j=1}^N \sum_{[C]\leq[A_j]} \mathcal{C}_{A_g A_j}^C(y,x_j) G^{\Lambda,\Lambda_0}(\otimes_{\substack{i=1\\ i\neq j}}^N \mathcal{O}_{A_i}(x_i) \otimes \mathcal{O}_C(x_j)) \Big]
\end{aligned} \tag{3.6.8}
$$

where $A_g := \{n = 4, w = 0\}$, i.e. $\mathcal{O}_{A_g} = \varphi^4$ is the interaction operator.

Remark 13: With the help of lemma 2 one can check that equation (3.6.8) reduces to proposition 3 in the case $N = 1$.

Proof. Again we argue that the integral on the right hand side converges absolutely for finite Λ_0. Note that the first two terms on the right hand side give

$$
G^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i} \otimes \varphi^4) - G^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}) L^{\Lambda,\Lambda_0}(\varphi^4) = \hbar H^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}; \varphi^4) \tag{3.6.9}
$$

by definition. We know from remark 8 that these H^{Λ,Λ_0}-functionals decay more rapidly than any power for large $|y|$, so the integral over these terms is infrared safe. To estimate the infrared behaviour of the OPE coefficients $\mathcal{C}_{A_g A_j}^C(y,x_j)$, we recall their definition in terms of AG's with insertions, see def. 3. The contribution where φ^4 appears as a single operator insertion vanishes in the case $[C] < [A_g] + [A_j]$ due to the boundary conditions of the CAG's with one insertion, and the contribution including the bilocal insertion is found to decay rapidly for large $|y|$ by the same arguments as in proposition 3.

Let us now discuss the UV behaviour of the integral, i.e. the regions where y is close to one of the x_j. The two contributions which are potentially singular in this region are

$$
\begin{aligned}
G^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}(x_i) \otimes \varphi^4(y)) - \sum_{[C]\leq[A_j]} \mathcal{C}_{A_g A_j}^C(y,x_j) G^{\Lambda,\Lambda_0}(\otimes_{\substack{i=1\\ i\neq j}}^N \mathcal{O}_{A_i}(x_i) \otimes \mathcal{O}_C(x_j)) \\
= G^{\Lambda,\Lambda_0}\big([\mathcal{O}_{A_j}(x_j) \otimes \varphi^4(y)]_{[A_j]}; \otimes_{\substack{i=1\\ i\neq j}}^N \mathcal{O}_{A_i}(x_i)\big)
\end{aligned} \tag{3.6.10}
$$

where we used lemma 3 in the second line. To see that we can safely integrate over the region $y \approx x_j$ as long as we keep the UV-cutoff finite, we recall from remark 5 that we have bounds on the functionals $F^{\Lambda,\Lambda_0}_{[A_j]}(\mathcal{O}_{A_j}(x_j)\otimes\varphi^4(y))$, which do not depend on $|x_j - y|$, but instead contain factors of Λ_0. If we use these estimates in the proof of bound 3, we obtain an alternative bound on the functionals $G^{\Lambda,\Lambda_0}\big([\mathcal{O}_{A_j}(x_j)\otimes\varphi^4(y)]_{[A_j]};\otimes_{\substack{i=1\\i\neq j}}^{N}\mathcal{O}_{A_i}(x_i)\big)$, which shows no dependence on $|x_j - y|$ either, but which contains factors of Λ_0. This alternative bound shows that the integral over the UV-regions, where y is close to one of the x_j, converges absolutely for finite Λ_0.

Thus, combining all these bounds for different spacetime regions, we conclude that the integral on the right hand side of equation (3.6.8) is absolutely convergent for finite Λ_0. Following a similar strategy as in the proof of proposition 3, it is then also not hard to find a function $g(y)$ which satisfies the analog of (3.6.6) for the case at hand.

Hence, we are allowed to exchange the order of the y-integral with the Λ-derivative if we want to determine the flow equation for the right hand side of equation (3.6.8). A proof that both sides of equation (3.6.8) then indeed satisfy the same boundary conditions and flow equations can be found in appendix A.5. Finally we argue, based on our bounds on the AG's with insertions [see corollary 1], that the left hand side of equation (3.6.8) has a finite limit $\Lambda_0 \to \infty$. Again, the same, then, has to be true for the right hand side, provided we remove the UV-cutoff after performing the k-integral. □

3.6.2. Variation of OPE coefficients with respect to the coupling constant

The OPE coefficients have been defined in def. 3 in terms of amputated Green's functions with insertions. The results of the previous section can be used to derive the following formula for the deformation of the OPE algebra:

Theorem 4 (OPE deformation): *Let $A_g := \{n = 4, v = 0\}$, i.e. $\mathcal{O}_{A_g} = \varphi^4$. The derivative of the OPE coefficients w.r.t. the coupling constant g can be expressed as*

$$\hbar\partial_g\, \mathcal{C}^{B}_{A_1\ldots A_N}(x_1,\ldots,x_N) = \frac{-1}{4!}\int \mathrm{d}^4y \times\Bigg[\mathcal{C}^{B}_{A_g A_1\ldots A_N}(y,x_1,\ldots,x_N) - \sum_{i=1}^{N}\sum_{[C]\le[A_i]}\mathcal{C}^{C}_{A_g A_i}(y,x_i)\,\mathcal{C}^{B}_{A_1\ldots\widehat{A_i}\,C\ldots A_N}(x_1,\ldots,x_N) - \sum_{[C]<[B]}\mathcal{C}^{C}_{A_1\ldots A_N}(x_1,\ldots,x_N)\,\mathcal{C}^{B}_{A_g C}(y,x_N)\Bigg]. \tag{3.6.11}$$

Here $\widehat{A_i}$ denotes omission of the corresponding index. The relation holds to arbitrary finite *perturbation order in Euclidean $g\varphi^4$-theory with BPHZ renormalisation conditions. It is understood here that we perform the y-integral before removing the UV-cutoff on the right hand side.*

Remark 14: A few observations are in order:

1. We suspect that the expressions which appear with a negative sign in the brackets on the right hand side of equation (3.6.11) may be interpreted as "counter terms", i.e. they cancel possible UV- and IR-divergent contributions from the first term on the right hand side in the theory without cutoffs. More precisely, if the integration variable y is close to one of the arguments x_j, then we can factorise the first coefficient on the right hand side using theorem 3:

$$\mathcal{C}^{B}_{A_g A_1 \dots A_N}(y, x_1, \dots, x_N) = \sum_C \mathcal{C}^{C}_{A_g A_j}(y, x_j) \mathcal{C}^{B}_{A_1 \dots \widehat{A_j}\, C \dots A_N}(x_1, \dots, x_N) \tag{3.6.12}$$

The corresponding counter term subtracts all terms from the sum over C with $[C] \leq [A_j]$. Recall that the OPE coefficients were given in def. 3 in terms of the AG's with operator insertions. Assuming, for the moment, that the AG's with insertions $G^{0,\infty}_D(\otimes_{i=1}^N \mathcal{O}_{A_i}(\varepsilon x_i))$ scale as $\varepsilon^{D - \sum [A_i] + 1 - \delta}$ for $\varepsilon, \delta > 0$ [see our discussion in remarks 5 and 12], we find that the remaining terms in the sum over C, which contain coefficients $\mathcal{C}^{C}_{A_g A_j}(y, x_j)$ with $[C] > [A_j]$, diverge at most like $|x_j - y|^{-3-\delta}$ and are thus indeed integrable on the domain with y close to x_j. Similarly, if $|y|$ is large compared to the $|x_i|$, then we can factorise the first term on the right hand side of (3.6.11) using theorem 3:

$$\mathcal{C}^{B}_{A_g A_1 \dots A_N}(y, x_1, \dots, x_N) = \sum_C \mathcal{C}^{C}_{A_1 \dots A_N}(x_1, \dots, x_N)\, \mathcal{C}^{B}_{A_g C}(y, x_N) \tag{3.6.13}$$

Here the corresponding counter term subtracts all summands with $[C] < [B]$. One can check, using arguments from the derivation of proposition 4, that the remaining terms decay faster than any power $|y - x_N|^{-r}$ for arbitrary $r \in \mathbb{N}$.

We will observe this cancellation of divergences in a concrete example in section 3.6.4.

2. Since OPE coefficients in perturbative quantum field theory are generally subject to renormalisation ambiguities, we expect the formula (3.6.11) to be sensitive to the renormalisation conditions.

3. Expanding the OPE coefficients as formal power series in the coupling constant,i.e.

$$\mathcal{C}^{B}_{A_1 \dots A_N}(x_1, \dots, x_N) =: \sum_{i=0}^{\infty} (\mathcal{C}_i)^{B}_{A_1 \dots A_N}(x_1, \dots, x_N)\, g^i\,, \tag{3.6.14}$$

and fixing our auxiliary loop parameter $\hbar = 1$, the theorem implies the relation

$$
\begin{aligned}
&(\mathcal{C}_{r+1})^{B}_{A_1\dots A_N}(x_1,\dots,x_N) = \\
&\frac{-1}{4!\,(r+1)}\int \mathrm{d}^4y \Bigg[(\mathcal{C}_r)^{B}_{A_g A_1\dots A_N}(y,x_1,\dots,x_N) \\
&\quad -\sum_{i=1}^{N}\sum_{[C]\leq[A_i]}\sum_{s=0}^{r}(\mathcal{C}_s)^{C}_{A_g A_i}(y,x_i)\,(\mathcal{C}_{r-s})^{B}_{A_1\dots\widehat{A_i}\,C\dots A_N}(x_1,\dots,x_N) \\
&\quad -\sum_{[C]<[B]}\sum_{s=0}^{r}(\mathcal{C}_s)^{C}_{A_1\dots A_N}(x_1,\dots,x_N)\,(\mathcal{C}_{r-s})^{B}_{A_g C}(y,x_N)\Bigg].
\end{aligned}
\tag{3.6.15}
$$

This equation allows us to determine the coefficients at order $(r+1)$ from those of lower perturbation order. In particular, given the OPE coefficients of the free theory, we can in principle iterate this equation to construct the coefficients to arbitrary order in g. Again, this iteration should be performed with a finite cutoff Λ_0, which can be removed, i.e. $\Lambda_0 \to \infty$, in the very end. Our bounds in section 3.2 guarantee that taking this limit is safe up to any finite order in perturbation theory.

Proof of theorem 4: Recall from eq.(3.5.4) that we may rewrite our definition of the OPE coefficients as (suppressing the dependence on the spacetime arguments for the moment)

$$
\mathcal{C}^{B}_{A_1\dots A_N} = \mathcal{D}^B\left\{ G^{0,\Lambda_0}(\otimes_{i=1}^{N}\mathcal{O}_{A_i}) - \sum_{[C]<[B]}\mathcal{C}^{C}_{A_1\dots A_N}L^{0,\Lambda_0}(\mathcal{O}_C)\right\}. \tag{3.6.16}
$$

Applying the g-derivative and using proposition 4 yields

$$
\begin{aligned}
\hbar\partial_g\mathcal{C}^{B}_{A_1\dots A_N} \;=\;& \mathcal{D}^B\frac{-1}{4!}\int \mathrm{d}^4y\Big[G^{0,\Lambda_0}(\otimes_{i=1}^{N}\mathcal{O}_{A_i}\otimes\mathcal{O}_g) - G^{0,\Lambda_0}(\otimes_{i=1}^{N}\mathcal{O}_{A_i})L^{0,\Lambda_0}(\mathcal{O}_g) \\
&-\sum_{j=1}^{N}\sum_{[C]\leq[A_j]}\mathcal{C}^{C}_{A_g A_j}G^{0,\Lambda_0}(\otimes_{\substack{i=1\\ i\neq j}}^{N}\mathcal{O}_{A_i}\otimes\mathcal{O}_C) + \hbar\sum_{[C]<[B]}\mathcal{C}^{C}_{A_1\dots A_N}L^{0,\Lambda_0}_{[C]}(\mathcal{O}_{A_g}\otimes\mathcal{O}_C)\Big] \\
&-\hbar\mathcal{D}^B\sum_{[\tilde{C}]<[B]}\left(\partial_g\mathcal{C}^{\tilde{C}}_{A_1\dots A_N}\right)L^{0,\Lambda_0}(\mathcal{O}_{\tilde{C}})
\end{aligned}
\tag{3.6.17}
$$

Now note that, in view of lemma 2, we have for $D > [B]$

$$
\begin{aligned}
&\mathcal{D}^B\partial_g\left[G^{0,\Lambda_0}(\otimes_{i=1}^{N}\mathcal{O}_{A_i}) - \sum_{[C]\leq D}\mathcal{C}^{C}_{A_1\dots A_N}L^{0,\Lambda_0}(\mathcal{O}_C)\right] = \mathcal{D}^B\partial_g R^{0,\Lambda_0}_{D}(\otimes_{i=1}^{N}\mathcal{O}_{A_i}) \\
&= \partial_g\mathcal{D}^B(1-\sum_{j\leq\Delta}\mathbb{T}^j)\,G^{0,\Lambda_0}_{D}(\otimes_{i=1}^{N}\mathcal{O}_{A_i}) = 0
\end{aligned}
\tag{3.6.18}
$$

where we recalled the boundary conditions for the AG's with insertions, see def.1, in the last line. This allows us to replace the AG's in eq.(3.6.17) by sums over OPE coefficients. Using also

$$L^{0,\Lambda_0}(\mathcal{O}_{A_g})L^{0,\Lambda_0}(\mathcal{O}_C) - \hbar L^{0,\Lambda_0}_{[C]}(\mathcal{O}_{A_g}\otimes\mathcal{O}_C) = G^{0,\Lambda_0}_{[C]}(\mathcal{O}_{A_g}\otimes\mathcal{O}_C)\,, \tag{3.6.19}$$

we arrive at the form

$$\begin{aligned}
&\mathcal{D}^B \sum_{[C]\le[B]} \left(\hbar\partial_g \mathcal{C}^C_{A_1\dots A_N}\right) L^{0,\Lambda_0}(\mathcal{O}_C) \\
&= \mathcal{D}^B \sum_{[C]\le[B]} L^{0,\Lambda_0}(\mathcal{O}_C)\frac{-1}{4!}\int \mathrm{d}^4y \\
&\times\left[\mathcal{C}^C_{A_gA_1\dots A_N} - \sum_{j=1}^N \sum_{[C']\le[A_j]} \mathcal{C}^{C'}_{A_gA_j}\mathcal{C}^C_{A_1\dots\widehat{A_j}C'\dots A_N} - \sum_{[C']<[C]} \mathcal{C}^{C'}_{A_1\dots A_N}\mathcal{C}^C_{A_gC'}\right].
\end{aligned} \tag{3.6.20}$$

Since this relation holds for any choice of index B, we can ascend inductively in [B]:

- Let $[B]=0$, i.e $B=\mathbb{1}$. The boundary conditions for the CAG's with one insertion then imply $\mathcal{D}^B L^{0,\Lambda_0}(\mathcal{O}_B)=1$, which immediately yields

$$\begin{aligned}
&\hbar\partial_g\mathcal{C}^B_{A_1\dots A_N} = \\
&-\int\frac{\mathrm{d}^4y}{4!}\left[\mathcal{C}^B_{A_gA_1\dots A_N} - \sum_{j=1}^N\sum_{[C']\le[A_j]}\mathcal{C}^{C'}_{A_gA_j}\mathcal{C}^B_{A_1\dots\widehat{A_j}C\dots A_N} - \sum_{[C']<[B]}\mathcal{C}^{C'}_{A_1\dots A_N}\mathcal{C}^B_{A_gC'}\right]
\end{aligned} \tag{3.6.21}$$

  in accordance with the theorem (here the sum over $[C']<[B]=0$ actually vanishes).

- Assume the theorem holds for all B with $[B] < D$. Pick a B' with $[B'] = D$. Then we obtain

$$
\begin{aligned}
&\mathfrak{D}^{B'} \sum_{[C]\leq[B']} \left(\hbar\partial_g \mathcal{C}^C_{A_1\ldots A_N}\right) L^{0,\Lambda_0}(\mathcal{O}_C) = \mathfrak{D}^{B'} \sum_{[C]<[B']} L^{0,\Lambda_0}(\mathcal{O}_C)\frac{-1}{4!}\int \mathrm{d}^4 y \\
&\times \left[\mathcal{C}^C_{A_g A_1\ldots A_N} - \sum_{j=1}^N \sum_{[C']\leq[A_j]} \mathcal{C}^{C'}_{A_g A_j}\mathcal{C}^C_{A_1\ldots\widehat{A_j}C\ldots A_N} - \sum_{[C']<[C]} \mathcal{C}^{C'}_{A_1\ldots A_N}\mathcal{C}^C_{A_g C'}\right] \\
&-\int\frac{\mathrm{d}^4 y}{4!}\left[\mathcal{C}^{B'}_{A_g A_1\ldots A_N} - \sum_{j=1}^N \sum_{[C']\leq[A_j]} \mathcal{C}^{C'}_{A_g A_j}\mathcal{C}^{B'}_{A_1\ldots\widehat{A_j}C\ldots A_N} - \sum_{[C']<[B']} \mathcal{C}^{C'}_{A_1\ldots A_N}\mathcal{C}^{B'}_{A_g C'}\right]
\end{aligned}
\tag{3.6.22}
$$

where we made use of the boundary conditions for the CAG's with one insertion in the last line, which imply for $[B] = [C]$ that $\mathfrak{D}^B L^{0,\Lambda_0}(\mathcal{O}_C) = \delta_{B,C}$. By assumption, we can apply theorem 4 on the left hand side for the terms with $[C] < [B']$, which yields exactly the same expressions as the sum over $[C] < [B']$ on the right hand side. Subtracting these contributions from both sides of equation (3.6.22), we are again left with the claim of the theorem, which closes the induction. □

3.6.3. Zeroth order OPE coefficients

We have seen in the previous section that it is possible to express the OPE coefficients in perturbation theory entirely in terms of the OPE coefficients of the corresponding free theory. The purpose of the present section is to give explicit formulae for these zeroth order coefficients, which we will denote by $(\mathcal{C}_0)^B_{A_1\ldots A_N}$. In the following we will set $\hbar = 1$ for the sake of simplicity, since we are not interested in the "loop-expansion" here.

Recall again our general definition of the OPE coefficients in terms of the amputated Green's functions:

$$
\mathcal{C}^B_{A_1\ldots A_N}(x_1,\ldots,x_N) = \mathfrak{D}^B\left\{\left(1-\sum_{j<\Delta}\mathbb{T}^j\right) G^{0,\Lambda_0}_{[B]-1}(\otimes_{i=1}^N \mathcal{O}_{A_i}(x_i))\right\} \tag{3.6.23}
$$

At zeroth perturbation order, one would expect to be able to obtain a more explicit formula. It will be our aim to derive such a formula in the following. Let us first introduce some notation. We find it most convenient to express the zeroth order coefficients diagrammatically. The following definition will be helpful:

Definition 4 (Decorated graphs): *Let $A_1, \ldots, A_N$ be multi-indices with $A_i = \{n_{A_i}, v_{A_i}\}$. By $\mathcal{P}(A_1, \ldots, A_N)$ we denote a collection of decorated graphs, which are characterised as follows:*

- *The elements of $\mathcal{P}(A_1, \ldots, A_N)$ are (not necessarily connected) graphs with N vertices, labelled A_i and of degree n_{A_i}, respectively.*
- *Every edge connects two different vertices. The total number of edges is thus $(n_{A_1} + \ldots + n_{A_N})/2$ (if $n_{A_1} + \ldots + n_{A_N}$ is odd, then $\mathcal{P}(A_1, \ldots, A_N) = \emptyset$).*
- *To every edge attached to the vertex A_i we associate one of the indices $v \in \mathbb{N}^4$ from the multi-index $v_{A_i} \in \mathbb{N}^{4n_{A_i}}$. This process is required to be "injective", in the sense that no index is associated to more than one edge. In this way, all the indices from $v_{A_1}, \ldots, v_{A_N}$ are distributed over the edges, and every edge is decorated with two indices $(v, w) \in (\mathbb{N}^4 \times \mathbb{N}^4)$.*

For any such graph $\mathbf{P} \in \mathcal{P}$, we further denote by $\mathcal{E}(\mathbf{P})$ the set of all decorations attached to edges in $\mathbf{P}$, i.e.

$$\mathcal{E}(\mathbf{P}) := \{(v, w) \in (\mathbb{N}^4 \times \mathbb{N}^4) \,|\, (v, w) \text{ is the decoration of an edge in } \mathbf{P}\}\,. \tag{3.6.24}$$

An example graph is given in fig.3.3 below. The set of decorations corresponding to the graph displayed in that figure is

$$\mathcal{E}(\mathbf{P}) = \{(v^2_{A_1}, v^2_{A_2}), (v^1_{A_1}, v^1_{A_2}), (v^3_{A_1}, v^1_{A_3}), (v^2_{A_3}, v^3_{A_2}), (v^1_{A_4}, v^1_{A_5})\}\,. \tag{3.6.25}$$

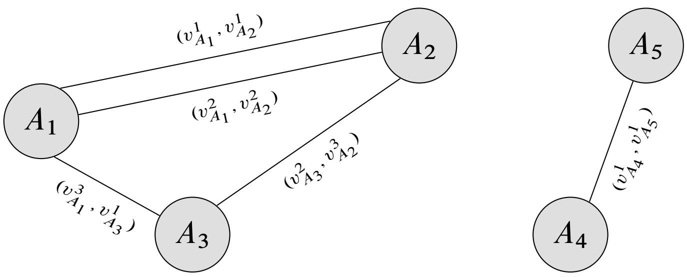


Figure 3.3.: A graph $\mathbf{P} \in \mathcal{P}(A_1, \ldots, A_5)$ with $n_{A_1} = 3 = n_{A_2}, n_{A_3} = 2$ and $n_{A_4} = 1 = n_{A_5}$.

In view of equation (3.6.23), the following lemma will allow us to derive an explicit formula for the zeroth order OPE coefficients:

Lemma 4: *The amputated Green's functions with insertions at zeroth perturbation order, $(G_0)^{0,\infty}(\otimes_{i=1}^N \mathcal{O}_{A_i})$, are given by the formula*[12]

$$\mathcal{D}^B (G_0)^{0,\infty}(\otimes_{i=1}^N \mathcal{O}_{A_i}(x_i)) = \sum_{\mathrm{P}\in\mathcal{P}(A_1,\ldots,A_N,B)} \prod_{\substack{(v,w)\in\mathcal{E}(\mathrm{P})\\ v,w\notin v_B}} f_{v,w}\big((x_l)_{v\in v_{A_l}},(x_k)_{w\in v_{A_k}}\big) \prod_{\substack{(v,w)\in\mathcal{E}(\mathrm{P})\\ w\in v_B}} g_{v,w}\big((x_l)_{v\in v_{A_l}},0\big) \tag{3.6.26}$$

where we used the notation ($v,w\in\mathbb{N}^4$)

$$f_{v,w}(x_1,x_2) := \partial^v_{x_1}\partial^w_{x_2}\hat{C}^{0,\infty}(x_1-x_2) \tag{3.6.27}$$

$$g_{v,w}(x_1,x_2) := \prod_{\mu=1}^4 \theta(w_\mu - v_\mu)(x_1-x_2)_\mu^{w_\mu - v_\mu}/(w_\mu - v_\mu)!\,. \tag{3.6.28}$$

Here θ is the Heavyside step-function (with convention $\theta(0)=1$) and $\hat{C}^{0,\infty}(x)$ is the coordinate space propagator (without cutoffs), i.e.

$$\hat{C}^{0,\infty}(x) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\frac{\mathrm{e}^{ipx}}{p^2+m^2}\,. \tag{3.6.29}$$

Proof. To begin with, recall equation (3.1.30) (we do not expand in powers of $\hbar$ here),

$$\Big\langle \prod_{i=1}^N \mathcal{O}_{A_i}(x_i) \prod_{j=1}^n \hat{\varphi}(p_i)\Big\rangle \prod_{k=1}^n (C^{\Lambda,\Lambda_0}(p_k))^{-1}$$
$$= \sum_{j=1}^n \sum_{\substack{I_1\cup\ldots\cup I_j=\{1,\ldots,n\}\\ I_a\cap I_b=\emptyset}} \mathcal{G}^{\Lambda,\Lambda_0}_{|I_1|}(\otimes_{i=1}^N \mathcal{O}_{A_i}(x_i),\vec{p}_{I_1})\,\bar{\mathcal{L}}^{\Lambda,\Lambda_0}_{|I_2|}(\vec{p}_{I_2})\cdots\bar{\mathcal{L}}^{\Lambda,\Lambda_0}_{|I_j|}(\vec{p}_{I_j})\,, \tag{3.6.30}$$

where $\bar{\mathcal{L}}^{\Lambda,\Lambda_0}_n$ are the expansion coefficients of the generating functional $\bar{L}^{\Lambda,\Lambda_0}(\varphi) = -L^{\Lambda,\Lambda_0}(\varphi)+\frac{1}{2}\langle\varphi,\,(C^{\Lambda,\Lambda_0})^{-1}\star\varphi\rangle$ without the momentum conservation delta functions taken out. It follows by definition that the CAG's without insertion vanish at zeroth perturbation order (see e.g. [46]). Therefore, the r.h.s. of eq.(3.6.30) simplifies in the case

[12] By $v\in v_A$ we mean that the index $v\in\mathbb{N}^4$ is part of the multi-index $v_A\in\mathbb{N}^{4n_A}$.

at hand. We obtain:

$$\Big\langle \prod_{i=1}^{N} \mathcal{O}_{A_i}(x_i) \prod_{j=1}^{n} \hat{\varphi}(p_i) \Big\rangle_{\text{0th-order}} \prod_{k=1}^{n} (C^{0,\infty}(p_k))^{-1}$$
$$= \sum_{\substack{I_1 \cup I_2 = \{1,\dots,n\} \\ I_1 \cap I_2 = \emptyset}} (\mathcal{G}_0)^{0,\infty}_{|I_1|}(\otimes_{i=1}^{N} \mathcal{O}_{A_i}(x_i), \vec{p}_{I_1}) \prod_{\substack{(i,j)\in I_2 \times I_2 \\ i \neq j}} \delta(p_i - p_j)\,(C^{0,\infty}(p_i))^{-1} \tag{3.6.31}$$

Here $\delta(p)$ is the Dirac delta function. We read off that $(\mathcal{G}_0)^{0,\infty}_n(\otimes_{i=1}^{N}\mathcal{O}_{A_i}; \vec{p})$ is simply the contribution to the zeroth order amputated Schwinger function $\langle \prod_{i=1}^{N} \mathcal{O}_{A_i}(x_i) \prod_{j=1}^{n} \hat{\varphi}(p_i) \rangle$ that contains no contractions between the spectator fields $\hat{\varphi}$. To determine the l.h.s. of eq.(3.6.31), we can use standard methods for non-interacting quantum fields. In particular, we can apply *Wick's theorem*. It is then not hard to convince oneself that the corresponding Feynman diagrams are precisely the graphs in $\mathcal{P}(A_1, \dots, A_N, B)$ with $B = \{n, v = 0\}$: As usual in Feynman diagram expansions, each edge in these diagrams corresponds to a propagator, and the labels on the edges correspond to spacetime derivatives. The edges connected to the vertex B represent the Wick-contractions with the spectator fields $\hat{\varphi}(p_i)$. Note that these propagators are amputated by the factor $\prod_{k=1}^{n}(C^{0,\infty}(p_k))^{-1}$ on the l.h.s. of (3.6.31). Therefore, any edge that connects the vertex A_i to B contributes a factor $\partial^v_{x_i} e^{ip_j x_i}$ instead of a propagator, where $v \in v_{A_i}$ and $v_{B,j} = 0$ are the decorations of the edge. If we take momentum derivatives $\partial^w_{\vec{p}}$ on both sides of eq.(3.6.31), we find that, in summary, $\partial^w_{\vec{p}}(\mathcal{G}_0)^{0,\infty}_n(\otimes_{i=1}^{N}\mathcal{O}_{A_i}; \vec{p})$ is given by the r.h.s. of equation (3.6.26) with $B = \{n, w\}$, but with $g_{v,w}$ replaced by $\partial^w_{p_j}\partial^v_{x_i} e^{ip_j x_i}$. Setting $\vec{p} = \vec{0}$, dividing by $w!$ and multiplying by $(-i)^{|w|}$, we find that, indeed, $\mathcal{D}^B(G_0)^{0,\infty}_n(\otimes_{i=1}^{N}\mathcal{O}_{A_i})$ is given exactly by the r.h.s. of (3.6.26). □

We are now ready to state our result for the zeroth order OPE coefficients:

Lemma 5: *The OPE coefficients of the free, scalar quantum field in four dimensions are given by*

$$(\mathcal{C}_0)^B_{A_1\dots A_N}(x_1, \dots, x_N) = \mathcal{D}^B(G_0)^{0,\infty}(\otimes_{i=1}^{N}\mathcal{O}_{A_i}(x_i))$$
$$= \sum_{P \in \mathcal{P}(A_1,\dots,A_N,B)} \left(\prod_{\substack{(v,w)\in\mathcal{E}(P) \\ v,w \notin v_B}} f_{v,w}\big((x_l)_{v\in v_{A_l}}, (x_k)_{w\in v_{A_k}}\big) \right) \left(\prod_{\substack{(v,w)\in\mathcal{E}(P) \\ w \in v_B}} g_{v,w}\big((x_l)_{v\in v_{A_l}}, x_N\big) \right) \tag{3.6.32}$$

where f, g are the functions defined in eqs.(3.6.27) and (3.6.28).

Remark 15: The lemma suggests the following procedure to obtain the OPE coefficients $(\mathcal{C}_0)^B_{A_1\dots A_N}$ in terms of graphs:

- Draw all decorated graphs $\mathbf{P} \in \mathcal{P}(A_1, \dots, A_N, B)$ as explained in definition 4.

- For a given graph $\mathbf{P}$, associate the function $f_{v,w}(x_i, x_j)$ to edges connecting the vertices A_i and A_j with decorations v and w, and associate $g_{v,w}(x_i, x_N)$ to every edge connected to the vertices A_i and B with decorations $v \in v_{A_i}$ and $w \in v_B$.

- Each graph contributes to the OPE coefficient the product of these functions associated to its edges. The contributions from all the different graphs in $\mathcal{P}$ are summed over.

Proof. Recall from equation (3.5.4) that we may generally write the OPE coefficients as

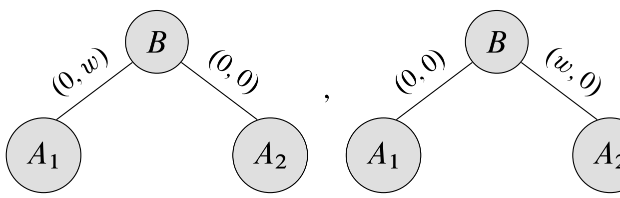
$$\mathcal{C}^B_{A_1 \dots A_N}(x_1, \dots, x_N) = \mathcal{D}^B \left\{ G^{0,\infty}(\otimes_{i=1}^N \mathcal{O}_{A_i}(x_i)) - \sum_{[C]<[B]} \mathcal{C}^C_{A_1 \dots A_N}(x_1, \dots, x_N) L^{0,\infty}(\mathcal{O}_C(x_N)) \right\} . \tag{3.6.33}$$

If we assume $x_N = 0$ for the moment, then equation (3.6.32) follows directly from eq.(3.6.33) and lemma 4. Namely, in that case the expression $\mathcal{D}^B(L_0)^{0,\infty}(\mathcal{O}_C(0))$ with $[C] < [B]$ vanishes according to lemma 4 (the CAG's with one insertion are covered by the case $N = 1$ in that lemma), and the remaining contribution $\mathcal{D}^B(G_0)^{0,\infty}(\otimes_{i=1}^N \mathcal{O}_{A_i})$ yields exactly the r.h.s. of eq.(3.6.32). The general case, $x_N \neq 0$, then follows simply from the inherent translation invariance of the OPE coefficients. □

To conclude this section, we give a few examples of zeroth order OPE coefficients, which should help to clarify the graphical definition above:

- Let $\mathcal{O}_{A_1} = \varphi$, $\mathcal{O}_{A_2} = \varphi$ and $\mathcal{O}_B = \varphi\, \partial^w \varphi$. We will refer to the coefficients $(\mathcal{C}_0)^B_{A_1 A_2}$ also as $(\mathcal{C}_0)^{\varphi\, \partial^w \varphi}_{\varphi\, \varphi}$. The graphs in $\mathcal{P}(A_1, A_2, B)$ are in the case at hand[13]

$\mathcal{P}(A_1, A_2, B) = \Big\{$



$\Big\}$ (3.6.34)

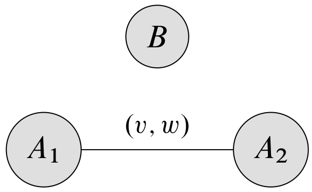
Translating these graphs into equations as defined in eq.(3.6.32), we obtain

$$(\mathcal{C}_0)^{\varphi\, \partial^w \varphi}_{\varphi\, \varphi}(x_1, x_2) = g_{0,w}(x_1, x_2) + g_{0,w}(x_2, x_2) = \frac{(x_1 - x_2)^w}{w!} \tag{3.6.35}$$

[13]Note that in the case $w = 0$ the two diagrams in eq.(3.6.34) coincide. Thus, the set $\mathcal{P}(A_1, A_2, B)$ contains only one element in that case, and we find $(\mathcal{C}_0)^{\varphi^2}_{\varphi\,\varphi} = 1$.

