The Ising Model and Beyond

Mathew Pugh [1]

Supervised by Professor David E. Evans

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Doctor of Philosophy

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[1]School of Mathematics, Cardiff University, CF24 4AG, United Kingdom

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I thank God for the ability He has given me and for all His manifold goodness to me. May His be the glory.

Summary

We study the $SU(3)$ $\mathcal{ADE}$ graphs, which appear in the classification of modular invariant partition functions from numerous viewpoints, including determination of their Boltzmann weights, representations of Hecke algebras, a new notion of A_2 planar algebras and their modules, various Hilbert series of dimensions and spectral measures, and the K-theory of associated Cuntz-Krieger algebras.

We compute the K-theory of the of the Cuntz-Krieger algebras associated to the $SU(3)$ $\mathcal{ADE}$ graphs.

We compute the numerical values of the Ocneanu cells, and consequently representations of the Hecke algebra, for the $\mathcal{ADE}$ graphs. Some such representations have appeared in the literature and we compare our results. We use these cells to define an $SU(3)$ analogue of the Goodman-de la Harpe-Jones construction of a subfactor, where we embed the A_2-Temperley-Lieb algebra in an AF path-algebra of the $SU(3)$ $\mathcal{ADE}$ graphs. Using this construction, we realize all $SU(3)$ modular invariants by subfactors previously announced by Ocneanu.

We give a diagrammatic representation of the A_2-Temperley-Lieb algebra, and show that it is isomorphic to Wenzl's representation of a Hecke algebra. Generalizing Jones's notion of a planar algebra, we construct an A_2-planar algebra which captures the structure contained in the $SU(3)$ $\mathcal{ADE}$ subfactors. We show that the subfactor for an $\mathcal{ADE}$ graph with a flat connection has a description as a flat A_2-planar algebra. We introduce the notion of modules over an A_2-planar algebra, and describe certain irreducible Hilbert A_2-Temperley-Lieb-modules. A partial decomposition of the A_2-planar algebras for the $\mathcal{ADE}$ graphs is achieved.

We compare various Hilbert series of dimensions associated to ADE models for $SU(2)$, and the Hilbert series of certain Calabi-Yau algebras of dimension 3. We also consider spectral measures for the ADE graphs and generalize to $SU(3)$, and in particular obtain spectral measures for the infinite $SU(3)$ graphs.

Contents

Chapter 1

Introduction

In this thesis we study the $SU(3)$ $\mathcal{ADE}$ graphs, which appear in the classification of modular invariant partition functions from numerous viewpoints including the determination of their Boltzmann weights, representations of A_2-Temperley-Lieb or Hecke algebra, a new notion of A_2 planar algebras and their modules, endomorphisms of infinite factors, and K-theory of associated Cuntz-Krieger algebras.

In this preliminary Chapter 1, we introduce the background, notions, notation and definitions which we need from operator algebras, particularly the theory of subfactors in von Neumann algebras and modular invariant partition functions in statistical mechanics and conformal field theory.

Then in Chapter 2, we warm up by explicitly constructing endomorphisms on Cuntz algebras for the inclusion of infinite factors associated to some very basic statistical mechanical models.

In Chapter 3, we study the K-theory of the Cuntz-Krieger algebras $\mathcal{O}_{\mathcal{G}}$ where $\mathcal{G}$ is one of the Dynkin diagrams. We completely derive $(K_0(\mathcal{O}_{\mathcal{G}}), [\mathbf{1}])$ and compute its K_1 group. For the $SU(3)$ $\mathcal{ADE}$ graphs, we compute the K_0, K_1 groups for the $\mathcal{ADE}$ and their 01-parts.

We compute in Chapter 4 the numerical values of the Ocneanu cells, and consequently representations of the Hecke algebra, for the $\mathcal{ADE}$ graphs. Some of the representations of the Hecke algebra have appeared in the literature and we compare our results.

We use these cells in Chapter 5 to define an $SU(3)$ analogue of the Goodman-de la Harpe-Jones construction of a subfactor, where we embed the A_2-Temperley-Lieb or Hecke algebra in an AF path algebra of the $SU(3)$ $\mathcal{ADE}$ graphs. Using this construction, we realize all the $SU(3)$ modular invariants by subfactors.

Chapter 6 looks at the A_2-Temperley-Lieb algebra and the subfactors of Chapter 5 from the viewpoint of planar algebras. We give a diagrammatic representation of the A_2-Temperley-Lieb algebra, and show that it is isomorphic to Wenzl's representation of a Hecke algebra. Generalizing Jones's notion of a planar algebra, we construct an A_2-planar algebra which will capture the structure contained in the $SU(3)$ $\mathcal{ADE}$ subfactors. We show that the subfactor for an $\mathcal{ADE}$ graph with a flat connection has a description as a flat A_2-planar algebra. We introduce the notion of modules over an A_2-planar algebra, and describe certain irreducible Hilbert A_2-TL-modules. A partial decomposition of the A_2-planar algebras for the $\mathcal{ADE}$ graphs is achieved.

In the final Chapter 7, we compare various Hilbert series of dimensions associated to ADE models for $SU(2)$, and compute the Hilbert series of certain q-deformed Calabi-Yau algebras of dimension 3. We also consider spectral measures for the ADE graphs in terms of probability measures on the circle $\mathbb{T}$. We generalize this to $SU(3)$, and in particular obtain spectral measures for the infinite $SU(3)$ graphs.

1.1 Statistical Mechanical Models

1.1.1 The Ising Model

The Ising model is a lattice model in the plane, with sites constrained to be + or − corresponding to a particle at that site with a positive or negative spin. This model is given by the Dynkin diagram A_3, with the endpoints labelled by + and −, and the other vertex is a dummy spin. A configuration is a distribution of the edges of A_3 on the edges of the square lattice and the energy function, or Hamiltonian, H is for nearest neighbour interactions, i.e. $H(\sigma) = -\sum_{\alpha,\beta} J\sigma(\alpha)\sigma(\beta)$, where σ is a configuration and the summation is over all nearest neighbours α, β, with J given by the interaction between nearest neighbours. Ising [55] introduced his model for a ferromagnet in an external magnetic field for the one-dimensional model with n lattice sites. As the external magnetic field tended to zero, he found that the solution admitted no phase transition, i.e. no sudden change from negative to positive magnetization (or vice versa), and concluded that his model did not exhibit phase behaviour in any dimension. But this is not true, since in higher dimensions it is possible for the model to have non-zero magnetization when the external field goes to 0. This is called non-zero spontaneous magnetization. Whilst the Ising model is a simplified description of ferromagnetism, other systems can be mapped exactly or approximately to the Ising system, which allows the use of simulation and analytical results of the Ising model to answer questions about the related models.

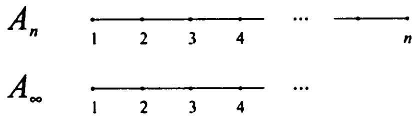


Figure 1.1: Dynkin diagrams A_n, $n = 2, 3, \ldots$, and A_∞

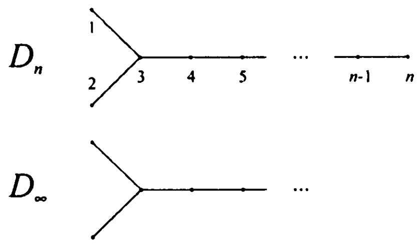


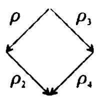
Figure 1.2: Dynkin diagrams D_n, $n = 4, 5, \ldots$, and D_∞

1.1.2 Generalized Models via other Graphs

This model can be generalized to other lattice models using other graphs $\mathcal{G}$. The Dynkin diagrams A_n of Figure 1.1 give the ABF models of Andrews, Baxter and Forrester [1]. Or one could use the other Dynkin diagrams of Figures 1.2-1.3, or more general graphs such as the $SU(3)$ $\mathcal{ADE}$ graphs of Section 1.5.

The graphs D_{n+2} are the $\mathbb{Z}_2$-orbifolds of the graphs A_{2n+1}, $n = 2, 3, \ldots$, whilst the tadpole graphs Tad_n of Figure 1.4 are the $\mathbb{Z}_2$-orbifolds of the graphs A_{2n}, $n = 1, 2, \ldots$.

A configuration is now a distribution of the edges of $\mathcal{G}$ on the edges of the lattice, and the Hamiltonian H is again an energy function from the configuration space to $\mathbb{R}$. We associate to each local configuration a Boltzmann weight $X_i(u)$:

$X_i(u) =$ 

for edges ρ_i of $\mathcal{G}$, $i = 1, 2, 3, 4$. The integrability condition is a sufficient condition for

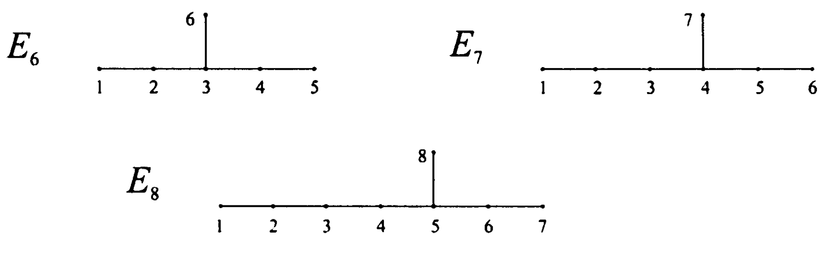


Figure 1.3: Dynkin diagrams E_6, E_7 and E_8

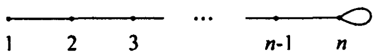


Figure 1.4: Graph Tad_n

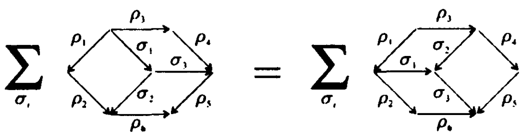


Figure 1.5: The Yang-Baxter equation

the model to be solvable, namely that there exists an infinite set of commuting transfer matrices $T(u)$, where the u are in some interval. This is equivalent to requiring the Boltzmann weights $X_i(u)$ to satisfy the Yang-Baxter equation:

$$X_i(u)X_{i+1}(u+v)X_i(v) = X_{i+1}(v)X_i(u+v)X_{i+1}(u), \tag{1.1}$$

which is given pictorially in Figure 1.5.

In the context of critical lattice $SU(N)$ models, di Francesco and Zuber take the following ansatz for $X_i(u)$:

$$X_i(u) = \sin(\pi(\hat{\lambda} - u))\mathbf{1}_i + \sin \pi u U_i,$$

where $\mathbf{1}_i$ is the identity operator acting on site i and $\hat{\lambda}$ is a real parameter. The Boltzmann weight $X_i(u)$ satisfies (1.1) provided the U_i's satisfy the Hecke algebra relations H1-H3, where δ is related to $\hat{\lambda}$ by $\delta = 2\cos \pi\hat{\lambda}$.

At criticality, with

$$g_i = q^{-1} - U_i, \tag{1.2}$$

the Boltzmann weights reduce to the natural braid generators g_i which satisfy

$$\begin{aligned} g_i g_j &= g_j g_i, \quad \text{if } |j-i| > 1, & (1.3) \\ g_i g_{i+1} g_i &= g_{i+1} g_i g_{i+1}. & (1.4) \end{aligned}$$

When $q = 1$, the g_i, $i = 1, \ldots, N$, give a representation of the permutation group S_{N+1} acting on a line of $N+1$ sites, where, for $i = 1, \ldots, N$, g_i is associated to the transposition $\tau_{i,i+1}$. To any $\sigma \in S_{N+1}$, decomposed into $|I_\sigma|$ transpositions of nearest neighbours $\sigma = \prod_{i \in I_\sigma} \tau_{i,i+1}$, we associate the operator

$$g_\sigma = \prod_{i \in I_\sigma} g_i,$$

which is well defined because of the braiding relation (1.4). Then the commutant of the quantum group $SU(N)_q$ is obtained from the Hecke algebra by imposing an extra condition, which is the vanishing of the q-antisymmetrizer

$$\sum_{\sigma \in S_{N+1}} (-q)^{|I_\alpha|} g_\sigma = 0. \tag{1.5}$$

For $SU(2)$ it reduces to the Temperley-Lieb condition

$$U_i U_{i\pm 1} U_i - U_i = 0, \tag{1.6}$$

and for $SU(3)$ it is

$$(U_i - U_{i+2}U_{i+1}U_i + U_{i+1})(U_{i+1}U_{i+2}U_{i+1} - U_{i+1}) = 0. \tag{1.7}$$

1.2 Hecke Algebras

1.2.1 Temperley-Lieb Algebra

The algebraic structure behind the Ising model is the Temperley-Lieb algebra. For integers $n > 0$ and any non-zero $\delta \in \mathbb{C}$, the abstract Temperley-Lieb algebra $TL_n(\delta)$ is defined to be the $*$-algebra generated by the identity $\mathbf{1}$ and projections e_i, $i = 1, \dots, n-1$, which satisfy the Jones-Temperley-Lieb relations:

$$\begin{aligned} \text{TL1}: \quad & e_i e_j = e_j e_i, \qquad \text{if } |j - i| > 1, \\ \text{TL2}: \quad & e_i e_{i\pm 1} e_i = \delta^{-2} e_i. \end{aligned}$$

We see that (1.6) is satisfied with $U_i = \delta e_i$, $i = 1, \dots, n-1$. There is a standard pictorial representation of $TL_n(\delta)$ given by the $*$-algebra over $\mathbb{C}$ with basis consisting of all planar n-diagrams on a rectangle with n vertices along the top and bottom, due to Kauffman [69]. These n-diagrams consist of disjoint curves, called strings, whose endpoints are the vertices along the top and bottom edges of the rectangle such that every vertex is the endpoint for one string. Multiplication ST of two n-diagrams S, T is defined by placing S on top of T in such a way that the n vertices along the bottom of S and along the top of T coincide. These vertices are then removed, and the strings smoothed if necessary. The new diagram may contain closed loops ○, which are removed, each one contributing a scalar factor δ. The adjoint T^* of a planar n-diagram T is given by reflecting T about the horizontal line passing midway between its top and bottom edges. The element E_i is shown in Figure 1.6. The elements $\delta^{-1} E_i$ are projections which satisfy TL1 and TL2, and in fact any planar n-diagram can be written as a product of the E_i, $i = 1, \dots n-1$, so that $TL_n(\delta)$ is isomorphic to the algebra of all planar n-diagrams.

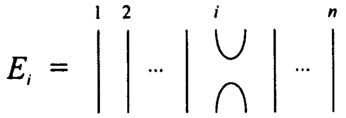


Figure 1.6: n-diagram E_i

1.2.2 Hecke Algebras for $SU(N)$

The algebraic structure behind the $SU(N)$ models are the Hecke algebras $H_n(q)$ of type A_{n-1}, for $q \in \mathbb{C}$, since the Boltzmann weights lie in $(\bigotimes_{\mathrm{N}} M_N)^{SU(N)}$ or $(\bigotimes_{\mathrm{N}} M_N)^{SU(N)_q}$. The Hecke algebra $H_n(q)$ is the algebra generated by operators $\mathbf{1}, U_1, U_2, \ldots, U_{n-1}$ which satisfy

$$
\begin{aligned}
&\text{H1:} & U_i^2 &= \delta U_i, \\
&\text{H2:} & U_i U_j &= U_j U_i, \quad |i-j| > 1, \\
&\text{H3:} & U_i U_{i+1} U_i - U_i &= U_{i+1} U_i U_{i+1} - U_{i+1},
\end{aligned}
$$

where $\delta = q + q^{-1}$. We will say that a family of operators $\{U_m\}$ satisfy the A_2-Temperley-Lieb relations if they satisfy the Hecke relations H1-H3 and the extra condition (1.7). If we let $g_j = q^{-1} - U_j$, then H1-H3 are equivalent to the relations

$$
\begin{aligned}
(q^{-1} - g_j)(q + g_j) &= 0, \\
g_i g_j &= g_j g_i, \quad |i-j| > 1, \\
g_i g_{i+1} g_i &= g_{i+1} g_i g_{i+1}.
\end{aligned}
$$

When $q = 1$, the first relation becomes $g_j^2 = 1$, so that $H_n(1)$ reduces to the symmetric, or permutation, group S_n, where g_j represents a transition $(j, j+1)$.

It is well known that the irreducible representations of the symmetric group S_n can be labelled by Young diagrams with n boxes such that $m_1 \geq m_2 \geq \cdots \geq m_n \geq 0$, where m_i denotes the number of boxes in row i. A Young tableau is obtained by inserting the numbers $1, \ldots, n$ in the boxes of a Young diagram, and a Young tableau is called standard if the entries in each row and each column are increasing. The dimension of the irreducible representation labelled by a Young diagram λ is equal to the number of different standard Young tableaux that can be obtained from the diagram. In a similar way as for the symmetric group S_n, Wenzl [112] defined representations of the Hecke algebra $H_n(q)$ on Young diagrams with at most N rows. Let $H_\infty(q)$ denote the inductive limit of the $H_n(q)$. When q is not a root of unity, there is a representation π_λ corresponding to each Young diagram λ. For $x \in H_n(q)$, the direct sum $\pi(x) := \bigoplus_\lambda \pi_\lambda(x)$, where the summation is over all Young diagrams λ with n boxes, defines a faithful representation of $H_n(q)$. Then if $B_n = \bigoplus_\lambda \pi_\lambda(H_n(q))$, where the summation is again over all Young diagrams λ with n boxes, $B_n \subset B_{n+1}$ has inclusion matrix given by the adjacency matrix for the $SU(N)$ graph $\mathcal{A}^{(\infty)}$, where $B_0 = \mathbb{C}$ is identified with the apex vertex $(0,0,\dots,0)$. When $q = e^{2\pi i/l}$ there are representations corresponding to a special class of Young diagrams, called (N,l)-diagrams, for which $\lambda_1 - \lambda_N \leq l - N$, given by $\pi_\lambda^{(N,l)}$. For $x \in H_n(q)$, the direct sum $\pi^{(N,l)}(x) := \bigoplus_\lambda \pi_\lambda(x)$, where the summation is over all (N,l)-diagrams λ with n boxes, defines a faithful representation of $H_n(q)$. If B_n is the algebra defined by $B_n = \bigoplus_\lambda \pi_\lambda^{(N,l)}(H_n(q))$, then Wenzl showed that $B_n \subset B_{n+1}$ with inclusion matrix given by the adjacency matrix for the $SU(N)$ graph $\mathcal{A}^{(l)}$, where $B_0 = \mathbb{C}$ is identified with the apex vertex $(0,0,\dots,0)$ of $\mathcal{A}^{(l)}$. A representation ρ of $H_n(q)$ is called a C^*-representation if $\rho(\delta^{-1}U_i)$ is a self-adjoint projection for $i = 1,\dots,n-1$. The following result is contained in [112, Theroem 3.6]:

Theorem 1.2.1 *(a) For $q \in \mathbb{R}$, $q \geq 1$, if ρ is a C^*-representation of $H_\infty(q)$ with trace* tr *such that* $\mathrm{tr}(\rho(\delta^{-1}U_i)) = (1-q^{-N+1})/(1+q)(1-q^{-N})$, *then* tr *is a Markov trace and the traces are non-zero on those representations of $H_n(q)$ which belong to Young diagrams with at most N columns.*

(b) For $l > N$ and $q = e^{\pm 2\pi i/l}$, if ρ is a C^*-representation of $H_\infty(q)$ with Markov trace tr *such that* $\mathrm{tr}(\rho(\delta^{-1}U_i)) = (1-q^{-N+1})/(1+q)(1-q^{-N})$, *then the representation corresponding to the GNS construction with respect to this trace is equivalent to $\pi^{(N,l)}$, and has positive definite trace.*

1.3 Representation Theory

1.3.1 Representation Theory of $SU(N)$

Let $\rho : SU(N) \to M_N$ denote the fundamental representation of $SU(N)$. The restriction of ρ to the $(N-1)$-torus $\mathbb{T}^{N-1}$ is given by

$$(\rho|_{\mathbb{T}^{N-1}})(t_1, t_2, \dots, t_{N-1}) = \mathrm{diag}(t_1, t_2, \dots, t_{N-1}, \bar{t}), \tag{1.8}$$

where $t = t_1 t_2 \cdots t_{N-1}$, and $t_i \in \mathbb{T}$.

Every irreducible representation ρ_m is classified by a signature, or highest weight, $m = (m_1, m_2, \dots, m_{N-1})$, where m_i are integers such that $m_1 \geq m_2 \geq \cdots \geq m_{N-1} \geq 0$, for $i = 1,\dots,N-1$. A signature m can be represented by a Young diagram with at most

$N-1$ rows, and m_i boxes in the i^{th} row, $i = 1, \dots, N-1$. The trivial representation has signature $(0, 0, \dots, 0)$ and corresponds to the empty Young diagram $\emptyset$ with no boxes at all, whilst the fundamental representation has signature $(1, 0, 0, \dots, 0)$ and corresponds to the Young diagram $\Box$. Given a Young diagram m, we can obtain a new Young diagram by adding a box to one of the rows (including the N^{th} row if $m_{N-1} > 0$). Suppose the Young diagram m has $N-1$ rows, so that $m_{N-1} > 0$. Then if we add a box in the N^{th} row, we obtain a diagram with N rows and we delete all the boxes in the first column. The fusion rules of the irreducible representations ρ_m of $SU(N)$ with respect to the fundamental representation ρ are given by

$$\rho_m \otimes \rho = \bigoplus_{m' > m} \rho_{m'}, \tag{1.9}$$

where on the right hand side we have a direct sum of all irreducible representations $\rho_{m'}$ for which the Young diagram of m' can be obtained from the Young diagram of m by adding one box. Let e_j, $j = 1, \dots, N$, be the unit vector given by the edge on the fusion graph from a vertex labelled by a Young diagram f to the vertex labelled by the Young diagram obtained by adding a box in the j^{th} row. The fusion graph for $SU(2)$ is the infinite Dynkin diagram A_∞ (see Figure 1.1), where the signatures are just the integers $k \geq 0$. It is well known that the k^{th} symmetric product of $\mathbb{C}^2$ gives the irreducible level k representation.

For $SU(3)$ the fusion graph is the infinite graph $\mathcal{A}^{(\infty)}$. The graph $\mathcal{A}^{(\infty)}$ is illustrated in Figure 1.7, where the vertices are labelled by "Dynkin labels" (λ_1, λ_2): if p, q denote the number of boxes in the first, second row respectively of a Young diagram, the corresponding Dynkin label is $(p-q, q)$. There are then edges on $\mathcal{A}^{(\infty)}$ from the vertex (λ_1, λ_2) to the vertices $(\lambda_1 + 1, \lambda_2)$, $(\lambda_1 - 1, \lambda_2 + 1)$ and $(\lambda_1, \lambda_2 - 1)$, i.e. edges in the directions of the vectors e_1, e_2, e_3 respectively.

1.3.2 Loop Groups

The loop group $LSU(N)$ is the group of smooth maps from S^1 into $SU(N)$ under pointwise multiplication. The projective representations of $LSU(N) \rtimes \mathrm{Rot}(S^1)$, where the rotation group acts on the maps of S^1 in a natural way such that the infinitesimal generator L_0 of the rotation group is positive, are called positive energy representations and are classified by irreducible representations of $SU(N)$ and a level k. To obtain positive energy, the admissible irreducible representations at level k are those labelled by signatures g such that $g_1 \leq k$ and $g_1 + g_2 + \cdots + g_{N-1} \leq k$. For $N = 3$, these correspond to the vertices (λ_1, λ_2) of the infinite graph $\mathcal{A}^{(\infty)}$ where $\lambda_1 + \lambda_2 \leq n - 3$. For a level k we have finite

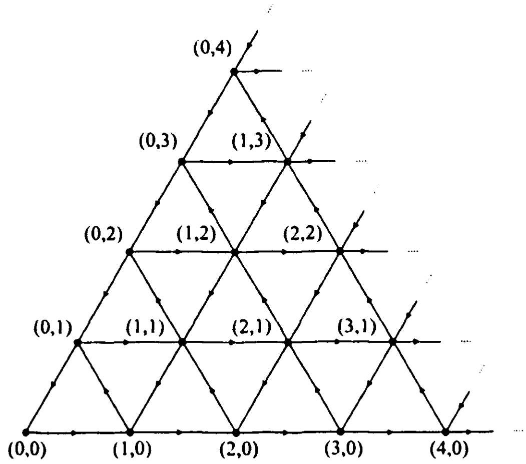


Figure 1.7: The infinite $SU(3)$ fusion graph $\mathcal{A}^{(\infty)}$

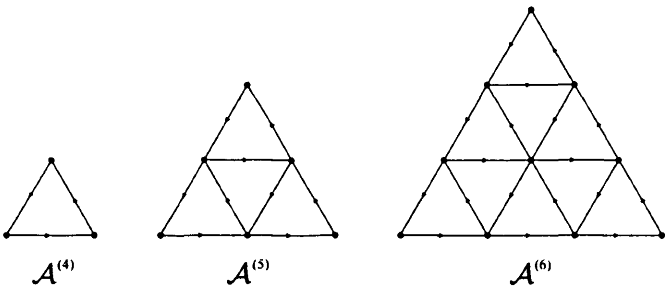


Figure 1.8: $\mathcal{A}^{(n)}$ for $n = 4, 5, 6$

graphs $\mathcal{A}^{(n)}$, where $n = k + 3$. These are illustrated in Figure 1.8 for $n = 4, 5, 6$.

For a level k, let $\lambda = (\lambda_1, \lambda_2)$ be the irreducible representations of $SU(3)$ which label the vertices of $\mathcal{A}^{(n)}$, $n = k + 3$. These representations obey the fusion rules:

$$\lambda \otimes \mu = \sum_{\nu \in \mathcal{A}^{(n)}} N_{\lambda\mu}^{\nu}\, \nu, \tag{1.10}$$

where the numbers $N_{\lambda\mu}^{\nu}$ are non-negative integers. For a group G, if π is a representation on the complex vector space V then the conjugate representation $\overline{\pi}$ is defined on the conjugate vector space V^* by $\overline{\pi}(g) = \overline{\pi(g)}$ for all $g \in G$. The conjugation of a representation $\lambda = (\lambda_1, \lambda_2)$ is given by $\overline{\lambda} = (\lambda_2, \lambda_1)$, and the fusion rules are invariant under conjugation: $N_{\overline{\lambda}\overline{\mu}}^{\overline{\nu}} = N_{\lambda\mu}^{\nu}$. The colour (sometimes called the triality in the literature) $\tau(\lambda)$ of a representation λ is given by $\tau(\lambda) = \lambda_1 - \lambda_2 \bmod 3$. For the fundamental representation $\rho = (1, 0)$, the fusion rules define the adjacency matrix $\Delta_{\mathcal{A}^{(n)}}$ of the graph $\mathcal{A}^{(n)}$, that is, $(\Delta_{\mathcal{A}^{(n)}})_{\lambda\mu} = N_{f\lambda}^{\mu}$ [110].

1.4 Modular Invariant Partition Functions

Modular invariant partition functions come about as continuum limits in statistical mechanics, i.e. letting the lattice spacing tend to zero whilst simultaneously approaching the critical temperature. They play a fundamental role in conformal field theory. Let $\chi_\lambda = \mathrm{tr}(q^{L_0 - c/24})$ denote the character of the irreducible representation λ, which is the trace in the positive energy representation of a loop group, where $q = e^{2\pi i\tau}$, $\mathrm{Im}(\tau) > 0$. Here L_0 is the conformal Hamiltonian which is the infinitesimal generator of the rotation group on the circle. Typically, the characters are transformed linearly under the action of $SL(2;\mathbb{Z})$, e.g. $\chi_a(-1/\tau) = \sum_b S_{a,b}\chi_b(\tau)$, $\chi_a(\tau+1) = \sum_b T_{a,b}\chi_b(\tau)$, where S is a symmetric unitary matrix which diagonalizes the fusion rules (see (1.11)), with $S_{\lambda,0} \geq S_{0,0} > 0$, and T is a diagonal matrix. Then a modular invariant partition function is of the form

$$Z(\tau) = \sum_{\lambda,\mu} Z_{\lambda,\mu}\chi_\lambda(\tau)\chi_\mu(\tau)^*.$$

The problem of the classification of modular invariants is to find all non-negative integer valued matrices Z such that $ZS = SZ$, $ZT = TZ$, subject to the constraint $Z_{0,0} = 1$ (which reflects the physical concept of the uniqueness of the vacuum state). The non-negative integer requirement on the entries of Z comes from the understanding of the entries as multiplicities of the decomposition of the underlying Hilbert space. The trivial modular invariant, given by $Z_{\lambda,\mu} = \delta_{\lambda,\mu}$ or $Z = \sum_\lambda |\chi_\lambda|^2$ is always a solution. There may also be permutation invariants $Z = \sum_\lambda \chi_\lambda\chi^*_{\omega(\lambda)}$, where ω is a permutation of the labels which preserves the fusion rules and $\omega(0) = 0$. For a Rational Conformal Field Theory (RCFT) the partition function is at most a permutation matrix $Z^{\mathrm{ext}}_{\tau,\tau'} = \delta_{\tau,\omega(\tau')}$, where τ, τ' label the representations of an extended chiral algebra and ω is now a permutation of these labels (see [89]). The extended characters χ^{ext}_τ can be decomposed in terms of the original characters χ_λ as $\chi^{\mathrm{ext}}_\tau = \sum_\lambda b_{\tau,\lambda}\chi_\lambda$ for some non-negative coefficients $b_{\tau,\lambda}$, and $Z_{\lambda,\mu} = \sum_\tau b_{\tau,\lambda}b_{\omega(\tau),\mu}$. The modular invariants are of two types: those for which ω is trivial are called type I, i.e. $Z^{\mathrm{ext}} = \sum_\tau |\chi^{\mathrm{ext}}_\tau|^2$, whereas those corresponding to non-trivial ω are called type II, i.e. $Z^{\mathrm{ext}} = \sum_\tau \chi^{\mathrm{ext}}_\tau(\chi^{\mathrm{ext}}_{\omega(\tau)})^*$. The matrices $Z_{\lambda,\mu}$ for these type I invariants are symmetric, whereas for the type II invariants only "vacuum coupling" is necessarily symmetric: $Z_{0,\lambda} = Z_{\lambda,0}$ for all λ. However, we will modify this notion of classifying type in Section 5.2.7, where type will instead refer to an inclusion $N \subset M$ of factors.

1.4.1 Classification of Z for $SU(2)$

The $SU(2)$ modular invariants at level k are:

$$
\begin{aligned}
Z_{A_{k+1}} &= \sum_{\lambda=1}^{k+1} |\chi_\lambda|^2, \qquad k \geq 1, \\
Z_{D_{2\rho+2}} &= \sum_{\lambda\,\mathrm{odd}=1,\lambda\neq 2\rho+1}^{4\rho+1} |\chi_\lambda|^2 + 2|\chi_{2\rho+1}|^2 + \sum_{\lambda\,\mathrm{odd}=2}^{2\rho-1} \left(\chi_\lambda\chi^*_{4\rho+2-\lambda} + \chi_{4\rho+2-\lambda}\chi^*_\lambda\right) \\
&= \sum_{\lambda\,\mathrm{odd}=1}^{2\rho-1} |\chi_\lambda + \chi_{4\rho+2-\lambda}|^2 + 2|\chi_{2\rho+1}|^2, \qquad k = 4\rho, \rho \geq 1, \\
Z_{D_{2\rho+1}} &= \sum_{\lambda\,\mathrm{odd}=1}^{4\rho-1} |\chi_\lambda|^2 + 2|\chi_{2\rho}|^2 + \sum_{\lambda\,\mathrm{even}=2}^{2\rho-2} \left(\chi_\lambda\chi^*_{4\rho-\lambda} + \chi_{4\rho-\lambda}\chi^*_\lambda\right), \qquad k = 4\rho-2, \rho \geq 2, \\
Z_{E_6} &= |\chi_1+\chi_7|^2 + |\chi_4+\chi_8|^2 + |\chi_5+\chi_{11}|^2, \qquad k+2 = 12, \\
Z_{E_7} &= |\chi_1+\chi_{17}|^2 + |\chi_5+\chi_{13}|^2 + |\chi_7+\chi_{11}|^2 + |\chi_9|^2 \\
&\quad +(\chi_3+\chi_{15})\chi_9^* + \chi_9(\chi_3+\chi_{15})^*, \qquad k+2 = 18, \\
Z_{E_8} &= |\chi_1+\chi_{11}+\chi_{19}+\chi_{29}|^2 + |\chi_7+\chi_{13}+\chi_{17}+\chi_{23}|^2, \qquad k+2 = 30.
\end{aligned}
$$

The ADE classification of the $SU(2)$ modular invariants is due to Cappelli, Itzykson and Zuber [18], and the list was shown to be complete in [19] and independently by [68]. The original reason for the ADE classification was because the diagonal entries $Z_{\lambda,\lambda}$ of the modular invariant $Z_{\mathcal{G}}$ at level k are given exactly by the multiplicity of the eigenvalue $2\cos(\pi\lambda/h)$ for the Dynkin diagram $\mathcal{G}$ with Coxeter number $h = k+2$. The trivial modular invariant is $Z_{A_{k+1}}$. Let $N_\lambda = [N_{\lambda,\mu}^{\nu}]$, where the $N_{\lambda,\mu}^{\nu}$ are the fusion coefficients of $SU(2)_k$ which are related to the S-matrix by the Verlinde formula [110]

$$
N_{\lambda,\mu}^{\nu} = \sum_{\sigma} \frac{S_{\sigma,\lambda}}{S_{\sigma,1}} S_{\sigma,\mu} S^*_{\sigma,\nu}. \tag{1.11}
$$

The matrices N_λ satisfy $N_\lambda N_\mu = \sum_\nu N_{\lambda,\mu}^{\nu} N_\nu$. Since the adjacency matrix Δ_A of A_{k+1} is given by the level k fusion matrix N_ρ of the fundamental representation ρ, Δ_A can be interpreted as the fundamental representation matrix in the regular representation of the fusion rules. The D and E graphs turn out to be the fundamental matrices G_ρ of a whole family of non-negative integer valued matrices (nimreps for short) G_λ which provide a representation of the original $SU(2)_k$ fusion rules, i.e. $G_\lambda G_\mu = \sum_\nu N_{\lambda,\mu}^{\nu} G_\nu$. For the graphs A_n, the eigenvalues λ_n^j, $j = 1,\ldots,n$, are given by

$$
\lambda_n^j = 2\cos(j\pi/(n+1)). \tag{1.12}
$$

Writing the modular invariant associated to a graph $\mathcal{G}$ as $Z = \sum_i |\chi_i|^2 +$ remainder, the integers i for which the diagonal term $|\chi_i|^2$ appears are called the Coxeter exponents of the graph. The eigenvalues λ^j of the graph $\mathcal{G}$ are given by

$$\lambda^j = 2\cos(\pi m_j/h), \tag{1.13}$$

where m_j are the Coxeter exponents of $\mathcal{G}$ and h is the Coxeter number.

The modular invariants arising from $SU(2)_k$ conformal embeddings are (see [34]):

- E_6: $SU(2)_{10} \subset SO(5)_1$,
- E_8: $SU(2)_{28} \subset (\mathrm{G}_2)_1$,
- E_7: automorphism or twist of the orbifold invariant $\mathrm{D}_{10} = SU(2)_{16}/\mathbb{Z}_2$.

1.4.2 Classification of Z for $SU(3)$

The list below of all $SU(3)$ modular invariants was shown to be complete by Gannon [45]. Each one has a corresponding $\mathcal{ADE}$ graph. We label the vertices of $\mathcal{A}^{(n)}$ by $\mu = (\mu_1, \mu_2)$ for $\mu_1, \mu_2 = 0, \ldots, n-3$ such that $\mu_1 + \mu_2 \leq n-3$. We define the automorphism A of order 3 on the weights $\mu = (\mu_1, \mu_2)$ of $\mathcal{A}^{(n)}$ (where the apex is denoted $(0,0)$) by $A(\mu_1, \mu_2) = (n-3-\mu_1-\mu_2, \mu_1)$.

The identity invariant is

$$Z_{\mathcal{A}^{(n)}} = \sum_{\mu \in \mathcal{A}^{(n)}} |\chi_\mu|^2, \qquad n \geq 4, \tag{1.14}$$

and its orbifold invariant is given by

$$Z_{\mathcal{D}^{(3k)}} = \frac{1}{3} \sum_{\substack{\mu \in \mathcal{A}^{(3k)} \\ \mu_1 - \mu_2 \equiv 0 \bmod 3}} |\chi_\mu + \chi_{A\mu} + \chi_{A^2\mu}|^2, \qquad k \geq 2, \tag{1.15}$$

$$Z_{\mathcal{D}^{(n)}} = \sum_{\mu \in \mathcal{A}^{(n)}} \chi_\mu \chi^*_{A^{(n-3)(\mu_1-\mu_2)}\mu}, \qquad n \geq 5, n \not\equiv 0 \bmod 3. \tag{1.16}$$

The conjugate invariant $Z_{\mathcal{A}^{(n)*}} = C$ and the conjugate orbifold invariants $Z_{\mathcal{D}^{(n)*}} = Z_{\mathcal{D}^{(n)}}C$ are

$$Z_{\mathcal{A}^{(n)*}} = \sum_{\mu \in \mathcal{A}^{(n)}} \chi_\mu \chi^*_{\overline{\mu}}, \qquad n \geq 4, \tag{1.17}$$

$$Z_{\mathcal{D}^{(3k)*}} = \frac{1}{3} \sum_{\substack{\mu \in \mathcal{A}^{(3k)} \\ \mu_1 - \mu_2 \equiv 0 \bmod 3}} (\chi_\mu + \chi_{A\mu} + \chi_{A^2\mu})(\chi^*_{\overline{\mu}} + \chi^*_{\overline{A\mu}} + \chi^*_{\overline{A^2\mu}}), \qquad k \geq 2, \tag{1.18}$$

$$Z_{\mathcal{D}^{(n)*}} = \sum_{\mu \in \mathcal{A}^{(n)}} \chi_\mu \chi^*_{\overline{A^{(n-3)(\mu_1-\mu_2)}\mu}}, \qquad n \geq 5, n \not\equiv 0 \bmod 3. \tag{1.19}$$

There are also exceptional invariants, i.e. invariants which are not diagonal or orbifold, or their conjugates,

$$
\begin{aligned}
Z_{\mathcal{E}^{(8)}} &= |\chi_{(0,0)}+\chi_{(2,2)}|^2+|\chi_{(0,2)}+\chi_{(3,2)}|^2+|\chi_{(2,0)}+\chi_{(2,3)}|^2+|\chi_{(2,1)}+\chi_{(0,5)}|^2 \\
&\quad +|\chi_{(3,0)}+\chi_{(0,3)}|^2+|\chi_{(1,2)}+\chi_{(5,0)}|^2, \qquad (1.20)\\
Z_{\mathcal{E}^{(8)*}} &= |\chi_{(0,0)}+\chi_{(2,2)}|^2+(\chi_{(0,2)}+\chi_{(3,2)})(\chi^*_{(2,0)}+\chi^*_{(2,3)}) \\
&\quad +(\chi_{(2,0)}+\chi_{(2,3)})(\chi^*_{(0,2)}+\chi^*_{(3,2)})+(\chi_{(2,1)}+\chi_{(0,5)})(\chi^*_{(1,2)}+\chi^*_{(5,0)}) \\
&\quad +|\chi_{(3,0)}+\chi_{(0,3)}|^2+(\chi_{(1,2)}+\chi_{(5,0)})(\chi^*_{(2,1)}+\chi^*_{(0,5)}), \qquad (1.21)\\
Z_{\mathcal{E}_1^{(12)}} &= |\chi_{(0,0)}+\chi_{(0,9)}+\chi_{(9,0)}+\chi_{(4,4)}+\chi_{(4,1)}+\chi_{(1,4)}|^2 \\
&\quad +2|\chi_{(2,2)}+\chi_{(2,5)}+\chi_{(5,2)}|^2 \;=\; Z_{\mathcal{E}_2^{(12)}}, \qquad (1.22)\\
Z_{\mathcal{E}_4^{(12)}} &= |\chi_{(0,0)}+\chi_{(0,9)}+\chi_{(9,0)}|^2+|\chi_{(2,2)}+\chi_{(2,5)}+\chi_{(5,2)}|^2+2|\chi_{(3,3)}|^2 \\
&\quad +(\chi_{(0,3)}+\chi_{(6,0)}+\chi_{(3,6)})(\chi^*_{(3,0)}+\chi^*_{(0,6)}+\chi^*_{(6,3)}) \\
&\quad +(\chi_{(3,0)}+\chi_{(0,6)}+\chi_{(6,3)})(\chi^*_{(0,3)}+\chi^*_{(6,0)}+\chi^*_{(3,6)})+|\chi_{(4,4)}+\chi_{(4,1)}+\chi_{(1,4)}|^2 \\
&\quad +(\chi_{(1,1)}+\chi_{(1,7)}+\chi_{(7,1)})\chi^*_{(3,3)}+\chi_{(3,3)}(\chi^*_{(1,1)}+\chi^*_{(1,7)}+\chi^*_{(7,1)}), \qquad (1.23)\\
Z_{\mathcal{E}_5^{(12)}} &= |\chi_{(0,0)}+\chi_{(0,9)}+\chi_{(9,0)}|^2+|\chi_{(2,2)}+\chi_{(2,5)}+\chi_{(5,2)}|^2+2|\chi_{(3,3)}|^2 \\
&\quad +|\chi_{(0,3)}+\chi_{(6,0)}+\chi_{(3,6)}|^2+|\chi_{(3,0)}+\chi_{(0,6)}+\chi_{(6,3)}|^2+|\chi_{(4,4)}+\chi_{(4,1)}+\chi_{(1,4)}|^2 \\
&\quad +(\chi_{(1,1)}+\chi_{(1,7)}+\chi_{(7,1)})\chi^*_{(3,3)}+\chi_{(3,3)}(\chi^*_{(1,1)}+\chi^*_{(1,7)}+\chi^*_{(7,1)}), \qquad (1.24)\\
Z_{\mathcal{E}^{(24)}} &= |\chi_{(0,0)}+\chi_{(4,4)}+\chi_{(6,6)}+\chi_{(10,10)}+\chi_{(21,0)}+\chi_{(0,21)}+\chi_{(13,4)}+\chi_{(4,13)} \\
&\quad +\chi_{(10,1)}+\chi_{(1,10)}+\chi_{(9,6)}+\chi_{(6,9)}|^2 \\
&\quad +|\chi_{(15,6)}+\chi_{(6,15)}+\chi_{(15,0)}+\chi_{(0,15)}+\chi_{(10,7)}+\chi_{(7,10)}+\chi_{(10,4)} \\
&\qquad +\chi_{(4,10)}+\chi_{(7,4)}+\chi_{(4,7)}+\chi_{(6,0)}+\chi_{(0,6)}|^2, \qquad (1.25)
\end{aligned}
$$

where $Z_{\mathcal{E}_1^{(12)}}$, $Z_{\mathcal{E}_2^{(12)}}$ and $Z_{\mathcal{E}^{(24)}}$ are self-conjugate, and $Z_{\mathcal{E}_4^{(12)}} = Z_{\mathcal{E}_5^{(12)}}C$. The modular invariant $Z_{\mathcal{E}_5^{(12)}}$ is the Moore-Seiberg invariant [89]. The modular invariants arising from $SU(3)_k$ conformal embeddings are (see [34]):

- $\mathcal{D}^{(6)}$: $SU(3)_3 \subset SO(8)_1$, also realised as an orbifold $SU(3)_3/\mathbb{Z}_3$,
- $\mathcal{E}^{(8)}$: $SU(3)_5 \subset SU(6)_1$, plus its conjugate $\mathcal{E}^{(8)*} = \mathcal{E}^{(8)}/\mathbb{Z}_3$,
- $\mathcal{E}_1^{(12)}$: $SU(3)_9 \subset (\mathrm{E}_6)_1$, with two nimreps $\mathcal{E}_1^{(12)}$ and $\mathcal{E}_2^{(12)} = \mathcal{E}_1^{(12)}/\mathbb{Z}_3$,
- $\mathcal{E}_5^{(12)}$: Moore-Seiberg invariant, automorphism of the orbifold invariant $SU(3)_9/\mathbb{Z}_3$, plus its conjugate $\mathcal{E}_4^{(12)} = \mathcal{E}_5^{(12)*}$,
- $\mathcal{E}^{(24)}$: $SU(3)_{21} \subset (\mathrm{E}_7)_1$.

1.5 $SU(3)$ $\mathcal{ADE}$ graphs

The $SU(3)$ graphs $\mathcal{A}^{(n)}$, $n = 4, 5, \dots$, were introduced in Section 1.3.2. There is another infinite series of graphs $\mathcal{D}^{(n)}$, where $\mathcal{D}^{(n)}$ is obtained from $\mathcal{A}^{(n)}$ by an orbifold procedure [76, 27]. The graph $\mathcal{A}^{(n)}$ is left invariant by the $\mathbb{Z}/3\mathbb{Z}$ automorphism σ defined by rotation of the graph by $2\pi/3$. When n is not a multiple of 3, there are no fixed points under the rotation. Then the graph $\mathcal{D}^{(n)}$ is given by the "fundamental domain" of σ, i.e. if a vertex v is mapped to $\sigma(v) = v'$ then the vertices v and v' are identified, and similarly for the edges of $\mathcal{A}^{(n)}$. The the graph $\mathcal{D}^{(n)}$ is not three-colourable. For $n = 3k$ for some positive integer k, the vertex $(k-1, k-1)$ of $\mathcal{A}^{(3k)}$ is a fixed point. This vertex is split into three distinct vertices, and we also split the edges which joined the vertex $(k-1, k-1)$ to other vertices. For the other vertices and edges of $\mathcal{A}^{(3k)}$ we have the same procedure as for $n \not\equiv 0 \bmod 3$. The graphs $\mathcal{D}^{(3k)}$ are three-colourable. The graphs $\mathcal{D}^{(n)}$ are illustrated in Figure 1.9 for $n = 5, 6, 7, 8, 9$.

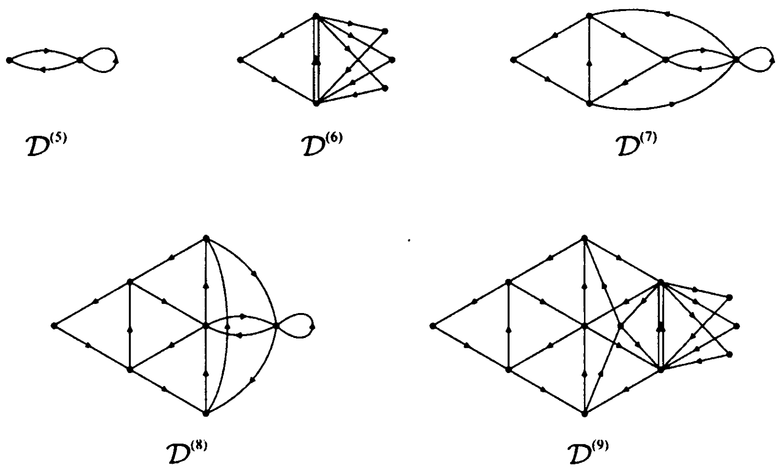


Figure 1.9: $\mathcal{D}^{(n)}$ for $n = 5, 6, 7, 8, 9$

Two more infinite series of graphs are given by $\mathcal{A}^{(n)*}$, $\mathcal{D}^{(n)*}$ respectively, which are "conjugations" of the graphs $\mathcal{A}^{(n)}$, $\mathcal{D}^{(n)}$ respectively. The term conjugation here is due to the fact that the modular invariants corresponding to $\mathcal{A}^{(n)*}$ and $\mathcal{D}^{(n)*}$are the conjugate modular invariants of those corresponding to $\mathcal{A}^{(n)}$ and $\mathcal{D}^{(n)}$. The graph $\mathcal{A}^{(n)*}$ can be obtained from $\mathcal{D}^{(n)*}$ by an orbifold procedure as described above. These graphs are illustrated in Figures 1.10 and 1.11. For the graphs $\mathcal{A}^{(n)}$, $\mathcal{D}^{(n)}$, $\mathcal{A}^{(n)*}$ and $\mathcal{D}^{(n)*}$, we call n the Coxeter number of the graph.

There are also a number of exceptional graphs: $\mathcal{E}^{(8)}$ and its conjugate $\mathcal{E}^{(8)*}$ (which both have Coxeter number 8), $\mathcal{E}_i^{(12)}$, $i = 1, \dots, 5$, (with Coxeter number 12) and $\mathcal{E}^{(24)}$

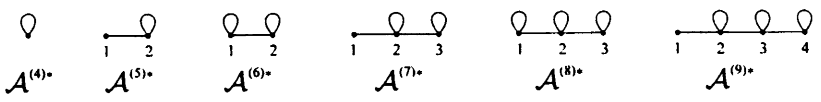


Figure 1.10: $\mathcal{A}^{(n)*}$ for $n = 4, 5, 6, 7, 8, 9$

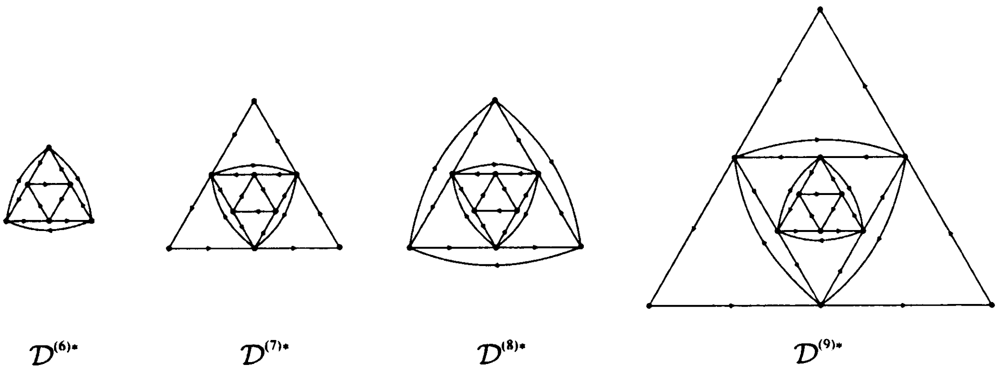


Figure 1.11: $\mathcal{D}^{(n)*}$ for $n = 6, 7, 8, 9$

(which has Coxeter number 24). These are illustrated in Figures 1.12 - 1.15. The graphs $\mathcal{E}_i^{(12)}$, $i = 1, 2, 3$, are isospectral, and indeed $\mathcal{E}_2^{(12)}$ may be constructed as a $\mathbb{Z}/3\mathbb{Z}$ orbifold of $\mathcal{E}_1^{(12)}$, and vice versa. The graph $\mathcal{E}_4^{(12)}$ is the conjugate of the graph $\mathcal{E}_5^{(12)}$ which is associated with the Moore-Seiberg invariant (1.24). Since we are looking for graphs which represent modular invariants through nimreps (M-N) or from subfactors, the graph $\mathcal{E}_3^{(12)}$ is discarded as it does not appear in these descriptions.

All the graphs except $\mathcal{D}^{(n)}$, $n \not\equiv 0 \bmod 3$, $\mathcal{A}^{(n)*}$, $n = 4, 5, \ldots,$ are three-colourable. For these graphs (except for the graph $\mathcal{E}_5^{(12)}$) the distinguished vertex $*$, with the lowest

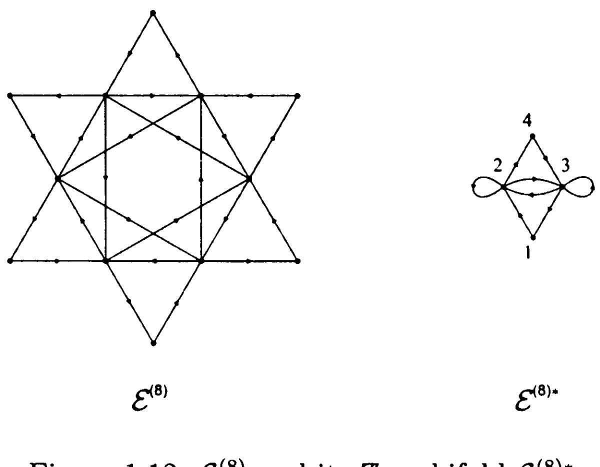
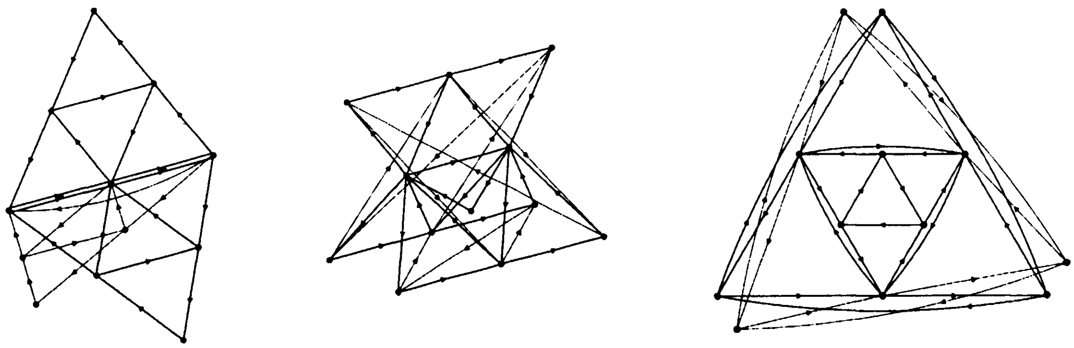
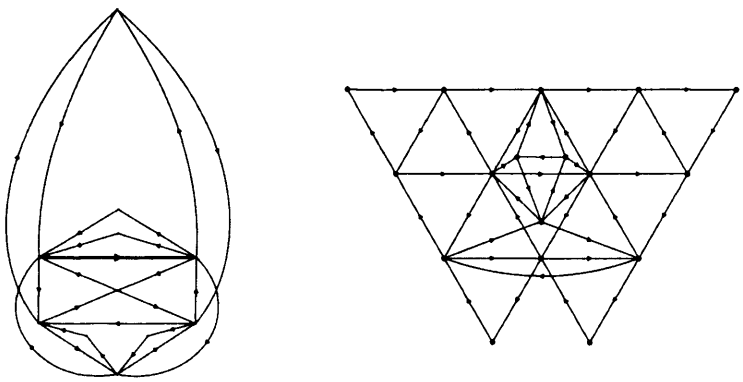


Figure 1.12: $\mathcal{E}^{(8)}$ and its $\mathbb{Z}_3$ orbifold $\mathcal{E}^{(8)*}$



$\mathcal{E}_1^{(12)}$ $\mathcal{E}_2^{(12)}$ $\mathcal{E}_3^{(12)}$

Figure 1.13: $\mathcal{E}_1^{(12)}$, $\mathcal{E}_2^{(12)}$ and the virtual graph $\mathcal{E}_3^{(12)}$



$\mathcal{E}_4^{(12)}$ $\mathcal{E}_5^{(12)}$

Figure 1.14: Moore-Seiberg graph $\mathcal{E}_5^{(12)}$ and conjugate Moore-Seiberg graph $\mathcal{E}_4^{(12)}$

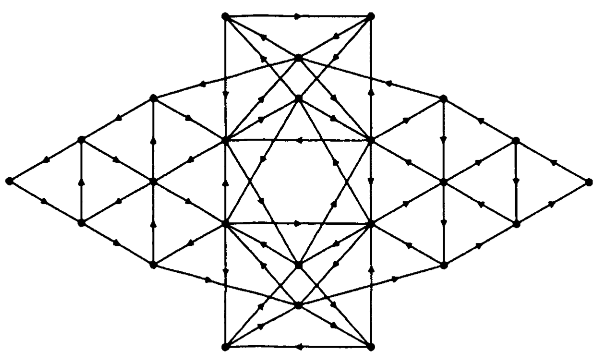


Figure 1.15: $\mathcal{E}^{(24)}$

Perron-Frobenius weight, is said to have colour 0, and there is a conjugation on the graphs which switches the vertices of colours 1 and 2. For $\mathcal{E}_5^{(12)}$ there are two vertices with lowest Perron-Frobenius weight, one of colour 1 and one of colour 2, which are conjugate to each other. For the non-three-colourable graphs we define the j-coloured vertices to be all the vertices of the graph, for any $j = 0, 1, 2$.

The Perron-Frobenius theorem for an irreducible square matrix A with non-negative entries states that the spectral radius of A is an eigenvalue, called the Perron-Frobenius eigenvalue, and that the corresponding (Perron-Frobenius) eigenvector has all entries positive. All other eigenvectors of A have at least one negative entry. Any irreducible graph has an adjacency matrix of this form. The eigenvalue $\beta^{(\sigma)}$ corresponding to the vertex $\sigma = (\sigma_1, \sigma_2)$ of $\mathcal{A}^{(n)}$ is given by [27]:

$$\beta^{(\sigma)} = \exp\left(\frac{2i\pi}{3n}(\sigma_1 + 2\sigma_2 + 3)\right) + \exp\left(-\frac{2i\pi}{3n}(2\sigma_1 + \sigma_2 + 3)\right) + \exp\left(\frac{2i\pi}{3n}(\sigma_1 - \sigma_2)\right) \tag{1.26}$$

and has a corresponding eigenvector $\phi^{(\sigma)} = (\phi_\lambda^{(\sigma)})_\lambda$. In terms of the S-matrix, we have an orthonormal basis of eigenvectors $\phi_\lambda^{(\sigma)} = S_{\lambda\sigma}$ for the eigenvalues

$$\beta^{(\sigma)} = S_{\rho\sigma}/S_{0\sigma}, \tag{1.27}$$

where we denote by 0 the apex $(0,0)$ of the graph $\mathcal{A}^{(n)}$. In particular, the Perron-Frobenius eigenvector $\phi \equiv \phi^{(0)}$ has the form [24]:

$$\phi_\lambda = \frac{\sin((\lambda_1+1)\pi/n)\sin((\lambda_2+1)\pi/n)\sin((\lambda_1+\lambda_2+2)\pi/n)}{\sin^2(\pi/n)\sin(2\pi/n)}. \tag{1.28}$$

Definition 1.5.1 (Quantum Numbers) *For $m \in \mathbb{Z}$ and $q \in \mathbb{C}\setminus\{0, 1, -1\}$, the* ***quantum number*** *$[m]_q$ is defined as*

$$[m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}.$$

Note that $[m]_q = q^{m-1} + q^{m-3} + q^{m-5} + \cdots + q^{-(m-1)}$, so that when $q = 1$ the quantum numbers recover the integers: $[m]_1 = m$.

Then, in terms of quantum numbers, the Perron-Frobenius eigenvalue is $[3]_q$, where $q = \exp(i\pi/n)$, and the eigenvector ϕ is

$$\phi_\lambda = \frac{[\lambda_1+1]_q[\lambda_2+1]_q[\lambda_1+\lambda_2+2]_q}{[2]}. \tag{1.29}$$

Note, the quantum numbers $[m]_q$ satisfy the fusion rules for the irreducible representations of the quantum group $SU(2)_n$, i.e.

$$[a]_q[b]_q = \sum_{c=|b-a|:\, a+b+c \text{ even}}^{\min(a+b, 2n-a-b)} [c]_q, \tag{1.30}$$

whilst the entries of the Perron-Frobenius eigenvector (1.29) give a one-dimensional representation of the fusion rules for the irreducible representations of the quantum group $SU(3)_n$.

It will be shown in Section 5.2 that the above graphs all appear as the M-N graphs for certain subfactors $N \subset M$, and we have an associated modular invariant as in Section 1.4.2. Then by the Verlinde formula (1.11), the eigenvalues $\beta^{(\lambda)}$ of these graphs are again ratios of the S-matrix given by (1.27) for vertices λ of $\mathcal{A}^{(n)}$ which are the Coxeter exponents of the graph $\mathcal{G}$, where n is the Coxeter number of $\mathcal{G}$. The multiplicity of the eigenvalue $\beta^{(\lambda)}$ is given by the entry $Z_{\lambda,\lambda}$ of the corresponding modular invariant [13, Theorem 4.16]. For an $\mathcal{ADE}$ graph $\mathcal{G}$ with Coxeter number n we will often write $[m]$ for $[m]_q$ when $q = \exp(i\pi/n)$.

1.6 Subfactors

1.6.1 AF algebras

Let $C_0 \subset C_1 \subset C_2 \subset \cdots$ be a sequence of finite dimensional C^*-algebras with inclusion maps $\jmath_n : C_n \to C_{n+1}$. An AF algebra is an inductive limit $C_\infty = \lim_{\to} C_n$. To each AF algebra we can associate a Bratteli diagram. The Bratteli diagram associated with an AF algebra is not unique, however the AF algebra given by any Bratteli diagram is unique. For the inductive limit of multi-matrix algebras C_n the Bratteli diagram is given as follows. Let the embedding of C_n in C_{n+1} be given by the multiplicity $\Lambda_n = (\lambda_{i,j}^{(n)})$. For all $n > 0$ let $q^{(n)}$ denote the number of simple subalgebras or minimal central projections of C_n, and let the sequence $\Omega[n] = (i_1^{(n)}, i_2^{(n)}, \dots, i_{q^{(n)}}^{(n)})$ represent the minimal central projections of C_n. Then we draw a graph consisting of two parallel horizontal rows of vertices, where the top row has $q^{(n)}$ vertices indexed by $\Omega[n]$ representing the minimal central projections of C_n, and the bottom row has $q^{(n+1)}$ vertices indexed by $\Omega[n+1]$, and we draw $\lambda_{j,i}^{(n)}$ edges between the i^{th} vertex along the top row and the j^{th} vertex along the bottom row. It is convenient to adjoin an additional stage given by the unital embedding $\mathbb{C} = C_0 \to C_1$ with multiplicity graph Λ_0. The Bratteli diagram for the AF algebra is obtained by concatenating the multiplicity graphs Λ_n for $n \geq 0$, identifying the vertices along the bottom of the graph for Λ_n with the corresponding vertices along the top of the graph for Λ_{n+1}.