- Let $\mathcal{O}_{A_1} = \partial^v \varphi$, $\mathcal{O}_{A_2} = \partial^w \varphi$ and $\mathcal{O}_B = \mathbb{1}$, i.e. we are now interested in the coefficient $(\mathcal{C}_0)_{\partial^v\varphi\,\partial^w\varphi}^{\mathbb{1}}$. This time, the set $\mathcal{P}(A_1, A_2, B)$ is given by

$$\mathcal{P}(A_1, A_2, B) = \Bigg\{$$



$$\Bigg\} \tag{3.6.36}$$

which implies for the corresponding OPE coefficient

$$(\mathcal{C}_0)_{\partial^v\varphi\,\partial^w\varphi}^{\mathbb{1}}(x_1, x_2) = \partial_{x_1}^v \partial_{x_2}^w \hat{C}^{0,\infty}(x_1 - x_2) \tag{3.6.37}$$

- Let $\mathcal{O}_{A_1} = \varphi^n$,$\mathcal{O}_{A_2} = \varphi^m$ and $\mathcal{O}_B = \varphi^{n+m-2k}$. By straightforward combinatorics, one checks

$$(\mathcal{C}_0)_{\varphi^n\,\varphi^m}^{\varphi^{n+m-2k}}(x_1, x_2) = \frac{n!m!}{(n-k)!(m-k)!k!}[\hat{C}^{0,\infty}(x_1 - x_2)]^k \tag{3.6.38}$$

If instead we had $\mathcal{O}_B = \varphi^{n+m-2k+1}$, we would find

$$(\mathcal{C}_0)_{\varphi^n\,\varphi^m}^{\varphi^{n+m-2k+1}}(x_1, x_2) = 0\,, \tag{3.6.39}$$

as in this case $n_{A_1} + n_{A_2} + n_B$ is odd, which implies $\mathcal{P}(A_1, A_2, B) = \emptyset$.

We see that our graphical notation reproduces the expected results for simple zeroth order OPE coefficients (see for example [22, 39]).

3.6.4. Examples of first order OPE coefficients

The OPE coefficients at first perturbation order can now be determined from the zeroth order ones with the help of our integral formula, eq.(3.6.15). To give the reader an impression of this algorithm, we will calculate a few simple examples below.

We note that in the case at hand we can, in fact, remove the cutoff under the integral sign in equation (3.6.15). This can be seen from the fact that at zeroth order the OPE coefficients $(\mathcal{C}_0)_{A_1 A_2}^B(x)$ have the scaling behaviour $|x|^{[B]-[A_1]-[A_2]}$, which implies that the integral in (3.6.15) converges absolutely, as discussed in remark 14.

The coefficient $(C_1)_{\varphi\,\varphi}^{\varphi^4}$ **:** Equation (3.6.15) allows us to write this coefficient as

$$(\mathcal{C}_1)_{\varphi\,\varphi}^{\varphi^4}(x_1,x_2) = \frac{-1}{4!}\int \mathrm{d}^4 y\,\Bigg[(\mathcal{C}_0)_{\varphi^4\,\varphi\,\varphi}^{\varphi^4}(y,x_1,x_2) - \sum_{[C]\le 1}(\mathcal{C}_0)_{\varphi^4\,\varphi}^{\mathcal{O}_C}(y,x_1)\,(\mathcal{C}_0)_{\mathcal{O}_C\,\varphi}^{\varphi^4}(x_1,x_2) - \sum_{[C]\le 1}(\mathcal{C}_0)_{\varphi^4\,\varphi}^{\mathcal{O}_C}(y,x_2)\,(\mathcal{C}_0)_{\varphi\,\mathcal{O}_C}^{\varphi^4}(x_1,x_2) - \sum_{[C]<4}(\mathcal{C}_0)_{\varphi\,\varphi}^{\mathcal{O}_C}(x_1,x_2)\,(\mathcal{C}_0)_{\varphi^4\,\mathcal{O}_C}^{\varphi^4}(y,x_2)\Bigg]. \tag{3.6.40}$$

On the right hand side we can now substitute our explicit expression for the zeroth order coefficients, see lemma 5. We find that many terms in the sums over C in equation (3.6.40) actually vanish. First, note that the condition $[C]\le 1$ restricts $\mathcal{O}_C$ to be either $\mathbb{1}$ or φ. In either case, our results of the previous section imply that the coefficient $(\mathcal{C}_0)_{\varphi^4\,\varphi}^{\mathcal{O}_C}$ vanishes. Concerning the last term on the r.h.s. of eq.(3.6.40), the results of the previous section suggest that only the terms with $\mathcal{O}_C=\mathbb{1}, \mathcal{O}_C=\varphi^2$ and $\mathcal{O}_C=\varphi\partial_\mu\varphi$ are non-zero. Hence, we arrive at the simpler equation

$$(\mathcal{C}_1)_{\varphi\,\varphi}^{\varphi^4}(x_1,x_2) = \frac{-1}{4!}\int \mathrm{d}^4 y\,\Bigg[(\mathcal{C}_0)_{\varphi^4\,\varphi\,\varphi}^{\varphi^4}(y,x_1,x_2) - (\mathcal{C}_0)_{\varphi\,\varphi}^{\mathbb{1}}(x_1,x_2)\,(\mathcal{C}_0)_{\varphi^4\,\mathbb{1}}^{\varphi^4}(y,x_2) - \sum_{\mu=1}^{4}(\mathcal{C}_0)_{\varphi\,\varphi}^{\varphi\partial_\mu\varphi}(x_1,x_2)\,(\mathcal{C}_0)_{\varphi^4\,\varphi\partial_\mu\varphi}^{\varphi^4}(y,x_2) - (\mathcal{C}_0)_{\varphi\,\varphi}^{\varphi^2}(x_1,x_2)\,(\mathcal{C}_0)_{\varphi^4\,\varphi^2}^{\varphi^4}(y,x_2)\Bigg]. \tag{3.6.41}$$

It is now an easy exercise to compute the zeroth order coefficients on the r.h.s. of this equation. Our final result is

$$(\mathcal{C}_1)_{\varphi\,\varphi}^{\varphi^4}(x_1,x_2) = -\frac{\int \mathrm{d}^4 y}{4!}\Big[4\hat{C}^{0,\infty}(x_1-y) + 4\hat{C}^{0,\infty}(x_2-y) + \hat{C}^{0,\infty}(x_1-x_2) - \hat{C}^{0,\infty}(x_1-x_2) - 4(x_1-x_2)^\mu\partial_\mu\hat{C}^{0,\infty}(x_2-y) - 8\hat{C}^{0,\infty}(x_2-y)\Big] = 0 \tag{3.6.42}$$

The coefficient $(C_1)_{\varphi\,\varphi}^{\varphi^2}$ **:** We again use eq.(3.6.15) to write

$$(\mathcal{C}_1)_{\varphi\,\varphi}^{\varphi^2}(x_1,x_2) = \frac{-1}{4!}\int \mathrm{d}^4 y\,\Bigg[(\mathcal{C}_0)_{\varphi^4\,\varphi\,\varphi}^{\varphi^2}(y,x_1,x_2) - \sum_{[C]\le 1}(\mathcal{C}_0)_{\varphi^4\,\varphi}^{\mathcal{O}_C}(y,x_1)\,(\mathcal{C}_0)_{\mathcal{O}_C\,\varphi}^{\varphi^2}(x_1,x_2) - \sum_{[C]\le 1}(\mathcal{C}_0)_{\varphi^4\,\varphi}^{\mathcal{O}_C}(y,x_2)\,(\mathcal{C}_0)_{\varphi\,\mathcal{O}_C}^{\varphi^2}(x_1,x_2) - \sum_{[C]<2}(\mathcal{C}_0)_{\varphi\,\varphi}^{\mathcal{O}_C}(x_1,x_2)\,(\mathcal{C}_0)_{\varphi^2\,\mathcal{O}_C}^{\varphi^2}(y,x_2)\Bigg]. \tag{3.6.43}$$

Using the results of the previous section, one verifies that the sums over C on the right hand side actually vanish. Our result for this coefficient is

$$
\begin{aligned}
(\mathcal{C}_1)_{\varphi\varphi}^{\varphi^2}(x_1,x_2) &= -\int \frac{\mathrm{d}^4 y}{4!}(\mathcal{C}_0)_{\varphi^4\varphi\varphi}^{\varphi^2}(y,x_1,x_2) = -\int \frac{\mathrm{d}^4 y}{2}\hat{C}^{0,\infty}(x_1-y)\hat{C}^{0,\infty}(x_2-y) \\
&= \frac{-1}{16\pi^2}K_0(\sqrt{(x_1-x_2)^2m^2}) = \frac{1}{16\pi^2}\left[\log(\sqrt{(x_1-x_2)^2m^2}/2)+\Gamma_{\mathrm{E}}\right]+\dots\,,
\end{aligned}
\tag{3.6.44}
$$

where K_0 is a modified Bessel function of the second kind, see eq.(B.1.2) in the appendix, and where Γ_{E} is the *Euler-Mascheroni constant*. This particular coefficient is the standard example computed in the textbooks [7, 3]. While these authors perform their calculations on Minkowski space, it is not hard to translate their results into the Euclidean context via Wick rotation. The explicit form of this coefficient given in the standard reference [7, p. 262] then corresponds to

$$
(\mathcal{C}_1)_{\varphi\varphi}^{\varphi^2}(x_1,x_2) = \frac{1}{16\pi^2}\cdot\frac{1}{2}\left[\Gamma_{\mathrm{E}}+\log(\pi^2\mu^2(x_1-x_2)^2)\right] \tag{3.6.45}
$$

where μ is a free renormalisation parameter. The appearance of an extra free parameter in this result as compared to ours, eq.(3.6.44), can be traced back to the fact that we have fixed our renormalisation conditions through our choice of boundary conditions for the CAG's. It is crucial, however, that both results agree on the leading short distance behaviour of the coefficient, and we find that this is indeed the case.

The coefficient $(C_1)_{\varphi\varphi^3}^{\varphi^2}$: We use the same strategy as above.

$$
\begin{aligned}
(\mathcal{C}_1)_{\varphi\varphi^3}^{\varphi^2}(x_1,x_2) = -\int \frac{\mathrm{d}^4 y}{4!}\Bigg[&(\mathcal{C}_0)_{\varphi^4\varphi\varphi^3}^{\varphi^2}(y,x_1,x_2) - \sum_{[C]\le 1}(\mathcal{C}_0)_{\varphi^4\varphi}^{\mathcal{O}_C}(y,x_1)\,(\mathcal{C}_0)_{\mathcal{O}_C\varphi^3}^{\varphi^2}(x_1,x_2) \\
&- \sum_{[C]\le 3}(\mathcal{C}_0)_{\varphi^4\varphi^3}^{\mathcal{O}_C}(y,x_2)\,(\mathcal{C}_0)_{\varphi\mathcal{O}_C}^{\varphi^2}(x_1,x_2) - \sum_{[C]<2}(\mathcal{C}_0)_{\varphi\varphi^3}^{\mathcal{O}_C}(x_1,x_2)\,(\mathcal{C}_0)_{\varphi^2\mathcal{O}_C}^{\varphi^2}(y,x_2)\Bigg].
\end{aligned}
\tag{3.6.46}
$$

Dropping vanishing terms in the summations, we obtain the formula

$$
\begin{aligned}
(\mathcal{C}_1)_{\varphi\varphi^3}^{\varphi^2}(x_1,x_2) = \frac{-1}{4!}\int \mathrm{d}^4 y\,\Bigg[&(\mathcal{C}_0)_{\varphi^4\varphi\varphi^3}^{\varphi^2}(y,x_1,x_2) - (\mathcal{C}_0)_{\varphi^4\varphi^3}^{\varphi}(y,x_2)\,(\mathcal{C}_0)_{\varphi\varphi}^{\varphi^2}(x_1,x_2) \\
&- (\mathcal{C}_0)_{\varphi^4\varphi^3}^{\varphi^3}(y,x_2)\,(\mathcal{C}_0)_{\varphi\varphi^3}^{\varphi^2}(x_1,x_2)\Bigg].
\end{aligned}
\tag{3.6.47}
$$

Substituting the explicit form of the zeroth order coefficients on the right hand side, we arrive at the result

$$(\mathcal{C}_1)_{\varphi\varphi^3}^{\varphi^2}(x_1, x_2) = -3 \int \mathrm{d}^4 y \, \Big[(\hat{C}^{0,\infty}(y - x_2))^2 \left(\hat{C}^{0,\infty}(y - x_1) - \hat{C}^{0,\infty}(x_1 - x_2)\right)\Big] \tag{3.6.48}$$

We do not bother here to compute the remaining loop integral. An interesting point about this particular coefficient is that the integral over y diverges (in the UV-region $y \to x_2$) if we integrate over the two summands on the right hand side separately. It is not hard to check, however, that the divergent contributions cancel between the two terms, so that the total expression on the right side is indeed finite, as it should be. Here we see in a concrete example how the sums over C, which are subtracted on the right hand side of our perturbation formula (3.6.15), act as counter-terms, which guarantee (UV-) finiteness.

4

The OPE, non-perturbative effects and Dark Energy

One of the most significant cosmological discoveries in recent history has been the observation that the expansion of the universe is *accelerating*. This finding was based on observational data of Supernovae accumulated up to the year 1998 [51, 52], and it has been confirmed by various other experiments since then. The common explanation for such an accelerated expansion is that there has to be some, as of yet unknown, form of energy, dubbed now as "Dark Energy", exerting a negative vacuum pressure. Recent high precision measurements imply that Dark Energy constitutes about 70% of the total energy in the universe [53].

The simplest and most natural candidate for Dark Energy is the *cosmological constant* Λ_{Cosm}. Originally, this quantity was introduced by Einstein in 1917, shortly after his formulation of the theory of General Relativity, as an additional parameter to model a static universe. When in the 1920s observations of recession speeds of distant galaxies suggested that the universe is actually *expanding*, the cosmological constant appeared to be a superfluous and inelegant modification of Einstein's equations. This led to Einstein's famous assertion that the idea may have been his "biggest blunder". Interest in the cosmological constant was re-invigorated, however, after the mentioned 1998 discovery.

The cosmological constant can be interpreted as *energy density* of the universe. Therefore, it should be given by the expectation value of the quantum field theoretic stress energy operator, $\langle T_{\mu\nu}\rangle$, of the Standard Model of particle physics. The quantum state should in principle contain the approximately 10^{80} hadronic particles in the universe distributed onto stars, galaxies, dust clouds, etc. But for the problem at hand, we are not really interested in the detailed functional form of $\langle T_{\mu\nu}\rangle$ on smaller scales arising from these features, but rather in the contribution from the vacuum itself, in particular since the universe is mostly empty. Hence, one may take the state to be the vacuum state. Also, although our universe is expanding, its expansion rate is so small compared to the scales occurring in particle physics that we may safely do our analysis in Minkowski spacetime. Since the Minkowski vacuum state is Poincare invariant, the vacuum expectation value (VEV) must automatically have the form $\langle T_{\mu\nu}\rangle = \rho_{\text{vac}}\,\eta_{\mu\nu}$ of a cosmological constant.

Rough estimates of ρ_{vac} within the Standard Model of particle physics, based on dimensional analysis, suggest a value of the order of magnitude $M^4_{\text{Higgs}} = (125\,\text{GeV})^4$. On the other hand, astrophysical measurements suggest an upper bound of $\rho_{\text{vac}} \leq (10^{-12}\text{GeV})^4$ [25]. This spectacular failure of theoretical prediction has been known for a long time as the *cosmological constant problem*. Prior to 1998 it therefore seemed most likely that there had to be some unknown mechanism which causes the vacuum energy to vanish. The discovery of Dark Energy, whose value is estimated to be of the order 10^{-47}GeV^4 by modern experiments [54, 55], makes this problem even more difficult to resolve, since it seems harder to imagine some mechanism which causes the vacuum energy to be extremely small, but non-zero. Nevertheless, many, and very diverse, possible explanations have been proposed in this direction, see e.g. [25] for a review. Many of these proposals involve highly speculative features such as hypothetical new fields or dynamical mechanisms that have neither been observed, nor have been explored thoroughly from the theoretical viewpoint.

In this thesis we will follow a strategy, put forward by Hollands and Wald [13], which may provide a very natural explanation for the small value of the cosmological constant. To put the general idea into context, let us first elaborate on the main difficulties one has to face when trying to determine $\langle T_{\mu\nu}\rangle$:

1. The huge complexity of the Standard Model, and in particular the difficulty of making non-perturbative calculations.

2. The fact that, as is well-known, 'the' stress energy operator, like any other composite operator in quantum field theory, is an intrinsically ambiguous object.

To circumvent the problems associated with point one, we will consider a simpler toy model in this thesis, which allows us to substantiate the proposal of [13] in a clean setup. Ideally, our toy model should be mathematically tractable and at the same time display some of the non-perturbative effects characteristic of the Standard Model. One of the few candidate theories fitting these requirements is the *Gross-Neveu* model in two dimensions [14], which will be our toy model of choice in the following[1].

Our focus in this thesis will be on the second problem mentioned above, i.e. on the renormalisation ambiguities affecting the computation of the vacuum energy within quantum field theory. In order to be able to predict the value of $\langle T_{\mu\nu}\rangle$, one inevitably has to introduce some additional, reasonable conditions to reduce these ambiguities. The key idea put forward in [13] is to impose the requirement that the OPE coefficients in quantum field theory should depend analytically on the coupling constant(s) of the model. As we will explain in more detail in the next section, this condition has the potential to permit unique predictions for the non-perturbative contributions to expectation values of quantum field observables. It was speculated in [13] that, in the presence of *exponentially small* non-perturbative effects, which occur quite typically in quantum field theory, these effects could account for a deviation from the naive dimensional analysis result by a dimensionless, exponentially small factor. Potentially, this could explain the unexpectedly small observed value of the cosmological constant.

It is our aim to apply the strategy proposed in [13] to the Gross-Neveu model in a rigorous fashion. As a result, we obtain a unique prediction for the vacuum expectation value of the stress-energy operator to leading order in the "large flavour expansion" (see below). The corresponding vacuum energy density is found to be of the order $\rho_{\text{vac}} \sim \mathrm{e}^{-2\pi/g^2}$, where g is the coupling constant of the model. The results presented in this chapter are based on the paper [27] by Hollands and the present author.

The present chapter is organised as follows: First, we will elaborate on the general idea behind the proposed analyticity condition in section 4.1. In section 4.2 we will then apply this strategy to our specific toy model, the Gross-Neveu model. Finally, we will discuss possible implications of our findings on more realistic cosmological models in section 4.3.

4.1 General Idea: Restriction of renormalisation ambiguities

We have already encountered in the previous chapters that in order to give a sensible definition of composite operators in renormalised quantum field theory, one inevitably has to make some choice of *renormalisation conditions*. It was mentioned at the end of section 2.1 that this ambiguity can be expressed in terms of field redefinitions

$$\mathcal{O}_A(x) \to Z_A^B \cdot \mathcal{O}_B(x)\,, \tag{4.1.1}$$

[1] We expect very similar results to hold also for the two dimensional O(N) sigma model, treated along the lines of [56]

where $\mathcal{O}_A, \mathcal{O}_B$ are composite operators and Z_A^B is a matrix of complex numbers. Recall that in the flow equation framework of the previous chapter, the ambiguity in the definition of composite operators corresponds to our freedom in the choice of the boundary conditions (3.1.27). As we have mentioned previously in section 2.1, this mixing matrix Z_A^B is generally restricted by various conditions, such as for example consistency with symmetry properties and scaling behaviour. In the more specific setting of a Lagrangian theory, additional conditions arise through field equations and conservation laws.

Let us consider an example to illustrate the effect of these restrictions. Let $\mathcal{O}_A$ be a conserved current, called J^μ, associated with a symmetry of the theory. If there is no other conserved current in the theory, then the only possible field redefinition is $J^\mu \to Z\, J^\mu$ for $Z \in \mathbb{R}$. The corresponding conserved charge $Q = \int J^0 \mathrm{d}^3x$ should furthermore generate the symmetry, $[Q, \mathcal{O}_B(x)] = i q_B \mathcal{O}_B(x)$, where q_B is the charge quantum number of the operator $\mathcal{O}_B$. Since q_B is fixed, we must have $Z = 1$ in this example. Thus, the current J^μ is uniquely defined as an operator.

In this thesis we are mainly interested in the stress energy operator $T_{\mu\nu}$. A crucial requirement on this tensor is that it should be conserved, $\partial^\mu T_{\mu\nu} = 0$, which implies that the stress energy operator can only mix with other conserved operators that are also symmetric tensors. A possible field redefinition is then

$$T_{\mu\nu} \to Z\, T_{\mu\nu} + c\, \eta_{\mu\nu}\, \mathbb{1}\,, \tag{4.1.2}$$

where $\mathbb{1}$ is the identity operator and c is a dimensionful constant (if the Lagrangian of the theory contains only one mass parameter M, and if we consider four spacetime dimensions, then $c \propto M^4$). As above, we argue that $Z = 1$. This is due to the fact the we want the operator $P_\mu = \int T^0{}_\mu \mathrm{d}^3x$ to generate translations, i.e. $[P_\mu, \mathcal{O}_B(x)] = i\,\partial_\mu \mathcal{O}_B(x)$, which requires $Z = 1$. Unfortunately, however, the constant c remains unconstrained. If we normalise $\langle \mathbb{1} \rangle = 1$, our remaining freedom to redefine $\langle T_{\mu\nu} \rangle \to \langle T_{\mu\nu} \rangle + c\eta_{\mu\nu}$ allows us to set the vacuum energy ρ_{vac} to any value we want. In the presence of a coupling constant g, we can even make the vacuum energy into an arbitrary smooth function of g.

In the usual areas of application for quantum field theory, i.e. particle- or condensed matter physics, this ambiguity in the value of $\langle T_{\mu\nu} \rangle$ is not a serious problem, since one generally measures only *differences* of expectation values between different states (like e.g. vacuum and many-particle states). Gravity, however, is sensitive to the *absolute* value of $\langle T_{\mu\nu} \rangle$ via Einstein's equation. In order to apply quantum field theory also to cosmological questions, it therefore appears to be necessary to impose additional conditions of some kind to reduce the mentioned ambiguities in the definition of composite operators.

In the present work, we will study one such condition, which was first proposed in [13]. Very much in the spirit of the approach to quantum field theory outlined in chapter 2, this additional restriction is stated as a requirement on the OPE coefficients. In view of the general form of the operator product expansion,

$$\left\langle \mathcal{O}_A(x)\mathcal{O}_B(0)\prod_i \phi(z_i)\right\rangle \sim \mathcal{C}_{AB}^C(x)\left\langle \mathcal{O}_C(0)\prod_i \phi(z_i)\right\rangle \tag{4.1.3}$$

it is quite clear that a field redefinition (4.1.1) would cause a corresponding transformation of the OPE coefficients. If our theory contains a coupling constant g, then this redefinition $Z_A^B(g)$, as well as the OPE coefficients $\mathcal{C}_{AB}^C(x,g)$, will depend (smoothly) on this coupling constant. In this setting, the proposal of [13] is as follows:

Suppose there exists a definition of the composite operators such that $\mathcal{C}_{AB}^C(x,g)$ is an analytic function of g, i.e. has a convergent Taylor expansion in g for small, but finite, g. Then we allow only field redefinitions $Z_A^B(g)$ preserving this property, i.e. ones which are likewise analytic in g.

The proposal appears to be quite natural. As a consequence, we would, for example, only be allowed to make a redefinition with an analytic function $c(g)$ in eq.(4.1.2). Such a redefinition could not cancel any non-analytic (= non-perturbative) contributions to the vacuum energy. In other words, if the VEV of the stress energy operator contains non-perturbative contributions, then these will in fact be unique. This procedure could in principle allow for an unambiguous prediction of a non-perturbatively small value for the vacuum energy. We will substantiate this idea in the following section within the Gross-Neveu model.

Remark 16: The methods employed within the present chapter differ from those of the previous one. Namely, we will not use the flow equation approach to the theory in the following. Instead, we will make use of standard Feynman diagram expansions combined with the so called *large flavour* (or also $1/N$-) expansion.

4.2 The Gross-Neveu model

The quantum field theory described by the Lagrangian (here $\lambda \in \mathbb{R}$)

$$\mathcal{L}_{\mathrm{GN}} = i\,\overline{\psi}\partial\!\!\!/\psi + \frac{\lambda^2}{2}\,(\overline{\psi}\psi)^2 \tag{4.2.1}$$

on two dimensional Minkowski space[2] is known as the *Gross-Neveu model*. Here

$$\overline{\psi} = (\overline{\psi}_1, \ldots, \overline{\psi}_N) , \qquad \psi = \begin{pmatrix} \psi^1 \\ \vdots \\ \psi^N \end{pmatrix} \tag{4.2.2}$$

are row/column vectors of N flavours of a 2-component spinor field, and we use the standard 'slash'-notation, $\partial\!\!\!/ = \gamma_\mu \partial^\mu$ (our conventions for gamma matrices, metric signature, etc. can be found in the notation and conventions section at the beginning of this thesis). The model was the first example of a quantum field theory featuring the phenomenon of *dynamical symmetry breaking* (see below). A crucial tool in the study of the Gross-Neveu model has been the $1/N$- or large flavour- expansion, which makes it possible to obtain non-perturbative information. Such non-perturbative results are otherwise hard to come by. The model has also been of interest due to the fact that it shares some basic features with quantum chromodynamics (QCD), namely asymptotic freedom and symmetry breaking. Our interest in this particular model is rooted in the fact that is sufficiently complex to feature interesting non-perturbative effects, but at the same time sufficiently simple to still be mathematically tractable.

For the purposes of this thesis, it will be more convenient to use a slightly modified version of the Lagrangian. Namely, we rescale the fields $\overline{\psi}, \psi \to \overline{\psi}/\sqrt{N}, \psi/\sqrt{N}$, and we introduce the t'Hooft coupling $g^2 := \lambda^2 N$. Within the large N expansion it is then understood that g is kept fixed while quantities of interest are expanded in powers of $1/N$. The Lagrangian used in this thesis therefore takes the form

$$\mathscr{L}_{\mathrm{GN}} = N \left[i\, \overline{\psi} \partial\!\!\!/ \psi + \frac{g^2}{2} (\overline{\psi}\psi)^2 \right] . \tag{4.2.3}$$

A crucial feature of the Gross-Neveu Lagrangian is its invariance under the discrete transformation

$$\psi \to \gamma_5 \psi , \tag{4.2.4}$$

which ensures masslessness of the Fermions (to any order in perturbation theory).

Our ultimate aim in this chapter is to determine the vacuum expectation value of the stress-energy operator in the Gross-Neveu model (to leading order in $1/N$). As a starting point, we will need the classical form of the stress-energy tensor *per flavour*, i.e. divided

[2] Recall from the notation and conventions section that $\mu, \nu \in \{0, 1\}$ denote Minkowski space indices from now on.

by N:

$$T_{\mu\nu} = \frac{i}{4}\left(\overline{\psi}\gamma_\mu\partial_\nu\psi + \overline{\psi}\gamma_\nu\partial_\mu\psi - \partial_\nu\overline{\psi}\gamma_\mu\psi - \partial_\mu\overline{\psi}\gamma_\nu\psi\right) - \eta_{\mu\nu}\left(i\,\overline{\psi}\partial\!\!\!/\psi + \frac{g^2}{2}\,(\overline{\psi}\psi)^2\right) \tag{4.2.5}$$

With the help of the classical equations of motion,

$$i\,\partial\!\!\!/\psi = -g^2\,(\overline{\psi}\psi)\,\psi \qquad i\,\partial\!\!\!/\overline{\psi} = g^2\,(\overline{\psi}\psi)\,\overline{\psi} \tag{4.2.6}$$

the stress tensor can be rewritten in the compact form

$$T_{\mu\nu} = i\,\overline{\psi}\gamma_{(\mu}\overset{\leftrightarrow}{\partial}_{\nu)}\psi \tag{4.2.7}$$

where we used the notation

$$f\overset{\leftrightarrow}{\partial}g := \frac{1}{2}\Big[f\,\partial g - (\partial f)g\Big] \tag{4.2.8}$$

and where

$$t_{(\mu\nu)} = \frac{1}{2}\Big[t_{\mu\nu} + t_{\nu\mu} - \eta_{\mu\nu}t^{\mu}{}_{\mu}\Big] \tag{4.2.9}$$

denotes the symmetric, trace-free part of a tensor. Using the equations of motion, it is easy to verify that the stress tensor is conserved, i.e. $\partial^\mu T_{\mu\nu} = 0$, and traceless, i.e. $T^{\mu}{}_{\mu} = 0$, in the classical theory. The vanishing trace is a consequence of the conformal invariance of the classical Gross-Neveu model.

4.2.1. The $1/N$ expansion and dynamical symmetry breaking

It was shown in the work of Gross and Neveu that, despite the fact that the underlying Lagrangian is conformally invariant, the fermions in the GN-model obtain a mass dynamically. In the original work [14] this was shown to be true in the large N limit (i.e. for $N \to \infty$), and it was later also rigorously proven for sufficiently large, but finite N by [57]. By "dynamical mass generation" we mean concretely that, at large space like separation, the 2-point correlation function falls off exponentially, $\langle\overline{\psi}(x)\psi(0)\rangle \sim \exp(-K(g)\sqrt{-x^2/\ell^2})$. Here $K(g) > 0$ is a numerical constant and ℓ is a constant with the dimension of [length]. The dynamically generated mass is hence given by $m(g) := K(g)/\ell$. The constant ℓ may be viewed as a choice of units (fm, mm, km, etc.), which are obviously not provided by the classical Lagrangian (this phenomenon is known as *dimensional transmutation* in the literature). Its value may best be viewed as part of the definition of the quantum field theory.

In the following we will present the derivation of dynamical symmetry breaking to leading order in the $1/N$ expansion as originally given by Gross and Neveu [14]. We will then state the Feynman rules for the perturbation theory around the physical vacuum, and we will explicitly compute the 2- and 4-point functions of the basic field, which will be needed later in our construction of the stress energy operator.

We start out by introducing an auxiliary field, σ, and by adding to the Gross-Neveu Lagrangian a term that does not affect the dynamics of the model:

$$\mathcal{L}_\sigma := \mathcal{L}_{\mathrm{GN}} - \frac{N}{2g^2}\left(\sigma - g^2\overline{\psi}\psi\right)^2 = N\left[i\,\overline{\psi}\partial\!\!\!/\psi - \frac{1}{2g^2}\sigma^2 + \sigma\,\overline{\psi}\psi\right] \tag{4.2.10}$$

The functional integral over σ is simply a Gaussian integral which only causes the generating functional of the theory to be multiplied by some constant. Also one easily checks that the Euler-Lagrange equation for σ reads $\sigma = g^2\,\overline{\psi}\psi$. Substituting this constraint into the Lagrangian leads us back to the original form (4.2.3), which confirms that the additional term has no effect on the physics of the model.

The motivation to introduce such an auxiliary field is the following: The effective potential, $V_{\mathrm{eff}}(\sigma)$, defined as the sum of the one-particle-irreducible (1PI) Feynman diagrams with external σ-lines carrying zero momentum, see fig. 4.1, can be interpreted at stationary points as the vacuum energy density to leading order in $1/N$ [14, 58].

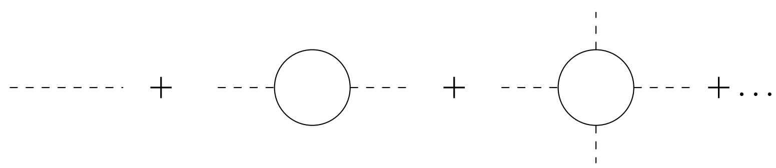


Figure 4.1.: Feynman graphs contributing to $V_{\mathrm{eff}}(\sigma)$ at leading order in $1/N$ (solid lines correspond to the fermion propagator, dashed ones to the σ-propagator)

In the presence of multiple stationary points, only those minimising the energy are physically realised. The effective potential can be computed explicitly as follows:

$$V_{\mathrm{eff}}(\sigma) = \frac{N}{2g^2}\sigma^2 + i\sum_{n=1}^{\infty}\frac{N}{2n}\int^{\Lambda}\frac{\mathrm{d}^2 p}{(2\pi)^2}\left(\frac{\sigma^2}{p^2}\right)^n = \frac{N}{2g^2}\sigma^2 + \frac{N}{4\pi}\sigma^2\left[\log\frac{\sigma^2}{\Lambda^2} - 1\right], \tag{4.2.11}$$

where Λ is a UV-cutoff here. The theory can be renormalised by introducing a running coupling $g(\ell)$, which we define as

$$\frac{1}{g^2(\ell)} := \frac{1}{g^2(\Lambda)} - \frac{1}{2\pi}\log(\Lambda^2\ell^2)\,, \tag{4.2.12}$$

where $g(\Lambda)$ is the coupling constant of the unrenormalised theory (i.e. the *bare* coupling constant). Hence, the effective potential of the renormalised theory takes the form

$$V_{\text{eff}}(\sigma) = \frac{N}{2g^2(\ell)}\sigma^2 + \frac{N}{4\pi}\sigma^2\left[\log(\sigma^2\ell^2) - 1\right]. \tag{4.2.13}$$

As mentioned above, we are interested in the minima of this potential. A straightforward calculation reveals that $V_{\text{eff}}(\sigma)$ has a local maximum at the origin and two global minima at the values

$$\pm m := \pm \ell^{-1}\exp(-\pi/g^2), \tag{4.2.14}$$

see fig. 4.2. Thus, we have found that the discrete symmetry (4.2.4) is spontaneously broken. The value of the effective potential at its minima is $V(m) = -Nm^2/(4\pi)$.[3]

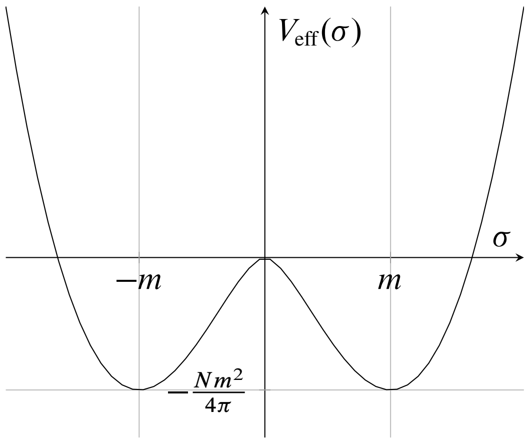


Figure 4.2.: The effective potential $V_{\text{eff}}(\sigma)$

In order to formulate perturbation theory around the physical vacuum, we introduce a shifted auxiliary field $\tilde{\sigma} := \sigma + m$, in terms of which the Lagrangian takes the form

$$\mathcal{L}_{\tilde{\sigma}} = N\left[\overline{\psi}\left(i\,\partial\!\!\!/ - m\right)\psi - \frac{1}{2g^2}\tilde{\sigma}^2 + \tilde{\sigma}\,\overline{\psi}\psi + \frac{m}{g^2}\tilde{\sigma} - \frac{m^2}{2g^2}\right]. \tag{4.2.15}$$

The last term on the r.h.s. simply yields an overall multiplicative constant and can again be safely neglected. We are now interested in the Feynman rules corresponding to the Lagrangian (4.2.15). Here it is important to note that all *tadpole diagrams*, which result from the third term in the Lagrangian, are cancelled by the fourth term. This follows from

[3] This expression can also be interpreted as the vacuum energy density to leading order in $1/N$ [58]. Below in section 4.2.2 we will re-derive this result via an explicit construction, based on the operator product expansion, of the energy momentum operator. This method will allow us to discuss the renormalisation ambiguities, as well as possible restrictions by the requirement to keep the OPE coefficients analytic in g, as outlined in section 4.1.

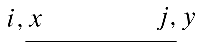
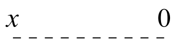
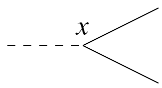
the straightforward computation

$$\frac{iNm}{g^2}\int \mathrm{d}^2x\,\tilde{\sigma}(x) - iN\int \mathrm{d}^2x\,\tilde{\sigma}(x)\ \mathrm{Tr}\int \frac{\mathrm{d}k^2}{(2\pi)^2}\hat{S}(k) = 0\,, \tag{4.2.16}$$

where $\hat{S}(k)$ is the (massive) Fermion propagator in momentum space, i.e. $\hat{S}(p) := \frac{i(\not{p}+m)}{p^2-m^2}$. Equation (4.2.16) is verified by performing the momentum integral with a UV-cutoff Λ and using equation (4.2.14) with $\ell = \Lambda$. It also implies that the shifted auxiliary field has vanishing vacuum expectation value.

One can now determine the Feynman rules corresponding to the Lagrangian (4.2.15). These diagrammatic rules are depicted in table 4.1. With these Feynman rules at hand we

Table 4.1.: Feynman rules for the GN-model (here K_0 is a modified Bessel function of the second kind, see eq.(B.1.2), i, j are flavour indices and $\delta(x)$ is the Dirac delta-function)

ψ-propagator		$\langle T\psi_i(x)\overline{\psi}_j(0)\rangle = \delta_{ij}\,\frac{(i\not{\partial}+m)}{2\pi N}K_0(m\sqrt{-x^2+i\epsilon})$
$\tilde{\sigma}$-propagator		$\frac{-ig^2}{N}\delta(x)$
$\overline{\psi}\psi\tilde{\sigma}$-vertex		$iN\int \mathrm{d}^2x$

could in principle compute any n-point correlation function of the basic field. In this thesis, we will only need the 2- and 4-point functions. The 2-point function is (here and below we assume x to be space like):

$$\langle\overline{\psi}_\alpha(x)\psi_\beta(0)\rangle = -\frac{(i\not{\partial}_x + m)_{\beta\alpha}}{2\pi}K_0(\sqrt{-x^2m^2}) + O(\frac{1}{N}) \tag{4.2.17}$$

where α, β are spinor indices and where K_0 is a modified Bessel function, see eq.(B.1.2). Here and in the following we write $O(x)$ for contributions of order x or higher. The 4-point function is written most conveniently for our purposes as (see appendix B.1 for the derivation of this formula)

$$
\begin{aligned}
&\langle\overline{\psi}_\alpha(x)\psi_\beta(0)\overline{\psi}_\gamma(z_1)\psi_\delta(z_2)\rangle = \\
&\frac{-1}{2N}\int\frac{\mathrm{d}^2p\,\mathrm{d}^2q}{(2\pi)^4}\frac{[(\not p+m)(\not q+m)]_{\gamma\delta}\,\mathrm{e}^{i(z_1-x)p+i(x-z_2)q}}{(p^2-m^2)(q^2-m^2)\,B(q-p)}\,[(\not q-\not p+m-i\not\partial_x)(m-i\not\partial_x)]_{\alpha\beta} \\
&\quad\times\int_0^1\mathrm{d}\alpha\,\frac{\sqrt{-x^2}K_1[\sqrt{-x^2(m^2-\alpha(1-\alpha)(q-p)^2)}]\mathrm{e}^{ix(q-p)(\alpha-1)}}{\sqrt{m^2-\alpha(1-\alpha)(q-p)^2}} \\
&+\langle\overline{\psi}_\alpha(x)\psi_\beta(0)\rangle\,\langle\overline{\psi}_\gamma(z_1)\psi_\delta(z_2)\rangle-\frac{1}{N}\langle\overline{\psi}_\alpha(x)\psi_\delta(z_2)\rangle\,\langle\overline{\psi}_\gamma(z_1)\psi_\beta(0)\rangle+O(\frac{1}{N^2})\,,
\end{aligned}
\tag{4.2.18}
$$

where again K_1 is a modified Bessel function, and where we use the short-hand

$$
B(k) := \sqrt{\frac{4m^2-k^2}{-k^2}}\log\frac{\sqrt{4m^2-k^2}+\sqrt{-k^2}}{\sqrt{4m^2-k^2}-\sqrt{-k^2}}\,. \tag{4.2.19}
$$

Note that the correlation functions have a *non-analytic* dependence on g through $m = \mathrm{e}^{-\pi/g^2}/\ell$.

4.2.2. Construction of the stress-energy operator

Our ultimate aim in this chapter is to determine the vacuum expectation value (VEV) of the stress energy operator, which is evidently a composite operator. Expectation values of composite operators, subject to the intrinsic renormalisation ambiguities discussed in section 4.1, can be obtained form the operator product expansion of the n-point correlation functions of the basic fields, $\psi, \overline{\psi}$.

As an example, let us start off with the computation of the VEV $\langle\overline{\psi}\psi\rangle$. Our general strategy works as follows: Assume we know the first two expansion coefficients of the operator product expansion

$$
\overline{\psi}(x)\psi(0) = \mathcal{C}_{\overline{\psi}\psi}^{\mathbb{1}}(x)\,\mathbb{1} + \mathcal{C}_{\overline{\psi}\psi}^{\overline{\psi}\psi}(x)\,\overline{\psi}\psi(0) + O(x) \quad . \tag{4.2.20}
$$

Then we can take the expectation value of this equation, solve for $\langle\overline{\psi}\psi\rangle$ and take the limit $x \to 0$:

$$
\langle\overline{\psi}\psi(0)\rangle = \lim_{x\to 0}\left(\frac{\langle\overline{\psi}(x)\psi(0)\rangle - \mathcal{C}_{\overline{\psi}\psi}^{\mathbb{1}}(x)\,\langle\mathbb{1}\rangle}{\mathcal{C}_{\overline{\psi}\psi}^{\overline{\psi}\psi}(x)}\right) \tag{4.2.21}
$$

We would like to find an explicit expression for the right hand side. For the two point function such an expression can be found in eq.(4.2.17). It remains to determine the two OPE coefficients. In the following, we will show that the most general form of these coefficients is

$$\mathcal{C}_{\overline{\psi}\psi}^{\overline{\psi}\psi}(x) = Z\left[1 - \frac{g^2}{2\pi}\log\left(\frac{-x^2 e^{2\Gamma_{\mathrm{E}}}}{4\ell^2}\right)\right] + O(x) + O(\frac{1}{N}) \tag{4.2.22}$$

and

$$\mathcal{C}_{\overline{\psi}\psi}^{\mathbb{1}}(x) = c(g)\left[1 - \frac{g^2}{2\pi}\log\left(\frac{-x^2 e^{2\Gamma_{\mathrm{E}}}}{4\ell^2}\right)\right] + O(x) + O(\frac{1}{N}), \tag{4.2.23}$$

where $c(g)$ is of dimension $[\text{length}]^{-1}$, $Z \in \mathbb{R}$ and Γ_{E} is the Euler-Mascheroni constant. In principle we would have to show that the expansion (4.2.20) holds as an operator insertion into an arbitrary correlation function. However, in view of the identity

$$\begin{aligned}\left\langle\prod_{i=1}^{n}\overline{\psi}(y_i)\psi(z_i)\right\rangle = &\sum_{1\le i<j\le n}\langle\overline{\psi}(y_i)\psi(z_i)\overline{\psi}(y_j)\psi(z_j)\rangle\langle\prod_{k\in\{1,\dots,N\}\setminus\{i,j\}}\overline{\psi}(y_k)\psi(z_k)\rangle\\ &+ O(1/N^2)\end{aligned} \tag{4.2.24}$$

which holds for $n \geq 2$, we see that it suffices to consider the OPE only inside the 2- and 4-point functions, if one is willing to neglect contributions of order $1/N^2$. Since we are mainly interested in the large N-limit within this thesis, this will be exactly our strategy.

The coefficient $\mathcal{C}_{\overline{\psi}\psi}^{\overline{\psi}\psi}$ is then determined with the help of the 4-point function (4.2.18). Contracting the spinor indices, we find the expansion (see appendix B.2 for the derivation of this formula)

$$\begin{aligned}&\langle\overline{\psi}_\alpha(x)\psi^\alpha(0)\,\overline{\psi}_\beta(z_1)\psi^\beta(z_2)\rangle - \langle\overline{\psi}_\alpha(x)\psi^\alpha(0)\rangle\,\langle\overline{\psi}_\beta(z_1)\psi^\beta(z_2)\rangle\\ &= \frac{4\pi}{g^2N}\int\frac{d^2p\,d^2q}{(2\pi)^4}\frac{(p^\mu q_\mu + m^2)\,e^{iz_1p - iz_2q}}{(p^2-m^2)(q^2-m^2)\,B(q-p)}\left[1 - \frac{g^2}{2\pi}\log\left(\frac{-x^2e^{2\Gamma_{\mathrm{E}}}}{4\ell^2}\right) + O(x)\right]\\ &\quad + O(\frac{1}{N^2})\\ &= \mathcal{C}_{\overline{\psi}\psi}^{\overline{\psi}\psi}(x)\left[\langle\overline{\psi}\psi(0)\,\overline{\psi}_\beta(z_1)\psi^\beta(z_2)\rangle - \langle\overline{\psi}\psi(0)\rangle\,\langle\overline{\psi}_\beta(z_1)\psi^\beta(z_2)\rangle\right] + O(x)\end{aligned} \tag{4.2.25}$$

which confirms eq.(4.2.22). In the last line we have substituted the OPE (4.2.20). The coefficient $\mathcal{C}_{\overline{\psi}\psi}^{\mathbb{1}}$ appears in the expansion of the two point function:

$$\begin{aligned}\langle\overline{\psi}_\alpha(x)\psi^\alpha(0)\rangle &= \frac{-m}{g^2}\left[1 - \frac{g^2}{2\pi}\log\left(\frac{-x^2e^{2\Gamma_{\mathrm{E}}}}{4\ell^2}\right) + O(x)\right] + O(1/N)\\ &= \mathcal{C}_{\overline{\psi}\psi}^{\mathbb{1}}(x) + \mathcal{C}_{\overline{\psi}\psi}^{\overline{\psi}\psi}(x)\,\langle\overline{\psi}\psi(0)\rangle + O(x)\end{aligned} \tag{4.2.26}$$

This equation is solved by eq.(4.2.23) under the condition that

$$\langle\overline{\psi}\psi(0)\rangle = -\frac{m/Z + c(g)}{g^2} + O(\frac{1}{N}) = \frac{-1}{Zg^2\ell}\mathrm{e}^{-\pi/g^2} - \frac{c(g)}{g^2} + O(\frac{1}{N}) \qquad (4.2.27)$$

for the vacuum condensate. The freedom to choose the functions $Z(g), c(g)$ amounts to field redefinitions of the type $\overline{\psi}\psi \to Z\overline{\psi}\psi + c\mathbb{1}$. At this point our analyticity condition on the OPE coefficients effectively restricts such renormalisation ambiguities. Namely, since we require $Z(g), c(g)$ to be analytic, it follows that the non-perturbative contribution to $\langle\overline{\psi}\psi\rangle$ cannot be cancelled by a field redefinition!

Further, it is easy to see that different choices for the parameter Z simply rescale the constant ℓ. We have already mentioned above that one is free to choose this constant, so we can absorb the factor Z by rescaling $\ell \to \ell/Z$. The constant $c(g)$, on the other hand, causes a purely perturbative contribution to the vacuum condensate $\langle\overline{\psi}\psi\rangle$. In the following we will set $c(g) = 0$, which is the same as requiring that $\langle\overline{\psi}\psi\rangle = 0$ to all orders in perturbation theory. This appears to be a reasonable requirement, since we would not expect dynamical symmetry breaking to occur within perturbation theory. To summarise, we have found

$$\langle\overline{\psi}\psi(0)\rangle = -\frac{m}{g^2} + O(\frac{1}{N}) = \frac{-1}{g^2\ell}\mathrm{e}^{-\pi/g^2} + O(\frac{1}{N}) \, . \qquad (4.2.28)$$

Using the strategy outlined above, one can in principle compute the VEV for any composite operator, provided that all OPE coefficients and n-point functions are known. We will focus on the VEV of the stress energy operator in the following. One has to be a little more careful in this case, because we also want to make sure that our definition of this composite operator obeys the conservation law, $\partial^\mu T_{\mu\nu} = 0$, as an operator equation. For this purpose, we will proceed as follows: First, we will define separately the composite operators corresponding to the classical expressions in eq.(4.2.5). The resulting tensor will turn out to be *not conserved*. But fortunately we can add another operator of the same dimension (field redefinition) to it such that it now is conserved [up to order $O(\frac{1}{N^2})$]. The resulting conserved operator is then the physical stress energy operator, which is seen to have a non-zero VEV.

Let us now describe this procedure in some more detail. Since we can consistently assume that

$$i\,\overline{\psi}\partial\!\!\!/\psi = (g\,\overline{\psi}\psi)^2 \qquad (4.2.29)$$

as an operator equation, we may simply write $T_{\mu\nu} = i\overline{\psi}\gamma_{(\mu}\overleftrightarrow{\partial}_{\nu)}\psi$ again. We want to define this composite operator using our method based on the OPE. For this purpose, we will need the following two OPE's[4]:

$$\begin{aligned}\overline{\psi}(x)\partial\!\!\!/\psi(0) &= O(\frac{1}{N})\mathbb{1} + O(\frac{1}{N})\,\overline{\psi}\psi(0)\\ &+ \frac{g^2}{i}\left[1 - \frac{g^2}{2\pi}\log\left(\frac{-x^2 e^{2\Gamma_E}}{4\ell^2}\right) + O(\frac{1}{N})\right](\overline{\psi}\psi)^2(0) + O(x)\end{aligned} \tag{4.2.30}$$

$$\begin{aligned}\overline{\psi}(x)\gamma_{(\mu}\partial_{\nu)}\psi(0) &= \left[\frac{-2x_{(\mu}x_{\nu)}}{i\pi\, x^4} + O(\frac{1}{N})\right]\mathbb{1} + O(\frac{1}{N})\,\overline{\psi}\psi(0)\\ &+ \left[1 + O(\frac{1}{N})\right]\overline{\psi}\gamma_{(\mu}\partial_{\nu)}\psi(0) - \left[\frac{g^4 x_{(\mu}x_{\nu)}}{2\pi i\; x^2} + O(\frac{1}{N})\right](\overline{\psi}\psi)^2(0) + O(x)\end{aligned} \tag{4.2.31}$$

In order to verify these expansions, we have to make sure that they hold as insertions into arbitrary correlation functions. By the same arguments as above, we will only need to consider the 2- and 4-point functions in order to show that our expansions hold up to terms of the order $1/N^2$.

Verification of the OPE (4.2.30): We find for the short distance expansion of the 4-point function (see again appendix B.2 for the derivation of this formula)

$$\begin{aligned}&\langle\overline{\psi}(x)\partial\!\!\!/\psi(0)\,\overline{\psi}(z_1)\psi(z_2)\rangle - \langle\overline{\psi}(x)\partial\!\!\!/\psi(0)\rangle\,\langle\overline{\psi}(z_1)\psi(z_2)\rangle\\ &= \frac{8\pi m}{ig^2N}\int\frac{d^2p\,d^2q}{(2\pi)^4}\frac{(p^\mu q_\mu + m^2)\,e^{iz_1p - iz_2q}}{(p^2-m^2)(q^2-m^2)\,B(q-p)}\left[1 - \frac{g^2}{2\pi}\log\left(\frac{-x^2e^{2\Gamma_E}}{4\ell^2}\right) + O(x)\right]\\ &+ O(\frac{1}{N^2})\end{aligned} \tag{4.2.32}$$

When combined with the OPE (4.2.30), this equation allows us to determine the following insertion of the composite operator $\overline{\psi}\partial\!\!\!/\psi$:

$$\begin{aligned}&\langle\overline{\psi}\partial\!\!\!/\psi(0)\,\overline{\psi}(z_1)\psi(z_2)\rangle - \langle\overline{\psi}\partial\!\!\!/\psi(0)\rangle\,\langle\overline{\psi}(z_1)\psi(z_2)\rangle\\ &= \lim_{x\to 0}\left(\frac{\langle\overline{\psi}(x)\partial\!\!\!/\psi(0)\,\overline{\psi}(z_1)\psi(z_2)\rangle - \langle\overline{\psi}(x)\partial\!\!\!/\psi(0)\rangle\,\langle\overline{\psi}(z_1)\psi(z_2)\rangle + O(1/N^2)}{\left[1 - \frac{g^2}{2\pi}\log\left(\frac{-x^2e^{2\Gamma_E}}{4\ell^2}\right)\right] + O(1/N)}\right)\\ &= \frac{8\pi m}{ig^2N}\int\frac{d^2p\,d^2q}{(2\pi)^4}\frac{(p^\mu q_\mu + m^2)\,e^{iz_1p - iz_2q}}{(p^2-m^2)(q^2-m^2)\,B(q-p)} + O(\frac{1}{N^2})\end{aligned} \tag{4.2.33}$$

[4]Here we do not give the most the most general form of the OPE coefficients, but choose a particularly convenient set of coefficients for the sake of brevity. These coefficients along with the composite operators defined below are then subject to the renormalisation ambiguities outlined in section 4.1.

For the 2-point function we find the short distance expansion

$$\langle\overline{\psi}(x)\partial\!\!\!/\psi(0)\rangle = m^2\left[1-\frac{g^2}{2\pi}\log\left(\frac{-x^2\mathrm{e}^{2\Gamma_\mathrm{E}}}{4\ell^2}\right)+O(x)\right]+O(\frac{1}{N}) \tag{4.2.34}$$

which allows us to determine the VEV

$$\langle\overline{\psi}\partial\!\!\!/\psi(0)\rangle = \lim_{x\to 0}\left(\frac{\langle\overline{\psi}(x)\partial\!\!\!/\psi(0)\rangle+O(1/N)}{\left[1-\frac{g^2}{2\pi}\log\left(\frac{-x^2\mathrm{e}^{2\Gamma_\mathrm{E}}}{4\ell^2}\right)\right]+O(\frac{1}{N})}\right) = \frac{m^2}{ig^2}+O(\frac{1}{N}) \tag{4.2.35}$$

In view of eqs.(4.2.32) to (4.2.35), we see that the OPE (4.2.30) does indeed hold when inserted into correlators with up to two spectator fields. Due to the fact that the higher n-point functions factorise (up to contributions of order $1/N^2$), see eq.(4.2.24), we deduce that our OPE in fact holds as an insertion into an arbitrary correlator.

Verification of the OPE (4.2.31): We can repeat the game for the OPE (4.2.31). Again, we first compute the 4-point function (see appendix B.2 for details)

$$\begin{aligned}
&\langle\overline{\psi}(x)\gamma_{(\mu}\partial_{\nu)}\psi(0)\,\overline{\psi}(z_1)\psi(z_2)\rangle - \langle\overline{\psi}(x)\gamma_{(\mu}\partial_{\nu)}\psi(0)\rangle\,\langle\overline{\psi}(z_1)\psi(z_2)\rangle\\
&= \frac{2m}{iN}\int\frac{\mathrm{d}^2p\,\mathrm{d}^2q}{(2\pi)^4}\frac{(p^\mu q_\mu+m^2)\,\mathrm{e}^{iz_1p-iz_2q}}{(p^2-m^2)(q^2-m^2)\,B(q-p)}\\
&\qquad\qquad\times\left[\frac{-2x_{(\mu}x_{\nu)}}{x^2}+\frac{(q-p)_{(\mu}(q-p)_{\nu)}}{q^2}\,(B(q-p)+2)\right]\\
&+\frac{2m}{iN}\int\frac{\mathrm{d}^2p\,\mathrm{d}^2q}{(2\pi)^4}\frac{(p_{(\mu}q_{\nu)}+p_{(\mu}p_{\nu)})\,\mathrm{e}^{iz_1p-iz_2q}}{(p^2-m^2)(q^2-m^2)}+O(x)+O(1/N^2)
\end{aligned} \tag{4.2.36}$$

which then allows us to determine the following insertion of the composite operator $\overline{\psi}\gamma_{(\mu}\partial_{\nu)}\psi$:

$$
\begin{aligned}
&\langle\overline{\psi}\gamma_{(\mu}\partial_{\nu)}\psi(0)\overline{\psi}(z_1)\psi(z_2)\rangle - \langle\overline{\psi}\gamma_{(\mu}\partial_{\nu)}\psi(0)\rangle\langle\overline{\psi}(z_1)\psi(z_2)\rangle \\
&= \lim_{x\to 0}\Bigg(\frac{\langle i\overline{\psi}(x)\gamma_{(\mu}\partial_{\nu)}\psi(0)\overline{\psi}(z_1)\psi(z_2)\rangle - \langle i\overline{\psi}(x)\gamma_{(\mu}\partial_{\nu)}\psi(0)\rangle\langle\overline{\psi}(z_1)\psi(z_2)\rangle}{1+O(\frac{1}{N})} \\
&\quad - \frac{O(\frac{1}{N})\,\langle(\overline{\psi}\psi)(0)\overline{\psi}(z_1)\psi(z_2)\rangle}{1+O(\frac{1}{N})} \\
&+ \frac{\left[-\frac{g^4}{2\pi}\frac{x_{(\mu}x_{\nu)}}{x^2}+O(\frac{1}{N})\right]\left[\langle(\overline{\psi}\psi)^2(0)\overline{\psi}(z_1)\psi(z_2)\rangle - \langle(\overline{\psi}\psi)^2(0)\rangle\langle\overline{\psi}(z_1)\psi(z_2)\rangle\right]}{1+O(\frac{1}{N})}\Bigg) \\
&= \frac{2m}{iN}\int\frac{\mathrm{d}^2p\,\mathrm{d}^2q}{(2\pi)^4}\frac{(p^\mu q_\mu+m^2)\,\mathrm{e}^{iz_1p-iz_2q}}{(p^2-m^2)(q^2-m^2)\,B(q-p)}\left[\frac{(q-p)_{(\mu}(q-p)_{\nu)}}{(q-p)^2}\,(B(q-p)+2)\right] \\
&+ \frac{2m}{iN}\int\frac{\mathrm{d}^2p\,\mathrm{d}^2q}{(2\pi)^4}\frac{p_{(\mu}(q+p)_{\nu)}\,\mathrm{e}^{iz_1p-iz_2q}}{(p^2-m^2)(q^2-m^2)} + O(1/N^2)
\end{aligned}
\tag{4.2.37}
$$

The short distance expansion of the 2-point function is

$$
\langle\overline{\psi}(x)\gamma_{(\mu}\partial_{\nu)}\psi(0)\rangle = \frac{-2x_{(\mu}x_{\nu)}}{i\pi\,x^4} + m^2\frac{g^4x_{(\mu}x_{\nu)}}{2\pi i\,x^2} + O(x) + O(1/N) \tag{4.2.38}
$$

which yields the VEV

$$
\begin{aligned}
&\langle\overline{\psi}\gamma_{(\mu}\partial_{\nu)}\psi(0)\rangle \\
&= \lim_{x\to 0}\left(\frac{\langle\overline{\psi}(x)\gamma_{(\mu}\partial_{\nu)}\psi(0)\rangle - \left[\frac{2}{\pi}\frac{x_{(\mu}x_{\nu)}}{x^4}+O(\frac{1}{N})\right] - \left[-\frac{g^4}{2\pi}\frac{x_{(\mu}x_{\nu)}}{x^2}+O(\frac{1}{N})\right]\langle(\overline{\psi}\psi)^2(0)\rangle}{1+O(\frac{1}{N})}\right) \\
&= O(\frac{1}{N})
\end{aligned}
\tag{4.2.39}
$$

Inserting these equations into the OPE (4.2.31) confirms that this OPE holds as an insertion into correlators with up to two spectator fields. Again, we argue based on eq.(4.2.24) that this implies that the OPE also holds, up to terms of order $1/N^2$, when inserted into any other correlator.

Having defined the composite operator $T_{\mu\nu} = i\,\overline{\psi}\gamma_{(\mu}\overleftrightarrow{\partial}_{\nu)}\psi$, we are now ready to check the conservation law. Since we would like to check whether the stress tensor is conserved *as an operator*, we need to calculate the divergence $\langle\partial^\mu T_{\mu\nu}(0)\prod\overline{\psi}(y_i)\prod\psi(z_j)\rangle$ inside a correlation function. Actually, it suffices for our purpose to consider two spectator fields

$\overline{\psi}(z_1)\psi(z_2)$ inside the correlator. Taking the divergence of eq.(4.2.37), we find explicitly[5]

$$\begin{aligned}
&\langle \partial^\mu T_{\mu\nu}(0)\overline{\psi}(z_1)\psi(z_2)\rangle = i\,\partial^\mu \langle \overline{\psi}\gamma_{(\mu}\overleftrightarrow{\partial}_{\nu)}\psi(0)\overline{\psi}(z_1)\psi(z_2)\rangle \\
&= \frac{2mi}{N}\int \frac{\mathrm{d}^2p\,\mathrm{d}^2q}{(2\pi)^4}\frac{(p^\mu q_\mu + m^2)\,\mathrm{e}^{iz_1p-iz_2q}}{(p^2-m^2)(q^2-m^2)\,B(q-p)} \\
&\qquad\qquad \times\left[\frac{(q-p)^\mu(q-p)_{(\mu}(q-p)_{\nu)}}{(q-p)^2}\,(B(q-p)+2)\right] \\
&+ \frac{2mi}{N}\int \frac{\mathrm{d}^2p\,\mathrm{d}^2q}{(2\pi)^4}\frac{(q-p)^\mu(q+p)_{(\mu}(q+p)_{\nu)}\,\mathrm{e}^{iz_1p-iz_2q}}{(p^2-m^2)(q^2-m^2)} + O(1/N^2) \\
&= \frac{2mi}{N}\int \frac{\mathrm{d}^2p\,\mathrm{d}^2q}{(2\pi)^4}\frac{(p^\mu q_\mu + m^2)\,\mathrm{e}^{iz_1p-iz_2q}}{(p^2-m^2)(q^2-m^2)\,B(q-p)}\cdot(q-p)_\nu + O(1/N^2) \\
&= \frac{ig^2}{4\pi}\,\partial_\nu\langle\overline{\psi}\partial\!\!\!/\psi(0)\,\overline{\psi}(z_1)\psi(z_2)\rangle + O(1/N^2)
\end{aligned} \tag{4.2.41}$$

Since the r.h.s. is not zero, it follows that the composite operator $T_{\mu\nu}$, as defined, is *not* conserved. However, it follows that the operator $\theta_{\mu\nu} := T_{\mu\nu} - (g^2 i/4\pi)\,\eta_{\mu\nu}\overline{\psi}\partial\!\!\!/\psi$ *is* conserved [up to order $O(\frac{1}{N^2})$]. We consequently define $\theta_{\mu\nu}$ to be the *physical* stress energy tensor up to that order. Using equations (4.2.35) and (4.2.39), its VEV is found to be

$$\langle\theta_{\mu\nu}\rangle = -\frac{m^2}{4\pi}\,\eta_{\mu\nu} + O(\frac{1}{N}) = -\frac{1}{4\pi\ell^2}\,\mathrm{e}^{-2\pi/g^2}\,\eta_{\mu\nu} + O(\frac{1}{N})\,. \tag{4.2.42}$$

This corresponds to a negative vacuum energy of $\rho_{\text{vac}} = -1/(4\pi\ell^2)\,\mathrm{e}^{-2\pi/g^2}$ to leading order in $1/N$. The negative sign is related to the negative sign of the β-function in the Gross-Neveu model.

We must finally discuss the *ambiguity* of our result. According to the general discussion in section 4.1 we are still free to change $\theta_{\mu\nu} \to \theta_{\mu\nu} + \ell^{-2}c(g)\,\eta_{\mu\nu}\mathbb{1}$, where $c(g) = c_0 + c_1 g + c_2 g^2 + \ldots$ is *analytic* [cf. equation (4.1.2)]. This will result in a corresponding change $\rho_{\text{vac}} \to \rho_{\text{vac}} + \ell^{-2}c(g)$. We can eliminate this remaining ambiguity by making the, seemingly-reasonable, assumption, that ρ_{vac} should vanish to all orders in perturbation theory. This is the same as demanding that, at the perturbative level, Minkowski space is a solution to the semi-classical Einstein equations. Under this assumption, $\rho_{\text{vac}} = -1/(4\pi\ell^2)\,\mathrm{e}^{-2\pi/g^2}$ is *unique*. This is the main result of this section.

[5]Note that the contribution $\langle\partial_{(\mu}\overline{\psi}\gamma_{\nu)}\psi(0)\overline{\psi}(z_1)\psi(z_2)\rangle$ can be obtained from eq.(4.2.37) through the relation

$$\langle\partial_{(\mu}\overline{\psi}\gamma_{\nu)}\psi(0)\overline{\psi}(z_1)\psi(z_2)\rangle = \langle\psi\gamma_{(\nu}\partial_{\mu)}\overline{\psi}(0)\psi(z_2)\overline{\psi}(z_1)\rangle = \overline{\langle\overline{\psi}\gamma_{(\nu}\partial_{\mu)}\psi(0)\overline{\psi}(z_2)\psi(z_1)\rangle} \tag{4.2.40}$$

4.3 A possible explanation for Dark Energy

In the previous section we have defined the stress energy operator of the Gross-Neveu model to leading order in the large flavour expansion, and we have made sure that our definition respects the energy-momentum conservation law. Under the condition that the OPE coefficients are analytic in the coupling constant, we found that the non-perturbative contribution to the vacuum energy is uniquely given by $\rho_{\mathrm{vac}} = -1/(4\pi\ell^2)\ \mathrm{e}^{-2\pi/g^2}$. Unfortunately, the definition of our model at the quantum level includes the scale ℓ, which corresponds to a choice of units of length. This constant is not provided by the classical Lagrangian, which makes it difficult to determine the exact value of the vacuum energy. It would certainly be preferable to have a theory wherein all dimensionful parameters are already part of the fundamental Lagrangian defining the theory in the ultra-violet. In order to achieve this, one could couple our model to other quantum fields with dimensionful couplings. As an example, one could consider the Lagrangian

$$\mathcal{L}_\varphi := \mathcal{L}_{\mathrm{GN}} + \frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi + \frac{1}{2}M^2\varphi^2 + yM\ \varphi\overline{\psi}\psi \tag{4.3.1}$$

where $\mathcal{L}_{\mathrm{GN}}$ is the Gross-Neveu Lagrangian (4.2.3) and where $y, M \in \mathbb{R}$ are additional coupling constants. In this model it is possible to relate the constant ℓ to the dimensionful parameter M by a renormalisation condition, such as e.g. the condition that the physical (renormalised) mass of ψ (as determined by the exponential decay of the fermion 2-point function) at some value $g = O(1) = y$ equals M, where M is the physical (renormalised) mass of φ (as determined by the exponential decay of the φ 2-point function). In this model we would expect that our result for the vacuum energy would be modified to $\rho_{\mathrm{vac}} \sim M^2\mathrm{e}^{-O(1)/g^2}$, i.e. the scale ℓ is simply set by the new parameter M, which is now a parameter that appears in the Lagrangian. As mentioned above, the analyticity condition on the OPE coefficients only fixes the non-perturbative contribution to the stress-energy operator. In the case at hand, the remaining freedom amounts to addition of a term of the form $c(g, y)M^2\eta_{\mu\nu}$, where $c(g, y)$ is analytic in the coupling constants. Again, by demanding that Minkowski space is a solution to the semi-classical Einstein equations to all perturbation orders in y, g gets rid of this perturbative contribution and leads to a unique result for the vacuum energy.

In order to make a realistic prediction for the vacuum energy of the universe, we would of course have to use the Standard Model of particle physics instead of the Gross-Neveu model. If we take a bold leap and pursue the analogy with the model outlined above, one would expect the parameter M to be replaced by a mass scale of the Standard Model, such as e.g. the Higgs mass, $M \to M_H$. Furthermore, the coupling should be replaced by a gauge coupling, such as $g^2/4\pi \to \alpha_{\mathrm{EW}} \sim 1/137$. These speculations would result in a vacuum energy of the form $\rho_{\text{vac}} \sim M_H^4 \mathrm{e}^{-O(1)/\alpha_{EW}}$. The *unnatural* smallness of ρ_{vac} would thus be achieved through the exponential factor. We believe that, as proposed in [13], such a mechanism could possibly explain the observed value of Dark Energy.