For any two matrix algebras M_n, M_m, the embedding of $M_n \to M_m$ is given by

$$x \to u \begin{pmatrix} x_1 & & & \\ & \ddots & & \\ & & x_k & \\ & & & 0 \end{pmatrix} u^*,$$

for some unitary $u \in M_m$ and unique integer k, where $x_i = x$ for $i = 1, \ldots, k$ and 0 is a $p \times p$ zero-matrix for $p = m - kn$. Then suppose A, B, A', B' are multi-matrix algebras and that the Bratteli diagrams for inclusions $\pi_1 : A \to B$, $\pi_2 : A' \to B'$ are the same, then π_1 and π_2 are unitarily equivalent and hence isomorphic. Hence a one-dimensional Bratteli diagram has a unique limit which determines an AF algebra.

However, if we consider a two-dimensional Bratteli diagram we obtain a double sequence of finite dimensional algebras:

$$\begin{array}{ccccccc} C_{0,0} & \subset & C_{0,1} & \subset & C_{0,2} & \subset & \cdots \\ \cap & & \cap & & \cap & & \\ C_{1,0} & \subset & C_{1,1} & \subset & C_{1,2} & \subset & \\ \cap & & \cap & & \cap & & \\ C_{2,0} & \subset & C_{2,1} & \subset & C_{2,2} & \subset & \\ \cap & & \cap & & \cap & & \\ \vdots & & \vdots & & \vdots & & . \end{array}$$

The Bratteli diagram alone is no longer sufficient to determine the embeddings of different horizontal or vertical AF algebras. Consider for example the squares given by

$$\begin{array}{ccc} \mathbb{C} & \to & M_2 \otimes \mathbb{C} \\ \downarrow & & \downarrow \\ A_i & \to & M_2 \otimes M_2 \end{array},$$

for $i = 1, 2$, where $A_1 = \mathbb{C} \otimes M_2$, $A_2 = M_2 \otimes \mathbb{C}$. These two squares are not isomorphic since for $i = 1$ any element of the algebra $M_2 \otimes \mathbb{C}$ in the upper right corner of the square will commute with any element A_1 in $M_2 \otimes M_2$, however this is not true for $i = 2$. The extra ingredient needed to measure this freedom is the connection.

A connection is the assignment of a complex number

$$X^{\rho_1,\rho_2}_{\rho_3,\rho_4} = \begin{array}{ccc} l & \xrightarrow{\rho_1} & i \\ {\scriptstyle\rho_3}\downarrow & & \downarrow{\scriptstyle\rho_2} \\ k & \xrightarrow[\rho_4]{} & j \end{array} \in \mathbb{C}$$

to each square $\begin{array}{ccc} l & \xrightarrow{\rho_1} & i \\ \rho_3\downarrow & & \downarrow\rho_2 \\ k & \overrightarrow{\rho_4} & j \end{array}$ in the two-dimensional Bratteli diagram.

The unitarity property of connections is given by

$$\sum_{\rho_3,\rho_4} X^{\rho_1,\rho_2}_{\rho_3,\rho_4} \overline{X^{\rho_1',\rho_2'}_{\rho_3,\rho_4}} = \delta_{\rho_1,\rho_1'}\delta_{\rho_2,\rho_2'}, \tag{1.31}$$

whilst the Yang-Baxter equation for connections is

$$\sum_{\sigma_1,\sigma_2,\sigma_3} X^{\sigma_1,\sigma_2}_{\rho_1,\rho_2} X^{\rho_3,\rho_4}_{\sigma_1,\sigma_3} X^{\sigma_3,\rho_5}_{\sigma_2,\rho_6} = \sum_{\sigma_1,\sigma_2,\sigma_3} X^{\rho_3,\sigma_2}_{\rho_1,\sigma_1} X^{\sigma_1,\sigma_3}_{\rho_2,\rho_6} X^{\rho_4,\rho_5}_{\sigma_2,\sigma_3}. \tag{1.32}$$

The Yang-Baxter equation (1.32) is represented graphically as in Figure 1.5.

1.6.2 Path Algebra Model of an AF Algebra

We describe the path algebra model for a Bratteli diagram which describes unital embeddings. The vertices at the n^{th} level of the Bratteli diagram are those which correspond to the simple subalgebras of C_n, i.e. the vertices $\Omega[n]$. For $i \in \Omega[m]$, $j \in \Omega[n]$ with $m < n$, we denote by $\text{Path}(i,j)$ the space of all paths in the Bratteli diagram from i to j. For a path $\gamma \in \text{Path}(i,j)$, i is called the source of γ, denoted by $s(\gamma) = i$, and j is called the range of γ, denoted by $r(\gamma) = j$, and $|\gamma|$, the length of the path γ, is $n - m$. The space $\Omega[m,n]$ of all paths from level m to level n is given by $\Omega[m,n] = \sqcup_{i\in\Omega[m],j\in\Omega[n]}\text{Path}(i,j)$. For paths of length zero we let $\text{Path}(i,i) = \{i\}$ and $\text{Path}(i,i') = \emptyset$ if $i,i' \in \Omega[m]$ such that $i \neq i'$. Then $\Omega[m,m] = \Omega[m]$. Let $m \leq n \leq m'$, $i \in \Omega[m]$, $j \in \Omega[n]$ and $k \in \Omega[m']$. Then for any paths $\mu \in \text{Path}(i,j)$, $\nu \in \text{Path}(j,k)$, the path $\mu \cdot \nu \in \text{Path}(i,k)$ is defined by concatenating the paths μ and ν. For $m < n$ and any $i \in \Omega[m]$, $j \in \Omega[n]$, let $A_{i,j} = \text{End}(\ell^2(\text{Path}(i,j)))$ generated by matrix units (γ_1,γ_2) indexed by paths $\gamma_1,\gamma_2 \in \text{Path}(i,j)$, and $A[m,n] = \oplus A_{i,j}$ where the summation is over all $i \in \Omega[m]$, $j \in \Omega[n]$. Thus $A[m,n]$ is generated by matrix units (γ_1,γ_2) where $\gamma_1,\gamma_2 \in \Omega[m,n]$ where $s(\gamma_1) = s(\gamma_2)$, $r(\gamma_1) = r(\gamma_2)$. For $m \geq m'$, $n \leq n'$, we embed $A[m,n]$ in $A[m',n']$ by

$$(\gamma_1,\gamma_2) \to \sum_{\mu,\nu}(\mu\cdot\gamma_1\cdot\nu, \mu\cdot\gamma_2\cdot\nu), \tag{1.33}$$

for $\gamma_1,\gamma_2 \in \text{Path}(i,j)$, where the summation is over all $\mu \in \text{Path}(i',i)$, $\nu \in \text{Path}(j,j')$ and all $i' \in \Omega[m']$, $j' \in \Omega[n']$. For $m_1 \leq n_1 \leq m_2 \leq n_2$, any $a_1 \in A[m_1,n_1]$ and $a_2 \in A[m_2,n_2]$ commute,

$$a_1a_2 = a_2a_1. \tag{1.34}$$

Then $A[m,n] = A[0,m]' \cap A[0,n]$, and in particular the centre of $A_n = A[0,n]$ is identified with $A[n,n] = C(\Omega[n]) = \mathbb{C}^{\Omega[n]}$, and the minimal central projections of $A[0,n]$ can be identified with $\Omega[n]$ by $(i,i) \leftrightarrow i$. The AF algebra associated with the Bratteli diagram is then $A = \lim_{\rightarrow} A_n$ where the embedding of $A_n = A[0,n]$ in A_{n+1} is given in (1.33). We will write $A(\mathcal{G})$ for the path algebra A where the embeddings on the Bratteli diagram are given by the graph $\mathcal{G}$, and will denote the finite dimensional algebra A_k at the k^{th} level of the Bratteli diagram by $A(\mathcal{G})_k$.

1.6.3 Von Neumann algebras

Let H be a Hilbert space and $B(H)$ the space of all bounded linear operators on H. For a subset $S \subset B(H)$ the commutant of S is $S' = \{x \in B(H) | xy = yx \text{ for all } y \in S\}$. A von Neumann algebra is a $*$-subalgebra M of $B(H)$ which contains the identity operator $\mathbf{1}$ and satisfies $M'' = M$. A finite dimensional von Neumann algebra is $*$-isomorphic to a multi-matrix algebra since it is a C^*-algebra.

A trace on a finite dimensional matrix algebra A is uniquely determined up to a scalar multiple of the canonical un-normalized trace given by $\mathrm{tr}(x) = \sum_i x_{i,i}$ for a matrix $x = (x_{i,j}) \in A$. Then a trace on a finite dimensional multi-matrix algebra A is determined by a sequence (s_i), indexed by the minimal central projections of A, with $s_i \in \mathbb{C}$, called a trace vector, given by $\mathrm{tr}(\oplus_i x_i) = \sum_i s_i \mathrm{tr}_i(x_i)$, where tr_i is the canonical un-normalized trace on the simple subalgebra A_i of A. The trace is positive if $s_i \geq 0$ for all i and faithful if $s_i > 0$ for all i. Let A and B be finite dimensional C^*-algebras such that A is embedded in B with the embedding given by matrix Λ, and let tr_A, tr_B respectively, be the traces on A, B respectively, with trace vectors s, t respectively. The traces are compatible under the embedding given by Λ if and only if $s = \Lambda^T t$, where Λ^T denotes the transpose of Λ. Then we just write tr for the trace. We have an inner-product defined by the trace by $\langle x, y \rangle = \mathrm{tr}(y^* x)$ for $x, y \in B$, and we can regard A as a subspace of B. For any inclusion of (possibly infinite dimensional) von Neumann algebras $A \subset B$ with a finite faithful normal trace on B which coincides with a finite faithful normal trace on A, the conditional expectation $E_A : B \to A$ is the projection E_A onto A such that $E_A(b)$ is the unique element $b' \in A$ which satisfies $\mathrm{tr}(b'a) = \mathrm{tr}(ba)$ for all $a \in A$. We call E_A the conditional expectation of B onto A with respect to the trace.

Definition 1.6.1 *Let*

$$\begin{array}{ccc} M_1 & \subset & M_2 \\ \cup & & \cup \\ M_3 & \subset & M_4 \end{array}$$

be four von Neumann algebras with a finite faithful normal trace on M_4. We say they form a ***commuting square*** *if they satisfy one of the following equivalent conditions, where the conditional expectations are relative to the trace:*

1. $E_{M_2}(M_3) \subset M_1$.
2. $E_{M_3}(M_2) \subset M_1$.
3. $E_{M_2}E_{M_3} = E_{M_1}$.
4. $E_{M_3}E_{M_2} = E_{M_1}$.
5. $E_{M_2}E_{M_3} = E_{M_3}E_{M_2}$ *and* $M_1 = M_2 \cap M_3$.
6. $E_{M_1}(x) = E_{M_2}(x)$ *for* $x \in M_3$.
7. $E_{M_1}(x) = E_{M_3}(x)$ *for* $x \in M_2$.

The following proposition regarding von Neumann inclusions is found in [51]:

Proposition 1.6.2 *Let $N \subset M$ be a pair of von Neumann algebras, with a finite faithful normal trace* tr *on M and let S be a self-adjoint subset of N. Then*

$$\begin{array}{ccc} S' \cap M & \subset & M \\ \cup & & \cup \\ S' \cap N & \subset & N \end{array}$$

is a commuting square.

1.6.4 Factors

A factor is an infinite dimensional von Neumann algebra M which has trivial center, i.e. $M \cap M' = \mathbb{C}$. Factors are classified into types I_n ($n = 1, 2, \ldots, \infty$), II_1, II_∞ and III. Type I_n factors are matrix algebras $M_n(\mathbb{C})$, and a type I_∞ factor is $B(H)$ on an infinite-dimensional Hilbert space H. A II_1 factor is an infinite dimensional von Neumann algebra M which has a unique σ-weakly continuous linear functional tr, called a trace on M, satisfying

1. $\mathrm{tr}(\mathbf{1}) = 1$,
2. $\mathrm{tr}(x^*x) \geq 0$ for all $x \in M$,
3. If $\mathrm{tr}(x^*x) = 0$, then $x = 0$,

4. $\mathrm{tr}(xy) = \mathrm{tr}(yx)$ for all $x, y \in M$.

A type II_∞ is a tensor product of a type II_1 factor and $B(H)$. All other factors are called type III. A factor M is said to be finite if the multiplicative identity $\mathbf{1}$ of M is a finite projection in M, or equivalently, if M does not contain any non-unitary isometry, and is called infinite otherwise. If M is a factor and N is a von Neumann subalgebra of M which is also a factor then we call N a subfactor.

A hyperfinite von Neumann algebra is defined to be the weak closure of a union of an increasing sequence of finite dimensional von Neumann algebras. If A is the union of a sequence $M_0 \subset M_1 \subset \cdots$ of finite dimensional von Neumann algebras with a trace tr which is compatible with the inclusions. Then we define an inner-product on A by $\langle x, y\rangle = \mathrm{tr}(y^* x)$, and completing A with respect to this inner-product we obtain a Hilbert space H. If $\pi(a) \in B(H)$ is the extension of the left multiplication by a on A, we can regard $\pi(A)$ as a $*$-subalgebra of $B(H)$. We can extend tr to the weak closure M of A by $\mathrm{tr}(x) = \langle x\widehat{1}, \widehat{1}\rangle$, where $\widehat{1} \in H$ is the image of $1 \in A$. This trace satisfies conditions 1-4 above, and hence the hyperfinite von Neumann algebra M is a II_1 factor. All hyperfinite II_1 factors are isomorphic, however the position of one hyperfinite II_1 embedded as a subalgebra of another hyperfinite II_1 has a rich structure. The investigation of this structure is the main point of subfactor theory.

Suppose M is a II_1 factor acting on a Hilbert space H, where the action of M on H is isomorphic to the action of M on $(\bigoplus_{j=1}^n L^2(M))p$ by left multiplication, where $p = (p_{jk})$ is some projection in $M_n(\mathbb{C}) \otimes M$ with $p_{j,k} \in M$. The coupling constant $\dim_M H$ of M in H is defined as $\sum_{j=1}^n \mathrm{tr}(p_{jj})$. If the action of M on H is not of this form then the coupling constant is set to be ∞. For $\dim_M H < \infty$, the commutant M' of M in $B(H)$ is a II_1 factor with a unique trace $\mathrm{tr}_{M'}$. The Jones index $[M : N]$ of a subfactor N in M was introduced by Jones in [61]. It is defined to be $\dim_N L^2(M)$, where N acts on $L^2(M)$ by left multiplication. It was shown in [61] that the Jones index has value τ if and only if $\tau \in \{4\cos^2(\pi/n) | n = 3, 4, 5, \dots\} \cup [4, \infty]$.

For a subfactor $N \subset M$ with finite index, the relative commutant $N' \cap M$ is finite dimensional, and in particular, if $[M : N] < 4$ then $N' \cap M = \mathbb{C}$. For a subfactor with finite index, the conditional expectation $E_N : M \to N$ naturally extends to a projection e_1 of $L^2(M)$ onto $L^2(N)$. The algebra $M_1 = \langle M, e_1\rangle$ is the von Neumann algebra generated by M and e_1, and this construction is called the basic construction. Let J be the conjugation on $L^2(M)$ defined by $J(\widehat{x}) = \widehat{x^*}$ for $x \in M$, where we use the notation $\widehat{x}$ to denote the image of x in $L^2(M)$. The algebra M_1 is also a II_1 factor since $M_1 = JN'J$. The basic construction can be repeated for $M \subset M_1$ to obtain $M_2 = \langle M_1, e_2\rangle$, and continuing in this way gives the Jones tower $N \subset M \subset M_1 \subset M_2 \subset \cdots$ of II_1 factors. The Jones projections e_j, $j = 1, 2, \dots$, satisfy the Temperley-Lieb relations TL1, TL2, with $\delta = [M : N]^{1/2}$. Since the Jones index satisfies $[M : N] = [M : P][P : n]$ for factors $N \subset P \subset M$ with $[M : N] < \infty$, and $[M_1 : M] = [M : N]$, the Jones index $[M_k : N] = [M : N^{k+1}] < \infty$, and the higher relative commutant $N' \cap M_k$ is finite dimensional.

For a finite index subfactor $N \subset M$, the map $\Phi : N' \cap M \to M' \cap M_1$ given by $\Phi(x) = J_M x^* J_M$, where J_M is the canonical conjugation on $L^2(M)$, is an anti-isomorphism which is not trace-preserving in general. Trace-preserving means that the normalized trace $\text{tr}_{N'}$ on N' coincides on $N' \cap M$ with the trace tr on M. By [100, Cor. 4.5], Φ is trace-preserving if and only if $E_{N' \cap M}(e_1) \in \mathbb{C}$ for a Jones projection $e_1 \in M_1$. Such a subfactor $N \subset M$ is called extremal.

For a inclusion of type II_1 factors with finite index, the lattice of higher relative commutants $M_i' \cap M_j$ is called the standard invariant of the subfactor. The standard invariant can be described as a certain category of bimodules [91]. In [102], Popa obtained an axiomatization of lattices of inclusions $(A_{ij})_{0 \le i \le j}$, which he called standard λ-lattices, which is the standard invariant for extremal subfactors. More recently, the standard invariant has been described as a planar algebra [64].

1.6.5 Sectors

Let M, N be type III factors. We denote by $\text{Hom}(M, N)$ the set of all unital morphisms from M to N, and $\text{End}(M) = \text{Hom}(M, M)$. For $\rho \in \text{Hom}(M, N)$, the positive number $d_\rho = [N : \rho(M)]^{1/2}$ is called the statistical dimension of ρ, where $[N : \rho(M)]$ is the Jones index of the subfactor $\rho(M) \subset N$. For $\rho_1, \rho_2 \in \text{Hom}(M, N)$, we denote by (ρ_1, ρ_2) the set of intertwiners between ρ_1 and ρ_2, that is

$$(\rho_1, \rho_2) := \{y \in N | y\rho_1(x) = \rho_2(x)y \quad \text{for all } x \in M\}.$$

Let $\rho_1, \rho_2 \in \text{Hom}(M, N)$. We say that ρ_1 and ρ_2 are unitarily equivalent if and only if there exists a unitary $u \in M$ such that $\rho_1 = \text{Ad}(u) \circ \rho_2$.

We call the equivalence class $[\rho]$ of a morphism ρ a sector, and denote by $\text{Sect}(M, N)$ the quotient of $\text{Hom}(M, N)$ by the unitary equivalence. We can define sum and product of sectors on $\text{Sect}(M) := \text{Sect}(M, M)$ which satisfy associativity and distributivity in the following way: Since we assume M to be a type III factor, there exist non-zero projections $p_1, p_2 \in M$ and isometries $v_1, v_2 \in M$ such that $v_i v_i^* = p_i$, $i = 1, 2$, and $p_1 + p_2 = 1$. For $\rho_1, \rho_2 \in \text{Hom}(M, N)$ we define $\rho \in \text{Hom}(M, N)$ by $\rho(x) = v_1 \rho_1(x) v_1^* + v_2 \rho_2(x) v_2^*$ so that $[\rho_1] \oplus [\rho_2] = [\rho]$. This sum is well-defined as it does not depend on the choice of ρ_1

and ρ_2 in their classes or on the choice of v_1 and v_2- if t_1, t_2 are two other isometries in M satisfying the $t_1t_1^* + t_2t_2^* = 1$ then $u = t_1v_1^* + t_2v_2^*$ is a unitary in (ρ, ρ') where $\rho'(x) = t_1\rho_1(x)t_1^* + t_2\rho_2(x)t_2^*$. The sector $[\rho]$ contains the sector $[\rho_i]$, $i = 1, 2$, and we will write $[\rho] \supset [\rho_i]$. Product is defined by the composition of morphisms $\rho_1 \in \mathrm{Hom}(M, N)$ and $\rho_2 \in \mathrm{Hom}(N, P)$ $[\rho_1][\rho_2] = [\rho_1\rho_2]$, so that $[\rho_1\rho_2] \in \mathrm{Sect}(M, P)$.

We say that $\rho \in \mathrm{Hom}(M, N)$ is irreducible if $\langle\rho, \rho\rangle = 1$. This is equivalent to the subfactor $\rho(M) \subset N$ having a trivial relative commutant, i.e. $N \cap \rho(M)' = \mathbb{C}$. We call ρ self-conjugate if and only if $[\rho] = \overline{[\rho]}$, where $\overline{[\rho]}$ is the conjugate sector of $[\rho]$, as given by Longo [80]. For irreducible $\rho \in \mathrm{Hom}(M, N)$, the conjugate sector $\overline{[\rho]} \in \mathrm{Sect}(N, M)$ is the irreducible sector such that $[\rho]\overline{[\rho]} \supset [\mathrm{id}_M]$, i.e. the sector product $[\rho]\overline{[\rho]}$ contains the identity sector $[\mathrm{id}_M]$. Then the multiplicity of $[\mathrm{id}_M]$ in the decomposition of $[\rho]\overline{[\rho]}$ is one, and the product $\overline{[\rho]}[\rho]$ also contains $[\mathrm{id}_N]$ with multiplicity one. We denote a representative of the sector $\overline{[\rho]}$ by $\overline{\rho}$. Note that $\alpha \in \mathrm{Aut}(M)$ is self-conjugate if and only if α^2 is inner, i.e. α^2 is given by $\alpha^2(x) = uxu^*$ for some unitary $u \in M$.

Let M, N be infinite factors with $N \subset M$. A vector ξ in a Hilbert space H is a cyclic vector if $H = \overline{M\xi}$, and separating if $x\xi = 0$ for $x \in M$ implies $x = 0$. We can represent M on H where there is a vector $\xi \in H$ which is cyclic and separating for both M and N. Let J_N, J_M be the corresponding Tomita-Takesaki modular conjugations where $J_NNJ_N = N'$, $J_MMJ_M = M'$, and define the canonical endomorphism γ from M to N as in [79] by $\gamma(x) = J_NJ_MxJ_MJ_N$. Different choices of Hilbert spaces and cyclic and separating vectors give a unitarily equivalent endomorphism, hence the sector $[\gamma]$ is well-defined. We can then obtain a sequence of inclusions of factors

$$\cdots \subset \gamma\gamma(N) \subset \gamma\gamma(M) \subset \gamma(N) \subset \gamma(M) \subset N \subset M.$$

This sequence called the Jones tunnel. By using the endomorphism $\zeta(x) = J_MJ_NxJ_NJ_M$ instead we can also extend the sequence to the right by

$$N \subset M \subset \zeta(N) \subset \zeta(M) \subset \zeta\zeta(N) \subset \cdots$$

which is called the Jones tower. However the sequence has period two in the sense that the inclusion $\gamma(N) \subset \gamma(M)$ is isomorphic to the inclusion $N \subset M$, and the inclusion $\gamma(M) \subset N$ is isomorphic to $M \subset \zeta(N)$. The restriction of γ to N is called the dual canonical endomorphism $\theta = \gamma|_N$ for $N \subset M$.

If we denote the inclusion homomorphism by $\iota : N \hookrightarrow M$, a conjugate homomorphism $\overline{\iota} : M \to N$ is given by $\overline{\iota}(x) = \gamma(x)$ for $x \in M$. Then the canonical and dual canonical endomorphism are $\gamma = \iota\overline{\iota}$ and $\theta = \overline{\iota}\iota$. Let $\mathrm{Irr}(N) \subset \mathrm{Sect}(N)$, $\mathrm{Irr}(M, N) \subset \mathrm{Sect}(M, N)$

denote the set of all irreducible subsectors of $[\theta^i]$, $[\theta^i\bar{\iota}]$, respectively, $i = 1, 2, \dots$. The principal graph of the inclusion $N \subset M$ is given by labelling the even vertices by the elements of $\mathrm{Irr}(N)$ and the odd vertices by the elements of $\mathrm{Irr}(M, N)$, and there are $\langle \lambda\bar{\iota}, \mu\rangle$ edge connecting the vertex labelled by $[\lambda] \in \mathrm{Irr}(N)$ to the vertex labelled by $[\mu] \in \mathrm{Irr}(M, N)$, where $\langle \lambda\bar{\iota}, \mu\rangle$ is computed by decomposing $[\lambda\bar{\iota}]$ into irreducible sectors of $\mathrm{Irr}(M, N)$. By Frobenius reciprocity we have $\langle \rho\lambda, \sigma\rangle = \langle \lambda, \bar{\rho}\sigma\rangle = \langle \rho, \sigma\bar{\lambda}\rangle$. It is well known that for index $[M : N] < 4$, the principal graph is one of the Dynkin diagrams A_n, D_n, E_6, E_7, E_8 [51, Cor. 4.6.6]. Ocneanu proved that D_{odd} and E_7 cannot appear as the principal graphs for any subfactor. This was later proved by Izumi in [57] using the fusion rules of sectors.

1.6.6 α-induction

For a type III factor N, let ${}_N\mathcal{X}_N$ denote a finite system of irreducible inequivalent endomorphisms of N, that is, for any $\lambda \in {}_N\mathcal{X}_N$ there is a representative $\bar{\lambda} \in {}_N\mathcal{X}_N$ of the conjugate sector $[\bar{\lambda}]$, and for any $\lambda, \mu \in {}_N\mathcal{X}_N$, a representative of each sector in the irreducible decomposition of $[\lambda\mu]$ is in ${}_N\mathcal{X}_N$. Then in particular $\mathrm{id} \in {}_N\mathcal{X}_N$. Let $\sum({}_N\mathcal{X}_N)$ denote the set of finite sums of endomorphisms in ${}_N\mathcal{X}_N$. A system ${}_N\mathcal{X}_N$ is braided if for any $\lambda, \mu \in {}_N\mathcal{X}_N$ there is a unitary operator $\varepsilon(\lambda, \mu) \in (\lambda, \mu)$, called a braiding operator, subject to the initial conditions $\varepsilon(\mathrm{id}, \mu) = \varepsilon(\lambda, \mathrm{id}) = 1$ and which satisfy the Braiding Fusion Equations (BFE) [12, Def. 2.2]. For every braiding $\varepsilon^+ \equiv \varepsilon$ there is an "opposite" braiding ε^- defined by $\varepsilon^-(\lambda, \mu) = (\varepsilon^+(\mu, \lambda))^*$. A braiding is said to be non-degenerate if $\varepsilon^+(\lambda, \mu) = \varepsilon^-(\lambda, \mu)$ for all $\mu \in {}_N\mathcal{X}_N$ implies $\lambda = \mathrm{id}$. If we have an inclusion $\iota : N \hookrightarrow M$ of type III factors together with a non-degenerately braided finite system ${}_N\mathcal{X}_N$ such that the dual canonical endomorphism $\theta = \bar{\iota}\iota \in \sum {}_N\mathcal{X}_N$, then we call $N \subset M$ a braided subfactor. The α-induced morphisms $\alpha^\pm_\lambda \in \mathrm{End}(M)$, which extend $\lambda \in {}_N\mathcal{X}_N$, are defined by the Longo-Rehren formula [81] $\alpha^\pm_\lambda = \bar{\iota}^{-1} \circ \mathrm{Ad}(\varepsilon^\pm(\lambda, \theta)) \circ \lambda \circ \bar{\iota}$, and satisfy $\alpha^\pm_\lambda \iota = \iota\lambda$. A "coupling matrix" Z can be defined [12] by $Z_{\lambda,\mu} = \langle \alpha^+_\lambda, \alpha^-_\mu\rangle$, where $\lambda, \mu \in\in {}_N\mathcal{X}_N$, normalized so that $Z_{0,0} = 1$. By [11, 34], this matrix Z commutes with the modular S- and T-matrices, and therefore Z is a modular invariant. We let ${}_M\mathcal{X}_M \subset \mathrm{End}(M)$ denote a system of endomorphisms which are representatives of the irreducible sectors $[\iota\lambda\bar{\iota}]$, $\lambda \in {}_N\mathcal{X}_N$. We define the chiral induced systems as the subsystems ${}_M\mathcal{X}^\pm_M \subset {}_M\mathcal{X}_M$ of all β such that $[\beta]$ is an irreducible subsector of $[\alpha^\pm_\lambda]$, and the ambichiral, or neutral, system ${}_M\mathcal{X}^0_M = {}_M\mathcal{X}^+_M \cap {}_M\mathcal{X}^-_M$. The modular invariant Z is a permutation matrix if and only if ${}_M\mathcal{X}_M = {}_M\mathcal{X}^0_M$.

A modular invariant Z associated to a subfactor $N \subset M$ is said to be of type I if

$Z_{0,\lambda} = \langle\theta,\lambda\rangle$ for all $\lambda \in {}_N\mathcal{X}_N$ [11]. This is equivalent to chiral locality holding, i.e. $\varepsilon^+(\theta,\theta)v^2 = v^2$, where $v \in (\mathrm{id}, \iota\bar{\iota})$ is an isometry. It was also shown in [11], using results on intermediate subfactors, that if a braided subfactor $N \subset M$ has an associated modular invariant Z, there are intermediate subfactors $N \subset M_\pm \subset M$ such that $N \subset M_\pm$ satisfy chiral locality. The modular invariants $Z^\pm$ associated to the subfactors $N \subset M_\pm$ are called the type I parents of Z. We have $Z^+_{\lambda,0} = Z_{\lambda,0}$ and $Z^-_{0,\lambda} = Z_{0,\lambda}$, so that Z^+, Z^- is the type I modular invariant which has the same first row, column respectively as Z.

Evans and Pinto [40, Theorem 3.6] showed that if $N \subset M_a$, $N \subset M_b$ are braided inclusions with associated modular invariants Z_a, Z_b respectively, then the product $Z_a Z_b^T$ also arises from a braided inclusion through a process of α-induction.

Chapter 2

The Ising Model

In this chapter we present the two-dimensional Ising model as a model on the Dynkin diagram A_3, and its generalization to other graphs. We explicitly construct the extension of the Kramers-Wannier endomorphism to ρ on $\mathcal{O}_2 \cong \mathcal{O}_{A_3}$. For the inclusion of purely infinite factors $M \supset \rho(M)$ with finite index $\sqrt{3}$ and principal graph A_5, we construct the endomorphism ρ_2 on $\mathcal{O}_3$. We also show that the crossed product of $\mathcal{O}_{A_{2n+1}}$ by the $\mathbb{Z}_2$ action induced by the $\mathbb{Z}_2$ action on A_{2n+1} is stably isomorphic to $\mathcal{O}_{D_{n+2}}$, and similarly $\mathcal{O}_{D_{n+2}} \rtimes \mathbb{Z}_2$ is stably isomorphic to $\mathcal{O}_{A_{2n+1}}$.

2.1 The Classical Ising Model

Ising introduced his model for a ferromagnet in an external magnetic field for the one-dimensional case in his PhD thesis [55]. He found that the solution admitted no phase transition, i.e. a sudden change from negative to positive magnetization (or vice-versa), and concluded that his model did not exhibit phase behaviour in any dimension. But this is not true, since in higher dimensions it is possible for the model to have non-zero spontaneous magnetization. Whilst the Ising model is a simplified description of ferromagnetism, other systems can be mapped exactly or approximately to the Ising system, which allows the use of simulation and analytical results of the Ising model to answer questions about the related models.

The two-dimensional Ising model of a magnet is modelled on a square lattice, where each site on the lattice represents a particle which has a spin, or magnetization. We restrict the spin to be either positive or negative, with values +1 and -1 respectively.

In the case of a general discrete model on a lattice the Dynkin diagram shows which values sites that are connected may take. If we use a diagonal lattice and the Dynkin

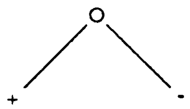


Figure 2.1: Dynkin diagram A_3

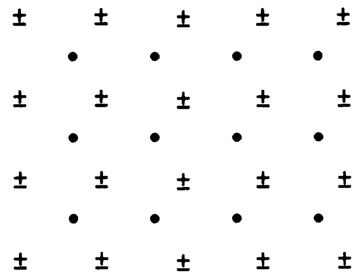


Figure 2.2: Ising Configuration Space

diagram A_3 (Figure 2.1), we have the configuration space of Figure 2.2 where $\pm$ indicates that the site may take either the value +1 or -1. Here, the odd sublattice merely has frozen spins $\bullet$, and we get a copy of the Ising model on the even sublattice. To complete the description of the model, we need to specify the Hamiltonian, or energy, in each configuration. Let us consider a square lattice Λ_{LM} with L rows and M columns, and we impose periodic boundary conditions. Then the Hamiltonian for a configuration $\sigma \in P_{LM}$ where P_{LM} is the space of all configurations on the lattice, is

$$H^{LM}(\sigma) = -\sum_{i=1}^{M}\sum_{j=1}^{L}\Big(J_1\sigma(i,j)\sigma(i+1,j) + J_2\sigma(i,j)\sigma(i,j+1)\Big)$$

where J_1, J_2 are the interaction energies between neighbouring sites in the horizontal and vertical direction respectively.

For an observable F of the system, the expectation value $\langle F\rangle$ of F is given by $Z_\beta^{-1}\sum_\sigma F(\sigma)e^{-\beta H(\sigma)}$, where Z_β is the partition function $Z_\beta = \sum_{\sigma\in P_{LM}} e^{-\beta H(\sigma)}$. Non-zero spontaneous magnetization exists in the two-dimensional case, or at least,

$$\lim_{|\Lambda|\to\infty} \inf \frac{1}{\Lambda}\sum_x \langle\sigma(x)\rangle^+ > 0,$$

where $\langle\sigma(x)\rangle^+$ is the expectation value of the magnetization with + boundary conditions, since Peierls' estimate [98] says that for fixed $a \in (0,1)$ there exists β_0 such that for all $\beta > \beta_0$ we have $\langle\sigma(x)\rangle_\Lambda^+ \geq a$, independent of x in Λ, with + boundary conditions.

The spontaneous magnetization was computed by Onsager [97] as

$$m^* = \begin{cases} [1-(\sinh 2\beta J)^{-4}]^{\frac{1}{8}} & T < T_c \\ 0 & T > T_c, \end{cases}$$

where the critical temperature T_c satisfies $\sinh 2\beta_c J = 1$, $\beta_c = 1/kT_c$, as shown by Kramers and Wannier [77].

The existence of this spontaneous magnetization is related to the non-differentiability of the free energy. For arbitrary boundary conditions, the free energy per lattice site in finite volume is $f^b = -\frac{1}{\beta|\Lambda|}\log Z^b$. In the thermodynamic limit, $f(H) = \lim_{|\Lambda|\to\infty} f^b_\Lambda(H)$, where H is an external magnetic field. f exists and is independent of boundary conditions, but is not differentiable at $H = 0$. This can be shown without computing $f(H)$, by using Peierls' estimate. Onsager [96] computed $f(0)$ in the two-dimensional case using the transfer matrix formalism, but we cannot explicitly compute $f(H)$ in higher dimensions, even for $H = 0$. The existence of phase transitions in higher dimensions of the nearest-neighbour Ising model can still be proved by putting all interactions to be zero, except in parallel two-dimensional planes, using Peierls' estimate, then turning on the interactions in new dimensions and using the GKS inequalities.

2.2 Algebraic Approach

If we call the configuration along the j^{th} row $\sigma^j = (\sigma(1,j),\dots,\sigma(M,j)) \in \{\pm 1\}^M$, then a configuration σ will be given by

$$\sigma = \begin{pmatrix} \sigma^L \\ \vdots \\ \sigma^1 \end{pmatrix}.$$

Then the Hamiltonian is

$$H^{LM}(\sigma) = \sum_{j=1}^{L} S(\sigma^j) + \sum_{j=1}^{L} I(\sigma^j,\sigma^{j+1})$$

where

$$S(\overline{\sigma}) = -J_2\sum_{i=1}^{M}\overline{\sigma}(j)\overline{\sigma}(j+1), \qquad I(\overline{\sigma},\overline{\sigma}') = -J_1\sum_{i=1}^{M}\overline{\sigma}(j)\overline{\sigma}'(j)$$

and $\overline{\sigma},\overline{\sigma}' \in \{\pm\}^M$ are row configurations.

This gives the partition function

$$Z_\beta = \mathrm{tr}T^L$$

where the transfer matrix T is a symmetric $2^M \times 2^M$ matrix with rows and columns labelled by the $\overline{\sigma} \in \{\pm\}^M$. For $\overline{\sigma},\overline{\sigma}' \in \{\pm\}^M$,

$$T(\overline{\sigma},\overline{\sigma}') = \exp\Big(-\beta\Big\{\frac{S(\overline{\sigma})+S(\overline{\sigma}')}{2} + I(\overline{\sigma},\overline{\sigma}')\Big\}\Big).$$

The transfer matrix T can be identified with an element of the Pauli algebra $A^P_M = C^*(\sigma^i_\alpha | i = 1,\dots,M, \quad \alpha = x,y,z) = \bigotimes_1^M M_2$, where, for $i = 1,\dots M$, σ^i_α is given by

$$\sigma^i_\alpha = \mathbf{1}\otimes\cdots\otimes\mathbf{1}\otimes\sigma_\alpha\otimes\mathbf{1}\otimes\cdots\otimes\mathbf{1},$$

where σ_α is in the i^{th} place. The Pauli matrices σ_α are given by

$$\sigma_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \sigma_y = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \quad \sigma_z = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

These satisfy $\sigma_\alpha^2 = 1$, and $\sigma_\alpha\sigma_\beta = i\sigma_\gamma$ where (α,β,γ) is some cyclic permutation of (x,y,z). Then $T = (2\sinh 2K_1)^{\frac{M}{2}} V^{\frac{1}{2}} W V^{\frac{1}{2}}$, where

$$V = \exp K_2 \sum_{j=1}^{M} \sigma_x^j \sigma_x^{j+1}, \qquad W = \exp K_1^* \sum_{j=1}^{M} \sigma_z^j,$$

and $K_j = \beta J_j$, $\sinh 2K_1 \sinh 2K_1^* = 1$.

The matrix V is diagonal (since the σ_x^j are diagonal matrices), and comes from the interactions along horizontal rows. The scalar factor $(2\sinh 2K_1)^{\frac{M}{2}} W$ comes from the interactions between neighbouring rows.

The expectation value of a local observable F on Λ_{lm}, where $l < L$, $m < M$, is given by

$$\langle F\rangle_\beta^{LM} = \frac{\operatorname{tr}(T^L F_\beta^M)}{\operatorname{tr} T^L}$$

where F_β^M is an operator in $\bigotimes_1^M M_2$ independent of L. Here the transfer matrix T has strictly positive entries providing K_1 is finite (or, equivalently, $K_1^* \neq 0$). By the Perron-Frobenius theorem T has a unique eigenvector Ω^M associated with the largest eigenvalue, with $\Omega^M(\overline{\sigma}) > 0$, $\overline{\sigma} \in \{\pm\}^M$. Letting $L \to \infty$ we pick out the eigenspace associated with the largest eigenvalue:

$$\langle F\rangle_\beta^M = \langle F_\beta^M \Omega^M, \Omega^M\rangle.$$

As $M \to \infty$, F_β^M is eventually constant, and the states $\varphi_\beta^M = \langle \cdot\Omega^M, \Omega^M\rangle$ on $\bigotimes_1^M M_2$ converge to a state φ_β on $A^P = \bigotimes_1^\infty M_2$.

The factor $(2\sinh 2K_1)^{\frac{M}{2}}$ will cancel in $\langle F\rangle_\beta^{LM}$, and so we can regard K_2 and K_1^* as independent parameters. The extreme temperature $\beta = 0$ corresponds to $K_2 = \beta J_2 = 0$, and the extreme temperature $\beta = \infty$ corresponds to $K_1 = \beta J_1 = \infty$, or $K_1^* = 0$.

After removing the scalar $(2\sinh 2K_1)^{\frac{M}{2}}$, the transfer matrix is now $T = V^{\frac{1}{2}} W V^{\frac{1}{2}}$. So T is essentially W when $K_2 = 0$, and V when $K_1^* = 0$.

If we let $e_{2i} = (1+\sigma_z^i)/2$, $e_{2i+1} = (1+\sigma_x^i\sigma_x^{i+1})/2$, then these e_i are projections satisfying the Temperley-Lieb relations TL1, TL2.

The transfer matrix T is then described by

$$V = \exp K_2 \sum_{i=1}^{M} e_{2i+1}, \qquad W = \exp K_1^* \sum_{i=1}^{M} e_{2i}.$$

2.2.1 The Kramers-Wannier Endomorphism

To extract further information we make use of the high temperature-low temperature duality used by Kramers and Wannier to locate the value of the critical temperature [77]. The Kramers-Wannier high temperature-low temperature duality, which, roughly speaking, interchanges the role of V and W in T, is effected by the shift $e_i \longrightarrow e_{i+1}$.

The Pauli algebra is graded by the symmetry $\theta = \bigotimes_j \mathrm{Ad}(\sigma_z^j)$, a period two automorphism, so that $\theta(\sigma_z^i) = \sigma_z^i$, $\theta(\sigma_x^i) = -\sigma_x^i$, $\theta(\sigma_y^i) = -\sigma_y^i$. Then the even part $A_+^P = \{x \in A^P | \theta(x) = x\}$ of the Pauli algebra $A^P = \bigotimes_{\mathbb{N}} M_2$ is generated by σ_z^i and $\sigma_x^i\sigma_x^{i+1}$. The Kramers-Wannier automorphism κ on A_+^P is then given by

$$\kappa(\sigma_z^j) = \sigma_x^j\sigma_x^{j+1}, \qquad \kappa(\sigma_x^j\sigma_x^{j+1}) = \sigma_z^{j+1},$$

so that $\kappa(e_i) = e_{i+1}$.

Although $\kappa^2|_{A_+^P} = \sigma$, where σ is the unilateral shift:

$$\kappa^2(\sigma_z^j) = \sigma_z^{j+1}, \qquad \kappa^2(\sigma_x^j\sigma_x^{j+1}) = \sigma_x^{j+1}\sigma_x^{j+2},$$

it was shown in [39, Cor. 7.11] that κ does not extend to an automorphism of A^P.

But κ can be extended to an endomorphism of the Pauli Algebra. Let $\mathbb{Z}/2$ act on $\mathbb{C}^2$ by transposition α. Then the crossed product $\mathbb{C}^2 \rtimes (\mathbb{Z}/2) \cong M_2$, which is generated by unitaries u_1, u_2 satisfying $u_i^2 = 1, u_1u_2 = -u_2u_1$, and $\bigotimes_{\mathbb{N}} M_2$ is generated by self-adjoint unitaries $u_1, u_2, \ldots$ satisfying $u_iu_j = u_ju_i$, if $|i-j| > 1$, and $u_iu_{i+1} = -u_{i+1}u_i$. We can then identify A^P with $C^*(u_j | j \in \mathbb{N})$ by

$$\begin{aligned} \{\sigma_x^j\}_{j=1}^\infty &= \{u_1, u_1u_3, u_1u_3u_5, u_1u_3u_5u_7, \ldots\}, \\ \{\sigma_z^j\}_{j=1}^\infty &= \{u_2, u_4, u_6, u_8, \ldots\}. \end{aligned}$$

Then $\{\sigma_x^j\sigma_x^{j+1}\}_{j=1}^\infty = \{u_3, u_5, u_7, u_9, \ldots\}$. So the even part A_+^P is generated by $u_2, u_3, u_4, \ldots$, and A^P by A_+^P and u_1. By the universal property of a crossed product, there exists a unique endomorphism $\nu : A^P \longrightarrow A^P$ which sends u_i to u_{i+1}. Then κ is extended to ν, since $\nu|_{A_+^P} = \kappa|_{A_+^P}$. Then $\nu^2|_{A_+^P} = \sigma$, but $\nu^2 \neq \sigma$ since $\nu^2(\sigma_x^j) = \sigma_x^1\sigma_x^{j+1}$.

2.3 The Cuntz Algebra $\mathcal{O}_n$

The Cuntz algebras $\mathcal{O}_n$ were studied by Cuntz in [20]. They are the simple C^*-algebras generated by $n \geq 2$ isometries $S_1, \ldots S_n$ which satisfy the Cuntz relations:

$$S_i^*S_j = \delta_{ij}1, \qquad \sum_i S_iS_i^* = 1. \tag{2.1}$$

Any C^*-algebra generated by n isometries $S'_1, \ldots S'_n$ on a Hilbert space, which also satisfy (2.1) is canonically isomorphic to $\mathcal{O}_n$. Isometries $S_1, \ldots S_n$ satisfying (2.1) may be constructed as follows [39, §2.8]: The (full) Fock space F on the Hilbert space $\mathbb{C}^n$ is given by

$$F(\mathbb{C}^n) = \bigoplus_{m=0}^{\infty} \left(\otimes^m \mathbb{C}^n \right)$$

where $\otimes^m \mathbb{C}^n = \mathbb{C}^n \otimes \mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n$, m copies of $\mathbb{C}^n$, and $\otimes^0 \mathbb{C}^n$ is the one-dimensional Hilbert space spanned by vacuum vector Ω. We define $t(\xi) \in B(F)$ by $t(\xi) = \bigoplus_{m=0}^{\infty} t^m(\xi)$, where $t^m(\xi) : \otimes^m \mathbb{C}^n \longrightarrow \otimes^{m+1} \mathbb{C}^n$ and are given by $t^m(\xi) f = \xi \otimes f$, $t^0(\xi)\Omega = \xi$. We write $t_i = t(e_i)$ where $\{e_1, \ldots, e_n\}$ is an orthonormal basis for $\mathbb{C}^n$. These t_i satisfy

$$t_i^* t_j = \delta_{ij} 1, \qquad \sum_i t_i t_i^* + T_{\Omega,\Omega} = 1,$$

where T is the linear map $T_{\xi,\eta} f = \langle f, \eta \rangle \xi$. The rank one operator $T_{\Omega,\Omega} \in \mathcal{K}$, the compact operators on $\mathbb{C}^n$, which is an ideal in $B(F)$. Let π be the quotient map $\pi : B(F) \longrightarrow B(F)/\mathcal{K}$. We have that $t_\mu [1 - \sum_{i=1}^n t_i t_i^*] t_{\mu'}^* = T_{e_\mu, e_{\mu'}} f$, so $\mathcal{T}_n = C^*(t_1, \ldots, t_n)$ contains every rank one operator and so contains $\mathcal{K}$. Then $\pi(\mathcal{T}_n) = C^*(s_1, \ldots, s_n) \equiv \mathcal{O}_n$, where $s_i = \pi(t_i)$ and satisfy the Cuntz relations (2.1).

2.3.1 Extending ν to the Cuntz Algebra $\mathcal{O}_2$

To get a better understanding of the Kramers-Wannier endomorphism we extend it to the Cuntz algebra $\mathcal{O}_2$. The extension of ν to ρ on $\mathcal{O}_2$ is given by Evans [35] as

$$\rho[(S_+ + \sigma S_-)/\sqrt{2}] = S_+ S_\sigma S_\sigma^* + S_- S_{-\sigma} S_{-\sigma}^*, \tag{2.2}$$

where $\sigma = \pm$. Then

$$\begin{aligned} \rho^2(S_\sigma) &= S_+ S_\sigma S_+^* + S_- S_{-\sigma} S_-^* \\ &= S_+ \mathrm{id}(S_\sigma) S_+^* + S_- \alpha(S_\sigma) S_-^*, \end{aligned}$$

where α is the automorphism of $\mathcal{O}_2$ which is the switch $S_\sigma \to S_{-\sigma}$, i.e. $\alpha = \alpha_u$ where $u = \sigma_z$. This means that

$$[\rho]^2 = [\mathrm{id}] \oplus [\alpha] \tag{2.3}$$

as sectors on $\mathcal{O}_2$. Our notion of sectors on type III von Neumann algebras clearly makes sense in the C^*-setting of $\mathcal{O}_2$.

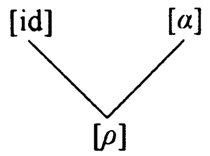
We give an explicit construction of the endomorphism ρ defined above (2.2) satisfying the Ising fusion rules (2.3), using a similar method to that which Izumi used to construct endomorphisms given by the fusion rules for the principal graphs of certain inclusions of factors [58, §3].

Let ρ be an endomorphism of a purely infinite factor M such that $M \supset \rho(M)$ has finite index $\sqrt{2}$ and principal graph A_3. Then, by [57, §3.2], ρ satisfies the following fusion rules of sectors:

$$[\rho^2] = [\mathrm{id}] \oplus [\alpha], \tag{2.4}$$

$$[\alpha][\rho] = [\rho], \qquad [\alpha^2] = [\mathrm{id}].$$

We have the following diagram of the descendant sectors:



We take representatives ρ, α such that

$$\alpha \cdot \rho = \rho, \qquad \rho \cdot \alpha = \mathrm{Ad}(U) \cdot \rho, \tag{2.5}$$

where U is a unitary in (ρ^2, ρ^2) (since $\rho^2 = \rho\alpha\rho = \mathrm{Ad}(U)\rho^2$ and so $\rho^2 U = U\rho^2$). Equation (2.4) means that there exist isometries $S_1, S_2 \in M$ which generate $\mathcal{O}_2$ and satisfy

$$\begin{aligned} S_1 x &= \rho^2(x) S_1, \qquad x \in M, & (2.6)\\ S_2 \alpha(x) &= \rho^2(x) S_2, \qquad x \in M, & (2.7) \end{aligned}$$

i.e. $S_1 \in (\mathrm{id}, \rho^2), S_2 \in (\alpha, \rho^2)$. By [80, Cor. 5.8], $\dim(\mathrm{id}, \rho\overline{\rho}) = 1$ if ρ, $\overline{\rho}$ are irreducible conjugate endomorphisms with finite index. Then $\dim(\mathrm{id}, \rho^2) = 1$ since ρ is self-conjugate. Theorem 5.2 of [80] says that for every isometry $v \in (\mathrm{id}, \rho\overline{\rho})$ there exists a unique isometry $\overline{v} \in (\mathrm{id}, \overline{\rho}\rho)$ such that $v^*\rho(\overline{v}) = 1/\sqrt{2}$, where $\sqrt{2}$ is the index of $M \supset \rho(M)$. Then we have $S_1^*\rho(S_1) = \pm 1/\sqrt{2}$. Then from (2.5) we obtain $\alpha(S_2) \in \alpha\left((\alpha, \rho^2)\right) = (\mathrm{id}, \rho^2)$, and since (id, ρ^2) is one-dimensional and $S_1 \in (\mathrm{id}, \rho^2)$, we have that $\alpha(S_2) = cS_1$, with $c \in \mathbb{C}$. Since $\alpha(S_2)^*\alpha(S_2) = \alpha(1) = 1$, we require $c\overline{c} = 1$, and so $\alpha(S_2) = cS_1, \in \mathbb{T}$. Changing the relative phase between S_1 and S_2 if necessary (i.e. letting $S_1' = c'S_1$, $c' \in \mathbb{T}$), we may assume $\alpha(S_2) = S_1$ and $\alpha(S_1) = \alpha^2(S_2) = S_2$.

From (2.6), (2.7) we obtain

$$S_2^*\rho(S_1)\rho(x) = S_2^*\rho(S_1 x) = S_2^*\rho(\rho^2(x)S_1) = \alpha(\rho(x))S_2^*\rho(S_1) = \rho(x)S_2^*\rho(S_1),$$

and so $S_2^* \rho(S_1) \in (\rho, \rho) = \mathbb{C}$ (since (ρ, ρ) is every element in M which commutes with $\rho(x)$ for all $x \in M$, and so $(\rho, \rho) = \rho(M)' \cap M = \mathbb{C}$ for $[M : \rho(M)] < 4$). Then we have

$$\rho(S_1) = (S_1S_1^* + S_2S_2^*)\rho(S_1) = \pm\frac{1}{\sqrt{2}}S_1 + fS_2, \qquad f \in \mathbb{C}.$$

Since $\alpha\rho(S_1) = \rho(S_1)$, we have $\pm 1/\sqrt{2}S_1 + fS_2 = \pm 1/\sqrt{2}S_2 + fS_1$. Pre-multiplying by S_1^* gives $f = \pm 1/\sqrt{2}$. And so we have

$$\rho(S_1) = \pm\frac{1}{\sqrt{2}}(S_1 + S_2).$$

Since ρ^2 is reducible as $[\rho^2] = [\mathrm{id}] \oplus [\alpha]$, the space of intertwiners between $[\mathrm{id}] \oplus [\alpha]$ and $[\mathrm{id}] \oplus [\alpha]$ is two-dimensional. By (2.6), $S_1S_1^* \in (\rho^2, \rho^2)$ and similarly $S_2S_2^* \in (\rho^2, \rho^2)$ using (2.7). Then we have $(\rho^2, \rho^2) = \mathbb{C}S_1S_1^* + \mathbb{C}S_2S_2^*$, where the unitary U in (2.5) is given by $U = gS_1S_1^* + hS_2S_2^*$, $g, h \in \mathbb{C}$, and since $UU^* = g\bar{g}S_1S_1^* + h\bar{h}S_2S_2^* = 1$, we have that $g, h \in \mathbb{T}$. Then since $\rho(S_2) = \rho(\alpha(S_1))$ we obtain

$$\rho(S_2) = U\rho(S_1)U^* = (gS_1S_1^* + hS_2S_2^*)(\pm\frac{1}{\sqrt{2}}(S_1 + S_2))U^* = \pm\frac{1}{\sqrt{2}}(gS_1 + hS_2)U^*.$$

Due to orthogonality of $\rho(S_1)$ and $\rho(S_2)$ we have that

$$\rho(S_1)^*\rho(S_2) = \frac{1}{2}(S_1^* + S_2^*)(gS_1 + hS_2)U^* = \frac{1}{2} = \frac{1}{2}(g + h)U^* = 0,$$

and so $h = -g$. Then

$$\begin{aligned} \rho(S_2) &= \pm\frac{1}{\sqrt{2}}g\bar{g}(S_1 - S_2)(S_1S_1^* - S_2S_2^*) \\ &= \pm\frac{1}{\sqrt{2}}(S_1 - S_2)(S_1S_1^* - S_2S_2^*). \end{aligned}$$

We can still change $S_1, S_2 \to tS_1, tS_2$ $(t \in \mathbb{T})$ if necessary, without changing the relative phase between S_1 and S_2, and so we can take $\rho(S_1), \rho(S_2)$ to be:

$$\begin{aligned} \rho(S_1) &= \frac{1}{\sqrt{2}}(S_1 + S_2), \\ \rho(S_2) &= \frac{1}{\sqrt{2}}(S_1 - S_2)(S_1S_1^* - S_2S_2^*). \end{aligned}$$

This gives

$$\begin{aligned} \rho[(S_1 + S_2)/\sqrt{2}] &= \frac{1}{2}(S_1 + S_2 + (S_1 - S_2)(S_1S_1^* - S_2S_2^*)) \\ &= \frac{1}{2}(S_1(S_1S_1^* + S_2S_2^*) + S_2(S_1S_1^* + S_2S_2^*) + S_1S_1S_1^* - S_1S_2S_2^* \\ &\quad -S_2S_1S_1^* + S_2S_2S_2^*) \\ &= S_1S_1S_1^* + S_2S_2S_2^*, \end{aligned}$$

and

$$\begin{aligned}\rho[(S_1 - S_2)/\sqrt{2}] &= \frac{1}{2}(S_1(S_1S_1^* + S_2S_2^*) + S_2(S_1S_1^* + S_2S_2^*) - S_1S_1S_1^* + S_1S_2S_2^* \\ &\quad +S_2S_1S_1^* - S_2S_2S_2^*) \\ &= S_1S_2S_2^* + S_2S_1S_1^*.\end{aligned}$$

Setting $S_1 = S_+$, $S_2 = S_-$ and $\sigma = \pm$, we have

$$\rho[(S_+ + \sigma S_-)/\sqrt{2}] = S_+S_\sigma S_\sigma^* + S_-S_{-\sigma}S_{-\sigma}^*,$$

as given earlier (2.2).

2.4 The Cuntz-Krieger Algebra $\mathcal{O}_A$

The Cuntz-Krieger Algebras $\mathcal{O}_A$ were introduced by Cuntz and Krieger in [22]. They were a generalization of the Cuntz algebras and had a close relationship with topological Markov chains, whose theory is part of symbolic dynamics. Let Σ be a finite set of cardinality m. If $A = (A(i,j))_{i,j\in\Sigma}$ is a finite $\{0,1\}$-matrix the Cuntz-Krieger algebra $\mathcal{O}_A$ is the C^*-algebra generated by a family of (non-zero) partial isometries $\{S_i | i \in \Sigma\}$ satisfying

$$S_i^*S_j = 0 \text{ if } i \neq j, \qquad S_i^*S_i = \sum_{j\in\Sigma} A(i,j)S_jS_j^*. \tag{2.8}$$

If the matrix A is irreducible then the Cuntz-Krieger algebra $\mathcal{O}_A$ is unique up to isomorphism [22, Theorem 2.13] and is simple [22, Theorem 2.14]. An irreducible $\{0,1\}$-matrix A defines a connected graph $\mathcal{G}$ with m vertices labelled by the elements of Σ, and for $i,j \in \Sigma$ there is an edge from vertex i of $\mathcal{G}$ to vertex j if $A(i,j) = 1$. Then A is the adjacency matrix of the graph $\mathcal{G}$. We will often write $\mathcal{O}_\mathcal{G}$ for the Cuntz-Krieger algebra $\mathcal{O}_A$. Note, we can obtain a different graph by indexing the matrix A by m edges labelled by the elements of Σ, where for any two edges $i,j \in \Sigma$ there is a vertex v such that $r(i) = v = s(j)$ if $A(i,j) = 1$.

The partial isometries $\{S_i | i \in \Sigma\}$ may be constructed in a similar way to the construction of the isometries that generate the Cuntz algebra (see §2.3) as follows [39, §2.10]:

Let $\mathfrak{V}^\mathcal{G}$ denote the vertex set of $\mathcal{G}$. If $\widetilde{\Lambda}_n(\mathcal{G})$ denotes the paths of vertices of length n in $\mathcal{G}$, we define $F_A = \bigoplus_{n=0}^{\infty} \ell^2(\widetilde{\Lambda}_n(\mathcal{G}))$. Then for each vertex $i \in \mathfrak{V}^\mathcal{G}$, we again define a partial isometry on F_A by a shift action:

$$T_i\widetilde{e}_x = \begin{cases} \widetilde{e}_{ix} & \text{if } A(i,s(x)) \neq 0 \\ 0 & \text{otherwise,} \end{cases}$$

for any path $x \in \widetilde{\Lambda}_n(\mathcal{G})$, where $\{\widetilde{e}_\nu | \nu \in \widetilde{\Lambda}_n(\mathcal{G})\}$ is a canonical orthonormal basis for $\ell^2(\widetilde{\Lambda}_n(\mathcal{G}))$. Then $\{T_i | i \in \mathfrak{V}^{\mathcal{G}}\}$ satisfy the relations

$$t_i^* t_i = \sum_j A(i,j) t_j t_j^* + \Omega \otimes \overline{\Omega}$$

where $\ell^2(\Lambda_0(\mathcal{G}))$ is defined to be a one-dimensional space spanned by a vacuum vector Ω. Then, $\mathcal{T}_A \equiv C^*(T_i | i \in \mathfrak{V}^{\mathcal{G}})$ contains the compact operators $\mathcal{K}$, and we have the exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T}_A \longrightarrow \mathcal{O}_A \longrightarrow 0$$

where $\mathcal{O}_A \cong \mathcal{T}_A / \mathcal{K}$ is generated by partial isometries $\{S_i | i \in \mathfrak{V}^{\mathcal{G}}\}$ satisfying the relations (2.8).

These partial isometries were indexed by the set $\mathfrak{V}^{\mathcal{G}}$ of vertices of $\mathcal{G}$. But partial isometries satisfying Cuntz-Krieger relations may also be indexed by the set $\mathfrak{E}^{\mathcal{G}}$ of edges of $\mathcal{G}$. A path $\alpha \in \Lambda$ given by $\alpha = i_0 i_1 \dots i_n$, where i_j are vertices of $\mathcal{G}$, $j = 1, \dots, n$ may easily be written as a path of edges of $\mathcal{G}$ by $\alpha = \alpha_1 \dots \alpha_n$, where $s(\alpha_1) = i_0$ and $r(\alpha_j) = i_j$, $j = 1, \dots, n$. Then $s(\alpha) = i_0$, $r(\alpha) = i_n$ and $|\alpha| = n$ as before.

We define B-Fock space F_B, where B is the adjacency matrix of the edges of a $\mathcal{G}$ given by

$$B(x,y) = \begin{cases} 1 & \text{if } r(x) = s(y) \\ 0 & \text{otherwise,} \end{cases}$$

as $F_B \bigoplus_{n=0}^{\infty} \ell^2(\Lambda_n(\mathcal{G}))$, where Λ_n denotes the paths of length n in $\mathcal{G}$. For each edge $x \in \mathfrak{E}^{\mathcal{G}}$, define a partial isometry t_x on F_B by a shift action:

$$t_x e_\lambda = \begin{cases} e_{x\lambda} & \text{if } r(x) = s(\lambda) \\ 0 & \text{otherwise,} \end{cases}$$

where $\{e_\mu | \mu \in \Lambda_n(\mathcal{G})\}$ is a canonical orthonormal basis for $\ell^2(\Lambda_n(\mathcal{G}))$. Then, as before, $\mathcal{T}_B \equiv C^*(t_x | x \in \mathfrak{E}^{\mathcal{G}})$ contains the compact operators $\mathcal{K} = \mathcal{K}(F_B)$, and we have the exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T}_B \longrightarrow \mathcal{O}_B \longrightarrow 0$$

where $\mathcal{O}_B$ is generated by partial isometries $\{s_x | x \in \mathfrak{E}^{\mathcal{G}}\}$ satisfying the Cuntz-Krieger relations (2.8):

$$s_x^* s_x = \sum_{y: s(y) = r(x)} s_y s_y^* = \sum_{y \in \mathfrak{E}^{\mathcal{G}}} B(x,y) s_y s_y^*.$$

Theorem 6.5 of [106] states that two simple Cuntz-Krieger algebras $\mathcal{O}_A$, $\mathcal{O}_B$ are stably isomorphic, i.e. $\mathcal{K} \otimes \mathcal{O}_A \cong \mathcal{K} \otimes \mathcal{O}_B$, if and only if $K_0(\mathcal{O}_A) \cong K_0(\mathcal{O}_B)$. They are isomorphic if and only if $(K_0(\mathcal{O}_A), [1])$ and $(K_0(\mathcal{O}_B), [1])$ are isomorphic, i.e. there is a

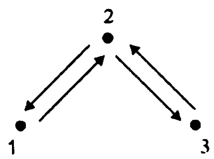


Figure 2.3: Directed A_3

group isomorphism between the K_0-groups that maps the class of the unit [1] of $\mathcal{O}_A$ to the class of the unit [1] of $\mathcal{O}_B$. So K_0 is a complete invariant for the stable isomorphism class of Cuntz-Krieger algebras, and together with the position of the unit it is a complete invariant for the isomorphism class of Cuntz-Krieger algebras.