5
Conclusions and Outlook

Our focus in the present work has been on fundamental properties as well as applications of the operator product expansion in both perturbative- and non-perturbative quantum field theory. In the perturbative setting on Euclidean space we have shown that the OPE of any number of fields *converges* (generalising the previous convergence result for the OPE of two fields [15]) and *factorises* at long distances. As we have argued above, the convergence result offers the important insight that the model is entirely determined through its OPE coefficients together with the 1-point functions, in the sense that all n-point functions can be constructed from these data. The factorisation property imposes non-trivial algebraic relations between the OPE coefficients, which can lead to strong restrictions on the latter. Both results also yield support to the axiomatic framework proposed in [21]. The explicit bounds on various quantities of interest, which were necessary to prove our results, were derived within the flow equation framework of perturbative quantum field theory. For this purpose, we had to get a grasp on the regularisation of subdivergences of Green's functions with multiple insertions of composite operators, which is probably the main technical advance to the flow equation framework provided by this thesis.

We further introduced an *explicit formula for perturbations* of the OPE coefficients, which provides an algorithm for the computation of perturbed OPE coefficients in terms of the zeroth order ones. It was shown that this formula follows directly from our definition of the OPE coefficients in the flow equation framework. The possibility to compute OPE coefficients in perturbation theory without reference to any other objects, such as for example correlation functions, is quite satisfying conceptually. To our knowledge, our formula constitutes the first result in this regard that does not require any additional assumptions (such as factorisation or bootstrap conditions).

In the last part of this thesis we determined the vacuum expectation value of the stress energy operator in the two dimensional Gross-Neveu model, to leading order in the large flavour expansion. Our construction was based in the operator product expansion. We found, following a proposal by Hollands and Wald, that one can impose an analyticity condition on the OPE coefficients, which leads to a unique prediction for the non-perturbative contribution to the VEV of the stress energy operator. This non-perturbative contribution is ”exponentially small”. We finally discussed possible cosmological implications of our findings. Namely, if a similar mechanism is present in the Standard Model of particle physics, which seems conceivable, the mentioned smallness of the effect could offer a natural explanation for the observed value of Dark Energy.

We conclude this thesis with a discussion of some possible lines of future research linked to the work presented here:

- We expect that convergence and long distance factorisation of the OPE also hold in other perturbative models. One could for example try to extend our results to theories with different symmetry properties, to massless models or even to theories on curved manifolds.

- It would be desirable to improve our factorisation result. In particular, one would like to have a version of theorem 3 without the second factor on the l.h.s. and with $\tilde{K}$ as close to 1 as possible. This would validate the axioms of [21] in perturbative quantum field theory, and confirm the relation between the OPE and vertex operators [22]. It seems difficult, however, to improve much on the constant $\tilde{K}$ if one follows the strategy presented in section 3.5, since we expect it to be hard, if at all possible, to derive significantly stronger bounds on the relevant Schwinger functions with insertions. We believe that it might be possible to exploit our OPE deformation result, theorem 4, in order to attack the problem from a different angle.

- The perturbation formula for the OPE coefficients presented in theorem 4 should depend on the renormalisation conditions. It would be interesting to study this dependence in more detail.

- Concerning the proposed mechanism to reduce renormalisation ambiguities via an analyticity condition on the OPE coefficients, a similar non-perturbative analysis of more complex models could be fruitful. In particular, one should be looking for models (most likely in two dimensions), which share as many features with the Standard Model as possible. One could also look for more arguments on how a non-perturbative effect like the one described in chapter 4 could appear in the Standard Model.

- The motivation for the research presented in chapter 4 lies primarily in cosmological applications. Therefore, it would be more ”realistic” to apply the same strategy to the Gross-Neveu model on a curved spacetime, especially on Robertson-Walker space. It would be interesting to see how far curvature effects influence our result.

A

Derivation of bounds on Schwinger functions in $g\varphi^4$-theory

In this appendix we collect the somewhat lengthy and technical proofs of the bounds presented in section 3.2.

A.1 Proof of bound 1

Using the inductive scheme sketched in section 3.2, we will first derive a bound on the functionals F_D^{Λ,Λ_0} that holds for arbitrary Λ. Under the condition $\Lambda \leq m$ this result then implies our bound 1, which is uniform in Λ and thus allows for the removal of this cutoff (i.e. $\Lambda \to 0$).

Proposition 5: *Let $x_N = 0$ and $D \leq D' = [A_1] + \ldots + [A_N]$. There exists a constant $K > 0$ such that*

$$|\partial_{\vec{p}}^{w} \mathcal{F}_{D,2n,l}^{\Lambda,\Lambda_0}(\otimes_{i=1}^{N} \mathcal{O}_{A_i}(x_i); \vec{p})| \leq \Lambda^{D-2n-|w|}\, K^{(4n+8l-3)|w|+D'(n+2l)^3} \sqrt{(|w| + D' - D)!}$$
$$\times \frac{\sqrt{D'!}\, \max_{1\leq i\leq N} |x_i|^{\max(|w|,D+1)}}{\min\limits_{1\leq i<j\leq N} |x_i - x_j|^{D'-D+\max(|w|,D+1)}} \sum_{\mu=0}^{d(N,n,l,w,D')} \frac{1}{\sqrt{\mu!}} \left(\frac{|\vec{p}|}{\Lambda}\right)^{\mu} \sum_{\lambda=0}^{2l+n} \frac{\log^{\lambda}\left(\sup(\frac{|\vec{p}|}{\kappa}, \frac{\kappa}{m})\right)}{2^{\lambda}\lambda!} \tag{A.1.1}$$

with $d(N,n,l,w,D') := 2D'(n + l + 2(N-1)) + \sup(D' + 1 - 2n - |w|, 0)$.

Proof of proposition 5. The strategy is to integrate the differentiated flow equation

$$
\begin{aligned}
&\partial_\Lambda \partial^w_{\vec p} \mathcal{F}^{\Lambda,\Lambda_0}_{D,2n,l}(\otimes_{i=1}^N \mathcal{O}_{A_i}; p_1,\dots,p_{2n}) = \\
&= \binom{2n+2}{2} \int_k \dot C^\Lambda(k)\, \partial^w_{\vec p} \mathcal{F}^{\Lambda,\Lambda_0}_{D,2n+2,l-1}(\otimes_{i=1}^N \mathcal{O}_{A_i}; k,-k,p_1,\dots,p_{2n}) \\
&- \mathbb{S}\Bigg[\sum_{\substack{l_1+l_2=l\\ n_1+n_2=n+1}} 4n_1 n_2 \sum_{w_1+w_2+w_3=w} c_{\{w_j\}} \partial^{w_1}_{\vec p} \mathcal{F}^{\Lambda,\Lambda_0}_{D,2n_1,l_1}(\otimes_{i=1}^N \mathcal{O}_{A_i}; q, p_1,\dots,p_{2n_1-1}) \\
&\qquad\qquad \times \partial^{w_2}_{\vec p} \dot C^\Lambda(q)\, \partial^{w_3}_{\vec p} \mathcal{L}^{\Lambda,\Lambda_0}_{2n_2,l_2}(p_{2n_1},\dots,p_{2n}) \\
&- \sum_{\substack{l_1+\ldots+l_N=l\\ n_1+\ldots+n_N=n+1}} \sum_{1\le a<b\le N} 4 n_a n_b \partial^w_{\vec p} \int_k \mathcal{L}^{\Lambda,\Lambda_0}_{2n_a,l_a}(\mathcal{O}_{A_a}; k, p_{2n_{a-1}},\dots,p_{2n_a-1})\, \dot C^\Lambda(k) \\
&\quad \times \mathcal{L}^{\Lambda,\Lambda_0}_{2n_b,l_b}(\mathcal{O}_{A_b}; -k, p_{2n_{b-1}},\dots,p_{2n_b-1}) \prod_{r\in\{1,\dots,N\}\setminus\{a,b\}} \mathcal{L}^{\Lambda,\Lambda_0}_{2n_r,l_r}(\mathcal{O}_{A_r}; p_{2n_{r-1}},\dots,p_{2n_r-1}) \Bigg]
\end{aligned}
\tag{A.1.2}
$$

over Λ and bound each term on the right hand side separately. For the first two terms on the right hand side of this equation, the bound is verified to hold inductively as one goes up in $n+l$ and for fixed $n+l$ goes up in l. This general procedure is very similar to the one employed in [15]. To bound the third term on the r.h.s. of eq.(A.1.2), we will make use of the known bounds on the CAG's with one insertion, (3.2.6).

When integrating eq.(A.1.2) over Λ, we have to distinguish three cases:

(A) Contributions with $2n+|w|>D$ are referred to as *irrelevant*. Here the boundary conditions are given at $\Lambda=\Lambda_0$, see eq.(3.1.37). Therefore, we integrate over Λ' from Λ to Λ_0 in this case.

(B) Contributions with $2n+|w|\le D$ are referred to as *relevant*. The boundary conditions for relevant terms, eq.(3.1.36), are given at $\Lambda=0$ and at vanishing external momentum, $\vec p=\vec 0$. Thus, we will integrate over Λ' from 0 to Λ in this case.

(C) Contributions with $2n+|w|\le D$ and $\vec p\neq\vec 0$ will be obtained from (A),(B) with the help of a Taylor expansion in $\vec p$.

(A) Irrelevant terms ($2n+|w|>D$):

First term on the r.h.s. of the flow equation: In the following we will use the shorthand

$$
\mathcal{X}(\vec x, D', D, w) := \frac{\max_{1\le i\le N} |x_i|^{\max(|w|,D+1)}}{\min_{1\le i<j\le N} |x_i-x_j|^{D'-D+\max(|w|,D+1)}}
\tag{A.1.3}
$$

for the sake of brevity. Substituting our inductive bound, (A.1.1), as well as

$$\dot{C}^{\Lambda}(k) = -\frac{2}{\Lambda^3}\,\mathrm{e}^{-\frac{k^2+m^2}{\Lambda^2}} \tag{A.1.4}$$

into the first term on the r.h.s. of eq.(A.1.2) and integrating over Λ' from Λ to Λ_0 yields the inequality [recall the definition of $|\vec{p}|_{2n+2}$ from eq.(0.5)]

$$\begin{aligned}
&\Big|\int_{\Lambda}^{\Lambda_0} \mathrm{d}\Lambda' \binom{2n+2}{2} \int \mathrm{d}^4k\; \dot{C}^{\Lambda'}(k)\; \partial^w_{\vec{p}}\, \mathcal{F}^{\Lambda',\Lambda_0}_{D,2n+2,l-1}(\otimes_{i=1}^N \mathcal{O}_{A_i}; k, -k, p_1, \ldots, p_{2n})\Big| \\
&\le \int_{\Lambda}^{\Lambda_0} \mathrm{d}\Lambda' \binom{2n+2}{2} \int \mathrm{d}^4k\, \frac{2}{\Lambda'^3}\,\mathrm{e}^{-\frac{k^2+m^2}{\Lambda'^2}}\, \Lambda'^{D-(2n+2)-|w|}\; K^{(4n+8l-7)|w|+D'(n+2l-1)^3} \\
&\times \sqrt{D'!\,(|w|+D'-D)!}\;\mathcal{X}(\vec{x},D',D,w) \sum_{\mu=0}^{d(N,n+1,l-1,w,D')} \frac{1}{\sqrt{\mu!}} \left(\frac{|\vec{p}|_{2n+2}}{\Lambda'}\right)^{\mu} \\
&\times \sum_{\lambda=0}^{2l+n-1} \frac{\log^{\lambda}\left(\sup(\frac{|\vec{p}|_{2n+2}}{\kappa'}, \frac{\kappa'}{m})\right)}{2^{\lambda}\lambda!} \\
&\le \binom{2n+2}{2} K^{(4n+8l-7)|w|+D'(n+2l-1)^3} \sqrt{D'!\,(|w|+D'-D)!}\;\mathcal{X}(\vec{x},D',D,w) \\
&\times \sum_{\lambda=0}^{2l+n-1} \frac{2}{2^{\lambda}\lambda!} \int_{\Lambda}^{\Lambda_0} \mathrm{d}\Lambda'\, \Lambda'^{D-2n-1-|w|} \mathrm{e}^{-\frac{m^2}{\Lambda'^2}} \\
&\times \sum_{\mu=0}^{d(N,n,l,w,D')} \frac{1}{\sqrt{\mu!}} \int \mathrm{d}^4(\frac{k}{\Lambda'}) \left(\frac{|\vec{p}|_{2n+2}}{\Lambda'}\right)^{\mu} \log^{\lambda}\left(\sup(\frac{|\vec{p}|_{2n+2}}{\kappa'}, \frac{\kappa'}{m})\right) \mathrm{e}^{-\frac{k^2}{\Lambda'^2}}
\end{aligned} \tag{A.1.5}$$

Here we made use of the inequality

$$d(N, n+1, l-1, w, D') \le d(N, n, l, w, D')\,, \tag{A.1.6}$$

which follows directly from the definition of d stated in the proposition. One can show [48] that the momentum integral in the last line of (A.1.5) is bounded by

$$\begin{aligned}
&\sum_{\mu=0}^{d} \frac{1}{\sqrt{\mu!}} \int \mathrm{d}^4(k/\Lambda') \left(\frac{|\vec{p}|_{2n+2}}{\Lambda'}\right)^{\mu} \log^{\lambda}\left(\sup(\frac{|\vec{p}|_{2n+2}}{\kappa'}, \frac{\kappa'}{m})\right) \mathrm{e}^{-\frac{k^2}{\Lambda'^2}} \\
&\le 2^d \sum_{\mu=0}^{d} \frac{1}{\sqrt{\mu!}} \left(\frac{|\vec{p}|}{\Lambda'}\right)^{\mu} \left(\log^{\lambda}\left(\sup(\frac{|\vec{p}|}{\kappa'}, \frac{\kappa'}{m})\right) + \sqrt{\lambda!}\right) \quad .
\end{aligned} \tag{A.1.7}$$

The Λ' integral can then be estimated using the formula [48]

$$\sum_{\lambda=0}^{2l+n-1} \frac{1}{2^\lambda \lambda!} \int_\Lambda^{\Lambda_0} \mathrm{d}\Lambda' \, \Lambda'^{-s-1} \left(\log^\lambda \left(\sup(\frac{|\vec{p}|}{\kappa'}, \frac{\kappa'}{m}) \right) + \sqrt{\lambda!} \right) \\ \leq 5 \frac{\Lambda^{-s}}{s} \sum_{\lambda=0}^{2l+n-1} \frac{1}{2^\lambda \lambda!} \log^\lambda \left(\sup(\frac{|\vec{p}|}{\kappa}, \frac{\kappa}{m}) \right) \tag{A.1.8}$$

which holds for any $s \in \mathbb{N}$. Using (A.1.7) and (A.1.8) in formula (A.1.5) and also noting the relation $(2l+n-1)^3 = (2l+n)^3 - 3(2l+n)(2l+n-1) - 1$, we find the bound

$$\Big| \int_\Lambda^{\Lambda_0} \mathrm{d}\Lambda' \binom{2n+2}{2} \int \mathrm{d}^4k \; \dot{C}^{\Lambda'}(k) \, \partial_{\vec{p}}^w \mathcal{F}^{\Lambda',\Lambda_0}_{D,2n+2,l-1}(\otimes_{i=1}^N \mathcal{O}_{A_i}; k, -k, p_1, \dots, p_{2n}) \Big| \\ \leq \Lambda^{D-2n-|w|} \, K^{D'(2l+n)^3+(4n+8l-3)|w|} \, \sqrt{D'! \, (|w|+D'-D)!} \; \mathcal{X}(\vec{x}, D', D, w) \\ \times \sum_{\mu=0}^{d(N,n,l,w,D')} \frac{1}{\sqrt{\mu!}} \left(\frac{|\vec{p}|}{\Lambda} \right)^\mu \sum_{\lambda=0}^{2l+n-1} \frac{\log^\lambda \left(\sup(\frac{|\vec{p}|}{\kappa}, \frac{\kappa}{m}) \right)}{2^\lambda \lambda!} \\ \times \; K^{-3D'(2l+n)(2l+n-1)-D'-4|w|} \binom{2n+2}{2} \, 2^{d(N,n,l,w,D')} \, \frac{10}{2n+|w|-D} \tag{A.1.9}$$

We see that this contribution satisfies the inductive bound, (A.1.1) multiplied by [1] $1/8 \; \mathcal{A}$,

$$\mathcal{A}(D', D, n, l, w) \quad := K^{-3|w|-2D'(n+2l)(n+2l-1)-D'} \tag{A.1.10}$$

if K is chosen large enough such that

$$10(n+1)(2n+1) \; 2^{d(N,n,l,w,D')} K^{-D'(n+2l)(n+2l-1)-|w|} \leq 1/8 \quad , \tag{A.1.11}$$

for all n, l. For large n, we see that the left hand side of (A.1.11) behaves as $n^2 4^{D'n} K^{-D'n^2}$, so a K satisfying the inequality is easy to find in that case. The case of large $l, |w|$ is similar. To see that a large K also satisfies the inequality for small $n, l, |w|$, it is crucial that K always appears with a negative power on the left hand side of the inequality (A.1.11). For this purpose, it is helpful to note that $l \geq 1$ here, since otherwise the first term on the right side of the flow equation is simply zero.

Second term on the r.h.s. of the flow equation: We integrate the second term in the flow equation (A.1.2) over Λ' from Λ to Λ_0 and insert our inductive bound as well as

[1] This factor will be crucial below in the discussion of the relevant terms at non-zero momentum, see in particular eq.(A.1.50).

the known bound for the CAG's without insertions [see (3.2.3)] to obtain

$$\begin{aligned}
&\Big|\int_{\Lambda}^{\Lambda_0} \mathrm{d}\Lambda' \sum_{\substack{l_1+l_2=l\\ n_1+n_2=n+1\\ w_1+w_2+w_3=w}} 4n_1n_2 c_{\{w_j\}} \\
&\times\, \partial_{\vec p}^{w_1}\mathcal{F}^{\Lambda',\Lambda_0}_{D,2n_1,l_1}(\otimes_{i=1}^N\mathcal{O}_{A_i};q,p_1,\ldots,p_{2n_1-1})\,\partial_{\vec p}^{w_2}\dot{C}^{\Lambda'}(q)\,\partial_{\vec p}^{w_3}\mathcal{L}^{\Lambda',\Lambda_0}_{2n_2,l_2}(p_{2n_1},\ldots,p_{2n})\Big| \\
\leq\ & \sum_{\substack{l_1+l_2=l\\ n_1+n_2=n+1\\ w_1+w_2+w_3=w}} 4n_1n_2\, c_{\{w_j\}} \int_{\Lambda}^{\Lambda_0} d\Lambda'\ \Lambda'^{D-2n_1-|w_1|+4-2n_2-|w_2|}\ \mathcal{X}(\vec x,D',D,w_1) \\
&\times K^{D'(n_1+2l_1)^3+(4n_1+8l_1-3+2n_2+4l_2-4)(|w_1|+|w_2|+1)}\sqrt{D'!\,(|w_1|+D'-D)!} \\
&\times \sum_{\mu=0}^{d(N,n_1,l_1,w_1,D')}\frac{1}{\sqrt{\mu!}}\,(\frac{|\vec p|}{\Lambda'})^{\mu}\sum_{\lambda_1=0}^{2l_1+n_1}\frac{\log^{\lambda_1}\big(\sup(\frac{|\vec p|}{\kappa'},\frac{\kappa'}{m})\big)}{2^{\lambda_1}\,\lambda_1!}\,\frac{2}{\Lambda'^3}\,|\partial^{w_3}\,\mathrm{e}^{-\frac{q^2+m^2}{\Lambda'^2}}| \\
&\times \sqrt{|w_2|!}\,(n_2+l_2-2)!\sum_{\lambda_2=0}^{l_2}\frac{\log^{\lambda_2}\big(\sup(\frac{|\vec p|}{\kappa'},\frac{\kappa'}{m})\big)}{2^{\lambda_2}\,\lambda_2!}
\end{aligned} \tag{A.1.12}$$

Here we implicitly assumed that the constant K in proposition 5 is greater or equal to the constant K in the bound for the CAG's without insertion, (3.2.3). It can of course always be chosen that way. We now use the inequality $2l_1+n_1\leq 2l+n-1$, which holds due to the fact that the CAG without insertion vanishes unless $n_2+2l_2\geq 2$, to obtain

$$\begin{aligned}
&\text{r.h.s. of (A.1.12)} \leq K^{D'(2l+n-1)^3+(4n+8l-7)(|w_1|+|w_2|+1)}\sqrt{D'!\,(|w_1|+D'-D)!|w_2|!} \\
&\times \sum_{\substack{l_1+l_2=l\\ n_1+n_2=n+1\\ w_1+w_2+w_3=w}} 4n_1n_2\, c_{\{w_j\}}\,(n_2+l_2-2)!\,\mathcal{X}(\vec x,D',D,w) \\
&\times \int_{\Lambda}^{\Lambda_0}\mathrm{d}\Lambda'\Lambda'^{D-2n-|w_1|-|w_2|-1}\ 2|\partial^{w_3}\,\mathrm{e}^{-\frac{q^2+m^2}{\Lambda'^2}}| \\
&\times \sum_{\mu=0}^{d(N,n_1,l_1,w_1,D')}\frac{1}{\sqrt{\mu!}}\,(\frac{|\vec p|}{\Lambda'})^{\mu}\sum_{\lambda_1=0}^{2l_1+n_1}\sum_{\lambda_2=0}^{l_2}\frac{\log^{\lambda_1+\lambda_2}\big(\sup(\frac{|\vec p|}{\kappa'},\frac{\kappa'}{m})\big)}{2^{\lambda_1+\lambda_2}\,(\lambda_1+\lambda_2)!}\,\frac{(\lambda_1+\lambda_2)!}{\lambda_1!\,\lambda_2!}\,.
\end{aligned} \tag{A.1.13}$$

Here we also made use of the fact that $\mathcal{X}(\vec x,D',D,w_1)\leq\mathcal{X}(\vec x,D',D,w)$ for $|w_1|\leq|w|$. Then, using the bound [15, 59]

$$\Big|\partial^{w}\mathrm{e}^{-\frac{q^2+m^2}{\Lambda^2}}\Big| \leq \mathscr{k}\ \Lambda^{-|w|}\ \sqrt{|w|!}\,2^{\frac{|w|}{2}}\ \mathrm{e}^{-\frac{q^2}{2\Lambda^2}}\ \mathrm{e}^{-\frac{m^2}{\Lambda^2}}\quad,\ \mathscr{k}=1.086\ldots \tag{A.1.14}$$

and the identity $\lambda_1 + \lambda_2 \le 2l_1 + n_1 + l_2 \le 2l + n$ (again since $n_2 + 2l_2 \ge 2$), we find the bound

$$\text{r.h.s. of (A.1.13)} \le \sum_{\substack{l_1+l_2=l,\\ n_1+n_2=n+1}} (n_2+l_2)!\, 4n_1(2l+n)2^{2l+n} K^{(4n+8l-7)(|w|+1)+D'(n+2l-1)^3}$$
$$\times \sum_{w_i} c_{\{w_i\}}\, 2^{\frac{1}{2}|w_3|}\, \mathcal{k}\, \sqrt{(|w_1|+D'-D)!|w_3|!\,|w_2|!\,D'!}\; \mathcal{X}(\vec{x}, D', D, w)$$
$$\times \sum_{\mu=0}^{d(N,n_1,l_1,w_1,D')} \frac{1}{\sqrt{\mu!}} \int_\Lambda^{\Lambda_0} d\Lambda'\, \Lambda'^{D-2n-|w|-1} \left(\frac{|\vec{p}|}{\Lambda}\right)^{\mu} \sum_{\lambda=0}^{2l+n} \frac{\log^\lambda\left(\sup\left(\frac{|\vec{p}|}{\kappa'}, \frac{\kappa'}{m}\right)\right)}{2^\lambda\, \lambda!}\,. \tag{A.1.15}$$

Now the Λ' integral can again be estimated using the inequality (A.1.8). Noting that also

$$d(N, n_1, l_1, w_1, D') \le d(N, n, l, w, D') \tag{A.1.16}$$

which holds as a result of the inequality $2(n_1+l_1) \le 2(n+l)-2$ (using $n_1+n_2 = n+1$, and $n_2 + l_2 \ge 2$), and using also $\sum c_{\{w_i\}} 2^{|w_3|/2} = (2+\sqrt{2})^{|w|}$, one obtains the bound claimed in (A.1.1) multiplied by $1/8\ \mathcal{A}$, which was defined in eq.(A.1.10), provided that K is chosen large enough that

$$K^{-D'(n+2l)(n+2l-1)+(4n+8l-7)-|w|}\, 5\mathcal{k} \sum_{\substack{l_1+l_2=l\\ n_1+n_2=n+1}} (n_2+l_2)!\, 4\, n_1\, (2l+n)\, 2^{2l+n}\, (2+\sqrt{2})^{|w|} \le \frac{1}{8} \tag{A.1.17}$$

Again, it is easy to convince oneself that such a K can be found for large $n, l, |w|$. To see that this is also true for small values of these parameters, it is helpful to note the inequality $(n+2l)(n+2l-1) > 4n+8l-7$, which implies that the exponent of K on the l.h.s. is always negative.

Third term on the r.h.s. of the flow equation: Note that this term is a momentum integral over the CAG's with one insertion, for which we have a bound already, see (3.2.6). To keep formulas at a reasonable length, we will use the notation

$$\vec{p}_i = (p_{2n_{i-1}}, \ldots, p_{2n_i-1}) \tag{A.1.18}$$

in the following, where i takes values between 1 and N and where we set $p_{2n_0} := p_1$ and $p_{2n_N-1} := p_{2n}$. The momentum integral can then be estimated as follows:

Lemma 6: *Let $n_1+\ldots+n_N = n+1$ and $l = l_1+\ldots+l_N$ with $n+l+1 \geq N$. Then we have for any $D \leq D' = [A_1]+\ldots+[A_N]$ and $1 \leq a < b \leq N$*

$$\Big| \partial_{\vec{p}}^{w} \int_k \mathcal{L}^{\Lambda,\Lambda_0}_{2n_a,l_a}(\mathcal{O}_{A_a}(x_a);k,\vec{p}_a)\,\dot{C}^{\Lambda}(k)\,\mathcal{L}^{\Lambda,\Lambda_0}_{2n_b,l_b}(\mathcal{O}_{A_b}(x_b);-k,\vec{p}_b) \times \prod_{r\in\{1,\ldots,N\}\setminus\{a,b\}} \mathcal{L}^{\Lambda,\Lambda_0}_{2n_r,l_r}(\mathcal{O}_{A_r}(x_r);\vec{p}_r) \Big|$$
$$\leq Q\, K_0^{(4n+8l-4)(|w|+D'-D)+D'(n+2l)^3} \sqrt{D'!}\,\Lambda^{D-2n-|w|-1}\, \mathrm{e}^{-m^2/\Lambda^2} \frac{\sqrt{(|w|+D'-D)!}}{|x_a-x_b|^{D'-D}}$$
$$\times \sup\left(\frac{\max_{1\leq i\leq N}|x_i|}{|x_a-x_b|},1\right)^{|w|} \sum_{\mu=0}^{d(N,n,l,w,D')} \frac{\left(\frac{|\vec{p}|}{\Lambda}\right)^{\mu}}{\sqrt{\mu!}} \sum_{\lambda=0}^{2l+n+1-N} \frac{\log^{\lambda}\left(\sup(\frac{|\vec{p}|}{\kappa},\frac{\kappa}{m})\right)+\sqrt{\lambda!}}{2^{\lambda}\lambda!} \tag{A.1.19}$$

where $Q = (N+1)^{|w|}\, 6^{|w|+D'-D}\, N^{2l+n+2-N} N^{d}(2l+n+2-N)\, d\, 2^{d}$ and where K_0 is the constant (called K there) appearing in our bound on the CAG's with one insertion, see (3.2.6). For $n+l+1 < N$ the left hand side vanishes.

Remark 17: For $N = 2$, $D = [A_1]+[A_2]$ and $|w| \leq D+1$ this integral was first estimated by Hollands and Kopper in [15].

Proof. Recall that $\mathcal{L}^{\Lambda,\Lambda_0}_{0,0}(\mathcal{O}_A(0);\vec{p}) = 0$, which implies that the left side of (A.1.19) vanishes for $n+l+1 < N$. Assuming now $n+l+1 \geq N$ and using the translation property of the CAG's with one insertion, eq.(3.1.22), we can rewrite our integrand as

$$\mathcal{L}^{\Lambda,\Lambda_0}_{2n_a,l_a}(\mathcal{O}_{A_a}(x_a);k,\vec{p}_a)\,\dot{C}^{\Lambda}(k)\,\mathcal{L}^{\Lambda,\Lambda_0}_{2n_b,l_b}(\mathcal{O}_{A_b}(x_b);-k,\vec{p}_b)\prod_{r\neq a,b}\mathcal{L}^{\Lambda,\Lambda_0}_{2n_r,l_r}(\mathcal{O}_{A_r}(x_r);\vec{p}_r)$$
$$= \mathrm{e}^{i(p_1+\ldots+p_{2n_1-1})x_1+\ldots+i(p_{2n_{N-1}}+\ldots+p_{2n_N-1})x_N+ik(x_a-x_b)}$$
$$\times \mathcal{L}^{\Lambda,\Lambda_0}_{2n_a,l_a}(\mathcal{O}_{A_a}(0);k,\vec{p}_a)\,\dot{C}^{\Lambda}(k)\,\mathcal{L}^{\Lambda,\Lambda_0}_{2n_b,l_b}(\mathcal{O}_{A_b}(0);-k,\vec{p}_b)\prod_{r\neq a,b}\mathcal{L}^{\Lambda,\Lambda_0}_{2n_r,l_r}(\mathcal{O}_{A_r}(0);\vec{p}_r) \tag{A.1.20}$$

Applying the momentum derivatives then yields

$$\partial_{\vec{p}}^{w}\left(\mathcal{L}^{\Lambda,\Lambda_0}_{2n_a,l_a}(\mathcal{O}_{A_a}(x_a);k,\vec{p}_a)\,\dot{C}^{\Lambda}(k)\,\mathcal{L}^{\Lambda,\Lambda_0}_{2n_b,l_b}(\mathcal{O}_{A_b}(x_b);-k,\vec{p}_j)\prod_{r\neq a,b}\mathcal{L}^{\Lambda,\Lambda_0}_{2n_r,l_r}(\mathcal{O}_{A_r}(x_r);\vec{p}_r)\right)$$
$$= \sum_{w_1+w_2=w} c_{\{w_i\}}\partial_{\vec{p}}^{w_1}\mathrm{e}^{i(p_1+\ldots+p_{2n_1-1})x_1+\ldots+i(p_{2n_{N-1}}+\ldots+p_{2n_N-1})x_N+ik(x_a-x_b)}$$
$$\times\,\partial_{\vec{p}}^{w_2}\left(\mathcal{L}^{\Lambda,\Lambda_0}_{2n_a,l_a}(\mathcal{O}_{A_a}(0);k,\vec{p}_a)\,\dot{C}^{\Lambda}(k)\,\mathcal{L}^{\Lambda,\Lambda_0}_{2n_b,l_b}(\mathcal{O}_{A_b}(0);-k,\vec{p}_b)\prod_{r\neq a,b}\mathcal{L}^{\Lambda,\Lambda_0}_{2n_r,l_r}(\mathcal{O}_{A_r}(0);\vec{p}_r)\right) \tag{A.1.21}$$

The momentum derivatives on the exponentials can be estimated simply via

$$\begin{aligned}&\left|\partial_{\vec{p}}^{w_1} \mathrm{e}^{i(p_1+\ldots+p_{2n_1-1})x_1+\ldots+i(p_{2n_{N-1}}+\ldots+p_{2n_N-1})x_N+ik(x_a-x_b)}\ldots\right| \\ &\leq \max_{1\leq i\leq N}|x_i|^{|w_1|}\left|\mathrm{e}^{i(p_1+\ldots+p_{2n_1-1})x_1+\ldots+i(p_{2n_{N-1}}+\ldots+p_{2n_N-1})x_N+ik(x_a-x_b)}\ldots\right|\end{aligned} \tag{A.1.22}$$

Thus, we have arrived at the bound

$$\begin{aligned}&\left|\,\partial_{\vec{p}}^{w}\int_k \mathcal{L}_{2n_i,l_i}^{\Lambda,\Lambda_0}(\mathcal{O}_{A_a}(x_a);k,\vec{p}_a)\,\dot{C}^{\Lambda}(k)\,\mathcal{L}_{2n_b,l_b}^{\Lambda,\Lambda_0}(\mathcal{O}_{A_b}(x_b);-k,\vec{p}_b)\prod_{r\neq a,b}\mathcal{L}_{2n_r,l_r}^{\Lambda,\Lambda_0}(\mathcal{O}_{A_r}(x_r);\vec{p}_r)\right| \\ &\leq \sum_{w_1+w_2=w} c_{\{w_i\}}\max_{1\leq i\leq N}|x_i|^{|w_1|} \\ &\times\left|\int_k \mathrm{e}^{i(p_1+\ldots+p_{2n_1-1})x_1+\ldots+i(p_{2n_{N-1}}+\ldots+p_{2n_N-1})x_N+ik(x_a-x_b)}\right. \\ &\left.\times\,\partial_{\vec{p}}^{w_2}\left(\mathcal{L}_{2n_a,l_a}^{\Lambda,\Lambda_0}(\mathcal{O}_{A_a}(0);k,\vec{p}_a)\,\dot{C}^{\Lambda}(k)\,\mathcal{L}_{2n_b,l_b}^{\Lambda,\Lambda_0}(\mathcal{O}_{A_b}(0);-k,\vec{p}_b)\prod_{r\neq a,b}\mathcal{L}_{2n_r,l_r}^{\Lambda,\Lambda_0}(\mathcal{O}_{A_r}(0);\vec{p}_r)\right)\right|\end{aligned} \tag{A.1.23}$$

In order to obtain the desired dependence on the IR-cutoff, which is $\Lambda^{D-2n-|w|-1}$ in lemma 6, we now introduce additional momentum derivatives at the cost of inverse powers of $|x_a - x_b|$ through the following procedure: Denote by $||x|| = \max_{\mu\in\{1,\ldots,4\}}|x_\mu|$ the maximal component of x. We can then use the elementary relation

$$\Big|\,||x||\cdot\exp(ikx)\ldots\Big| = \Big|\,\partial_{k_\alpha}\exp(ikx)\ldots\Big| \tag{A.1.24}$$

where $\alpha \in \{1,\ldots,4\}$ is defined via $||x|| = |x_\alpha|$. We use this method to introduce $D' - D + |w_1|$ additional k-derivatives into eq.(A.1.23). These derivatives can then be shifted from the exponential onto the CAG's via partial integration[2], and we arrive at

$$\begin{aligned}
&\Bigg| \int_k \mathrm{e}^{i(p_1+\ldots+p_{2n_1-1})x_1+\ldots+i(p_{2n_{N-1}}+\ldots+p_{2n_N-1})x_N+ik(x_a-x_b)} \\
&\times \partial_{\vec{p}}^{w_2}\Big(\mathcal{L}^{\Lambda,\Lambda_0}_{2n_a,l_a}(\mathcal{O}_{A_a}(0);k,\vec{p}_a)\,\dot{C}^{\Lambda}(k)\,\mathcal{L}^{\Lambda,\Lambda_0}_{2n_b,l_b}(\mathcal{O}_{A_b}(0);-k,\vec{p}_b) \prod_{r\neq a,b} \mathcal{L}^{\Lambda,\Lambda_0}_{2n_r,l_r}(\mathcal{O}_{A_r}(0);\vec{p}_r)\Big)\Bigg| \\
&\leq \Bigg| \int_k \mathrm{e}^{i(p_1+\ldots+p_{2n_1-1})x_1+\ldots+i(p_{2n_{N-1}}+\ldots+p_{2n_N-1})x_N+ik(x_a-x_b)} \\
&\times \frac{1}{||x_a-x_b||^{D'-D+|w_1|}}\,\partial_{k_\alpha}^{D'-D+|w_1|}\partial_{\vec{p}}^{w_2}\Big(\mathcal{L}^{\Lambda,\Lambda_0}_{2n_a,l_a}(\mathcal{O}_{A_a}(0);k,\vec{p}_a)\,\dot{C}^{\Lambda}(k) \\
&\qquad\qquad \times \mathcal{L}^{\Lambda,\Lambda_0}_{2n_b,l_b}(\mathcal{O}_{A_b}(0);-k,\vec{p}_b) \prod_{r\neq a,b} \mathcal{L}^{\Lambda,\Lambda_0}_{2n_r,l_r}(\mathcal{O}_{A_r}(0);\vec{p}_r)\Big)\Bigg| .
\end{aligned} \tag{A.1.25}$$

Here the index α on the k-derivatives is defined through the condition $||x_a - x_b|| = |(x_a - x_b)_\alpha|$. We can now use our bounds for the CAG's with one insertion in order to estimate the integrand. Note also that $|x| \leq 2||x||$ for any $x \in \mathbb{R}^4$. Substituting these estimates along with the bounds for the CAG's with one insertion, we find after some bookkeeping [note also that $D'(n+2l)^3 \geq \sum_i [A_i](n_i+2l_i)^3$]

$$\begin{aligned}
&\Bigg| \partial_{\vec{p}}^{w} \int_k \mathcal{L}^{\Lambda,\Lambda_0}_{2n_a,l_a}(\mathcal{O}_{A_a}(x_a);k,\vec{p}_a)\,\dot{C}^{\Lambda}(k)\,\mathcal{L}^{\Lambda,\Lambda_0}_{2n_b,l_b}(\mathcal{O}_{A_b}(x_b);-k,\vec{p}_b) \\
&\qquad \times \prod_{r\in\{1,\ldots,N\}\setminus\{a,b\}} \mathcal{L}^{\Lambda,\Lambda_0}_{2n_r,l_r}(\mathcal{O}_{A_r}(x_r);\vec{p}_r)\Bigg| \\
&\leq \frac{\sqrt{(|w|+D'-D)!}}{|x_a-x_b|^{D'-D}} \sup\left(\frac{\max_{1\leq i\leq N}|x_i|}{|x_a-x_b|},1\right)^{|w|} Q_0\, K_0^{(4n+8l-4)(|w|+D'-D)+D'(n+2l)^3}\sqrt{D'!} \\
&\times \Lambda^{D-2n-|w|-5}\,\mathrm{e}^{-m^2/\Lambda^2}\int_k \mathrm{e}^{-k^2/2\Lambda^2} \sum_{\mu=0}^{d_1+\ldots+d_N} \frac{\left(\frac{|\vec{p}|+|k|}{\Lambda}\right)^{\mu}}{\sqrt{\mu!}} \sum_{\lambda=0}^{2l+n+1-N} \frac{\log^{\lambda}\left(\sup(\frac{|\vec{p}|+|k|}{\kappa},\frac{\kappa}{m})\right)}{2^\lambda \lambda!}
\end{aligned} \tag{A.1.26}$$

where $Q_0 = (N+1)^{|w|}\,6^{|w|+D'-D}\,N^{2l+n+2-N}\,N^{d_1+\ldots+d_N}\,(2l+n+2-N)(d_1+\ldots+d_N)$ and where we used the shorthand $d_i = 2[A_i](n_i+l_i) + \sup([A_i]+1-2n_i-|w_i|,0)$. We can replace the upper limit for the summation over μ, which is $d_1+\ldots+d_N$, by $d(N,n,l,w,D')$, since

$$d_1+\ldots+d_N \leq 2D'(n+1+l)+D'+N-2 \leq d(N,n,l,w,D')\,. \tag{A.1.27}$$

[2]The exponential damping factor $\dot{C}^{\Lambda}(k)$ ensures that the integrand decays sufficiently rapid for large k to allow for this partial integration. Recall that the bounds on the CAG's with one insertion, (3.2.6), only grow polynomially in the momenta.

The momentum integral can then be bounded by an application of the inequality (A.1.7).

$$
\begin{aligned}
&\sum_{\mu=0}^{d} \frac{1}{\sqrt{\mu!}} \int_{k/\Lambda} e^{-|k|^2/2\Lambda^2} \left(\frac{|\vec{p}|+|k|}{\Lambda}\right)^{\mu} \sum_{\lambda=0}^{2l+n+1-N} \frac{\log^{\lambda}(\sup(\frac{|\vec{p}|+|k|}{\kappa}, \frac{\kappa}{m}))}{2^{\lambda}\lambda!} \\
\leq\ & 2^d \sum_{\mu=0}^{d} \frac{1}{\sqrt{\mu!}} \left(\frac{|\vec{p}|}{\Lambda}\right)^{\mu} \sum_{\lambda=0}^{2l+n+1-N} \frac{\log^{\lambda}(\sup(\frac{|\vec{p}|}{\kappa}, \frac{\kappa}{m})) + \sqrt{\lambda!}}{2^{\lambda}\lambda!}
\end{aligned} \tag{A.1.28}
$$

Substitution into (A.1.26) confirms the bound stated in the lemma. □

Now, to obtain a bound for the third term in the flow equation we have to integrate the bound derived in lemma 6 over Λ' between Λ_0 and Λ. Using the identity (A.1.8) for the Λ' integral, we obtain the bound

$$
\begin{aligned}
&\left| \int_{\Lambda}^{\Lambda_0} d\Lambda' \partial_{\vec{p}}^{w} \int_k \mathcal{L}_{2n_a,l_a}^{\Lambda',\Lambda_0}(\mathcal{O}_{A_a}(x_a); k, \vec{p}_a)\, \dot{C}^{\Lambda'}(k)\, \mathcal{L}_{2n_b,l_b}^{\Lambda',\Lambda_0}(\mathcal{O}_{A_b}(x_b); -k, \vec{p}_b) \right. \\
&\qquad \left. \times \prod_{r\in\{1,\dots,N\}\setminus\{a,b\}} \mathcal{L}_{2n_r,l_r}^{\Lambda',\Lambda_0}(\mathcal{O}_{A_r}(x_r); \vec{p}_r) \right| \\
&\leq Q K_0^{(4n+8l-4)(|w|+D'-D)+D'(n+2l)^3} \frac{\sqrt{D'!\,(|w|+D'-D)!}}{|x_a-x_b|^{D'-D}} \sup\left(\frac{\max_{1\leq i\leq N}|x_i|}{|x_a-x_b|}, 1\right)^{|w|} \\
&\times \int_{\Lambda}^{\Lambda_0} \mathrm{d}\Lambda' \Lambda'^{-1+D-2n-|w|}\, e^{-m^2/\Lambda'^2} \sum_{\mu=0}^{d(N,n,l,w,D')} \frac{1}{\sqrt{\mu!}} \left(\frac{|\vec{p}|}{\Lambda'}\right)^{\mu} \\
&\times \sum_{\hat{a}=0}^{2l+n+1-N} \frac{\log^{\lambda}\left(\sup(\frac{|\vec{p}|}{\kappa'}, \frac{\kappa'}{m})\right) + \sqrt{\lambda!}}{2^{\lambda}\,\lambda!} \\
&\leq 5Q K_0^{(4n+8l-4)(|w|+D'-D)+D'(n+2l)^3} \frac{\sqrt{D'!\,(|w|+D'-D)!}}{|x_a-x_b|^{D'-D}} \sup\left(\frac{\max_{1\leq i\leq N}|x_i|}{|x_a-x_b|}, 1\right)^{|w|} \\
&\times\ \Lambda^{D-2n-|w|} \sum_{\mu=0}^{d(N,n,l,w,D')} \frac{1}{\sqrt{\mu!}} \left(\frac{|\vec{p}|}{\Lambda}\right)^{\mu} \sum_{\lambda=0}^{2l+n+1-N} \frac{\log^{\lambda}\left(\sup(\frac{|\vec{p}|}{\kappa}, \frac{\kappa}{m})\right)}{2^{\lambda}\,\lambda!}
\end{aligned} \tag{A.1.29}
$$

where K_0 is the constant from lemma 6. Recall from the flow equation, eq.(A.1.2), that this term is to be multiplied by $4n_a n_b$, and we also have to apply the operator $\mathbb{S}$ and sum over the configurations $n_1 + \ldots + n_N = n+1$ and $l_1 + \ldots + l_N = l$ as well as over $1 \leq a < b \leq N$. Note that the assumption $x_N = 0$ guarantees that $\max_{1\leq k\leq N}|x_k| / \min_{1\leq i<j\leq N}|x_i - x_j| \geq 1$. In total we find that the inductive bound (A.1.1) multiplied by $1/8$ is reproduced under the condition that $K \geq K_0$ satisfies

$$
20N^2(l+1)^{N-1}(n+1)^{N+1} Q K_0^{(4n+8l-4)(|w|+D'-D)+D'(n+2l)^3} \leq \frac{1}{8} K^{(4n+8l-3)|w|+D'(n+2l)^3}. \tag{A.1.30}
$$

Here it is useful to note that both $(2l+n)$ and $(4n+8l-3)$ are always positive (since $n+l \geq 1$), which helps one to see that the inequality can be satisfied by making K large enough.

(B) Relevant terms ($2n+|w| \leq D$) at $\vec{p}=\vec{0}$:

As the boundary conditions for the relevant terms are given at zero momentum, we will first derive bounds for $\vec{p}=\vec{0}$ and then proceed to arbitrary momentum with the help of the Taylor expansion formula

$$\begin{aligned}\partial^{w}_{\vec{p}}\, f_{2n}(\vec{p}) \;=& \sum_{|\tilde{w}|\leq D-2n-|w|} \frac{\vec{p}^{\,\tilde{w}}}{\tilde{w}!}\,[\partial^{\tilde{w}+w}_{\vec{p}}\, f_{2n}](0) \\ &+ \sum_{|\tilde{w}|=D+1-2n-|w|} \vec{p}^{\,\tilde{w}} \int_0^1 \mathrm{d}\tau \frac{|\tilde{w}|}{\tilde{w}!}\,(1-\tau)^{|\tilde{w}|-1}[\partial^{\tilde{w}+w}_{\vec{p}}\, f_{2n}](\tau\vec{p})\,.\end{aligned} \tag{A.1.31}$$

First term on the r.h.s. of the FE: In view of eq.(A.1.31), let us consider the first term on the r.h.s. of the flow equation with momentum derivatives $\partial^{\tilde{w}+w}_{\vec{p}}$ and with $2n+|\tilde{w}+w| \leq D$ at zero momentum. Integrating over Λ' from 0 to Λ we find

$$\begin{aligned}&\Big|\int_0^{\Lambda} \mathrm{d}\Lambda' \binom{2n+2}{2} \int \mathrm{d}^4k\; \dot{C}^{\Lambda'}(k)\, \partial^{w+\tilde{w}}_{\vec{p}}\, \mathcal{F}^{\Lambda',\Lambda_0}_{D,2n+2,l-1}(\otimes_{i=1}^{N}\mathcal{O}_{A_i};k,-k,0,\ldots,0)\Big| \\ &\leq \binom{2n+2}{2} K^{(4n+8l-7)|\tilde{w}+w|+D'(n+2l-1)^3}\sqrt{D'!\,(|w+\tilde{w}|+D'-D)!} \\ &\times \int_0^{\Lambda} d\Lambda'\; \Lambda'^{D-(2n+2)-|\tilde{w}+w|}\,\frac{2}{\Lambda'^3}\,\mathcal{X}(\vec{x},D',D,w+\tilde{w}) \int \mathrm{d}^4k\; \mathrm{e}^{-\frac{k^2+m^2}{\Lambda'^2}} \\ &\times \sum_{\mu=0}^{d(N,n+1,l-1,\tilde{w}+w,D')} \frac{1}{\sqrt{\mu!}}\left(\frac{|k|}{\Lambda'}\right)^{\mu} \sum_{\lambda=0}^{2l+n-1} \frac{\log^{\lambda}(\sup(\frac{|k|}{\kappa'},\frac{\kappa'}{m}))}{2^{\lambda}\,\lambda!} \\ &\leq \binom{2n+2}{2} K^{(4n+8l-7)|\tilde{w}+w|+D'(n+2l-1)^3}\sqrt{D'!\,(|\tilde{w}+w|+D'-D)!} \\ &\times \mathcal{X}(\vec{x},D',D,w+\tilde{w}) \sum_{\mu=0}^{d(N,n,l,\tilde{w}+w,D')} \sum_{\lambda=0}^{2l+n-1} \frac{2}{2^{\lambda}\lambda!} \int_0^{\Lambda} d\Lambda'\; \Lambda'^{D-(2n+1)-|\tilde{w}+w|} \\ &\times \mathrm{e}^{-\frac{m^2}{\Lambda'^2}} \int \mathrm{d}^4(k/\Lambda')\, \frac{1}{\sqrt{\mu!}}\left(\frac{|k|}{\Lambda'}\right)^{\mu} \log^{\lambda}(\sup(\frac{|k|}{\kappa'},\frac{\kappa'}{m}))\, \mathrm{e}^{-\frac{k^2}{\Lambda'^2}}\end{aligned} \tag{A.1.32}$$

The momentum integral in the last line can be estimated by (cf. inequality (76) in [15])

$$\int \mathrm{d}^4(k/\Lambda')\, \frac{\left(\frac{|k|}{\Lambda'}\right)^{\mu}}{\sqrt{\mu!}} \log^{\lambda}(\sup(\frac{|k|}{\kappa'},\frac{\kappa'}{m}))\, \mathrm{e}^{-\frac{k^2}{\Lambda'^2}} \leq 2^{\frac{\mu}{2}}\, \frac{(\frac{\mu}{2})!}{\sqrt{\mu!}}\, [\log^{\lambda}(\frac{\kappa'}{m}) \;+\; (\lambda!)^{1/2}] \tag{A.1.33}$$

and for the subsequent sum over μ we can use the bound (cf. inequality (88) in [15])

$$\sum_{\mu=0}^{d} 2^{\frac{\mu}{2}} \, \frac{1}{\sqrt{\mu!}} (\frac{\mu}{2})! \, \leq \, 2\, d^{3/2} \, . \tag{A.1.34}$$

For the Λ' integral we make use of the inequality (cf. (89) in [15])

$$\begin{aligned} &\int_0^{\Lambda} \mathrm{d}\Lambda' \; \Lambda'^{D-(2n+1)-|\tilde{w}+w|} \; \log^{\lambda}(\frac{\kappa'}{m}) \, \mathrm{e}^{-\frac{m^2}{\Lambda'^2}} \\ &\leq \; \Lambda^{D-2n-|\tilde{w}+w|} \begin{cases} \log^{\lambda}(\frac{\kappa}{m}) & \text{if } D-2n-|\tilde{w}+w|>0, \\ 2(\lambda+1)^{-1} \log^{\lambda+1}(\frac{\kappa}{m}) & \text{if } D-2n-|\tilde{w}+w|=0. \end{cases} \end{aligned} \tag{A.1.35}$$

Using these bounds to estimate the r.h.s. of (A.1.32) shows that this contribution satisfies the inductive bound, formula (A.1.1), multiplied by $1/8\ \mathcal{A}$, defined in eq.(A.1.10), provided that K is chosen such that

$$12 \binom{2n+2}{2} \; d(N,n,l,\tilde{w}+w,D')^{3/2} \; K^{-D'(n+2l)(n+2l-1)-|w+\tilde{w}|} \; \leq \; 1/8 \, . \tag{A.1.36}$$

It can be seen that it is indeed possible to find such a K. This is easy to see for large values of $n+l$. To see that it is also true for small $n+l$, it is useful to recall that for the first term on the r.h.s. of the flow equation, (A.1.2), we can assume $l \geq 1$, so K will always have a negative exponent.

Second term on the r.h.s of the FE: Inserting the induction hypothesis for the AG's with N insertion and the known bounds for the CAG's without insertion, formula (3.2.3), yields

$$
\begin{aligned}
&\Big|\int_0^{\Lambda} \mathrm{d}\Lambda' \sum_{\substack{l_1+l_2=l\\ n_1+n_2=n+1\\ w_1+w_2+w_3=w+\tilde{w}}} 4n_1 n_2\, c_{\{w_j\}} \\
&\times\ \partial_{\vec{p}}^{w_1} \mathcal{F}^{\Lambda',\Lambda_0}_{D,2n_1,l_1}(\otimes_{i=1}^{N}\mathcal{O}_{A_i}; q, p_1,\ldots,p_{2n_1-1})\, \partial_{\vec{p}}^{w_3} \dot{C}^{\Lambda'}(q)\, \partial_{\vec{p}}^{w_2} \mathcal{L}^{\Lambda',\Lambda_0}_{2n_2,l_2}(p_{2n_1},\ldots,p_{2n})\Big|_{\vec{p}=\vec{0}} \\
&\le \sum_{\substack{l_1+l_2=l,\\ w_1+w_2+w_3=w+\tilde{w},\\ n_1+n_2=n+1}} 4\,n_1 n_2 \int_0^{\Lambda} \mathrm{d}\Lambda'\ \Lambda'^{D+4-2n-2-|w_1|-|w_2|}\ K^{(4n_1+8l_1-3+2n_2+4l_2-4)(|w_1|+|w_2|+1)} \\
&\quad \times\ K^{D'(n_1+2l_1)^3}\sqrt{D'!\,(|w_1|+D'-D)!}\ \mathcal{X}(\vec{x},D',D,w_1)\ c_{\{w_i\}} \sum_{\lambda_1=0}^{2l_1+n_1} \frac{\log^{\lambda_1}(\frac{\kappa'}{m})}{2^{\lambda_1}\,\lambda_1!} \\
&\quad \times\ \frac{2}{\Lambda'^3}\left|\partial^{w_3}\, \mathrm{e}^{-\frac{q^2+m^2}{\Lambda'^2}}\right|_{q=0} \sqrt{|w_2|!}\ (n_2+l_2-2)! \sum_{\lambda_2=0}^{l_2} \frac{\log^{\lambda_2}(\frac{\kappa'}{m})}{2^{\lambda_2}\,\lambda_2!}
\end{aligned}
\tag{A.1.37}
$$

We again use the inequality (A.1.14) and proceed as in (A.1.35) to estimate the Λ' integral. Recall also that $\mathcal{X}(\vec{x},D',D,w_1) \le \mathcal{X}(\vec{x},D',D,w)$ for $|w_1| \le |w|$. As a result we arrive at the bound (A.1.1) multiplied by $1/8\ \mathcal{A}$, under the condition that K satisfies the lower bound

$$
\begin{aligned}
&6\,K^{-D'(n+2l)(n+2l-1)+(4n+8l-7)-|w+\tilde{w}|}\ 5\mathscr{k} \\
&\times \sum_{\substack{l_1+l_2=l,\\ n_1+n_2=n+1}} (n_2+l_2)!\,4\,n_1\,(2l+n)\ 2^{2l+n}\ (2+\sqrt{2})^{|w+\tilde{w}|} \le \frac{1}{8}\,.
\end{aligned}
\tag{A.1.38}
$$

The inequality $(n+2l)(n+2l-1) \ge (4n+8l-6)$ ensures that K always appears with a negative exponent on the right side, which is helpful in order to see that such a K can be found.

Third term on the r.h.s. of the FE : Using lemma 6, we find

$$
\begin{aligned}
&\Bigg| \int_0^\Lambda \mathrm{d}\Lambda' \partial_{\vec{p}}^{w+\tilde{w}} \int_k \mathcal{L}_{2n_a,l_a}^{\Lambda',\Lambda_0}(\mathcal{O}_{A_a}(x_a);k,0,\ldots,0)\, \dot{C}^{\Lambda'}(k)\, \mathcal{L}_{2n_b,l_b}^{\Lambda',\Lambda_0}(\mathcal{O}_{A_b}(x_b);-k,0,\ldots,0) \\
&\qquad\qquad \times \prod_{r\in\{1,\ldots,N\}\setminus\{a,b\}} \mathcal{L}_{2n_r,l_r}^{\Lambda',\Lambda_0}(\mathcal{O}_{A_r}(x_r);\vec{0}) \Bigg| \\
&\leq\ Q\ K_0^{(4n+8l-4)(|w|+|\tilde{w}|+D'-D)+D'(n+2l)^3}\ \sqrt{D'!\,(|w|+|\tilde{w}|+D'-D)!} \\
&\times \mathcal{X}(\vec{x},D',D,w+\tilde{w}) \int_0^\Lambda \mathrm{d}\Lambda' \Lambda'^{D-2n-|w|-|\tilde{w}|-1}\ \mathrm{e}^{-m^2/\Lambda'^2} \sum_{\lambda=0}^{2l+n+1-N} \frac{\log^\lambda(\frac{\kappa'}{m})+\sqrt{\lambda!}}{2^\lambda\,\lambda!}\,.
\end{aligned}
\tag{A.1.39}
$$

Since we assume $N \geq 2$, we can replace the upper limit for the summation over λ by $2l+n-1$. For the integral over Λ' we then use the inequality [15]

$$
\begin{aligned}
&\int_0^\Lambda \mathrm{d}\Lambda' \Lambda'^{D-2n-|w|-|\tilde{w}|-1}\ \mathrm{e}^{-m^2/\Lambda'^2} \sum_{\hat{a}=0}^{2l+n-1} \frac{\log^\lambda(\frac{\kappa'}{m})+\sqrt{\lambda!}}{2^\lambda\,\lambda!} \\
&\leq\ \Lambda^{D-2n-|w|-|\tilde{w}|} \sum_{\lambda=0}^{2l+n-1} \frac{1}{2^\lambda\,\lambda!} \begin{cases} \frac{\log^{\lambda+1}(\frac{\kappa}{m})}{\lambda+1} + \sqrt{\lambda!}\log(\frac{\kappa}{m})+1 & \text{if } D-2n-|w|=|\tilde{w}|, \\ \frac{\log^\lambda(\frac{\kappa}{m})+\sqrt{\lambda!}}{D-2n-|w|-|\tilde{w}|} & \text{if } D-2n-|w|>|\tilde{w}| \end{cases} \\
&\leq\ 6(2l+n)\Lambda^{D-2n-|w|-|\tilde{w}|} \sum_{\lambda=0}^{2l+n} \frac{\log^\lambda(\frac{\kappa}{m})}{2^\lambda\,\lambda!}\,.
\end{aligned}
\tag{A.1.40}
$$

Recalling that we have to multiply (A.1.39) by $4n_a n_b$ and that we also have to apply the symmetrization operator $\mathbb{S}$ and sum over the indices $n_1+\ldots+n_N = n+1$ and $l_1+\ldots+l_N = l$, as well as over $1 \leq a < b \leq N$, we reproduce the inductive bound, (A.1.1), multiplied by $1/8$ under the condition

$$
\begin{aligned}
&4\cdot 6N^2(l+1)^{N-1}(2l+n)\ (n+1)^{N+1} Q K_0^{D'(n+2l)^3} K_0^{(4n+8l-4)(|w|+|\tilde{w}|+D'-D)} \\
&\qquad\qquad\qquad\qquad \leq \frac{1}{8} K^{D'(n+2l)^3} K^{(4n+8l-3)(|w|+|\tilde{w}|)},
\end{aligned}
\tag{A.1.41}
$$

on K. Again, this condition can be satisfied by a sufficiently large K.

(C) Relevant terms ($2n + |w| \leq D$) at arbitrary momentum:

In order to proceed to non-zero momentum we now make use of the Taylor series.

$$
\begin{aligned}
|\partial^w_{\vec{p}} \mathcal{F}^{\Lambda,\Lambda_0}_{D,2n,l}(\otimes_{i=1}^N \mathcal{O}_{A_i};\vec{p})| \;&= \Bigg| \sum_{|\tilde{w}|\leq D-2n-|w|} \frac{\vec{p}^{\,\tilde{w}}}{\tilde{w}!}\, \partial^{\tilde{w}+w}_{\vec{p}} \mathcal{F}^{\Lambda,\Lambda_0}_{D,2n,l}(\otimes_{i=1}^N \mathcal{O}_{A_i};\vec{0}) \\
&+ \sum_{|\tilde{w}|=D+1-2n-|w|} \vec{p}^{\,\tilde{w}} \int_0^1 \mathrm{d}\tau \frac{|\tilde{w}|}{\tilde{w}!} (1-\tau)^{|\tilde{w}|-1}\, \partial^{\tilde{w}+w}_{\tau\vec{p}} \mathcal{F}^{\Lambda,\Lambda_0}_{D,2n,l}(\otimes_{i=1}^N \mathcal{O}_{A_i};\tau\vec{p}) \Bigg|
\end{aligned}
\tag{A.1.42}
$$

On the right hand side we can use the bounds previously derived in **(A)** and **(B)**.

$$
\begin{aligned}
&|\partial^w_{\vec{p}} \mathcal{F}^{\Lambda,\Lambda_0}_{D,2n,l}(\otimes_{i=1}^N \mathcal{O}_{A_i};\vec{p})| \\
&\leq \sum_{|\tilde{w}|\leq D-2n-|w|} \Lambda^{D-2n-|w|} \left(\frac{|\vec{p}|}{\Lambda}\right)^{|\tilde{w}|} \frac{\sqrt{D'!\,(|w|+|\tilde{w}|+D'-D)!}}{\tilde{w}!} \sum_{\lambda=0}^{2l+n} \frac{\log^\lambda(\frac{\kappa}{m})}{2^\lambda \lambda!} \\
&\quad \times \mathcal{X}(\vec{x},D',D,w+\tilde{w}) \Big[1/4\, \mathcal{A}(D',D,n,l,w+\tilde{w})\, K^{(4n+8l-3)(|w|+|\tilde{w}|)+D'(2l+n)^3} \\
&\quad + 24N^2 Q(2l+n)(l+1)^{N-1}(n+1)^{N+1}\, K_0^{(4n+8l-4)(|w|+|\tilde{w}|+D'-D)+D'(2l+n)^3} \Big] \\
&+ \sum_{|\tilde{w}|=D+1-2n-|w|} \Lambda^{D-2n-|w|} \left(\frac{|\vec{p}|}{\Lambda}\right)^{|\tilde{w}|} \frac{\sqrt{D'!\,(|w|+|\tilde{w}|+D'-D)!}}{|\tilde{w}|!} \mathcal{X}(\vec{x},D',D,w+\tilde{w}) \\
&\quad \times |\tilde{w}| \int_0^1 \mathrm{d}\tau (1-\tau)^{|\tilde{w}|-1} \sum_{\mu=0}^{d(N,n,l,w+\tilde{w},D')} \frac{1}{\sqrt{\mu!}} \left(\frac{\tau|\vec{p}|}{\Lambda}\right)^{\mu} \sum_{\lambda=0}^{2l+n} \frac{\log^\lambda(\sup(\frac{\tau|\vec{p}|}{\kappa},\frac{\kappa}{m}))}{2^\lambda \lambda!} \\
&\quad \times \Big[1/4\, \mathcal{A}(D',D,n,l,w+\tilde{w})\, K^{(4n+8l-3)(|w|+|\tilde{w}|)+D'(2l+n)^3} \\
&\qquad + 20N^2 Q(l+1)^{N-1}(n+1)^{N+1}\, K_0^{(4n+8l-4)(|w|+|\tilde{w}|+D'-D)+D'(2l+n)^3} \Big]
\end{aligned}
\tag{A.1.43}
$$

Here K_0 is the constant (called K there) from the bound on the CAG's with one insertion, (3.2.9), and Q is the constant defined in the statement of lemma 6. To obtain the r.h.s. of the inequality above, we used the fact that the first and second term on the r.h.s. of the flow equation satisfy the bound (A.1.1) multiplied by $1/8\ \mathcal{A}$ in the cases **(A)** and **(B)**. Hence the expressions include the factor $\mathcal{A}$. For the contribution from the third term in the flow equation we also used the bounds derived above for the cases **(A)** and **(B)**, but expressed in terms of the constant K_0 instead of K, see the inequalities (A.1.29) and (A.1.39) and the discussion following them.

To simplify the inequality (A.1.43), we make use of the estimates:

$$\mathcal{X}(\vec{x}, D', D, w+\tilde{w}) = \mathcal{X}(\vec{x}, D', D, w) \quad \text{for } |w|+|\tilde{w}| \leq D+1 \tag{A.1.44}$$

$$\sqrt{(|w|+|\tilde{w}|+D'-D)!} \leq \sqrt{|\tilde{w}|!(|w|+D'-D)!}\, 2^{(|w|+|\tilde{w}|+D'-D)/2} \tag{A.1.45}$$

$$\frac{|\tilde{w}|!}{\tilde{w}!} = (8n)^{|w|} \quad \text{for } w \in \mathbb{N}^{8n} \tag{A.1.46}$$

$$\frac{1}{\sqrt{|\tilde{w}|!}} \sum_{\mu=0}^{d} \frac{1}{\sqrt{\mu!}} \left(\frac{|\vec{p}|}{\Lambda}\right)^{\mu+|\tilde{w}|} \leq \sum_{\mu=0}^{d+\tilde{w}} \frac{1}{\sqrt{\mu!}} \left(\frac{|\vec{p}|}{\Lambda}\right)^{\mu} 2^{\mu/2} \tag{A.1.47}$$

$$d(N,n,l,\tilde{w}+w,D') + |\tilde{w}| \leq d(N,n,l,w,D') \quad \text{for } |\tilde{w}| \leq D'-2n-|w|+1 \tag{A.1.48}$$

We then obtain the bound

$$\begin{aligned}
&|\partial_{\vec{p}}^{w} \mathcal{F}_{D,2n,l}^{\Lambda,\Lambda_0}(\otimes_{i=1}^{N} \mathcal{O}_{A_i};\vec{p})| \\
&\leq \Lambda^{D-2n-|w|} \sqrt{D'!\,(|w|+D'-D)!}\; \mathcal{X}(\vec{x},D',D,w) \\
&\times \sum_{\mu=0}^{d(N,n,l,w,D')} \frac{1}{\sqrt{\mu!}} \left(\frac{|\vec{p}|}{\Lambda}\right)^{\mu} \sum_{\lambda=0}^{2l+n} \frac{\log^{\lambda}(\sup(\frac{|\vec{p}|}{\kappa},\frac{\kappa}{m}))}{2^{\lambda}\lambda!} 2^{(|w|+|\tilde{w}|+D'-D)/2} \\
&\times \Bigg(\sum_{|\tilde{w}|\leq D-2n-|w|} \Big[1/4\; \mathcal{A}(D',D,n,l,w+\tilde{w})\; K^{(4n+8l-3)(|w|+|\tilde{w}|)+D'(2l+n)^3} \\
&+ 24N^2 Q(2l+n)(l+1)^{N-1}(n+1)^{N+1}\; K_0^{(4n+8l-4)(|w|+|\tilde{w}|+D'-D)+D'(2l+n)^3} \Big] \frac{(8n)^{|\tilde{w}|}}{\sqrt{\tilde{w}!}} \Bigg) \\
&\times \Bigg(\sum_{|\tilde{w}|\leq D+1-2n-|w|} \Big[1/4\; \mathcal{A}(D',D,n,l,w+\tilde{w})\; K^{(4n+8l-3)(|w|+|\tilde{w}|)+D'(2l+n)^3} \\
&\quad + 20N^2 Q(l+1)^{N-1}(n+1)^{N+1}\; K_0^{(4n+8l-4)(|w|+|\tilde{w}|+D'-D)+D'(2l+n)^3} \Big] |\tilde{w}|\, 2^{d/2} \Bigg)
\end{aligned} \tag{A.1.49}$$

It follows that this contribution satisfies the bound claimed in the theorem, (A.1.1), provided that K is chosen sufficiently large such that the following two conditions are satisfied:

$$\sum_{j\leq D+1-2n-|w|} (8n)^j\, j\, 2^{\frac{d(N,n,l,w,D')}{2}}\, 2^{\frac{j+|w|+D'-D}{2}}\, K^{(4n+8l-3)j}\; \mathcal{A}(D',D,n,l,|w|+j) \leq 1 \tag{A.1.50}$$

$$\begin{aligned}
&\sum_{j\leq D+1-2n-|w|} 24N^2 Q(2l+n)(l+1)^{N-1}(n+1)^{N+1}\; K_0^{(4n+8l-4)(|w|+j+D'-D)+D'(2l+n)^3} \\
&\qquad \times (8n)^j\, j\, 2^{\frac{d(N,n,l,w,D')}{2}}\, 2^{\frac{j+|w|+D'-D}{2}} \leq \frac{1}{4} K^{(4n+8l-3)|w|+D'(2l+n)^3}
\end{aligned} \tag{A.1.51}$$

It can be seen that it is indeed possible to choose K such that the conditions (A.1.50) and (A.1.51) are satisfied (recall also that we can always assume $n+l \geq 1$, since otherwise the contributions vanish), which finishes the proof of proposition 5. □

With proposition 5 at hand, we are now in a position to verify bound 1 without much effort.