It is well-known that if the matrix A given by the edges of $\mathcal{G}$ and the adjacency matrix B of $\mathcal{G}$ are both $\{0,1\}$-matrices then they give isomorphic Cuntz-Krieger algebras, i.e. $\mathcal{O}_A \cong \mathcal{O}_B$. Mann, Raeburn and Sutherland give a proof of this in [84].

2.4.1 Different Representations of $\mathcal{O}_2$

The Cuntz algebra $\mathcal{O}_n$ is a particular case of the Cuntz-Krieger algebras $\mathcal{O}_A$ introduced by Cuntz and Krieger in [22, §1], where A is the $n \times n$ matrix where every entry is 1. In the case of $\mathcal{O}_2$ we then have that $\mathcal{O}_2 \cong \mathcal{O}_C$, where

$$C = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

$\mathcal{O}_2$ may also be identified with the algebras $\mathcal{O}_A \cong \mathcal{O}_B$, where A is the adjacency matrix of $\mathcal{G}$ and B is the matrix indexed by the edges of $\mathcal{G}$ given by

$$B(x,y) = \delta_{r(x),s(y)},$$

where $\mathcal{G}$ is the directed A_3 graph (Figure 2.3). Then A, B are given by

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}. \tag{2.9}$$

From the K-theory of $\mathcal{O}_B$, where B is the adjacency matrix of the bipartite graph A_3, and $\mathcal{O}_2$ we get $K_0(\mathcal{O}_B) = K_1(\mathcal{O}_B) = K_0(\mathcal{O}_2) = K_1(\mathcal{O}_2) = 0$, and hence we have $\mathcal{O}_A \cong \mathcal{O}_B \cong \mathcal{O}_C \cong \mathcal{O}_2$ [106]. We give another proof that $\mathcal{O}_A \cong \mathcal{O}_2$, similar to that to [84, Prop. 4.1], which gives an explicit isomorphism.

Proposition 2.4.1 *Let A be the adjacency matrix of A_3 defined in (2.9). Then $\mathcal{O}_A \cong \mathcal{O}_2$.*

Proof. Let s_+, s_- be isometries that generate $\mathcal{O}_2$, i.e. they satisfy $s_\mu^* s_\nu = \delta_{\mu,\nu} 1$, $s_+ s_+^* + s_- s_-^* = 1$. We define mutually orthogonal partial isometries $S_1, S_2, S_3 \in C^*(s_\mu)$ by $S_1 = s_+ s_- s_-^*$, $S_2 = s_- s_+^*$, $S_3 = s_+ s_+ s_-^*$. Then $s_+ = s_+(s_+ s_+^* + s_- s_-^*) = S_3\, s_- s_+^* + S_1 = S_1 + S_3 S_2$, and $s_- = s_-(s_+ s_+^* + s_- s_-^*) = s_- s_+^* S_3\, s_- s_+^* + s_- s_+^* S_1 = S_2 S_1 + S_2 S_3 S_2$. Thus $C^*(s_\mu) = C^*(S_i)$.

We now need to verify that the S_i generate $\mathcal{O}_A$. We have $S_1^* S_1 = s_- s_-^* s_+^* s_+ s_- s_-^* = s_- s_-^*$, $S_3^* S_3 = s_- s_+^* s_+^* s_+ s_+ s_-^* = s_- s_-^*$, $S_2 S_2^* = s_- s_+^* s_+ s_-^* = s_- s_-^*$, and so $S_1^* S_1 = S_2 S_2^*$, $S_3^* S_3 = S_2 S_2^*$. Also, $S_2^* S_2 = s_+ s_-^* s_- s_+^* = s_+ s_+^*$, $S_1 S_1^* + S_3 S_3^* = s_+ s_- s_-^* s_- s_-^* s_+^* + s_+ s_+ s_-^* s_- s_+^* s_+^* = s_+(s_- s_-^* + s_+ s_+^*) s_+^* = s_+ s_+^*$, and so $S_2^* S_2 = S_1 S_1^* + S_3 S_3^*$. Then these S_i do satisfy the Cuntz-Krieger relations $S_i^* S_i = \sum_j A(i,j) S_j S_j^*$. Thus by the uniqueness of $\mathcal{O}_2$ we have $\mathcal{O}_A \cong C^*(S_i) = C^*(s_\mu) \cong \mathcal{O}_2$. □

The automorphism α on $\mathcal{O}_2$ given in (2.3) acts by switching $s_+ \leftrightarrow s_-$. On $\mathcal{O}_A$ the corresponding action is given by the involution which switches $S_1 \leftrightarrow S_2 S_3 S_2$, $S_3 \leftrightarrow S_2 S_1 S_2$ and $S_2 \leftrightarrow S_2^*$. Then α leaves the AF-part of $\mathcal{O}_A$ invariant. However the isomorphism constructed in the proof above does not identify the AF-parts $\mathcal{F}_2$, $\mathcal{F}_A$ of $\mathcal{O}_2$, $\mathcal{O}_A$ respectively, since it sends $s_- s_+^* \in \mathcal{F}_2$ to $S_2 \notin \mathcal{F}_A$.

2.5 The Generalization of the Ising Model to other graphs

The Ising model is constructed using the graph A_3. Other lattice models may be constructed using other graphs, such as the Q-state Potts model using the graph in Figure 2, whilst the Dynkin diagrams A_n of Figure 1.1 give the ABF models of Andrews, Baxter and Forrester [1].

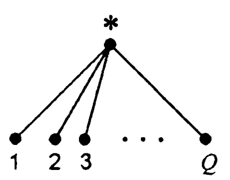


Figure 2.4: Q-State Potts Model

The Potts model was described by Potts [103] and is a generalization of the Ising model. Ashkin and Teller had earlier considered a four-component version [2]. As with

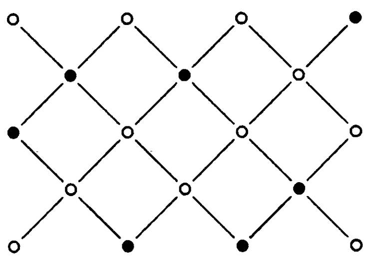


Figure 2.5: An Ising model configuration

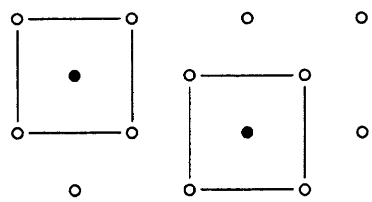


Figure 2.6: A Hard square configuration

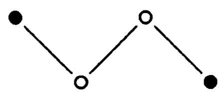


Figure 2.7: Dynkin diagram A_4

the Ising model, it is useful for gaining insight into the behaviour of ferromagnets and certain other phenomena in solid state physics. Potts defined two models: one is now known as the clock model, the other is the (standard) Potts model. In the clock model, the spin at each sight may take one of Q possible values, distributed uniformly about the circle at angles $\theta_n = 2\pi n/Q$. The Hamiltonian of the interactions between nearest neighbours is given by

$$H = J \sum_{(i,j)} \cos(\theta_{s_i} - \theta_{s_j}),$$

where the summation is over nearest neighbour pairs (i, j) over all lattice sites, and the site colours $s_i \in \{1, \dots, Q\}$. The Potts model uses a simpler Hamiltonian

$$H = -J \sum_{(i,j)} \delta_{s_i, s_j}.$$

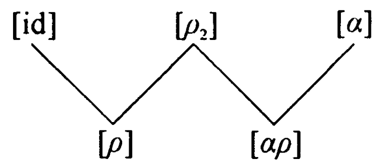
The $Q = 2$ Potts model is equivalent to the two-dimensional Ising model.

The Ising model could be used to describe a lattice gas as well as a magnet, where $+$ means an occupied site $\bullet$ and $-$ an unoccupied site $\circ$, as in Figure 2.5. The hard square model is used to represent particles with non-zero volume, where a similar state space is used and a square is drawn around each occupied site as in Figure 2.6. Then these squares must not overlap so that distinct particles do not occupy the same portion of space. If we use the distributions of the vertices of the Dynkin diagram A_4 (Figure 2.7), we get the state space of the hard square model.

2.5.1 Construction of ρ_2 for Principal Graph A_5

For a purely infinite factor M and $\rho \in \mathrm{End}(M)$ with $d_\rho < \infty$, Izumi [57] computed the fusion rules of descendant sectors of ρ, which are the rules of the irreducible decomposition of sectors, for the cases where the principal graph of $M \supset \rho(M)$ is one of the Dynkin diagrams.

In a similar way to the construction of ρ for principal graph A_3 in §2.3.1, we construct the endomorphism ρ_2 for the case where the inclusion $M \supset \rho(M)$, for M, $\rho(M)$ infinite factors, has finite index $\sqrt{3}$ and principal graph A_5. From [57], the diagram for the multiplication of the sectors by $[\rho]$ is:



Since $[\rho_2][\rho]^2 = [\rho_2]([\mathrm{id}] \oplus [\rho_2]) = [\rho_2] \oplus [\rho_2]^2$ but also $[\rho_2][\rho]^2 = ([\rho] \oplus [\alpha\rho])[\rho] = [\mathrm{id}] \oplus 2[\rho_2] \oplus [\alpha]$ then we obtain

$$[\rho_2]^2 = [\mathrm{id}] \oplus [\rho_2] \oplus [\alpha], \tag{2.10}$$

and similarly, since $[\alpha][\rho]^2 = [\alpha]([\mathrm{id}] \oplus [\rho_2]) = [\alpha] \oplus [\alpha][\rho_2]$ and $[\alpha][\rho]^2 = [\alpha\rho][\rho] = [\rho_2] \oplus [\alpha]$, we have $[\alpha][\rho_2] = [\rho_2]$. Proposition 3.3 of [57] says that $[\alpha^2] = [\mathrm{id}]$.

Since

$$[\rho_2]^2[\rho_2] = ([\mathrm{id}] \oplus [\rho_2] \oplus [\alpha])\,[\rho_2] = [\rho_2] \oplus [\rho_2]^2 \oplus [\alpha][\rho_2],$$

and

$$[\rho_2][\rho_2]^2 = [\rho_2]\,([\mathrm{id}] \oplus [\rho_2] \oplus [\alpha]) = [\rho_2] \oplus [\rho_2]^2 \oplus [\rho_2][\alpha],$$

we have

$$[\rho_2][\alpha] = [\alpha][\rho_2] = [\rho_2]. \tag{2.11}$$

From (2.10) we have that there exist Cuntz isometries S_1, S_2, S_3 which generate $\mathcal{O}_3$ and satisfy

$$\begin{aligned} S_1 x &= \rho_2^2(x) S_1, & x \in M, \quad (2.12)\\ S_2 \rho_2(x) &= \rho_2^2(x) S_2, & x \in M, \quad (2.13)\\ S_3 \alpha(x) &= \rho_2^2(x) S_3, & x \in M, \quad (2.14) \end{aligned}$$

i.e. $S_1 \in (\mathrm{id}, \rho_2^2), S_2 \in (\rho_2, \rho_2^2), S_3 \in (\alpha, \rho_2^2)$.

From (2.11) we may choose representatives of α, ρ_2 such that

$$\alpha \cdot \rho_2 = \rho_2, \qquad \rho_2 \cdot \alpha = \mathrm{Ad}(U) \cdot \rho_2 \tag{2.15}$$

for U a unitary in (ρ_2^2, ρ_2^2).

By (2.10), $[\rho_2]^2$ contains [id], $[\rho_2]$ and $[\alpha]$ each with multiplicity one, and hence $\dim((\mathrm{id}, \rho_2^2)) = \dim((\rho_2, \rho_2^2)) = \dim((\alpha, \rho_2^2)) = 1$. Then from (2.15) we obtain $\alpha(S_3) \in \alpha\left((\alpha, \rho_2^2)\right) = (\mathrm{id}, \rho_2^2)$, and so $\alpha(S_3) = cS_1$, $c \in \mathbb{T}$. So changing the relative phase between S_1 and S_3 we have

$$\alpha(S_3) = S_1, \qquad \alpha(S_1) = \alpha^2(S_3) = S_3. \tag{2.16}$$

From (2.15), $\alpha(S_2) \in \alpha\left((\rho_2, \rho_2^2)\right) = (\rho_2, \rho_2^2)$, so $\alpha(S_2) = \varepsilon_1 S_2$, $\varepsilon_1 \in \mathbb{T}$. Now $S_2 = \alpha^2(S_2) = \varepsilon_1^2 S_2$ giving $\varepsilon_1^2 = 1$, so we have

$$\alpha(S_2) = \varepsilon_1 S_2, \quad \varepsilon_1 \in \{\pm 1\}. \tag{2.17}$$

Since ρ_2 is self-conjugate, we have $S_1^* \rho_2(S_1) = \pm 1/d_{\rho_2} = \pm 1/2 = \varepsilon_2/2$, $\varepsilon_2 \in \{\pm 1\}$, as in Section 2.3.1. Then using (2.15), (2.16) $S_3^* \rho_2(S_1) = \alpha(S_1)^* \rho_2(S_1) = \alpha(S_1^* \rho_2(S_1)) = \alpha(\varepsilon_2/2) = \varepsilon_2/2$.

From (2.12),(2.13) we obtain

$$S_2^* \rho_2(S_1 x) = S_2^* \rho_2(\rho_2^2(x) S_1) = \rho_2(\rho_2(x)) S_2^* \rho_2(S_1) = \rho_2^2(x) S_2^* \rho_2(S_1),$$

so $S_2^* \rho_2(S_1) \in (\rho_2, \rho_2^2)$. Then $S_2^* \rho_2(S_1) = f S_2$, $f \in \mathbb{C}$.

Then $\rho_2(S_1) = (S_1 S_1^* + S_2 S_2^* + S_3 S_3^*) \rho_2(S_1) = \varepsilon_2 (S_1 + S_3)/2 + f S_2 S_2$. Since we have $\rho_2(S_1)^* \rho_2(S_1) = 1$ we find that $f\bar{f} = 1/2$, and so $f = f'/\sqrt{2}$, $f' \in \mathbb{T}$. Changing the relative phase between S_1, S_3 and S_2 if necessary, we may assume $f' = 1$, and so we have

$$\rho_2(S_1) = \frac{1}{2} \varepsilon_2 (S_1 + S_3) + \frac{1}{\sqrt{2}} S_2 S_2, \qquad \varepsilon_2 \in \{\pm 1\}.$$

From $U \in (\rho_2^2, \rho_2^2) = \mathbb{C} S_1 S_1^* + \mathbb{C} S_2 S_2^* + \mathbb{C} S_3 S_3^*$, we have $U = h S_1 S_1^* + j S_2 S_2^* + k S_3 S_3^*$. Then $UU^* = h\bar{h} S_1 S_1^* + j\bar{j} S_2 S_2^* + k\bar{k} S_3 S_3^* = 1$ implies that $h, j, k \in \mathbb{T}$. We may take $h = 1$, since $\mathrm{Ad}(U) \cdot \rho_2 = \mathrm{Ad}(U') \cdot \rho_2$, where $U' = U/t$, $t \in \mathbb{T}$. Then we have

$$U = S_1 S_1^* + \varepsilon_3 S_2 S_2^* + \varepsilon_4 S_3 S_3^*, \qquad \varepsilon_3, \varepsilon_4 \in \mathbb{T}.$$

Now from (2.15), (2.16)

$$\rho_2(S_3) = \rho_2(\alpha(S_1)) = U \rho_2(S_1) U^* = \left(\frac{1}{2} \varepsilon_2 S_1 + \frac{1}{\sqrt{2}} \varepsilon_3 S_2 S_2 + \frac{1}{2} \varepsilon_2 \varepsilon_4 S_3\right) U^*$$

. Due to orthogonality of $\rho_2(S_1)$ and $\rho_2(S_3)$ we find that $\varepsilon_4 = 1, \varepsilon_3 = -1$. So we have $U = S_1S_1^* - S_2S_2^* + S_3S_3^*$, and

$$\rho_2(S_3) = \left(\frac{1}{2}\varepsilon_2(S_1 + S_3) - \frac{1}{\sqrt{2}}S_2S_2\right)(S_1S_1^* - S_2S_2^* + S_3S_3^*).$$

From (2.12), (2.13), (2.14) we obtain

$$S_1^*\rho_2(S_2\rho_2(x)) = S_1^*\rho_2(\rho_2^2(x)S_2) = \rho_2(x)S_1^*\rho_2(S_2),$$

and so $S_1^*\rho_2(S_2) \in (\rho_2^2, \rho_2) = ((\rho_2, \rho_2^2))^*$. Then $S_1^*\rho_2(S_2) = lS_2^*$, $l \in \mathbb{C}$.

From (2.12), (2.13), (2.14) we also obtain

$$S_2^*\rho_2(S_2\rho_2(x)) = S_2^*\rho_2(\rho_2^2(x)S_2) = \rho_2(\rho_2(x))S_2^*\rho_2(S_2) = \rho_2^2(x)S_2^*\rho_2(S_2),$$

and so $S_2^*\rho_2(S_2) \in (\rho_2^2, \rho_2^2) = \mathbb{C}S_1S_1^* + \mathbb{C}S_2S_2^* + \mathbb{C}S_3S_3^*$. Then $S_2^*\rho_2(S_2) = m_1S_1S_1^* + m_2S_2S_2^* + m_3S_3S_3^*$, $m_1, m_2, m_3 \in \mathbb{C}$.

And again, from (2.12), (2.13), (2.14) we obtain

$$S_3^*\rho_2(S_2\rho_2(x)) = S_3^*\rho_2^2(\rho_2(x))\rho_2(S_2) = \alpha(\rho_2(x))S_3^*\rho_2(S_2) = \rho_2(x)S_3^*\rho_2(S_2),$$

and so $S_3^*\rho_2(S_2) \in (\rho_2^2, \rho_2) = ((\rho_2, \rho_2^2))^*$. Then $S_3^*\rho_2(S_2) = nS_2^*$, $n \in \mathbb{C}$.

Then $\rho_2(S_2) = (S_1S_1^* + S_2S_2^* + S_3S_3^*)\rho_2(S_2) = lS_1S_2^* + m_1S_2S_1S_1^* + m_2S_2S_2S_2^* + m_3S_2S_3S_3^* + nS_3S_2^*$.

From the Cuntz relations of $\rho_2(S_1)$, $\rho_2(S_2)$ and $\rho_2(S_3)$ we find that $l = p/\sqrt{2} = -n$, $m_2 = 0$ and $p, m_1, m_3 \in \mathbb{T}$. Then

$$\rho_2(S_2) = \frac{1}{\sqrt{2}}pS_1S_2^* + m_1S_2S_1S_1^* + m_3S_2S_3S_3^* - \frac{1}{\sqrt{2}}pS_3S_2^*. \tag{2.18}$$

Now using (2.15) we have

$$\rho_2(S_2) = \alpha(\rho_2(S_2)) = \frac{1}{\sqrt{2}}p\varepsilon_1S_3S_2^* + m_1\varepsilon_1S_2S_3S_3^* + m_3\varepsilon_1S_2S_1S_1^* - \frac{1}{\sqrt{2}}p\varepsilon_1S_1S_2^*. \tag{2.19}$$

Then comparing (2.18) and (2.19) yields $\varepsilon_1 = -1$ and $m_1 = -m_3$.

Computing $\rho_2^2(S_1)$ and using (2.12), (2.13) we find that $p = \bar{m}, \varepsilon_2 = 1$, and that m satisfies $m^3 = 1$. Then we conclude that

$$\begin{aligned}
\rho_2(S_1) &= \frac{1}{2}(S_1 + S_3) + \frac{1}{\sqrt{2}}S_2S_2,\\
\rho_2(S_2) &= \frac{1}{\sqrt{2}}\bar{a}(S_1 - S_3)S_2^* + aS_2(S_1S_1^* - S_3S_3^*),\\
\rho_2(S_3) &= \left(\frac{1}{2}(S_1 + S_3) - \frac{1}{2}S_2S_2\right)(S_1S_1^* - S_2S_2^* + S_3S_3^*),\\
\alpha(S_1) &= S_3, \qquad \alpha(S_2) = -S_2, \qquad \alpha(S_3) = S_1,
\end{aligned}$$

and there are three non-conjugate solutions, given by $a = e^{2\pi ki/3} \in \mathbb{T}$, for $k \in \{0, 1, 2\}$.

2.5.2 $\mathcal{O}_A$ as a Crossed Product

Cuntz and Krieger [22] showed that the stable algebra $\mathcal{K} \otimes \mathcal{O}_A$ also arises as the crossed product of an AF algebra by a shift operator as follows.

Let Σ again be a finite set of cardinality m and $A = (A(i,j))_{i,j\in\Sigma}$ a finite $\{0,1\}$-matrix. The matrix A is used in topological dynamics to construct one-sided and two-sided subshifts. Let $X_A = \{(x_i)_{i\in\mathbf{Z}} \in \Sigma^{\mathbf{Z}} | A(x_j, x_{j+1}) = 1, j \in \mathbb{Z}\}$. The two-sided subshift σ_A acts on the compact spaces X_A and is defined by $\sigma_A(x)_i = x_{i+1}$, $x \in X_A$. The pair (X_A, σ_A) is called a topological Markov chain.

For $m \leq n$, let $X_{m,n} = \{(x_i)_{i=m}^n \in \Sigma^{n-m} | A(x_j, x_j) = 1, j = m, m+1, \dots, n-1\}$ so that $X_A = X_{\infty,\infty}$. For $i, j \in \Sigma$, let $X_{m,n}^{i,j} = \{(x_i)_{i=m}^n \in X_{m,n} | x_m = i, x_n = j\}$, so that $|X_{m,n}^{i,j}| = A^{n-m}(i,j)$ and $X_{m,n} = \bigcup_{i,j} X_{m,n}^{i,j}$. For $m \geq 0$ we use the notation $X_m^{i,j}$ for $X_{-m,m}^{i,j}$ so that $X_m = \bigcup_{i,j} X_m^{i,j}$. Let $\mathcal{F}_m = \bigoplus_{i,j} \mathcal{F}_m^{i,j}$, where $\mathcal{F}_m^{i,j} = M(|X_m^{i,j}|)$ is the full $n \times n$ complex matrix algebra, $n = |X_m^{i,j}|$, and is generated by matrix units $e_{\mu,\nu}$, $\mu, \nu \in X_m^{i,j}$. We define a homomorphism $\phi_m : \mathcal{F}_m \to \mathcal{F}_{m+1}$ by $\phi_m(e_{\mu,\nu}) = \sum_{p,q\in\Sigma} A(p,i)A(j,q)e_{p\mu q, p\nu q}$ for $\mu, \nu \in X_m^{i,j}$. Then we can construct an AF algebra $\mathcal{F}_A = \lim_{\to}(\mathcal{F}_m, \phi_m)$.

For a subset Y of X_A, we define its unstable manifold by

$$W(Y) = \{x = (x_i) \in X_A | x_j = y_j \text{ for some } y = (y_i) \in Y, \text{ for all } j \leq j_0 \text{ for some } j_0 \in \mathbb{Z}\},$$

which has the inductive limit topology inherited from the shift space. If $x \in X_A$, let $F(x) = \{\sigma_A^k x | k \in \mathbb{Z}\}$, which is a countable shift invariant subset of X_A. If $x, x' \in X_A$, there exists a homeomorphism $h : W(F(x)) \to W(F(x'))$ such that $h(y)_i = y_i$ for all $i \geq 0$, $y \in W(F(x))$. Let $G(x)$ denote the uniformly finite dimensional homeomorphisms of $W(F(x))$, i.e. the homeomorphisms g of $W(F(x))$ such that $g(y)_i = y_i$ for all $i \geq 0$. Then $hG(x)h^{-1} = G(x')$ and the following construction doesn't depend on the choice of x. Let $\overline{\mathcal{D}}_A = C_0(W(F(x)))$ be the algebra of continuous complex-valued functions on $W(F(x))$ that vanish at infinity, and let $\mathcal{A}$ be the crossed product of $\overline{\mathcal{D}}_A$ by $G(x)$, i.e. $\mathcal{A} = C^*(\overline{\mathcal{D}}_A, G(x))$. Let U be the canonical representation of $G(x)$ in the multiplier algebra of $\mathcal{A}$, so that for every $g \in G(x)$ there is a corresponding unitary $U(g)$ in the multiplier algebra of $\mathcal{A}$. If $\mathcal{J}$ is the closed ideal of $\mathcal{A}$ generated by all elements of the form $U(g)P_B - U(g')P_B$, where u, v are uniformly finite dimensional homeomorphisms which agree on the compact open set B of $W(F(x))$, and P_B is the characteristic function of B. Then we define the AF algebra $\overline{\mathcal{F}}_A = \mathcal{A}/\mathcal{J}$. The shift induces an automorphism σ of $\overline{\mathcal{F}}_A$, and let $\overline{\mathcal{O}}_A$ denote the corresponding crossed product $\overline{\mathcal{O}}_A = \overline{\mathcal{F}}_A \rtimes \mathbb{Z} = C^*(\overline{\mathcal{F}}_A, \mathbb{Z})$. Cuntz and Krieger [22, Theorem 3.8] proved that there is an isomorphism so that $\overline{\mathcal{O}}_A \cong \mathcal{K} \otimes \mathcal{O}_A$, and as a consequence of [21, Theorem 2.3], the same isomorphism gives $\overline{\mathcal{F}}_A \cong \mathcal{K} \otimes \mathcal{F}_A$. It should be noted however that the decomposition of $\overline{\mathcal{O}}_A$ as the crossed product of an AF algebra by the integers is not unique.

For the Dynkin diagrams A_{2n+1}, D_{n+2}, $n \geq 2$, illustrated in Figures 1.1, 1.2, there are $\mathbb{Z}_2$ actions on both A_{2n+1} and D_{n+2}. For A_{2n+1} this is given by reflecting the graph horizontally about the vertex $n+1$, sending the vertex i to the vertex $2n+2-i$ (the vertex $n+1$ is invariant under this action). For D_{n+2} the $\mathbb{Z}_2$ action is given by interchanging the vertices 1 and 2, and leaving the others invariant.

Proposition 2.5.1 *The stable algebra $\mathcal{K} \otimes \mathcal{O}_{D_{n+2}}$ is isomorphic to the crossed product of the stable algebra $\mathcal{K} \otimes \mathcal{O}_{A_{2n+1}}$ by the $\mathbb{Z}_2$ action on $\mathcal{O}_{A_{2n+1}}$ induced by the above $\mathbb{Z}_2$ action on the graph A_{2n+1}. The reverse is also true, that $\mathcal{K} \otimes \mathcal{O}_{A_{2n+1}}$ is isomorphic to the crossed product of $\mathcal{K} \otimes \mathcal{O}_{D_{n+2}}$ by the $\mathbb{Z}_2$ action on $\mathcal{O}_{D_{n+2}}$ induced by the above $\mathbb{Z}_2$ action on the graph D_{n+2}.*

Proof

The $\mathbb{Z}_2$ actions on A_{2n+1}, D_{n+2} induce actions on the algebras $\mathcal{O}_{A_{2n+1}}$, $\mathcal{O}_{D_{n+2}}$ by the uniqueness of Cuntz-Krieger algebras. Let ϕ be the $\mathbb{Z}_2$ action on $\mathcal{O}_{A_{2n+1}}$, and let $W = W(F(x)) \cup W(F(\phi(x)))$. Then with $\overline{\mathcal{D}}_{A_{2n+1}} = C_0(W)$ and $\mathcal{A} = C^*(\overline{\mathcal{D}}_{A_{2n+1}}, G(x))$, let $\overline{\mathcal{F}}_{A_{2n+1}}$ be the AF algebra $\mathcal{A}/\mathcal{J}$ as above. Then $\mathcal{K} \otimes \mathcal{O}_{A_{2n+1}} \rtimes \mathbb{Z}_2 \cong \overline{\mathcal{F}}_{A_{2n+1}} \rtimes \mathbb{Z} \rtimes \mathbb{Z}_2$. Since the crossed product $\overline{\mathcal{F}}_{A_{2n+1}} \rtimes \mathbb{Z}_2$ of the AF algebra $\overline{\mathcal{F}}_{A_{2n+1}}$ is isomorphic to $\overline{\mathcal{F}}_{D_{n+2}}$, the AF algebra for D_{n+2}, which is the $\mathbb{Z}_2$-orbifold of A_{2n+1}, and the $\mathbb{Z}$ and $\mathbb{Z}_2$ actions commute, $\mathcal{K} \otimes \mathcal{O}_{A_{2n+1}} \rtimes \mathbb{Z}_2 \cong \overline{\mathcal{F}}_{A_{2n+1}} \rtimes \mathbb{Z}_2 \rtimes \mathbb{Z} \cong \overline{\mathcal{F}}_{D_{n+2}} \rtimes \mathbb{Z} \cong \mathcal{K} \otimes \mathcal{O}_{D_{n+2}}$. The reverse statement follows similarly. $\square$

Chapter 3

The K-Theory of Cuntz-Krieger Algebras of the $SU(N)$-Models

For the K-theory of the Cuntz-Krieger algebras $\mathcal{O}_\mathcal{G}$ where $\mathcal{G}$ is one of the Dynkin diagrams we completely derive $(K_0(\mathcal{O}_\mathcal{G}), [\mathbf{1}])$ and compute its K_1 group. For the $SU(3)$ $\mathcal{ADE}$ graphs, we compute $K_0(\mathcal{O}_\mathcal{G})$ for the graphs $\mathcal{A}^{(n)}$, $\mathcal{D}^{(n)}$, $n \leq 20$, the exceptional $\mathcal{E}$ graphs, and the 01-parts $\mathcal{G}_{01}$ of these graphs. We also compute their K_1 groups explicitly.

K-theory provides invariants which may be used to classify C^*-algebras. The classification of C^*-algebras began with the classification of AF algebras by Elliott in [30] in terms of the ordered group K_0. Murray-von Neumann equivalence on projections in a unital C^*-algebra A is given by the equivalence relation e f if there exists an element $v \in A$ such that $e = v^*v$ and $f = vv^*$. Let $D(A)$ denotes the space of equivalence classes of projections in A. For the matrix algebra $M_n(A)$ over A, there is a $*$-homomorphism ψ_n which embeds $M_n(A)$ in $M_{n+1}(A)$ given by $\psi_n(a) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$. Then there is an induced map $\psi_n : D(M_n(A)) \to D(M_{n+1}(A))$, and we let $D_\infty(A) = \lim_\to (D(M_n(A)), \psi_n)$. Then $K_0(A)$ is the enveloping group of the semigroup $D_\infty(A)$, which is formed in the same was as going from the semigroup $S = \{0, 1, \ldots, N\}$ to $\mathbb{Z}$ by taking the equivalence classes of differences in S. The K-group $K_1(A)$ is defined similarly in terms of unitaries rather than projections: The unitary group $U_n(A)$ of $M_n(A)$ is embedded in $U_{n+1}(A)$ by $u \to \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$, and we let $U_\infty(A) = \lim_\to U_n(A)$. Then $K_1(A) = U_\infty(A)/U_\infty(A)_0$ where $U_\infty(A)_0$ is the connected component of the identity.

It is well known from [21] that for an $n \times n$ matrix A, $K_0(\mathcal{O}_A) = \mathbb{Z}^n/(\mathbf{1} - A^T)\mathbb{Z}^n$, whilst $K_1(\mathcal{O}_A) = \mathrm{Ker}(\mathbf{1} - A^T) = \{v \in \mathbb{Z}^n | \ (\mathbf{1} - A^T)v = 0\}$, i.e. $K_1(\mathcal{O}_A) = \mathbb{Z}^p$, where p is the multiplicity of 1 as an eigenvalue of A.

3.1 $SU(2)$

By a classification theorem of Rørdam [106, Theorem 6.5], two simple Cuntz-Krieger algebras $\mathcal{O}_A$ and $\mathcal{O}_{A'}$ are stably isomorphic , i.e. $\mathcal{K}\otimes\mathcal{O}_A \cong \mathcal{K}\otimes\mathcal{O}_{A'}$, if and only if $K_0(\mathcal{O}_A)$ is isomorphic to $K_0(\mathcal{O}_{A'})$, and $\mathcal{O}_A$ is isomorphic to $\mathcal{O}_{A'}$ if and only if $(K_0(\mathcal{O}_A),[\mathbf{1}])$ and $(K_0(\mathcal{O}_{A'}),[\mathbf{1}])$ are isomorphic. The K-theory of the Cuntz-Krieger algebras $\mathcal{O}_\mathcal{G}$ where $\mathcal{G}$ is one of the Dynkin diagrams was given by Izumi in [59]. However, in order to compute the position of the unit $[\mathbf{1}]$ we completely derive $(K_0(\mathcal{O}_\mathcal{G}),[\mathbf{1}])$ here.

3.1.1 Dynkin Diagram A_n

The $n\times n$ adjacency matrix Δ_{A_n} of A_n is given by

$$\Delta_{A_n} = \begin{pmatrix} 0 & 1 & 0 & \cdots & & 0 \\ 1 & 0 & 1 & & & \\ 0 & 1 & 0 & 1 & & \\ \vdots & & 1 & \ddots & \ddots & \\ & & & \ddots & 0 & 1 \\ 0 & & & & 1 & 0 \end{pmatrix}. \tag{3.1}$$

We will use the notation $\Lambda_n = \mathbf{1}-\Delta_{A_n}^T$ $(=\mathbf{1}-\Delta_{A_n})$.

First we consider $n=2$: Let $w=(w_i)\in\mathbb{Z}^2$ be a column vector. Since

$$\Lambda_2 w = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} w = (w_1-w_2,-w_1+w_2)^T,$$

the space $\Lambda_2\mathbb{Z}^2$ is one-dimensional, generated by the single vector, $(-1,1)^T$. Then $\mathbb{Z}^2/\Lambda_2\mathbb{Z}^2$ can be described as the space of equivalence classes of $\mathbb{Z}^2$, with equivalence relation defined by $v\sim w$ if $v_1+v_2=w_1+w_2$, for $v=(v_i)$, $w=(w_i)$. There is an isomorphism $\nu:\mathbb{Z}^2/\Lambda_2\mathbb{Z}^2 \longrightarrow \mathbb{Z}$ given by $\nu\left((\alpha_1,\alpha_2)^T+\Lambda_2\mathbb{Z}^2\right)=\alpha_1+\alpha_2$.

The equivalence class $[\mathbf{1}]$ of the identity is $[\mathbf{1}]=\pi\left((1,1)^T\right)=\left((1+k,1-k)^T\right)+\Lambda_2\mathbb{Z}^2$, where π is the quotient map $\pi:\mathbb{Z}^2\longrightarrow\mathbb{Z}^2/\Lambda_2\mathbb{Z}^2$, which is mapped to $(1+k)+(1-k)=2$ in $\mathbb{Z}$. So for A_2, $(K_0(\mathcal{O}_{A_2}),[\mathbf{1}])=(\mathbb{Z},2)$.

Next we consider $n=3$. We have

$$\Lambda_3 = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

Then

$$\Lambda_3\mathbb{Z}^3 = \left\{(w_1 - w_2, -w_1 + w_2 - w_3, -w_2 + w_3)^T \middle|\, w_1, w_2, w_3 \in \mathbb{Z}\right\}.$$

By the successive changes of variables $w_1 - w_2 \longrightarrow a_1$ and $-w_2 + w_3 \longrightarrow a_3$, then $-a_1 - a_3 + w_2 \longrightarrow a_2$ we can rewrite $\Lambda_3\mathbb{Z}^3$ as $\Lambda_3\mathbb{Z}^3 = \left\{(a_1, a_2, a_3)^T \middle|\, a_1, a_2, a_3 \in \mathbb{Z}\right\} = \mathbb{Z}^3$. So $\mathbb{Z}^3/\Lambda_3\mathbb{Z}^3 = \mathbb{Z}^3/\mathbb{Z}^3 \cong 0$, and the equivalence class of the identity is trivial, $[\mathbf{1}] = 0$. Then we have, for A_3, $(K_0(\mathcal{O}_{A_3}), [\mathbf{1}]) = (0, 0)$. The same holds for A_4.

We now move to the case when $n \geq 5$.

$$\Lambda_n\mathbb{Z}^n = \left\{(w_1 - w_2, -w_1 + w_2 - w_3, -w_2 + w_3 - w_4, \ldots, -w_{n-1} + w_n)^T \middle|\, w_i \in \mathbb{Z}\right\}.$$

By the changes of variables $w_1 - w_2 \longrightarrow a_1$, $a_1 - w_3 \longrightarrow a_2$ and $-w_2 - w_4 + a_1 - a_2 \longrightarrow a_3$, we can rewrite $\Lambda_n\mathbb{Z}^n$ as

$$\Lambda_n\mathbb{Z}^n = \left\{(a_1, a_2, a_3, w_4 - w_5 - a_1 + a_2, -w_4 + w_5 - w_6, \ldots, -w_{n-1} + w_n)^T \middle|\, a_i, w_j \in \mathbb{Z}\right\}.$$

So we can see that a vector $w \in \Lambda_n\mathbb{Z}^n$ can be written as $w = u \oplus u' \in \mathbb{Z}^3 \oplus \Lambda'_{n-3}\mathbb{Z}^{n-3}$, where $\Lambda'_{n-3}\mathbb{Z}^{n-3}$ is the space of vectors $u' = \widetilde{u}^{(0)} + \widetilde{u}^{(1)}$, where $\widetilde{u}^{(0)}$ is any vector in $\Lambda_{n-3}\mathbb{Z}^{n-3}$, and $\widetilde{u}^{(1)} = (\widetilde{u}_i^{(1)}) \in \mathbb{Z}^{n-3}$ is the vector with $\widetilde{u}_1^{(1)} = -u_1 + u_2$ and $\widetilde{u}_j^{(1)} = 0$ for all other j. Since $\widetilde{u}^{(1)}$ is fixed by the choice of $u = (u_i) \in \mathbb{Z}^3$, u' relates $\Lambda'_{n-3}\mathbb{Z}^{n-3}$ and $\Lambda_{n-3}\mathbb{Z}^{n-3}$ as isomorphic vector spaces. Then

$$\begin{aligned}
\mathbb{Z}^n/\Lambda_n\mathbb{Z}^n &= \mathbb{Z}^n/\left(\mathbb{Z}^3 \oplus \Lambda'_{n-3}\mathbb{Z}^{n-3}\right) \cong \mathbb{Z}^3/\mathbb{Z}^3 \oplus \mathbb{Z}^{n-3}/\Lambda'_{n-3}\mathbb{Z}^{n-3} \\
&\cong \mathbb{Z}^{n-3}/\Lambda'_{n-3}\mathbb{Z}^{n-3} \cong \mathbb{Z}^{n-3}/\Lambda_{n-3}\mathbb{Z}^{n-3}.
\end{aligned}$$

When $n = 5$, $\Lambda'_{n-3}\mathbb{Z}^{n-3} = \Lambda'_2\mathbb{Z}^2 = \left\{(-a - u_1 + u_2, a)^T \middle|\, a \in \mathbb{Z}\right\}$. Since u_1, u_2 are fixed by the choice of u, in a similar way to the case when $n = 2$ above, there is an isomorphism $\mu : \mathbb{Z}^2/\Lambda'_2\mathbb{Z}^2 \longrightarrow \mathbb{Z}$ given by $\mu\left((\alpha_1, \alpha_2)^T + \Lambda'_2\mathbb{Z}^2\right) = \alpha_1 + \alpha_2$. This extends to an isomorphism $\nu : \mathbb{Z}^5/\Lambda_5\mathbb{Z}^5 \longrightarrow \mathbb{Z}$, given by

$$\nu\left(\pi\left(w\right)\right) = \nu\left((0, 0, 0, w_4 + w_1 - w_2 + k, w_5 - k)^T + \Lambda_5\mathbb{Z}^5\right) = w_5 + w_4 - w_2 + w_1,$$

where π is the quotient map $\pi : \mathbb{Z}^5 \longrightarrow \mathbb{Z}^5/\Lambda_5\mathbb{Z}^5$, $w = (w_i) \in \mathbb{Z}^5$.

When $n = 8$, $\Lambda'_5\mathbb{Z}^5 = \left\{(a_1 - u_1 + u_2, a_2, a_3, -k + a_1 + a_2, k)^T \middle|\, a_1, a_2, a_3, k \in \mathbb{Z}\right\}$. Performing the change of variable $a_1 - u_1 + u_2 \longrightarrow a'_1$, we get

$$\Lambda'_5\mathbb{Z}^5 = \left\{(a'_1, a_2, a_3, -k + a_2 - a'_1 - u_1 + u_2, k)^T \middle|\, a'_1, a_2, a_3, k \in \mathbb{Z}\right\}.$$

The isomorphism $\nu : \mathbb{Z}^8/\Lambda_8\mathbb{Z}^8 \longrightarrow \mathbb{Z}$ is now given by

$$\begin{aligned}
\nu\left(\pi\left(w\right)\right) &= \nu\left((0, \ldots, 0, w_7 + w_4 - w_5 + w_1 - w_2 + k, w_8 - k)^T + \Lambda_8\mathbb{Z}^8\right) \\
&= w_8 + w_7 - w_5 + w_4 - w_2 + w_1,
\end{aligned}$$

where π is again the quotient map.

Consider now $n \equiv 2 \bmod 3$, for $n \geq 8$. We have an isomorphism $\nu : \mathbb{Z}^n/\Lambda_n\mathbb{Z}^n \longrightarrow \mathbb{Z}$, given for $w = (w_i) \in \mathbb{Z}^n$ by

$$\begin{aligned}\nu(\pi(w)) &= w_n + w_{n-1} - w_{n-3} + w_{n-4} - w_{n-6} + w_{n-7} - w_{n-9} + w_{n-10} \\ &\quad - \ \ldots \quad - w_2 + w_1.\end{aligned}$$

The identity $\mathbf{1} = (1)_i$ is the column vector with 1 as every entry. Its equivalence class $[\mathbf{1}]$ in $\mathbb{Z}$ is again 2. So, for A_n, $n \equiv 2 \bmod 3$, $(K_0(\mathcal{O}_{A_n}), [\mathbf{1}]) = (\mathbb{Z}, 2)$.

For $n \not\equiv 2 \bmod 3, n > 5$, we have $\mathbb{Z}^n/\Lambda_n\mathbb{Z}^n \cong \mathbb{Z}^{n-3}/\Lambda_{n-3}\mathbb{Z}^{n-3} \cong 0$.

Summarizing, we have the following for A_n:

$$(K_0(\mathcal{O}_{A_n}), [\mathbf{1}]) = \begin{cases} (\mathbb{Z}, 2) & \text{if} \quad n \equiv 2 \bmod 3, \\ (0, 0) & \text{if} \quad n \not\equiv 2 \bmod 3. \end{cases} \tag{3.2}$$

Next we compute K_1 for $\mathcal{O}_{A_n}$. For the adjacency matrix Δ_{A_n} of A_n, the eigenvalues λ_n^k, $k = 1, \ldots, n$, are given by (1.12). Then $\lambda_n^k = 1$, for some k, if

$$\frac{k\pi}{n+1} = \frac{\pi}{3} + 2\pi r, \qquad r \in \mathbb{Z}. \tag{3.3}$$

Since $|k| \leq n$ and $k, n > 0$, the left hand side satisfies $0 < k\pi/(n+1) < \pi$. Then the right hand side of the equation must also lie within these bounds, giving the constraint $0 < (1 + 6r)/3 < 1$, so that $-1/6 < r < 1/3$. Then $r = 0$, and (3.3) becomes $3k\pi = (n+1)\pi$, and we get $k = (n+1)/3$. Here k is only an integer when $n = 3q + 2$, for $q \in \mathbb{Z}$, and hence $\lambda_n^k = 1$ only when $n \equiv 2 \bmod 3$ and has multiplicity one.

Hence we have the following result for A_n:

$$K_1(\mathcal{O}_{A_n}) = \begin{cases} \mathbb{Z} & \text{if} \quad n \equiv 2 \bmod 3, \\ 0 & \text{if} \quad n \not\equiv 2 \bmod 3. \end{cases}$$

3.1.2 Dynkin Diagram D_n

We first compute K_0 of $\mathcal{O}_{D_n}$. The $n \times n$ adjacency matrix Δ_{D_n} of D_n is given by

$$\Delta_{D_n} = \begin{pmatrix} 0 & 1 & 0 & & \cdots & & 0 \\ 1 & 0 & 1 & & & & \\ 0 & 1 & \ddots & \ddots & & & \\ \vdots & & \ddots & & 1 & & \\ & & & 1 & 0 & 1 & 1 \\ & & & & 1 & 0 & 0 \\ 0 & & & & 1 & 0 & 0 \end{pmatrix}, \tag{3.4}$$

and we will again denote by Λ_n the matrix $\Lambda_n = \mathbf{1} - \Delta_{D_n}^T$ $(= \mathbf{1} - \Delta_{D_n})$.

When $n = 4$,

$$\Lambda_4 = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & -1 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix},$$

and the vector space $\Lambda_4\mathbb{Z}^4$ is

$$\Lambda_4\mathbb{Z}^4 = \left\{(w_1 - w_2, -w_1 + w_2 - w_3 - w_4, -w_2 + w_3, -w_2 + w_4)^T \middle| \, w_1, \dots, w_4 \in \mathbb{Z}\right\},$$

which, by the change of variables $w_1 - w_2 \to a_1$, $-w_2 + w_3 \to a_3$, $-w_2 + w_4 \to a_4$ and $w_2 \to k$, $\Lambda_4\mathbb{Z}^4$ can be rewritten as

$$\Lambda_4\mathbb{Z}^4 = \left\{(a_1, -a_1 - a_3 - a_4 - 2k, a_3, a_4)^T \middle| \, a_1, a_3, a_4, k \in \mathbb{Z}\right\}.$$

Then the quotient map $\pi : \mathbb{Z}^4 \longrightarrow \mathbb{Z}^4/\Lambda_4\mathbb{Z}^4$ sends

$$v = (v_i) \longrightarrow (0, v_1 + v_2 + v_3 + v_4 + 2k, 0, 0)^T + \Lambda_4\mathbb{Z}^4,$$

where k is the unique integer such that $v_1 + v_2 + v_3 + v_4 + 2k \in \{0, 1\}$. Then there is an isomorphism $\nu : \mathbb{Z}^4/\Lambda_4\mathbb{Z}^4 \longrightarrow \mathbb{Z}_2$ given by $\nu\left((0, b, 0, 0)^T + \Lambda_4\mathbb{Z}^4\right) = b$, or, equivalently, for $w = (w_i) \in \mathbb{Z}^4$,

$$\nu(\pi(w)) = \begin{cases} 1 & \text{if } \sum_i w_i \text{ is odd,} \\ 0 & \text{if } \sum_i w_i \text{ is even.} \end{cases}$$

So the equivalence class $[\mathbf{1}]$ of the identity is mapped to 0 in $\mathbb{Z}_2$ for D_4. Then we have $(K_0(\mathcal{O}_{D_4}), [\mathbf{1}]) = (\mathbb{Z}_2, 0)$.

When $n = 5$, $\Lambda_5\mathbb{Z}^5$ is given by

$$\left\{(w_1 - w_2, -w_1 + w_2 - w_3, -w_2 + w_3 - w_4 - w_5, -w_3 + w_4, -w_3 + w_5)^T \middle| \, w_i \in \mathbb{Z}\right\}.$$

Performing the successive changes of variables $w_1 - w_2 \longrightarrow a_1$, $-a_1 - w_3 \longrightarrow a_2$, $w_4 + a_1 + a_2 \longrightarrow a_4$, $w_5 + a_1 + a_2 \longrightarrow a_5$, and lastly $-w_2 + a_1 + a_2 - a_4 - a_5 \longrightarrow a_3$, we find that $\Lambda_5\mathbb{Z}^5 = \left\{(a_1, a_2, a_3, a_4, a_5)^T \middle| \, a_1, \dots, a_5 \in \mathbb{Z}\right\} = \mathbb{Z}^5$, and $\mathbb{Z}^5/\Lambda_5\mathbb{Z}^5 = \mathbb{Z}^5/\mathbb{Z}^5 \cong 0$. So, for D_5, $(K_0(\mathcal{O}_{D_5}), [\mathbf{1}]) = (0, 0)$. The same holds for D_6.

Since, for $n \geq 7$, D_n is just the graph D_4 with the graph A_{n-4} added as a tail, then, in the same way as for A_n, we have $\Lambda_n\mathbb{Z}^n = \mathbb{Z}^3 \oplus \Lambda'_{n-3}\mathbb{Z}^{n-3}$, and $\Lambda'_{n-3}\mathbb{Z}^{n-3} \cong \Lambda_{n-3}\mathbb{Z}^{n-3}$.

When $n = 7$, $\Lambda'_4\mathbb{Z}^4 = \left\{(a_1 - u_1 + u_2, -a_1 - a_3 - a_4 - 2k, a_3, a_4)^T \middle| \, a_1, a_3, a_4, k \in \mathbb{Z}\right\}$. Performing the change of variable $a_1 - u_1 + u_2 \longrightarrow a'_1$, we get

$$\Lambda'_4\mathbb{Z}^4 = \left\{(a'_1, -a'_1 - a_3 - a_4 - u_1 + u_2 - 2k, a_3, a_4)^T \middle| \, a'_1, a_3, a_4, k \in \mathbb{Z}\right\}.$$

Then the quotient map $\pi : \mathbb{Z}^7 \longrightarrow \mathbb{Z}^7/\Lambda_7\mathbb{Z}^7$ sends

$$v = (v_i) \longrightarrow (0,0,0,0,v_5+v_4+v_6+v_7+v_1-v_2+2k,0,0)^T + \Lambda_7\mathbb{Z}^7,$$

where, again, $k \in \mathbb{Z}$ such that $v_5+v_4+v_6+v_7+v_1-v_2+2k \in \{0,1\}$. Then there is an isomorphism $\nu : \mathbb{Z}^7/\Lambda_7\mathbb{Z}^7 \longrightarrow \mathbb{Z}_2$, given, for $w = (w_i) \in \mathbb{Z}^7$, by

$$\nu(\pi(w)) = \begin{cases} 1 & \text{if } w_7+w_6+w_5+w_4-w_2+w_1 \text{ is odd,} \\ 0 & \text{if } w_7+w_6+w_5+w_4-w_2+w_1 \text{ is even.} \end{cases}$$

For general $n \equiv 1 \bmod 3$, $n \geq 7$, the isomorphism $\nu : \mathbb{Z}^n/\Lambda_n\mathbb{Z}^n \longrightarrow \mathbb{Z}_2$, is given, for $w = (w_i) \in \mathbb{Z}^n$, by

$$\nu(\pi(w)) = \begin{cases} 1 & \text{if } \widetilde{w} \text{ is odd,} \\ 0 & \text{if } \widetilde{w} \text{ is even,} \end{cases}$$

where

$$\begin{aligned} \widetilde{w} &= w_n + w_{n-1} + w_{n-2} + w_{n-3} - w_{n-5} + w_{n-6} - w_{n-8} + w_{n-9} \\ &\quad - \ \dots \quad - w_2 + w_1. \end{aligned}$$

The sum $\widetilde{w}$ is even when w is the identity $\mathbf{1}$, so the equivalence class of the identity is always 0. Then, for all $n \equiv 1 \bmod 3$, $(K_0(\mathcal{O}_{D_n}), [\mathbf{1}]) = (\mathbb{Z}_2, 0)$.

When $n \not\equiv 1 \bmod 3, n > 7$, $\mathbb{Z}^n/\Lambda_n\mathbb{Z}^n \cong \mathbb{Z}^{n-3}/\Lambda_{n-3}\mathbb{Z}^{n-3} \cong 0$.

Summarizing, we have the following for D_n:

$$(K_0(\mathcal{O}_{D_n}), [\mathbf{1}]) = \begin{cases} (\mathbb{Z}_2, 0) & \text{if} \quad n \equiv 1 \bmod 3, \\ (0,0) & \text{if} \quad n \not\equiv 1 \bmod 3. \end{cases} \tag{3.5}$$

We now compute K_1 for $\mathcal{O}_{D_n}$. The eigenvalues λ_n^k, $k = 0, 1, \dots, n-1$, of the adjacency matrix Δ_{D_n} are given by (1.13). Then $\lambda_n^k = 1$ for some k if

$$\frac{(2k+1)\pi}{2n-2} = \frac{\pi}{3} + 2\pi r, \qquad r \in \mathbb{Z}.$$

Using the same argument as for A_n we find that $r = 0$, and we find that k must satisfy $k = (2n-5)/6$. For k to be an integer we require that $n = 3q + 5/2$, for some $q \in \mathbb{Z}$, so n cannot be an integer. Then 1 is never an eigenvalue for any n. Hence, for D_n, $K_1(\mathcal{O}_{D_n}) = 0$, for all $n \geq 4$.

Since $\mathcal{O}_{A_{2n+1}}$ is a subalgebra of $\mathcal{O}_{A_{2n+1}} \rtimes \mathbb{Z}_2$, there is a map $\mathcal{O}_{A_{2n+1}} \hookrightarrow \mathcal{O}_{A_{2n+1}} \rtimes \mathbb{Z}_2$. As a result of Proposition 2.5.1, this should give a map $\mathcal{O}_{A_{2n+1}} \hookrightarrow \mathcal{O}_{D_{n+2}}$, and $K_0(\mathcal{O}_{A_{2n+1}})$ maps into $K_0(\mathcal{O}_{D_{n+2}})$. By the results (3.2), (3.5) we have $K_0(\mathcal{O}_{A_{2n+1}}) = \mathbb{Z}$ and $K_0(\mathcal{O}_{D_{n+2}}) = \mathbb{Z}_2$, so that this claim is true. Similarly, there is a map $K_0(\mathcal{O}_{D_{n+2}}) \hookrightarrow K_0(\mathcal{O}_{A_{2n+1}})$. However, we see that the unit $[1] = 0$ in $K_0(\mathcal{O}_{D_{n+2}})$ does not map to the unit $[1] = 2$ in $K_0(\mathcal{O}_{A_{2n+1}})$, hence the algebras $\mathcal{O}_{A_{2n+1}} \rtimes \mathbb{Z}_2$ and $\mathcal{O}_{D_{n+2}}$ are only isomorphic when we tensor with the compact operators. We expect similar results for the K-theory of the algebras $\mathcal{O}_{\mathcal{A}^{(n)}}$ and $\mathcal{O}_{\mathcal{D}^{(n)}}$, since the graph $\mathcal{D}^{(n)}$ is a $\mathbb{Z}_3$-orbifold of the graph $\mathcal{A}^{(n)}$, and vice versa, and hence we expect that $\mathcal{K} \otimes \mathcal{O}_{\mathcal{A}^{(n)}} \rtimes \mathbb{Z}_3 \cong \mathcal{K} \otimes \mathcal{O}_{\mathcal{D}^{(n)}}$ and $\mathcal{K} \otimes \mathcal{O}_{\mathcal{D}^{(n)}} \rtimes \mathbb{Z}_3 \cong \mathcal{K} \otimes \mathcal{O}_{\mathcal{A}^{(n)}}$. There should then be maps $K_0(\mathcal{O}_{\mathcal{A}^{(n)}}) \to K_0(\mathcal{O}_{\mathcal{D}^{(n)}})$ and $K_0(\mathcal{O}_{\mathcal{D}^{(n)}}) \to K_0(\mathcal{O}_{\mathcal{A}^{(n)}})$.

3.1.3 Exceptional Dynkin Diagram E_6, E_7 and E_8

For E_6, the adjacency matrix Δ_{E_6} is given by

$$\Delta_{E_6} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}. \tag{3.6}$$

We let $\Lambda = \mathbf{1} - \Delta_{E_6}^T$. Then

$$\Lambda\mathbb{Z}^6 = \{(w_1 - w_2, -w_1 + w_2 - w_3, -w_2 + w_3 - w_4 - w_6, -w_3 + w_4 - w_5, -w_4 + w_5, -w_3 + w_6)^T | w_i \in \mathbb{Z}\}.$$

By the successive changes of variables $w_1 - w_2 \to a_1$, $-a_1 - w_3 \to a_2$, $w_5 - w_4 \to a_5$, $a_1 + a_2 + w_6 \to a_6$ and $-w_2 - w_4 - a_6 \to a_3$, we can write $\Lambda\mathbb{Z}^6$ as

$$\Lambda\mathbb{Z}^6 = \left\{(a_1, a_2, a_3, a_1 + a_2 - a_5, a_5, a_6)^T | a_i \in \mathbb{Z}\right\}.$$

Then the quotient map $\pi : \mathbb{Z}^6 \to \mathbb{Z}^6/\Lambda\mathbb{Z}^6$ sends

$$v = (v_i) \to (0, 0, 0, v_4 - v_1 - v_2 + v_5, 0, 0)^T + \Lambda\mathbb{Z}^6,$$

and there is an isomorphism $\nu : \mathbb{Z}^6/\Lambda\mathbb{Z}^6 \to \mathbb{Z}$ given, for $w = (w_i) \in \mathbb{Z}^6$, by

$$\nu(\pi(w)) = \nu((0, 0, 0, v_4 - v_1 - v_2 + v_5, 0, 0)^T + \Lambda\mathbb{Z}^6) = v_4 - v_1 - v_2 + v_5.$$

Then for the identity $\mathbf{1} = (1)_i \in \mathbb{Z}^6$, $\nu(\pi(\mathbf{1})) = 1 - 1 - 1 + 1 = 0$. Then for E_6 we have

$$(K_0(\mathcal{O}_{E_6}), [\mathbf{1}]) = (\mathbb{Z}, 0).$$

Next we compute K_1 for $\mathcal{O}_{E_6}$. From (1.13) we see that the adjacency matrix Δ_{E_6} of E_6 has the eigenvalue 1 once, and hence $K_1(\mathcal{O}_{E_6}) = \mathbb{Z}$.

For $\mathcal{G}$ either of the Dynkin diagrams E_7 or E_8, $K_0(\mathcal{O}_{\mathcal{G}}) = 0$ [59], and hence we have $(K_0(\mathcal{O}_{E_7}), [\mathbf{1}]) = (K_0(\mathcal{O}_{E_8}), [\mathbf{1}]) = (0, 0)$, and $(K_1(\mathcal{O}_{E_7}) = (K_1(\mathcal{O}_{E_8}) = 0$.

3.2 K-Theory for $\mathcal{O}_{\mathcal{G}}$ where $\mathcal{G}$ is an $SU(3)$ $\mathcal{ADE}$ graph.

To obtain the following results for $K_0(\mathcal{O}_{\mathcal{G}})$ where $\mathcal{G}$ is an $SU(3)$ $\mathcal{ADE}$ graph we wrote a code in Visual Basic to reduce the matrix $\mathbf{1} - \Delta_{\mathcal{G}}$ to Smith normal form- a diagonal form where the diagonal elements are the elementary divisors of the matrix. This is achieved by using the following elementary (determinant) row and column operations:

1. Add a multiple of one row to another row, or a multiple of one column to another column
2. Interchange two rows or two columns, and multiply one of them by a factor -1.

If $a_1, a_2, \ldots, a_k$ are the (moduli of the) elementary divisors that appear along the diagonal, then $K_0(\mathcal{O}_{\mathcal{G}}) = \mathbb{Z}_{a_1} \oplus \ldots \oplus \mathbb{Z}_{a_k}$. If a has prime decomposition $a = p_1^{c_1} p_2^{c_2} \cdots p_m^{c_m}$ then $\mathbb{Z}_a \cong \mathbb{Z}_{q_1} \oplus \mathbb{Z}_{q_2} \oplus \cdots \oplus \mathbb{Z}_{q_m}$, where $q_i = p_i^{c_i}$. We list the prime decomposition of the integers a_j in the following tables. The computations of the K_1 groups are shown in Section 3.2.6.