Proof of bound 1. We can insert the bounds from proposition 5 into the flow equation (A.1.2) once more and integrate over Λ from 0 to m. Note that there is a damping factor $\exp(-m^2/\Lambda^2)$ in each of the terms on the r.h.s. of the flow equation, so we can bound negative powers of Λ through the estimate

$$\begin{aligned}\int_0^m \mathrm{d}\Lambda' \exp(-m^2/\Lambda'^2)\, \frac{\Lambda'^{D-2n-|w|-\mu-1}}{\sqrt{\mu!}} &\leq m^{D-2n-|w|-\mu}\, \frac{\sqrt{(2n+|w|+\mu-D)_+!}}{\sqrt{\mu!}} \\ &\leq 2^{(2n+|w|+\mu-D)_+}\, m^{D-2n-|w|-\mu} \sqrt{(2n-D)_+!|w|!}\end{aligned} \tag{A.1.52}$$

Choosing a somewhat larger constant K to accommodate for the additional powers of 2 and for the factor $\sqrt{(2n-D)_+!}$, we obtain bound 1. □

A.1.1. Alternative bound with explicit Λ_0-dependence

It was mentioned in remark 5 that we can derive a version of bound 1 where the factor $\min_{1\leq i<j\leq N} |x_i - x_j|^{D-D'}$ is replaced by a factor $\Lambda_0^{D'-D}/\sqrt{(D'-D)!}$. This is achieved as follows: We follow the same inductive strategy as in the proof of bound 1. For the first two terms on the right hand side of the flow equation (A.1.2), we can follow exactly the same steps as before, making the adjustment $\min_{1\leq i<j\leq N} |x_i - x_j|^{D-D'} \to \Lambda_0^{D'-D}/\sqrt{(D'-D)!}$ in the definition of the placeholder $\mathcal{X}$. Concerning the source term, we again make use of lemma 6, but we choose $D = D'$ for the constant appearing in that lemma. The resulting bound differs from the one used in the proof of proposition 5 by the replacement $\min_{1\leq i<j\leq N} |x_i - x_j|^{D-D'} \to \Lambda^{D'-D}/\sqrt{(D'-D)!}$. We recall that by assumption $\Lambda \leq \Lambda_0$, so we can replace the extra powers of Λ by powers of Λ_0, which confirms that also the third term on the right hand side of the flow equation satisfies the alternative bound.

A.1.2. Alternative bound with improved IR-behaviour

Finally, we would like to substantiate our claim in remark 5 concerning the IR-behaviour of the bound. Recall that we introduced additional momentum derivatives at the cost of inverse powers of $|x_i - x_j|$ in the proof of lemma 6. Nothing prevents us from going further and introducing $r \in \mathbb{N}$ additional momentum derivatives this way. The resulting bound, then, contains an additional factor $\Lambda^{-r}\sqrt{r!}$ and some additional powers of K_0, which are not relevant here. This bound would of course not be consistent with the induction hypothesis (A.1.1) due to the extra powers of Λ^{-1}. However, we can get rid of this factor with the help of the exponential $\mathrm{e}^{-m^2/\Lambda^2}$, which is still at our disposal, i.e. we use the estimate $\Lambda^{-r}\mathrm{e}^{-m^2/\Lambda^2} \le m^{-r}\sqrt{r!}$. We can then bound the Λ integral as before in (A.1.29) and (A.1.40) and adjust the definition of $\mathcal{X}$ accordingly to obtain proposition 5 with an additional factor of $r!\,(m \cdot \min_{1\le i<j\le N} |x_i - x_j|)^{-r}$ on the right hand side of the bound, which is just the behaviour claimed in remark 5.

A.2 Proof of bound 2

We will assume $x_N = 0$ for the remainder of this proof, which allows us to keep formulas relatively compact. The general case, $x_N \neq 0$, follows quite easily in the end with the help of the translation properties of the AG's with insertions, eq.(3.1.48). Recalling the decomposition (3.1.38) of the functionals G_D^{Λ,Λ_0},

$$G_D^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}) := \hbar F_D^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}) + \prod_{i=1}^N L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_i})\,, \tag{A.2.1}$$

we first verify bound 2 for the factorised contribution. Estimating the remainder of the Taylor expansion with the help of eq.(3.2.17) and using the Lowenstein rule (3.1.45) for the spacetime derivatives, we arrive at the formula

$$\begin{aligned}
&\Big|(1 - \sum_{j\le\Delta-1} \mathbb{T}^j_{\vec{x}\to\vec{0}} - \mathbb{T}^\Delta_{\vec{x}\to\vec{0}})\, \partial^w_{\vec{p}} \prod_{i=1}^N \mathcal{L}^{\Lambda,\Lambda_0}(\mathcal{O}_{A_i}(x_i);\vec{p}_i)\Big| \\
&= \Big|\int_0^1 \mathrm{d}\tau \sum_{v:|v_1|+\ldots+|v_N|=\Delta} \frac{\Delta}{v!}(1-\tau)^{\Delta-1}\vec{x}^{\,v}\partial^w_{\vec{p}} \prod_{i=1}^N \mathcal{L}^{\Lambda,\Lambda_0}(\partial^{v_i}\mathcal{O}_{A_i}(\tau(x_i));\vec{p}_i) \\
&\quad - \sum_{v:|v_1|+\ldots+|v_N|=\Delta} \frac{\vec{x}^{\,v}}{v!}\partial^w_{\vec{p}} \prod_{i=1}^N \mathcal{L}^{\Lambda,\Lambda_0}(\partial^{v_i}\mathcal{O}_{A_i}(0);\vec{p}_i)\Big| \\
&\le 2 \sum_{w_1+w_2=w} c_{\{w_j\}} \frac{\max_{1\le i\le N}|x_i|^{\Delta+|w_1|}\, 2^\Delta}{(\Delta-1)!} \sum_{|v_1|+\ldots+|v_N|=\Delta} \partial^{w_2}_{\vec{p}} \prod_{i=1}^N \Big|\mathcal{L}^{\Lambda,\Lambda_0}(\partial^{v_i}\mathcal{O}_{A_i}(0);\vec{p}_i)\Big|
\end{aligned} \tag{A.2.2}$$

The CAG's with one insertion on the right hand side can now be estimated using the inequality (3.2.9). It is then not hard to check that this contribution indeed satisfies the claimed bound, (3.2.16), for a suitably large constant K.

To prove that the remaining contribution containing the F_D^{Λ,Λ_0}-functionals also satisfies bound 2, we first derive the following result:

Proposition 6: *Let* $D = D' + \Delta$, *where* $D' = [A_1] + \ldots + [A_N]$ *and* $\Delta > 0$, *and assume* $x_N = 0$. *There exists a constant* $K > 0$ *such that*

$$
\begin{aligned}
&|\partial^w_{\vec{p}}(1 - \sum_{j\le\Delta} \mathbb{T}^j_{\vec{x}\to\vec{0}}) \mathcal{F}^{\Lambda,\Lambda_0}_{D,2n,l}(\otimes^N_{i=1}\mathcal{O}_{A_i}(x_i);\vec{p})| \\
&\le \Lambda^{D-2n-|w|}\, K^{(4n+8l-3)|w|}\, K^{D(n+2l)^3}\sqrt{D'!|w|!}\sum_{\mu=0}^{d(N,n,l,w,D)}\frac{1}{\sqrt{\mu!}}\left(\frac{|\vec{p}|}{\Lambda}\right)^{\mu} \\
&\times \sum_{\lambda=0}^{2l+n}\frac{\log^{\lambda}\left(\sup(\frac{|\vec{p}|}{\kappa},\frac{\kappa}{m})\right)}{2^\lambda \lambda!}\left(\frac{\max\limits_{1\le i\le N}|x_i|}{\min\limits_{1\le i<j\le N}|x_i - x_j|}\right)^{\max(|w|,D+1)-\Delta}\frac{\max\limits_{1\le i\le N}|x_i|^\Delta}{\sqrt{\Delta!}}
\end{aligned}
\tag{A.2.3}
$$

with $d(N,n,l,w,D') := 2D'(n + l + 2(N-1)) + \sup(D' + 1 - 2n - |w|, 0)$.

Proof. The proof is very similar to that of proposition 5. Applying the Taylor expansion $(1 - \sum_{j\le\Delta}\mathbb{T}^j_{\vec{x}\to\vec{0}})$ to the flow equation (A.1.2), we can in fact follow exactly the same inductive procedure as in the proof of proposition 5 to show that the first two terms on the r.h.s. of the flow equation satisfy the bound (A.2.3). The only significant difference is that in the proof we replace $\mathcal{X}(\vec{x}, D', D, w)$, as defined in (A.1.3), by

$$
\tilde{\mathcal{X}}(\vec{x}, D', D, w) := \left(\frac{\max\limits_{1\le i\le N}|x_i|}{\min\limits_{1\le i<j\le N}|x_i - x_j|}\right)^{\max(|w|,D+1)-\Delta}\frac{\max\limits_{1\le i\le N}|x_i|^\Delta}{\sqrt{\Delta!}}\,. \tag{A.2.4}
$$

The only properties of this function that were used in the proof are

$$
\mathcal{X}(\vec{x}, D', D, w_1) \le \mathcal{X}(\vec{x}, D', D, w) \quad \text{for } |w_1| \le |w| \tag{A.2.5}
$$

$$
\mathcal{X}(\vec{x}, D', D, w + \tilde{w}) = \mathcal{X}(\vec{x}, D', D, w) \quad \text{for } |w| + |\tilde{w}| \le D + 1\,. \tag{A.2.6}
$$

Clearly, both these conditions are fulfilled also by $\tilde{\mathcal{X}}$. Thus, with the adjustment $\mathcal{X} \to \tilde{\mathcal{X}}$ one simply has to follow the same steps as in the proof of proposition 5. The main effort in proving proposition 6 therefore goes into showing that the "source term", i.e. the third term on the r.h.s. of the flow equation, satisfies the claimed bound. In order to verify our improved bound (3.2.16), we will need, instead of lemma 6, the following estimate:

Lemma 7: *Let* $n_1 + \ldots + n_N = n + 1$, $l = l_1 + \ldots + l_N$ *with* $n + l + 1 \geq N$ *and* $x_N = 0$. *Further, pick* $D = D' + \Delta$, *where* $D' = [A_1] + \ldots + [A_N]$ *and* $\Delta \geq 0$. *Then we have for any* $1 \leq a < b \leq N$

$$\begin{aligned}
&\Big| \partial^w_{\vec{p}} (1 - \sum_{j \leq \Delta} \mathbb{T}^j_{\vec{x} \to \vec{0}}) \int_k \mathcal{L}^{\Lambda,\Lambda_0}_{2n_a,l_a}(\mathcal{O}_{A_a}(x_a); k, \vec{p}_a)\, \dot{C}^\Lambda(k) \\
&\quad \times \mathcal{L}^{\Lambda,\Lambda_0}_{2n_b,l_b}(\mathcal{O}_{A_b}(x_b); -k, \vec{p}_b) \prod_{r \in \{1,\ldots,N\} \setminus \{a,b\}} \mathcal{L}^{\Lambda,\Lambda_0}_{2n_r,l_r}(\mathcal{O}_{A_r}(x_r); \vec{p}_r) \Big| \\
&\leq Q\, K_0^{(4n+8l-4)|w|} K_0^{D(n+2l)^3} \sqrt{D'!|w|!}\, \Lambda^{D-2n-|w|-1}\, \mathrm{e}^{-m^2/\Lambda^2} \frac{\max_{1\leq i\leq N} |x_i|^\Delta}{\sqrt{\Delta!}} \\
&\times \sup \left(\frac{\max\limits_{1\leq i \leq N} |x_i|}{|x_a - x_b|}, 1 \right)^{(|w|-\Delta)_+} \sum_{\mu=0}^{d(N,n,l,w,D)} \frac{\left(\frac{|\vec{p}|}{\Lambda}\right)^\mu}{\sqrt{\mu!}} \sum_{\lambda=0}^{2l+n+1-N} \frac{\log^\lambda\left(\sup(\frac{|\vec{p}|}{\kappa}, \frac{\kappa}{m})\right) + \sqrt{\lambda!}}{2^\lambda \lambda!}
\end{aligned} \tag{A.2.7}$$

where $Q = \Delta\, 4^{\Delta+|w|} (N+1)^{|w|}\, 6^{|w|}\, N^{2l+n+1-N} N^d (2l+n+1-N)\, d\, 2^d$, *where* $(|w| - \Delta)_+ = \sup(|w| - \Delta, 0)$, *and where* K_0 *is the constant (called* K *there) appearing in our bound on the CAG's with one insertion, see* (3.2.6). *For* $n + l + 1 < N$ *the left hand side vanishes.*

Proof. Vanishing of the l.h.s. for $n + l + 1 < N$ follows again from $\mathcal{L}^{\Lambda,\Lambda_0}_{0,0}(\mathcal{O}_A(0)) = 0$. We start off by decomposing the Taylor expansion into two steps. For later convenience, we define the tuple

$$\vec{\xi} = (\xi_1, \ldots, \xi_N) := \Big(x_1, \ldots, \underbrace{\frac{x_a + x_b}{\sqrt{2}}}_{a\text{-th entry}}, \ldots, \underbrace{\frac{x_a - x_b}{\sqrt{2}}}_{b\text{-th entry}}, \ldots, x_N \Big) \in \mathbb{R}^{4N}, \tag{A.2.8}$$

i.e. $\vec{\xi}$ is equal to $\vec{x}$ except for the a-th and b-th entry (geometrically, $\vec{\xi}$ is obtained from $\vec{x}$ by a $\pi/4$-rotation in the (a, b)-plane). The Taylor expansion can equivalently be expressed in terms of the $\vec{\xi}$ coordinates:

$$\begin{aligned}
(1 - \sum_{j\leq\Delta} \mathbb{T}^j_{\vec{x}\to\vec{0}}) &= (1 - \sum_{j \leq \Delta} \mathbb{T}^j_{\vec{\xi} \to (0,\ldots,\xi_b,\ldots,0)}) \\
&+ \sum_{j_1 \leq \Delta} \mathbb{T}^{j_1}_{\vec{\xi} \to (0,\ldots,\xi_b,\ldots,0)} (1 - \sum_{j_2 \leq \Delta - j_1} \mathbb{T}^{j_2}_{\vec{\xi} \to (\xi_1,\ldots,\xi_{b-1},0,\xi_{b+1},\ldots,\xi_N)})
\end{aligned} \tag{A.2.9}$$

This equation can be checked explicitly using the definition of the operators $\mathbb{T}$, eq.(3.1.61). Substituting the decomposition (A.2.9) on the left hand side of (A.2.7), we will show that the contributions from both summands on the r.h.s. of (A.2.9) satisfy the claimed bound. For the contribution corresponding to the second term on the r.h.s. of eq.(A.2.9), this simply follows from lemma 6 combined with the remainder formula (3.2.17). Note in particular that in this case

$$\sup\left(\frac{\max_{1\leq i\leq N}|x_i|}{|x_a-x_b|},1\right)_{\vec{\xi}=(0,\ldots,\xi_b,\ldots,0)}=\sup\left(\frac{|\xi_b|}{2|\xi_b|},1\right)=1\,, \tag{A.2.10}$$

so this particular factor can be neglected in the bounds. Using the formula (3.2.17) for the remainder of the Taylor expansion, we can write the contribution from the first term on the r.h.s. of eq.(A.2.9) as

$$\begin{aligned}
&\partial^w_{\vec{p}}\int_k\Bigg[\int_0^1 \mathrm{d}\tau\,\frac{(1-\tau)^{\Delta-1}}{(\Delta-1)!}\partial^\Delta_\tau\mathcal{L}^{\Lambda,\Lambda_0}_{2n_a,l_a}(\mathcal{O}_{A_a}(\frac{\tau\xi_a+\xi_b}{\sqrt{2}});k,\vec{p}_a)\,\dot{C}^\Lambda(k)\\
&\quad\times\mathcal{L}^{\Lambda,\Lambda_0}_{2n_b,l_b}(\mathcal{O}_{A_b}(\frac{\tau\xi_a-\xi_b}{\sqrt{2}});-k,\vec{p}_b)\prod_{r\in\{1,\ldots,N\}\setminus\{a,b\}}\mathcal{L}^{\Lambda,\Lambda_0}_{2n_r,l_r}(\mathcal{O}_{A_r}(\tau\xi_r);\vec{p}_r)\\
&-\mathbb{T}^\Delta_{\vec{\xi}\to(0,\ldots,\xi_b,\ldots,0)}\mathcal{L}^{\Lambda,\Lambda_0}_{2n_a,l_a}(\mathcal{O}_{A_a}(\frac{\xi_a+\xi_b}{\sqrt{2}});k,\vec{p}_a)\,\dot{C}^\Lambda(k)\\
&\quad\times\mathcal{L}^{\Lambda,\Lambda_0}_{2n_b,l_b}(\mathcal{O}_{A_b}(\frac{\xi_a-\xi_b}{\sqrt{2}});-k,\vec{p}_b)\prod_{r\in\{1,\ldots,N\}\setminus\{a,b\}}\mathcal{L}^{\Lambda,\Lambda_0}_{2n_r,l_r}(\mathcal{O}_{A_r}(\xi_r);\vec{p}_r)\Bigg]
\end{aligned} \tag{A.2.11}$$

The second term on the right hand side of this expression is again easily found to be consistent with the claimed bound (A.2.7) with the help of lemma 6 together with the Lowenstein rule (3.1.45). Using the translation properties of the CAG's with insertions, eq.(3.1.22), the first term on the right hand side can also be written in the form

$$\begin{aligned}
&\sum_{w_1+w_2=w}c_{\{w_i\}}\int_k\int_0^1\mathrm{d}\tau\,\frac{(1-\tau)^{\Delta-1}}{(\Delta-1)!}\partial^\Delta_\tau\\
&\times\partial^{w_1}_{\vec{p}}\mathrm{e}^{i(p_{2n_a-1}+\ldots+k)\frac{\tau\xi_a+\xi_b}{\sqrt{2}}+i(p_{2n_b-1}+\ldots-k)\frac{\tau\xi_a-\xi_b}{\sqrt{2}}+i\sum_{r\neq a,b}(p_{2n_r-1}+\ldots+p_{2n_r-1})\tau\xi_r}\\
&\partial^{w_2}_{\vec{p}}\left(\mathcal{L}^{\Lambda,\Lambda_0}_{2n_a,l_a}(\mathcal{O}_{A_a}(0);k,\vec{p}_a)\dot{C}^\Lambda(k)\mathcal{L}^{\Lambda,\Lambda_0}_{2n_b,l_b}(\mathcal{O}_{A_b}(0);-k,\vec{p}_b)\prod_{r\neq a,b}\mathcal{L}^{\Lambda,\Lambda_0}_{2n_r,l_r}(\mathcal{O}_{A_r}(0);\vec{p}_r)\right)
\end{aligned} \tag{A.2.12}$$

Our aim is now to *get rid of* the momentum derivatives on the exponential in the second line. Let us first consider derivatives with respect to $\vec{p}_r$ with $r\neq a,b$. Every such derivative on the exponential yields a factor $i\tau\xi_r$. Each of those factors of τ, in turn, allows us to perform a partial integration in τ without picking up a boundary contribution, i.e. we can use

$$\int_0^1\mathrm{d}\tau\,(1-\tau)^{\Delta-1}\cdot\tau^n\,\partial^\Delta_\tau\ldots=(-1)^n\int_0^1\mathrm{d}\tau\,\partial^n_\tau\left((1-\tau)^{\Delta-1}\cdot\tau^n\right)\partial^{\Delta-n}_\tau\ldots\,, \tag{A.2.13}$$

which holds for $n < \Delta$. The momentum derivatives with respect to $\vec{p}_a$ and $\vec{p}_b$ yield factors of $i(\tau\xi_a \pm \xi_b)/\sqrt{2}$, respectively. The additional factors of τ can again be used to perform partial integrations as above. The powers of ξ_b, on the other hand, can be transformed into k-derivatives, which can then be moved to the term in the last line of (A.2.12) via partial integration in k. Summing up, we can estimate (A.2.12) via [recall that we use the notation $(c)_+ = \sup(c, 0)$]

$$
\begin{aligned}
\Big|(\text{A.2.12})\Big| \leq & \sum_{w_1+w_1=w} c_{\{w_i\}} \frac{\max\limits_{1\leq i\leq N} |x_i|^\Delta}{(\Delta-1)!} \int_0^1 \mathrm{d}\tau \left|\partial_\tau^{\min(|w_1|,\Delta-1)} \left((1-\tau)^{\Delta-1}\tau^{|w_1|}\right)\right| |\vec{p}|^{\max(\Delta-|w_1|,1)} \\
& \times \sup\left(\frac{\max_{1\leq i\leq N}|x_i|}{|x_a-x_b|/2}, 1\right)^{(|w_1|-\Delta+1)_+} \int_k \partial_{\vec{p}}^{w_2} \partial_{k_\alpha}^{(|w_1|-\Delta+1)_+} \Big| \mathcal{L}^{\Lambda,\Lambda_0}_{2n_a,l_a}(\mathcal{O}_{A_a}(0); k, \vec{p}_a)\, \dot{C}^\Lambda(k) \\
& \times \mathcal{L}^{\Lambda,\Lambda_0}_{2n_b,l_b}(\mathcal{O}_{A_b}(0); -k, \vec{p}_b) \prod_{r\neq a,b} \mathcal{L}^{\Lambda,\Lambda_0}_{2n_r,l_r}(\mathcal{O}_{A_r}(0); \vec{p}_r)\Big|
\end{aligned}
\tag{A.2.14}
$$

where the index $\alpha \in \{1, \ldots, 4\}$ on the k-derivatives satisfies $||x_a - x_b|| = |(x_a - x_b)_\alpha|$. Using the bounds on the CAG's with one insertion, (3.2.6), as well as the estimate

$$
\int_0^1 \mathrm{d}\tau \left|\partial_\tau^n \left((1-\tau)^\Delta \cdot \tau^n\right)\right| \leq 2^n \int_0^1 \mathrm{d}\tau \sup_{0\leq m\leq n} \left((1-\tau)^{\Delta-m} \cdot \tau^m \frac{\Delta! n!}{(\Delta-m)! m!}\right) \leq 2^n n!\,,
\tag{A.2.15}
$$

the claimed bound, (A.2.7), is verified after some straightforward algebraic manipulations and estimates. □

Returning to the proof of proposition 6, recall that our final task to finish the proof is to make sure that the source term, i.e. the third term on the right hand side of the flow equation, satisfies the claimed bound. For this purpose, we have to estimate the Λ integral over the bound we have just derived in lemma 7. To achieve this, one can again follow the same computational steps as in the proof of proposition 5, i.e. we make use of the estimate (A.1.8). We spare the reader the repetition of these calculations. □

With proposition 6 at hand, we can now remove the dependence on the IR-cutoff just as in the estimate (A.1.52). This finishes the proof of bound 2 for the case $x_N = 0$. As mentioned above, one can use the translation properties of the amputated Green's functions with insertions, eq.(3.1.48), in order to generalise the bound to $x_N \neq 0$.

A.3 Proof of bound 3

Our strategy here will be similar to that of the previous two sections. First, we will derive a bound on the H^{Λ,Λ_0}-functionals for arbitrary Λ with the help of the inductive scheme based on the renormalisation flow equations. To verify bound 3 we will then remove the Λ dependence under the assumption $\Lambda \leq m$.

Proposition 7: *Let $D \leq [A_1]+\ldots+[A_M]$ and $x_M = 0$. There exists a constant $K > 0$ such that*

$$
\begin{aligned}
&|\partial^w_{\vec{p}} \mathcal{H}^{\Lambda,\Lambda_0}_{2n,l}\left([\otimes_{i=1}^M \mathcal{O}_{A_i}(x_i)]_D; \otimes_{i=M+1}^N \mathcal{O}_{A_i}(x_i); \vec{p}\right)| \leq \frac{\Lambda^{-1-2n-|w|}}{\min\limits_{\substack{1\leq i\leq M\\ M+1\leq j\leq N}} |x_i - x_j|^D} \\
&\times \frac{\sqrt{(|w|+D'+1)!\,D'!}\,K^{(4n+8l-3)|w|+D'(n+2l)^3}}{\min\limits_{\substack{1\leq i\leq N, i\neq j\\ 1\leq j\leq M}} |x_i - x_j|^{[A_1]+\ldots+[A_M]-D} \min\limits_{\substack{1\leq i\leq N, i\neq j\\ M+1\leq j\leq N}} |x_i - x_j|^{[A_{M+1}]+\ldots+[A_N]+1}} \\
&\times \left(\frac{\max_{1\leq i\leq M}|x_i|}{\min_{1\leq i<j\leq M}|x_i - x_j|} \cdot \frac{\max_{M+1\leq i\leq N}|x_i - x_N|}{\min_{M+1\leq i<j\leq N}|x_i - x_j|}\right)^{D'+|w|+1} \\
&\times \left(\frac{\max\limits_{1\leq i\leq N}|x_i|}{\min\limits_{1\leq i\leq M<j\leq N}|x_i - x_j|}\right)^{|w|} \sum_{\mu=0}^{d(N,n,l,w,D')} \frac{1}{\sqrt{\mu!}}\left(\frac{|\vec{p}|}{\Lambda}\right)^{\mu} \sum_{\lambda=0}^{2l+n} \frac{\log^\lambda\left(\sup(\frac{|\vec{p}|}{\kappa}, \frac{\kappa}{m})\right)}{2^\lambda \lambda!}
\end{aligned}
\tag{A.3.1}
$$

with $D' = [A_1]+\ldots+[A_N]$ and $d(N,n,l,w,D') := 2D'(n+l+2(N-1)) + \sup(D'+1-2n-|w|, 0)$.

Proof of proposition 7. Our strategy is to use the same inductive scheme as in the proof of proposition 5. Our task is to integrate the flow equation for $\mathcal{H}$, which can be written as [cf. eq.(3.1.51)]

$$
\begin{aligned}
&\partial_\Lambda \partial^w_{\vec{p}} \mathcal{H}^{\Lambda,\Lambda_0}_{2n,l}\left([\otimes_{i=1}^M \mathcal{O}_{A_i}]_D ; \otimes_{j=M+1}^N \mathcal{O}_{A_j} ; p_1,\dots,p_{2n}\right) = \\
&= \binom{2n+2}{2} \int_k \dot{C}^\Lambda(k)\, \partial^w_{\vec{p}} \mathcal{H}^{\Lambda,\Lambda_0}_{2n+2,l-1}\left([\otimes_{i=1}^M \mathcal{O}_{A_i}]_D ; \otimes_{j=M+1}^N \mathcal{O}_{A_j} ; k,-k,p_1,\dots,p_{2n}\right) \\
&- \mathbb{S}\Bigg[\partial^w_{\vec{p}} \sum_{\substack{l_1+l_2=l\\ n_1+n_2=n+1}} 4n_1n_2 \mathcal{H}^{\Lambda,\Lambda_0}_{D,2n_1,l_1}\left([\otimes_{i=1}^M \mathcal{O}_{A_i}]_D ; \otimes_{j=M+1}^N \mathcal{O}_{A_j} ; q,p_1,\dots,p_{2n_1-1}\right) \\
&\qquad\qquad \times \dot{C}^\Lambda(q)\, \mathcal{L}^{\Lambda,\Lambda_0}_{2n_2,l_2}(p_{2n_1},\dots,p_{2n}) \\
&- \sum_{\substack{l_1+l_2=l-2\\ n_1+n_2=n+1}} 4n_1n_2 \partial^w_{\vec{p}} \int_k \mathcal{F}^{\Lambda,\Lambda_0}_{D,2n_1,l_1}\left(\otimes_{i=1}^M \mathcal{O}_{A_i} ; k,p_1,\dots,p_{2n_1-1}\right) \\
&\qquad\qquad \times \dot{C}^\Lambda(k)\, \mathcal{F}^{\Lambda,\Lambda_0}_{2n_2,l_2}\left(\otimes_{i=M+1}^N \mathcal{O}_{A_i} ; -k,p_{2n_1},\dots,p_{2n}\right) \\
&- \sum_{\substack{l_1+\dots+l_{M+1}=l-1\\ n_1+\dots+n_{M+1}=n+1\\ 1\le a\le M}} 4n_a n_{M+1} \partial^w_{\vec{p}} \int_k \mathcal{L}^{\Lambda,\Lambda_0}_{2n_a,l_a}(\mathcal{O}_{A_a};k,\vec{p}_a) \prod_{r\in\{1,\dots,M\}\setminus\{a\}} \mathcal{L}^{\Lambda,\Lambda_0}_{2n_r,l_r}(\mathcal{O}_{A_r};\vec{p}_r) \\
&\qquad\qquad \times \dot{C}^\Lambda(k) \mathcal{F}^{\Lambda,\Lambda_0}_{2n_{M+1},l_{M+1}}\left(\otimes_{i=M+1}^N \mathcal{O}_{A_i} ; -k,p_{2n_M},\dots,p_{2n}\right) \\
&- \sum_{\substack{l_M+\dots+l_N=l-1\\ n_M+\dots+n_N=n+1\\ M+1\le a\le N}} 4n_a n_M \partial^w_{\vec{p}} \int_k \mathcal{L}^{\Lambda,\Lambda_0}_{2n_a,l_a}(\mathcal{O}_{A_a};k,\vec{p}_a) \prod_{r\in\{M+1,\dots,N\}\setminus\{a\}} \mathcal{L}^{\Lambda,\Lambda_0}_{2n_r,l_r}(\mathcal{O}_{A_r};\vec{p}_r) \\
&\qquad\qquad \times \dot{C}^\Lambda(k) \mathcal{F}^{\Lambda,\Lambda_0}_{D,2n_M,l_M}\left(\otimes_{i=1}^M \mathcal{O}_{A_i} ; -k,p_1,\dots,p_{2n_M-1}\right) \\
&- \sum_{\substack{l_1+\dots+l_N=l\\ n_1+\dots+n_N=n+1\\ 1\le a\le M<b\le N}} 4n_a n_b \partial^w_{\vec{p}} \int_k \mathcal{L}^{\Lambda,\Lambda_0}_{2n_a,l_a}(\mathcal{O}_{A_a};k,\vec{p}_a)\, \dot{C}^\Lambda(k) \mathcal{L}^{\Lambda,\Lambda_0}_{2n_b,l_b}(\mathcal{O}_{A_b};-k,\vec{p}_b) \prod_{r\in\{1,\dots,N\}\setminus\{a,b\}} \mathcal{L}^{\Lambda,\Lambda_0}_{2n_r,l_r}(\mathcal{O}_{A_r};\vec{p}_r)\Bigg]
\end{aligned}
\tag{A.3.2}
$$

Note that, in contrast to the proof of proposition 5, we do not have to distinguish between *relevant* and *irrelevant* contributions. Here the boundary conditions for the H^{Λ,Λ_0} functionals, eq.(3.1.52), imply that we always integrate from Λ to Λ_0. In the following we will estimate each of the terms on the r.h.s. of eq.(A.3.2) separately.

First and second term: The first two terms on the r.h.s. of the flow equation, which are linear in $\mathcal{H}^{\Lambda,\Lambda_0}$, can be estimated using the same inductive scheme used in the proof of proposition 5. As mentioned above, we only have to consider the part of the induction which refers to *irrelevant contributions*. Collecting all the $\vec{x}$ dependent terms in $\bar{\mathcal{X}}$,

$$\bar{\mathcal{X}}(\vec{x}, D', D, w) := \frac{\left(\frac{\max_{1\le i\le M}|x_i|}{\min_{1\le i<j\le M}|x_i-x_j|}\cdot\frac{\max_{M+1\le i\le N}|x_i-x_N|}{\min_{M+1\le i<j\le N}|x_i-x_j|}\right)^{D'+|w|+1}}{\min\limits_{\substack{1\le i\le M\\ M+1\le j\le N}}|x_i-x_j|^D \min\limits_{\substack{1\le i\le N, i\neq j\\ 1\le j\le M}}|x_i-x_j|^{[A_1]+\ldots+[A_M]-D}}$$
$$\times \frac{1}{\min\limits_{\substack{1\le i\le N, i\neq j\\ M+1\le j\le N}}|x_i-x_j|^{[A_{M+1}]+\ldots+[A_N]+1}}\left(\frac{\max\limits_{1\le i\le N}|x_i|}{\min\limits_{1\le i\le M<j\le N}|x_i-x_j|}\right)^{|w|} \tag{A.3.3}$$

we can follow exactly the same steps as in the proof of proposition 5 (our boundary conditions correspond to the case $D = -1$ there). Note that, crucially, $\bar{\mathcal{X}}$ satisfies the condition (A.2.5), and that the condition (A.2.6), which is not satisfied by $\bar{\mathcal{X}}$, is not used in the part of the proof dealing with irrelevant contributions. We will not repeat the lengthy estimates here.