3.2.1 $\mathcal{A}$ graphs

n	$a_1, \ldots, a_k$	$K_1(\mathcal{O}_{\mathcal{A}^{(n)}})$
4	0	$\mathbb{Z}$
5	2, 2	0
6	7	0
7	2^2, 13	0
8	0, 2, 3^2, 7	$\mathbb{Z}$
9	37, 109	0
10	2, 2, 2^4, 2^4, 41	0
11	67, 109, 199	0
12	0, 3^2, 3^2, 5, 7, 37, 109	$\mathbb{Z}$
13	3^2, 5, 5, 13, 13, 13, 13, 433	0
14	2, 2^4, 13, 43, 43, 239, 757	0
15	2, 2, 61, 241, 271, 271, 5851	0
16	0, 2, 2, 2, 2^3, 3^2, 7, 7, 17, 17, 79, 241, 4561	$\mathbb{Z}$
17	103, 137, 919, 2857, 4591, 6971	0
18	17, 19, 19, 37, 109, 199, 271, 11719, 44281	0
19	7, 7, 7, 37, 229, 419, 15581, 77863, 175447	0
20	0, 2, 2, 2, 2, 2^3, 2^3, 2^6, 2^6, 5, 5, 5^2, 11, 11, 19, 29, 41, 41, 61, 241, 1321	$\mathbb{Z}$

01-part of $\mathcal{A}$

n	$K_0(\mathcal{O}_{\mathcal{A}_{01}^{(n)}})$	$K_1(\mathcal{O}_{\mathcal{A}_{01}^{(n)}})$
4	$\mathbb{Z}$	
5	0	0
6	$\mathbb{Z}^2$	
7	0	0
8	$\mathbb{Z}^3$	
9	0	0
10	$\mathbb{Z}^4$	
11	0	0
12	$\mathbb{Z}^5$	
13	0	0
14	$\mathbb{Z}^6$	

3.2.2 $\mathcal{D}$ graphs

n	$a_1, \ldots, a_k$	$K_1(\mathcal{O}_{\mathcal{D}^{(n)}})$
5	1	0
6	7	0
7	2^2	0
8	0, 2, 3	$\mathbb{Z}$
9	37	0
10	41	0
11	67	0
12	0, 3, 73	$\mathbb{Z}$
13	3, 241	0
14	2, 2^4, 239	0
15	2, 2, 61, 241	0
16	0, 2, 2, 2, 2^3, 3, 17, 17	$\mathbb{Z}$
17	137, 6971	0
18	7, 19, 19, 37, 199	0
19	9, 4613701	0
20	0, 5, 5, 5, 5^2, 29, 41, 41	$\mathbb{Z}$
21	2, 2, 2, 2^2, 13, 43, 421, 1933	0
22	2^4, 23^2, 31, 199, 1163	0
23	47, 47, 47, 1657, 5521	0
24	0, 2, 3, 3, 3, 3^2, 3^2, 5, 5, 5, 11, 97, 337	$\mathbb{Z}$
27	3^3, 5, 13, 2789321, 4353169	0

3.2.3 $\mathcal{A}^*$ graphs

n	$K_0(\mathcal{O}_{\mathcal{A}^{(n)*}})$	$K_1(\mathcal{O}_{\mathcal{A}^{(n)*}})$
$n \equiv 0 \bmod 4$	$\mathbb{Z}$	$\mathbb{Z}$
$n \not\equiv 0 \bmod 4$	0	0

3.2.4 $\mathcal{D}^*$ graphs

n	$a_1, \ldots, a_k$	$K_1(\mathcal{O}_{\mathcal{D}^{(n)*}})$
6	7	0
7	13	0
8	0, 7	$\mathbb{Z}$
9	37	0
10	2^3, 2^3	0
11	109	0
12	0, 3, 3, 7	$\mathbb{Z}$
13	5, 5, 13	0
14	13, 41	0
15	2, 2, 241	0
16	0, 7, 79	$\mathbb{Z}$
17	2857	0
18	7, 19, 37	0
19	37, 229	0
20	0, 2^4, 2^4, 19	$\mathbb{Z}$
21	13, 1933	0
22	109, 397	0
23	74521	0
24	0, 3, 3, 7, 7, 97	$\mathbb{Z}$

01-part of $\mathcal{D}^*$

n	$K_0(\mathcal{O}_{\mathcal{D}_{01}^{(n)*}})$	$K_1(\mathcal{O}_{\mathcal{D}_{01}^{(n)*}})$
$2k+1$	0	0
$4k$	$\mathbb{Z}_k \oplus \mathbb{Z}$	$\mathbb{Z}$
$4k+2$	$\mathbb{Z}_k$	0

3.2.5 $\mathcal{E}$ graphs

$\mathcal{E}$	$K_0(\mathcal{O}_{\mathcal{E}})$	$K_1(\mathcal{O}_{\mathcal{E}})$
$\mathcal{E}^{(8)}$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_7$	0
$\mathcal{E}_1^{(12)}$	$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$	$\mathbb{Z}^2$
$\mathcal{E}_2^{(12)}$	$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$	$\mathbb{Z}^2$
$\mathcal{E}_3^{(12)}$	$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$	$\mathbb{Z}^2$
$\mathcal{E}_4^{(12)}$	$\mathbb{Z} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$	$\mathbb{Z}$
$\mathcal{E}_5^{(12)}$	$\mathbb{Z} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$	$\mathbb{Z}$
$\mathcal{E}^{(24)}$	$\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{97}$	0

01-part of $\mathcal{E}$

$\mathcal{E}$	$K_0(\mathcal{O}_{\mathcal{E}_{01}})$	$K_1(\mathcal{O}_{\mathcal{E}_{01}})$
$\mathcal{E}^{(8)}$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
$\mathcal{E}_1^{(12)}$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
$\mathcal{E}_2^{(12)}$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
$\mathcal{E}_3^{(12)}$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
$\mathcal{E}_4^{(12)}$	$\mathbb{Z}$	$\mathbb{Z}$
$\mathcal{E}_5^{(12)}$	$\mathbb{Z}$	$\mathbb{Z}$
$\mathcal{E}^{(24)}$	0	0

3.2.6 $K_1(\mathcal{O}_{\mathcal{G}})$

We begin by computing $K_1(\mathcal{O}_{\mathcal{G}})$ for the graphs $\mathcal{A}^{(n)}$:

Lemma 3.2.1

$$K_1(\mathcal{O}_{\mathcal{A}^{(n)}}) = \begin{cases} \mathbb{Z} & if \quad n \equiv 0 \ mod\ 4, \\ 0 & if \quad n \not\equiv 0 \ mod\ 4. \end{cases}$$

Proof

The eigenvalues of the adjacency matrix Δ_n of $\mathcal{A}^{(n)}$ are given by (1.26). Then $\beta^{(\rho)} = 1$ yields the following equations:

$$\cos\left(\frac{2\pi}{3n}(\rho_1 + 2\rho_2)\right) + \cos\left(\frac{2\pi}{3n}(2\rho_1 + \rho_2)\right) + \cos\left(\frac{2\pi}{3n}(\rho_1 - \rho_2)\right) = 1, \qquad (3.7)$$

$$\sin\left(\frac{2\pi}{3n}(\rho_1 + 2\rho_2)\right) - \sin\left(\frac{2\pi}{3n}(2\rho_1 + \rho_2)\right) + \sin\left(\frac{2\pi}{3n}(\rho_1 - \rho_2)\right) = 0. \qquad (3.8)$$

Using trigonometry sum-to-product identities we can rewrite equation (3.8) in the following three ways:

$$\begin{aligned}
\sin\left(\frac{2\pi}{3n}(\rho_1-\rho_2)\right) - 2\sin\left(\frac{\pi}{3n}(\rho_1-\rho_2)\right)\cos\left(\frac{\pi}{n}(\rho_1+\rho_2)\right) &= 0, && (3.9)\\
-\sin\left(\frac{2\pi}{3n}(2\rho_1+\rho_2)\right) + 2\sin\left(\frac{\pi}{3n}(2\rho_1+\rho_2)\right)\cos\left(\frac{\pi}{n}\rho_2\right) &= 0, && (3.10)\\
\sin\left(\frac{2\pi}{3n}(\rho_1+2\rho_2)\right) - 2\sin\left(\frac{\pi}{3n}(\rho_1+2\rho_2)\right)\cos\left(\frac{\pi}{n}\rho_1\right) &= 0. && (3.11)
\end{aligned}$$

From equations (3.10), (3.11) we see that if $\beta^{(\rho)} = 1$, then $\beta^{(\overline{\rho})} = 1$, where $\overline{\rho} = (\rho_2, \rho_1)$. Using the double angle formula, we can write (3.9) as

$$2\sin\left(\frac{\pi}{3n}(\rho_1-\rho_2)\right)\left[\cos\left(\frac{\pi}{3n}(\rho_1-\rho_2)\right) - \cos\left(\frac{\pi}{n}(\rho_1+\rho_2)\right)\right] = 0.$$

We have two cases:

Case (1): $\cos(\pi(\rho_1-\rho_2)/3n) - \cos(\pi(\rho_1+\rho_2)/n) = 0$.

Then $\pi(\rho_1-\rho_2)/3n = \pm\pi(\rho_1+\rho_2)/n - 2\pi r$, $r \in \mathbb{Z}$. For the positive case we have

$$\begin{aligned}
\rho_1-\rho_2 &= 3\rho_1+3\rho_2-6rn\\
6rn &= 2\rho_1+4\rho_2.
\end{aligned}$$

Since the R.H.S. is positive, the L.H.S. must also be positive, so we have $r \in \mathbb{N}$, and using the restriction to the Weyl Alcove $\rho_1+\rho_2 \le n-1$, we find $3rn - 2\rho_2 + \rho_2 \le n-1$, $3rn - n + 1 \le \rho_2$. Since $\rho_2 \le n-2$, we get $3r \le 2 - 3/n \le 2$. But $r \in \mathbb{N}$ so we have a contradiction. We arrive at a similar contradiction when we consider $\pi(\rho_1-\rho_2)/3n = -\pi(\rho_1+\rho_2)/n - 2\pi r$.

Case (2): $\sin(\pi(\rho_1-\rho_2)/3n) = 0$. Then $\rho_1-\rho_2 = 3an$ for some $a \in \mathbb{Z}$. If $a > 0$ we have $\rho_1 = 3an + \rho_2 > n$, whilst for $a < 0$ we have $\rho_2 = \rho_1 - 3an > n$. So we must have $a = 0$ and $\rho_1 = \rho_2$. Putting $\rho_1 = \rho_2$ back into equation (3.7) gives $0 = 0$, and equation (3.8) becomes $2\cos(2\pi\rho_1/n) + 1 = 1$. Solving $\cos(2\pi\rho_1/n) = 0$ gives $2\pi\rho_1/n = \pi/2 + b\pi$, $b \in \mathbb{Z}$, so $\rho_1 = n(2b+1)/4$. Since $\rho_1, n > 0$, b must be non-negative, and either n or $(2b+1)$ must equal $4k$ for some positive integer k. The latter is impossible since $b = 2k - 1/2$ cannot be an integer for any $k \in \mathbb{N}$. So for 1 to be an eigenvalue, we must have $n \equiv 0 \bmod 4$.

We now show that the eigenvalue 1 only occurs once. Let $n = 4k$, and $\rho_1 = \rho_2$. Solving equation (3.7) gives $\pi\rho_1/2k = \pi/2 + \pi m$, for some $m \in \mathbb{Z}$, so

$$\rho_1 = k(2m+1). \qquad (3.12)$$

Now $m \geq 0$ since $\rho_1, k > 0$, so $\rho_1 + \rho_2 \leq n - 1$ gives

$$\rho_1 \leq 2k - \frac{1}{2}. \tag{3.13}$$

Combining equations (3.12) and (3.13) we get $2m \leq 1 - 1/2k \leq 1$. Since $m \geq 0$ must be an integer, $m = 0$. Then $\rho_1 = k(2m+1) = k$, and the only vertex for which 1 is an eigenvalue is $\rho = (\rho_1, \rho_1) = (k, k)$. $\square$

Corollary 3.2.2 *(i) For* $n = 5, 6, \dots$,

$$K_1(\mathcal{O}_{\mathcal{D}^{(n)}}) = \begin{cases} \mathbb{Z} & \text{if} \quad n \equiv 0 \ mod\ 4, \\ 0 & \text{if} \quad n \not\equiv 0 \ mod\ 4, \end{cases}$$

(ii) For $n = 5, 6, \dots$,

$$K_1(\mathcal{O}_{\mathcal{A}^{(n)*}}) = \begin{cases} \mathbb{Z} & \text{if} \quad n \equiv 0 \ mod\ 4, \\ 0 & \text{if} \quad n \not\equiv 0 \ mod\ 4, \end{cases}$$

(iii) For $n = 5, 6, \dots$,

$$K_1(\mathcal{O}_{\mathcal{D}^{(n)*}}) = \begin{cases} \mathbb{Z} & \text{if} \quad n \equiv 0 \ mod\ 4, \\ 0 & \text{if} \quad n \not\equiv 0 \ mod\ 4, \end{cases}$$

(iv) $K_1(\mathcal{O}_{\mathcal{E}^{(8)}}) = K_1(\mathcal{O}_{\mathcal{E}^{(24)}}) = 0$,

(v) $K_1(\mathcal{O}_{\mathcal{E}_1^{(12)}}) = K_1(\mathcal{O}_{\mathcal{E}_2^{(12)}}) = K_1(\mathcal{O}_{\mathcal{E}_3^{(12)}}) = \mathbb{Z}^2$,

(vi) $K_1(\mathcal{O}_{\mathcal{E}_4^{(12)}}) = K_1(\mathcal{O}_{\mathcal{E}_5^{(12)}}) = \mathbb{Z}$.

Proof. The results follow from the multiplicity of the exponent $(n/4, n/4)$ (which gives the eigenvalue 1) in the corresponding modular invariant for each graph, as given in (1.14)-(1.25). $\square$

Chapter 4

Ocneanu Cells and Boltzmann Weights

Here we will compute the numerical values of the Ocneanu cells, and consequently representations of the Hecke algebra, for the $\mathcal{ADE}$ graphs. However we have been unable thus far to compute the cells for the exceptional graph $\mathcal{E}_4^{(12)}$. For the graphs $\mathcal{D}^{(3k)}$, $k = 2, 3, \dots$, $\mathcal{D}^{(n)*}$, $n = 6, 7, \dots$, and $\mathcal{E}_1^{(12)}$ we compute solutions which satisfy some additional condition, but for the other graphs we compute all the Ocneanu cells, up to equivalence. The existence of these cells has been announced by Ocneanu, although the numerical values have remained unpublished. Some of the representations of the Hecke algebra have appeared in the literature and we compare our results.

For the $\mathcal{A}$ graphs, our solution for the Ocneanu cells W gives an identical representation of the Hecke algebra to that of Jimbo et al. [60] given in (4.15). Our cells for the $\mathcal{A}^{(n)*}$ graphs give equivalent Boltzmann weights to those given by Behrend and Evans in [4]. In [27], di Francesco and Zuber give a representation of the Hecke algebra for the graphs $\mathcal{D}^{(6)*}$ and $\mathcal{E}^{(8)}$, whilst in [108] a representation of the Hecke algebra is computed for the graphs $\mathcal{E}_1^{(12)}$ and $\mathcal{E}^{(24)}$. Our solutions for the cells W give an identical Hecke representation for $\mathcal{E}^{(8)}$ and an equivalent Hecke representation for $\mathcal{E}_1^{(12)}$. However, for $\mathcal{E}^{(24)}$, our cells give inequivalent Boltzmann weights. In [43], Fendley gives Boltzmann weights for $\mathcal{D}^{(6)}$ which are not equivalent to those we obtain, but which are equivalent if we take one of the weights in [43] to be the complex conjugate of what is given.

4.1 Ocneanu Cells

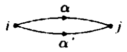
Let Γ be a subgroup of $SU(3)$ and denote by $\widehat{\Gamma}$ its irreducible representations. One can associate to Γ a graph $\mathcal{G}_\Gamma$ whose vertices are labelled by the irreducible representations of Γ, where for any pair of vertices $i, j \in \widehat{\Gamma}$ the number of edges from i to j are given by the the multiplicity of j in the decomposition of $i \otimes \rho$ into irreducible representations, where ρ is the fundamental irreducible representation of $SU(3)$, and which, along with its conjugate representation $\overline{\rho}$, generates $\widehat{SU(3)}$, the irreducible representations of $SU(3)$. The graph $\mathcal{G}_\Gamma$ of a subgroup Γ of $SU(3)$ or $SU(3)_n$ is made of triangles, corresponding to the fact that the fundamental representation ρ satisfies $\rho \otimes \rho \otimes \rho \ni \mathbf{1}$. For a graph $\mathcal{G}$, a triangle $\triangle_{ijk}^{(\alpha\beta\gamma)} = i \xrightarrow{\alpha} j \xrightarrow{\beta} k \xrightarrow{\gamma} i$ is a closed path of length 3 on $\mathcal{G}$, consisting of edges α, β, γ of $\mathcal{G}$ such that $s(\alpha) = r(\gamma) = i$, $s(\beta) = r(\alpha) = j$ and $s(\gamma) = r(\beta) = k$. For each triangle $\triangle_{ijk}^{(\alpha\beta\gamma)}$, the maps α, β and γ are composed:

$$i \xrightarrow{\mathrm{id}\otimes\det^*} i \otimes \rho \otimes \rho \otimes \rho \xrightarrow{\gamma\otimes\mathrm{id}} k \otimes \rho \otimes \rho \xrightarrow{\beta\otimes\mathrm{id}} j \otimes \rho \xrightarrow{\alpha\otimes\mathrm{id}} i,$$

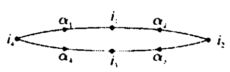
and since i is irreducible, the composition is a scalar. Then for every such triangle on $\mathcal{G}_\Gamma$ there is a complex number, called an Ocneanu cell. There is a gauge freedom on the cells, which comes from a unitary change of basis in $\mathrm{Hom}[i \otimes \rho, j]$ for every pair i, j.

Let $\mathcal{G}$ be one of the finite $\mathcal{ADE}$ graphs with Coxeter number n, let $q = e^{\pi i/n}$ so that the Perron-Frobenius eigenvalue of $\mathcal{G}$ is $[3]_q$. We will denote the quantum number $[n]_q$ simply by $[n]$, $n \in \mathbb{N}$. A type I frame in the graph $\mathcal{G}$ is a pair of edges α, α' which have the same start and end points. A type II frame is given by four edges α_i, $i = 1, 2, 3, 4$, such that $s(\alpha_1) = s(\alpha_4)$, $s(\alpha_2) = s(\alpha_3)$, $r(\alpha_1) = r(\alpha_2)$ and $r(\alpha_3) = r(\alpha_4)$.

Definition 4.1.1 ([94]) *A **cell system** W on $\mathcal{G}$ is a map that associates to each oriented triangle $\triangle_{ijk}^{(\alpha\beta\gamma)}$ in $\mathcal{G}$ a complex number $W\left(\triangle_{ijk}^{(\alpha\beta\gamma)}\right)$ with the following properties:*

(1) *for any type I frame*  *in $\mathcal{G}$ we have*

$$\sum_{k,\beta_1,\beta_2} W\left(\begin{smallmatrix} & k & \\ \beta_1 & & \beta_2 \\ j & \alpha & i \end{smallmatrix}\right) \overline{W\left(\begin{smallmatrix} & k & \\ \beta_1 & & \beta_2 \\ j & \alpha' & i \end{smallmatrix}\right)} = \delta_{\alpha,\alpha'} [2]\, \phi_i\, \phi_j \tag{4.1}$$

(2) *for any type II frame*  *in $\mathcal{G}$ we have*

$$\begin{aligned} \sum_{\substack{k,\beta_1,\beta_2,\\ \beta_3,\beta_4}} \phi_k^{-1}\, W\left(\begin{smallmatrix} & k & \\ \beta_1 & & \beta_2 \\ i_1 & \alpha_2 & i_2 \end{smallmatrix}\right) \overline{W\left(\begin{smallmatrix} & k & \\ \beta_2 & & \beta_3 \\ i_2 & \alpha_3 & i_3 \end{smallmatrix}\right)}\, W\left(\begin{smallmatrix} & k & \\ \beta_3 & & \beta_4 \\ i_3 & \alpha_4 & i_4 \end{smallmatrix}\right) \overline{W\left(\begin{smallmatrix} & k & \\ \beta_4 & & \beta_1 \\ i_4 & \alpha_1 & i_1 \end{smallmatrix}\right)} \\ = \delta_{\alpha_1,\alpha_4}\, \delta_{\alpha_2,\alpha_3}\, \phi_{i_4} \phi_{i_1} \phi_{i_2} + \delta_{\alpha_1,\alpha_2}\, \delta_{\alpha_3,\alpha_4}\, \phi_{i_1} \phi_{i_2} \phi_{i_3} \end{aligned} \tag{4.2}$$

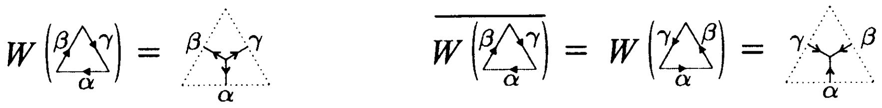


Figure 4.1: Cells associated to trivalent vertices

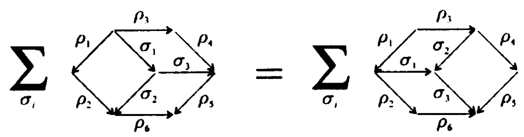


Figure 4.2: The Yang-Baxter equation

These rules correspond precisely to evaluating the Kuperberg relations K2, K3 respectively (see Section 6.1.2), associating a cell $W(\triangle_{\alpha,\beta,\gamma})$ to an incoming trivalent vertex, and $\overline{W(\triangle_{\alpha,\beta,\gamma})}$ to an outgoing trivalent vertex, as in Figure 4.1.

We define the connection

$$X^{\rho_1,\rho_2}_{\rho_3,\rho_4} = \begin{array}{ccc} l & \xrightarrow{\rho_1} & i \\ {\scriptstyle \rho_3}\downarrow & & \downarrow{\scriptstyle \rho_2} \\ k & \xrightarrow[\rho_4]{} & j \end{array}$$

for $\mathcal{G}$ by

$$X^{\rho_1,\rho_2}_{\rho_3,\rho_4} = q^{\frac{2}{3}}\delta_{\rho_1,\rho_3}\delta_{\rho_2,\rho_4} - q^{-\frac{1}{3}}\mathcal{U}^{\rho_1,\rho_2}_{\rho_3,\rho_4}, \tag{4.3}$$

where $\mathcal{U}^{\rho_1,\rho_2}_{\rho_3,\rho_4}$ is given by the representation of the Hecke algebra, and is defined by

$$\mathcal{U}^{\rho_1,\rho_2}_{\rho_3,\rho_4} = \sum_{\lambda} \phi^{-1}_{s(\rho_1)}\phi^{-1}_{r(\rho_2)} W(\triangle^{(\lambda,\rho_3,\rho_4)}_{j,l,k})\overline{W(\triangle^{(\lambda,\rho_1,\rho_2)}_{j,l,i})}. \tag{4.4}$$

This definition of the connection is really Kuperberg's braiding of (6.1).

The above connection corresponds to the braid element g_i (1.2), which is the Boltzmann weight at criticality. It was claimed in [93] that it satisfies the unitarity property of connections (1.31) and the Yang-Baxter equation (1.32). The Yang-Baxter equation (1.32) is represented graphically in Figure 4.2. We give a proof that the connection (4.3) satisfies these two properties.

Lemma 4.1.2 *The connection defined in (4.3) satisfies the unitarity property (1.31) and the Yang-Baxter equation (1.32).*

Proof

We first show unitarity.

$$\begin{aligned}
\sum_{\rho_3,\rho_4} X^{\rho_1,\rho_2}_{\rho_3,\rho_4}\overline{X^{\rho_1',\rho_2'}_{\rho_3,\rho_4}} &= \sum_{\rho_3,\rho_4}\left(q^{\frac{2}{3}}\delta_{\rho_1,\rho_3}\delta_{\rho_2,\rho_4} - q^{-\frac{1}{3}}\sum_{\lambda}\frac{1}{\phi_{s(\rho_1)}\phi_{r(\rho_2)}}W_{\rho_3,\rho_4,\lambda}\overline{W_{\rho_1,\rho_2,\lambda}}\right)\\
&\quad\left(q^{-\frac{2}{3}}\delta_{\rho_1',\rho_3}\delta_{\rho_2',\rho_4} - q^{\frac{1}{3}}\sum_{\lambda}\frac{1}{\phi_{s(\rho_1)}\phi_{r(\rho_2)}}W_{\rho_1',\rho_2',\lambda}\overline{W_{\rho_3,\rho_4,\lambda}}\right)\\
&= \delta_{\rho_1,\rho_1'}\delta_{\rho_3,\rho_3'} + \sum_{\substack{\rho_3,\rho_4\\ \lambda,\lambda'}}\frac{1}{\phi^2_{s(\rho_1)}\phi^2_{r(\rho_2)}}W_{\rho_3,\rho_4,\lambda}\overline{W_{\rho_1,\rho_2,\lambda}}W_{\rho_1',\rho_2',\lambda}\overline{W_{\rho_3,\rho_4,\lambda}}\\
&\quad - \sum_{\rho_3,\rho_4,\lambda}\frac{1}{\phi_{s(\rho_1)}\phi_{r(\rho_2)}}\left(q\delta_{\rho_1,\rho_3}\delta_{\rho_2,\rho_4}W_{\rho_1',\rho_2',\lambda}\overline{W_{\rho_3,\rho_4,\lambda}} + q^{-1}\delta_{\rho_1',\rho_3}\delta_{\rho_2',\rho_4}W_{\rho_3,\rho_4,\lambda}\overline{W_{\rho_1,\rho_2,\lambda}}\right)\\
&= \delta_{\rho_1,\rho_1'}\delta_{\rho_3,\rho_3'} + \sum_{\lambda,\lambda'}\frac{1}{\phi^2_{s(\rho_1)}\phi^2_{r(\rho_2)}}\overline{W_{\rho_1,\rho_2,\lambda}}W_{\rho_1',\rho_2',\lambda}[2]\phi_{s(\rho_3)}\phi_{r(\rho_4)}\delta_{\lambda,\lambda'}\\
&\quad -(q+q^{-1})\sum_{\lambda}\frac{1}{\phi_{s(\rho_1)}\phi_{r(\rho_2)}}W_{\rho_1',\rho_2',\lambda}\overline{W_{\rho_1,\rho_2,\lambda}}\\
&= \delta_{\rho_1,\rho_1'}\delta_{\rho_3,\rho_3'},
\end{aligned}$$

since $q+q^{-1}=[2]$, where we have used Ocneanu's type I equation (4.2) in the penultimate equality.

We now show that the connection satisfies the Yang-Baxter equation. For the left hand side of (1.32) we have

$$\begin{aligned}
&\sum_{\sigma_1,\sigma_2,\sigma_3} X^{\sigma_1,\sigma_2}_{\rho_1,\rho_2}X^{\rho_3,\rho_4}_{\sigma_1,\sigma_3}X^{\sigma_3,\rho_5}_{\sigma_2,\rho_6}\\
&= \sum_{\sigma_1,\sigma_2,\sigma_3}\left(q^{\frac{2}{3}}\delta_{\rho_1,\sigma_1}\delta_{\rho_2,\sigma_2} - q^{-\frac{1}{3}}\mathcal{U}^{\sigma_1,\sigma_2}_{\rho_1,\rho_1}\right)\left(q^{-\frac{2}{3}}\delta_{\sigma_1,\rho_3}\delta_{\sigma_3,\rho_4} - q^{\frac{1}{3}}\mathcal{U}^{\rho_3,\rho_4}_{\sigma_1,\sigma_3}\right)\\
&\qquad\times\left(q^{-\frac{2}{3}}\delta_{\sigma_2,\sigma_3}\delta_{\rho_6,\rho_5} - q^{\frac{1}{3}}\mathcal{U}^{\sigma_3,\rho_5}_{\sigma_2,\rho_6}\right)\\
&= q^2\delta_{\rho_1,\rho_3}\delta_{\rho_2,\rho_4}\delta_{\rho_5,\rho_6} - q\delta_{\rho_1,\rho_3}\,\mathcal{U}^{\rho_4,\rho_5}_{\rho_2,\rho_6} - q\delta_{\rho_5,\rho_6}\,\mathcal{U}^{\rho_3,\rho_4}_{\rho_1,\rho_2} - q\delta_{\rho_5,\rho_6}\,\mathcal{U}^{\rho_3,\rho_4}_{\rho_1,\rho_2} + \sum_{\sigma_3}\mathcal{U}^{\rho_3,\rho_4}_{\rho_1,\sigma_3}\,\mathcal{U}^{\sigma_3,\rho_5}_{\rho_2,\rho_6}\\
&\quad + \sum_{\sigma_2}\mathcal{U}^{\rho_3,\sigma_2}_{\rho_1,\rho_2}\,\mathcal{U}^{\rho_4,\rho_5}_{\sigma_2,\rho_6} + \delta_{\rho_5,\rho_6}\sum_{\sigma_1,\sigma_2}\mathcal{U}^{\sigma_1,\sigma_2}_{\rho_1,\rho_2}\,\mathcal{U}^{\rho_3,\rho_4}_{\sigma_1,\sigma_2} - q^{-1}\sum_{\sigma_i}\mathcal{U}^{\sigma_1,\sigma_2}_{\rho_1,\rho_2}\,\mathcal{U}^{\rho_3,\rho_4}_{\sigma_1,\sigma_3}\,\mathcal{U}^{\sigma_3,\rho_5}_{\sigma_2,\rho_6}\\
&= q^2\delta_{\rho_1,\rho_3}\delta_{\rho_2,\rho_4}\delta_{\rho_5,\rho_6} - q\delta_{\rho_1,\rho_3}\,\mathcal{U}^{\rho_4,\rho_5}_{\rho_2,\rho_6} - 2q\delta_{\rho_5,\rho_6}\,\mathcal{U}^{\rho_3,\rho_4}_{\rho_1,\rho_2} + \sum_{\sigma_3}\mathcal{U}^{\rho_3,\rho_4}_{\rho_1,\sigma_3}\,\mathcal{U}^{\sigma_3,\rho_5}_{\rho_2,\rho_6} + \sum_{\sigma_2}\mathcal{U}^{\rho_3,\sigma_2}_{\rho_1,\rho_2}\,\mathcal{U}^{\rho_4,\rho_5}_{\sigma_2,\rho_6}\\
&\quad + \delta_{\rho_5,\rho_6}\sum_{\substack{\sigma_1,\sigma_2\\ \lambda,\lambda'}}\frac{1}{\phi_{s(\rho_1)}\phi_{r(\rho_2)}\phi_{s(\rho_3)}\phi_{r(\rho_4)}}W_{\rho_1,\rho_2,\lambda}\overline{W_{\sigma_1,\sigma_2,\lambda}}W_{\sigma_1,\sigma_2,\lambda'}\overline{W_{\rho_3,\rho_4,\lambda'}}\\
&\quad - q^{-1}\sum_{\substack{\sigma_i,\lambda\\ \lambda',\lambda''}}\frac{1}{\phi^2_{s(\rho_1)}\phi_{r(\rho_2)}\phi_{r(\rho_4)}\phi_{s(\sigma_2)}\phi_{r(\rho_6)}}W_{\rho_1,\rho_2,\lambda}\overline{W_{\sigma_1,\sigma_2,\lambda}}W_{\sigma_1,\sigma_3,\lambda'}\overline{W_{\rho_3,\rho_4,\lambda'}}W_{\sigma_2,\rho_6,\lambda''}\overline{W_{\sigma_1,\rho_5,\lambda''}}
\end{aligned}$$

$$
\begin{aligned}
= {} & q^2\delta_{\rho_1,\rho_3}\delta_{\rho_2,\rho_4}\delta_{\rho_5,\rho_6} - q\delta_{\rho_1,\rho_3}\,\mathcal{U}^{\rho_4,\rho_5}_{\rho_2,\rho_6} - 2q\delta_{\rho_5,\rho_6}\,\mathcal{U}^{\rho_3,\rho_4}_{\rho_1,\rho_2} + \sum_{\sigma_3}\mathcal{U}^{\rho_3,\rho_4}_{\rho_1,\sigma_3}\,\mathcal{U}^{\sigma_3,\rho_5}_{\rho_2,\rho_6} + \sum_{\sigma_2}\mathcal{U}^{\rho_3,\sigma_2}_{\rho_1,\rho_2}\,\mathcal{U}^{\rho_4,\rho_5}_{\sigma_2,\rho_6} \\
& +\delta_{\rho_5,\rho_6}\sum_{\lambda,\lambda'}\frac{1}{\phi_{s(\rho_1)}\phi_{r(\rho_2)}\phi_{s(\rho_3)}\phi_{r(\rho_4)}}W_{\rho_1,\rho_2,\lambda}\overline{W_{\rho_3,\rho_4,\lambda'}}[2]\phi_{r(\rho_2)}\phi_{s(\rho_1)}\delta_{\lambda,\lambda'} \\
& -q^{-1}\sum_{\lambda,\lambda'}\frac{1}{\phi^2_{s(\rho_1)}\phi_{r(\rho_2)}\phi_{r(\rho_4)}\phi_{r(\rho_6)}}W_{\rho_1,\rho_2,\lambda}\overline{W_{\rho_3,\rho_4,\lambda'}}\Big(\delta_{\lambda,\rho_6}\delta_{\lambda',\rho_5}\phi_{r(\rho_2)}\phi_{r(\rho_6)}\phi_{r(\rho_4)} \\
& \qquad\qquad +\delta_{\lambda,\lambda'}\delta_{\rho_5,\rho_6}\phi_{s(\rho_1)}\phi_{r(\rho_2)}\phi_{r(\rho_6)}\Big) \\
= {} & q^2\delta_{\rho_1,\rho_3}\delta_{\rho_2,\rho_4}\delta_{\rho_5,\rho_6} - q\delta_{\rho_1,\rho_3}\,\mathcal{U}^{\rho_4,\rho_5}_{\rho_2,\rho_6} - 2q\delta_{\rho_5,\rho_6}\,\mathcal{U}^{\rho_3,\rho_4}_{\rho_1,\rho_2} + \sum_{\sigma_3}\mathcal{U}^{\rho_3,\rho_4}_{\rho_1,\sigma_3}\,\mathcal{U}^{\sigma_3,\rho_5}_{\rho_2,\rho_6} + \sum_{\sigma_2}\mathcal{U}^{\rho_3,\sigma_2}_{\rho_1,\rho_2}\,\mathcal{U}^{\rho_4,\rho_5}_{\sigma_2,\rho_6} \\
& +[2]\delta_{\rho_5,\rho_6}\,\mathcal{U}^{\rho_3,\rho_4}_{\rho_1,\rho_2} - q^{-1}\frac{1}{\phi_{s(\rho_1)}}W_{\rho_1,\rho_2,\rho_6}\overline{W_{\rho_3,\rho_4,\rho_5}} - q^{-1}\delta_{\rho_5,\rho_6}\,\mathcal{U}^{\rho_3,\rho_4}_{\rho_1,\rho_2} \\
= {} & q^2\delta_{\rho_1,\rho_3}\delta_{\rho_2,\rho_4}\delta_{\rho_5,\rho_6} - q\delta_{\rho_1,\rho_3}\,\mathcal{U}^{\rho_4,\rho_5}_{\rho_2,\rho_6} - q\delta_{\rho_5,\rho_6}\,\mathcal{U}^{\rho_3,\rho_4}_{\rho_1,\rho_2} + \sum_{\sigma_3}\mathcal{U}^{\rho_3,\rho_4}_{\rho_1,\sigma_3}\,\mathcal{U}^{\sigma_3,\rho_5}_{\rho_2,\rho_6} + \sum_{\sigma_2}\mathcal{U}^{\rho_3,\sigma_2}_{\rho_1,\rho_2}\,\mathcal{U}^{\rho_4,\rho_5}_{\sigma_2,\rho_6} \\
& -q^{-1}\frac{1}{\phi_{s(\rho_1)}}W_{\rho_1,\rho_2,\rho_6}\overline{W_{\rho_3,\rho_4,\rho_5}}.
\end{aligned}
$$

Computing the right hand side of (1.32) in the same way, we arrive at the same expression.

□

4.2 Computation of the cells W for $\mathcal{ADE}$ graphs

In this section we will compute cells systems W for each $\mathcal{ADE}$ graph $\mathcal{G}$, with the exception of the graph $\mathcal{E}_4^{(12)}$.

Let $\triangle^{(\alpha,\beta,\gamma)}_{i,j,k}$ be the triangle $i \xrightarrow{\alpha} j \xrightarrow{\beta} k \xrightarrow{\gamma} i$ in $\mathcal{G}$. For most of the $\mathcal{ADE}$ graph, using the equations (4.1) and (4.2) only, we can compute the cells up to choice of phase $W(\triangle^{(\alpha,\beta,\gamma)}_{i,j,k}) = \lambda^{\alpha,\beta,\gamma}_{i,j,k}|W(\triangle^{(\alpha,\beta,\gamma)}_{i,j,k})|$ for some $\lambda^{\alpha,\beta,\gamma}_{i,j,k} \in \mathbb{T}$, and also obtain some restrictions on the values which the phases $\lambda^{\alpha,\beta,\gamma}_{i,j,k}$ may take. However, for the graph $\mathcal{D}^{(n)*}$, $n = 5, 6, \dots$, we impose a $\mathbb{Z}_3$ symmetry on our solutions, whilst for the graphs $\mathcal{D}^{(3k)}$, $k = 2, 3, \dots$, and $\mathcal{E}_1^{(12)}$ we seek an orbifold solution obtained using the identification of the graphs $\mathcal{D}^{(3k)}$, $\mathcal{E}_1^{(12)}$ as $\mathbb{Z}_3$ orbifolds of $\mathcal{A}^{(3k)}$, $\mathcal{E}_2^{(12)}$ respectively. There is still much freedom in the actual choice of phases, so that the cell system is not unique. We therefore define an equivalence relation between two cell systems:

Definition 4.2.1 *Two families of cells W_1, W_2 which give a cell system for $\mathcal{G}$ are equivalent if, for each pair of adjacent vertices i, j of $\mathcal{G}$, we can find a family unitary matrices $(u(\sigma_1,\sigma_2))_{\sigma_1,\sigma_2}$, where σ_1, σ_2 are any pair of edges from i to j, such that*

$$W_1(\triangle^{(\sigma,\rho,\gamma)}_{i,j,k}) = \sum_{\sigma',\rho',\gamma'} u(\sigma,\sigma')u(\rho,\rho')u(\gamma,\gamma')W_2(\triangle^{(\sigma',\rho',\gamma')}_{i,j,k}), \tag{4.5}$$

where the sum is over all edges σ' from i to j, ρ' from j to k, and γ' from k to i.

Lemma 4.2.2 *Let W_1, W_2 be two equivalent families of cells, and $X^{(1)}$, $X^{(2)}$ the corresponding connections defined using cells W_1, W_2 respectively. Then $X^{(1)}$ and $X^{(2)}$ are equivalent in the sense of [39, p.542], i.e. there exists a set of unitary matrices $(u(\rho,\sigma))_{\rho,\sigma}$ such that*

$$X^{(1)\rho_1,\rho_2}_{\rho_3,\rho_4} = \sum_{\sigma_i} u(\rho_3,\sigma_3)u(\rho_4,\sigma_4)\overline{u(\rho_1,\sigma_1)u(\rho_2,\sigma_2)}X^{(2)\sigma_1,\sigma_2}_{\sigma_3,\sigma_4}.$$

Let $W_l(\triangle^{(\sigma,\rho,\gamma)}_{i,j,k}) = \lambda^{(l)\sigma,\rho,\gamma}_{i,j,k}|W_l(\triangle^{(\sigma,\rho,\gamma)}_{i,j,k})|$, for $l=1,2$, be two families of cells which give cell systems. If $|W_1(\triangle^{(\sigma,\rho,\gamma)}_{i,j,k})| = |W_2(\triangle^{(\sigma,\rho,\gamma)}_{i,j,k})|$, so that W_1 and W_2 differ only up to phase choice, then the equation (4.5) becomes

$$\lambda^{(1)\sigma,\rho,\gamma}_{i,j,k} = \sum_{\sigma',\rho',\gamma'} u(\sigma,\sigma')u(\rho,\rho')u(\gamma,\gamma')\lambda^{(2)\sigma,\rho,\gamma}_{i,j,k}. \tag{4.6}$$

For graphs with no multiple edges we write $\triangle_{i,j,k}$ for the triangle $\triangle^{(\alpha,\beta,\gamma)}_{i,j,k}$. For such graphs, two solutions W_1 and W_2 differ only up to phase choice, and (4.6) becomes

$$\lambda^{(1)}_{i,j,k} = u_\sigma u_\rho u_\gamma \lambda^{(2)}_{i,j,k}, \tag{4.7}$$

where $u_\sigma, u_\rho, u_\gamma \in \mathbb{T}$ and σ is the edge from i to j, ρ the edge from j to k and γ the edge from k to i.

We will write $U^{(x,y)}$ for the matrix indexed by the vertices of $\mathcal{G}$, with entries given by $\mathcal{U}^{\rho_1,\rho_2}_{\rho_3,\rho_4}$ for all edges ρ_i, $i=1,2,3,4$ on $\mathcal{G}$ such that $s(\rho_1)=s(\rho_3)=x$, $r(\rho_2)=r(\rho_4)=y$, i.e. $[U^{(s(\rho_1),r(\rho_2))}]_{r(\rho_1),r(\rho_3)} = \mathcal{U}^{\rho_1,\rho_2}_{\rho_3,\rho_4}$.

We first present some relations that the quantum numbers $[a]_q$ satisfy, which are easily checked:

Lemma 4.2.3 *(i) If $q=\exp(i\pi/n)$ then $[a]_q=[n-a]_q$, for any $a=1,2,\dots,n-1$,*

(ii) For any q, $[a]_q-[a-2]_q=[2a-2]_q/[a-1]_q$, for any $a\in\mathbb{N}$,

(iii) For any q, $[a]_q^2-[a-1]_q[a+1]_q=1$ and $[a]_q[a+b]_q-[a-1]_q[a+b+1]_q=[b+1]_q$, for any $a\in\mathbb{N}$.

4.2.1 $\mathcal{A}$ graphs

Let the vertices of the graph $\mathcal{A}^{(n)}$ be labelled by (λ_1,λ_2), $\lambda_1,\lambda_2\geq 0$, $\lambda_1+\lambda_2\leq n-3$, as in §1.5, with $(0,0)$ as the distinguished apex. For the triangle $\triangle_{(i_1,j_1)(i_2,j_2)(i_3,j_3)} = (i_1,j_1)\to(i_2,j_2)\to(i_3,j_3)\to(i_1,j_1)$ in $\mathcal{A}^{(n)}$ we will use the notation $W_{\triangle(i,j)}$ for the cell $W(\triangle_{(i,j)(i+1,j)(i,j+1)})$ and $W_{\nabla(i,j)}$ for the cell $W(\triangle_{(i+1,j)(i,j+1)(i+1,j+1)})$.

Theorem 4.2.4 *There is up to equivalence a unique set of cells for $\mathcal{A}^{(n)}$, $n < \infty$, given by:*

$$\begin{aligned} W_{\triangle(k,m)} &= \sqrt{[k+1][k+2][m+1][m+2][k+m+1][k+m+2]}/[2], && (4.8) \\ W_{\nabla(k,m)} &= \sqrt{[k+1][k+2][m+1][m+2][k+m+2][k+m+3]}/[2], && (4.9) \end{aligned}$$

for all $k, m \geq 0$. For the graph $\mathcal{A}^{(\infty)}$ with Perron-Frobenius eigenvalue $\alpha \geq 3$, there is a solution given by replacing $[j]$ by $[j]_q$ where $q = e^x$ for any $x \in \mathbb{R}$ such that $\alpha = [3]_q$.

Proof

Let $n < \infty$. We first prove the equalities

$$\begin{aligned} |W_{\triangle(k,m)}| &= \sqrt{[k+1][k+2][m+1][m+2][k+m+1][k+m+2]}/[2], && (4.10) \\ |W_{\nabla(k,m)}| &= \sqrt{[k+1][k+2][m+1][m+2][k+m+2][k+m+3]}/[2], && (4.11) \end{aligned}$$

by induction on k, m. The Perron-Frobenius eigenvector for $\mathcal{A}^{(n)}$ is given in (1.29). By considering the type I frame $\overset{(0,0)}{\bullet} \rightarrow \overset{(1,0)}{\bullet}$ equation (4.1) gives $|W_{\triangle(0,0)}|^2 = [2][3]$, whilst from the type I frame $\overset{(1,0)}{\bullet} \rightarrow \overset{(0,1)}{\bullet}$ we obtain $|W_{\triangle(0,0)}|^2 + |W_{\nabla(0,0)}|^2 = [2][3]^2$, giving $|W_{\nabla(0,0)}|^2 = [3][4]$. We assume (4.10) and (4.11) are true for $(k,m) = (p,q)$. We first show (4.10) is true for $(k,m) = (p+1,q)$ and $(k,m) = (p,q+1)$ (see Figure 4.3). From the type I frame $\overset{(p+1,q+1)}{\bullet} \rightarrow \overset{(p+1,q)}{\bullet}$ we get

$$|W_{\triangle(p+1,q)}|^2 + |W_{\nabla(p,q)}|^2 = [p+2]^2[q+1][q+2][p+q+2][p+q+3]/[2],$$

and substituting in for $|W_{\nabla(p,q)}|^2$ we obtain

$$\begin{aligned} |W_{\triangle(p+1,q)}|^2 &= [p+2][q+1][q+2][p+q+2][p+q+3]([2][p+2]-[p+1])/[2]^2 \\ &= [p+2][p+2][q+1][q+2][p+q+2][p+q+3]/[2]^2. \end{aligned}$$

Similarly, from the type I frame $\overset{(p,q+1)}{\bullet} \rightarrow \overset{(p+1,q+1)}{\bullet}$ we get

$$|W_{\triangle(p,q+1)}|^2 = [p+1][p+2][q+2][q+3][p+q+2][p+q+3]/[2]^2,$$

as required.

For $k, m \geq 0$, (4.11) follows from (4.10): consider the type I frame $\overset{(k+1,m)}{\bullet} \rightarrow \overset{(k,m+1)}{\bullet}$. We get

$$|W_{\triangle(k,m)}|^2 + |W_{\nabla(k,m)}|^2 = [k+1][k+2][m+1][m+2][k+m+2]^2/[2],$$

and substituting in for $|W_{\triangle(k,m)}|^2$ we obtain

$$\begin{aligned} |W_{\nabla(k,m)}|^2 &= [k+1][k+2][m+1][m+2][k+m+2]([2][k+m+2]-[k+m+1])/[2]^2 \\ &= [k+1][k+2][m+1][m+2][k+m+2][k+m+3]/[2]. \end{aligned}$$

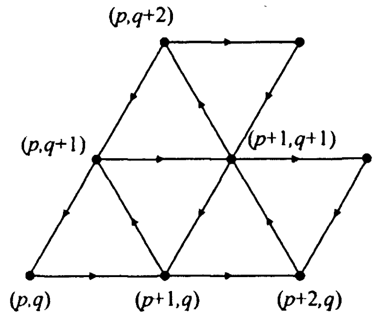


Figure 4.3: Triangles in $\mathcal{A}^{(n)}$

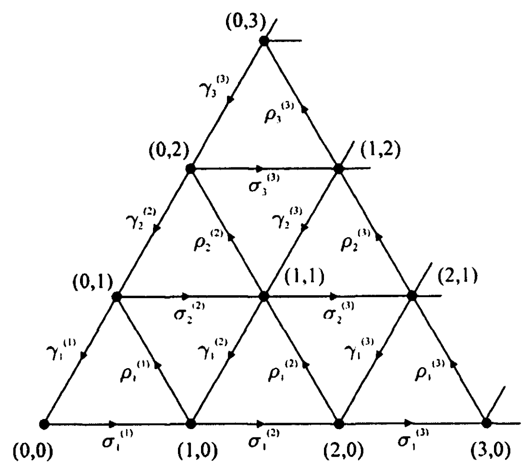


Figure 4.4: Labels for the vertices and edges of $\mathcal{A}^{(n)}$

Hence (4.10) and (4.11) are true for all $k, m \geq 0$.

There is no restriction on the choice of phase for $\mathcal{A}^{(n)}$, so any choice is a solution. We now turn to the uniqueness of these cells. Let $W^\sharp$ be another family of cells, with $W^\sharp_{\triangle(k,m)} = \lambda_{(k,m)}|W_{\triangle(k,m)}|$ and $W^\sharp_{\nabla(k,m)} = \lambda'_{(k,m)}|W_{\nabla(k,m)}|$ (any other solution must be of this form since there are no double edges on $\mathcal{A}^{(n)}$). We label the edges of $\mathcal{A}^{(n)}$ by $\sigma_i^{(j)}$, $\rho_i^{(j)}$, $\gamma_i^{(j)}$, $j = 1, \ldots, n-3$, $i = 1, \ldots, j$, as shown in Figure 4.4.

Let us start with the triangle $\triangle_{(0,0)(1,0)(0,1)}$, equation (4.7) gives $1 = u_{\sigma_1^{(1)}} u_{\rho_1^{(1)}} u_{\gamma_1^{(1)}} \lambda_{(0,0)}$. Choose $u_{\sigma_1^{(1)}} = u_{\gamma_1^{(1)}} = 1$ and $u_{\rho_1^{(1)}} = \overline{\lambda_{(0,0)}}$.

Next consider the triangle $\triangle_{(1,0)(0,1)(1,1)}$. We have $1 = u_{\sigma_2^{(2)}} u_{\gamma_1^{(2)}} \overline{\lambda_{(0,0)}}\lambda'_{(0,0)}$, so choose $u_{\sigma_2^{(2)}} = 1$ and $u_{\gamma_1^{(2)}} = \lambda_{(0,0)}\overline{\lambda'_{(0,0)}}$. Similarly, setting $u_{\sigma_1^{(2)}} = u_{\gamma_2^{(2)}} = 1$, $u_{\rho_1^{(2)}} = \lambda'_{(0,0)}\overline{\lambda_{(0,0)}\lambda_{(1,0)}}$ and $u_{\rho_2^{(2)}} = \overline{\lambda_{(0,1)}}$ then equation (4.7) is satisfied for the triangles $\triangle_{(1,0)(2,0)(1,1)}$ and $\triangle_{(0,1)(1,1)(0,2)}$.

Continuing in this way we set, for each k, $u_{\sigma_i^{(k)}} = 1$, $(i = 1, \ldots, k)$, $u_{\gamma_k^{(k)}} = 1$, $u_{\gamma_i^{(k)}} = \overline{u_{\rho_i^{(k-1)}}\lambda'_{(k-i-1,i-1)}}$, $(i = 1, \ldots, k-1)$ and $u_{\rho_i^{(k)}} = u_{\rho_i^{(k-1)}}\lambda'_{(k-i-1,i-1)}\overline{\lambda_{(k-i,i-1)}}$, $(i = 1, \ldots, k)$. Hence, any choice of λ and λ' will give an equivalent solution to (4.8), (4.9).

For $\mathcal{A}^{(\infty)}$, we have Perron-Frobenius eigenvectors $\phi = (\phi_{\lambda_1,\lambda_2})$ given by

$$\phi_{(\lambda_1,\lambda_2)} = \frac{[\lambda_1+1]_q[\lambda_2+1]_q[\lambda_1+\lambda_2+2]_q}{[2]_q}.$$

Then the rest of the proof follows as for finite n. $\square$

Using these cells W we obtain the following representation of the Hecke algebra for $\mathcal{A}^{(n)}$. We have written the label for the rows (and columns) in front of each matrix.

$$U^{((\lambda_1,\lambda_2),(\lambda_1,\lambda_2+1))} = \begin{array}{c} (\lambda_1+1,\lambda_2) \\ (\lambda_1-1,\lambda_2+1) \end{array} \left(\begin{array}{cc} \frac{[\lambda_1+2]}{[\lambda_1+1]} & \frac{\sqrt{[\lambda_1][\lambda_1+2]}}{[\lambda_1+1]} \\ \frac{\sqrt{[\lambda_1][\lambda_1+2]}}{[\lambda_1+1]} & \frac{[\lambda_1]}{[\lambda_1+1]} \end{array} \right), \tag{4.12}$$

$$U^{((\lambda_1,\lambda_2),(\lambda_1-1,\lambda_2))} = \begin{array}{c} (\lambda_1-1,\lambda_2+1) \\ (\lambda_1,\lambda_2-1) \end{array} \begin{pmatrix} \frac{[\lambda_2+2]}{[\lambda_2+1]} & \frac{\sqrt{[\lambda_2][\lambda_2+2]}}{[\lambda_2+1]} \\ \frac{\sqrt{[\lambda_2][\lambda_2+2]}}{[\lambda_2+1]} & \frac{[\lambda_2]}{[\lambda_2+1]} \end{pmatrix}, \tag{4.13}$$

$$U^{((\lambda_1,\lambda_2),(\lambda_1+1,\lambda_2-1))} = \begin{array}{c} (\lambda_1+1,\lambda_2) \\ (\lambda_1,\lambda_2-1) \end{array} \begin{pmatrix} \frac{[\lambda_1+\lambda_2+3]}{[\lambda_1+\lambda_2+2]} & \frac{\sqrt{[\lambda_1+\lambda_2+1][\lambda_1+\lambda_2+3]}}{[\lambda_1+\lambda_2+2]} \\ \frac{\sqrt{[\lambda_1+\lambda_2+1][\lambda_1+\lambda_2+3]}}{[\lambda_1+\lambda_2+2]} & \frac{[\lambda_1+\lambda_2+1]}{[\lambda_1+\lambda_2+2]} \end{pmatrix} \tag{4.14}$$

Wenzl [112] constructed representations of the Hecke algebra, which are given in [27] as:

$$\begin{array}{ccc} \lambda & \longrightarrow & \lambda+e_k \\ \downarrow & & \downarrow \\ \lambda+e_j & \longrightarrow & \lambda+e_j+e_k \end{array} = (1-\delta_{jl})\frac{\sqrt{s_{jl}(\lambda'+e_j)s_{jl}(\lambda'+e_k)}}{s_{jl}(\lambda')}, \tag{4.15}$$

where $\lambda=(\lambda_1,\lambda_2)$ is a vertex on $\mathcal{A}^{(n)}$, $\lambda'=(\lambda_1+1,\lambda_2+1)$, the vectors e_j are defined in Section 1.3.2, $s_{jl}(\lambda)=\sin((\pi/n)(e_j-e_l)\cdot\lambda)$ and the inner-product is given by $e_j\cdot e_k=\delta_{j,k}-1/N$. Note that this weight is 0 when $j=l$.

Lemma 4.2.5 *The weights in the representation of the Hecke algebra given above for $\mathcal{A}^{(n)}$ are identical to those in (4.15).*

Proof

For $j=l$ the result is immediate since there is no triangle $\lambda\to\lambda+e_j\to\lambda+2e_j\to\lambda$ on $\mathcal{A}^{(n)}$, and hence the weight in our representation of the Hecke algebra will be zero also. For an arbitrary vertex $\lambda=(\lambda_1,\lambda_2)$ of $\mathcal{A}^{(n)}$, $s_{jl}(\lambda')=\sin((\pi/n)(e_j-e_l)\cdot((\lambda_1+1)e_1-(\lambda_2+1)e_3))$. We will show the result for $j=1$, $l=2$ (the other cases follow similarly). We have $s_{12}(\lambda')=\sin((\lambda_1+1)\pi/n)$ and $s_{12}(\lambda'+e_j)=s_{12}(\lambda'+e_1)=\sin((\lambda_1+2)\pi/n)$. We also have $s_{12}(\lambda'+e_2)=\sin(\lambda_1\pi/n)$. Then for $k=1$, (4.15) becomes

$$\frac{\sqrt{\sin^2((\lambda_1+2)\pi/n)}}{\sin((\lambda_1+1)\pi/n)}=\frac{[\lambda_1+2]}{[\lambda_1+1]}=(U^{((\lambda_1,\lambda_2),(\lambda_1,\lambda_2+1))})_{(\lambda_1+1,\lambda_2),(\lambda_1+1,\lambda_2)}.$$

For $k=2$, (4.15) becomes

$$\frac{\sqrt{\sin((\lambda_1+2)\pi/n)\sin(\lambda_1\pi/n)}}{\sin((\lambda_1+1)\pi/n)}=\frac{\sqrt{[\lambda_1][\lambda_1+2]}}{[\lambda_1+1]}=(U^{((\lambda_1,\lambda_2),(\lambda_1,\lambda_2+1))})_{(\lambda_1+1,\lambda_2),(\lambda_1-1,\lambda_2+1)},$$

as required. □

4.2.2 $\mathcal{D}$ graphs

The Perron-Frobenius weights for the vertices of $\mathcal{A}^{(n)}$ are invariant under the $\mathbb{Z}_3$ symmetry given by rotation by $2\pi/3$. Since $\mathcal{D}^{(n)}$ comes from an orbifold of $\mathcal{A}^{(n)}$ (as illustrated in

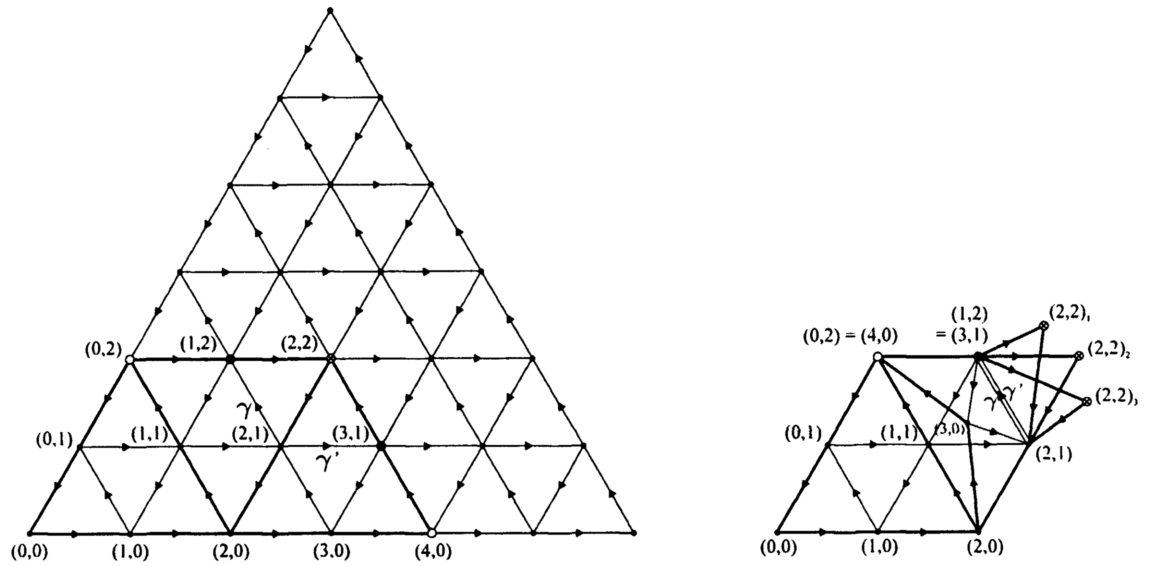


Figure 4.5: $\mathcal{A}^{(9)}$ and its $\mathbb{Z}_3$ orbifold $\mathcal{D}^{(9)}$

Figure 4.5 for $n = 9$), the Perron-Frobenius weights for the vertices of $\mathcal{D}^{(n)}$ are equal to the corresponding weights in $\mathcal{A}^{(n)}$, except that for $n = 3k+3$, for integer $k \geq 1$, the vertices $(k,k)_1$, $(k,k)_2$ and $(k,k)_3$ (see Figure 4.6) which come from the fixed point (k,k) of $\mathcal{A}^{(3k+3)}$ under the rotation whose Perron-Frobenius weights are a third of the weight for the vertex (k,k) of $\mathcal{A}^{(3k+3)}$. The absolute values $|W^{\mathcal{A}}|$ of the cells for $\mathcal{A}^{(n)}$ are also invariant under the rotation.

Let $n \geq 5$, $n \not\equiv 0 \bmod 3$. We will find one solution (up to a choice of phase) for the cells of $\mathcal{D}^{(n)}$ by identifying the absolute values $|W^{(\mathcal{A})}|$ for the cells in $\mathcal{A}^{(n)}$ with the absolute values $|W^{(\mathcal{D})}|$ for the corresponding cells in $\mathcal{D}^{(n)}$ when taking the orbifold. Each type I frame in $\mathcal{D}^{(n)}$ has a corresponding type I frame in $\mathcal{A}^{(n)}$, and similarly for the type II frames. Since the Perron-Frobenius weights are the same for $\mathcal{A}^{(n)}$ and $\mathcal{D}^{(n)}$, these $|W^{\mathcal{D}}|$ will certainly satisfy (4.1) and (4.2) since the $|W^{\mathcal{A}}|$ do. As in the case of $\mathcal{A}^{(n)}$, there are no restrictions on the choice of phase. Then we have the following theorem:

Theorem 4.2.6 *Every orbifold solution for the cells of $\mathcal{D}^{(n)}$, $n \not\equiv 0$ mod 3, is equivalent to the solution for which the cells in $\mathcal{D}^{(n)}$ are equal to the corresponding cells in $\mathcal{A}^{(n)}$ given in (4.8), (4.9).*

Proof

The unitaries $u_{i,j} \in \mathbb{T}$, for i,j vertices on $\mathcal{D}^{(n)}$, may be chosen systematically as in the proof of Theorem 4.2.4, beginning with $u_{(k,k),(k,k)} = \overline{\lambda_{(k,k),(k,k),(k,k)}}^{1/3}$ if $n = 3k+4$ or

$u_{(k+1,k),(k+1,k)} = \overline{\lambda_{(k+1,k),(k+1,k),(k+1,k)}}^{-1/3}$ if $n = 3k+5$, and proceeding triangle by triangle.