Third term: In order to estimate the third term on the r.h.s. of the flow equation, we will make use of he following lemma.

Lemma 8: *Let $n_1 + n_2 = n + 1$ and $l - 2 = l_1 + l_2 \geq 0$. Further, let $\vec{p}_1 = (p_1, \ldots, p_{2n_1-1})$ and $\vec{p}_2 = (p_{2n_1}, \ldots, p_{2n})$. Then we have for any $D \leq [A_1]+\ldots+[A_M]$*

$$\left|\partial_{\vec{p}}^w \int_k \mathcal{F}^{\Lambda,\Lambda_0}_{D,2n_1,l_1}(\otimes_{i=1}^M \mathcal{O}_{A_i}(x_i); k, \vec{p}_1)\, \dot{C}^\Lambda(k)\, \mathcal{F}^{\Lambda,\Lambda_0}_{2n_2,l_2}(\otimes_{j=M+1}^N \mathcal{O}_{A_j}(x_j); -k, \vec{p}_2)\right|$$
$$\leq \quad Q\, K_0^{(4n+8l-3)(|w|+D)} K_0^{D'(n+2l)^3}\, \Lambda^{-2n-|w|-2}\, \mathrm{e}^{-m^2/\Lambda^2} \sqrt{(|w|+D'+1)!\, D'!}$$
$$\times \frac{\left(\frac{\max_{1\le i\le M}|x_i|}{\min_{1\le i<j\le M}|x_i-x_j|}\cdot\frac{\max_{M+1\le i\le N}|x_i-x_N|}{\min_{M+1\le i<j\le N}|x_i-x_j|}\right)^{D+|w|+1}}{\min\limits_{1\le i<j\le M}|x_i-x_j|^{[A_1]+\ldots+[A_M]-D} \min\limits_{M+1\le i<j\le N}|x_i-x_j|^{[A_{M+1}]+\ldots+[A_N]+1}|x_N|^D}$$
$$\times \sum_{\mu=0}^{d(N,n,l,w,D')} \frac{1}{\sqrt{\mu!}}\left(\frac{|\vec{p}|}{\Lambda}\right)^\mu \sum_{\lambda=0}^{2l+n-3} \frac{\log^\lambda\left(\sup(\frac{|\vec{p}|}{\kappa}, \frac{\kappa}{m})\right) + \sqrt{\lambda!}}{2^\lambda \lambda!} \tag{A.3.4}$$

where $Q = 18^{|w|+D}\, 2^{2l+n} 4^d (2l+n)\, d$ and where K_0 is the constant (called K there) appearing in our bound on the F-functionals, see proposition 5.

Proof. Using the translation properties of the F-functionals, the integral can be written as

$$
\begin{aligned}
&\left| \partial_{\vec{p}}^{w} \int_k \mathcal{F}_{D,2n_1,l_1}^{\Lambda,\Lambda_0}\left(\otimes_{i=1}^{M} \mathcal{O}_{A_i}(x_i); k, \vec{p}_1\right) \dot{C}^{\Lambda}(k)\, \mathcal{F}_{2n_2,l_2}^{\Lambda,\Lambda_0}\left(\otimes_{i=M+1}^{N} \mathcal{O}_{A_i}(x_i); -k, \vec{p}_2\right) \right| \\
&\leq \Bigg| \int_k \sum_{w_1+w_2+w_3=w} c_{\{w_i\}}\, \partial_{\vec{p}}^{w_3} \mathrm{e}^{i x_N (p_{2n_1}+\ldots+p_{2n}) - i k x_N} \\
&\quad \times \partial_{\vec{p}}^{w_1} \mathcal{F}_{D,2n_1,l_1}^{\Lambda,\Lambda_0}\left(\otimes_{i=1}^{M} \mathcal{O}_{A_i}(x_i); k, \vec{p}_1\right) \dot{C}^{\Lambda}(k) \\
&\qquad\qquad \times \partial_{\vec{p}}^{w_2} \mathcal{F}_{2n_2,l_2}^{\Lambda,\Lambda_0}\left(\otimes_{i=M+1}^{N} \mathcal{O}_{A_i}(x_i - x_N); -k, \vec{p}_2\right) \Bigg|
\end{aligned}
\tag{A.3.5}
$$

The momentum derivatives on the exponential can turned into k-derivatives and moved onto the moments of the F-functionals via partial integration. Just as in the proof of lemma 6, we now introduce D additional k-derivatives at the cost of a factor $(2/|x_N|)^D$, in order to obtain the desired dependence on Λ. Hence

$$
\begin{aligned}
&\left| \partial_{\vec{p}}^{w} \int_k \mathcal{F}_{D,2n_1,l_1}^{\Lambda,\Lambda_0}\left(\otimes_{i=1}^{M} \mathcal{O}_{A_i}(x_i); k, \vec{p}_1\right) \dot{C}^{\Lambda}(k)\, \mathcal{F}_{2n_2,l_2}^{\Lambda,\Lambda_0}\left(\otimes_{i=M+1}^{N} \mathcal{O}_{A_i}(x_i); -k, \vec{p}_2\right) \right| \\
&\leq \Bigg| \int_k \sum_{w_1+w_2+w_3=w} c_{\{w_i\}} \mathrm{e}^{i x_N (p_{2n_1}+\ldots+p_{2n}) - i k x_N} \left(\frac{2}{|x_N|}\right)^D \\
&\times \partial_{k_\alpha}^{D} \partial_k^{w_3} \Bigg(\partial_{\vec{p}}^{w_1} \mathcal{F}_{D,2n_1,l_1}^{\Lambda,\Lambda_0}\left(\otimes_{i=1}^{M} \mathcal{O}_{A_i}(x_i); k, \vec{p}_1\right) \dot{C}^{\Lambda}(k) \\
&\qquad\qquad \times \partial_{\vec{p}}^{w_2} \mathcal{F}_{2n_2,l_2}^{\Lambda,\Lambda_0}\left(\otimes_{i=M+1}^{N} \mathcal{O}_{A_i}(x_i - x_N); -k, \vec{p}_2\right) \Bigg) \Bigg|
\end{aligned}
\tag{A.3.6}
$$

where $\alpha \in \{1, \ldots, 4\}$ corresponds to the maximal component of x_N, i.e. $||x_N|| =: |x_{N,\alpha}|$. Distributing the k-derivatives over the three factors, substituting our bounds for the F-functionals [see proposition 5] and estimating the k-integral as in (A.1.28) then yields the lemma after some elementary estimates for binomial factors. □

Continuing the proof of proposition 7, we can now estimate the Λ-integral over the third term on the r.h.s. of the flow equation. Using lemma 8 and the bound (A.1.7), we find

$$\left| \int_{\Lambda}^{\Lambda_0} d\Lambda' \partial_{\vec{p}}^{w} \int_k \mathcal{F}^{\Lambda',\Lambda_0}_{D,2n_1,l_1}\left(\otimes_{i=1}^{M}\mathcal{O}_{A_i}; k, \vec{p}_1\right) \dot{C}^{\Lambda'}(k)\, \mathcal{F}^{\Lambda',\Lambda_0}_{2n_2,l_2}\left(\otimes_{i=M+1}^{N}\mathcal{O}_{A_i}; -k, \vec{p}_2\right) \right|$$
$$\leq \Lambda^{-1-2n-|w|}\, 5QK_0^{(4n+8l-3)(|w|+D)+D'(n+2l)^3} \sum_{\mu=0}^{d} \frac{\left(\frac{|\vec{p}|}{\Lambda}\right)^{\mu}}{\sqrt{\mu!}} \sum_{\lambda=0}^{2l+n-3} \frac{\log^{\lambda}(\sup(\frac{|\vec{p}|}{\kappa}, \frac{\kappa}{m}))}{2^{\lambda}\,\lambda!}$$
$$\times \frac{\sqrt{(|w|+D'+1)!\, D'!}\left(\frac{\max_{1\leq i\leq M}|x_i|}{\min_{1\leq i<j\leq M}|x_i-x_j|}\cdot\frac{\max_{M<i\leq N}|x_i-x_N|}{\min_{M<i<j\leq N}|x_i-x_j|}\right)^{D+1+|w|}}{\min_{1\leq i<j\leq M}|x_i-x_j|^{[A_1]+\ldots+[A_M]-D}\min_{M+1\leq i<j\leq N}|x_i-x_j|^{[A_{M+1}]+\ldots+[A_N]+1}|x_N|^{D}} \tag{A.3.7}$$

where K_0 is the constant from lemma 8 (called K there). Multiplying this bound by $4n_1n_2$, applying the operator $\mathbb{S}$ and summing over the configurations $n_1 + n_2 = n + 1$ and $l_1 + l_2 = l - 2$ we find that the inductive bound (A.3.1), multiplied by $1/6$, is fulfilled under the condition that K satisfies (recall also that we assumed $D \leq D'$ in the proposition)

$$20l\,(n+1)^3\, Q\, K_0^{(4n+8l-3)(|w|+D)}\, K_0^{D'(n+2l)^3} \leq \frac{1}{6} K^{(4n+8l-3)|w|}\, K^{D'(n+2l)^3}\,. \tag{A.3.8}$$

Here it is useful to note that both $(2l + n)$ and $(4n + 8l - 3)$ are always positive (since $l \geq 2$ for this contribution), which helps one to see that the inequality can be satisfied by making K large enough.

Fourth and fifth term: The following lemma will help us to estimate both these contributions:

Lemma 9: *Let $n_1 + \ldots + n_{M+1} = n + 1$ and $l - 1 = l_1 + \ldots + l_{M+1}$. Further, let $\vec{p}_i$ be defined as in eq.(A.1.18). Then we have for any $D \le [A_{M+1}] + \ldots + [A_N]$ and $a \in \{1, \ldots, M\}$*

$$
\begin{aligned}
&\Bigg| \partial^w_{\vec{p}} \int_k \mathcal{L}^{\Lambda,\Lambda_0}_{2n_a,l_a}(\mathcal{O}_{A_a}; k, \vec{p}_a) \prod_{r\in\{1,\ldots,M\}\setminus\{a\}} \mathcal{L}^{\Lambda,\Lambda_0}_{2n_r,l_r}(\mathcal{O}_{A_r}; \vec{p}_r) \\
&\qquad \times \dot{C}^\Lambda(k) \mathcal{F}^{\Lambda,\Lambda_0}_{D,2n_{M+1},l_{M+1}}\left(\otimes_{i=M+1}^N \mathcal{O}_{A_i}; -k, p_{2n_M}, \ldots, p_{2n}\right) \Bigg| \\
&\le Q\, K_0^{(4n+8l-3)(|w|+D+[A_1]+\ldots+[A_M]+1)} K_0^{D'(n+2l)^3}\, \Lambda^{-2n-|w|-2}\, \mathrm{e}^{-m^2/\Lambda^2} \\
&\times \sum_{\mu=0}^{d(N,n,l,w,D')} \frac{1}{\sqrt{\mu!}} \left(\frac{|\vec{p}|}{\Lambda}\right)^\mu \sum_{\lambda=0}^{2l+n-M} \frac{\log^\lambda\left(\sup(\frac{|\vec{p}|}{\kappa}, \frac{\kappa}{m})\right) + \sqrt{\lambda!}}{2^\lambda \lambda!} \\
&\times \frac{\sqrt{(|w| + D' + 1)!\, D'!}\, \left(\frac{\max_{M+1\le i\le N}|x_i - x_N|}{\min_{M+1\le i<j\le N}|x_i - x_j|}\right)^{D+[A_1]+\ldots+[A_M]+|w|+1} \left(\frac{\max_{1\le i\le M}|x_i|}{|x_N - x_a|}\right)^{|w|}}{|x_N - x_a|^{[A_1]+\ldots+[A_M]+D+1} \min_{M+1\le i<j\le N}|x_i - x_j|^{[A_{M+1}]+\ldots+[A_N]-D}}
\end{aligned}
\tag{A.3.9}
$$

where $Q_0 = (M+2)^{|w|}\, 6^{|w|+D+[A_1]+\ldots+[A_M]+1} (M+1)^{2l+n-M+1} (M+1)^d (2l+n-M+1)\, d$ and where K_0 is the constant (called K there) appearing in our bound on the F-functionals, see proposition 5.

Proof. Using the same strategy as in the proof of lemma 8, we can estimate the l.h.s. of eq.(A.3.9) as

$$
\begin{aligned}
&\Bigg| \partial^w_{\vec{p}} \int_k \mathcal{L}^{\Lambda,\Lambda_0}_{2n_a,l_a}(\mathcal{O}_{A_a}; k, \vec{p}_a) \prod_{r\in\{1,\ldots,M\}\setminus\{a\}} \mathcal{L}^{\Lambda,\Lambda_0}_{2n_r,l_r}(\mathcal{O}_{A_r}; \vec{p}_r) \\
&\qquad \times \dot{C}^\Lambda(k) \mathcal{F}^{\Lambda,\Lambda_0}_{D,2n_{M+1},l_{M+1}}\left(\otimes_{i=M+1}^N \mathcal{O}_{A_i}; -k, p_{2n_M}, \ldots, p_{2n}\right) \Bigg| \\
&\le \sum_{w_1+w_2=w} c_{\{w_i\}} \left(\frac{\max_{1\le i\le M}|x_i|}{|x_a - x_N|/2}\right)^{|w_1|} \frac{1}{(|x_a - x_N|/2)^{[A_1]+\ldots+[A_M]+D+1}} \\
&\times \Bigg| \int_k \partial_{k_\alpha}^{[A_1]+\ldots+[A_M]+D+1+|w_1|} \partial^{w_2}_{\vec{p}} \Bigg(\mathcal{L}^{\Lambda,\Lambda_0}_{2n_a,l_a}(\mathcal{O}_{A_a}(0); k, \vec{p}_j) \prod_{r\in\{1,\ldots,M\}\setminus\{a\}} \mathcal{L}^{\Lambda,\Lambda_0}_{2n_r,l_r}(\mathcal{O}_{A_r}(0); \vec{p}_r) \\
&\qquad \times \dot{C}^\Lambda(k) \mathcal{F}^{\Lambda,\Lambda_0}_{D,2n_{M+1},l_{M+1}}\left(\otimes_{i=M+1}^N \mathcal{O}_{A_i}(x_i - x_N); -k, p_{2n_M}, \ldots, p_{2n}\right) \Bigg) \Bigg|
\end{aligned}
\tag{A.3.10}
$$

We now distribute the momentum derivatives over the factors, pull the modulus inside the integral and substitute our previous bounds from (3.2.6) and (A.1.1). Using also (A.1.28) to estimate the momentum integral, we arrive at the lemma, after some straightforward algebraic manipulations and elementary estimates. □

This lemma allows us to find a bound for the Λ integral of both the fourth and the fifth term on the r.h.s. of the flow equation, (A.3.2). Using the case $D = -1$ in the lemma, we can estimate the Λ-integral over the fourth term on the r.h.s. of the flow equation as (here we choose again $a \in \{1, \dots, M\}$)

$$\begin{aligned}
&\Bigg| \int_{\Lambda}^{\Lambda_0} \mathrm{d}\Lambda' \partial_{\vec{p}}^{w} \int_k \mathcal{L}^{\Lambda',\Lambda_0}_{2n_a,l_a}(\mathcal{O}_{A_a}; k, \vec{p}_a) \prod_{r\in\{1,\dots,M\}\setminus\{a\}} \mathcal{L}^{\Lambda',\Lambda_0}_{2n_r,l_r}(\mathcal{O}_{A_r}; \vec{p}_r) \\
&\qquad \times \dot{C}^{\Lambda'}(k) \mathcal{F}^{\Lambda',\Lambda_0}_{2n_{M+1},l_{M+1}}\left(\otimes_{i=M+1}^{N} \mathcal{O}_{A_i}; -k, p_{2n_M}, \dots, p_{2n}\right) \Bigg| \\
&\le \frac{\left(\frac{\max_{M+1\le i\le N}|x_i - x_N|}{\min_{M+1\le i<j\le N}|x_i-x_j|}\right)^{[A_1]+\dots+[A_M]} \left(\frac{\max_{M+1\le i\le N}|x_i - x_N|}{\min_{M+1\le i<j\le N}|x_i-x_j|} \cdot \frac{\max_{1\le i\le M}|x_i|}{|x_N - x_a|}\right)^{|w|}}{|x_a - x_N|^{[A_1]+\dots+[A_M]} \min\limits_{M<i<j\le N} |x_i - x_j|^{[A_{M+1}]+\dots+[A_N]+1}} \\
&\times\ 5\ Q\ K_0^{(4n+8l-3)(|w|+[A_1]+\dots+[A_M])}\ K_0^{D'(n+2l)^3}\ \Lambda^{-1-2n-|w|} \sqrt{(|w| + D' + 1)!\, D'!} \\
&\times \sum_{\mu=0}^{d(N,n,l,w,D')} \frac{1}{\sqrt{\mu!}} \left(\frac{|\vec{p}|}{\Lambda}\right)^{\mu} \sum_{\lambda=0}^{2l+n-M} \frac{\log^{\lambda}(\sup(\frac{|\vec{p}|}{\kappa}, \frac{\kappa}{m}))}{2^{\lambda}\, \lambda!}
\end{aligned} \tag{A.3.11}$$

where K_0 is the constant from lemma 9. Multiplying the bound (A.3.11) by $4n_a n_{M+1}$, applying the operator $\mathbb{S}$ and summing over the configurations $n_1 + \dots + n_{M+1} = n + 1$ and $l_1 + \dots + l_{M+1} = l - 1$ as well as over $1 \le a \le M$, we find that the inductive bound is reproduced under the condition that K satisfies

$$\begin{aligned}
&20 M l^M (n+1)^{M+3} Q K_0^{(4n+8l-3)(|w|+[A_1]+\dots+[A_M])} K_0^{D'(n+2l)^3} \\
&\qquad\qquad \le \frac{1}{8} K^{(4n+8l-3)|w| + D'(n+2l)^3} ,
\end{aligned} \tag{A.3.12}$$

which can always be achieved by a large enough K.

For the fifth term on the r.h.s. of the flow equation we use lemma 9 again, but we exchange the role of the indices $(1, \dots, M) \leftrightarrow (M+1, \dots, N)$ to obtain the bound (here we fix $a \in \{M+1, \ldots, N\}$)

$$
\begin{aligned}
&\Bigg| \int_{\Lambda}^{\Lambda_0} \mathrm{d}\Lambda' \partial_{\vec{p}}^{w} \int_k \mathcal{L}^{\Lambda',\Lambda_0}_{2n_a,l_a}(\mathcal{O}_{A_a}; k, \vec{p}_a) \prod_{r \in \{M+1,\ldots,N\}\setminus\{a\}} \mathcal{L}^{\Lambda',\Lambda_0}_{2n_r,l_r}(\mathcal{O}_{A_r}; \vec{p}_r) \\
&\qquad \times \dot{C}^{\Lambda'}(k) \mathcal{F}^{\Lambda',\Lambda_0}_{D,2n_M,l_M}\left(\otimes_{i=1}^{M} \mathcal{O}_{A_i}; -k, p_{2n_1}, \ldots, p_{2n_M-1}\right) \Bigg| \\
&\leq \frac{\left(\frac{\max_{1\leq i\leq M}|x_i - x_M|}{\min_{1\leq i<j\leq M}|x_i-x_j|}\right)^{D+[A_{M+1}]+\ldots+[A_N]+1} \left(\frac{\max_{1\leq i\leq M}|x_i - x_M|}{\min_{1\leq i<j\leq M}|x_i-x_j|} \cdot \frac{\max_{M+1\leq i\leq N}|x_i|}{|x_a - x_M|}\right)^{|w|}}{\min_{1\leq i<j\leq M}|x_i - x_j|^{[A_1]+\ldots+[A_M]-D} |x_a - x_M|^{[A_{M+1}]+\ldots+[A_N]+D+1}} \\
&\times 5\, Q\, K_0^{(4n+8l-3)(|w|+D+[A_{M+1}]+\ldots+[A_N]+1)}\; K_0^{D'(n+2l)^3}\; \Lambda^{-1-2n-|w|} \\
&\times \frac{\sqrt{(|w|+D'+1)!\, D'!}}{\min_{\substack{1\leq i\leq M\\ M+1\leq j\leq N}} |x_i - x_j|^D} \sum_{\mu=0}^{d(N,n,l,w,D')} \frac{1}{\sqrt{\mu!}} \left(\frac{|\vec{p}|}{\Lambda}\right)^{\mu} \sum_{\lambda=0}^{2l+n-(N-M)} \frac{\log^{\lambda}(\sup(\frac{|\vec{p}|}{\kappa}, \frac{\kappa}{m}))}{2^{\lambda}\, \lambda!}
\end{aligned}
\tag{A.3.13}
$$

By the same arguments as above, and also recalling that we assumed $x_M = 0$ in the proposition, we find that the inductive bound is reproduced under the condition that K satisfies

$$
\begin{aligned}
20(N-M) l^{N-M} (n+1)^{N-M+3}\, Q\, K_0^{(4n+8l-3)(|w|+[A_{M+1}]+\ldots+[A_N]+1)}\, K_0^{D'(n+2l)^3} \\
\leq \frac{1}{8} K^{(4n+8l-3)|w|}\, K^{D'(n+2l)^3}\,,
\end{aligned}
\tag{A.3.14}
$$

which can always be achieved by a large enough K.

Sixth term: The last term on the r.h.s. of the flow equation (A.3.2) can be estimated with the help of lemma 6. If we take the $D = -1$ in the inequality (A.1.19), multiply by $4n_a n_b$, apply the operator $\mathbb{S}$ and sum over the configurations $n_1 + \ldots + n_N = n+1$ and $l_1 + \ldots + l_N = l$ as well as over $1 \leq a \leq M$ and $M+1 \leq b \leq N$, we find that this bound is consistent with the inequality (A.3.1), provided K is chosen large enough that

$$
\begin{aligned}
20N^2(l+1)^{N-1}(n+1)^{N+1}\, Q\, K_0^{(4n+8l-4)(|w|+D'+1)}\, K_0^{D'(n+2l)^3} \\
\leq \frac{1}{8} K^{(4n+8l-3)|w|+D'(n+2l)^3}\,,
\end{aligned}
\tag{A.3.15}
$$

which, again, can always be satisfied by an appropriate choice of K. □

Bound 3 is obtained from proposition 7 by inserting the inequality (A.3.1) into the flow equation (A.3.2) and integrating over Λ from 0 to m. The procedure is analogous to the derivation of bound 1, see (A.1.52).

A.4 Proof of bound 4

To arrive at the claimed bound, we follow the same steps as in the proof of proposition 7. The main difference is that in the case at hand we make use of proposition 6 to estimate the Taylor expansion of the regularised AG's.

Proposition 8: *Let $D = [A_1] + \ldots + [A_M] + \Delta$ and $x_M = 0$ and assume $|x_i| \leq |x_j|$ for all $1 \leq i \leq M < j \leq N$. There exists a constant $K > 0$ such that*

$$\Big|\partial^w_{\vec{p}}(1 - \sum_{j\leq\Delta} \mathbb{T}^j_{(x_1,\ldots,x_M)\to\vec{0}})\mathcal{H}^{\Lambda,\Lambda_0}_{2n,l}\left([\otimes^M_{i=1}\mathcal{O}_{A_i}(x_i)]_D; \otimes^N_{i=M+1}\mathcal{O}_{A_i}(x_i); \vec{p}\right)\Big|$$
$$\leq \frac{\sqrt{|w|!}D'!K^{(4n+8l-3)|w|+(D'+\Delta)(n+2l)^3}}{\Lambda^{1+2n+|w|}\min_{1\leq i<j\leq N}|x_i - x_j|^{D'+2}} \sum_{\mu=0}^{d(N,n,l,w,D'+\Delta)} \frac{\left(\frac{|\vec{p}|}{\Lambda}\right)^\mu}{\sqrt{\mu!}} \sum_{\lambda=0}^{2l+n} \frac{\log^\lambda(\sup(\frac{|\vec{p}|}{\kappa}, \frac{\kappa}{m}))}{2^\lambda\lambda!}$$
$$\times \frac{\left(\max\limits_{1\leq i\leq M}|x_i| \cdot \max\limits_{M<i\leq N}|x_i - x_N|\right)^\Delta}{\min\limits_{M<j\leq N}|x_j|^{\Delta-1}\cdot\min\limits_{M<i<j\leq N}|x_i - x_j|^\Delta} \cdot \left(\frac{\max\limits_{1\leq i<j\leq N}|x_i - x_j|}{\min\limits_{1\leq i<j\leq N}|x_i - x_j|}\right)^{2D'+3|w|+2} \tag{A.4.1}$$

with $D' = [A_1] + \ldots + [A_N]$ and $d(N,n,l,w,D) := 2D'(n + l + 2(N-1)) + \sup(D + 1 - 2n - |w|, 0)$.

Proof. We follow the same strategy as in the proof of proposition 7. Applying the operator $(1 - \sum_{j\leq\Delta}\mathbb{T}^j)$ to both sides of the flow equation (A.3.2), we again bound all six terms on the right hand side separately. Since the procedure is very close to that described in the previous section, we only focus on the differences in the proof.

- For the first two terms on the right hand side of the flow equation we again verify the bound (A.4.1) inductively. Adjusting the expression $\tilde{\mathcal{X}}(\vec{x}, D', D, w)$ accordingly, this part of the proof works just the same way as in propositions 5 and 7.

- For the third and fifth term, the only difference to the proof presented in the previous section is that we make use of proposition 6 to bound the Taylor expansion of the moments of $F^{\Lambda,\Lambda_0}_D(\otimes^M_{i=1}\mathcal{O}_{A_i})$.

- To estimate the fourth term, we make use of the expression (3.2.17) for the remainder of the Taylor expansion and pull the resulting spacetime derivatives in the CAG's with the help of the Lowenstein rule (3.1.45). We can then proceed as before, i.e. use lemma 9 and integrate over Λ as in (A.3.11). We also make use of the inequality

$$\frac{(1-\tau)}{|x_N - \tau x_a|} \leq \frac{1}{|x_N|}\frac{(1-\tau)}{|\frac{x_N}{|x_N|} - \tau\frac{x_a}{|x_N|}|} \leq \frac{1}{|x_N|} \tag{A.4.2}$$

which holds for all $1 \leq a \leq M$ under the assumption $|x_a| \leq |x_N|$.

- Finally, to verify our bound for the sixth term on the r.h.s. of the flow equation, we again make use of eq.(3.2.17) for the remainder of the Taylor expansion and pull spacetime derivatives into CAG's using (3.1.45). We can then use lemma 6 and estimate the Λ integral as before. Here we also make use of the inequality

$$\frac{(1-\tau)}{|x_b - \tau x_a|} \leq \frac{1}{|x_b|} \tag{A.4.3}$$

which holds for all $1 \leq a \leq M < b \leq N$ under the assumption $|x_a| \leq |x_b|$. $\square$

We are now ready to finish the proof of bound 4. Recall the definition of the partially regularised AG's with insertions,

$$\begin{aligned} G^{\Lambda,\Lambda_0}\left([\otimes_{i=1}^M \mathcal{O}_{A_i}]_D \otimes_{M+1}^N \mathcal{O}_{A_i}\right) := \; & G_D^{\Lambda,\Lambda_0}(\otimes_{i=1}^M \mathcal{O}_{A_i})\, G^{\Lambda,\Lambda_0}(\otimes_{i=M+1}^N \mathcal{O}_{A_i}) \\ & + \hbar H^{\Lambda,\Lambda_0}\left([\otimes_{i=1}^M \mathcal{O}_{A_i}]_D ; \otimes_{i=M+1}^N \mathcal{O}_{A_i}\right) . \end{aligned} \tag{A.4.4}$$

Combing corollary 1 and bound 2, we find that the Taylor expansion of first term on the r.h.s. of the equation above satisfies the claimed inequality, (3.2.21). To estimate the contribution from the H^{Λ,Λ_0} functional, we use proposition 8. We can remove the cutoff dependence from that bound as in (A.1.52) and arrive at an estimate consistent with bound 4 in the case $x_M = 0$. The general case $x_M \neq 0$ is obtained with the help of the translation property (3.1.59).

A.5 Proof of proposition 4

We make use of the decomposition $G^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}) = \hbar F^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}) + \prod_{i=1}^N L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_i})$ and study the g-derivative of the expressions on the right side of this equation. To begin with, consider the derivative of the factorised contribution to $\hbar\partial_g G^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}(x_i))$. Using proposition 3 we find

$$\hbar\, \partial_g \prod_{i=1}^N L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_i}) = \frac{\hbar}{4!} \int \mathrm{d}^4 y \sum_{j=1}^N L^{\Lambda,\Lambda_0}_{D=[A_j]}(\mathcal{O}_{A_j} \otimes \varphi^4(y)) \prod_{\substack{i=1 \\ i \neq j}}^N L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_i}) \tag{A.5.1}$$

We find a similar contribution on the r.h.s. of eq.(3.6.8):

$$
\begin{aligned}
&\frac{1}{4!}\int \mathrm{d}^4y \sum_{j=1}^{N}\sum_{[C]\le[A_j]} \mathcal{C}^{C}_{A_g A_j}(y,x_j) L^{\Lambda,\Lambda_0}(\mathcal{O}_C(y)) \prod_{\substack{i=1\\ i\neq j}}^{N} L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_i}) \\
=&\frac{\hbar}{4!}\int \mathrm{d}^4y \sum_{j=1}^{N}\Big(L^{\Lambda,\Lambda_0}_{D=[A_j]}(\mathcal{O}_{A_j}\otimes\varphi^4(y)) - L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_j}\otimes\varphi^4(y))\Big)\prod_{\substack{i=1\\ i\neq j}}^{N} L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_i})
\end{aligned}
\tag{A.5.2}
$$

In the second line we made use of lemma 2. We see that the first term on the r.h.s. of (A.5.2) coincides with the r.h.s. of equation (A.5.1), which means that these terms cancel in equation (3.6.8). Note also that the factorised contributions (i.e. terms containing only CAG's with one insertion) to the first two terms on the r.h.s of equation (3.6.8) cancel each other. Let us now come to the various contributions from the F^{Λ,Λ_0}-functionals. Consider first the g-derivative of the flow equation for F^{Λ,Λ_0}

$$
\begin{aligned}
&\partial_\Lambda \partial_g F^{\Lambda,\Lambda_0}(\otimes_{i=1}^{N}\mathcal{O}_{A_i}) \\
=&\frac{\hbar}{2}\langle\frac{\delta}{\delta\varphi},\dot{C}^\Lambda \star \frac{\delta}{\delta\varphi}\rangle\,\partial_g F^{\Lambda,\Lambda_0}(\otimes_{i=1}^{N}\mathcal{O}_{A_i}) - \langle\frac{\delta}{\delta\varphi}\partial_g F^{\Lambda,\Lambda_0}(\otimes_{i=1}^{N}\mathcal{O}_{A_i}),\dot{C}^\Lambda\star\frac{\delta}{\delta\varphi}L^{\Lambda,\Lambda_0}\rangle \\
&-\langle\frac{\delta}{\delta\varphi}F^{\Lambda,\Lambda_0}(\otimes_{i=1}^{N}\mathcal{O}_{A_i}),\dot{C}^\Lambda\star\frac{\delta}{\delta\varphi}\frac{1}{4!}\int \mathrm{d}^4y\, L^{\Lambda,\Lambda_0}(\varphi^4(y))\rangle \\
&+\sum_{1\le i<j\le N}\langle\frac{\delta}{\delta\varphi}\frac{1}{4!}\int \mathrm{d}^4y\, L^{\Lambda,\Lambda_0}_{[A_i]}(\mathcal{O}_{A_i}\otimes\varphi^4(y)),\dot{C}^\Lambda\star\frac{\delta}{\delta\varphi}L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_j})\rangle\prod_{r\in\{1,\dots,N\}\setminus\{i,j\}} L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_r}) \\
&+\sum_{1\le i<j\le N}\langle\frac{\delta}{\delta\varphi}L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_i}),\dot{C}^\Lambda\star\frac{\delta}{\delta\varphi}L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_j})\rangle \\
&\quad\times\sum_{k\in\{1,\dots,N\}\setminus\{i,j\}}\frac{1}{4!}\int \mathrm{d}^4y\, L^{\Lambda,\Lambda_0}_{[A_k]}(\mathcal{O}_{A_k}\otimes\varphi^4(y)),\prod_{r\in\{1,\dots,N\}\setminus\{i,j,k\}} L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_r})\,,
\end{aligned}
\tag{A.5.3}
$$

where we made use of proposition 3. The boundary conditions read

$$
\partial^w_{\vec{p}}\,\partial_g \mathcal{F}^{\Lambda_0,\Lambda_0}_{n,l}(\otimes_{i=1}^{N}\mathcal{O}_{A_i};\vec{p}) = 0 \quad \text{for all } n,l,w. \tag{A.5.4}
$$

We want to compare this to the flow equations for the terms on the r.h.s. of eq.(3.6.8). To start with, we have

$$
\begin{aligned}
&\partial_\Lambda \frac{-1}{4!} \int \mathrm{d}^4 y \; F^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i} \otimes \varphi^4(y)) \\
&= \frac{\hbar}{2} \langle \frac{\delta}{\delta\varphi}, \dot{C}^\Lambda \star \frac{\delta}{\delta\varphi} \rangle \frac{-1}{4!} \int \mathrm{d}^4 y \; F^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i} \otimes \varphi^4(y)) \\
&- \langle \frac{\delta}{\delta\varphi} \frac{-1}{4!} \int \mathrm{d}^4 y \; F^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i} \otimes \varphi^4(y)), \dot{C}^\Lambda \star \frac{\delta}{\delta\varphi} L^{\Lambda,\Lambda_0} \rangle \\
&- \sum_{1\le i<j\le N} \langle \frac{\delta}{\delta\varphi} L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_i}), \dot{C}^\Lambda \star \frac{\delta}{\delta\varphi} L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_j}) \rangle \prod_{r\in\{1,\dots,N\}\setminus\{i,j\}} L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_r}) \int \frac{\mathrm{d}^4 y}{4!} L^{\Lambda,\Lambda_0}(\varphi^4(y)) \\
&- \sum_{j\in\{1,\dots,N\}} \langle \frac{\delta}{\delta\varphi} L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_j}), \dot{C}^\Lambda \star \frac{\delta}{\delta\varphi} \frac{1}{4!} \int \mathrm{d}^4 y L^{\Lambda,\Lambda_0}(\varphi^4(y)) \rangle \prod_{\substack{r=1\\ r\neq j}}^N L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_r})
\end{aligned}
\tag{A.5.5}
$$