$\square$

Now let $n = 3k+3$ for some integer $k \geq 1$. For $q = e^{i\pi/(3k+3)}$, we have $[(3k+3)/2+i]_q = [(3k+3)/2-i]_q$ where $i \in \mathbb{Z}$ for k even and $i \in \mathbb{Z}+\frac{1}{2}$ for k odd. In particular we will use $[2k+2+j] = [k+1-j]$ for $j \in \mathbb{Z}$. The Perron-Frobenius weights $\phi_{(k,k)_i} = \phi_{(k,k)}/3 = [k+1]^2[2k+2]/(3[2]) = [k+1]^3/(3[2])$, $i = 1,2,3$. We again find an orbifold solution for the cells for $\mathcal{D}^{(3k+3)}$, except for those which involve the vertices $(k,k)_i$, $i = 1,2,3$, which correspond to the fixed point (k,k) on the graph $\mathcal{A}^{(3k+3)}$. Let γ, γ' be the two edges in the double edge of $\mathcal{D}^{(3k+3)}$, where γ is the edge from $(k,k-1)$ to $(k-1,k)$ and γ' is the edge from $(k,k-1)$ to $(k+1,k-1)$ in $\mathcal{A}^{(3k+3)}$ (see Figure 4.5). We will use the notation $W^{(\xi)}_{v,(k,k-1),(k-1,k)}$ to denote the cell for the triangle $\triangle_{v,(k,k-1),(k-1,k)}$ where the edge $\xi \in \{\gamma,\gamma'\}$ is used, for $v = (k-1,k-1)$, $(k+1,k-2)$ or $(k,k)_i$, $i = 1,2,3$. Then in particular we have the following:

$$\begin{aligned}
|W^{(\gamma)}_{(k-1,k-1),(k,k-1),(k-1,k)}|^2 &= \frac{[k]^2[k+1]^2[2k][2k+1]}{[2]^2} = \frac{[k]^2[k+1]^2[k+2][k+3]}{[2]^2}, \\
|W^{(\gamma')}_{(k+1,k-2),(k,k-1),(k-1,k)}|^2 &= \frac{[k-1][k][k+1][k+2][2k+1][2k+2]}{[2]^2} \\
&= \frac{[k-1][k][k+1]^2[k+2]^2}{[2]^2}.
\end{aligned}$$

Since γ' is not an edge used to form the triangle $\triangle_{(k-1,k-1),(k,k-1),(k-1,k)}$ in $\mathcal{A}^{(3k+3)}$, we have $|W^{(\gamma')}_{(k-1,k-1),(k,k-1),(k-1,k)}|^2 = 0$, and similarly $|W^{(\gamma)}_{(k+1,k-2),(k,k-1),(k-1,k)}|^2 = 0$. The cells involving the vertices $(k,k)_i$ coming from the triplicated vertex (k,k) in $\mathcal{A}^{(3k+3)}$ will then be a third of the corresponding cells for $\mathcal{A}^{(3k+3)}$, since the type I frames $\overset{(k-1,k)}{\bullet} \longrightarrow \overset{(k,k)_i}{\bullet}$ give $|W^{(\gamma)}_{(k-1,k),(k,k)_i,(k,k-1)}|^2 + |W^{(\gamma')}_{(k-1,k),(k,k)_i,(k,k-1)}|^2 = [k][k+1]^4[k+2]/(3[2])$ for $i = 1,2,3$. So

$$\begin{aligned}
|W^{(\gamma)}_{(k-1,k),(k,k)_i,(k,k-1)}|^2 &= \frac{1}{3}|W_{(k-1,k),(k,k),(k,k-1)}|^2 = \frac{1}{3}\frac{[k]^2[k+1]^2[2k+1][2k+2]}{[2]^2} \\
&= \frac{1}{3}\frac{[k]^2[k+1]^3[k+2]}{[2]^2}, \\
|W^{(\gamma')}_{(k-1,k),(k,k)_i,(k,k-1)}|^2 &= \frac{1}{3}|W_{(k+1,k-1),(k,k),(k,k-1)}|^2 \\
&= \frac{1}{3}\frac{[k][k+1]^2[k+2][2k+1][2k+2]}{[2]^2} = \frac{1}{3}\frac{[k][k+1]^3[k+2]^2}{[2]^2}.
\end{aligned}$$

For $\lambda_i, \lambda'_i \in \mathbb{T}$, be the choice of phase for $W^{(\gamma)}_{(k-1,k),(k,k)_i,(k,k-1)}$, $W^{(\gamma')}_{(k-1,k),(k,k)_i,(k,k-1)}$ respectively, so that $W^{(\gamma)}_{(k-1,k),(k,k)_i,(k,k-1)} = \lambda_i|W^{(\gamma)}_{(k-1,k),(k,k)_i,(k,k-1)}|$ and $W^{(\gamma')}_{(k-1,k),(k,k)_i,(k,k-1)} = \lambda'_i|W^{(\gamma')}_{(k-1,k),(k,k)_i,(k,k-1)}|$, for $i = 1,2,3$. Similarly, let

$$W^{(\xi)}_{(k-1,k-1),(k,k-1),(k-1,k)} = \lambda^{(\xi)}_{(k-1,k-1),(k,k-1),(k-1,k)}|W^{(\xi)}_{(k-1,k-1),(k,k-1),(k-1,k)}|,$$

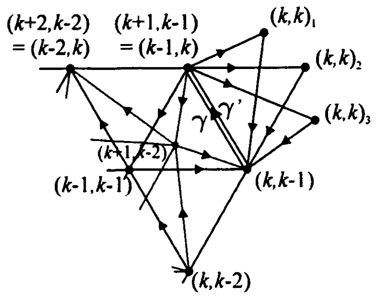


Figure 4.6: Labels for the graph $\mathcal{D}^{(3k+3)}$

where $\xi \in \{\gamma, \gamma'\}$, and $W_{v_1,v_2,v_3} = \lambda_{v_1,v_2,v_3}|W_{v_1,v_2,v_3}|$ for all other triangles $\triangle_{v_1,v_2,v_3}$ of $\mathcal{D}^{(3k+3)}$. The type II frame $\overset{(k,k-1)}{\bullet} \rightrightarrows \overset{(k-1,k)}{\bullet}$ gives the following restriction on the phases λ_i, λ_i':

$$\lambda_1\overline{\lambda_1'} + \lambda_2\overline{\lambda_2'} + \lambda_3\overline{\lambda_3'} = 0. \tag{4.16}$$

From the type II frame $\overset{(k,k)_i}{\bullet} \rightarrow \overset{(k,k-1)}{\bullet} \leftarrow \overset{(k,k)_j}{\bullet}$ we obtain $\mathrm{Re}(\lambda_i\lambda_j'\overline{\lambda_i'\lambda_j}) = -1/2$ for $i \neq j$, giving $\lambda_i\overline{\lambda_i'} = (-1/2 + \varepsilon_{ij}\sqrt{3}i/2)\lambda_j\overline{\lambda_j'}$, $\varepsilon_{ij} \in \{\pm 1\}$. Note that $\varepsilon_{ji} = -\varepsilon_{ij}$, and substituting for $\lambda_i\overline{\lambda_i'}$ with $j = i+1$ into (4.16) we find $\varepsilon_{12} = \varepsilon_{23} = \varepsilon_{31}$. Then we have

$$\lambda_i\overline{\lambda_i'} = (-\frac{1}{2} + \varepsilon\frac{\sqrt{3}i}{2})\lambda_{i+1}\overline{\lambda_{i+1}'}, \tag{4.17}$$

for $\varepsilon \in \{\pm 1\}$, $i = 1,2,3$ (mod 3). Then there are two solutions for the cells of $\mathcal{D}^{(3k+3)}$, W and its complex conjugate $\overline{W}$. The solution $\overline{W}$ is the solution to the graph where we switch vertices $(k,k)_2 \leftrightarrow (k,k)_3$.

Theorem 4.2.7 *Every orbifold solution for the cells of $\mathcal{D}^{(3k+3)}$ is given, up to equivalence, by the inequivalent solutions W or its complex conjugate $\overline{W}$, where W is given by:*

$$W^{(\gamma)}_{(k-1,k),(k,k)_i,(k,k-1)} = \epsilon_i\frac{[k]\sqrt{[k+1]^3[k+2]}}{\sqrt{3}\,[2]},$$

$$W^{(\gamma')}_{(k-1,k),(k,k)_i,(k,k-1)} = \overline{\epsilon_i}\frac{[k+2]\sqrt{[k][k+1]^3}}{\sqrt{3}\,[2]},$$

$$W^{(\gamma)}_{(k-1,k-1),(k,k-1),(k-1,k)} = \frac{[k][k+1]\sqrt{[k+2][k+3]]}}{[2]},$$

$$W^{(\gamma')}_{(k+1,k-2),(k,k-1),(k-1,k)} = \frac{[k+1][k+2]\sqrt{[k-1][k]}}{[2]},$$

$$W^{(\gamma')}_{(k-1,k-1),(k,k-1),(k-1,k)} = W^{(\gamma)}_{(k+1,k-2),(k,k-1),(k-1,k)} = 0,$$

where $\epsilon_1 = 1$, $\epsilon_2 = e^{2\pi i/3} = \overline{\epsilon_3}$, and all other cells are equal to the corresponding cells in $\mathcal{A}^{(3k+3)}$ given in (4.8), (4.9).

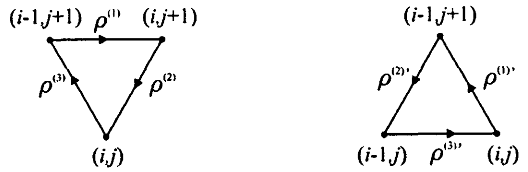


Figure 4.7: Triangles $\triangle^{(\rho^{(1)},\rho^{(2)},\rho^{(3)})}_{(i,j),(i-1,j+1),(i,j+1)}$ and $\triangle^{(\rho^{(1)\prime},\rho^{(2)\prime},\rho^{(3)\prime})}_{(i-1,j),(i,j),(i-1,j+1)}$

Proof

Let $W^\natural$ be any orbifold solution for the cells of $\mathcal{D}^{(3k+3)}$. Then $W^\natural$ is given, for $i=1,2,3$, by

$$\begin{aligned}
W^{\natural(\gamma)}_{(k-1,k),(k,k)_i,(k,k-1)} &= \lambda^\natural_i |W^{(\gamma)}_{(k-1,k),(k,k)_i,(k,k-1)}|, \\
W^{\natural(\gamma')}_{(k-1,k),(k,k)_i,(k,k-1)} &= \lambda^{\natural\prime}_i |W^{(\gamma')}_{(k-1,k),(k,k)_i,(k,k-1)}|, \\
W^{\natural(\xi)}_{(k-1,k-1),(k,k-1),(k-1,k)} &= \lambda^{\natural(\xi)}_{(k-1,k-1),(k,k-1),(k-1,k)} |W^{\natural(\xi)}_{(k-1,k-1),(k,k-1),(k-1,k)}|,
\end{aligned}$$

where $\xi \in \{\gamma,\gamma'\}$, and $W^\natural_{v_1,v_2,v_3} = \lambda^\natural_{v_1,v_2,v_3}|W_{v_1,v_2,v_3}|$ for all other triangles $\triangle_{v_1,v_2,v_3}$ of $\mathcal{D}^{(3k+3)}$, and where the choice of $\lambda^\natural_i$, $\lambda^{\natural\prime}_i$ satisfy condition (4.17) with $\varepsilon = 1$. We need to find a family of unitaries $\{u_\rho\}$ for edges $\rho \neq \gamma'$ of $\mathcal{D}^{(3k+3)}$, where $u_\gamma = (u_\gamma(\xi,\xi'))$, $\xi,\xi' \in \{\gamma,\gamma'\}$, is a 2×2 unitary matrix, and $u_\rho \in \mathbb{T}$ for all other ρ. These unitaries must satisfy (4.6) and (4.7), i.e. $\epsilon_l = u_{\mu_l}u_{\mu'_l}(u_\gamma(\gamma,\gamma)\lambda^\natural_l + u_\gamma(\gamma,\gamma')\lambda^{\natural\prime}_l)$ and $\overline{\epsilon_l} = u_{\mu_l}u_{\mu'_l}(u_\gamma(\gamma',\gamma)\lambda^\natural_l + u_\gamma(\gamma',\gamma')\lambda^{\natural\prime}_l)$, for $l=1,2,3$, and

$$\begin{aligned}
1 &= u_{\sigma_1}u_{\sigma_2}\sum_{\xi'} u(\xi,\xi')\lambda^{\natural(\xi')}_{(k-1,k-1),(k,k-1),(k-1,k)}, \\
1 &= u_{\sigma'_1}u_{\sigma'_2}\sum_{\xi'} u(\xi,\xi')\lambda^{\natural(\xi')}_{(k+1,k-2),(k,k-1),(k-1,k)}.
\end{aligned}$$

For all other triangles $\triangle^{(\rho_1,\rho_2,\rho_3)}_{p_1,p_2,p_3}$ of $\mathcal{D}^{(3k+3)}$ we require $1 = u_{\rho_1}u_{\rho_2}u_{\rho_3}\lambda^\natural_{p_1,p_2,p_3}$.

For u_γ we choose $u_\gamma(\gamma,\gamma)=1$, $u_\gamma(\gamma,\gamma')=u_\gamma(\gamma',\gamma)=0$ and $u_\gamma(\gamma'\gamma') = \lambda^\natural_1\overline{\lambda^\natural_1}$. We set $u_{\mu'_l}=1$ and $u_{\mu_l} = \epsilon_l\overline{\lambda^\natural_l}$, for $l=1,2,3$, and $u_{\sigma_1}=u_{\sigma'_1}=1$, $u_{\sigma_2} = \overline{\lambda^{\natural(\gamma)}_{(k-1,k-1),(k,k-1),(k-1,k)}}$ and $u_{\sigma'_2} = \overline{\lambda^{\natural(\gamma')}_{(k+1,k-2),(k,k-1),(k-1,k)}}$.

For the remaining triangles we proceed as follows. Let $m = 2k-2$. For each triangle $\triangle^{(\rho^{(1)},\rho^{(2)},\rho^{(3)})}_{(i,j),(i-1,j+1),(i,j+1)}$ as in Figure 4.7 (and similarly for triangles $\triangle_{(i,j),(i-1,j+1),(i,j+1)}$) such that $i+j=m$, if either $u_{\rho^{(1)}}$ or $u_{\rho^{(2)}}$ hasn't yet been assigned a value we set it to be 1, and set $u_{\rho^{(3)}} = \overline{u_{\rho^{(1)}}u_{\rho^{(2)}}\lambda^\natural_{(i,j),(i-1,j+1),(i,j+1)}}$. Next, for each triangle $\triangle^{(\rho^{(1)\prime},\rho^{(2)\prime},\rho^{(3)\prime})}_{(i-1,j),(i,j),(i-1,j+1)}$ as in Figure 4.7 (and similarly for triangles $\triangle_{(i+1,j-1),(i,j),(i+1,j)}$) such that $i+j=m$, if either $u_{\rho^{(1)\prime}}$ or $u_{\rho^{(2)\prime}}$ hasn't yet been assigned a value we set it to be 1, and set $u_{\rho^{(3)\prime}} = \overline{u_{\rho^{(1)\prime}}u_{\rho^{(2)\prime}}\lambda^\natural_{(i-1,j),(i,j),(i-1,j+1)}}$. We then set $m = 2k-3$ and repeat the above steps. Continuing in this way, for $m = 2k-4, \ldots, 3$, we find the required unitaries $\{u_\rho\}$. The proof for the uniqueness of the complex conjugate solution can be shown similarly.

For the solutions W and $\overline{W}$ to be equivalent, we require unitaries as above such that

$$\begin{aligned}
\epsilon_l &= u_{\mu_l} u_{\mu_l'} (u_\gamma(\gamma,\gamma)\overline{\epsilon}_l + \frac{\sqrt{[k+2]}}{\sqrt{[k]}} u_\gamma(\gamma,\gamma')\epsilon_l), \\
\overline{\epsilon}_l &= u_{\mu_l} u_{\mu_l'} (\frac{\sqrt{[k]}}{\sqrt{[k+2]}} u_\gamma(\gamma',\gamma)\overline{\epsilon}_l + u_\gamma(\gamma',\gamma')\epsilon_l),
\end{aligned}$$

for $l = 1,2,3$. This forces $u_\gamma(\gamma,\gamma) = u_\gamma(\gamma',\gamma') = 0$, $u_\gamma(\gamma,\gamma') = \sqrt{[k]}/\sqrt{[k+2]}$ and $u_\gamma(\gamma',\gamma) = \sqrt{[k+2]}/\sqrt{[k]}$. But then u_γ isn't a unitary. $\square$

Using the cells W we obtain the following representation of the Hecke algebra for $\mathcal{D}^{(3k+3)}$:

$$U^{((k-1,k-1),(k-1,k))} = \begin{array}{c} (k,k-1)^{(\gamma)} \\ (k,k-1)^{(\gamma')} \\ (k-2,k) \end{array} \left(\begin{array}{ccc} \frac{[k+1]}{[k]} & 0 & \frac{\sqrt{[k-1][k+1]}}{[k]} \\ 0 & 0 & 0 \\ \frac{\sqrt{[k-1][k+1]}}{[k]} & 0 & \frac{[k-1]}{[k]} \end{array} \right),$$

$$= U^{((k,k-1),(k-1,k-1))} \text{ with rows labelled by } (k-1,k)^{(\gamma)}, (k-1,k)^{(\gamma')},$$

$$U^{((k+1,k-2),(k-1,k))} = \begin{array}{c} (k,k-1)^{(\gamma)} \\ (k,k-1)^{(\gamma')} \\ (k-2,k) \end{array} \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & \frac{[k+1]}{[k+2]} & \frac{\sqrt{[k+1][k+3]}}{[k+2]} \\ 0 & \frac{\sqrt{[k+1][k+3]}}{[k+2]} & \frac{[k+3]}{[k+2]} \end{array} \right),$$

$$= U^{((k,k-1),(k+1,k-2))} \text{ with rows labelled by } (k-1,k)^{(\gamma)}, (k-1,k)^{(\gamma')}, (k,k-2),$$

$$U^{((k,k-1),(k,k)_i)} = \begin{array}{c} (k-1,k)^{(\gamma)} \\ (k-1,k)^{(\gamma')} \end{array} \left(\begin{array}{cc} \frac{[k]}{[k+1]} & \overline{\epsilon}_i \frac{\sqrt{[k][k+2]}}{[k+1]} \\ \epsilon_i \frac{\sqrt{[k][k+2]}}{[k+1]} & \frac{[k+2]}{[k+1]} \end{array} \right),$$

$$= U^{((k,k)_i,(k-1,k))} \text{ with rows labelled by } (k,k-1)^{(\gamma)}, (k,k-1)^{(\gamma')},$$

$$U^{((k-1,k),(k,k-1))}$$

$$= \begin{array}{c} (k,k)_1 \\ (k,k)_2 \\ (k,k)_3 \\ (k-1,k-1) \\ (k+1,k-2) \end{array} \left(\begin{array}{ccccc}
\frac{[2][k+1]^2}{3[k][k+2]} & \frac{[k+1]\overline{a}}{3[k][k+2]} & \frac{[k+1]a}{3[k][k+2]} & \frac{\sqrt{[k+1][k+3]}}{\sqrt{3}[k+2]} & \frac{\sqrt{[k-1][k+1]}}{\sqrt{3}\,[k]} \\
\frac{[k+1]a}{3[k][k+2]} & \frac{[2][k+1]^2}{3[k][k+2]} & \frac{[k+1]\overline{a}}{3[k][k+2]} & \frac{\epsilon_2\sqrt{[k+1][k+3]}}{\sqrt{3}\,[k+2]} & \frac{\overline{\epsilon}_2\sqrt{[k-1][k+1]}}{\sqrt{3}\,[k]} \\
\frac{[k+1]\overline{a}}{3[k][k+2]} & \frac{[k+1]a}{3[k][k+2]} & \frac{[2][k+1]^2}{3[k][k+2]} & \frac{\overline{\epsilon}_2\sqrt{[k+1][k+3]}}{\sqrt{3}\,[k+2]} & \frac{\epsilon_2\sqrt{[k-1][k+1]}}{\sqrt{3}[k]} \\
\frac{\sqrt{[k+1][k+3]}}{\sqrt{3}\,[k+2]} & \frac{\overline{\epsilon}_2\sqrt{[k+1][k+3]}}{\sqrt{3}\,[k+2]} & \frac{\epsilon_2\sqrt{[k+1][k+3]}}{\sqrt{3}\,[k+2]} & \frac{[k+3]}{[k+2]} & 0 \\
\frac{\sqrt{[k-1][k+1]}}{\sqrt{3}\,[k]} & \frac{\epsilon_2\sqrt{[k-1][k+1]}}{\sqrt{3}\,[k]} & \frac{\overline{\epsilon}_2\sqrt{[k-1][k+1]}}{\sqrt{3}\,[k]} & 0 & \frac{[k-1]}{[k]}
\end{array} \right),$$

where $a = \epsilon_2[k] + \overline{\epsilon}_2[k+2]$, and we use the notation $v^{(\gamma)}$ if the path uses the edge γ, where v is a vertex of $\mathcal{D}^{(3k+3)}$. Another representation of the Hecke algebra is given by taking the complex conjugates of the weights in the representation above.

In [43], Fendley gives Boltzmann weights for $\mathcal{D}^{(6)}$, which at criticality and with the parameter $u = 1$, give a representation of the Hecke algebra. However these Boltzmann weights are not equivalent to the representation of the Hecke algebra using the cells W or $\overline{W}$. To see this, we use a similar labelling for the graph $\mathcal{D}^{(6)}$ as in [43]- see Figure 4.8.

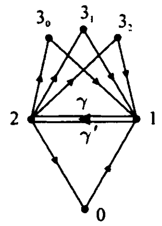


Figure 4.8: Labelling the graph $\mathcal{D}^{(6)}$

Consider the weight $[\overline{U}^{(3_r,2)}]_{\gamma,\gamma'}$, where we label the rows and columns by γ, γ' to denote which edge from 1 to 2 is used for the path of length 2 from 3_r to 2, $r = 0, 1, 2$, and the weight $\overline{U}$ is the complex conjugate of that given above, i.e. it is the weight given by the solution $\overline{W}$ for the cells of $\mathcal{D}^{(6)}$. Then for equivalence we require a unitary $u_{3_r,1} \in \mathbb{T}$ and a 2×2 unitary matrix u_γ such that

$$\begin{aligned}\epsilon_2^r \frac{\sqrt{[3]}}{[2]} &= |u_{3_r,1}|^2 \Bigg(u_\gamma(\gamma,\gamma)\overline{u_\gamma(\gamma',\gamma)}\frac{1}{[2]} + u_\gamma(\gamma,\gamma)\overline{u_\gamma(\gamma',\gamma')}\epsilon_2^r\frac{\sqrt{[3]}}{[2]} \\ &\qquad + u_\gamma(\gamma,\gamma')\overline{u_\gamma(\gamma',\gamma)}\overline{\epsilon_2^r}\frac{\sqrt{[3]}}{[2]} + u_\gamma(\gamma,\gamma')\overline{u_\gamma(\gamma',\gamma')}\frac{[3]}{[2]}\Bigg). \qquad (4.18)\end{aligned}$$

Since u_γ is independent of r, for (4.18) to be satisfied for each $r = 0, 1, 2$, we require $u_\gamma(\gamma,\gamma)\overline{u_\gamma(\gamma',\gamma')} = 1$ and the other terms to be zero, which gives $u_\gamma(\gamma,\gamma') = u_\gamma(\gamma',\gamma) = 0$ and $u_\gamma(\gamma',\gamma') = (u_\gamma(\gamma,\gamma))^{-1}$. But now if we consider the weight $[\overline{U}^{(1,3_r)}]_{\gamma,\gamma'}$, with $u_{2,3_r} \in \mathbb{T}$, we have

$$\begin{aligned}\overline{\epsilon_2^r} \frac{\sqrt{[3]}}{[2]} &= |u_{2,3_r}|^2 \Bigg(u_\gamma(\gamma,\gamma)\overline{u_\gamma(\gamma',\gamma)}\frac{1}{[2]} + u_\gamma(\gamma,\gamma)\overline{u_\gamma(\gamma',\gamma')}\overline{\epsilon_2^r}\frac{\sqrt{[3]}}{[2]} \\ &\qquad + u_\gamma(\gamma,\gamma')\overline{u_\gamma(\gamma',\gamma)}\epsilon_2^r\frac{\sqrt{[3]}}{[2]} + u_\gamma(\gamma,\gamma')\overline{u_\gamma(\gamma',\gamma')}\frac{[3]}{[2]}\Bigg),\end{aligned}$$

but $[\overline{U}^{(1,3_r)}]_{\gamma,\gamma'} = \epsilon_2^r \frac{\sqrt{[3]}}{[2]}$, for $r = 0, 1, 2$. We obtain a similar contradiction when considering the weights U defined using the solution W for the cells.

Suppose however that the Boltzmann weight denoted by $\widetilde{W}^{(\tilde{1},\tilde{3}_r)}_{\tilde{2},\tilde{2}}$ in [43] is the complex conjugate of that given. Then the Boltzmann weights at criticality of Fendley [43] are equivalent to the representation of the Hecke algebra given by the solution $\overline{W}$ for the cells of $\mathcal{D}^{(6)}$. We choose a family of unitaries $u_{0,1} = u_{2,0} = u_{2,3_r} = 1$, $u_{3_r,1} = \overline{\epsilon_2^r}$, $r = 0,1,2$, and choose u_γ to be the 2×2 identity matrix.

4.2.3 $\mathcal{A}^*$ graphs

First we consider the graphs $\mathcal{A}^{(2n+1)*}$, illustrated in Figure 1.10. The Perron-Frobenius weights on the vertices are given by $\phi_i = [2i-1]$, $i = 1, \ldots, n$.

Theorem 4.2.8 *There is up to equivalence a unique set of cells for $\mathcal{A}^{(2n+1)*}$, $n < \infty$, given by:*

$$\begin{aligned} W_{i-1,i,i} &= \frac{\sqrt{[i][2i-3][2i-1]}}{\sqrt{[i-1]}}, & i &= 2,\ldots,n, \\ W_{i,i,i+1} &= \frac{\sqrt{[i-1][2i-1][2i+1]}}{\sqrt{[i]}}, & i &= 2,\ldots,n-1, \\ W_{i,i,i} &= (-1)^{i+1}\frac{[2i-1]}{\sqrt{[i-1][i]}}, & i &= 2,\ldots,n. \end{aligned}$$

Proof

We first prove the equalities

$$|W_{i-1,i,i}|^2 = \frac{[i][2i-3][2i-1]}{[i-1]}, \qquad i = 2,\ldots,n, \tag{4.19}$$

$$|W_{i,i,i+1}|^2 = \frac{[i-1][2i-1][2i+1]}{[i]}, \qquad i = 2,\ldots,n-1, \tag{4.20}$$

$$|W_{i,i,i}|^2 = \frac{[2i-1]^2}{[i-1][i]}, \qquad i = 2,\ldots,n. \tag{4.21}$$

We have the following equations from type I frames:

$$|W_{1,2,2}|^2 = [2][3], \tag{4.22}$$

$$|W_{i,i,i+1}|^2 + |W_{i,i+1,i+1}|^2 = [2][2i-1][2i+1], \qquad i = 2,\ldots,n-1, \tag{4.23}$$

$$|W_{i-1,i,i}|^2 + |W_{i,i,i}|^2 + |W_{i,i,i+1}|^2 = [2][2i-1]^2, \qquad i = 2,\ldots,n-1, \tag{4.24}$$

$$|W_{n-1,n,n}|^2 + |W_{n,n,n}|^2 = [2]^3, \tag{4.25}$$

and from type II frames we have:

$$|W_{i-1,i,i}|^2|W_{i,i,i+1}|^2 = [2i-3][2i-1]^2[2i+1], \qquad i = 2,\ldots,n-1, \tag{4.26}$$

$$|W_{i-1,i,i}|^2\left(\frac{1}{[2i-3]}|W_{i-1,i-1,i}|^2 + \frac{1}{[2i-1]}|W_{i,i,i}|^2\right) = [2i-3][2i-1]^2, \qquad i = 2,\ldots,n. \tag{4.27}$$

For $i=2$, (4.19) is trivial by (4.22), and (4.21) follows from (4.27). From (4.24) we then have $|W_{2,2,3}|^2 = [2][3]^2 - |W_{1,2,2}|^2 - |W_{2,2,2}|^2 = [3][5]/[2]$, so (4.20) is true for $i=2$. We assume (4.19)-(4.21) are true for $i = k < n-1$. Then from (4.23),

$$|W_{k,k+1,k+1}|^2 = [2][2k-1][2k+1] - \frac{[k-1][2k-1][2k+1]}{[k]} = \frac{[k+1][2k-1][2k+1]}{[k]}.$$

From (4.27),

$$|W_{k+1,k+1,k+1}|^2 = \frac{[2k+1]^2}{[k][k+1]}([k]^2 - [k-1][k+1]) = \frac{[2k+1]^2}{[k][k+1]},$$

and finally, from (4.26),

$$|W_{k+1,k+1,k+2}|^2 = \frac{[k][2k+1][2k+3]}{[k+1]},$$

as required. So (4.19)-(4.21) are true for $i = 2,\ldots,n-1$. The result for $|W_{n-1,n,n}|^2$ follows from (4.23), and lastly, from (4.25),

$$\begin{aligned} |W_{n,n,n}|^2 &= [2][2n-1]^2 - \frac{[n][2n-3][2n-1]}{[n-1]} = [2]^3 - \frac{[n][4][2]}{[n-1]} \\ &= \frac{[2]}{[n-1]}([n-1]-[n+3]) = \frac{[2]}{[n-1]}([n+2]-[n-2]) = \frac{[2]^2}{[n-1][n]}. \end{aligned}$$

Let $W_{i,j,k} = \lambda_{i,j,k}|W_{i,j,k}|$ for $\lambda_{i,j,k} \in \mathbb{T}$. From type II frames we have the restriction

$$\lambda^3_{i,i,i+1}\lambda_{i+1,i+1,i+1} = -\lambda^3_{i,i+1,i+1}\lambda_{i,i,i}, \tag{4.28}$$

for $i = 2,\ldots,n-1$. Let $W^\sharp_{i,j,k} = \lambda^\sharp_{i,j,k}|W_{i,j,k}|$ be any other solution to the cells, where the $\lambda^\sharp$ satisfy (4.28). We need to find a family of unitaries $\{u_{i,j}\}$, where $u_{i,j}$ is the unitary for the edge from vertex i to vertex j on $\mathcal{A}^{(2n+1)*}$, which satisfy (4.7), i.e. $-1 = u^3_{2l,2l}\lambda^\sharp_{2l,2l,2l}$ for $l = 1,\ldots,\lfloor n/2 \rfloor$, and $1 = u_i u_j u_k \lambda^\sharp_{i,j,k}$ for all other triangles $\triangle_{i,j,k}$. We choose $u_{i,i+1} = 1$ $(i = 1,\ldots,n-1)$, $u_{2,1} = -(\lambda^\sharp_{2,2,2})^{1/3}\overline{\lambda^\sharp_{1,2,2}}$, $u_{i+1,i} = -(\overline{\lambda^\sharp_{2,2,2}})^{1/3}\lambda^\sharp_{2,3,3}\lambda^\sharp_{3,4,4}\cdots\lambda^\sharp_{i-1,i,i}\overline{\lambda^\sharp_{2,2,3}\lambda^\sharp_{3,3,4}\cdots\lambda^\sharp_{i,i,i+1}}$ $(i = 2,\ldots,n-1)$, $u_{2,2} = -(\overline{\lambda^\sharp_{2,2,2}})^{1/3}$ and $u_{i,i} = -(\overline{\lambda^\sharp_{2,2,2}})^{1/3}\lambda^\sharp_{2,2,3}\lambda^\sharp_{3,3,4}\cdots\lambda^\sharp_{i-1,i-1,i}\overline{\lambda^\sharp_{2,3,3}\lambda^\sharp_{3,4,4}\cdots}\,\overline{\lambda^\sharp_{i-1,i,i}}$ $(i = 3,\ldots,n)$. $\square$

For $\mathcal{A}^{(2n+1)*}$, the above cells W give the following representation of the Hecke algebra:

$$U^{(i,i+1)} = \begin{array}{c} i \\ i+1 \end{array} \left(\begin{array}{cc} \frac{[i-1]}{[i]} & \frac{\sqrt{[i-1][i+1]}}{[i]} \\ \frac{\sqrt{[i-1][i+1]}}{[i]} & \frac{[i+1]}{[i]} \end{array} \right),$$

$$U^{(i,i-1)} = \begin{array}{c} i-1 \\ i \end{array} \left(\begin{array}{cc} \frac{[i-2]}{[i-1]} & \frac{\sqrt{[i-2][i]}}{[i-1]} \\ \frac{\sqrt{[i-2][i]}}{[i-1]} & \frac{[i]}{[i-1]} \end{array} \right),$$

$$U^{(i,i)} \quad = \quad \begin{array}{c} i-1 \\ i \\ i+1 \end{array} \left(\begin{array}{ccc} \frac{[i][2i-3]}{[i-1][2i-1]} & \frac{(-1)^{i+1}\sqrt{[2i-3]}}{[i-1]\sqrt{[2i-1]}} & \frac{\sqrt{[2i-3][2i+1]}}{[2i-1]} \\ \frac{(-1)^{i+1}\sqrt{[2i-3]}}{[i-1]\sqrt{[2i-1]}} & \frac{1}{[i-1][i]} & \frac{(-1)^{i+1}\sqrt{[2i+1]}}{[i]\sqrt{[2i-1]}} \\ \frac{\sqrt{[2i-3][2i+1]}}{[2i-1]} & \frac{(-1)^{i+1}\sqrt{[2i+1]}}{[i]\sqrt{[2i-1]}} & \frac{[i-1][2i+1]}{[i][2i-1]} \end{array} \right).$$

In [4], Behrend and Evans give Boltzmann weights

$$W\left(\begin{array}{cc} a & d \\ b & c \end{array} \middle| \, u \right),$$

which at criticality, with $u=1$, give a representation of the Hecke algebra. (Note, these Boltzmann weights are not to be confused with the Ocneanu cells W.)

Lemma 4.2.9 *The weights in the representation of the Hecke algebra given above for $\mathcal{A}^{(2n+1)*}$ are equivalent to the Boltzmann weights at criticality given by Behrend-Evans in [4].*

Proof

To make our notation the same as that of [4] one replaces i with $(a+1)/2$. Then it is easily checked that the absolute values of our weights given above are equal to those for the Boltzmann weights in [4], setting $q=0$, in all but a few cases. We will show that the absolute values in these other cases are also equal. For $\left[U^{(i,i)}\right]_{i+1,i+1}$, the Boltzmann weight in [4] is

$$\frac{[a+2]-[a+2]/[a]}{[a+1]} = \frac{[a+2]}{[a][a+1]}([a]-[1]) = \frac{[a+2]}{[a][a+1]} \frac{[\frac{1}{2}(a-1)][a+1]}{[\frac{1}{2}(a+1)]},$$

which is equal to our weight, and similarly for $[U^{(i,i)}]_{i-1,i-1}$. For $[U^{(i,i)}]_{i,i}$ we have to do the most work. From [4] its value is

$$\frac{1}{[3]}\left([2] - \frac{[a+2][\frac{1}{2}(a-5)]}{[a][\frac{1}{2}(a+1)]} - \frac{[a-2][\frac{1}{2}(a+5)]}{[a][\frac{1}{2}(a-1)]}\right)$$
$$= \frac{[2][a][\frac{1}{2}(a-1)][\frac{1}{2}(a+1)] - [a+2][\frac{1}{2}(a-5)][\frac{1}{2}(a-1)] - [a-2][\frac{1}{2}(a+1)][\frac{1}{2}(a+5)]}{[3][a][\frac{1}{2}(a-1)][\frac{1}{2}(a+1)]}. \tag{4.29}$$

Using (1.30), we can write the numerator as

$$[2][a]([2]+[4]+\cdots+[a-1]) - [a+2]([3]+[5]+\cdots+[a-4]) - [a-2]([3]+[5]+\cdots+[a+2])$$

$$
\begin{aligned}
&= [a]([1]+[3]+[3]+[5]+\cdots+[a-2]+[a]) \\
&\quad -([a+2]+[a-2])([3]+[5]+\cdots+[a-4])-[a-2]([a-2]+[a]+[a+2]) \\
&= [a]+(2[a]-[a+2]-[a-2])([3]+[5]+\cdots+[a-4]+[a-2])+[a]^2-[a-2][a] \\
&= [a]+([a]-[a+2])([3]+[5]+\cdots+[a-2]) \\
&= +([a]-[a-2])([3]+[5]+\cdots+[a-2]+[a]) \\
&= [a]+[(a-3)/2][(a+1)/2]([a]-[a+2])+[(a-1)/2][(a+3)/2]([a]-[a-2]).
\end{aligned}
\tag{4.30}
$$

Now

$$
\begin{aligned}
&[(a-3)/2][(a+1)/2]([a]-[a+2]) \\
&\quad = [(a-3)/2]([(a+1)/2]+[(a+5)/2]+\cdots+[(3a-1)/2] \\
&\qquad\qquad -[(a+5)/2]-[(a+9)/2]-\cdots-[(3a+3)/2]) \\
&\quad = [(a-3)/2]([(a+1)/2]-[(3a+3)/2]) \\
&\quad = [3]+[5]+\cdots+[a-2]-[a+4]-[a+6]-\cdots-[2a-1],
\end{aligned}
$$

and

$$
\begin{aligned}
&[(a-1)/2][(a+3)/2]([a]-[a-2]) \\
&\quad = [(a-1)/2]([(a+1)/2]+[(a+3)/2]+\cdots+[(3a+1)/2] \\
&\qquad\qquad -[(a-5)/2]-[(a-1)/2]-\cdots-[(3a-3)/2]) \\
&\quad = [(a-1)/2]([(3a+1)/2]-[(a-5)/2]) \\
&\quad = [a+2]+[a+4]+\cdots+[2a-1]-[3]-[5]-\cdots-[a-4].
\end{aligned}
$$

Substituting back into (4.30) we obtain $[a]+[a-2]+[a+2]=[3][a]$, and (4.29) becomes

$$
\frac{[3][a]}{[3][a][\frac{1}{2}(a-1)][\frac{1}{2}(a+1)]}=\frac{1}{[\frac{1}{2}(a-1)][\frac{1}{2}(a+1)]}
$$

as required. To show equivalence, we need unitaries $u_{i,j}\in\mathbb{T}$, for vertices i, j of $\mathcal{A}^{(n)*}$ such that

$$
\begin{aligned}
1 &= u_{i,i}u_{i+1,i+1}, & 1 &= u_{i,i}u_{i-1,i-1}, & -1 &= u_{i,i-1}u_{i-1,i}\overline{u_{i,i+1}u_{i+1,i}}, \\
(-1)^i &= u_{i,i}^2\overline{u_{i,i+1}u_{i+1,i}}, & (-1)^{i+1} &= u_{i,i}^2\overline{u_{i,i-1}u_{i-1,i}}.
\end{aligned}
$$

Then we set $u_{i,i}=1$ for all i, and for $m=0,\ldots,(n-2)/2$, $u_{2m+1,2m}=u_{2m,2m+1}=u_{2m+2,2m+1}=1$ and $u_{2m+1,2m+2}=-1$. □

For the graphs $\mathcal{A}^{(4n)*}$ (illustrated in Figure 1.10) the Perron-Frobenius weights on the vertices are given by $\phi_i = [2i]/[2]$, $i = 1, \dots, 2n-1$. There are now two solutions W^+, W^- for the cells for $\mathcal{A}^{(4n)*}$, which are not equivalent since $|W^+| \neq |W^-|$ and the graph $\mathcal{A}^{(4n)*}$ does not contain any multiple edges.

Theorem 4.2.10 *The cells for $\mathcal{A}^{(4n)*}$, $n < \infty$, are given, up to equivalence, by the inequivalent solutions W^+, W^-:*

$$W^{\pm}_{i,i,i+1} = \frac{\sqrt{[2i][2i+2]}}{[2]\sqrt{[2i+1]}}\sqrt{[2i] \mp [1]}, \qquad i = 1, \dots, 2n-2,$$

$$W^{\pm}_{i,i+1,i+1} = \frac{\sqrt{[2i][2i+2]}}{[2]\sqrt{[2i+1]}}\sqrt{[2i+2] \pm [1]}, \qquad i = 1, \dots, 2n-2,$$

$$W^{\pm}_{i,i,i} = \begin{cases} (-1)^{i+1}\dfrac{\sqrt{[2i]}}{[2]\sqrt{[2i-1][2i+1]}}\sqrt{[2][2i] \pm [4i]}, & i = 1, \dots, n-1, \\ (-1)^{n+1}\dfrac{[2n]}{\sqrt{[2][2n-1][2n+1]}}, & i = n, \\ (-1)^{i+1}\dfrac{\sqrt{[2i]}}{[2]\sqrt{[2i-1][2i+1]}}\sqrt{[2][2i] \mp [8n-4i]}, & i = n+1, \dots, 2n-1. \end{cases}$$

Proof

Due to the symmetry on the graph $\mathcal{A}^{(4n)*}$, we only need to consider half of the type I frames. In so doing, we are not assuming that the cells are invariant under this symmetry. We have the following equations from type I frames:

$$|W_{1,1,1}|^2 + |W_{1,1,2}|^2 = [2], \tag{4.31}$$

$$|W_{i,i,i+1}|^2 + |W_{i,i+1,i+1}|^2 = \frac{[2i][2i+2]}{[2]}, \qquad i = 1, \dots, n-1, \tag{4.32}$$

$$|W_{i-1,i,i}|^2 + |W_{i,i,i}|^2 + |W_{i,i,i+1}|^2 = \frac{[2i]^2}{[2]}, \qquad i = 2, \dots, n-1, \tag{4.33}$$

and from type II frames we have:

$$|W_{i-1,i,i}|^2|W_{i,i,i+1}|^2 = \frac{[2i-2][2i]^2[2i+2]}{[2]^4}, \qquad i = 2, \dots, n, \tag{4.34}$$

$$|W_{i-1,i,i}|^2\left(\frac{[2]}{[2i-2]}|W_{i-1,i-1,i}|^2 + \frac{[2]}{[2i]}|W_{i,i,i}|^2\right) = \frac{[2i-2][2i]^2}{[2]^3}, \quad i = 2, \dots, n. \tag{4.35}$$

From (4.33) and (4.34) we have

$$|W_{i,i,i}|^2 = \frac{[2i]^2}{[2]} - |W_{i-1,i,i}|^2 - \frac{[2i-2][2i]^2[2i+2]}{[2]^4}|W_{i-1,i,i}|^{-2}, \tag{4.36}$$

and substituting for $|W_{i,i,i}|^2$, and also for $|W_{i-1,i-1,i}|^2$ from (4.32), into (4.35) we obtain the quadratic equation

$$\frac{[2]^2[2i-1]}{[2i-2][2i]}|W_{i-1,i,i}|^4 - 2[2i]|W_{i-1,i,i}|^2 + \frac{[2i-2][2i][2i+1]}{[2]^2} = 0.$$

Then, for $i = 2, \ldots, n$,

$$|W_{i-1,i,i}|^2 = \frac{[2i-2][2i]}{[2]^2[2i-1]}([2i] \pm [1]), \tag{4.37}$$

and, from (4.32),

$$|W_{i-1,i-1,i}|^2 = \frac{[2i-2][2i]}{[2]^2[2i-1]}([2i-2] \mp [1]). \tag{4.38}$$

For $i = 1, \ldots, n-1$, (4.33) gives

$$|W_{i,i,i}|^2 = \frac{[2i]}{[2]^2[2i-1][2i+1]}([2][2i] \pm [4i]). \tag{4.39}$$

Note that $|W_{i,i,i}|^2 > 0$ for all $i = 1, \ldots, n-1$: We have $[2][2i] - [4i] = [2i-1] + [2i+1] - [4i]$ and $\sin((2i-1)\pi/4n) + \sin((2i+1)\pi/4n) - \sin(4i\pi/4n) = 2\sin(4i\pi/4n)\cos(2\pi/4n) - \sin(4i\pi/4n) > 0$ since $\cos(2\pi/4n) > 1/2$.

From (4.36) with $i = n$ we have

$$\begin{aligned}
&|W_{n,n,n}|^2 \\
&= \frac{[2n]^2}{[2]} - \frac{[2n-2][2n]}{[2]^2[2n-1]}([2n] \pm [1]) - \frac{[2n][2n+2]}{[2]^2[2n+1]}([2n] \mp [1]) \\
&= \frac{[2n]^2}{[2]^2[2n-1][2n+1]}([2][2n-1][2n+1] - [2n-2][2n+1] - [2n-1][2n-2]) \\
&= \frac{[2n]^2}{[2]^2[2n-1][2n+1]}([2n][2n+1] - [2n-1][2n-2]) \;=\; \frac{[2n]^2}{[2][2n-1][2n+1]}.
\end{aligned}$$

From (4.34) with $i = n$, we have

$$|W_{n,n,n+1}|^2 = \frac{[2n-2][2n]}{[2]^2[2n-1]}([2n] \mp [1]),$$

and the equations for $|W_{i-1,i-1,i}|$, $|W_{i-1,i,i}|$ and $|W_{i,i,i}|$ $i = n+1, \ldots, 2n-1$ follow.

We again obtain the restriction on the phase given in (4.28). Let $W^{\sharp}_{i,j,k}$ be another solution for the cells of $\mathcal{A}^{(4n)*}$ such that $|W^{\sharp}_{i,j,k}| = |W^{+}_{i,j,k}|$. Then the equivalence of these solutions follows in a similar way to the $\mathcal{A}^{(2n+1)*}$ case. □

For the graphs $\mathcal{A}^{(4n+2)*}$ (again illustrated in Figure 1.10) the Perron-Frobenius weights on the vertices are again given by $\phi_i = [2i]/[2]$, $i = 1, \ldots, 2n$. There are again two inequivalent solutions W^+, W^- for the cells of $\mathcal{A}^{(4n+2)*}$.

Theorem 4.2.11 *The cells for $\mathcal{A}^{(4n+2)*}$, $n < \infty$, are given, up to equivalence, by the inequivalent solutions W^+, W^-:*

$$W^{\pm}_{i,i,i+1} = \frac{\sqrt{[2i][2i+2]}}{[2]\sqrt{[2i+1]}}\sqrt{[2i]\mp[1]}, \qquad i = 1,\dots,2n-1,$$

$$W^{\pm}_{i,i+1,i+1} = \frac{\sqrt{[2i][2i+2]}}{[2]\sqrt{[2i+1]}}\sqrt{[2i+2]\pm[1]}, \qquad i = 1,\dots,2n-1,$$

$$W^{\pm}_{i,i,i} = \begin{cases} (-1)^{i+1}\dfrac{\sqrt{[2i]}}{[2]\sqrt{[2i-1][2i+1]}}\sqrt{[2][2i]\pm[4i]}, & i = 1,\dots,n, \\ (-1)^{i+1}\dfrac{\sqrt{[2i]}}{[2]\sqrt{[2i-1][2i+1]}}\sqrt{[2][2i]\mp[8n+4-4i]}, & i = n+1,\dots,2n. \end{cases}$$

Proof

We again have (4.31)-(4.39), where equation (4.32) is now for $i = 1,\dots,n$. From (4.36) with $i = n$ we have

$$\begin{aligned}
&|W_{n,n,n}|^2 \\
&= \frac{[2n]^2}{[2]} - \frac{[2n-2][2n]}{[2]^2[2n-1]}([2n]\pm[1]) - \frac{[2n][2n+2]}{[2]^2[2n+1]}([2n]\mp[1]) \\
&= \frac{[2n]^2}{[2]^2[2n-1][2n+1]}([2][2n-1][2n+1] - [2n-2][2n+1] - [2n-1][2n+2]) \\
&\quad \mp\frac{[2n]}{[2]^2[2n-1][2n+1]}([2n-2][2n+1] - [2n-1][2n+2]) \\
&= \frac{[2n]^2}{[2]^2[2n-1][2n+1]}([2n][2n+1] - [2n-1][2n]) \\
&\quad \mp\frac{[2n]}{[2]^2[2n-1][2n+1]}([2n-2][2n+1] - [2n-1][2n]) \\
&= \frac{[2n]}{[2][2n-1][2n+1]}([2n]\pm[1]).
\end{aligned}$$

From (4.33) we find

$$|W_{n,n,n+1}|^2 = \frac{[2n]^2}{[2]^2[2n+1]}([2n]\mp[1]),$$

whilst from (4.32) we have

$$|W_{n,n+1,n+1}|^2 = \frac{[2n]^2}{[2]^2[2n+1]}([2n]\pm[1]).$$

Then the equations for $|W_{i-1,i-1,i}|$, $|W_{i-1,i,i}|$ and $|W_{i,i,i}|$ $i = n+1,\dots,2n$ follow. Equivalence of solutions follows as in the previous cases. $\square$

For $\mathcal{A}^{(2n)*}$, the cells W^+ above give the following representation of the Hecke algebra:

$$U^{(i,i+1)} = \begin{array}{c} i \\ \\ i+1 \end{array} \begin{pmatrix} \frac{[2i]-[1]}{[2i+1]} & \frac{\sqrt{([2i]-[1])([2i+2]+[1])}}{[2i+1]} \\ \frac{\sqrt{([2i]-[1])([2i+2]+[1])}}{[2i+1]} & \frac{[2i+2]+[1]}{[2i+1]} \end{pmatrix},$$

$$U^{(i,i-1)} = \begin{array}{c} i-1 \\ \\ i \end{array} \begin{pmatrix} \frac{[2i-2]-[1]}{[2i-1]} & \frac{\sqrt{([2i-2]-[1])([2i]+[1])}}{[2i-1]} \\ \frac{\sqrt{([2i-2]-[1])([2i]+[1])}}{[2i-1]} & \frac{[2i]+[1]}{[2i-1]} \end{pmatrix},$$

$$U^{(i,i)} = \begin{array}{c} i-1 \\ \\ i \\ \\ i+1 \end{array} \begin{pmatrix} \frac{[2i-2]([2i]+[1])}{[2i][2i+1]} & \frac{(-1)^{i+1}\sqrt{x[2i-2]([2i]+[1])}}{\sqrt{[2i][2i+1]}} & \frac{\sqrt{[2i-2][2i-1][2i+2]}}{[2i]\sqrt{[2i+1]}} \\ \frac{(-1)^{i+1}\sqrt{x[2i-2]([2i]+[1])}}{\sqrt{[2i][2i+1]}} & x & \frac{(-1)^{i+1}\sqrt{x[2i+2]([2i]-[1])}}{\sqrt{[2i][2i+1]}} \\ \frac{\sqrt{[2i-2][2i-1][2i+2]}}{[2i]\sqrt{[2i+1]}} & \frac{(-1)^{i+1}\sqrt{x[2i+2]([2i]-[1])}}{\sqrt{[2i][2i+1]}} & \frac{[2i+2]([2i]-[1])}{[2i][2i+1]} \end{pmatrix},$$

where, for positive integer m, if $n = 2m$,

$$x = \begin{cases} \frac{[2][2i]+[4i]}{[2i-1][2i][2i+1]} & \text{for } i = 1, \dots, m-1, \\ \frac{[2]}{[2m-1]^2} & \text{for } i = m, \\ \frac{[2][2i]-[4n-4i]}{[2i-1][2i][2i+1]} & \text{for } i = m+1, \dots, 2m-1, \end{cases},$$

and if $n = 2m + 1$,

$$x = \begin{cases} \frac{[2][2i]+[4i]}{[2i-1][2i][2i+1]} & \text{for } i = 1, \dots, m, \\ \frac{[2][2i]-[4n-4i]}{[2i-1][2i][2i+1]} & \text{for } i = m+1, \dots, 2m, \end{cases}.$$

Lemma 4.2.12 *The weights in the representation of the Hecke algebra given above for $\mathcal{A}^{(2n)*}$ are equivalent to the Boltzmann weights at criticality given by Behrend-Evans in [4].*

Proof

To make our notation the same as that of [4] one replaces i with $a/2$. To see that the absolute values of our weights are equal to those of the Boltzmann weights in [4] one needs the following relations on the quantum numbers:

$$[2i] + [1] = \frac{[2i+1]_{q'}[4i+2]_{q'}}{[2i-1]_{q'}}, \qquad [2i] - [1] = \frac{[2i-1]_{q'}[4i+2]_{q'}}{[2i+1]_{q'}},$$

where $q' = \sqrt{q}$ ($q = e^{i\pi/n}$). Again, a bit more work is required for $[U^{(i,i)}]_{i,i}$. For equivalence we make the same choice of $(u_{i,j})_{i,j}$ as for $\mathcal{A}^{(2n+1)*}$. □

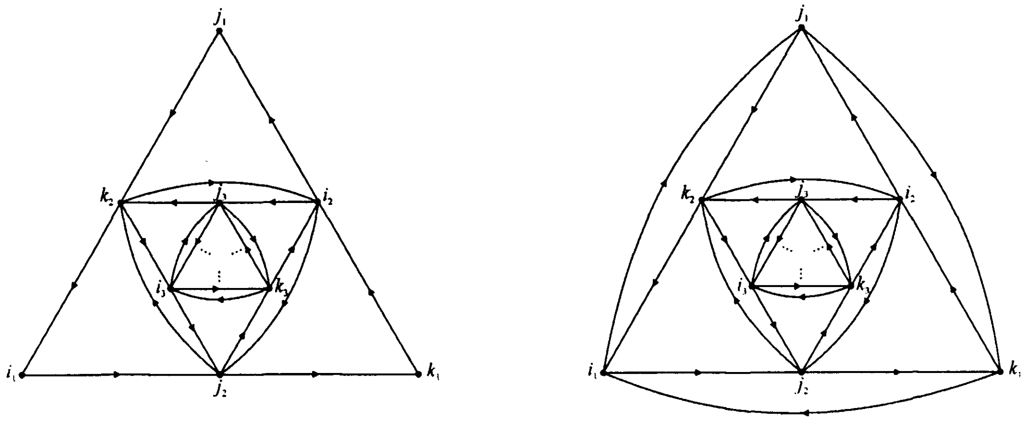


Figure 4.9: Labels for the vertices of $\mathcal{D}^{(2n+1)*}$ and $\mathcal{D}^{(2n)*}$

4.2.4 $\mathcal{D}^*$ graphs

We label the vertices of $\mathcal{D}^{(2n+1)*}$ by i_l, j_l and k_l, $l = 1, \dots, n$, as illustrated in Figure 4.9. The Perron-Frobenius weights are $\phi_{i_l} = \phi_{j_l} = \phi_{k_l} = [2l-1]$, $l = 1, \dots, n$. Since the graph has a $\mathbb{Z}_3$ symmetry, we will seek $\mathbb{Z}_3$-symmetric solutions (up to choice of phase), i.e. $|W_{i_p,j_q,k_r}|^2 = |W_{i_q,j_r,k_p}|^2 = |W_{i_r,j_p,k_q}|^2 =: |W_{p,q,r}|^2$, $p, q, r \in \{1, \dots, n\}$. Using this notation, we have the following equations from type I frames:

$$|W_{1,2,2}|^2 = [2][3], \tag{4.40}$$
$$|W_{l,l,l+1}|^2 + |W_{l,l+1,l+1}|^2 = [2][2l-1][2l+1], \qquad l = 2, \dots, n-1, \tag{4.41}$$
$$|W_{l-1,l,l}|^2 + |W_{l,l,l}|^2 + |W_{l,l,l+1}|^2 = [2][2l-1]^2, \qquad l = 2, \dots, n-1, \tag{4.42}$$
$$|W_{n-1,n,n}|^2 + |W_{n,n,n}|^2 = [2]^3, \tag{4.43}$$

and from type II frames we have:

$$|W_{l-1,l,l}|^2 |W_{l,l,l+1}|^2 = [2l-3][2l-1]^2[2l+1], \qquad l = 2, \dots, n-1, \tag{4.44}$$
$$|W_{l-1,l,l}|^2 \left(\frac{1}{[2l-3]} |W_{l-1,l-1,l}|^2 + \frac{1}{[2l-1]} |W_{l,l,l}|^2\right) = [2l-3][2l-1]^2, \quad l = 2, \dots, n, \tag{4.45}$$

which are exactly those for the type I and type II frames for the graph $\mathcal{A}^{(2n+1)*}$. Since the Perron-Frobenius weights and Coxeter number are also the same as for $\mathcal{A}^{(2n+1)*}$, the cells $|W_{p,q,r,}|$ follow.

From the type II frame consisting of the vertices i_l, j_l, i_{l+1} and j_{l+1} we have the following restriction on the choice of phase

$$\lambda_{i_l,j_l,k_{l+1}} \lambda_{i_l,j_{l+1},k_l} \lambda_{i_{l+1},j_l,k_l} \lambda_{i_{l+1},j_{l+1},k_{l+1}} = -\lambda_{i_l,j_l,k_l} \lambda_{i_l,j_{l+1},k_{l+1}} \lambda_{i_{l+1},j_l,k_{l+1}} \lambda_{i_{l+1},j_{l+1},k_l}. \tag{4.46}$$

Theorem 4.2.13 *Every $\mathbb{Z}_3$-symmetric solution for the cells W of $\mathcal{D}^{(2n+1)*}$, $n < \infty$, is equivalent to the solution:*

$$W_{i_{l-1},j_l,k_l} = W_{i_l,j_{l-1},k_l} = W_{i_l,j_l,k_{l-1}} = \frac{\sqrt{[l][2l-3][2l-1]}}{\sqrt{[l-1]}}, \quad l = 2,\dots,n,$$

$$W_{i_l,j_l,k_{l+1}} = W_{i_l,j_{l+1},k_l} = W_{i_{l+1},j_l,k_l} = \frac{\sqrt{[l-1][2l-1][2l+1]}}{\sqrt{[l]}}, \qquad l = 2,\dots,n-1,$$

$$W_{i_l,j_l,k_l} = (-1)^{l+1}\frac{[2l-1]}{\sqrt{[l-1][l]}}, \qquad l = 2,\dots,n.$$

Proof

Let $W^\sharp$ be any $\mathbb{Z}_3$-symmetric solution for the cells of $\mathcal{D}^{(2n+1)*}$, where the choice of phase satisfies the condition (4.46). Since $\mathcal{D}^{(2n+1)*}$ does not contain any multiple edges, we must have $|W^\sharp_{ijk}| = |W_{ijk}|$ for every triangle $\triangle_{ijk}$ of $\mathcal{D}^{(2n+1)*}$. We need to find a family of unitaries $\{u_{p,q}\}$, where $u_{p,q}$ is the unitary for the edge from vertex p to vertex q on $\mathcal{D}^{(2n+1)*}$, which satisfy (4.7), i.e. $-1 = u_{i_{2l},j_{2l}}u_{j_{2l},k_{2l}}u_{k_{2l},i_{2l}}\lambda^\sharp_{i_{2l},j_{2l},k_{2l}}$ for the triangle $\triangle_{i_{2l},j_{2l},k_{2l}}$, $l = 1,\dots,\lfloor n/2 \rfloor$, and $1 = u_{p_1}u_{p_2}u_{p_3}\lambda_{p_1,p_2,p_3}$ for all other triangles on $\mathcal{D}^{(2n+1)*}$. For triangles involving the outermost vertices, we require $1 = u_{i_1,j_2}u_{j_2,k_2}u_{k_2,i_1}\lambda^\sharp_{i_1,j_2,k_1}$, $1 = u_{i_2,j_1}u_{j_1,k_2}u_{k_2,i_2}\lambda^\sharp_{i_2,j_1,k_2}$, $1 = u_{i_2,j_2}u_{j_2,k_1}u_{k_1,i_2}\lambda^\sharp_{i_2,j_2,k_1}$ and $-1 = u_{i_2,j_2}u_{j_2,k_2}u_{k_2,i_2}\lambda^\sharp_{i_2,j_2,k_2}$. So we choose $u_{i_1,j_2} = u_{j_1,k_2} = u_{k_1,i_2} = u_{j_2,k_2} = u_{k_2,i_2} = 1$, $u_{i_2,j_1} = \overline{\lambda^\sharp_{i_2,j_1,k_2}}$, $u_{k_2,i_1} = \overline{\lambda^\sharp_{i_1,j_2,k_2}}$, $u_{i_2,j_2} = -\overline{\lambda^\sharp_{i_2,j_2,k_2}}$ and $u_{j_2,k_1} = -\lambda^\sharp_{i_2,j_2,k_2}\overline{\lambda^\sharp_{i_2,j_2,k_1}}$. Next consider the equations $1 = u_{i_2,j_3}u_{j_3,k_2}u_{k_2,i_2}\lambda^\sharp_{i_2,j_3,k_2}$, $1 = u_{i_3,j_2}u_{j_2,k_2}u_{k_2,i_3}\lambda^\sharp_{i_3,j_2,k_2}$ and $1 = u_{i_2,j_2}u_{j_2,k_3}u_{k_3,i_2}\lambda^\sharp_{i_2,j_2,k_3}$. We set $u_{i_2,j_3} = u_{j_2,k_3} = u_{k_2,i_3} = 1$, $u_{i_3,j_2} = \overline{\lambda^\sharp_{i_3,j_2,k_2}}$, $u_{j_3,k_2} = \overline{\lambda^\sharp_{i_2,j_3,k_2}}$ and $u_{k_3,i_2} = -\lambda^\sharp_{i_2,j_2,k_2}\overline{\lambda^\sharp_{i_2,j_2,k_3}}$. Next we consider the equations

$$1 = u_{i_2,j_3}u_{j_3,k_3}u_{k_3,i_2}\lambda^\sharp_{i_2,j_3,k_3} = -u_{j_3,k_3}\lambda^\sharp_{i_2,j_2,k_2}\overline{\lambda^\sharp_{i_2,j_2,k_3}}\lambda^\sharp_{i_2,j_3,k_3},$$

$$1 = u_{i_3,j_2}u_{j_2,k_3}u_{k_3,i_3}\lambda^\sharp_{i_3,j_2,k_3} = u_{k_3,i_3}\overline{\lambda^\sharp_{i_3,j_2,k_2}}\lambda^\sharp_{i_3,j_2,k_3},$$

$$1 = u_{i_3,j_3}u_{j_3,k_2}u_{k_2,i_3}\lambda^\sharp_{i_3,j_3,k_2} = u_{i_3,j_3}\overline{\lambda^\sharp_{i_2,j_3,k_2}}\lambda^\sharp_{i_3,j_3,k_2}.$$

We make the choices $u_{i_3,j_3} = \lambda^\sharp_{i_2,j_3,k_2}\overline{\lambda^\sharp_{i_3,j_3,k_2}}$, $u_{k_3,i_3} = \lambda^\sharp_{i_3,j_2,k_2}\overline{\lambda^\sharp_{i_3,j_2,k_3}}$ and $u_{j_3,k_3} = -\lambda^\sharp_{i_2,j_2,k_3}\overline{\lambda^\sharp_{i_2,j_2,k_2}\lambda^\sharp_{i_2,j_3,k_3}}$. Then

$$u_{i_3,j_3}u_{j_3,k_3}u_{k_3,i_3}\lambda^\sharp_{i_3,j_3,k_3} = -\lambda^\sharp_{i_2,j_3,k_2}\overline{\lambda^\sharp_{i_3,j_3,k_2}}\lambda^\sharp_{i_2,j_2,k_3}\overline{\lambda^\sharp_{i_2,j_2,k_2}\lambda^\sharp_{i_2,j_3,k_3}}\lambda^\sharp_{i_3,j_2,k_2}\overline{\lambda^\sharp_{i_3,j_2,k_3}} = -1,$$

by (4.46), as required. Continuing in this way we are done. □

For $\mathcal{D}^{(2n+1)*}$, the Hecke representation for the cells W above is given by the Hecke representation for $\mathcal{A}^{(2n+1)*}$, where $[U^{(i_l,k_r)}]_{j_m,j_p} = [U^{(j_l,i_r)}]_{k_m,k_p} = [U^{(k_l,j_r)}]_{i_m,i_p}$ are given by the weights $[U^{(l,r)}]_{m,p}$ for $\mathcal{A}^{(2n+1)*}$, for any l,m,p,r allowed by the graph.

We now consider the graphs $\mathcal{D}^{(2n)*}$. Labelling the vertices of $\mathcal{D}^{(2n)*}$ by i_l, j_l and k_l, for $l = 1, \ldots, n-1$ (as in Figure 4.9), the Perron-Frobenius weights are $\phi_{i_l} = \phi_{j_l} = \phi_{k_l} = [2l]/[2]$, and we again assume $|W_{i_p,j_q,k_r}|^2 = |W_{i_q,j_r,k_p}|^2 = |W_{i_r,j_p,k_q}|^2 =: |W_{p,q,r}|^2$, where $p, q, r \in \{1, \ldots, n-1\}$. Then as for $\mathcal{D}^{(2n+1)*}$, the $\mathbb{Z}_3$-symmetric solution for the cells follows from the solution for $\mathcal{A}^{(2n)*}$, and we have the same restriction (4.46) on the choice of phase. So we have

Theorem 4.2.14 *For $n < \infty$, the $\mathbb{Z}_3$-symmetric solution for the cells of $\mathcal{D}^{(4n)*}$ are given by:*

$$W^{\pm}_{i_l,j_l,k_{l+1}} = W^{\pm}_{i_l,j_{l+1},k_l} = W^{\pm}_{i_{l+1},j_l,k_l} = \frac{\sqrt{[2l][2l+2]}}{[2]\sqrt{[2l+1]}}\sqrt{[2l] \mp [1]}, \qquad l = 1, \ldots, 2n-2,$$

$$W^{\pm}_{i_l,j_{l+1},k_{l+1}} = W^{\pm}_{i_{l+1},j_l,k_{l+1}} = W^{\pm}_{i_{l+1},j_{l+1},k_l} = \frac{\sqrt{[2l][2l+2]}}{[2]\sqrt{[2l+1]}}\sqrt{[2l+2] \pm [1]}, \quad l = 1, \ldots, 2n-2,$$

$$W^{\pm}_{i_l,j_l,k_l} = \begin{cases} (-1)^{l+1}\dfrac{\sqrt{[2l]}}{[2]\sqrt{[2l-1][2l+1]}}\sqrt{[2][2l] \pm [4l]}, & l = 1, \ldots, n-1, \\ (-1)^{n+1}\dfrac{[2n]}{\sqrt{[2][2n-1][2n+1]}}, & l = n, \\ (-1)^{l+1}\dfrac{\sqrt{[2l]}}{[2]\sqrt{[2l-1][2l+1]}}\sqrt{[2][2l] \mp [8n-4l]}, & l = n+1, \ldots, 2n-1, \end{cases}$$

and the $\mathbb{Z}_3$-symmetric solution for the cells of $\mathcal{D}^{(4n+2)*}$ are:

$$W^{\pm}_{i_l,j_l,k_{l+1}} = W^{\pm}_{i_l,j_{l+1},k_l} = W^{\pm}_{i_{l+1},j_l,k_l} = \frac{\sqrt{[2l][2l+2]}}{[2]\sqrt{[2l+1]}}\sqrt{[2l] \mp [1]}, \qquad l = 1, \ldots, 2n-1,$$

$$W^{\pm}_{i_l,j_{l+1},k_{l+1}} = W^{\pm}_{i_{l+1},j_l,k_{l+1}} = W^{\pm}_{i_{l+1},j_{l+1},k_l} = \frac{\sqrt{[2l][2l+2]}}{[2]\sqrt{[2l+1]}}\sqrt{[2l+2] \pm [1]}, \quad l = 1, \ldots, 2n-1,$$

$$W^{\pm}_{i_l,j_l,k_l} = \begin{cases} (-1)^{l+1}\dfrac{\sqrt{[2l]}}{[2]\sqrt{[2l-1][2l+1]}}\sqrt{[2][2l] \pm [4l]}, & l = 1, \ldots, n, \\ (-1)^{l+1}\dfrac{\sqrt{[2l]}}{[2]\sqrt{[2l-1][2l+1]}}\sqrt{[2][2l] \mp [8n+4-4l]}, & l = n+1, \ldots, 2n. \end{cases}$$

The uniqueness of these solutions follows in the same way as for $\mathcal{D}^{(2n+1)*}$. If W^+ is a solution for the cells of $\mathcal{D}^{(2n)*}$, then W^- is a solution for the cells of the graph where we switch vertices $i_l \leftrightarrow i_{n-l}$, $j_l \leftrightarrow j_{n-l}$ and $k_l \leftrightarrow k_{n-l}$, for all $l = 1, \ldots, n-1$.

For $\mathcal{D}^{(2n)*}$, the Hecke representation for the cells W^+ above is given by the Hecke representation for $\mathcal{A}^{(2n)*}$, where $[U^{(i_l,k_r)}]_{j_m,j_p} = [U^{(j_l,i_r)}]_{k_m,k_p} = [U^{(k_l,j_r)}]_{i_m,i_p}$ are given by the weights $[U^{(l,r)}]_{m,p}$ for $\mathcal{A}^{(2n)*}$, for any l, m, p, r allowed by the graph.

In [27], di Francesco and Zuber gave a representation of the Hecke algebra for the graph $\mathcal{D}^{(6)*}$, with the absolute values of the weights there equal to those for our weights given above. The two Hecke representations are not identical as the weights in [27] involve the complex variable i. However it has not been possible to determine whether or not the two representations are equivalent as there are known to be a number of typographical errors in the representation in [27].

4.2.5 $\mathcal{E}^{(8)}$

We will label the graph $\mathcal{E}^{(8)}$ in the following way. We will label the six outmost vertices by i_l and the six inmost vertices by j_l, $l = 1, \dots, 6$, such that there are edges from i_l to j_l and from j_l to i_{l+1}. The Perron-Frobenius weights on the vertices are $\phi_{i_l} = 1$, $\phi_{j_l} = [3]$. With $[a] = [a]_q$, $q = e^{i\pi/8}$, we have $[4]/[2] = \sqrt{2}$.

We will again use the notation $W_{i,j,k}$ for $W(\triangle_{i,j,k})$. Then from the type I frames on the graph we have the following equations:

$$|W_{i_l,j_l,j_{l-1}}|^2 = [2]\phi_{i_l}\phi_{j_l} = [2][3],$$

$$|W_{i_l,j_l,j_{l-1}}|^2 + |W_{j_{l+1},j_l,j_{l-1}}|^2 + |W_{j_l,j_{l-1},j_{l-2}}|^2 = [2]\phi_{j_l}\phi_{j_{l-1}} = [2][3]^2.$$

Then $|W_{j_{l+1},j_l,j_{l-1}}|^2 + |W_{j_l,j_{l-1},j_{l-2}}|^2 = [3][4]$. Since there is a $\mathbb{Z}_6$ symmetry of $\mathcal{E}^{(8)}$ we assume $|W_{j_{l+1},j_l,j_{l-1}}|^2 = |W_{j_{k+1},j_k,j_{k-1}}|^2$ for all k, l, giving

$$|W_{j_{l+1},j_l,j_{l-1}}|^2 = \frac{1}{2}[3][4] = \frac{[2]^2[3]}{[4]}.$$

The $\mathbb{Z}_6$ symmetry of the cells can be deduced from equation (4.48). Finally, for the type I frames $\overset{j_l}{\bullet}\!\rightarrow\!\overset{j_{l+2}}{\bullet}$ we have $|W_{j_{l+2},j_{l+1},j_l}|^2 + |W_{j_l,j_{l+2},j_{l+4}}|^2 = [2][3]^2$ giving

$$|W_{j_l,j_{l+2},j_{l+4}}|^2 = [2][3]^2 - \frac{[2]^2[3]}{[4]} = \frac{[2]^2[3]^2}{[4]}.$$

Let

$$\begin{aligned} W_{i_l,j_l,j_{l-1}} &= \lambda_{i_l}\sqrt{[2][3]}, \quad l = 1, \dots, 6, \\ W_{j_l,j_{l-1},j_{l-2}} &= \lambda_{j_l}^{(1)}\frac{[2]\sqrt{[3]}}{\sqrt{[4]}}, \quad l = 1, \dots, 6, \\ W_{j_l,j_{l+2},j_{l+4}} &= \lambda_{j_l}^{(2)}\frac{[2][3]}{\sqrt{[4]}}, \quad l = 1, 2. \end{aligned} \tag{4.47}$$

The only type II frames that yield anything new are those for the frame involving the vertices j_{l-2}, $j_{l-3}(=j_{l+3})$, j_{l+1} and j_l:

$$\begin{aligned}
0 &= \phi_{j_{l-1}}^{-1} W_{j_{l-2},j_{l-1},j_l} \overline{W_{j_{l+1},j_l,j_{l-1}}} W_{j_{l-1},j_{l+1},j_{l+3}} \overline{W_{j_{l-1},j_{l-2},j_{l-3}}} \\
&\quad + \phi_{j_{l+2}}^{-1} W_{j_{l-2},j_l,j_{l+2}} \overline{W_{j_{l+2},j_{l+1},j_l}} W_{j_{l+3},j_{l+2},j_{l+1}} \overline{W_{j_{l-2},j_{l-3},j_{l+2}}} \\
&= \frac{[2]^4\sqrt{[3]^3}}{[4]^2} \lambda_{j_l}^{(1)} \lambda_{j_{l+2}}^{(1)} \lambda_{j_{l+4}}^{(1)} \lambda_{j_{l-1}}^{(2)} + \frac{[2]^4\sqrt{[3]^3}}{[4]^2} \lambda_{j_{l-1}}^{(1)} \lambda_{j_{l+1}}^{(1)} \lambda_{j_{l+3}}^{(1)} \lambda_{j_l}^{(2)},
\end{aligned} \tag{4.48}$$

which for any $l = 1, \ldots, 6$ gives

$$\lambda_{j_1}^{(1)} \lambda_{j_3}^{(1)} \lambda_{j_5}^{(1)} \lambda_{j_2}^{(2)} = -\lambda_{j_2}^{(1)} \lambda_{j_4}^{(1)} \lambda_{j_6}^{(1)} \lambda_{j_1}^{(2)}. \tag{4.49}$$

From the type II frame above we see that there must be a $\mathbb{Z}_6$ symmetry on the cells, $|W_{j_{l+1},j_l,j_{l-1}}|^2 = |W_{j_{k+1},j_k,j_{k-1}}|^2$ for all k, l, is correct since otherwise the coefficients of the two terms in equation (4.48) would be different, and (4.49) would be

$$\lambda_{j_1}^{(1)} \lambda_{j_3}^{(1)} \lambda_{j_5}^{(1)} \lambda_{j_2}^{(2)} = -c\lambda_{j_2}^{(1)} \lambda_{j_4}^{(1)} \lambda_{j_6}^{(1)} \lambda_{j_1}^{(2)},$$

for some constant $c \in \mathbb{R}$ with $|c| \neq 1$, which is impossible.

Theorem 4.2.15 *There is up to equivalence a unique set of cells for $\mathcal{E}^{(8)}$ given by:*

$$\begin{aligned}
W_{i_l,j_l,j_{l-1}} &= \sqrt{[2][3]}, & W_{j_l,j_{l-1},j_{l-2}} &= \frac{[2]\sqrt{[3]}}{\sqrt{[4]}}, & l = 1, \ldots, 6, \\
W_{j_1,j_3,j_5} &= \frac{[2][3]}{\sqrt{[4]}}, & W_{j_2,j_4,j_6} &= -\frac{[2][3]}{\sqrt{[4]}}
\end{aligned}$$

Proof

Let $W^\sharp$ be any solution for the cells of $\mathcal{E}^{(8)}$, where the choice of phase satisfies the condition (4.49). We need to find a family of unitaries $\{u_{p,q}\}$, where $u_{p,q}$ is the unitary for the edge from vertex p to vertex q on $\mathcal{E}^{(8)}$, which satisfy (4.7), i.e. $-1 = u_{j_2,j_4} u_{j_4,j_6} u_{j_6,j_2} \lambda_{j_2}^{(2)}$ for the triangle $\triangle_{j_2,j_4,j_6}$, and $1 = u_{p_1} u_{p_2} u_{p_3} \lambda_{p_1,p_2,p_3}$ for all other triangles, where λ_{p_1,p_2,p_3} is the phase associated to triangle $\triangle_{p_1,p_2,p_3}$. We choose $u_{i_l,j_l} = \overline{u_{j_l,j_{l-1}} \lambda_{i_l}}$, $u_{j_l,j_{l+1}} = 1$ for $l = 1, \ldots, 6$, $u_{j_2,j_1} = u_{j_5,j_4} = 1$, $u_{j_1,j_6} = \overline{\lambda_{j_2}^{(1)}}$, $u_{j_3,j_2} = \lambda_{j_2}^{(1)} \lambda_{j_6}^{(1)} \lambda_{j_1}^{(2)} \overline{\lambda_{j_1}^{(1)} \lambda_{j_3}^{(1)}}$, $u_{j_4,j_3} = \overline{\lambda_{j_5}^{(1)}}$, $u_{j_6,j_5} = \overline{\lambda_{j_6}^{(1)}}$, $u_{j_3,j_5} = u_{j_4,j_6} = u_{j_6,j_2} = 1$, $u_{j_1,j_3} = \lambda_{j_2}^{(1)} \overline{\lambda_{j_2}^{(1)} \lambda_{j_6}^{(1)} \lambda_{j_1}^{(2)}}$, $u_{j_2,j_4} = \lambda_{j_2}^{(1)} \lambda_{j_3}^{(1)} \lambda_{j_5}^{(1)} \overline{\lambda_{j_2}^{(1)} \lambda_{j_4}^{(1)} \lambda_{j_6}^{(1)} \lambda_{j_1}^{(2)}}$ and $u_{j_5,j_1} = \lambda_{j_2}^{(1)} \lambda_{j_6}^{(1)} \overline{\lambda_{j_1}^{(1)}}$. □

For $\mathcal{E}^{(8)}$, the above cells W give the following representation of the Hecke algebra:

$$U^{(i_l,j_{l-1})} = U^{(j_l,i_l)} = [2],$$

$$
\begin{array}{rcl}
U^{(j_l,j_{l-2})} & = & \begin{array}{c} j_{l-1} \\ j_{l+2} \end{array} \left(\begin{array}{cc} \frac{1}{[2]} & \frac{(-1)^{l+1}\sqrt{[3]}}{[2]} \\ \frac{(-1)^{l+1}\sqrt{[3]}}{[2]} & \frac{[3]}{[2]} \end{array} \right), \\
U^{(j_l,j_{l+1})} & = & \begin{array}{c} j_{l-1} \\ j_{l+2} \\ i_{l+1} \end{array} \left(\begin{array}{ccc} \frac{1}{[2]} & \frac{1}{[2]} & \frac{1}{\sqrt{[3]}} \\ \frac{1}{[2]} & \frac{1}{[2]} & \frac{1}{\sqrt{[3]}} \\ \frac{1}{\sqrt{[3]}} & \frac{1}{\sqrt{[3]}} & \frac{[2]}{[3]} \end{array} \right),
\end{array}
$$

for $l = 1, \dots, 6 \pmod 6$. This representation is identical to that given by di Francesco-Zuber in [27]. (The representation in [27] is given for the graph $\mathcal{E}^{(8)*}$, and the representation for $\mathcal{E}^{(8)}$ is obtained by an unfolding of the graph $\mathcal{E}^{(8)*}$.)

4.2.6 $\mathcal{E}^{(8)*}$

We will label the vertices of the graph $\mathcal{E}^{(8)*}$ as in Figure 1.12. The Perron-Frobenius weights for $\mathcal{E}^{(8)*}$ are $\phi_1 = \phi_4 = 1$, $\phi_2 = \phi_3 = [3]$. As with the graphs $\mathcal{A}^{(n)}$ and $\mathcal{E}^{(8)}$ we easily find $|W_{123}|^2 = [2][3]$ and $|W_{234}|^2 = [2][3]$. Then by the type II frame $\overset{1}{\bullet}\!\rightarrow\!\overset{2}{\bullet}\!\leftarrow\!\overset{2}{\bullet}$ we have $[3]^{-1}|W_{123}|^2|W_{223}|^2 = [3]^2$, and so $|W_{223}|^2 = [3]^2/[2]$. Similarly $|W_{233}|^2 = [3]^2/[2]$. From the type I frame $\overset{2}{\bullet}\!\rightarrow\!\overset{2}{\bullet}$ we get $|W_{222}|^2+|W_{223}|^2 = [2][3]^2$, giving $|W_{222}|^2 = [3]^3/[2]$, and similarly $|W_{333}|^2 = [3]^3/[2]$. Let $W_{ijk} = \lambda_{ijk}|W_{ijk}|$. Then from the type II frame consisting of the vertices 2,2,3,3 we obtain the following restriction on the choice of phase:

$$\lambda_{222}\lambda_{233}^3 = -\lambda_{333}\lambda_{223}^3. \tag{4.50}$$

Theorem 4.2.16 *There is up to equivalence a unique set of cells for $\mathcal{E}^{(8)*}$ given by:*

$$
\begin{array}{rclcl}
W_{123} & = & W_{234} & = & \sqrt{[2][3]}, \\
W_{223} & = & W_{233} & = & \frac{[3]}{\sqrt{[2]}}, \\
W_{222} & = & \frac{\sqrt{[3]^3}}{\sqrt{[2]}}, & W_{333} = & -\frac{\sqrt{[3]^3}}{\sqrt{[2]}}.
\end{array}
$$

Proof

Let $W^\sharp$ be any cell system for $\mathcal{E}^{(8)*}$, where the choice of phase satisfies the condition (4.50). We need to find a family of unitaries $\{u_{p,q}\}$, where $u_{p,q}$ is the unitary for the edge from vertex p to vertex q on $\mathcal{E}^{(8)*}$, which satisfy (4.7), i.e. $-1 = u_{3,3}^3\lambda_{333}$ for the triangle $\triangle_{3,3,3}$, and $1 = u_{i,j}u_{j,k}u_{k,i}\lambda_{ijk}$ for all other triangles, where λ_{ijk} is the phase associated to triangle $\triangle_{i,j,k}$. We choose $u_{3,1} = u_{3,2} = u_{4,3} = 1$, $u_{2,4} = \overline{\lambda_{234}}$, $u_{3,3} = -\overline{\lambda_{333}}^{\frac{1}{3}}$, $u_{2,3} = -\lambda_{333}^{\frac{1}{3}}\overline{\lambda_{233}}$, $u_{1,2} = -\lambda_{233}\overline{\lambda_{123}\lambda_{333}}^{\frac{1}{3}}$ and $u_{2,2} = -\lambda_{233}\overline{\lambda_{223}\lambda_{333}}^{\frac{1}{3}}$. $\square$

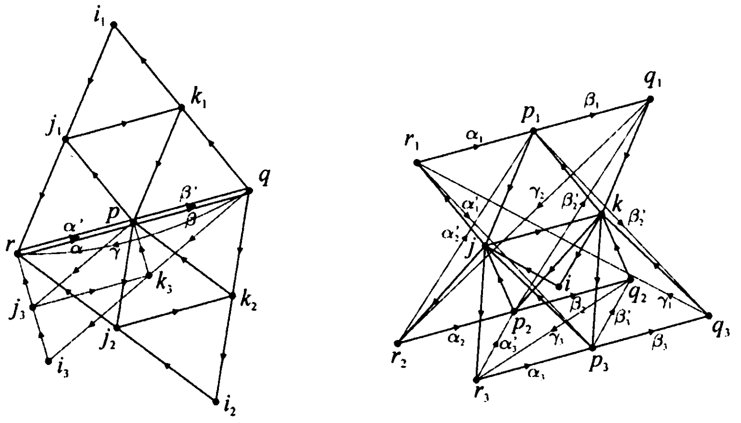


Figure 4.10: $\mathcal{E}_1^{(12)}$ and $\mathcal{E}_2^{(12)}$

For $\mathcal{E}^{(8)*}$, the above cells W give the following Hecke representation:

$$
\begin{aligned}
U^{(1,3)} &= U^{(2,1)} = U^{(3,4)} = U^{(4,2)} = [2], \\
U^{(2,2)} &= \begin{array}{c} 3 \\ 2 \end{array} \begin{pmatrix} \frac{1}{[2]} & \frac{\sqrt{[3]}}{[2]} \\ \frac{\sqrt{[3]}}{[2]} & \frac{[3]}{[2]} \end{pmatrix}, \\
U^{(3,3)} &= \begin{array}{c} 2 \\ 3 \end{array} \begin{pmatrix} \frac{1}{[2]} & -\frac{\sqrt{[3]}}{[2]} \\ -\frac{\sqrt{[3]}}{[2]} & \frac{[3]}{[2]} \end{pmatrix}, \\
U^{(2,3)} &= \begin{array}{c} 2 \\ 3 \\ 4 \end{array} \begin{pmatrix} \frac{1}{[2]} & \frac{1}{[2]} & \frac{1}{\sqrt{[3]}} \\ \frac{1}{[2]} & \frac{1}{[2]} & \frac{1}{\sqrt{[3]}} \\ \frac{1}{\sqrt{[3]}} & \frac{1}{\sqrt{[3]}} & \frac{[2]}{[3]} \end{pmatrix}. \\
&= U^{(3,2)} \text{ with rows labelled by } 2, 3, 1.
\end{aligned}
$$

This representation is identical to that given by di Francesco-Zuber in [27].

4.2.7 $\mathcal{E}_2^{(12)}$

We label the vertices and edges of the graph $\mathcal{E}_2^{(12)}$ as in Figure 4.10. The Perron-Frobenius weights for $\mathcal{E}_2^{(12)}$ are

$$\phi_i = 1, \qquad \phi_j = \phi_k = [3], \qquad \phi_{p_l} = \frac{[2]^3}{[4]}, \qquad \phi_{q_l} = \phi_{r_l} = \frac{[2][3]}{[4]}, \qquad l = 1, 2, 3.$$

With $[a] = [a]_q$, $q = e^{i\pi/12}$, we have $[2]^2 = [5]$ and $[3]^2 = [1] + [3] + [5] = [2]^2 + [7] = [5] + [7] = [2][6]$.

As in the $\mathcal{A}$ case, we have $|W_{i,j,k}|^2 = [2][3]$. Then from the type II frames $\overset{p_l}{\bullet}\rightarrow\overset{j}{\bullet}\leftarrow\overset{i}{\bullet}$ $(l = 1,2,3)$ we have $\phi_k^{-1}|W_{p_l,j,k}|^2|W_{i,j,k}|^2 = \phi_i\phi_j\phi_{p_l}$ giving $|W_{p_l,j,k}|^2 = [2]^2[3]/[4]$, $l = 1,2,3$. From the type I frames $\overset{p_l}{\bullet}\rightarrow\overset{j}{\bullet}$ $(l = 1,2,3)$ we have

$$|W_{p_l,j,r_l}|^2 + |W_{p_l,j,r_{l+1}}|^2 + |W_{p_l,j,k}|^2 = [2]\phi_{p_l}\phi_j = \frac{[2]^4[3]}{[4]},$$

giving

$$|W_{p_l,j,r_l}|^2 + |W_{p_l,j,r_{l+1}}|^2 = \frac{[2]^2[3]}{[4]}([2]^2-1) = \frac{[2]^2[3]^2}{[4]}, \quad l = 1,2,3. \tag{4.51}$$

Similarly,

$$|W_{p_l,q_l,k}|^2 + |W_{p_l,q_{l-1},k}|^2 = \frac{[2]^2[3]}{[4]}([2]^2-1) = \frac{[2]^2[3]^2}{[4]}, \quad l = 1,2,3. \tag{4.52}$$

From the type II frames $\overset{q_l}{\bullet}\rightarrow\overset{k}{\bullet}\leftarrow\overset{q_{l-1}}{\bullet}$ $(l = 1,2,3)$ we get

$$\phi_{p_l}^{-1}|W_{p_l,q_l,k}|^2|W_{p_l,q_{l-1},k}|^2 = \phi_{q_{l-1}}\phi_k\phi_{q_l} = \frac{[2]^2[3]^3}{[4]^2},$$

and substituting for $|W_{p_l,q_{l-1},k}|^2$ in equation (4.52) we obtain the quadratic equation

$$[4]^3|W_{p_l,q_l,k}|^4 - [2]^2[3]^3[4]^3|W_{p_l,q_l,k}|^2 + [2]^5[3]^3 = 0.$$

Solving, we get

$$|W_{p_l,q_l,k}|^2 = \frac{[2]^3}{[4]^2}([2][4] \pm \sqrt{[2][4]}), \quad l = 1,2,3.$$

Then from (4.52) we have

$$|W_{p_l,q_{l-1},k}|^2 = \frac{[2]^3}{[4]^2}([2][4] \mp \sqrt{[2][4]}), \quad l = 1,2,3.$$

Next consider the type I frames $\overset{q_l}{\bullet}\rightarrow\overset{k}{\bullet}$ $(l = 1,2,3)$: $|W_{p_l,q_{l-1},r_l}|^2 + |W_{p_l,q_{l-1},k}|^2 = [2]^5[3]/[4]^2$ giving

$$|W_{p_l,q_{l-1},r_l}|^2 = \frac{[2]^3}{[4]^2}([2]^2 \pm \sqrt{[2][4]}), \quad l = 1,2,3.$$

Then by considering the type I frames $\overset{r_l}{\bullet}\rightarrow\overset{p_l}{\bullet}$ for $l = 1,2,3$ we obtain

$$|W_{p_l,j,r_l}|^2 = \frac{[2]^3}{[4]^2}([2][4] \mp \sqrt{[2][4]}), \quad l = 1,2,3,$$

and from (4.51)

$$|W_{p_l,j,r_{l+1}}|^2 = \frac{[2]^3}{[4]^2}([2][4] \pm \sqrt{[2][4]}), \quad l = 1,2,3.$$

Finally, from the type I frames $\overset{r_l}{\bullet}\!\!\rightarrow\!\overset{p_{l-1}}{\bullet}$ ($l = 1,2,3$) we get

$$|W_{p_l,q_l,r_{l+1}}|^2 = \frac{[2]^3}{[4]^2}([2]^2 \mp \sqrt{[2][4]}), \quad l = 1,2,3.$$

Let $W_{v_1,v_2,v_3} = \lambda_{v_1,v_2,v_3}|W_{v_1,v_2,v_3}|$ for vertices v_1, v_2, v_3 of $\mathcal{E}_2^{(12)}$. The type II frames consisting of the vertices p_l, k, p_{l-1} and r_l give a restriction on the phases λ_{v_1,v_2,v_3}:

$$\begin{aligned}
0 &= \phi_{q_{l-1}}^{-1} W_{p_{l-1},q_{l-1},r_l}\overline{W_{p_{l-1},q_{l-1},k}}W_{p_l,q_{l-1},k}\overline{W_{p_l,q_{l-1},r_l}} \\
&\quad +\phi_j^{-1} W_{p_{l-1},j,r_l}\overline{W_{p_{l-1},j,k}}W_{p_l,j,k}\overline{W_{p_l,j,r_l}} \\
&= \sqrt{\frac{[2]^9[3]^3}{[4]^5}}\lambda_{p_{l-1},q_{l-1},r_l}\lambda_{p_l,q_{l-1},k}\overline{\lambda_{p_{l-1},q_{l-1},k}\lambda_{p_l,q_{l-1},r_l}} \\
&\quad +\sqrt{\frac{[2]^9[3]^3}{[4]^5}}\lambda_{p_{l-1},j,r_l}\lambda_{p_l,j,k}\overline{\lambda_{p_{l-1},j,k}\lambda_{p_l,j,r_l}},
\end{aligned}$$

so we have, for $l = 1,2,3$,

$$\lambda_{p_{l-1},q_{l-1},r_l}\lambda_{p_l,q_{l-1},k}\overline{\lambda_{p_{l-1},q_{l-1},k}\lambda_{p_l,q_{l-1},r_l}} = -\lambda_{p_{l-1},j,r_l}\lambda_{p_l,j,k}\overline{\lambda_{p_{l-1},j,k}\lambda_{p_l,j,r_l}}. \tag{4.53}$$

Then there are two solutions W^+, W^- for the cell system for $\mathcal{E}_2^{(12)}$.

Theorem 4.2.17 *Every solution for the cells of $\mathcal{E}_2^{(12)}$ is either equivalent to the solution W^+ or the inequivalent conjugate solution W^-, given by:*

$$W^{\pm}_{i,j,k} = \sqrt{[2][3]}, \qquad W^{\pm}_{p_l,j,k} = \frac{[2]\sqrt{[3]}}{\sqrt{[4]}},$$

$$W^{\pm}_{p_l,q_{l-1},r_l} = \frac{\sqrt{[2]}^3}{[4]}\sqrt{[2]^2 \pm \sqrt{[2][4]}}, \qquad W^{\pm}_{p_l,q_l,r_{l+1}} = -\frac{\sqrt{[2]}^3}{[4]}\sqrt{[2]^2 \mp \sqrt{[2][4]}},$$

$$W^{\pm}_{p_l,q_l,k} = W^{\pm}_{p_l,j,r_{l+1}} = \frac{\sqrt{[2]}^3}{[4]}\sqrt{[2][4] \pm \sqrt{[2][4]}},$$

$$W^{\pm}_{p_l,q_{l-1},k} = W^{\pm}_{p_l,j,r_l} = \frac{\sqrt{[2]}^3}{[4]}\sqrt{[2][4] \mp \sqrt{[2][4]}},$$

for $l = 1,2,3$.

Proof

Let $W^\sharp$ be another solution for the cells of $\mathcal{E}_2^{(12)}$, with $W^\sharp_{v_1,v_2,v_3} = \lambda^\sharp_{v_1,v_2,v_3}|W^+_{v_1,v_2,v_3}|$, and where the $\lambda^\sharp$'s satisfy the condition (4.53). We need to find unitaries $u_{v_1,v_2} \in \mathbb{T}$ such that $u_{p_l,q_l}u_{q_l,r_{l+1}}u_{r_{l+1},p_l}\lambda^\sharp_{p_l,q_l,r_{l+1}} = -1$, $l = 1,2,3$, and $u_{v_1,v_2}u_{v_2,v_3}u_{v_3,v_1}\lambda^\sharp_{v_1,v_2,v_3} = 1$ for all other triangles $\triangle_{v_1,v_2,v_3}$ on $\mathcal{E}_2^{(12)}$. We choose $u_{j,k} = u_{k,i} = u_{j,r_l} = u_{q_l,k} = u_{r_{l+1},p_l} = 1$, $u_{i,j} = \overline{\lambda^\sharp_{i,j,k}}$,

$u_{p_l,j} = \overline{\lambda^{\sharp}_{p_l,j,r_{l+1}}}$, $u_{k,p_l} = \lambda^{\sharp}_{p_l,j,r_{l+1}} \overline{\lambda^{\sharp}_{p_l,j,k}}$, $u_{r_l,p_l} = \lambda^{\sharp}_{p_l,j,r_{l+1}} \overline{\lambda^{\sharp}_{p_l,j,r_l}}$, $u_{p_l,q_l} = \lambda^{\sharp}_{p_l,j,k} \overline{\lambda^{\sharp}_{p_l,q_l,k} \lambda^{\sharp}_{p_l,j,r_{l+1}}}$, $u_{p_l,q_{l-1}} = \lambda^{\sharp}_{p_l,j,k} \overline{\lambda^{\sharp}_{p_l,q_{l-1},k} \lambda^{\sharp}_{p_l,j,r_{l+1}}}$, and $u_{q_l,r_{l+1}} = -\lambda^{\sharp}_{p_l,j,r_{l+1}} \lambda^{\sharp}_{p_l,q_l,k} \overline{\lambda^{\sharp}_{p_l,j,k} \lambda^{\sharp}_{p_l,j,r_{l+1}}}$, for $l = 1,2,3$.

Similarly, for any solution $W^{\sharp\sharp}$ with $|W^{\sharp\sharp}_{v_1,v_2,v_3}| = |W^{-}_{v_1,v_2,v_3}|$.

The solutions W^+ and W^- are not equivalent since $|W^+| \neq |W^-|$, and there are no double edges on $\mathcal{E}_2^{(12)}$. We remark that the complex conjugate solutions $\overline{W^{\pm}}$ are equivalent to the solutions $W^{\mp}$: we choose a family of unitaries which satisfy (4.5) by $u_{i_l,j_l} = u_{j_l,k_l} = u_{k_l,i_l} = u_{p,j_l} = u_{j_l,r} = u_{q,k_l} = u_{k_l,p} = 1$, $u_{q,r} = -1$, and 2×2 unitary matrices $u_\alpha = u_\beta = u$ where u is given by $u(i,j) = 1 - \delta_{i,j}$. $\square$

For $\mathcal{E}_2^{(12)}$, the cells W^+ above give the following representation of the Hecke algebra, where $l = 1,2,3$ (mod 3):

$$U^{(i,k)} = U^{(j,i)} = [2],$$

$$U^{(k,j)} = \begin{array}{c} i \\ p_l \end{array} \begin{pmatrix} \frac{[2]}{[3]} & \frac{\sqrt{[2]^3}}{[3]\sqrt{[4]}} \\ \frac{\sqrt{[2]^3}}{[3]\sqrt{[4]}} & \frac{[2]^2}{[3][4]} \end{pmatrix},$$

$$\begin{aligned} U^{(r_l,j)} &= \begin{array}{c} p_{l-1} \\ p_l \end{array} \begin{pmatrix} \frac{[2]^2([2][4]+\sqrt{[2][4]})}{[3]^2[4]} & \frac{\sqrt{[2]^3}}{\sqrt{[3][4]}} \\ \frac{\sqrt{[2]^3}}{\sqrt{[3][4]}} & \frac{[2]^2([2][4]-\sqrt{[2][4]})}{[3]^2[4]} \end{pmatrix}, \\ &= U^{(k,q_l)} \text{ with rows labelled by } p, p_{l+1}, \end{aligned}$$

$$\begin{aligned} U^{(q_l,p_l)} &= \begin{array}{c} k \\ r_{l+1} \end{array} \begin{pmatrix} \frac{[2][4]+\sqrt{[2][4]}}{[2][3]} & \frac{-\sqrt{[2][4]-\sqrt{[2][4]}}}{[2]\sqrt{[3]}} \\ \frac{-\sqrt{[2][4]-\sqrt{[2][4]}}}{[2]\sqrt{[3]}} & \frac{[2]^2-\sqrt{[2][4]}}{[2][3]} \end{pmatrix}, \\ &= U^{(p_l,r_{l+1})} \text{ with rows labelled by } j, q_l, \end{aligned}$$

$$\begin{aligned} U^{(p_l,r_l)} &= \begin{array}{c} j \\ q_{l-1} \end{array} \begin{pmatrix} \frac{[2][4]-\sqrt{[2][4]}}{[2][3]} & \frac{\sqrt{[2][4]-\sqrt{[2][4]}}}{[2]\sqrt{[3]}} \\ \frac{\sqrt{[2][4]-\sqrt{[2][4]}}}{[2]\sqrt{[3]}} & \frac{[2]^2+\sqrt{[2][4]}}{[2][3]} \end{pmatrix}, \\ &= U^{(q_{l-1},p_l)} \text{ with rows labelled by } k, r_l, \end{aligned}$$

$$U^{(r_{l+1},q_l)} = \begin{array}{c} p_l \\ p_{l+1} \end{array} \begin{pmatrix} \frac{[2]([2]^2-\sqrt{[2][4]})}{[3]^2} & \frac{-[2]}{\sqrt{[6]}} \\ \frac{-[2]}{\sqrt{[6]}} & \frac{[2]([2]^2+\sqrt{[2][4]})}{[3]^2} \end{pmatrix},$$

$$U^{(p_l,k)} = \begin{array}{c} j \\ q_{l-1} \\ q_l \end{array} \left(\begin{array}{ccc} \frac{1}{[2]} & \frac{\sqrt{[2][4]-\sqrt{[2][4]}}}{\sqrt{[2][3][4]}} & \frac{\sqrt{[2][4]+\sqrt{[2][4]}}}{\sqrt{[2][3][4]}} \\ \frac{\sqrt{[2][4]-\sqrt{[2][4]}}}{\sqrt{[2][3][4]}} & \frac{[2][4]-\sqrt{[2][4]}}{[3][4]} & \frac{\sqrt{[6]}}{\sqrt{[3][4]}} \\ \frac{\sqrt{[2][4]+\sqrt{[2][4]}}}{\sqrt{[2][3][4]}} & \frac{\sqrt{[6]}}{\sqrt{[3][4]}} & \frac{[2][4]+\sqrt{[2][4]}}{[3][4]} \end{array} \right).$$

4.2.8 $\mathcal{E}_1^{(12)}$

For the graph $\mathcal{E}_1^{(12)}$ (illustrated in Figure 4.10) we will use the notation $W^{(1)}_{v_1,v_2,v_3}$ for the cell of the triangle $\triangle_{v_1,v_2,v_3}$ where there are no double edges between any of the vertices v_1, v_2, v_3. For triangles that involve the double edges α, α' or β, β' we will specify which of the double edges is used by the notation $\triangle^{(\xi)}_{v_1,v_2,v_3}$, and $W_{v_1,v_2,v_3(\xi)} := W(\triangle^{\xi}_{v_1,v_2,v_3})$. Since the graph $\mathcal{E}_1^{(12)}$ is a $\mathbb{Z}_3$-orbifold of the graph $\mathcal{E}_2^{(12)}$, we can obtain an orbifold solution for the cells for $\mathcal{E}_1^{(12)}$as follows. We take the $\mathbb{Z}_3$-orbifold of $\mathcal{E}_2^{(12)}$ with the vertices i, j and k all fixed points- these are thus triplicated and become the vertices i_l, j_l and k_l, $l=1,2,3$, on $\mathcal{E}_1^{(12)}$. The vertices p_1, p_2 and p_3 on $\mathcal{E}_2^{(12)}$ are identified and become the vertex p on $\mathcal{E}_1^{(12)}$, and similarly the q_l and r_l become q and r. The edges α_1, α_2 and α_3 are identified and become the edge α on $\mathcal{E}_1^{(12)}$, also the edges α_1', α_2' and α_3' are identified and become the edge α'. Similarly the edges β_l, β_l' and γ_l become the edges β, β' and γ respectively on $\mathcal{E}_1^{(12)}$. The Perron-Frobenius weights for the vertices are $\phi_{i_l}=1$, $\phi_{j_l}=\phi_{k_l}=[3]$, $l=1,2,3$, $\phi_p=[2][4]$ and $\phi_q=\phi_r=[3][4]/[2]$. Note that these are equal to the Perron-Frobenius weights for the corresponding vertices of $\mathcal{E}_2^{(12)}$ up to a scalar factor of $[4]/[2]$.

From the type I frames $\overset{i_l}{\bullet}\!\rightarrow\!\overset{j_l}{\bullet}$, $l=1,2,3$, we have $|W^{(1)}_{i_l,j_l,k_l}|^2=[2][3]$ (which is equal to $([4]/[2])^2|W^{(2)}_{i,j,k}|^2/3$). Then the type I frame $\overset{j_l}{\bullet}\!\rightarrow\!\overset{k_l}{\bullet}$, $l=1,2,3$, gives $|W^{(1)}_{p,j_l,k_l}|^2=[3][4]$ $(=([4]/[2])^2|W^{(2)}_{p_l,j,k}|^2/3)$. Since the triangle $\triangle^{(\alpha)}_{p,j_l,r}$ in $\mathcal{E}_1^{(12)}$ comes from the triangle $\triangle_{p_l,j,r_l}$ in $\mathcal{E}_2^{(12)}$, then

$$|W^{(1)}_{p,j_l,r(\alpha)}|^2=\frac{[4]^2}{[2]^2}|W^{(2)}_{p_l,j,r_l}|^2=[2]([2][4]\mp\sqrt{[2][4]}).$$

The triangle $\triangle^{(\alpha')}_{p,j_l,r}$ in $\mathcal{E}_1^{(12)}$ comes from the triangle $\triangle_{p_l,j,r_{l+1}}$ in $\mathcal{E}_2^{(12)}$, giving

$$|W^{(1)}_{p,j_l,r(\alpha')}|^2=\frac{[4]^2}{[2]^2}|W^{(2)}_{p_l,j,r_{l+1}}|^2=[2]([2][4]\pm\sqrt{[2][4]}).$$

Similarly

$$|W^{(1)}_{p,q,k_l(\beta)}|^2=\frac{[4]^2}{[2]^2}|W^{(2)}_{p_l,q_l,k}|^2=[2]([2][4]\pm\sqrt{[2][4]}),$$

$$|W^{(1)}_{p,q,k_l(\beta')}|^2=\frac{[4]^2}{[2]^2}|W^{(2)}_{p_l,q_{l-1},k}|^2=[2]([2][4]\mp\sqrt{[2][4]}).$$

The three triangles $\triangle_{p_l,q_l,r_{l+1}}$, $l=1,2,3$, in $\mathcal{E}_2^{(12)}$ are identified in $\mathcal{E}_1^{(12)}$ and give the triangle $\triangle_{p,q,r}^{(\alpha',\beta)}$, so that $|W_{p,q,r(\alpha',\beta)}^{(1)}|^2 = 3([4]/[2])^2|W_{p_l,q_l,r_{l+1}}^{(2)}|^2 = ([4]/[2])^2([2]^2 \mp \sqrt{[2][4]})$. Similarly $|W_{p,q,r(\alpha,\beta')}^{(1)}|^2 = 3([4]/[2])^2|W_{p_l,q_{l-1},r_l}^{(2)}|^2 = ([4]/[2])^2([2]^2 \pm \sqrt{[2][4]})$. Then from the type I frame $\overset{q}{\bullet}\!\longrightarrow\!\overset{r}{\bullet}$ we have $|W_{p,q,r(\alpha,\beta)}^{(1)}|^2+|W_{p,q,r(\alpha,\beta')}^{(1)}|^2+|W_{p,q,r(\alpha',\beta)}^{(1)}|^2+|W_{p,q,r(\alpha',\beta')}^{(1)}|^2 = [3]^2[4]^2/[2]$. Substituting in for $|W_{p,q,r(\alpha',\beta)}^{(1)}|^2$ and $|W_{p,q,r(\alpha,\beta')}^{(1)}|^2$ we find $|W_{p,q,r(\alpha,\beta)}^{(1)}|^2 + |W_{p,q,r(\alpha',\beta')}^{(1)}|^2 = 0$, so that $|W_{p,q,r(\alpha,\beta)}^{(1)}|^2 = |W_{p,q,r(\alpha',\beta')}^{(1)}|^2 = 0$. The reason for this is that the triangle $\triangle_{p,q,r}^{(\alpha,\beta)}$ (and similarly for the triangle $\triangle_{p,q,r}^{(\alpha',\beta')}$) in $\mathcal{E}_1^{(12)}$ comes from the paths $p_l \to q_l \to r_{l+1} \to p_{l+1}$ in $\mathcal{E}_2^{(12)}$, which do not form a closed triangle.

From the type I frames $\overset{r}{\bullet}\!\rightrightarrows\!\overset{p}{\bullet}$ and $\overset{p}{\bullet}\!\rightrightarrows\!\overset{q}{\bullet}$, we obtain the equations

$$\begin{aligned}\lambda_{1(\alpha)}\overline{\lambda_{1(\alpha')}} + \lambda_{2(\alpha)}\overline{\lambda_{2(\alpha')}} + \lambda_{3(\alpha)}\overline{\lambda_{3(\alpha')}} &= 0, &(4.54)\\ \lambda_{1(\beta)}\overline{\lambda_{1(\beta')}} + \lambda_{2(\beta)}\overline{\lambda_{2(\beta')}} + \lambda_{3(\beta)}\overline{\lambda_{3(\beta')}} &= 0, &(4.55)\end{aligned}$$

where $W_{p,j_l,r(\xi)} = \lambda_{l(\xi)}|W_{p,j_l,r(\xi)}|$, for $\xi \in \{\alpha,\alpha',\beta,\beta'\}$, $l=1,2,3$. Another restriction on the choice of phase is found from the type II frames $\overset{j_l}{\bullet}\!\longrightarrow\!\overset{r}{\bullet}\!\longleftarrow\!\overset{j_m}{\bullet}$, for $l \neq m$, $\mathrm{Re}(\lambda_{l(\alpha)}\lambda_{m(\alpha')}\overline{\lambda_{l(\alpha')}\lambda_{m(\alpha)}}) = -1/2$, and similarly for the type II frames $\overset{k_l}{\bullet}\!\longrightarrow\!\overset{p}{\bullet}\!\longleftarrow\!\overset{k_m}{\bullet}$, $l \neq m$, giving

$$\begin{aligned}\lambda_{l(\alpha)}\lambda_{m(\alpha')}\overline{\lambda_{l(\alpha')}\lambda_{m(\alpha)}} &= -\frac{1}{2} + \varepsilon_{l,m}\frac{\sqrt{3}}{2}i, &(4.56)\\ \lambda_{l(\beta)}\lambda_{m(\beta')}\overline{\lambda_{l(\beta')}\lambda_{m(\beta)}} &= -\frac{1}{2} + \varepsilon'_{l,m}\frac{\sqrt{3}}{2}i, &(4.57)\end{aligned}$$

where $\varepsilon_{l,m}, \varepsilon'_{l,m} \in \{\pm 1\}$. Lastly, from the type II frame consisting of the vertices j_l, k_l, q and r $(l=1,2,3)$ we have

$$\lambda_{l(\alpha)}\lambda_{l(\beta')}\overline{\lambda_{l(\alpha')}\lambda_{l(\beta)}} = -\lambda_{(\alpha\beta')}\overline{\lambda_{(\alpha'\beta)}}, \tag{4.58}$$

where $W_{p,q,r(\xi_1,\xi_2)} = \lambda_{(\xi_1,\xi_2)}|W_{p,q,r(\xi_1,\xi_2)}|$, for $\xi_1 \in \{\alpha,\alpha'\}$, $\xi_2 \in \{\beta,\beta'\}$, $l=1,2,3$. Then for $l \neq m$,

$$\lambda_{l(\alpha)}\lambda_{m(\alpha')}\overline{\lambda_{l(\alpha')}\lambda_{m(\alpha)}} = \lambda_{l(\beta)}\lambda_{m(\beta')}\overline{\lambda_{l(\beta')}\lambda_{m(\beta)}},$$

and, from (4.56) and (4.57) we find $\varepsilon_{l,m} = \varepsilon'_{l,m}$. Substituting in for $\lambda_{l(\alpha)}\overline{\lambda_{l(\alpha')}}$ from (4.56) into (4.54), we see that $\varepsilon_{l,l+1} = \varepsilon_{m,m+1}$ for all $l,m = 1,2,3$, and that $\varepsilon_{l,l-1} = -\varepsilon_{l,l+1}$. Then the restrictions for the choice of phase are (4.58) and

$$\lambda_{l(\alpha)}\lambda_{l+1(\alpha')}\overline{\lambda_{l(\alpha')}\lambda_{l+1(\alpha)}} = \lambda_{l(\beta)}\lambda_{l+1(\beta')}\overline{\lambda_{l(\beta')}\lambda_{l+1(\beta)}} = -\frac{1}{2} + \varepsilon\frac{\sqrt{3}}{2}i = e^{\varepsilon\frac{2\pi i}{3}}, \tag{4.59}$$

where $\varepsilon \in \{\pm 1\}$.

Then we have obtained two orbifold solutions for the cell system for $\mathcal{E}_1^{(12)}$: W^+, W^-.

Theorem 4.2.18 *The following solutions W^+, W^- for the cells of $\mathcal{E}_1^{(12)}$ are inequivalent:*

$$W^{\pm}_{i_l,j_l,k_l} = \sqrt{[2][3]}, \qquad W^{\pm}_{p,j_l,k_l} = \sqrt{[3][4]},$$

$$W^{\pm}_{p,j_l,r(\alpha)} = \epsilon_l \sqrt{[2]}\sqrt{[2][4] \pm \sqrt{[2][4]}}, \qquad W^{\pm}_{p,j_l,r(\alpha')} = \overline{\epsilon_l} \sqrt{[2]}\sqrt{[2][4] \mp \sqrt{[2][4]}},$$

$$W^{\pm}_{p,q,k_l(\beta)} = \epsilon_l \sqrt{[2]}\sqrt{[2][4] \mp \sqrt{[2][4]}}, \qquad W^{\pm}_{p,q,k_l(\beta')} = \overline{\epsilon_l} \sqrt{[2]}\sqrt{[2][4] \pm \sqrt{[2][4]}},$$

$$W^{\pm}_{p,q,r(\alpha\beta')} = \frac{[4]}{\sqrt{[2]}}\sqrt{[2]^2 \mp \sqrt{[2][4]}}, \qquad W^{\pm}_{p,q,r(\alpha'\beta)} = -\frac{[4]}{\sqrt{[2]}}\sqrt{[2]^2 \pm \sqrt{[2][4]}},$$

$$W^{\pm}_{p,q,r(\alpha\beta)} = W^{\pm}_{p,q,r(\alpha'\beta')} = 0,$$

for $l = 1,2,3$, where $\epsilon_1 = 1$ and $\epsilon_2 = e^{2\pi i/3} = \overline{\epsilon_3}$.

Proof

The solutions W^+, W^- are not equivalent, as can be seen by considering (4.5) for the triangle $\triangle_{p,j_l,r}$. We have the following two equations, for $l = 1,2,3$:

$$\begin{aligned} W^+_{p,j_l,r(\alpha)} &= u_{p,j_l} u_{j_l,r} \left(u_\alpha(\alpha,\alpha) W^-_{p,j_l,r(\alpha)} + u_\alpha(\alpha,\alpha') W^-_{p,j_l,r(\alpha')} \right), \\ W^+_{p,j_l,r(\alpha')} &= u_{p,j_l} u_{j_l,r} \left(u_\alpha(\alpha',\alpha) W^-_{p,j_l,r(\alpha)} + u_\alpha(\alpha',\alpha') W^-_{p,j_l,r(\alpha')} \right). \end{aligned}$$

So we require $u_{p,j_l}, u_{j_l,r} \in \mathbb{T}$ and a 2×2 unitary matrix u_α such that, for $l = 1,2,3$,

$$\epsilon_l \sqrt{[2]}\, x_+ = u_{p,j_l} u_{j_l,r} \left(u_\alpha(\alpha,\alpha) \epsilon_l \sqrt{[2]}\, x_- + u_\alpha(\alpha,\alpha') \overline{\epsilon_l} \sqrt{[2]}\, x_+ \right), \tag{4.60}$$

$$\overline{\epsilon_l} \sqrt{[2]}\, x_- = u_{p,j_l} u_{j_l,r} \left(u_\alpha(\alpha',\alpha) \epsilon_l \sqrt{[2]}\, x_- + u_\alpha(\alpha',\alpha') \overline{\epsilon_l} \sqrt{[2]}\, x_+ \right). \tag{4.61}$$

where $x_\pm = \sqrt{[2][4] \pm \sqrt{[2][4]}}$. Equation (4.60) must hold for each $l = 1,2,3$. On the left hand side we have ϵ_l, hence we require $u_\alpha(\alpha,\alpha') = 0$ because u_α does not depend on l, and the difference in phase between ϵ_l and $\overline{\epsilon_l}$ is 0, $e^{-2\pi i/3}$, $e^{2\pi i/3}$ respectively for $l = 1,2,3$ respectively. This difference in phase for each l cannot come from $u_{p,j_l} u_{j_l,r}$ (although u_{p,j_l}, $u_{j_l,r}$ do depend on l) since in (4.61) the difference in phase is now 0, $e^{2\pi i/3}$, $e^{-2\pi i/3}$ respectively for $l = 1,2,3$ respectively, so we would need $\overline{u_{p,j_l} u_{j_l,r}}$ to take care of the phase difference here, not $u_{p,j_l} u_{j_l,r}$. Then we have $u_\alpha(\alpha,\alpha) = \overline{u_{p,j_l} u_{j_l,r}}\; x_+/x_-$, and similarly $u_\alpha(\alpha',\alpha) = 0$ and $u_\alpha(\alpha',\alpha') = \overline{u_{p,j_l} u_{j_l,r}}\; x_-/x_+$. But now u_α is not unitary. $\square$

For $\mathcal{E}_1^{(12)}$, the cells W^+ above give the following representation of the Hecke algebra, where $l = 1,2,3$ (mod 3):

$$U^{(i_l,k_l)} = U^{(j_l,i_l)} = [2], \qquad U^{(k_l,j_l)} = \begin{array}{c} i_l \\ p \end{array} \left(\begin{array}{cc} \frac{[2]}{[3]} & \frac{\sqrt{[2][4]}}{[3]} \\ \frac{\sqrt{[2][4]}}{[3]} & \frac{[4]}{[3]} \end{array} \right),$$

$$U^{(r,j_l)} = \begin{array}{c} p(\alpha) \\ p(\alpha') \end{array} \begin{pmatrix} \frac{[2]^2([2][4]+\sqrt{[2][4]})}{[3]^2[4]} & \frac{\overline{\epsilon_l}\sqrt{[2]^3}}{\sqrt{[3][4]}} \\ \frac{\epsilon_l\sqrt{[2]^3}}{\sqrt{[3][4]}} & \frac{[2]^2([2][4]-\sqrt{[2][4]})}{[3]^2[4]} \end{pmatrix},$$

$$= \; U^{(k_l,q)} \text{ with rows labelled by } p(\beta'), p(\beta),$$

$$U^{(j_l,p)} = \begin{array}{c} k_l \\ r(\alpha) \\ r(\alpha') \end{array} \begin{pmatrix} \frac{1}{[2]} & \frac{\overline{\epsilon_l}\sqrt{[2][4]+\sqrt{[2][4]}}}{\sqrt{[2][3][4]}} & \frac{\epsilon_l\sqrt{[2][4]-\sqrt{[2][4]}}}{\sqrt{[2][3][4]}} \\ \frac{\epsilon_l\sqrt{[2][4]+\sqrt{[2][4]}}}{\sqrt{[2][3][4]}} & \frac{[2][4]+\sqrt{[2][4]}}{[3][4]} & \frac{\overline{\epsilon_l}\sqrt{[6]}}{\sqrt{[3][4]}} \\ \frac{\overline{\epsilon_l}\sqrt{[2][4]-\sqrt{[2][4]}}}{\sqrt{[2][3][4]}} & \frac{\epsilon_l\sqrt{[6]}}{\sqrt{[3][4]}} & \frac{[2][4]-\sqrt{[2][4]}}{[3][4]} \end{pmatrix},$$

$$= \; U^{(p,k_l)} \text{ with rows labelled by } j_l, q(\beta'), q(\beta),$$

$$U^{(r,q)} = \begin{array}{c} p(\alpha\beta) \\ p(\alpha\beta') \\ p(\alpha'\beta) \\ p(\alpha'\beta') \end{array} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{[2]([2]^2-\sqrt{[2][4]})}{[3]^2} & -\frac{\sqrt{[2]}}{\sqrt{[6]}} & 0 \\ 0 & -\frac{\sqrt{[2]}}{\sqrt{[6]}} & \frac{[2]([2]^2+\sqrt{[2][4]})}{[3]^2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$U^{(p,r)} \text{ with labels } j_1, j_2, j_3, q(\beta), q(\beta') \; = \; U^{(q,p)} \text{ with labels } k_1, k_3, k_2, r(\alpha'), r(\alpha) =$$

$$\begin{pmatrix} \frac{[3]}{[4]} & \frac{-[2]^2-i\sqrt{[2][4]}}{[3][4]} & \frac{-[2]^2+i\sqrt{[2][4]}}{[3][4]} & \frac{-\sqrt{[2][4]+\sqrt{[2][4]}}}{[4]\sqrt{[3]}} & \frac{\sqrt{[2][4]-\sqrt{[2][4]}}}{[4]\sqrt{[3]}} \\ \frac{-[2]^2+i\sqrt{[2][4]}}{[3][4]} & \frac{[3]}{[4]} & \frac{-[2]^2-i\sqrt{[2][4]}}{[3][4]} & \frac{-\overline{\epsilon_2}\sqrt{[2][4]+\sqrt{[2][4]}}}{[4]\sqrt{[3]}} & \frac{\epsilon_2\sqrt{[2][4]-\sqrt{[2][4]}}}{[4]\sqrt{[3]}} \\ \frac{-[2]^2-i\sqrt{[2][4]}}{[3][4]} & \frac{-[2]^2+i\sqrt{[2][4]}}{[3][4]} & \frac{[3]}{[4]} & \frac{-\epsilon_2\sqrt{[2][4]+\sqrt{[2][4]}}}{[4]\sqrt{[3]}} & \frac{\overline{\epsilon_2}\sqrt{[2][4]-\sqrt{[2][4]}}}{[4]\sqrt{[3]}} \\ \frac{-\sqrt{[2][4]+\sqrt{[2][4]}}}{[4]\sqrt{[3]}} & \frac{-\epsilon_2\sqrt{[2][4]+\sqrt{[2][4]}}}{[4]\sqrt{[3]}} & \frac{-\overline{\epsilon_2}\sqrt{[2][4]+\sqrt{[2][4]}}}{[4]\sqrt{[3]}} & \frac{[2]^2+\sqrt{[2][4]}}{[2][3]} & 0 \\ \frac{\sqrt{[2][4]-\sqrt{[2][4]}}}{[4]\sqrt{[3]}} & \frac{\overline{\epsilon_2}\sqrt{[2][4]-\sqrt{[2][4]}}}{[4]\sqrt{[3]}} & \frac{\epsilon_2\sqrt{[2][4]-\sqrt{[2][4]}}}{[4]\sqrt{[3]}} & 0 & \frac{[2]^2-\sqrt{[2][4]}}{[2][3]} \end{pmatrix}.$$

Our representation of the Hecke algebra is not equivalent to that given by Sochen for $\mathcal{E}_1^{(12)}$ in [108], however we believe that there is a typographical error in Sochen's presentation and that the weights he denotes by $U^{(4,2_r)} = (U^{(3_r,6)})^*$ should be the complex conjugate of the one given. In this case, the representation of the Hecke algebra we give above can be shown to be equivalent by choosing a family of unitaries $u_{i_l,j_l} = u_{j_l,k_l} = u_{k_l,i_l} = u_{p,j_l} = u_{k_l,p} = u_{q,r} = 1$, $u_{j_l,r} = -\epsilon_l = \overline{u_{q,k_l}}$ and set the 2×2 unitary matrices u_α, u_β to be the identity matrix.

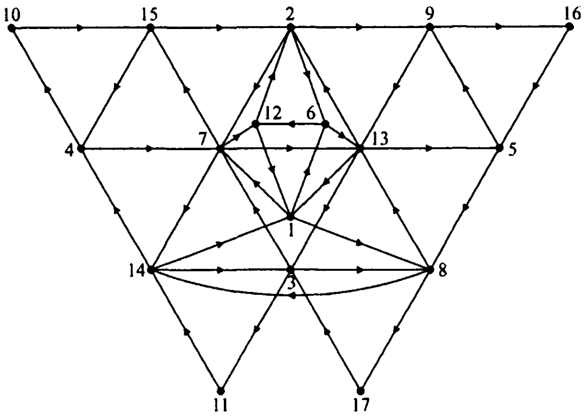


Figure 4.11: Labelled graph $\mathcal{E}_5^{(12)}$

4.2.9 $\mathcal{E}_5^{(12)}$

We label the vertices of $\mathcal{E}_5^{(12)}$ as in Figure 4.11. The Perron-Frobenius weights associated to the vertices are $\phi_1 = [3][6]/[2]$, $\phi_2 = \phi_3 = \phi_8 = \phi_{14} = [3][4]/[2]$, $\phi_4 = \phi_5 = \phi_9 = \phi_{15} = [3]$, $\phi_6 = \phi_{12} = [2][3]^2/[6] = [2]^2$, $\phi_7 = \phi_{13} = [3]^2[4]/[6] = [2][4]$, $\phi_{10} = \phi_{16} = 1$, $\phi_{11} = \phi_{17} = [4]/[2]$. The distinguished $*$-vertex is vertex 10. The following cells follow as in the $\mathcal{A}$ case: $|W_{4,10,15}|^2 = |W_{5,9,16}|^2 = [2][3]$, $|W_{4,7,15}|^2 = |W_{5,9,13}|^2 = [3][4]$, $|W_{2,7,15}|^2 = |W_{4,7,14}|^2 = |W_{2,9,13}|^2 = |W_{5,8,13}|^2 = [3]^2[4]$, whilst from the type I frames $\overset{17}{\bullet}\!\rightarrow\!\overset{3}{\bullet}$ and $\overset{3}{\bullet}\!\rightarrow\!\overset{11}{\bullet}$ we have $|W_{3,11,14}|^2 = |W_{3,8,17}|^2 = [3][4]^2/[2]$.