Next, we have

$$
\begin{aligned}
&\partial_\Lambda \left(F^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}) \frac{1}{4!} \int \mathrm{d}^4 y \; L^{\Lambda,\Lambda_0}(\varphi^4(y)) \right) \\
&= \frac{\hbar}{2} \langle \frac{\delta}{\delta\varphi}, \dot{C}^\Lambda \star \frac{\delta}{\delta\varphi} \rangle \; F^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}) \frac{1}{4!} \int \mathrm{d}^4 y \; L^{\Lambda,\Lambda_0}(\varphi^4(y)) \\
&- \langle \frac{\delta}{\delta\varphi} F^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}) \frac{1}{4!} \int \mathrm{d}^4 y \; L^{\Lambda,\Lambda_0}(\varphi^4(y)), \dot{C}^\Lambda \star \frac{\delta}{\delta\varphi} L^{\Lambda,\Lambda_0} \rangle \\
&- \hbar \langle \frac{\delta}{\delta\varphi} F^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i}), \dot{C}^\Lambda \star \frac{\delta}{\delta\varphi} \frac{1}{4!} \int \mathrm{d}^4 y \; L^{\Lambda,\Lambda_0}(\varphi^4(y)) \rangle \\
&+ \sum_{1\le i<j\le N} \langle \frac{\delta}{\delta\varphi} L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_i}), \dot{C}^\Lambda \star \frac{\delta}{\delta\varphi} L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_j}) \rangle \prod_{r\in\{1,\dots,N\}\setminus\{i,j\}} L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_r}) \int \frac{\mathrm{d}^4 y}{4!} L^{\Lambda,\Lambda_0}(\varphi^4(y))
\end{aligned}
\tag{A.5.6}
$$

Finally, for the last term in eq.(3.6.8):

$$
\begin{aligned}
&\partial_\Lambda \left(\frac{1}{4!} \int \mathrm{d}^4 y \sum_{j=1}^{N} \sum_{[C]\le[A_j]} \mathcal{C}^{C}_{A_g A_j}(y,x_j) F^{\Lambda,\Lambda_0}(\otimes_{\substack{i=1\\ i\neq j}}^{N} \mathcal{O}_{A_i}(x_i) \otimes \mathcal{O}_C(x_j)) \right) \\
&= \frac{\hbar}{2} \langle \frac{\delta}{\delta\varphi}, \dot{C}^\Lambda \star \frac{\delta}{\delta\varphi} \rangle \frac{1}{4!} \int \mathrm{d}^4 y \sum_{j=1}^{N} \sum_{[C]\le[A_j]} \mathcal{C}^{C}_{A_g A_j}(y,x_j) F^{\Lambda,\Lambda_0}(\otimes_{\substack{i=1\\ i\neq j}}^{N} \mathcal{O}_{A_i}(x_i) \otimes \mathcal{O}_C(x_j)) \\
&- \langle \frac{\delta}{\delta\varphi} \int \frac{\mathrm{d}^4 y}{4!} \sum_{j=1}^{N} \sum_{[C]\le[A_j]} \mathcal{C}^{C}_{A_g A_j}(y,x_j) F^{\Lambda,\Lambda_0}(\otimes_{\substack{i=1\\ i\neq j}}^{N} \mathcal{O}_{A_i}(x_i) \otimes \mathcal{O}_C(x_j)), \dot{C}^\Lambda \star \frac{\delta}{\delta\varphi} L^{\Lambda,\Lambda_0} \rangle \\
&- \hbar \sum_{1\le i<j\le N} \langle \frac{\delta}{\delta\varphi} L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_i}), \dot{C}^\Lambda \star \frac{\delta}{\delta\varphi} L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_j}) \rangle \\
&\qquad\qquad \times \sum_{k\in\{1,\dots,N\}\setminus\{i,j\}} \frac{1}{4!} \int \mathrm{d}^4 y L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_k} \otimes \varphi^4(y)) \prod_{r\in\{1,\dots,N\}\setminus\{i,j,k\}} L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_r}) \\
&+ \hbar \sum_{1\le i<j\le N} \langle \frac{\delta}{\delta\varphi} L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_i}), \dot{C}^\Lambda \star \frac{\delta}{\delta\varphi} L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_j}) \rangle \\
&\qquad\qquad \times \sum_{k\in\{1,\dots,N\}\setminus\{i,j\}} \frac{1}{4!} \int \mathrm{d}^4 y L^{\Lambda,\Lambda_0}_{[A_k]}(\mathcal{O}_{A_k} \otimes \varphi^4(y)) \prod_{r\in\{1,\dots,N\}\setminus\{i,j,k\}} L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_r}) \\
&- \hbar \sum_{1\le i<j\le N} \langle \frac{\delta}{\delta\varphi} L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_i}), \dot{C}^\Lambda \star \frac{\delta}{\delta\varphi} \frac{1}{4!} \int \mathrm{d}^4 y L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_j} \otimes \varphi^4(y)) \rangle \prod_{r\in\{1,\dots,N\}\setminus\{i,j\}} L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_r}) \\
&+ \hbar \sum_{1\le i<j\le N} \langle \frac{\delta}{\delta\varphi} L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_i}), \dot{C}^\Lambda \star \frac{\delta}{\delta\varphi} \frac{1}{4!} \int \mathrm{d}^4 y L^{\Lambda,\Lambda_0}_{[A_j]}(\mathcal{O}_{A_j} \otimes \varphi^4(y)) \rangle \prod_{r\in\{1,\dots,N\}\setminus\{i,j\}} L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_r})
\end{aligned}
\tag{A.5.7}
$$

Also recall the remaining term from eq.(A.5.2), which satisfies the flow equation

$$
\begin{aligned}
&\partial_\Lambda \left(\frac{-1}{4!} \int \mathrm{d}^4 y \sum_{j=1}^{N} L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_j} \otimes \varphi^4(y)) \prod_{\substack{i=1 \\ i\neq j}}^{N} L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_i}) \right) \\
&= \frac{\hbar}{2} \langle \frac{\delta}{\delta\varphi}, \dot{C}^\Lambda \star \frac{\delta}{\delta\varphi} \rangle \frac{-1}{4!} \int \mathrm{d}^4 y \sum_{j=1}^{N} L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_j} \otimes \varphi^4(y)) \prod_{\substack{i=1 \\ i\neq j}}^{N} L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_i}) \\
&- \langle \frac{\delta}{\delta\varphi} \frac{-1}{4!} \int \mathrm{d}^4 y \sum_{j=1}^{N} L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_j} \otimes \varphi^4(y)) \prod_{\substack{i=1 \\ i\neq j}}^{N} L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_i}), \dot{C}^\Lambda \star \frac{\delta}{\delta\varphi} L^{\Lambda,\Lambda_0} \rangle \\
&+ \hbar \sum_{1\le i<j\le N} \langle \frac{\delta}{\delta\varphi} L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_i}), \dot{C}^\Lambda \star \frac{\delta}{\delta\varphi} L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_j}) \rangle \\
&\qquad\qquad \times \sum_{k\in\{1,\dots,N\}\setminus\{i,j\}} \frac{1}{4!} \int \mathrm{d}^4 y L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_k} \otimes \varphi^4(y)) \prod_{r\in\{1,\dots,N\}\setminus\{i,j,k\}} L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_r}) \\
&+ \hbar \sum_{1\le i<j\le N} \langle \frac{\delta}{\delta\varphi} L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_i}), \dot{C}^\Lambda \star \frac{\delta}{\delta\varphi} \frac{1}{4!} \int \mathrm{d}^4 y L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_j} \otimes \varphi^4(y)) \rangle \prod_{r\in\{1,\dots,N\}\setminus\{i,j\}} L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_r}) \\
&+ \sum_{j\in\{1,\dots,N\}} \langle \frac{\delta}{\delta\varphi} L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_j}), \dot{C}^\Lambda \star \frac{\delta}{\delta\varphi} \frac{1}{4!} \int \mathrm{d}^4 y L^{\Lambda,\Lambda_0}(\varphi^4(y)) \rangle \prod_{\substack{r=1 \\ r\neq j}}^{N} L^{\Lambda,\Lambda_0}(\mathcal{O}_{A_r}) \, .
\end{aligned}
\tag{A.5.8}
$$

Now, summing up equations (A.5.5),(A.5.6),(A.5.7) and (A.5.8), one can check that these contributions satisfy the same flow equation as $\hbar\, \partial_g F^{\Lambda,\Lambda_0}(\otimes_{i=1}^N \mathcal{O}_{A_i})$, see eq.(A.5.3). Also note that all these contributions satisfy the boundary conditions of the form (A.5.4). It follows that the left and right hand side of equation (3.6.8) are equal.

B

Computations in the $1/N$-expansion of the Gross-Neveu model

In this appendix we collect some of the explicit, but somewhat lengthy, calculations that are used in our treatment of the $1/N$-expansion of the GN-model.

B.1 The 4-point function to order $1/N$

In eq.(4.2.18) we gave an explicit expression for the 4-point function of the basic field. Here we are going to present the calculation leading to that equation. Up to order $1/N$, the connected Feynman diagrams contributing to the 4-point function are

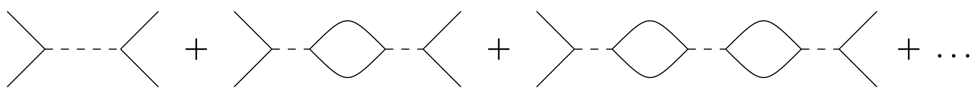


Figure B.1.: Connected contributions to the 4-point function

Let us denote the fermion propagator by

$$
\begin{aligned}
S_{\alpha\beta}(x) := \langle T\psi_\alpha(x)\overline{\psi}_\beta(0)\rangle &= \int \frac{\mathrm{d}^2 p}{(2\pi)^2}\,\frac{i(\not{p}+m)_{\alpha\beta}}{p^2-m^2+i\epsilon}\,\mathrm{e}^{-ipx} \\
&= \frac{(i\not{\partial}+m)_{\alpha\beta}}{2\pi}K_0(m\sqrt{-x^2+i\epsilon})\,,
\end{aligned} \tag{B.1.1}
$$

where α, β are spinor indices and where $K_0(x)$ is a modified Bessel function of the second kind, defined e.g. via the integral [60]

$$K_n(x) = \frac{x^n}{2^{n+1}} \int_0^\infty t^{-n-1} \mathrm{e}^{-t-x^2/4t}\, \mathrm{d}t \tag{B.1.2}$$

for $n \in \mathbb{Z}$. Using the rules given in table 4.1 to translate Feynman diagrams into equations, we obtain

$$\begin{aligned}\text{Fig. B.1} \;&= -\frac{1}{N}\sum_{n=1}^{\infty}\int \mathrm{d}^2\xi_1 \ldots \mathrm{d}^2\xi_n\; (-ig)^n\; S_{\lambda_1\alpha}(\xi_1 - x)\, S_\beta^{\lambda_1}(-\xi_1) \\ &\quad \times S_{\lambda_2\lambda_3}(\xi_1-\xi_2) S^{\lambda_2\lambda_3}(\xi_2-\xi_1)\cdots S_{\lambda_{2n}\gamma}(\xi_n - z_1) S_\delta^{\lambda_{2n}}(z_2-\xi_n)\end{aligned} \tag{B.1.3}$$

The integration over the fermion loops can be performed as follows[1].

$$\begin{aligned}&\int \mathrm{d}^2 y\; S_{\alpha\beta}(x-y) S^{\beta\alpha}(y-x) S_{\gamma\delta}(y-z) S^{\gamma\delta}(z-y) \\ &= 4\int \mathrm{d}^2 y \int \frac{\mathrm{d}p_1^2 \ldots \mathrm{d}p_4^2}{(2\pi)^8} \frac{(p_1^\mu p_{2,\mu}+m^2)(p_3^\mu p_{4,\mu}+m^2)}{(p_1^2-m^2)\cdots(p_4^2-m^2)} \\ &\qquad\qquad \times \mathrm{e}^{-ix(p_1-p_2)}\mathrm{e}^{-iy(p_2-p_1+p_3-p_4)}\mathrm{e}^{-iz(p_4-p_3)} \\ &= 4\int \frac{\mathrm{d}p_1^2\ldots \mathrm{d}p_3^2}{(2\pi)^6} \frac{(p_1^\mu p_{2,\mu}+m^2)(p_3^\mu(p_3+p_2-p_1)_\mu + m^2)e^{-i(x-z)(p_1-p_2)}}{(p_1^2-m^2)\cdots(p_3^2-m^2)((p_3+p_2-p_1)^2-m^2)} \\ &= \int \frac{\mathrm{d}p_1^2 \mathrm{d}p_2^2}{(2\pi)^4} \hat{S}_{\alpha\beta}(p_1)\hat{S}^{\beta\alpha}(p_2) e^{-i(x-z)(p_1-p_2)} \\ &\quad \times 2\int \frac{\mathrm{d}p_3^2}{(2\pi)^2} \frac{-(p_3^\mu(p_3+p_2-p_1)_\mu+m^2)}{(p_3^2-m^2)((p_3+p_2-p_1)^2-m^2)}\end{aligned} \tag{B.1.4}$$

Performing a Wick rotation and introducing a UV-cutoff Λ, the integral over p_3 can be computed (see also [61]):

$$\begin{aligned}&2\int \frac{\mathrm{d}k^2}{(2\pi)^2} \frac{-(k^\mu(k+q)_\mu + m^2)}{(k^2-m^2)((k+q)^2-m^2)} \\ &= \frac{i}{2\pi}\left(\sqrt{\frac{4m^2-q^2}{-q^2}} \log\frac{\sqrt{4m^2-q^2}-\sqrt{-q^2}}{\sqrt{4m^2-q^2}+\sqrt{-q^2}} - \log\frac{m^2}{\Lambda^2}\right)\end{aligned} \tag{B.1.5}$$

[1]The $i\epsilon$ terms are of no relevance to our calculations, so we suppress them in the following. Recall also that we assume x to be spacelike

Substituting this result for the fermion loops into eq.(B.1.3) yields

$$
\begin{aligned}
&-\int \frac{\mathrm{d}^2 p \mathrm{d}^2 q}{(2\pi)^4} \hat{S}_{\gamma\lambda}(p)\hat{S}^{\lambda}_{\delta}(q) e^{ipz_1} e^{-iqz_2} \frac{-ig}{N} \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{(\not{k}+\not{q}-\not{p}+m)_{\alpha\rho}(\not{k}+m)^{\rho}_{\beta}\, e^{ikx}}{(k^2-m^2)((k+q-p)^2-m^2)} \\
&\times \sum_{n=0}^{\infty} \left[\frac{g}{2\pi} \left(\sqrt{\frac{4m^2-(q-p)^2}{-(q-p)^2}} \log \frac{\sqrt{4m^2-(q-p)^2}-\sqrt{-(q-p)^2}}{\sqrt{4m^2-(q-p)^2}+\sqrt{-(q-p)^2}} - \log \frac{m^2}{\Lambda^2} \right) \right]^n
\end{aligned}
\tag{B.1.6}
$$

Performing the geometric series and recalling form eq.(4.2.14) that

$$
\frac{g(\Lambda)}{2\pi} \log \frac{m^2}{\Lambda^2} = 1 \tag{B.1.7}
$$

we arrive at the form

$$
\int \frac{\mathrm{d}^2 p \mathrm{d}^2 q}{(2\pi)^4} \frac{\hat{S}_{\gamma\lambda}(p)\hat{S}^{\lambda}_{\delta}(q) e^{ipz_1} e^{-iqz_2}}{B(q-p)} \frac{2\pi i}{N} \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{(\not{k}+\not{q}-\not{p}+m)_{\alpha\rho}(\not{k}+m)^{\rho}_{\beta}\, e^{ikx}}{(k^2-m^2)((k+q-p)^2-m^2)}, \tag{B.1.8}
$$

where $B(k)$ is the function defined in equation (4.2.19). Finally, rewriting the k-integral as

$$
\begin{aligned}
&\int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{(\not{k}+\not{q}-\not{p}+m)_{\alpha\rho}(\not{k}+m)^{\rho}_{\beta}\, e^{ikx}}{(k^2-m^2)((k+q-p)^2-m^2)} \\
&= \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{[(-i\not{\partial}_x+\not{q}-\not{p}+m)(-i\not{\partial}_x+m)]_{\alpha\beta}\, e^{ikx}}{(k^2-m^2)((k+q-p)^2-m^2)} \\
&= \frac{i}{4\pi}[(\not{q}-\not{p}+m-i\not{\partial}_x)(m-i\not{\partial}_x)]_{\alpha\beta} \\
&\quad \times \int_0^1 \mathrm{d}\alpha\, \frac{\sqrt{-x^2} K_1[\sqrt{-x^2(m^2-\alpha(1-\alpha)(q-p)^2)}] \mathrm{e}^{ix(q-p)(\alpha-1)}}{\sqrt{m^2-\alpha(1-\alpha)(q-p)^2}}
\end{aligned}
\tag{B.1.9}
$$

we arrive at eq.(4.2.18).

B.2 Short distance expansion of correlation functions

For the derivation of the OPE coefficients in section 4.2.2 we needed explicit expressions for the short distance expansion of various correlation functions. Here we will give the computations leading to those formulae.

Derivation of equation (4.2.25): Contracting the spinor indices in our formula for the 4-point function yields

$$
\begin{aligned}
&\langle\overline{\psi}_\alpha(x)\psi^\alpha(0)\,\overline{\psi}_\beta(z_1)\psi^\beta(z_2)\rangle - \langle\overline{\psi}_\alpha(x)\psi^\alpha(0)\rangle\,\langle\overline{\psi}_\beta(z_1)\psi^\beta(z_2)\rangle \\
&= \frac{-2}{N}\int \frac{\mathrm{d}^2p\,\mathrm{d}^2q}{(2\pi)^4}\frac{(p^\mu q_\mu + m^2)\,\mathrm{e}^{i(z_1-x)p+i(x-z_2)q}}{(p^2-m^2)(q^2-m^2)\,B(q-p)}\,[(q-p-i\partial_x)^\nu(-i\partial_x)_\nu + m^2] \\
&\quad\times\int_0^1 \mathrm{d}\alpha\,\frac{\sqrt{-x^2}K_1[\sqrt{-x^2(m^2-\alpha(1-\alpha)(q-p)^2)}]\mathrm{e}^{ix(q-p)(\alpha-1)}}{\sqrt{m^2-\alpha(1-\alpha)(q-p)^2}} \\
&- \frac{1}{N}\langle\overline{\psi}_\alpha(x)\psi_\beta(z_2)\rangle\,\langle\overline{\psi}^\beta(z_1)\psi^\alpha(0)\rangle + O(\frac{1}{N^2})
\end{aligned}
\tag{B.2.1}
$$

In the last line of this equation we can substitute the explicit expression for the two point functions, eq.(B.1.1), to obtain

$$
\begin{aligned}
-\frac{1}{N}\langle\overline{\psi}_\alpha(x)\psi_\beta(z_2)\rangle\,\langle\overline{\psi}^\beta(z_1)\psi^\alpha(0)\rangle &= \frac{2}{N}\int\frac{\mathrm{d}^2p\,\mathrm{d}^2q}{(2\pi)^4}\frac{(p^\mu q_\mu+m^2)\,\mathrm{e}^{iz_1p+i(x-z_2)q}}{(p^2-m^2)(q^2-m^2)} \\
&= \frac{2}{N}\int\frac{\mathrm{d}^2p\,\mathrm{d}^2q}{(2\pi)^4}\frac{(p^\mu q_\mu+m^2)\,\mathrm{e}^{iz_1p-iz_2q}}{(p^2-m^2)(q^2-m^2)} + O(x)
\end{aligned}
\tag{B.2.2}
$$

The expression in the third line of eq.(B.2.1) can be rewritten as follows (here we replace the difference of the two momenta, $(q-p)$, by k for the sake of brevity):

$$
\begin{aligned}
&[(k-i\partial_x)^\nu(-i\partial_x)_\nu + m^2]\int_0^1\mathrm{d}\alpha\,\frac{\sqrt{-x^2}K_1[\sqrt{-x^2(m^2-\alpha(1-\alpha)k^2)}]\mathrm{e}^{ixk(\alpha-1)}}{\sqrt{m^2-\alpha(1-\alpha)k^2}} \\
&= \int_0^1\mathrm{d}\alpha\,\mathrm{e}^{ixk(\alpha-1)}\,K_0[\sqrt{-x^2(m^2-\alpha(1-\alpha)k^2)}]\cdot\Big(-2-3ik^\nu x_\nu(\alpha-1)\Big) \\
&+2\int_0^1\mathrm{d}\alpha\,\sqrt{-x^2(m^2-\alpha(1-\alpha)k^2)}\cdot K_1[\sqrt{-x^2(m^2-\alpha(1-\alpha)k^2)}]\mathrm{e}^{ixk(\alpha-1)}
\end{aligned}
\tag{B.2.3}
$$

Using the short distance expansion for the modified Bessel functions,

$$
K_0(x) = -\log\frac{x\,\mathrm{e}^{\Gamma_\mathrm{E}}}{2} + O(x^2) \tag{B.2.4}
$$

$$
K_1(x) = \frac{1}{x} + O(x) \tag{B.2.5}
$$

where Γ_{E} is the Euler-Mascheroni constant, we arrive at the equation

$$
\begin{aligned}
&[(k - i\partial_x)^\nu(-i\partial_x)_\nu + m^2] \int_0^1 \mathrm{d}\alpha\, \frac{\sqrt{-x^2}K_1[\sqrt{-x^2(m^2-\alpha(1-\alpha)k^2)}]\mathrm{e}^{ixk(\alpha-1)}}{\sqrt{m^2-\alpha(1-\alpha)k^2}} \\
&= 2\int_0^1 \mathrm{d}\alpha \left(\log \frac{\sqrt{-x^2\mathrm{e}^{\Gamma_{\mathrm{E}}}(m^2-\alpha(1-\alpha)k^2)}}{2} + 1\right) + O(x) \\
&= -\frac{2\pi}{g^2}\left[1 - \frac{g^2}{2\pi}\log\left(\frac{-x^2\mathrm{e}^{2\Gamma_{\mathrm{E}}}}{4\ell^2}\right)\right] + B(k) + O(x)
\end{aligned}
\tag{B.2.6}
$$

The integral in the second line was performed using MATHEMATICA. We can now substitute equations (B.2.2) and (B.2.6) into equation (B.2.1), which yields the short distance expansion

$$
\begin{aligned}
&\langle\overline{\psi}_\alpha(x)\psi^\alpha(0)\,\overline{\psi}_\beta(z_1)\psi^\beta(z_2)\rangle - \langle\overline{\psi}_\alpha(x)\psi^\alpha(0)\rangle\,\langle\overline{\psi}_\beta(z_1)\psi^\beta(z_2)\rangle \\
&= \frac{4\pi}{g^2N}\int \frac{\mathrm{d}^2p\,\mathrm{d}^2q}{(2\pi)^4}\frac{(p^\mu q_\mu + m^2)\,\mathrm{e}^{iz_1p+iz_2q}}{(p^2-m^2)(q^2-m^2)\,B(q-p)}\left[1 - \frac{g^2}{2\pi}\log\left(\frac{-x^2\mathrm{e}^{2\Gamma_{\mathrm{E}}}}{4\ell^2}\right) + O(x)\right] \\
&+ O(\frac{1}{N^2})
\end{aligned}
\tag{B.2.7}
$$

as claimed in equation (4.2.25).

Derivation of equation (4.2.32): Our formula for the 4-point function implies

$$
\begin{aligned}
&\langle\overline{\psi}_\alpha(x)(\gamma_\mu)^{\alpha\beta}\partial^\mu\psi_\beta(0)\,\overline{\psi}_\gamma(z_1)\psi^\gamma(z_2)\rangle - \langle\overline{\psi}_\alpha(x)(\gamma_\mu)^{\alpha\beta}\partial^\mu\psi_\beta(0)\rangle\,\langle\overline{\psi}_\gamma(z_1)\psi^\gamma(z_2)\rangle \\
&= \frac{-2}{N}\int \frac{\mathrm{d}^2p\,\mathrm{d}^2q}{(2\pi)^4}\frac{(p^\mu q_\mu + m^2)\,\mathrm{e}^{i(z_1-x)p+i(x-z_2)q}}{(p^2-m^2)(q^2-m^2)\,B(q-p)}\, m[(q-p-2i\partial_x)^\nu(-\partial_x)_\nu] \\
&\quad\times \int_0^1 \mathrm{d}\alpha\, \frac{\sqrt{-x^2}K_1[\sqrt{-x^2(m^2-\alpha(1-\alpha)(q-p)^2)}]\mathrm{e}^{ix(q-p)(\alpha-1)}}{\sqrt{m^2-\alpha(1-\alpha)(q-p)^2}} \\
&- \frac{1}{N}\langle\overline{\psi}_\alpha(x)\psi_\gamma(z_2)\rangle\,(\gamma_\mu)^{\alpha\beta}\langle\overline{\psi}^\gamma(z_1)\partial^\mu\psi_\beta(0)\rangle + O(\frac{1}{N^2})
\end{aligned}
\tag{B.2.8}
$$

Inserting the known 2-point function in the last line, we find for the disconnected contribution

$$
\begin{aligned}
&\frac{1}{N}\langle\overline{\psi}_\alpha(x)\psi_\lambda(z_2)\rangle\,(\gamma_\mu)^{\alpha\beta}\,\langle\overline{\psi}^\lambda(z_1)\partial^\mu\psi_\beta(0)\rangle \\
&\qquad = \frac{-2mi}{N}\int \frac{\mathrm{d}^2p\,\mathrm{d}^2q}{(2\pi)^4}\frac{(p^\mu q_\mu + m^2)\,\mathrm{e}^{iz_1p-iz_2q}}{(p^2-m^2)(q^2-m^2)} + O(x)
\end{aligned}
\tag{B.2.9}
$$

For the integral in the third line of equation (B.2.8) we obtain

$$
\begin{aligned}
&[(q-p-2i\partial_x)^\nu(-\partial_x)_\nu]\int_0^1 \mathrm{d}\alpha\ \frac{\sqrt{-x^2}K_1[\sqrt{-x^2(m^2-\alpha(1-\alpha)(q-p)^2)}]\mathrm{e}^{ix(q-p)(\alpha-1)}}{\sqrt{m^2-\alpha(1-\alpha)(q-p)^2}} \\
&= i\int_0^1 \mathrm{d}\alpha \mathrm{e}^{ix(q-p)(\alpha-1)} K_0[\sqrt{-x^2(m^2-\alpha(1-\alpha)(q-p)^2)}]\Big(4+5i(q-p)^\nu x_\nu(\alpha-1)\Big) \\
&- i\int_0^1 \mathrm{d}\alpha\ K_1[\sqrt{-x^2(m^2-\alpha(1-\alpha)(q-p)^2)}]\mathrm{e}^{ix(q-p)(\alpha-1)} \\
&\quad\times\left(2\sqrt{-x^2(m^2-\alpha(1-\alpha)(q-p)^2)}+\frac{(q-p)^2[2(1-\alpha)^2+(\alpha-1)]\sqrt{-x^2}}{\sqrt{m^2-\alpha(1-\alpha)(q-p)^2}}\right)
\end{aligned}
\tag{B.2.10}
$$

Expanding around $x = 0$ the integrand simplifies to

$$
\begin{aligned}
&[(q-p-2i\partial_x)^\nu(-\partial_x)_\nu]\int_0^1 \mathrm{d}\alpha\ \frac{\sqrt{-x^2}K_1[\sqrt{-x^2(m^2-\alpha(1-\alpha)(q-p)^2)}]\mathrm{e}^{ix(q-p)(\alpha-1)}}{\sqrt{m^2-\alpha(1-\alpha)(q-p)^2}} \\
&= -i\int_0^1 \mathrm{d}\alpha\left(4\log\frac{\sqrt{-x^2\mathrm{e}^{\Gamma_\mathrm{E}}(m^2-\alpha(1-\alpha)(q-p)^2)}}{2}\right. \\
&\qquad\qquad\qquad \left.+2+\frac{(q-p)^2[2(1-\alpha)^2+(\alpha-1)]}{m^2-\alpha(1-\alpha)(q-p)^2}\right)+O(x) \\
&= \frac{4\pi i}{g^2}\left[1-\frac{g^2}{2\pi}\log\left(\frac{-x^2\mathrm{e}^{2\Gamma_\mathrm{E}}}{4\ell^2}\right)\right]-iB(q-p)+O(x)
\end{aligned}
\tag{B.2.11}
$$

Inserting equations (B.2.9) and (B.2.11) into equation (B.2.8) then yields equation (4.2.32).

Derivation of equation (4.2.36): Here we find with the help of equation (4.2.18)

$$
\begin{aligned}
&\langle\overline{\psi}_\alpha(x)(\gamma_{(\mu})^{\alpha\beta}\partial_{\nu)}\psi_\beta(0)\,\overline{\psi}_\gamma(z_1)\psi^\gamma(z_2)\rangle-\langle\overline{\psi}_\alpha(x)(\gamma_{(\mu})^{\alpha\beta}\partial_{\nu)}\psi_\beta(0)\rangle\,\langle\overline{\psi}_\gamma(z_1)\psi^\gamma(z_2)\rangle \\
&= \frac{-2}{N}\int\frac{\mathrm{d}^2p\,\mathrm{d}^2q}{(2\pi)^4}\frac{(p^\mu q_\mu+m^2)\,\mathrm{e}^{i(z_1-x)p+i(x-z_2)q}}{(p^2-m^2)(q^2-m^2)\,B(q-p)}\,m[(q-p-2i\partial_x)_{(\mu}(-\partial_x)_{\nu)}] \\
&\quad\times\int_0^1\mathrm{d}\alpha\ \frac{\sqrt{-x^2}K_1[\sqrt{-x^2(m^2-\alpha(1-\alpha)(q-p)^2)}]\mathrm{e}^{ix(q-p)(\alpha-1)}}{\sqrt{m^2-\alpha(1-\alpha)(q-p)^2}} \\
&-\frac{1}{N}\langle\overline{\psi}_\alpha(x)\psi_\gamma(z_2)\rangle\,(\gamma_{(\mu})^{\alpha\beta}\langle\overline{\psi}^\gamma(z_1)\partial_{\nu)}\psi_\beta(0)\rangle+O(\frac{1}{N^2})
\end{aligned}
\tag{B.2.12}
$$

For the disconnected part we can again insert the known 2-point function.

$$\begin{aligned}\frac{1}{N}\langle\overline{\psi}_\alpha(x)\psi_\gamma(z_2)\rangle\,(\gamma_{(\mu})^{\alpha\beta}\,\langle\overline{\psi}^\gamma(z_1)\partial_{\nu)}\psi_\beta(0)\rangle&\\ =\frac{-2mi}{N}\int\frac{\mathrm{d}^2p\,\mathrm{d}^2q}{(2\pi)^4}\frac{(p_{(\mu}q_{\nu)}+p_{(\mu}p_{\nu)})\,\mathrm{e}^{iz_1p-iz_2q}}{(p^2-m^2)(q^2-m^2)}+O(x)&\end{aligned} \tag{B.2.13}$$

The third line of eq.(B.2.12) can be written as follows:

$$\begin{aligned}&[(q-p-2i\partial_x)_{(\mu}(-\partial_x)_{\nu)}]\int_0^1\mathrm{d}\alpha\,\frac{\sqrt{-x^2}K_1[\sqrt{-x^2(m^2-\alpha(1-\alpha)(q-p)^2)}]\mathrm{e}^{ix(q-p)(\alpha-1)}}{\sqrt{m^2-\alpha(1-\alpha)(q-p)^2}}\\ &=i\int_0^1\mathrm{d}\alpha\;\mathrm{e}^{ix(q-p)(\alpha-1)}\,K_0[\sqrt{-x^2(m^2-\alpha(1-\alpha)(q-p)^2)}]\cdot\Big(5i(q-p)_{(\mu}x_{\nu)}\Big)(\alpha-1)\\ &-i\int_0^1\mathrm{d}\alpha\;K_1[\sqrt{-x^2(m^2-\alpha(1-\alpha)(q-p)^2)}]\;\mathrm{e}^{ix(q-p)(\alpha-1)}\\ &\qquad\times\Bigg(\frac{2x_{(\mu}x_{\nu)}\sqrt{-x^2(m^2-\alpha(1-\alpha)(q-p)^2)}}{x^2}\\ &\qquad\qquad+\frac{(q-p)_{(\mu}(q-p)_{\nu)}[2(1-\alpha)^2+(\alpha-1)]\sqrt{-x^2}}{\sqrt{m^2-\alpha(1-\alpha)(q-p)^2}}\Bigg)\\ &=-2i\frac{x_{(\mu}x_{\nu)}}{x^2}+i\frac{(q-p)_{(\mu}(q-p)_{\nu)}}{(q-p)^2}(2+B(q-p))+O(x)\end{aligned} \tag{B.2.14}$$

Substituting eqs.(B.2.13) and (B.2.14) into equation (B.2.12) then yields equation (4.2.36).

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List of Axioms, Definitions ,Theorems, etc.

List of Symbols