From the type I frames $\overset{2}{\bullet}\!\rightarrow\!\overset{7}{\bullet}$ and $\overset{13}{\bullet}\!\rightarrow\!\overset{2}{\bullet}$ we have: $|W_{2,7,12}|^2 + |W_{2,7,13}|^2 + |W_{2,7,15}|^2 = [3]^3[4]^2/[6]$ and $|W_{2,6,13}|^2 + |W_{2,7,13}|^2 + |W_{2,9,13}|^2 = [3]^3[4]^2/[6]$, giving

$$\begin{aligned} |W_{2,7,12}|^2 + |W_{2,7,13}|^2 &= \frac{[2][3]^3[4]}{[6]}, \qquad (4.62) \\ |W_{2,6,13}|^2 + |W_{2,7,13}|^2 &= \frac{[2][3]^3[4]}{[6]}, \end{aligned}$$

so $|W_{2,7,12}| = |W_{2,6,13}|$. The type I frames $\overset{7}{\bullet}\!\rightarrow\!\overset{12}{\bullet}$ and $\overset{6}{\bullet}\!\rightarrow\!\overset{13}{\bullet}$ then force $|W_{1,7,12}| = |W_{1,6,13}|$. Similarly, we find that $|W_{3,7,14}| = |W_{3,8,13}|$ and $|W_{1,7,14}| = |W_{1,8,13}|$. So there is an automatic $\mathbb{Z}_2$ symmetry for the cells of the graph $\mathcal{E}_5^{(12)}$.

From other type I frames we obtain the following equations

$$\begin{aligned} |W_{1,7,14}|^2 + |W_{3,7,14}|^2 &= \frac{[2][3]^3[4]}{[6]}, \qquad (4.63) \\ |W_{1,8,14}|^2 + |W_{3,8,14}|^2 &= \frac{[3]^2[4]^2}{[2]}, \qquad (4.64) \\ |W_{1,7,14}|^2 + |W_{1,8,14}|^2 &= \frac{[3]^2[4][6]}{[2]}, \qquad (4.65) \end{aligned}$$

$$|W_{3,7,13}|^2 + |W_{3,7,14}|^2 = \frac{[3]^3[4]^2}{[6]}, \tag{4.66}$$

$$|W_{1,7,12}|^2 + |W_{1,7,13}|^2 + |W_{1,7,14}|^2 = [3]^3[4], \tag{4.67}$$

$$|W_{1,7,12}|^2 + |W_{2,7,12}|^2 = \frac{[2]^2[3]^4[4]}{[6]^2}, \tag{4.68}$$

$$|W_{1,6,12}|^2 + |W_{2,6,12}|^2 = \frac{[2]^3[3]^4}{[6]^2}, \tag{4.69}$$

$$|W_{1,6,12}|^2 + |W_{1,7,12}|^2 = [2][3]^3. \tag{4.70}$$

Next, consider the type II frame $\overset{14}{\bullet}\!\rightarrow\!\overset{1}{\bullet}\!\leftarrow\!\overset{13}{\bullet}$, which gives

$$\phi_7^{-1}|W_{1,7,13}|^2|W_{1,7,14}|^2 + \phi_8^{-1}|W_{1,8,13}|^2|W_{1,8,14}|^2 = \phi_{14}\phi_1\phi_{13}.$$

Substituting in for $|W_{1,8,14}|^2$ from (4.65), and since $|W_{1,8,13}|^2 = |W_{1,7,14}|^2$, we get

$$\frac{[6]}{[3]^2[4]}|W_{1,7,13}|^2|W_{1,7,14}|^2 + [3][6]|W_{1,7,14}|^2 - \frac{[2]}{[3][4]}|W_{1,7,14}|^4 = \frac{[3]^4[4]^2}{[2]^2}. \tag{4.71}$$

Finally, from the type II frame $\overset{7}{\bullet}\!\rightarrow\!\overset{13}{\bullet}\!\leftarrow\!\overset{8}{\bullet}$ we have

$$\phi_1^{-1}|W_{1,7,13}|^2|W_{1,8,13}|^2 + \phi_3^{-1}|W_{3,7,13}|^2|W_{3,8,13}|^2 = \phi_7\phi_{13}\phi_8.$$

Substituting in for $|W_{3,7,13}|^2$ from (4.66) and $|W_{3,7,14}|^2$ from (4.63), and using the $\mathbb{Z}_2$ symmetry of the cells, we get

$$\frac{[6]}{[3]^2[4]}|W_{1,7,13}|^2|W_{1,7,14}|^2 + \frac{[3][6]}{[2][4]}|W_{1,7,14}|^2 - \frac{[6]^2}{[3]^2[4]^2}|W_{1,7,14}|^4 = \frac{[3]^4}{[2]^2}. \tag{4.72}$$

Then equating the $|W_{1,7,13}|^2|W_{1,7,14}|^2$ term in (4.71) and (4.72) we get the following quadratic for $|W_{1,7,14}|^2$:

$$[2]|W_{1,7,14}|^4 - [3][4][6]^3|W_{1,7,14}|^2 + [2][3]^6[4]^2 = 0,$$

which has solutions $|W_{1,7,14}|^2 = [2]^3[3][4][6]$ or $|W_{1,7,14}|^2 = [3][4][6]/[2]$. Substituting the first solution into (4.63) gives $|W_{3,7,14}|^2 < 0$, so we have $|W_{1,7,14}|^2 = |W_{1,8,13}|^2 = [3][4][6]/[2]$, and from (4.63) $|W_{3,7,14}|^2 = |W_{3,8,13}|^2 = [3][4]^2/[2]$. From (4.62)-(4.71) we obtain the following values for the remaining cells: $|W_{1,8,14}|^2 = [3][4]^2[6]/[2]^2$, $|W_{3,8,14}|^2 = [4]^2[6]/[2]^2$, $|W_{1,7,13}|^2 = [3][4]^2/[2]$, $|W_{1,7,12}|^2 = |W_{1,6,13}|^2 = [2]^2[3][4]$, $|W_{2,7,12}|^2 = |W_{2,6,13}|^2 = [2]^2[4]$, $|W_{1,6,12}|^2 = [2][3]$, $|W_{2,6,12}|^2 = [2][4]^2$, $|W_{2,7,13}|^2 = [2][4]^2$ and $|W_{3,7,13}|^2 = [4]^2[6]$.

With $W_{v_1,v_2,v_3} = \lambda_{v_1,v_2,v_3}|W_{v_1,v_2,v_3}|$, $\lambda_{v_1,v_2,v_3} \in \mathbb{T}$, we find two restrictions on the choice of phase

$$\lambda_{1,6,12}\lambda_{2,7,12}\overline{\lambda_{1,7,12}\lambda_{2,6,12}} = -\lambda_{1,6,13}\lambda_{2,7,13}\overline{\lambda_{1,7,13}\lambda_{2,6,13}}, \tag{4.73}$$

$$\lambda_{1,7,14}\lambda_{1,8,13}\overline{\lambda_{1,7,13}\lambda_{1,8,14}} = -\lambda_{3,7,14}\lambda_{3,8,13}\overline{\lambda_{3,7,13}\lambda_{3,8,14}}. \tag{4.74}$$

Theorem 4.2.19 *There is up to equivalence a unique set of cells for $\mathcal{E}_5^{(12)}$ given by:*

$$W_{1,6,12} = W_{4,10,15} = W_{5,9,16} = \sqrt{[2][3]},$$

$$W_{1,6,13} = W_{1,7,12} = [2]\sqrt{[3][4]},$$

$$W_{1,7,13} = W_{3,7,14} = W_{3,8,13} = W_{3,8,17} = W_{3,11,14} = W_{2,7,15} = W_{2,9,13} = W_{4,7,14} = W_{5,8,13} = \frac{[4]\sqrt{[3]}}{\sqrt{[2]}},$$

$$W_{1,8,14} = \frac{[4]\sqrt{[3][6]}}{\sqrt{[2]}}, \qquad W_{1,7,14} = W_{1,8,13} = \frac{\sqrt{[3][4][6]}}{\sqrt{[2]}},$$

$$W_{2,6,12} = [4]\sqrt{[2]}, \qquad W_{2,6,13} = W_{2,7,12} = [2]\sqrt{[4]},$$

$$W_{2,7,13} = -[4]\sqrt{[2]}, \qquad W_{3,7,13} = -[4]\sqrt{[6]},$$

$$W_{3,8,14} = \frac{[4]\sqrt{[6]}}{[2]}, \qquad W_{4,7,15} = W_{5,9,13} = \sqrt{[3][4]}.$$

Proof

Let $W^\sharp$ be any solution for the cells of $\mathcal{E}_5^{(12)}$, with $W^\sharp_{v_1,v_2,v_3} = \lambda^\sharp_{v_1,v_2,v_3}|W_{v_1,v_2,v_3}|$, and where the $\lambda^\sharp$'s satisfy the conditions (4.73) and (4.74). We need to find unitaries $u_{v_1,v_2} \in \mathbb{T}$ such that $u_{7,13}u_{13,2}u_{2,7}\lambda^\sharp_{2,7,13} = -1$, $u_{7,13}u_{13,3}u_{3,7}\lambda^\sharp_{3,7,13} = -1$ and $u_{v_1,v_2}u_{v_2,v_3}u_{v_3,v_1}\lambda^\sharp_{v_1,v_2,v_3} = 1$ for all other triangles $\triangle_{v_1,v_2,v_3}$ on $\mathcal{E}_5^{(12)}$. We choose $u_{2,7} = u_{2,9} = u_{3,8} = u_{3,11} = u_{6,13} = u_{7,13} = u_{7,14} = u_{8,13} = u_{8,17} = u_{9,16} = u_{10,15} = u_{12,1} = u_{12,2} = u_{13,5} = u_{14,7} = u_{15,2} = 1$, $u_{5,8} = \overline{\lambda^\sharp_{5,8,13}}$, $u_{7,12} = \overline{\lambda^\sharp_{2,7,12}}$, $u_{7,15} = \overline{\lambda^\sharp_{2,7,15}}$, $u_{11,14} = -\overline{\lambda^\sharp_{3,11,14}}$, $u_{13,1} = \overline{\lambda^\sharp_{1,6,13}}$, $u_{13,2} = -\overline{\lambda^\sharp_{2,7,13}}$, $u_{13,3} = \overline{\lambda^\sharp_{3,8,13}}$, $u_{14,4} = \overline{\lambda^\sharp_{4,7,14}}$, $u_{17,3} = \overline{\lambda^\sharp_{3,8,17}}$, $u_{1,7} = \lambda^\sharp_{2,7,12}\overline{\lambda^\sharp_{1,7,12}}$, $u_{2,6} = -\lambda^\sharp_{2,7,13}\overline{\lambda^\sharp_{2,6,13}}$, $u_{3,7} = -\lambda^\sharp_{3,8,13}\overline{\lambda^\sharp_{3,7,13}}$, $u_{9,13} = -\lambda^\sharp_{2,7,13}\overline{\lambda^\sharp_{2,9,13}}$, $u_{15,4} = \lambda^\sharp_{2,7,15}\overline{\lambda^\sharp_{4,7,15}}$, $u_{4,10} = \lambda^\sharp_{4,7,15}\overline{\lambda^\sharp_{2,7,15}\lambda^\sharp_{4,10,15}}$, $u_{5,9} = -\lambda^\sharp_{2,9,13}\overline{\lambda^\sharp_{2,7,13}\lambda^\sharp_{5,9,13}}$, $u_{6,12} = -\lambda^\sharp_{2,6,13}\overline{\lambda^\sharp_{2,6,12}\lambda^\sharp_{2,7,13}}$, $u_{14,1} = \lambda^\sharp_{1,7,12}\overline{\lambda^\sharp_{1,7,14}\lambda^\sharp_{2,7,12}}$, $u_{14,3} = -\lambda^\sharp_{3,7,13}\overline{\lambda^\sharp_{3,7,14}\lambda^\sharp_{3,8,13}}$, $u_{1,6} = -\lambda^\sharp_{2,6,12}\lambda^\sharp_{2,7,13}\overline{\lambda^\sharp_{1,6,12}\lambda^\sharp_{2,6,13}}$, $u_{1,8} = \lambda^\sharp_{1,7,12}\overline{\lambda^\sharp_{1,7,13}\lambda^\sharp_{1,8,13}\lambda^\sharp_{2,7,12}}$, $u_{8,14} = \lambda^\sharp_{1,7,14}\lambda^\sharp_{1,8,13}\overline{\lambda^\sharp_{1,7,13}\lambda^\sharp_{1,8,14}}$ and $u_{16,5} = -\lambda^\sharp_{2,7,13}\lambda^\sharp_{5,9,13}\overline{\lambda^\sharp_{2,9,13}\lambda^\sharp_{5,9,16}}$. □

For $\mathcal{E}_5^{(12)}$, we have the following representation of the Hecke algebra:

$$U^{(5,16)} = U^{(16,9)} = U^{(10,4)} = U^{(15,10)} = [2],$$

$$U^{(3,17)} = U^{(17,8)} = U^{(11,3)} = U^{(14,11)} = \frac{[2]}{[4]},$$

$$U^{(2,15)} = U^{(4,14)} = U^{(8,5)} = U^{(9,2)} = \frac{[4]}{[3]},$$

$$U^{(14,8)} = \begin{array}{c} 3 \\ 1 \end{array} \left(\begin{array}{cc} \frac{1}{[2]} & \frac{\sqrt{[3]}}{\sqrt{[2]}} \\ \frac{\sqrt{[3]}}{\sqrt{[2]}} & [3] \end{array} \right),$$

$$U^{(12,7)} = \begin{array}{c} 2 \\ 1 \end{array} \begin{pmatrix} \frac{1}{[2]} & \frac{\sqrt{[3]}}{[2]} \\ \frac{\sqrt{[3]}}{[2]} & \frac{[3]}{[2]} \end{pmatrix} = U^{(13,6)} \text{ with rows labelled by } 2, 1,$$

$$U^{(3,13)} = \begin{array}{c} 8 \\ 7 \end{array} \begin{pmatrix} \frac{1}{[2]} & -\frac{\sqrt{[3]}}{[2]} \\ -\frac{\sqrt{[3]}}{[2]} & \frac{[3]}{[2]} \end{pmatrix} = U^{(7,3)} \text{ with rows labelled by } 14, 13,$$

$$U^{(5,13)} = \begin{array}{c} 9 \\ 8 \end{array} \begin{pmatrix} \frac{1}{[2]} & \frac{\sqrt{[4]}}{\sqrt{[2]^3}} \\ \frac{\sqrt{[4]}}{\sqrt{[2]^3}} & \frac{[4]}{[2]^2} \end{pmatrix}$$
$$= U^{(13,9)} \text{ with labels 5,2} = U^{(7,4)} \text{ with labels 15,14} = U^{(15,7)} \text{ with labels 4,2},$$

$$U^{(2,12)} = \begin{array}{c} 7 \\ 6 \end{array} \begin{pmatrix} \frac{[2]}{[3]} & \frac{\sqrt{[2][4]}}{[3]} \\ \frac{\sqrt{[2][4]}}{[3]} & \frac{[4]}{[3]} \end{pmatrix}$$
$$= U^{(6,2)} \text{ with labels 13,12} = U^{(4,15)} \text{ with labels 10,7} = U^{(9,5)} \text{ with labels 16,13},$$

$$U^{(1,14)} = \begin{array}{c} 7 \\ 8 \end{array} \begin{pmatrix} \frac{[2]}{[3]} & \frac{[2]\sqrt{[4]}}{[3]} \\ \frac{[2]\sqrt{[4]}}{[3]} & \frac{[2][4]}{[3]} \end{pmatrix} = U^{(8,1)} \text{ with labels 13,14},$$

$$U^{(12,6)} = \begin{array}{c} 1 \\ 2 \end{array} \begin{pmatrix} \frac{[3]}{[2]^3} & \frac{[4]\sqrt{[3]}}{[2]^3} \\ \frac{[4]\sqrt{[3]}}{[2]^3} & \frac{[4]^2}{[2]^3} \end{pmatrix},$$

$$U^{(1,12)} = \begin{array}{c} 6 \\ 7 \end{array} \begin{pmatrix} \frac{1}{[6]} & \frac{\sqrt{[2][4]}}{[6]} \\ \frac{\sqrt{[2][4]}}{[6]} & \frac{[2][4]}{[6]} \end{pmatrix} = U^{(6,1)} \text{ with labels 12,13},$$

$$U^{(13,8)} = \begin{array}{c} 5 \\ 3 \\ 1 \end{array} \begin{pmatrix} \frac{1}{[2]} & \frac{1}{[2]} & \frac{\sqrt{[6]}}{[2]\sqrt{[4]}} \\ \frac{1}{[2]} & \frac{1}{[2]} & \frac{\sqrt{[6]}}{[2]\sqrt{[4]}} \\ \frac{\sqrt{[6]}}{[2]\sqrt{[4]}} & \frac{\sqrt{[6]}}{[2]\sqrt{[4]}} & \frac{[6]}{[2][4]} \end{pmatrix} = U^{(14,7)} \text{ with labels 4,3,1},$$

$$U^{(3,14)} = \begin{array}{c} 8 \\ 7 \\ 11 \end{array} \begin{pmatrix} \frac{1}{[2]} & \frac{1}{\sqrt{[3]}} & \frac{1}{\sqrt{[3]}} \\ \frac{1}{\sqrt{[3]}} & \frac{[2]}{[3]} & \frac{[2]}{[3]} \\ \frac{1}{\sqrt{[3]}} & \frac{[2]}{[3]} & \frac{[2]}{[3]} \end{pmatrix} = U^{(8,3)} \text{ with labels 14,13,17},$$

$$U^{(2,13)} = \begin{array}{c} 9 \\ 7 \\ 6 \end{array} \left(\begin{array}{ccc} \frac{1}{[2]} & -\frac{1}{\sqrt{[3]}} & \frac{\sqrt{[2]}}{\sqrt{[3][4]}} \\ -\frac{1}{\sqrt{[3]}} & \frac{[2]}{[3]} & -\frac{\sqrt{[2]^3}}{[3]\sqrt{[4]}} \\ \frac{\sqrt{[2]}}{\sqrt{[3][4]}} & -\frac{\sqrt{[2]^3}}{[3]\sqrt{[4]}} & \frac{[2]^2}{[3][4]} \end{array} \right) = U^{(7,2)} \text{ with labels 15,13,12,}$$

$$U^{(1,13)} = \begin{array}{c} 8 \\ 7 \\ 6 \end{array} \left(\begin{array}{ccc} \frac{1}{[2]} & \frac{\sqrt{[4]}}{[2]\sqrt{[6]}} & \frac{\sqrt{[2]^3}}{\sqrt{[6]}} \\ \frac{\sqrt{[4]}}{[2]\sqrt{[6]}} & \frac{[4]}{[2][6]} & \frac{\sqrt{[2]^3[4]}}{[6]} \\ \frac{\sqrt{[2]^3}}{\sqrt{[6]}} & \frac{\sqrt{[2]^3[4]}}{[6]} & \frac{[2]^2}{[6]} \end{array} \right) = U^{(7,1)} \text{ with labels 14,13,12,}$$

$$U^{(13,7)} = \begin{array}{c} 2 \\ 3 \\ 1 \end{array} \left(\begin{array}{ccc} \frac{1}{[2]} & \frac{\sqrt{[6]}}{\sqrt{[2]^3}} & -\frac{\sqrt{[3]}}{[2]^2} \\ \frac{\sqrt{[6]}}{\sqrt{[2]^3}} & \frac{[6]}{[2]^2} & -\frac{\sqrt{[3][6]}}{\sqrt{[2]^5}} \\ -\frac{\sqrt{[3]}}{[2]^2} & -\frac{\sqrt{[3][6]}}{\sqrt{[2]^5}} & \frac{[3]}{[2]^3} \end{array} \right).$$

4.2.10 $\mathcal{E}^{(24)}$

We label the vertices of the graph $\mathcal{E}^{(24)}$ as in Figure 4.12. The Perron-Frobenius weights are: $\phi_1 = \phi_8 = 1$, $\phi_2 = \phi_7 = [2][4]$, $\phi_3 = \phi_6 = [4][5]/[2]$, $\phi_4 = \phi_5 = [4][7]/[2]$, $\phi_9 = \phi_{16} = \phi_{17} = \phi_{24} = [3]$, $\phi_{10} = \phi_{15} = \phi_{18} = \phi_{23} = [3][4]/[2]$, $\phi_{11} = \phi_{14} = \phi_{19} = \phi_{22} = [3][5]$ and $\phi_{12} = \phi_{13} = \phi_{20} = \phi_{21} = [9]$. With $[a] = [a]_q$, $q = e^{i\pi/24}$, we have the relation $[4]^2 = [2][10]$.

The following cells follow from the $\mathcal{A}$ case: $|W_{1,9,17}|^2 = |W_{8,16,24}|^2 = [2][3]$, $|W_{2,9,17}|^2 = |W_{7,16,24}|^2 = [3][4]$, $|W_{2,9,18}|^2 = |W_{2,10,17}|^2 = |W_{7,15,24}|^2 = |W_{7,16,23}|^2 = [3]^2[4]$, $|W_{2,10,19}|^2 = |W_{2,11,18}|^2 = |W_{7,14,23}|^2 = |W_{7,15,22}|^2 = [3][4][5]$, $|W_{2,11,19}|^2 = |W_{7,14,22}|^2 = [3]^2[4][5]$ and $|W_{3,10,19}|^2 = |W_{3,14,23}|^2 = |W_{6,11,18}|^2 = |W_{6,15,22}|^2 = [3][4]^2[5]/[2]$.

The type II frame $\overset{2}{\bullet}\!\rightarrow\!\overset{19}{\bullet}\!\leftarrow\!\overset{4}{\bullet}$ gives $\phi_{11}^{-1}|W_{2,11,19}|^2|W_{4,11,19}|^2 = [3][4]^2[5][7]$, and so we obtain $|W_{4,11,19}|^2 = [4][5][7]$. From the type I frame $\overset{11}{\bullet}\!\rightarrow\!\overset{19}{\bullet}$ we have the equation $|W_{2,11,19}|^2 + |W_{4,11,19}|^2 + |W_{5,11,19}|^2 = [2][3]^2[5]^2$, giving $|W_{5,11,19}|^2 = [4][5][7] = |W_{4,11,19}|^2$. Then by considering the type I frames $\overset{4}{\bullet}\!\rightarrow\!\overset{11}{\bullet}$ and $\overset{22}{\bullet}\!\rightarrow\!\overset{4}{\bullet}$, we see that $|W_{4,14,22}|^2 = |W_{5,14,22}|^2 = |W_{4,11,19}|^2 = |W_{5,11,19}|^2$, and similarly $|W_{4,12,19}|^2 = |W_{4,14,21}|^2 = |W_{5,11,20}|^2 = |W_{5,13,22}|^2$ and $|W_{3,12,19}|^2 = |W_{3,14,21}|^2 = |W_{6,11,20}|^2 = |W_{6,13,22}|^2$, and the cells have a $\mathbb{Z}_2$ symmetry.

From type I frames we have the equations:

$$|W_{4,11,19}|^2 + |W_{4,12,19}|^2 + |W_{4,14,19}|^2 = [3][4][5][7], \tag{4.75}$$

$$|W_{3,12,19}|^2 + |W_{4,12,19}|^2 = [2][3][5][9], \tag{4.76}$$

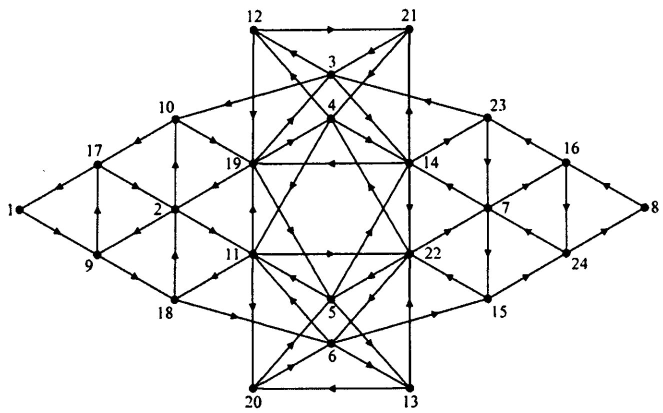


Figure 4.12: Labelled graph $\mathcal{E}^{(24)}$

$$|W_{3,10,19}|^2 + |W_{3,12,19}|^2 + |W_{3,14,19}|^2 = [3][4][5]^2, \tag{4.77}$$

$$|W_{3,14,19}|^2 + |W_{4,14,19}|^2 + |W_{5,14,19}|^2 = [2][3]^2[5]^2, \tag{4.78}$$

$$|W_{3,12,19}|^2 + |W_{3,12,21}|^2 = [4][5][9], \tag{4.79}$$

$$|W_{3,12,21}|^2 + |W_{4,12,21}|^2 = [2][9]^2. \tag{4.80}$$

Finally, the type II frame $\overset{11}{\bullet}\!\rightarrow\!\overset{19}{\bullet}\!\leftarrow\!\overset{12}{\bullet}$ we have $\phi_4^{-1}|W_{4,11,19}|^2|W_{4,12,19}|^2 = [3]^2[5]^2[9]$, giving $|W_{4,12,19}|^2 = [3]^2[5][9]/[2]$. Then using the equations (4.75)-(4.80) we obtain $|W_{4,14,19}|^2 = [5]^2[7]/[2]$, $|W_{3,12,19}|^2 = [3][5][9]/[2]$, $|W_{3,14,19}|^2 = [3]^2[5]^2/[2]$, $|W_{5,14,19}|^2 = [5][7][10]$, $|W_{3,12,21}|^2 = [5]^2[9]/[2]$ and $|W_{4,12,21}|^2 = [7][9]/[2]$.

With $W_{v_1,v_2,v_3} = \lambda_{v_1,v_2,v_3}|W_{v_1,v_2,v_3}|$, $\lambda_{v_1,v_2,v_3} \in \mathbb{T}$, we have the following restrictions on the λ's:

$$\lambda_{3,12,19}\lambda_{3,14,21}\overline{\lambda_{3,12,21}\lambda_{3,14,19}} = -\lambda_{4,12,19}\lambda_{4,14,21}\overline{\lambda_{4,12,21}\lambda_{4,14,19}}, \tag{4.81}$$

$$\lambda_{4,11,22}\lambda_{4,14,19}\overline{\lambda_{4,11,19}\lambda_{4,14,22}} = -\lambda_{5,11,22}\lambda_{5,14,19}\overline{\lambda_{5,11,19}\lambda_{5,14,22}}, \tag{4.82}$$

$$\lambda_{5,11,20}\lambda_{5,13,22}\overline{\lambda_{5,11,22}\lambda_{5,13,20}} = -\lambda_{6,11,20}\lambda_{6,13,22}\overline{\lambda_{6,11,22}\lambda_{6,13,20}}. \tag{4.83}$$

Theorem 4.2.20 *There is up to equivalence a unique set of cells for $\mathcal{E}^{(24)}$ given by:*

$$W_{1,9,17} = W_{8,16,24} = \sqrt{[2][3]}, \qquad W_{2,9,17} = W_{7,16,24} = \sqrt{[3][4]},$$

$$W_{2,9,18} = W_{2,10,17} = W_{7,15,24} = W_{7,16,23} = [3]\sqrt{[4]},$$

$$W_{2,10,19} = W_{2,11,18} = W_{7,14,23} = W_{7,15,22} = \sqrt{[3][4][5]},$$

$$W_{2,11,19} = W_{7,14,22} = [3]\sqrt{[4][5]},$$

$$W_{3,10,19} = W_{3,14,23} = W_{6,11,18} = W_{6,15,22} = \frac{[4]\sqrt{[3][5]}}{\sqrt{[2]}},$$

$$W_{4,11,19} = W_{4,14,22} = W_{5,11,19} = W_{5,14,22} = \sqrt{[4][5][7]},$$

$$W_{4,12,19} = W_{4,14,21} = W_{5,11,20} = W_{5,13,22} = \frac{[3]\sqrt{[5][9]}}{\sqrt{[2]}},$$

$$W_{3,12,19} = W_{3,14,21} = W_{6,11,20} = W_{6,13,22} = \frac{\sqrt{[3][5][9]}}{\sqrt{[2]}},$$

$$W_{3,14,19} = W_{6,11,22} = \frac{[3][5]}{\sqrt{[2]}}, \qquad W_{4,14,19} = W_{5,11,22} = \frac{[5]\sqrt{[7]}}{\sqrt{[2]}},$$

$$W_{5,14,19} = \sqrt{[5][7][10]}, \qquad W_{4,11,22} = -\sqrt{[5][7][10]},$$

$$W_{3,12,21} = W_{6,13,20} = -\frac{[5]\sqrt{[9]}}{\sqrt{[2]}}, \qquad W_{4,12,21} = W_{5,13,20} = \frac{\sqrt{[7][9]}}{\sqrt{[2]}}.$$

Proof

Let $W^{\sharp}$ be any solution for the cells of $\mathcal{E}^{(24)}$, with $W^{\sharp}_{v_1,v_2,v_3} = \lambda^{\sharp}_{v_1,v_2,v_3}|W_{v_1,v_2,v_3}|$, and where the $\lambda^{\sharp}$'s satisfy the conditions (4.81), (4.82) and (4.83). We need to find unitaries $u_{v_1,v_2} \in \mathbb{T}$ such that $u_{12,21}u_{21,3}u_{3,12}\lambda^{\sharp}_{3,12,21} = -1$, $u_{13,20}u_{20,6}u_{6,13}\lambda^{\sharp}_{6,13,20} = -1$, $u_{11,22}u_{22,4}u_{4,11}\lambda^{\sharp}_{4,11,22} = -1$ and $u_{v_1,v_2}u_{v_2,v_3}u_{v_3,v_1}\lambda^{\sharp}_{v_1,v_2,v_3} = 1$ for all other triangles $\triangle_{v_1,v_2,v_3}$ on $\mathcal{E}^{(24)}$. We make the following choices for the u_{v_1,v_2}:

$$u_{3,12} = u_{3,14} = u_{4,11} = u_{5,13} = u_{5,14} = u_{11,20} = u_{14,19} = u_{20,6} = u_{21,3} = u_{21,4} = u_{22,6} = 1,$$

$$u_{12,21} = -\overline{\lambda^{\sharp}_{3,12,21}}, \qquad u_{14,21} = \overline{\lambda^{\sharp}_{3,14,21}}, \qquad u_{19,3} = \overline{\lambda^{\sharp}_{3,14,19}}, \qquad u_{19,5} = -\overline{\lambda^{\sharp}_{5,14,19}},$$

$$u_{4,12} = -\lambda^{\sharp}_{3,12,21}\overline{\lambda^{\sharp}_{4,12,21}}, \qquad u_{4,14} = \lambda^{\sharp}_{3,14,21}\overline{\lambda^{\sharp}_{4,14,21}}, \qquad u_{6,11} = \lambda^{\sharp}_{5,14,22}\overline{\lambda^{\sharp}_{6,11,20}},$$

$$u_{12,19} = \lambda^{\sharp}_{3,14,19}\overline{\lambda^{\sharp}_{3,12,19}}, \qquad u_{11,22} = \lambda^{\sharp}_{6,11,20}\overline{\lambda^{\sharp}_{5,14,22}\lambda^{\sharp}_{6,11,22}},$$

$$u_{19,4} = \lambda^{\sharp}_{4,14,21}\overline{\lambda^{\sharp}_{3,14,21}\lambda^{\sharp}_{4,14,19}}, \qquad u_{22,4} = -\lambda^{\sharp}_{5,14,22}\lambda^{\sharp}_{6,11,22}\overline{\lambda^{\sharp}_{4,11,22}\lambda^{\sharp}_{6,11,20}},$$

$$u_{5,11} = -\lambda^{\sharp}_{4,11,22}\lambda^{\sharp}_{4,14,21}\lambda^{\sharp}_{5,14,22}\overline{\lambda^{\sharp}_{3,14,21}\lambda^{\sharp}_{4,14,22}\lambda^{\sharp}_{5,11,22}},$$

$$u_{11,19} = -\lambda^{\sharp}_{3,12,21}\lambda^{\sharp}_{3,14,19}\lambda^{\sharp}_{4,12,19}\overline{\lambda^{\sharp}_{3,12,19}\lambda^{\sharp}_{4,11,19}\lambda^{\sharp}_{4,12,21}},$$

$$u_{20,5} = -\lambda^{\sharp}_{3,14,21}\lambda^{\sharp}_{4,14,22}\lambda^{\sharp}_{5,11,22}\overline{\lambda^{\sharp}_{4,11,22}\lambda^{\sharp}_{4,14,21}\lambda^{\sharp}_{5,11,20}},$$

$$u_{22,5} = -\lambda^{\sharp}_{3,14,21}\lambda^{\sharp}_{4,14,22}\lambda^{\sharp}_{6,11,22}\overline{\lambda^{\sharp}_{4,11,22}\lambda^{\sharp}_{4,14,21}\lambda^{\sharp}_{6,11,20}},$$

$$u_{13,22} = -\lambda^{\sharp}_{4,11,22}\lambda^{\sharp}_{4,14,21}\lambda^{\sharp}_{6,11,20}\overline{\lambda^{\sharp}_{3,14,21}\lambda^{\sharp}_{4,14,22}\lambda^{\sharp}_{5,13,22}\lambda^{\sharp}_{6,11,22}},$$

$$u_{14,22} = -\lambda^{\sharp}_{4,11,22}\lambda^{\sharp}_{4,14,21}\lambda^{\sharp}_{6,11,20}\overline{\lambda^{\sharp}_{3,14,21}\lambda^{\sharp}_{4,14,22}\lambda^{\sharp}_{5,14,22}\lambda^{\sharp}_{6,11,22}},$$

$$u_{6,13} = -\lambda^{\sharp}_{3,14,21}\lambda^{\sharp}_{4,14,22}\lambda^{\sharp}_{5,13,22}\lambda^{\sharp}_{6,11,22}\overline{\lambda^{\sharp}_{4,11,22}\lambda^{\sharp}_{4,14,21}\lambda^{\sharp}_{6,11,20}\lambda^{\sharp}_{6,13,22}},$$

$$u_{13,20} = \lambda^{\sharp}_{4,11,22}\lambda^{\sharp}_{4,14,21}\lambda^{\sharp}_{6,11,20}\lambda^{\sharp}_{6,13,22}\overline{\lambda^{\sharp}_{3,14,21}\lambda^{\sharp}_{4,14,22}\lambda^{\sharp}_{5,13,22}\lambda^{\sharp}_{6,11,22}\lambda^{\sharp}_{6,13,20}}.$$

The u_{v_1,v_2} involving the vertices 1, 2, 7, 8, 9, 10, 15, 16, 17, 18, 23 and 24 are chosen in the same way as in the proof of uniqueness of the cells for the $\mathcal{A}$ graphs. $\square$

For $\mathcal{E}^{(24)}$, we have the following representation of the Hecke algebra (we omit those weights which come from the $\mathcal{A}^{(24)}$ graph):

$$U^{(3,21)} = \begin{array}{c} 12 \\ 14 \end{array} \left(\begin{array}{cc} \frac{[5]}{[4]} & -\frac{\sqrt{[3][5]}}{[4]} \\ -\frac{\sqrt{[3][5]}}{[4]} & \frac{[3]}{[4]} \end{array} \right)$$

$= U^{(12,3)}$ with labels 21,19 $= U^{(6,20)}$ with labels 13,11 $= U^{(13,6)}$ with labels 20,22,

$$U^{(19,12)} = \begin{array}{c} 3 \\ 4 \end{array} \left(\begin{array}{cc} \frac{1}{[2]} & \frac{\sqrt{[3]}}{[2]} \\ \frac{\sqrt{[3]}}{[2]} & \frac{[3]}{[2]} \end{array} \right),$$

$= U^{(21,14)}$ with labels 3,4 $= U^{(20,11)}$ with labels 6,5 $= U^{(22,13)}$ with labels 6,5,

$$U^{(5,19)} = \begin{array}{c} 11 \\ 14 \end{array} \left(\begin{array}{cc} \frac{[2]}{[3]} & \frac{\sqrt{[2][4]}}{[3]} \\ \frac{\sqrt{[2][4]}}{[3]} & \frac{[4]}{[3]} \end{array} \right), \quad = U^{(14,5)} \text{ with labels 22,19,}$$

$$U^{(4,22)} = \begin{array}{c} 14 \\ 11 \end{array} \left(\begin{array}{cc} \frac{[2]}{[3]} & -\frac{\sqrt{[2][4]}}{[3]} \\ -\frac{\sqrt{[2][4]}}{[3]} & \frac{[4]}{[3]} \end{array} \right), \quad = U^{(11,4)} \text{ with labels 19,22,}$$

$$U^{(20,13)} = \begin{array}{c} 6 \\ 5 \end{array} \left(\begin{array}{cc} \frac{[5]^2}{[2][9]} & -\frac{[5]\sqrt{[7]}}{[2][9]} \\ -\frac{[5]\sqrt{[7]}}{[2][9]} & \frac{[7]}{[2][9]} \end{array} \right), \quad = U^{(21,12)} \text{ with labels 3,4,}$$

$$U^{(4,21)} = \begin{array}{c} 12 \\ 14 \end{array} \left(\begin{array}{cc} \frac{1}{[4]} & \frac{[3]\sqrt{[5]}}{[4]\sqrt{[7]}} \\ \frac{[3]\sqrt{[5]}}{[4]\sqrt{[7]}} & \frac{[3]^2[5]}{[4][7]} \end{array} \right),$$

$= U^{(12,4)}$ with labels 21,19 $= U^{(5,20)}$ with labels 13,11 $= U^{(13,5)}$ with labels 20,22,

$$U^{(19,14)} = \begin{array}{c} 3 \\ 4 \\ 5 \end{array} \left(\begin{array}{ccc} \frac{1}{[2]} & \frac{\sqrt{[7]}}{[2][3]} & \frac{\sqrt{[7][10]}}{[3]\sqrt{[2][5]}} \\ \frac{\sqrt{[7]}}{[2][3]} & \frac{[7]}{[2][3]^2} & \frac{[7]\sqrt{[10]}}{[3]^2\sqrt{[2][5]}} \\ \frac{\sqrt{[7][10]}}{[3]\sqrt{[2][5]}} & \frac{[7]\sqrt{[10]}}{[3]^2\sqrt{[2][5]}} & \frac{[7][10]}{[3]^2[5]} \end{array} \right),$$

$$U^{(22,11)} = \begin{array}{c} 6 \\ 5 \\ 4 \end{array} \left(\begin{array}{ccc} \frac{1}{[2]} & \frac{\sqrt{[7]}}{[2][3]} & -\frac{\sqrt{[7][10]}}{[3]\sqrt{[2][5]}} \\ \frac{\sqrt{[7]}}{[2][3]} & \frac{[7]}{[2][3]^2} & -\frac{[7]\sqrt{[10]}}{[3]^2\sqrt{[2][5]}} \\ -\frac{\sqrt{[7][10]}}{[3]\sqrt{[2][5]}} & -\frac{[7]\sqrt{[10]}}{[3]^2\sqrt{[2][5]}} & \frac{[7][10]}{[3]^2[5]} \end{array} \right),$$

$$U^{(19,11)} = \begin{array}{c} 2 \\ 4 \\ 5 \end{array} \begin{pmatrix} \frac{[4]}{[5]} & \frac{[4]\sqrt{[7]}}{[3][5]} & \frac{[4]\sqrt{[7]}}{[3][5]} \\ \frac{[4]\sqrt{[7]}}{[3][5]} & \frac{[4][7]}{[3]^2[5]} & \frac{[4][7]}{[3]^2[5]} \\ \frac{[4]\sqrt{[7]}}{[3][5]} & \frac{[4][7]}{[3]^2[5]} & \frac{[4][7]}{[3]^2[5]} \end{pmatrix}, \quad = \quad U^{(22,14)} \text{ with labels 7,4,5,}$$

$$\begin{aligned} U^{(3,19)} &= \begin{array}{c} 10 \\ 14 \\ 12 \end{array} \begin{pmatrix} \frac{[4]}{[5]} & \frac{\sqrt{[3]}}{\sqrt{[5]}} & \frac{\sqrt{[9]}}{[5]} \\ \frac{\sqrt{[3]}}{\sqrt{[5]}} & \frac{[3]}{[4]} & \frac{\sqrt{[3][9]}}{[4]\sqrt{[5]}} \\ \frac{\sqrt{[9]}}{[5]} & \frac{\sqrt{[3][9]}}{[4]\sqrt{[5]}} & \frac{[9]}{[4][5]} \end{pmatrix}, \\ &= U^{(14,3)} \text{ with labels 23,19,21} \quad = \quad U^{(6,22)} \text{ with labels 15,11,13} \\ &= U^{(11,6)} \text{ with labels 18,22,20,} \end{aligned}$$

$$\begin{aligned} U^{(4,19)} &= \begin{array}{c} 11 \\ 14 \\ 12 \end{array} \begin{pmatrix} \frac{[2]}{[3]} & \frac{\sqrt{[2][5]}}{[3]\sqrt{[4]}} & \frac{\sqrt{[2][9]}}{\sqrt{[4][7]}} \\ \frac{\sqrt{[2][5]}}{[3]\sqrt{[4]}} & \frac{[5]}{[3][4]} & \frac{\sqrt{[5][9]}}{[4]\sqrt{[7]}} \\ \frac{\sqrt{[2][9]}}{\sqrt{[4][7]}} & \frac{\sqrt{[5][9]}}{[4]\sqrt{[7]}} & \frac{[3][9]}{[4][7]} \end{pmatrix} \\ &= U^{(14,4)} \text{ with labels 22,19,21} \quad = \quad U^{(5,22)} \text{ with labels 14,11,13} \\ &= U^{(11,5)} \text{ with labels 19,22,20.} \end{aligned}$$

The Hecke representation given above cannot be equivalent to that given by Sochen in [108] for $\mathcal{E}^{(24)}$ as our weights $[U^{(14,4)}]_{19,19}$, $[U^{(14,4)}]_{21,21}$, $[U^{(11,5)}]_{20,20}$, $[U^{(11,5)}]_{22,22}$ and $[U^{(19,11)}]_{2,2}$ (as well as the corresponding weights under the reflection of the graph which sends vertices $1 \leftrightarrow 8$) have different absolute values to those given by Sochen (and there are no double edges on the graph). We do not believe that there exists two inequivalent solutions for the Hecke representation for $\mathcal{E}^{(24)}$, and that the differences must be due to typographical errors in [108].

Chapter 5

A_2-Goodman-de la Harpe-Jones construction

In [51] Goodman, de la Harpe and Jones constructed a subfactor $B \subset C$ given by the embedding of the Temperley-Lieb algebra in the AF-algebra for an $SU(2)$ ADE Dynkin diagram G. We will present an $SU(3)$ analogue of this construction, where we embed the A_2-Temperley-Lieb or Hecke algebra in an AF path algebra of the $SU(3)$ $\mathcal{ADE}$ graphs. Using this construction, we are able to realize all the $SU(3)$ modular invariants by subfactors.

5.1 General construction

In this section we will construct the A_2-Goodman-de la Harpe-Jones subfactors. We first present some results that will be needed for this construction.

Let $U_1, U_2, \ldots U_{m-1}$ be operators which satisfy H1-H3 with parameter δ. We let

$$F_i = U_i U_{i+1} U_i - U_i = U_{i+1} U_i U_{i+1} - U_{i+1}, \tag{5.1}$$

for $i = 1, 2, \ldots, m-2$.

Lemma 5.1.1 *With F_i defined as above, $F_i F_{i+1} F_i = \delta^2 F_i$ if and only if the U_i satisfy the extra $SU(3)$ relation (1.7).*

Proof

The condition (1.7) can be written as

$$\begin{aligned} & U_{i+2}U_{i+1}U_iU_{i+1}U_{i+2}U_{i+1} - U_iU_{i+1} - U_iU_{i+1}U_{i+2}U_{i+1} - U_{i+2}U_{i+1}U_iU_{i+1} \\ &= \delta(U_{i+1}U_{i+2}U_{i+1} - U_{i+1}). \end{aligned} \tag{5.2}$$

We have

$$\begin{aligned}
F_iF_{i+1}F_i &= (U_{i+1}U_iU_{i+1}-U_{i+1})(U_{i+1}U_{i+2}U_{i+1}-U_{i+1})(U_{i+1}U_iU_{i+1}-U_{i+1}) \\
&= (U_{i+1}U_i-1)(U_{i+1}^2U_{i+2}U_{i+1}^2-U_{i+1}^3)(U_iU_{i+1}-1) \\
&= \delta(U_{i+1}U_iU_{i+1}-U_{i+1})(\delta U_{i+2}-1)(U_{i+1}U_iU_{i+1}-U_{i+1}) \\
&= \delta(U_iU_{i+1}U_i-U_i)(\delta U_{i+2}-1)(U_iU_{i+1}U_i-U_i) \\
&= \delta(\delta U_iU_{i+1}U_iU_{i+2}U_iU_{i+1}U_i-\delta U_iU_{i+1}U_iU_{i+2}U_i-\delta U_iU_{i+2}U_iU_{i+1}U_i \\
&\qquad +\delta U_iU_{i+2}U_i-U_iU_{i+1}U_i^2U_{i+1}U_i+U_iU_{i+1}U_i^2+U_i^2U_{i+1}U_i-U_i^2).
\end{aligned}$$

In the following we use relation H3 to transform each expression, and we indicate which terms have been replaced at each stage by enclosing them within square brackets [ ]. Since U_i, U_{i+2} commute by H1, we have

$$\begin{aligned}
&\delta^2(\delta U_iU_{i+1}U_{i+2}[U_iU_{i+1}U_i]-\delta U_iU_{i+1}U_{i+2}U_i-\delta U_{i+2}U_iU_{i+1}U_i+\delta U_iU_{i+2} \\
&\qquad -U_i[U_{i+1}U_iU_{i+1}]U_i+2U_iU_{i+1}U_i-U_i) \\
=\ &\delta^2(\delta U_i[U_{i+1}U_{i+2}U_{i+1}]U_iU_{i+1}-\delta U_iU_{i+1}U_{i+2}U_{i+1}+\delta U_iU_{i+1}U_{i+2}U_i-\delta U_iU_{i+1}U_{i+2}U_i \\
&\qquad -\delta U_{i+2}U_iU_{i+1}U_i+\delta U_iU_{i+2}-U_i^2U_{i+1}U_i^2-U_iU_{i+1}U_i+U_i^3+2U_iU_{i+1}U_i-U_i) \\
=\ &\delta^2(\delta U_{i+2}[U_iU_{i+1}U_i]U_{i+2}U_{i+1}-\delta U_iU_{i+2}U_iU_{i+1}+\delta U_i[U_{i+1}U_iU_{i+1}]-\delta U_iU_{i+1}U_{i+2}U_i \\
&\qquad -\delta U_{i+2}[U_iU_{i+1}U_i]+\delta U_iU_{i+2}-(\delta^2-1)(U_iU_{i+1}U_i-U_i)) \\
=\ &\delta^2(\delta U_{i+2}U_{i+1}U_iU_{i+1}U_{i+2}U_{i+1}-\delta[U_{i+2}U_{i+1}U_{i+2}]U_{i+1}+\delta U_{i+2}U_iU_{i+2}U_{i+1} \\
&\qquad -\delta^2U_{i+2}U_iU_{i+1}+\delta U_i^2U_{i+1}U_i-\delta U_i^2+\delta U_iU_{i+1}-\delta U_iU_{i+1}U_{i+2}U_{i+1} \\
&\qquad -\delta U_{i+2}U_{i+1}U_iU_{i+1}+\delta U_{i+2}U_{i+1}-\delta U_{i+2}U_i+\delta U_iU_{i+2}-(\delta^2-1)(U_iU_{i+1}U_i-U_i)) \\
=\ &\delta^2(\delta(U_{i+2}U_{i+1}U_iU_{i+1}U_{i+2}U_{i+1}+U_iU_{i+1}-U_iU_{i+1}U_{i+2}U_{i+1}-U_{i+2}U_{i+1}U_iU_{i+1}) \\
&\qquad -\delta U_{i+1}U_{i+2}U_{i+1}^2+\delta U_{i+1}^2-\delta U_{i+2}U_{i+1}+\delta U_{i+2}U_{i+1}+U_iU_{i+1}U_i-U_i) \\
=\ &\delta^2(\delta^2(U_{i+1}U_{i+2}U_{i+1}-U_{i+1})-\delta^2(U_{i+1}U_{i+2}U_{i+1}-U_{i+1})+U_iU_{i+1}U_i-U_i) \\
=\ &\delta^2F_i,
\end{aligned}$$

where the penultimate equality follows from (5.2). □

Note that if the condition (1.7) is satisfied, $\mathrm{alg}(\mathbf{1},F_i|i=1,\ldots,m-1)$ is not the Temperley-Lieb algebra, since although $F_iF_j=F_jF_i$ for $|i-j|>2$, it is not the case for $|i-j|=2$, indeed $F_iF_{i+2}F_i=\delta F_iU_{i+3}$ (cf. (6.7), (6.8)) so that F_i, $F_{i\pm2}$ do not commute.

Let $\mathcal{G}$ be a finite $\mathcal{ADE}$ graph with Coxeter number $n<\infty$. We write $[2]=[2]_q$, $[3]=[3]_q$, where $q=e^{i\pi/n}$. Let $M_0=\mathbb{C}^{n_0}$ where n_0 is the number of 0-coloured vertices of $\mathcal{G}$, and let $M_0\subset M_1\subset M_2\subset\cdots$ be finite dimensional von Neumann algebras, with the Bratteli diagram for the inclusion $M_j\subset M_{j+1}$ given by the graph $\mathcal{G}$, $j>0$. Let (μ,μ') be matrix units indexed by paths μ, μ' on $\mathcal{G}$, and denote by $\mathfrak{V}^{\mathcal{G}}$ the vertices of $\mathcal{G}$. We define operators $U_k \in M_{k+1}$, for $k = 1, 2, \ldots$, by

$$U_k = \sum_{\sigma,\beta_i,\gamma_i} \mathcal{U}_{\beta_1,\gamma_1}^{\beta_2,\gamma_2}(\sigma \cdot \beta_1 \cdot \gamma_1, \sigma \cdot \beta_2 \cdot \gamma_2), \tag{5.3}$$

where the summation is over all paths σ of length $k-1$ and edges $\beta_1, \beta_2, \gamma_1, \gamma_2$ of $\mathcal{G}$ such that $r(\sigma) = s(\beta_1) = s(\beta_2)$, $s(\gamma_i) = r(\beta_i)$ for $i = 1, 2$, and $r(\gamma_1) = r(\gamma_2)$, and $\mathcal{U}_{\beta_1,\gamma_1}^{\beta_2,\gamma_2}$ is defined in (4.4). We will use the notation W_{ρ_1,ρ_2,ρ_3} for $W(\triangle_{i_1,i_2,i_3}^{(\rho_1,\rho_2,\rho_3)})$, where $i_l = s(\rho_l)$, $l = 1, 2, 3$.

Lemma 5.1.2 *With $U_k \in M_{k+1}$ given as in (5.3), the operator $F_k \in M_{k+2}$ defined in (5.1) is given by*

$$F_k = \sum_{\sigma,\beta_i,\gamma_i} \frac{1}{\phi_{r(\beta_3)}^2} W_{\gamma_1,\gamma_2,\gamma_3} \overline{W_{\beta_1,\beta_2,\beta_3}} \, (\sigma \cdot \beta_1 \cdot \beta_2 \cdot \beta_3, \sigma \cdot \gamma_1 \cdot \gamma_2 \cdot \gamma_3), \tag{5.4}$$

where the summation is over all paths σ of length $k-1$ and edges β_i, γ_i of $\mathcal{G}$, $i = 1, 2, 3$.

Proof

We have

$$\begin{aligned}
&U_k U_{k+1} U_k \\
&= \sum_{\substack{\sigma_i,\beta_i,\\ \gamma_i,\mu_i}} \mathcal{U}_{\beta_1,\gamma_1}^{\beta_2,\gamma_2} \mathcal{U}_{\beta_3,\gamma_3}^{\beta_4,\gamma_4} \mathcal{U}_{\beta_5,\gamma_5}^{\beta_6,\gamma_6} \, (\sigma_1 \cdot \beta_1 \cdot \gamma_1 \cdot \mu_1, \sigma_1 \cdot \beta_2 \cdot \gamma_2 \cdot \mu_1) \\
&\qquad\qquad \times (\sigma_2 \cdot \mu_2 \cdot \beta_3 \cdot \gamma_3, \sigma_2 \cdot \mu_2 \cdot \beta_4 \cdot \gamma_4)(\sigma_3 \cdot \beta_5 \cdot \gamma_5 \cdot \mu_3, \sigma_3 \cdot \beta_6 \cdot \gamma_6 \cdot \mu_3) \\
&= \sum_{\substack{\sigma_i,\beta_i,\\ \gamma_i,\mu_i}} \mathcal{U}_{\beta_1,\gamma_1}^{\beta_2,\gamma_2} \mathcal{U}_{\beta_4,\mu_3}^{\beta_3,\mu_1} \mathcal{U}_{\beta_2,\gamma_4}^{\beta_6,\gamma_6} \, (\sigma_1 \cdot \beta_1 \cdot \gamma_1 \cdot \mu_1, \sigma_1 \cdot \beta_6 \cdot \gamma_6 \cdot \mu_3) \\
&= \sum_{\substack{\sigma_i,\beta_i,\gamma_i\\ \mu_i,\lambda_i}} \frac{1}{\phi_{s(\beta_6)}\phi_{r(\gamma_6)}\phi_{s(\beta_4)}\phi_{r(\mu_1)}\phi_{s(\beta_1)}\phi_{r(\gamma_1)}} W_{\beta_6,\gamma_6,\lambda_1} \overline{W_{\beta_2,\beta_4,\lambda_1}} W_{\beta_4,\mu_3,\lambda_2} \overline{W_{\beta_3,\mu_1,\lambda_2}} W_{\beta_2,\beta_3,\lambda_3} \\
&\qquad\qquad \times \overline{W_{\beta_1,\gamma_1,\lambda_3}} \, (\sigma_1 \cdot \beta_1 \cdot \gamma_1 \cdot \mu_1, \sigma_1 \cdot \beta_6 \cdot \gamma_6 \cdot \mu_3) \\
&= \sum_{\substack{\sigma_i,\beta_i,\gamma_i\\ \mu_i,\lambda_i}} \frac{1}{\phi_{s(\beta_6)}\phi_{r(\gamma_6)}\phi_{r(\mu_1)}\phi_{s(\beta_1)}\phi_{r(\gamma_1)}} W_{\beta_6,\gamma_6,\lambda_1} \overline{W_{\beta_1,\gamma_1,\lambda_3}} \Big(\delta_{\lambda_1,\mu_3}\delta_{\lambda_3,\mu_1}\phi_{s(\mu_3)}\phi_{r(\mu_3)}\phi_{s(\mu_1)} \\
&\qquad\qquad + \delta_{\lambda_1,\lambda_3}\delta_{\mu_1,\mu_3}\phi_{r(\lambda_1)}\phi_{s(\mu_3)}\phi_{r(\mu_3)} \Big) \, (\sigma_1 \cdot \beta_1 \cdot \gamma_1 \cdot \mu_1, \sigma_1 \cdot \beta_6 \cdot \gamma_6 \cdot \mu_3) \qquad (5.5) \\
&= \sum_{\substack{\sigma,\beta_i,\\ \gamma_i,\mu_i}} \frac{1}{\phi_{r(\mu_1)}^2} W_{\beta_6,\gamma_6,\mu_3} \overline{W_{\beta_1,\gamma_1,\mu_1}} \, (\sigma_1 \cdot \beta_1 \cdot \gamma_1 \cdot \mu_1, \sigma_1 \cdot \beta_6 \cdot \gamma_6 \cdot \mu_3) \; + \; U_k,
\end{aligned}$$

where we obtain (5.5) by Ocneanu's type II equation (4.2). □

Note that if p is a minimal projection in M_k corresponding to a vertex (v,k) of the Bratteli diagram $\widehat{\mathcal{G}}$ of $\mathcal{G}$, then $\alpha^{-1}\delta^{-1}F_{k+1}p$ is a projection in M_{k+3} corresponding to the vertex $(v,k+3)$ of $\widehat{\mathcal{G}}$, since from (5.4) we see that the last three edges in any pairs of paths in F_{k+1} form a closed loop of length 3 and hence the pairs of paths in $F_{k+1}p \in M_{k+3}$ must have the same end vertex as $p \in M_k$.

Lemma 5.1.3 *The operators U_k defined in (5.3) satisfy the A_2-Temperley-Lieb relations.*

Proof

These operators satisfy the Hecke relations H1-H3 since the connection defined in (4.3) satisfies the Yang-Baxter equation. We have left to show that they satisfy (1.7). By Lemma 5.1.1, we need only show that $F_kF_{k+1}F_k = [2]^2F_k$. We have

$$\begin{aligned}
&F_kF_{k+1}F_k \\
&= \sum_{\substack{\sigma_i,\beta_i,\\ \gamma_i,\mu_i}} \frac{1}{\phi^2_{r(\beta_3)}\phi^2_{r(\beta_6)}\phi^2_{r(\beta_9)}} W_{\gamma_7,\gamma_8,\gamma_9}\overline{W_{\beta_7,\beta_8,\beta_9}}W_{\gamma_4,\gamma_5,\gamma_6}\overline{W_{\beta_4,\beta_5,\beta_6}}W_{\gamma_1,\gamma_2,\gamma_3}\overline{W_{\beta_1,\beta_2,\beta_3}} \\
&\qquad (\sigma_1\cdot\beta_1\cdot\beta_2\cdot\beta_3\cdot\mu_1,\sigma_1\cdot\gamma_1\cdot\gamma_2\cdot\gamma_3\cdot\mu_1)(\sigma_2\cdot\mu_2\cdot\beta_4\cdot\beta_5\cdot\beta_6,\sigma_2\cdot\mu_2\cdot\gamma_4\cdot\gamma_5\cdot\gamma_6) \\
&\qquad\qquad \times(\sigma_3\cdot\beta_7\cdot\beta_8\cdot\beta_9\cdot\mu_3,\sigma_3\cdot\gamma_7\cdot\gamma_8\cdot\gamma_9\cdot\mu_3) \\
&= \sum_{\substack{\sigma_1,\beta_i,\\ \gamma_i,\mu_i}} \frac{1}{\phi^2_{r(\beta_3)}\phi^2_{r(\mu_1)}\phi^2_{s(\mu_3)}} W_{\gamma_7,\gamma_8,\gamma_9}\overline{W_{\beta_7,\beta_8,\beta_9}}W_{\beta_8,\beta_9,\mu_3}\overline{W_{\beta_4,\beta_5,\mu_1}}W_{\beta_7,\beta_4,\beta_5}\overline{W_{\beta_1,\beta_2,\beta_3}} \\
&\qquad (\sigma_1\cdot\beta_1\cdot\beta_2\cdot\beta_3\cdot\mu_1,\sigma_1\cdot\gamma_7\cdot\gamma_8\cdot\gamma_9\cdot\mu_3) \\
&= [2]^2\sum_{\substack{\sigma_1,\beta_i,\\ \gamma_i,\mu_i}} \frac{\phi_{s(\mu_1)}\phi_{r(\mu_1)}\phi_{s(\mu_3)}\phi_{r(\mu_3)}}{\phi^2_{r(\beta_3)}\phi^2_{r(\mu_1)}\phi^2_{s(\mu_3)}} W_{\gamma_7,\gamma_8,\gamma_9}\overline{W_{\beta_1,\beta_2,\beta_3}}\delta_{\mu_1,\beta_7}\delta_{\mu_1,\mu_3} \\
&\qquad (\sigma_1\cdot\beta_1\cdot\beta_2\cdot\beta_3\cdot\mu_1,\sigma_1\cdot\gamma_7\cdot\gamma_8\cdot\gamma_9\cdot\mu_3) \\
&= [2]^2\sum_{\substack{\sigma_1,\beta_i,\\ \gamma_i,\mu_1}} \frac{1}{\phi^2_{r(\beta_3)}} W_{\gamma_7,\gamma_8,\gamma_9}\overline{W_{\beta_1,\beta_2,\beta_3}}\,(\sigma_1\cdot\beta_1\cdot\beta_2\cdot\beta_3\cdot\mu_1,\sigma_1\cdot\gamma_7\cdot\gamma_8\cdot\gamma_9\cdot\mu_1) \\
&= [2]^2F_k.
\end{aligned}$$

□

By [28, Theorem 6.1] there is a unique normalized faithful trace on $\bigcup_k M_k$, defined as in [38] by

$$\mathrm{tr}((\sigma_1,\sigma_2)) = \delta_{\sigma_1,\sigma_2}[3]^{-k}\phi_{r(\sigma_1)}, \tag{5.6}$$

for paths σ_i of length k, $i=1,2$, $k=0,1,\ldots$. The conditional expectation of M_k onto M_{k-1} with respect to the trace is given by

$$E((\sigma_1\cdot\sigma_1',\sigma_2\cdot\sigma_2')) = \delta_{\sigma_1',\sigma_2'}[3]^{-1}\frac{\phi_{r(\sigma_1')}}{\phi_{r(\sigma_1)}}(\sigma_1,\sigma_2),$$

for paths σ_i of length $k-1$, and σ_i' of length 1, $i=1,2$, $k\geq 1$ (see e.g. [39, Lemma 11.7]).

Lemma 5.1.4 *For an $\mathcal{ADE}$ graph $\mathcal{G}$, let $M_0 = \mathbb{C}^{n_0}$ where n_0 is the number of 0-coloured vertices of $\mathcal{G}$. Let $M_0 \subset M_1 \subset M_2 \subset \cdots$ be a sequence of finite dimensional von Neumann algebras with normalized trace. Then for the operator $U_k \in M_{k+1}$ defined in (5.3),* tr *is a Markov trace in the sense that* $\mathrm{tr}(xU_k) = [2][3]^{-1}\mathrm{tr}(x)$ *for any $x \in M_k$, $k = 1, 2, \ldots$.*

Proof

Let $x \in M_k$ be the matrix unit $(\alpha_1 \cdot \alpha_1', \alpha_2 \cdot \alpha_2')$. Then

$$\begin{aligned} xU_k &= \sum_{\sigma,\beta_i,\gamma_i,\mu} \mathcal{U}^{\beta_2,\gamma_2}_{\beta_1,\gamma_1}(\alpha_1 \cdot \alpha_1' \cdot \mu, \alpha_2 \cdot \alpha_2' \cdot \mu) \cdot (\sigma \cdot \beta_1 \cdot \gamma_1, \sigma \cdot \beta_2 \cdot \gamma_2) \\ &= \sum_{\sigma,\beta_i,\gamma_i,\mu} \mathcal{U}^{\beta_2,\gamma_2}_{\beta_1,\gamma_1} \delta_{\alpha_2,\sigma}\delta_{\alpha_2',\beta_1}\delta_{\mu,\gamma_1}(\alpha_1 \cdot \alpha_1' \cdot \mu, \sigma \cdot \beta_2 \cdot \gamma_2) \\ &= \sum_{\beta_2,\gamma_2,\mu} \mathcal{U}^{\beta_2,\gamma_2}_{\alpha_2',\mu}(\alpha_1 \cdot \alpha_1' \cdot \mu, \alpha_2 \cdot \beta_2 \cdot \gamma_2), \end{aligned}$$

and

$$\begin{aligned} \mathrm{tr}(xU_k) &= \sum_{\beta_2,\gamma_2,\mu} \mathcal{U}^{\beta_2,\gamma_2}_{\alpha_2',\mu}\mathrm{tr}((\alpha_1 \cdot \alpha_1' \cdot \mu, \alpha_2 \cdot \beta_2 \cdot \gamma_2)) \\ &= \sum_{\beta_2,\gamma_2,\mu} \mathcal{U}^{\beta_2,\gamma_2}_{\alpha_2',\mu}\delta_{\alpha_1,\alpha_2}\delta_{\alpha_1',\beta_2}\delta_{\mu,\gamma_2}[3]^{-k+1}\phi_{r(\mu)} \quad = \quad \delta_{\alpha_1,\alpha_2}[3]^{-k+1}\sum_{\mu}\mathcal{U}^{\alpha_1',\mu}_{\alpha_2',\mu}\phi_{r(\mu)} \\ &= \delta_{\alpha_1,\alpha_2}[3]^{-k+1}\sum_{\mu}\frac{1}{\phi_{s(\alpha_1')}\phi_{r(\mu)}}W_{\lambda,\alpha_1',\mu}\overline{W_{\lambda,\alpha_2',\mu}}\phi_{r(\mu)} \\ &= \delta_{\alpha_1,\alpha_2}[3]^{-k+1}\frac{1}{\phi_{s(\alpha_1')}}[2]\phi_{s(\alpha_1')}\phi_{r(\alpha_1')}\delta_{\alpha_1',\alpha_2'} \quad = \quad [2][3]^{-1}\mathrm{tr}(x), \end{aligned}$$

where we have used Ocneanu's type I equation (4.1) in the penultimate equality. The result for any $x \in M_k$ follows by linearity of the trace. □

Then we have $\mathrm{tr}(U_k) = [2]/[3]$, and the conditional expectation of $U_k \in M_{k+1}$ onto M_k is $E(U_k) = [2]1_k/[3]$, for all $k \geq 1$. We will need the following result:

Lemma 5.1.5 *Let $F_i \in M_{i+2}$ be as above and* tr *a Markov trace on the M_i, $i = 1, 2, \ldots, ,$ then* $\mathrm{tr}(F_{k+1}x) = [2][3]^{-2}\mathrm{tr}(x)$, *for $x \in M_k$, $k \in \mathbb{N}$.*

Proof

Now $\mathrm{tr}(U_{k+1}U_{k+2}U_{k+1}x) = \mathrm{tr}(U_{k+2}U_{k+1}xU_{k+1}) = [2][3]^{-1}\mathrm{tr}(U_{k+1}xU_{k+1})$, since tr is a Markov trace. Then $\mathrm{tr}(U_{k+1}xU_{k+1}) = \mathrm{tr}(U_{k+1}^2 x) = [2]\mathrm{tr}(U_{k+1}x) = [2]^2[3]^{-1}\mathrm{tr}(x)$. We also have $\mathrm{tr}(U_{k+1}x) = [2][3]^{-1}\mathrm{tr}(x)$, so that

$$\mathrm{tr}((U_{k+1}U_{k+2}U_{k+1} - U_{k+1})x) \quad = \quad \left(\frac{[2]^3}{[3]^2} - \frac{[2]}{[3]}\right)\mathrm{tr}(x) = \frac{[2]}{[3]^2}\mathrm{tr}(x).$$

□

Proposition 5.1.6 *With $U_k \in M_{k+1}$ as above and $x \in M_k$, $k = 1, 2, \dots,$, x commutes with U_k if and only if $x \in M_{k-1}$, i.e. $M_{k-1} = \{U_k\}' \cap M_k$.*

Proof

In the notation of Section 1.6.2 we have $U_k \in A[k-1, k+1]$ and $x \in A[0, k-1]$, and hence by (1.34) x commutes with U_k.

We now check the converse. Let $x = \sum_{\alpha_i, \alpha_i'} \lambda_{\alpha_1 \cdot \alpha_2, \alpha_1' \cdot \alpha_2'} (\alpha_1 \cdot \alpha_2, \alpha_1' \cdot \alpha_2') \in M_k$, where the summation is over all $|\alpha_i| = k-1$, $|\alpha_i'| = 1$, $i = 1, 2$. Assume that x commutes with U_k. We have the inclusion of x in M_{k+1} given by $x = \sum_{\alpha_i, \alpha_i', \mu} \lambda_{\alpha_1 \cdot \alpha_2, \alpha_1' \cdot \alpha_2'} (\alpha_1 \cdot \alpha_2 \cdot \mu, \alpha_1' \cdot \alpha_2' \cdot \mu)$. Since x commutes with U_k we have $U_k^2 x = U_k x U_k$, and taking the conditional expectation onto M_k we have

$$[2] E(U_k x) = E(U_k x U_k). \tag{5.7}$$

By the Markov property of the trace on the M_k, the left hand side gives $[2]E(U_k x) = [2]E(U_k)x = [2]^2 x/[3]$, since $x \in M_k$. For the right hand side of (5.7) we have

$$\begin{aligned}
E(U_k x U_k) &= E\Bigg(\sum_{\substack{\sigma, \beta_3, \beta_4, \\ \gamma_3, \gamma_4}} \mathcal{U}_{\beta_3, \gamma_3}^{\beta_4, \gamma_4} (\sigma \cdot \beta_3 \cdot \gamma_3, \sigma \cdot \beta_4 \cdot \gamma_4) \\
&\qquad \times \sum_{\substack{\alpha_i, \alpha_i', \beta_2, \\ \gamma_2, \mu}} \mathcal{U}_{\alpha_2', \mu}^{\beta_2, \gamma_2} \lambda_{\alpha_1 \cdot \alpha_2, \alpha_1' \cdot \alpha_2'} (\alpha_1 \cdot \alpha_2 \cdot \mu, \alpha_1' \cdot \beta_2 \cdot \gamma_2) \Bigg) \\
&= E\left(\sum_{\substack{\sigma, \alpha_i, \alpha_i' \\ \beta_i, \gamma_i, \mu}} \mathcal{U}_{\beta_3, \gamma_3}^{\beta_4, \gamma_4} \mathcal{U}_{\alpha_2', \mu}^{\beta_2, \gamma_2} \lambda_{\alpha_1 \cdot \alpha_2, \alpha_1' \cdot \alpha_2'} \delta_{\alpha_1, \sigma} \delta_{\beta_4, \alpha_2} \delta_{\gamma_4, \mu} (\sigma \cdot \beta_3 \cdot \gamma_3, \alpha_1' \cdot \beta_2 \cdot \gamma_2) \right) \\
&= \sum_{\substack{\alpha_i, \alpha_i', \beta_i, \\ \gamma_i, \mu}} \mathcal{U}_{\beta_3, \gamma_3}^{\alpha_2, \mu} \mathcal{U}_{\alpha_2', \mu}^{\beta_2, \gamma_2} \lambda_{\alpha_1 \cdot \alpha_2, \alpha_1' \cdot \alpha_2'} E((\alpha_1 \cdot \beta_3 \cdot \gamma_3, \alpha_1' \cdot \beta_2 \cdot \gamma_2)) \\
&= [3]^{-1} \sum_{\substack{\alpha_1, \alpha_1' \\ \beta_3, \beta_2}} \left(\sum_{\substack{\alpha_2, \alpha_2' \\ \gamma_i, \mu}} \mathcal{U}_{\beta_3, \gamma_3}^{\alpha_2, \mu} \mathcal{U}_{\alpha_2', \mu}^{\beta_2, \gamma_2} \lambda_{\alpha_1 \cdot \alpha_2, \alpha_1' \cdot \alpha_2'} \delta_{\gamma_2, \gamma_3} \frac{\phi_{r(\gamma_2)}}{\phi_{r(\beta_3)}} \right) (\alpha_1 \cdot \beta_3, \alpha_1' \cdot \beta_2) \\
&= \alpha^{-1} \sum_{\substack{\alpha_1, \alpha_1' \\ \beta_1, \beta_2}} b_{\alpha_1 \cdot \beta_1, \alpha_1' \cdot \beta_2} (\alpha_1 \cdot \beta_1, \alpha_1' \cdot \beta_2),
\end{aligned}$$

where $b_{\alpha_1 \cdot \beta_1, \alpha_1' \cdot \beta_2} = \sum_{\substack{\alpha_2, \alpha_2' \\ \gamma, \mu}} \mathcal{U}_{\beta_1, \gamma}^{\alpha_2, \mu} \mathcal{U}_{\alpha_2', \mu}^{\beta_2, \gamma} \lambda_{\alpha_1 \cdot \alpha_2, \alpha_1' \cdot \alpha_2'} \frac{\phi_{r(\gamma)}}{\phi_{r(\beta_1)}}$. Then for any paths α_1, α_1' and edges β_1, β_2 on $\mathcal{G}$ we have

$$b_{\alpha_1 \cdot \beta_1, \alpha_1' \cdot \beta_2} = \sum_{\substack{\alpha_2, \alpha_2', \gamma \\ \mu, \zeta_i}} \frac{1}{\phi_{s(\alpha_1)} \phi_{r(\gamma)}} W_{\beta_1 \gamma \zeta_1} \overline{W_{\alpha_2 \mu \zeta_1}} \frac{1}{\phi_{s(\alpha_2)} \phi_{r(\gamma)}} W_{\alpha_2' \mu \zeta_2} \overline{W_{\beta_2 \gamma \zeta_2}} \lambda_{\alpha_1 \cdot \alpha_2, \alpha_1' \cdot \alpha_2'} \frac{\phi_{r(\gamma)}}{\phi_{r(\beta_1)}}$$

$$
\begin{aligned}
&= \sum_{\alpha_2,\alpha_2'} \frac{1}{\phi^2_{s(\alpha_2)}\phi_{r(\beta_1)}} \lambda_{\alpha_1\cdot\alpha_2,\alpha_1'\cdot\alpha_2'} \left(\sum_{\gamma,\mu,\zeta_i} \frac{1}{\phi_{r(\gamma)}} W_{\beta_1\gamma\zeta_1} \overline{W_{\alpha_2\mu\zeta_1}} W_{\alpha_2'\mu\zeta_2} \overline{W_{\beta_2\gamma\zeta_2}} \right) \\
&= \sum_{\alpha_2,\alpha_2'} \frac{1}{\phi^2_{s(\alpha_2)}\phi_{r(\beta_1)}} \lambda_{\alpha_1\cdot\alpha_2,\alpha_1'\cdot\alpha_2'} \Big(\phi_{r(\alpha_2)}\phi_{s(\alpha_2)}\phi_{r(\beta_1)}\delta_{\alpha_2,\alpha_2'}\delta_{\beta_1,\beta_2} \\
&\qquad\qquad + \phi_{s(\alpha_2)}\phi_{r(\beta_1)}\phi_{s(\alpha_2)}\delta_{\alpha_2,\beta_1}\delta_{\alpha_2',\beta_2} \Big) \qquad (5.8) \\
&= \sum_{\alpha_2} \frac{\phi_{r(\alpha_2)}}{\phi_{s(\alpha_2)}} \lambda_{\alpha_1\cdot\alpha_2,\alpha_1'\cdot\alpha_2'}\delta_{\beta_1,\beta_2} + \lambda_{\alpha_1\cdot\beta_1,\alpha_1'\cdot\beta_2},
\end{aligned}
$$

where equality (5.8) follows by Ocneanu's type II equation (4.2). We define

$$\lambda_{r(\alpha_1)} := \sum_{\beta'} \delta_{s(\beta'),r(\alpha_1)} \frac{\phi_{r(\beta')}}{\phi_{r(\alpha_1)}} \lambda_{\alpha_1\cdot\beta',\alpha_1'\cdot\beta'},$$

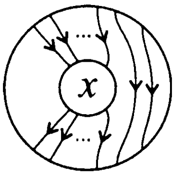
which only depends on the range of the paths α_1 and α_1'. Then we have for the right hand side of (5.7)

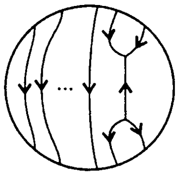
$$
\begin{aligned}
E(U_k x U_k) &= [3]^{-1} \Bigg(\sum_{\substack{\beta_1,\beta_2,\\ \alpha_i,\alpha_1'}} \frac{\phi_{r(\alpha_2)}}{\phi_{s(\alpha_2)}} \lambda_{\alpha_1\cdot\alpha_2,\alpha_1'\cdot\alpha_2'}\delta_{\beta_1,\beta_2}(\alpha_1\cdot\beta_1,\alpha_1'\cdot\beta_2) \\
&\qquad + \sum_{\substack{\beta_1,\beta_2,\\ \alpha_1,\alpha_1'}} \lambda_{\alpha_1\cdot\beta_1,\alpha_1'\cdot\beta_2}(\alpha_1\cdot\beta_1,\alpha_1'\cdot\beta_2) \Bigg) \\
&= [3]^{-1} \left(\sum_{\beta,\alpha_1,\alpha_1'} \lambda_{s(\beta)}(\alpha_1\cdot\beta,\alpha_1\cdot\beta) + \sum_{\substack{\beta_1,\beta_2,\\ \alpha_1,\alpha_1'}} \lambda_{\alpha_1\cdot\beta_1,\alpha_1'\cdot\beta_2}(\alpha_1\cdot\beta_1,\alpha_1'\cdot\beta_2) \right) \\
&= [3]^{-1}(w+x),
\end{aligned}
$$

where $w = \sum_{\alpha_1,\alpha_1'} \lambda_{r(\alpha_1)}(\alpha_1,\alpha_1') \in M_{k-1}$. Then (5.7) gives $([2]^2-1)x = w$, so $x \in M_{k-1}$. □

Remark. The above proof was motivated by the following pictorial argument, which uses concepts which will be introduced in Chapter 6.

Let $\jmath$ be the inclusion of M_{k-1} in M_k and $\imath$ the inclusion of M_k in M_{k+1}. For $x \in M_{k-1}$, we have the embedding $\imath\jmath(x)$ of x into M_{k+1}, and $U_1 \in M_{k+1}$ given by the tangles:





Then inserting x and U_1 into the discs of the multiplication tangle $M_{0,k+1}$, we have

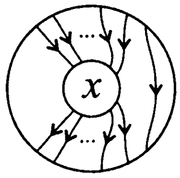
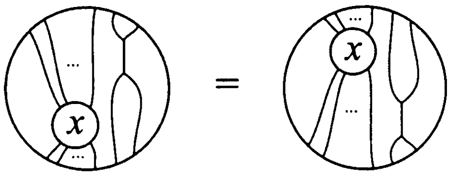
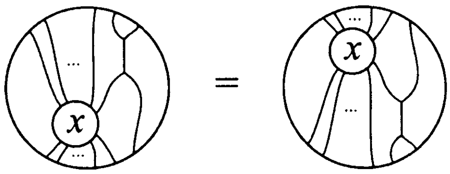


Figure 5.1: $\imath(x)$ for $x \in M_k$

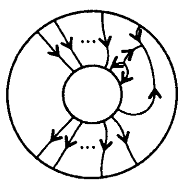


and clearly $U_1 \imath\jmath(x) = \imath\jmath(x) U_1$.

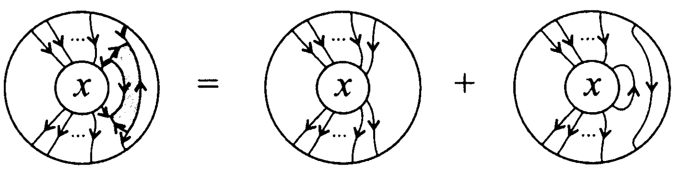
Conversely, if $x \in M_k$ we have $\imath(x) \in M_{k+1}$ as in Figure 5.1. Let $U_1 \imath(x) = \imath(x) U_1$, then we have the following equality of tangles:



Let T be the tangle



Enclosing both sides of $U_1 \imath(x) = \imath(x) U_1$ by the tangle T we obtain: $T(U_1 \imath(x)) = \delta^2 \imath(x)$, and $T(\imath(x) U_1)$ is



i.e. $T(\imath(x)U_1) = x + \jmath(v)$, were $v = E_{M_{k-1}}(x) \in M_{k-1}$ So $\delta^2 x = x + \jmath(v)$ which gives $x = (\delta^2 - 1)^{-1} \jmath(v)$, i.e. $x \in M_{k-1}$.

We define the **depth** of the graph $\mathcal{G}$ to be $d_{\mathcal{G}} = \max d_{v,v'}$, where we take the maximum value over all $v, v' \in \mathfrak{V}^{\mathcal{G}}$ and $d_{v,v'}$ is the length of the shortest path between any two vertices $v, v' \in \mathfrak{V}^{\mathcal{G}}$.

Lemma 5.1.7 *Let $\mathcal{G}$ be an $SU(3)$ $\mathcal{ADE}$ graph $\mathcal{G}$ (except $\mathcal{D}^{(n)}$ for $n \not\equiv 0 \bmod 3$, and $\mathcal{E}_4^{(12)}$). Then with $U_j \in M_{j+1}$ as above, any element of M_{m+1} can be written as a linear combination of elements of the form $aU_m b$ and c for $a, b, c \in M_m$, $m \geq d_{\mathcal{G}} + 3$.*

Proof

Let $a = (\lambda_1 \cdot \lambda_2, \zeta_1 \cdot \zeta_2), b = (\zeta_1 \cdot \zeta_2', \nu_1 \cdot \nu_2) \in M_m$ such that λ_1, ζ_1, ν_1 are paths of length $m-1$ on $\mathcal{G}$ starting from one of the 0-coloured vertices of $\mathcal{G}$, and λ_2, ζ_2, ζ_2', ν_2 are edges on $\mathcal{G}$. Then with U_m as in (5.3), and embedding a, b in M_{m+1}, we have

$$\begin{aligned} aU_m b &= \sum_{\sigma,\beta_i,\gamma_i,\mu,\mu'} \mathcal{U}_{\nu_1,\gamma_1}^{\nu_2,\gamma_2} \delta_{\zeta_1,\sigma} \delta_{\zeta_2,\nu_1} \delta_{\mu,\gamma_1} \delta_{\nu_2,\zeta_2'} \delta_{\gamma_2,\mu'} \, (\lambda_1 \cdot \lambda_2 \cdot \mu, \nu_1 \cdot \nu_2 \cdot \mu') \\ &= \sum_{\mu,\mu'} \mathcal{U}_{\zeta_2,\mu}^{\zeta_2',\mu'} \, (\lambda_1 \cdot \lambda_2 \cdot \mu, \nu_1 \cdot \nu_2 \cdot \mu') \\ &= \sum_{\mu,\mu',\xi} \frac{1}{\phi_{s(\zeta_2)} \phi_{r(\mu)}} W(\triangle^{(\xi,\zeta_2,\mu)}) \overline{W(\triangle^{(\xi,\zeta_2',\nu)})} \, (\lambda_1 \cdot \lambda_2 \cdot \mu, \nu_1 \cdot \nu_2 \cdot \mu'). \end{aligned} \tag{5.9}$$

The proof for each graph is similar, so we illustrate the general method by considering the graph $\mathcal{E}_1^{(12)}$, which contains double edges. Let $m \geq d_{\mathcal{G}} + 3$ be fixed. We denote by B the set of all linear combinations of elements of the form $aU_m b$ and c for $a, b, c \in M_m$. The elements in M_{m+1} will be written in the form

$$x = (\lambda_1 \cdot \lambda_2 \cdot \lambda_3, \nu_1 \cdot \nu_2 \cdot \nu_3) \tag{5.10}$$

where λ_1, ν_1 are paths of length $m-1$ on $\mathcal{G}$ with $s(\lambda_1) = s(\nu_1)$, and λ_1, λ_2, ν_1, ν_2 are edges of $\mathcal{G}$ with $r(\lambda_3) = r(\nu_3)$. Since the choice of the pair $(\lambda_1 \cdot \lambda_2, \nu_1 \cdot \nu_2)$ in a, b is arbitrary, the proof will depend on specific choices of ζ_2, ζ_2' in (5.9) in order to obtain the desired element. We let $\gamma_{v,v'}$ denote the edge on $\mathcal{E}_1^{(12)}$ from vertex v to v'.

We first consider any element (5.10) where $r(\lambda_2) = r(\nu_2)$. For any such pair $(\lambda_1 \cdot \lambda_2, \nu_1 \cdot \nu_2)$ with $r(\lambda_2) = i_l$, $l \in \{1,2,3\}$, there is only one element x, which is given by the embedding of $x' = (\lambda_1 \cdot \lambda_2, \nu_1 \cdot \nu_2) \in M_m$ in M_{m+1}. If $r(\lambda_2) = i_l$, $l \in \{1,2,3\}$, there are two possibilities for the edges $\lambda_3 = \nu_3$. If we choose $\zeta_2 = \zeta_2' = \gamma_{i_l,j_l}$ then (5.9) gives $x_l^{(1)} = (\lambda_1 \cdot \lambda_2 \cdot \gamma_{j_l,k_l}, \nu_1 \cdot \nu_2 \cdot \gamma_{j_l,k_l})$, so that $x_l^{(1)} \in B$, $l = 1,2,3$. Embedding x' in M_{m+1} we obtain $(\lambda_1 \cdot \lambda_2 \cdot \gamma_{j_l,r}, \nu_1 \cdot \nu_2 \cdot \gamma_{j_l,r}) = x' - x_l^{(1)} \in B$, for $l = 1,2,3$. A similar method gives the result for the case when $r(\lambda_2) = r(\nu_2) = k_l$, $l = 1,2,3$.

For any pair $(\lambda_1 \cdot \lambda_2, \nu_1 \cdot \nu_2)$ with $r(\lambda_2) = r(\nu_2) = p$, there are seven possibilities for λ_3, ν_3. We denote these elements by $x_l^{(2)}$, $x_{(\xi,\xi')}$, for $l = 1,2,3$, $\xi, \xi' \in \{\beta, \beta'\}$, where $x_l^{(2)} = (\lambda_1 \cdot \lambda_2 \cdot \gamma_{p,j_l}, \nu_1 \cdot \nu_2 \cdot \gamma_{p,j_l})$, $x_{(\xi,\xi')} = (\lambda_1 \cdot \lambda_2 \cdot \xi, \nu_1 \cdot \nu_2 \cdot \xi')$. First, choosing $\zeta_2 = \zeta_2' = \alpha$, equation (5.9) gives

$$y_0 = \frac{1}{\phi_r \phi_{j_1}} |W_{p,j_1,r(\alpha)}|^2 x_1^{(2)} + \frac{1}{\phi_r \phi_{j_2}} |W_{p,j_2,r(\alpha)}|^2 x_2^{(2)} + \frac{1}{\phi_r \phi_{j_3}} |W_{p,j_3,r(\alpha)}|^2 x_3^{(2)}$$

$$+\frac{1}{\phi_r\phi_q}|W_{p,q,r(\alpha\beta')}|^2 x_{(\beta',\beta')},$$

where y_0 is an element in B. Using the solution W^+ for the cells of $\mathcal{E}_1^{(12)}$ given in Theorem 4.2.18, we obtain

$$y_1 = [2]r_1^+\left(x_1^{(2)}+x_2^{(2)}+x_3^{(2)}\right)+[4]r_2^- \, x_{(\beta',\beta')}, \tag{5.11}$$

where $r_1^\pm = ([2][4] \pm \sqrt{[2][4]})$, $r_2^\pm = ([2]^2 \pm \sqrt{[2][4]})$ and $y_1 \in B$. Similarly, the choices $\zeta_2 = \zeta_2' = \alpha'$, $\zeta_2 = \alpha$, $\zeta_2' = \alpha'$ and $\zeta_2 = \alpha'$, $\zeta_2' = \alpha$ give

$$\begin{aligned} y_2 &= [2]r_1^-\left(x_1^{(2)}+x_2^{(2)}+x_3^{(2)}\right)+[4]r_2^+ \, x_{(\beta,\beta)}, && (5.12)\\ y_3 &= [2]\sqrt{r_1^+r_1^-}\left(x_1^{(2)}+\overline{\omega}x_2^{(2)}+\omega x_3^{(2)}\right)+[4]\sqrt{r_2^+r_2^-}\, x_{(\beta',\beta)}, && (5.13)\\ y_4 &= [2]\sqrt{r_1^+r_1^-}\left(x_1^{(2)}+\omega x_2^{(2)}+\overline{\omega}x_3^{(2)}\right)+[4]\sqrt{r_2^+r_2^-}\, x_{(\beta,\beta')}, && (5.14)\end{aligned}$$

where $\omega = e^{2\pi i/3}$ and $y_j \in B$, $j = 2,3,4$. We can obtain three more equations by choosing $\zeta_2 = \zeta_2' = \gamma_{k_l,p}$ for $l = 1,2,3$. Then (5.9) gives

$$\begin{aligned} y_5^{(l)} &= x_1^{(2)}+x_2^{(2)}+x_3^{(2)}+\frac{[2]^2}{[3][4]^2}r_1^-\, x_{(\beta,\beta)}+\overline{\epsilon}_l\frac{[2]^2}{[3][4]^2}\sqrt{r_1^+r_1^-}\, x_{(\beta,\beta')}\\ &\quad +\epsilon_l\frac{[2]^2}{[3][4]^2}\sqrt{r_1^+r_1^-}\, x_{(\beta',\beta)}+\frac{[2]^2}{[3][4]^2}r_1^+\, x_{(\beta',\beta')}, && (5.15)\end{aligned}$$

where $\epsilon_l = \omega^{l-1}$ and $y_5^{(l)} \in B$, $l = 1,2,3$. Equations (5.11)-(5.15) are linearly independent, and hence we can find $x_l^{(2)}$, $x_{(\xi,\xi')}$ in terms of y_j, $j = 1,\dots,4$, and $y_5^{(l)}$, for $l = 1,2,3$, $\xi,\xi' \in \{\beta,\beta'\}$; i.e. $x_l^{(2)}, x_{(\xi,\xi')} \in B$.

For any pair $(\lambda_1\cdot\lambda_2, \nu_1\cdot\nu_2)$ with $r(\lambda_2) = r(\nu_2) = q$, there are four possibilities for λ_3, ν_3. We denote these elements by $x_l^{(3)}$, x_r, for $l = 1,2,3$, where $x_l^{(3)} = (\lambda_1\cdot\lambda_2\cdot\gamma_{q,k_l}, \nu_1\cdot\nu_2\cdot\gamma_{q,k_l})$, $x_r = (\lambda_1\cdot\lambda_2\cdot\gamma, \nu_1\cdot\nu_2\cdot\gamma)$. Choosing $\zeta_2 = \zeta_2' = \beta$, equation (5.9) gives

$$y_6 = [2]r_1^-\left(x_1^{(3)}+x_2^{(3)}+x_3^{(3)}\right)+[4]r_2^+\, x_r, \tag{5.16}$$

where $y_6 \in B$. Similarly, the choices $\zeta_2 = \zeta_2' = \beta'$, $\zeta_2 = \beta$, $\zeta_2' = \beta'$ and $\zeta_2 = \beta'$, $\zeta_2' = \beta$ give

$$\begin{aligned} y_7 &= [2]r_1^+\left(x_1^{(3)}+x_2^{(3)}+x_3^{(3)}\right)+[4]r_2^-\, x_r, && (5.17)\\ y_8 &= [2]\sqrt{r_1^+r_1^-}\left(x_1^{(3)}+\overline{\omega}x_2^{(3)}+\omega x_3^{(3)}\right)+[4]\sqrt{r_2^+r_2^-}\, x_r, && (5.18)\\ y_9 &= [2]\sqrt{r_1^+r_1^-}\left(x_1^{(3)}+\omega x_2^{(3)}+\overline{\omega}x_3^{(3)}\right)+[4]\sqrt{r_2^+r_2^-}\, x_r, && (5.19)\end{aligned}$$

where $y_j \in B$, $j = 7,8,9$. Equations (5.16)-(5.19) are linearly independent, and we find $x_l^{(3)}, x_r \in B$ for $l = 1,2,3$.

For any pair $(\lambda_1 \cdot \lambda_2, \nu_1 \cdot \nu_2)$ with $r(\lambda_2) = r(\nu_2) = r$, there are four possibilities for λ_3, ν_3, and we denote these elements by $x_{(\xi,\xi')} = (\lambda_1 \cdot \lambda_2 \cdot \xi, \nu_1 \cdot \nu_2 \cdot \xi')$, $\xi, \xi' \in \{\alpha, \alpha'\}$. Choosing $\zeta_2 = \zeta_2' = \gamma$, equation (5.9) gives

$$y_{10} = r_2^- \, x_{(\alpha,\alpha)} + r_2^+ \, x_{(\alpha',\alpha')}, \tag{5.20}$$

where $y_{10} \in B$. We obtain three more equations by choosing $\zeta_2 = \zeta_2' = \gamma_{j_l,r}$, $l = 1,2,3$:

$$y_{11}^{(l)} = r_1^+ \, x_{(\alpha,\alpha)} + \bar{\epsilon}_l \sqrt{r_1^+ r_1^-} \, x_{(\alpha,\alpha')} + \epsilon_l \sqrt{r_1^+ r_1^-} \, x_{(\alpha',\alpha)} + r_1^- \, x_{(\alpha',\alpha')}, \tag{5.21}$$

where $y_{11}^{(l)} \in B$, $l = 1,2,3$. So from (5.20) and (5.21) for $l = 1,2,3$, we find that $x_{(\xi,\xi')} \in B$ for $\xi, \xi' \in \{\alpha, \alpha'\}$.

We now consider any element x in (5.10) where $r(\lambda_2) \neq r(\nu_2)$. When $r(\lambda_2) = i_l$, $r(\nu_2) = p$, there is only one possibility for λ_3, ν_3, which is $\lambda_3 = \gamma_{i_l,j_l}$, $\nu_3 = \gamma_{p,j_l}$, $l = 1,2,3$, given by choosing $\zeta_2 = \gamma_{k_l,i_l}$, $\zeta_2' = \gamma_{k_l,p}$. Then $x = (\lambda_1 \cdot \lambda_2 \cdot \gamma_{i_l,j_l}, \nu_1 \cdot \nu_2 \cdot \gamma_{p,j_l}) \in B$. When $r(\lambda_2) = j_l$, $r(\nu_2) = j_{l+1}$, $l = 1,2,3$, there is again only one possibility for λ_3, ν_3. So $x \in B$. Similarly when $r(\lambda_2) = k_l$, $r(\nu_2) = k_{l+1}$, $l = 1,2,3$.

Consider the pair $(\lambda_1 \cdot \lambda_2, \nu_1 \cdot \nu_2)$ where $r(\lambda_2) = j_l$, $l = 1,2,3$, and $r(\nu_2) = q$. For each $l = 1,2,3$, there are two possibilities for λ_3, ν_3. We denote these by $x_l^{(4)} = (\lambda_1 \cdot \lambda_2 \cdot \gamma_{j_l,k_l}, \nu_1 \cdot \nu_2 \cdot \gamma_{q,k_l})$, $x_l^{(5)} = (\lambda_1 \cdot \lambda_2 \cdot \gamma_{j_l,r}, \nu_1 \cdot \nu_2 \cdot \gamma)$. Choosing $\zeta_2 = \gamma_{p,j_l}$, $\gamma_2' = \beta$, we obtain

$$y_{12}^{(l)} = \sqrt{[3][4]}\, x_l^{(4)} - \sqrt{[2]}\sqrt{r_2^+}\, x_l^{(5)}, \tag{5.22}$$

where $y_{12}^{(l)} \in B$, $l = 1,2,3$. Similarly, choosing $\zeta_2 = \gamma_{p,j_l}$, $\gamma_2' = \beta'$, we obtain

$$y_{13}^{(l)} = \sqrt{[3][4]}\, x_l^{(4)} + \sqrt{[2]}\sqrt{r_2^-}\, x_l^{(5)}, \tag{5.23}$$

where $y_{13}^{(l)} \in B$, $l = 1,2,3$. Then for each $l = 1,2,3$, from (5.22), (5.23) we find that $x_l^{(4)}, x_l^{(5)} \in B$.

We now consider the pair $(\lambda_1 \cdot \lambda_2, \nu_1 \cdot \nu_2)$ where $r(\lambda_2) = k_l$, $l = 1,2,3$, and $r(\nu_2) = r$. For each $l = 1,2,3$, there are two possibilities for λ_3, ν_3. We denote these by $x_{(\xi),l} = (\lambda_1 \cdot \lambda_2 \cdot \gamma_{k_l,p}, \nu_1 \cdot \nu_2 \cdot \xi)$, $\xi \in \{\alpha, \alpha'\}$. Then for each $l = 1,2,3$, choosing $\zeta_2 = \gamma_{j_l,k_l}$, $\gamma_2' = \gamma_{j_l,r}$, we obtain

$$y_{14}^{(l)} = \bar{\epsilon}_l \sqrt{r_1^+} x_{(\alpha),l} + \epsilon_l \sqrt{r_1^-} x_{(\alpha'),l}, \tag{5.24}$$

where $y_{14}^{(l)} \in B$, $l = 1,2,3$. Similarly, choosing $\zeta_2 = \gamma_{q,k_l}$, $\gamma_2' = \gamma$, we obtain

$$y_{15}^{(l)} = \sqrt{r_2^-} x_{(\alpha),l} - \sqrt{r_2^+} x_{(\alpha'),l}, \tag{5.25}$$

where $y_{15}^{(l)} \in B$, $l = 1,2,3$. Then for each $l = 1,2,3$, from (5.24), (5.25) we find that $x_{(\alpha),l}, x_{(\alpha'),l} \in B$. All the other elements in M_{m+1} are in B since $y^* \in B$ if $y \in B$. $\square$

The following is an $SU(3)$ version of Skau's lemma. The proof is similar to the proof of Skau's lemma given in [51, Theorem 4.4.3].

Lemma 5.1.8 *For an $\mathcal{ADE}$ graph $\mathcal{G}$, let $M_0 = \mathbb{C}^{n_0}$ where n_0 is the number of 0-coloured vertices of $\mathcal{G}$, and let $M_0 \subset M_1 \subset M_2 \subset \cdots$ be a tower of finite dimensional von Neumann algebras with Markov trace* tr *on the M_i, with the inclusions $M_j \subset M_{j+1}$ given by an $SU(3)$ $\mathcal{ADE}$ graph $\mathcal{G}$ (except $\mathcal{E}_4^{(12)}$), and operators $U_m \in M_{m+1}$, $m \geq 1$, which satisfy the relation H1-H3 for $\delta \leq 2$, and such that U_m commutes with M_{m-1}. Let M_∞ be the GNS-completion of $\bigcup_{j\geq 0} M_j$ with respect to the trace. Then $\{U_1, U_2, \ldots\}' \cap M_\infty = M_0$.*

Proof

The first inclusion $M_0 \subset \{U_1, U_2, \ldots\}' \cap M_\infty$ is obvious, since M_0 commutes with U_m for all $m \geq 1$.

We now show the opposite inclusion $M_0 \supset \{U_1, U_2, \ldots\}' \cap M_\infty$. For each $k \geq 1$ let F_k be the conditional expectation of M_∞ onto $\{U_k, U_{k+1}, \ldots\}' \cap M_\infty$ with respect to the trace. Note that $F_k F_l = F_{\mathrm{min}(k,l)}$. So we want to show $F_1(M_\infty) \subset M_0$. We first show $F_2(M_\infty) \subset M_m$ for some sufficiently large m. By Proposition 1.6.2, the diagram

$$\begin{array}{ccc} \{U_{k+1}, U_{k+2}, \ldots\}' \cap M_\infty & \subset & M_\infty \\ \cup & & \cup \\ \{U_{k+1}, U_{k+2}, \ldots\}' \cap \{U_k, U_{k+1}, \ldots\}'' & \subset & \{U_k, U_{k+1}, \ldots\}'' \end{array}$$

is a commuting square, for $k \geq 1$. Since $\{U_{k+1}, U_{k+2}, \ldots\}'' \subset \{U_k, U_{k+1}, \ldots\}''$ is isomorphic to $R_2 \subset R_1$, where $R_1 = \{\mathbf{1}, U_1, U_2, \ldots\}''$, $R_2 = \{\mathbf{1}, U_2, U_3, \ldots\}''$, we may write the commuting square as

$$\begin{array}{ccc} R_2' \cap M_\infty & \subset & M_\infty \\ \cup & & \cup \\ R_2' \cap R_1 & \subset & R_1. \end{array}$$

Let E denote the conditional expectation from R_1 onto $R_2' \cap R_1$ with respect to the trace. Since F_{k+1} is the conditional expectation from M_∞ onto $R_2' \cap M_\infty$ and $U_k \in R_1$, we have $F_{k+1}(U_k) = E(U_k)$. Since by Theorem 6.3.3 the principal graph of $R_2 \subset R_1$ is the 01-part of $\mathcal{A}^{(n)}$, and there is only one vertex joined to the distinguished vertex $*$ of $\mathcal{A}^{(n)}$, the relative commutant $R_2' \cap R_1$ is trivial for $\alpha \leq 3$ (which corresponds to $\delta \leq 2$), and E is just the trace. Thus $F_{k+1}(U_k) \in \mathbb{C}$ for each $k \geq 1$. By Lemma 5.1.7, for sufficiently large m, any element of M_{m+1} can be written as a linear combination of elements of the form $aU_m b$ and c, for $a, b, c \in M_m$, and we have

$$F_2(aU_m b) = F_2 F_{m+1}(aU_m b) = F_2(aF_{m+1}(U_m)b) = F_2(\lambda ab) \;\; \in F_2(M_m),$$

where $\lambda \in \mathbb{C}$. So $F_2(M_{m+1}) \subset F_2(M_m)$, for sufficiently large m, and by induction we have $F_2(M_\infty) \subset F_2(M_r)$, where r is the smallest integer such that Lemma 5.1.7 holds. Then certainly $F_2(M_\infty) \subset F_{r+1}(M_r)$, and by Proposition 5.1.6, with $k = r$, any element x in M_r commutes with U_r if and only if $x \in M_{r-1}$, so $F_r F_{r+1}(M_r) \subset F_r(M_{r-1})$. Then by inductive use of Proposition 5.1.6 we obtain $F_2(M_\infty) \subset F_2(M_1) = M_1$, and so $F_1(M_\infty) = F_1 F_2(M_\infty) \subset F_1(M_1) = M_0$, by Proposition 5.1.6. □

We now construct the A_2 Goodman-de la Harpe-Jones subfactor for an $SU(3)$ $\mathcal{ADE}$ graph $\mathcal{G}$, following the idea of Goodman, de la Harpe and Jones for the ADE Dynkin diagrams [51]. Let n be the Coxeter number for $\mathcal{G}$, $*_{\mathcal{G}}$ a distinguished vertex and let n_0 be the number of 0-coloured vertices of $\mathcal{G}$. Let A_0 be the von Neumann algebra $\mathbb{C}^{n_0}$, and form a sequence of finite dimensional von Neumann algebras $A_0 \subset A_1 \subset A_2 \subset \cdots$ such that the Bratteli diagram for the inclusion $A_{l-1} \subset A_l$ is given by (part of) the graph $\mathcal{G}$. There are operators $U_m \in A_{m+1}$ which satisfy the Hecke relations H1-H3. Let $\widetilde{C}$ be the GNS-completion of $\bigcup_{m \geq 0} A_m$ with respect to the trace, and $\widetilde{B}$ its von Neumann subalgebra generated by $\{U_m\}_{m \geq 1}$. We have $\widetilde{B}' \cap \widetilde{C} = A_0$ by Lemma 5.1.8. Then for q the minimal projection in A_0 corresponding to the distinguished vertex $*_{\mathcal{G}}$ of $\mathcal{G}$, we have an A_2 Goodman-de la Harpe-Jones subfactor $B = q\widetilde{B} \subset q\widetilde{C}q = C$ for the graph $\mathcal{G}$. With $B_m = q\widetilde{B}_m$ and $C_m = q\widetilde{C}_m q$, the sequence $\{B_m \subset C_m\}_m$ is a periodic sequence of commuting squares of period 3, in the sense of Wenzl in [112], that is, for large enough m the Bratteli diagrams for the inclusions $B_m \subset B_{m+1}$, $C_m \subset C_{m+1}$ are the same as those for $B_{m+3} \subset B_{m+4}$, $C_{m+3} \subset C_{m+4}$, and the Bratteli diagrams for the inclusions $B_m \subset C_m$ and $B_{m+3} \subset C_{m+3}$ are the same. For such m the graph of the Bratteli diagram for $B_{3m} \subset C_{3m}$ is the intertwining graph, given by the intertwining matrix V computed in Proposition 5.1.10, whose rows are indexed by the vertices of $\mathcal{G}$ and columns are indexed by the vertices of $\mathcal{A}^{(n)}$, such that $V\Delta_{\mathcal{A}} = \Delta_{\mathcal{G}} V$. For sufficiently large m we can make a basic construction $B_m \subset C_m \subset D_m$. Then with $D = \bigvee_m D_m$, $B \subset C \subset D$ is also a basic construction. The graph of the Bratteli diagram for $C_m \subset D_m$ is the reflection of the graph for $B_m \subset C_m$, which is the intertwining graph. Then we can extend the definition of D_m to small m so that the graph $C_m \subset D_m$ is still given by the reflection of the intertwining graph. We see that $D_0 = \bigoplus_{\mu \in \mathcal{A}^{(n)}} VV^*(*_{\mathcal{A}}, \mu)\mathbb{C}$, where $*_{\mathcal{A}}$ is the distinguished vertex $(0,0)$ of $\mathcal{A}^{(n)}$. The minimal projections in D_0 correspond to the vertices μ' of $\mathcal{A}^{(n)}$ such that

$$VV^*(*, \mu') > 0, \tag{5.26}$$

and the Bratteli diagram for the inclusion $D_{m-1} \subset D_m$ is given by (part of) the graph $\mathcal{A}^{(n)}$. Each algebra B_m is generated by the $U_1, \ldots, U_{m-1}$ in D_m.

Now $\lambda_{(1,0)}(N) \subset N \cong P \subset Q$, where $P \subset Q$ is Wenzl's subfactor with principal graph given by the 01-part $\mathcal{A}^{(n)}_{01}$ of $\mathcal{A}^{(n)}$ (see Theorem 6.3.3). Then $(\lambda_{(1,0)}\overline{\lambda_{(1,0)}})^{d/2}(N) \cong P \subset Q_d$, where $P \subset Q \subset Q_1 \subset \cdots$ is the Jones tower. For any 0-coloured vertex μ of $\mathcal{A}^{(n)}_{01}$ let d_μ be the minimum number of edges in any path from $(0,0)$ to μ on $\mathcal{A}^{(n)}_{01}$, and let $d = \max\{d_\mu - 2|\ VV^*(*_\mathcal{A}, \mu) > 0\}$. Note that each d_μ is even since μ is a 0-coloured vertex. Let $[\theta] = \bigoplus_{\mu\in\mathcal{A}^{(n)}} VV^*(*_\mathcal{A}, \mu)[\lambda_\mu]$. Now $[(\lambda_{(1,0)}\overline{\lambda_{(1,0)}})^{d/2}]$ decomposes into irreducibles as $\bigoplus_\mu n_\mu[\lambda_\mu]$, where μ are the 0-coloured vertices of $\mathcal{A}^{(n)}$ and $n_\mu \in \mathbb{N}$. Then $\theta(N) \subset N$ is a restricted version of $(\lambda_{(1,0)}\overline{\lambda_{(1,0)}})^{d/2}(N)$, so that $\theta(N) \subset N \cong qP \subset q(Q_d)q$ where $q \in P' \cap Q_d$ is a sum of minimal projections corresponding to the vertices μ' such that $[\theta] \supset [\lambda_{\mu'}]$. We will show that $qP \subset q(Q_d)q$ is isomorphic to a subfactor obtained by a basic construction.

Following the example in [13, Lemma A.1] for E_7 in the $SU(2)$ case, we now do the same construction for the graph $\mathcal{A}^{(n)}$, where q is the projection corresponding to the distinguished vertex $*_\mathcal{A}$. We get a periodic sequence $\{E_m \subset F_m\}_m$ of commuting squares of period 3. Then the resulting subfactor $E \subset F$, where $E = \bigvee_m E_m$, $F = \bigvee_m F_m$, is Wenzl's subfactor [112].

If we make basic constructions of $E_m \subset F_m$ for $d-1$ times then we get a periodic sequence $\{E_m \subset G_m\}_m$ of commuting squares, and each E_m is generated by the Hecke operators in G_m. Let $\widetilde{q}$ be a sum of the minimal projections corresponding to the vertices μ' in G_0 given by (5.26). We set $\widetilde{E}_m = \widetilde{q}E_m$ and $\widetilde{G}_m = \widetilde{q}G_m\widetilde{q}$, and obtain a periodic sequence of commuting squares of period 3 such that the resulting subfactor is isomorphic to $qP \subset q(Q_d)q$. The Bratteli diagram for the sequence $\{\widetilde{G}_m\}_m$ is the same as that for $\{D_m\}_m$ since $D_0 = \widetilde{G}_0 = \mathbb{C}^r$ where the r minimal projections correspond to the vertices μ' of (5.26), where r is the number of such vertices μ', and the rest of the Bratteli diagram is given by the 01-part of the graph $\mathcal{A}^{(n)}$. Each $\widetilde{E}_m$ is generated by the Hecke operators $U_1, \ldots, U_{m-1} \in \widetilde{G}_m$. Then the sequence of commuting squares $\{B_m \subset D_m\}_m$ is isomorphic to the sequence of commuting squares $\{\widetilde{E}_m \subset \widetilde{G}_m\}_m$, and so the subfactors $B \subset D$ and $qP \subset q(Q_d)q$ are also isomorphic. Since $B \subset D$ is a basic construction of $B \subset C$, then the subfactor $qP \subset q(Q_d)q$ is also the basic construction of some subfactor. Since $\theta(N) \subset N$ is isomorphic to $qP \subset q(Q_d)q$,

$$[\theta] = \bigoplus_{\mu\in\mathcal{A}^{(n)}} VV^*(*_\mathcal{A}, \mu)[\lambda_\mu] \tag{5.27}$$

can be realised as the dual canonical endomorphism of some subfactor.

5.1.1 Computing the intertwining graphs.

Let $V(\mathcal{G})$ denote the free module over $\mathbb{Z}$ generated by the vertices of $\mathcal{G}$, identifying an element $a \in V(\mathcal{G})$ as $a = (a_v)$, $a_v \in \mathbb{Z}$, $v \in \mathfrak{V}^{\mathcal{G}}$. For graphs $\mathcal{G}_1$, $\mathcal{G}_2$, a map $V : V(\mathcal{G}_1) \longrightarrow V(\mathcal{G}_2)$ is *positive* if $V_{ij} \geq 0$ for all $i \in \mathfrak{V}^{\mathcal{G}_2}$, $j \in \mathfrak{V}^{\mathcal{G}_1}$. Let $A(\mathcal{G})$ be the path algebra for $\mathcal{G}$.

The following lemma and proposition are the $SU(3)$ versions of Proposition 4.5 and Corollary 4.7 in [37] (see also Lemma 11.26 and Proposition 11.27 in [39]).

Lemma 5.1.9 *Suppose that $\mathcal{G}_1$, $\mathcal{G}_2$ are locally finite connected graphs with Coxeter number n, adjacency matrices $\Delta_{\mathcal{G}_1}$, $\Delta_{\mathcal{G}_2}$ respectively and distinguished vertices $*_1$, $*_2$ respectively. Let $(U_m)_{m\in\mathbb{N}}$, $(W_m)_{m\in\mathbb{N}}$ denote canonical families of operators in $A(\mathcal{G}_1)$ and $A(\mathcal{G}_2)$ respectively, which satisfy the A_2-Temperley-Lieb relations such that $U_m^2 = [2]_q U_m$, $W_m^2 = [2]_q W_m$ for all $m \in \mathbb{N}$, $q = e^{2\pi i/n}$. Let $\pi : A(\mathcal{G}_1) \longrightarrow A(\mathcal{G}_2)$ be a unital embedding such that:*

(a) The diagram

$$\begin{array}{ccc} A(\mathcal{G}_1)_m & \xrightarrow{\pi_m} & A(\mathcal{G}_2)_m \\ \iota_m \downarrow & & \downarrow \jmath_m \\ A(\mathcal{G}_1)_{m+1} & \xrightarrow{\pi_{m+1}} & A(\mathcal{G}_2)_{m+1} \end{array}$$

commutes for all m, where $\pi_m = \pi|_{A(\mathcal{G}_1)_m}$, and ι_m, $\jmath_m$ are standard inclusions.

(b) $tr_1 \cdot \pi_m = tr_2$, where tr_i is a Markov trace on $A(\mathcal{G}_i)$, $i = 1, 2$.

(c) $\pi(U_m) = (W_m)$ for all $m \geq 1$ (so $\pi_{m+1}(U_m) = W_m$).

Then there exists a positive linear map $V : V(\mathcal{G}_1) \longrightarrow V(\mathcal{G}_2)$ such that:

(1) $V\Delta_{\mathcal{G}_1} = \Delta_{\mathcal{G}_2} V$,

(2) V has no zero rows or columns,

(3) $V *_1 = *_2$.

Proof

Let p_i^m denote a minimal projection in $A(\mathcal{G}_1)_m$ corresponding to the vertex (i, m) of the Bratteli diagram $\widehat{\mathcal{G}}_1$ of $\mathcal{G}_1$. Then $\pi_m(p_i^m)$ is a projection in $A(\mathcal{G}_2)_m$, and so there are families of equivalent minimal projections $\{q^m_{j,k(j)} | k(j) = 1, \dots, b^m_{ji}\}$ in $A(\mathcal{G}_2)_m$ corresponding to vertices (j,m) in $\widehat{\mathcal{G}}_2$, such that

$$\pi_m(p^m_i) = \sum_j \sum_{k(j)=1}^{b^m_{ji}} q^m_{j,k(j)}. \tag{5.28}$$

The numbers $\{b^m_{ji}\}_j$ are non-negative, are independent of the choice of p^m_i and are not all zero, since π_m is injective. Let $F^{(1)}_m = [2]^{-1}[3]^{-1}(U_m U_{m+1} U_m - U_m)$ in $A(\mathcal{G}_1)$, and $F^{(2)}_m = [2]^{-1}[3]^{-1}(W_m W_{m+1} W_m - W_m)$ in $A(\mathcal{G}_2)$. Now multiplying (5.28) on the left by $F^{(2)}_{m+1}$, we have

$$F^{(2)}_{m+1}\pi_m(p^m_i) = \sum_j \sum_{k(j)=1}^{b^m_{ji}} F^{(2)}_{m+1}\, q^m_{j,k(j)},$$

but by (a) and (c), $F^{(2)}_{m+1}\pi_m(p^m_i) = \pi_{m+3}(F^{(1)}_{m+1})\pi_m(p^m_i) = \pi_{m+3}(F^{(1)}_{m+1}p^m_i)$, so we have

$$\pi_{m+2}(F^{(1)}_{m+1}p^m_i) = \sum_j \sum_{k(j)=1}^{b^m_{ji}} F^{(2)}_{m+1}\, q^m_{j,k(j)}. \tag{5.29}$$

Since tr_1 and tr_2 are Markov traces, by Lemma 5.1.5 we have $\mathrm{tr}_1(F^{(1)}_{m+1}\, p^m_i) = [3]^{-3}\mathrm{tr}_1(p^m_i)$, and $\mathrm{tr}_2(F^{(2)}_{m+1}\, q^m_{j,k(j)}) = [3]^{-3}\mathrm{tr}_2(q^m_{j,k(j)})$. Since p^m_i, $q^m_{j,k(j)}$ are minimal projections, they have trace $[3]^{-k}\phi_i$, $[3]^{-k}\phi_j$ respectively. Then $F^{(1)}_{m+1}\, p^m_i$ has trace $[3]^{-k-3}\phi_i$, which shows that $F^{(1)}_{m+1}\, p^m_i$ is a minimal projection in $A(\mathcal{G}_1)_{m+3}$ corresponding to vertex $(i, m+3)$ of $\widehat{\mathcal{G}}_1$, and similarly $F^{(1)}_{m+1}\, q^m_{j,k(j)}$ is a minimal projection in $A(\mathcal{G}_2)_{m+3}$ corresponding to vertex $(j, m+3)$ of $\widehat{\mathcal{G}}_2$. It follows from (5.28) and (5.29) that the coefficients occurring in the decomposition of a minimal projection as in (5.28) corresponding to vertex (i,m) of $\widehat{\mathcal{G}}_1$, $m \geq 1$, is independent of the level m, i.e. $b^m_{ji} = b^l_{ji} =: b_{ji}$ for all $m, l \geq 0$.

Now put $V = (b_{ji})_{i \in \mathfrak{V}^{\mathcal{G}_1}, j \in \mathfrak{V}^{\mathcal{G}_2}}$, then since $A(\mathcal{G}_1)_0 \cong \mathbb{C} \cong A(\mathcal{G}_2)_0$, and $\pi_0 : A(\mathcal{G}_1)_0 \longrightarrow A(\mathcal{G}_2)_0$ we see that $V *_1 = *_2$. Note that since π is unital, the rows of V are non-zero. We need to show $V\Delta_{\mathcal{G}_1} = \Delta_{\mathcal{G}_2}V$.

Let $\Delta_{\mathcal{G}_k}(m)$, $k = 1,2$, be the finite submatrix of $\Delta_{\mathcal{G}_k}$, whose rows and columns are labelled by the vertices $v \in \mathcal{G}^{(0)}_k$ with $d(v) \leq m+1$, where $d(v)$ is the distance of vertex v from $*_k$, ie. the length of the shortest path on $\mathcal{G}_k$ from $*_k$ to v. Similarly let $V(m)$ denote the finite submatrix of V whose rows are labelled by $j \in \mathfrak{V}^{\mathcal{G}_2}$ with $d(j) \leq m+1$, and whose columns are labelled by $i \in \mathfrak{V}^{\mathcal{G}_1}$ with $d(i) \leq m+1$. It follows from (a) that for each m we have

$$K_0(\jmath_m)K_0(\pi_m) = K_0(\pi_{m+1})K_0(\iota_m). \tag{5.30}$$

Let M_1, M_2, be two multi-matrix algebras, with the embedding φ of M_1 in M_2 given by a matrix Λ, with p_1 columns corresponding to the minimal central projections in M_1 and

p_2 columns corresponding to the minimal central projections in M_2. Then $K_0(M_i) = \mathbb{Z}^{p_i}$, $i = 1, 2$, and $K_0(\varphi) : \mathbb{Z}^{p_1} \longrightarrow \mathbb{Z}^{p_2}$ is given by multiplication by the matrix Λ. For m of colour $\overline{j}$, we see that $K_0(\iota_m)$ is the submatrix of $\Delta_{\mathcal{G}_1}(m)$ mapping vertices of colour $\overline{j}$ to vertices of colour $\overline{j+1}$, and $K_0(\jmath_m)$ is the submatrix of $\Delta_{\mathcal{G}_2}(m)$ vertices of colour $\overline{j}$ to vertices of colour $\overline{j+1}$. Similarly, $K_0(\pi_m)$ is the submatrix of $V(m)$ mapping vertices of $\mathcal{G}_1$ of colour $\overline{j}$ to vertices of $\mathcal{G}_2$ of colour $\overline{j}$. Then (5.30) implies $\Delta_{\mathcal{G}_2}(m)V(m-1) = V(m)\Delta_{\mathcal{G}_1}(m)$ holds for all m. Hence $V\Delta_{\mathcal{G}_1} = \Delta_{\mathcal{G}_2}V$. $\square$

We define polynomials $S_\nu(x,y)$, for ν the vertices of $\mathcal{A}^{(n)}$, by $S_{(0,0)}(x,y) = 1$, and $xS_\nu(x,y) = \sum_\mu \Delta_{\mathcal{A}}(\nu,\mu)S_\mu(x,y)$, $yS_\nu(x,y) = \sum_\mu \Delta_{\mathcal{A}}^T(\nu,\mu)S_\mu(x,y)$. For concrete values of the first few $S_\mu(x,y)$ see [39, p. 610].

Proposition 5.1.10 *Let $\mathcal{G}$ be a finite $SU(3)$-$\mathcal{ADE}$ graph with distinguished vertex $*_\mathcal{G}$ and Coxeter number $n < \infty$. Let $\{U_m\}_{m\geq 0}$, $\{W_m\}_{m\geq 0}$ be the canonical family of operators satisfying the Hecke relations in $A(\mathcal{A}^{(n)})$, $A(\mathcal{G})$ respectively. We can identify $A(\mathcal{A}^{(n)})$ with the algebra generated by $\{\mathbf{1}, W_1, W_2, \ldots\}$. If we define $\pi : A(\mathcal{A}^{(n)}) \longrightarrow A(\mathcal{G})$ by $\pi(\mathbf{1}) = \mathbf{1}$, $\pi(U_m) = W_m$, then π is a unital embedding, and there exists a positive linear map $V : V(\mathcal{A}^{(n)}) \longrightarrow V(\mathcal{G})$ such that:*

(a) $V\Delta_{\mathcal{A}} = \Delta_{\mathcal{G}}V$,

(b) V has no zero rows or columns,

(c) $V*_{\mathcal{A}} = *_{\mathcal{G}}$, where $*_{\mathcal{A}} = (0,0)$ is the distinguished vertex of $\mathcal{A}^{(n)}$.

Let $V_{(0,0)}$ be the vector corresponding to the distinguished vertex $*_\mathcal{G}$, and for the other vertices define $V_{(\lambda_1,\lambda_2)} \in V(\mathcal{G})$ by $V_{(\lambda_1,\lambda_2)} = S_{(\lambda_1,\lambda_2)}\left(\Delta_{\mathcal{G}}^T, \Delta_{\mathcal{G}}\right)V_{(0,0)}$, for all vertices (λ_1,λ_2) of $\mathcal{A}^{(n)}$. Then $V = (V_{(0,0)}, V_{(1,0)}, V_{(0,1)}, V_{(2,0)}, \ldots, V_{(0,n-3)})$.

Proof

Now $\pi : A(\mathcal{A}^{(n)}) \longrightarrow A(\mathcal{G})$ defined by $\pi(\mathbf{1}) = \mathbf{1}$, $\pi(U_m) = W_m$ is a unital embedding which satisfies the condition of Lemma 5.1.9 with $*_1 = (0,0)$ and $*_2 = *_\mathcal{G}$. Hence when m is finite there exists $V = (V_{(\lambda_1,\lambda_2)})$, for (λ_1,λ_2) the vertices of $\mathcal{A}^{(n)}$, with the required properties. Now $V\Delta_{\mathcal{A}} = \left(V_{(\lambda_1-1,\lambda_2)} + V_{(\lambda_1+1,\lambda_2-1)} + V_{(\lambda_1,\lambda_2+1)}\right)_{(\lambda_1,\lambda_2)}$, where $V_{(\lambda_1,\lambda_2)}$ is understood to be zero if (λ_1,λ_2) is off the graph $\mathcal{A}^{(n)}$. Thus $V\Delta_{\mathcal{A}} = \Delta_{\mathcal{G}}V$ implies that $\Delta_{\mathcal{G}}V_{(\lambda_1,\lambda_2)} = V_{(\lambda_1-1,\lambda_2)} + V_{(\lambda_1+1,\lambda_2-1)} + V_{(\lambda_1,\lambda_2+1)}$. Then $V_{(\lambda_1,\lambda_2)} = S_{(\lambda_1,\lambda_2)}\left(\Delta_{\mathcal{G}}^T, \Delta_{\mathcal{G}}\right)V_{(0,0)}$, since

$$\begin{aligned}
\Delta_{\mathcal{G}}V_{(\lambda_1,\lambda_2)} &= \Delta_{\mathcal{G}}S_{(\lambda_1,\lambda_2)}\left(\Delta_{\mathcal{G}}^T, \Delta_{\mathcal{G}}\right)V_{(0,0)} \\
&= \sum_{(\mu_1,\mu_2)} \Delta_{\mathcal{A}}^T\left((\lambda_1,\lambda_2),(\mu_1,\mu_2)\right)S_{(\mu_1,\mu_2)}\left(\Delta_{\mathcal{G}}^T, \Delta_{\mathcal{G}}\right)V_{(0,0)} \\
&= V_{(\lambda_1-1,\lambda_2)} + V_{(\lambda_1+1,\lambda_2-1)} + V_{(\lambda_1,\lambda_2+1)},
\end{aligned}$$

and $V_{(0,0)} = S_{(0,0)}\left(\Delta_{\mathcal{G}}^T, \Delta_{\mathcal{G}}\right) V_{(0,0)}$. $\square$

For any $\mathcal{ADE}$ graph $\mathcal{G}$ the matrix V is the adjacency matrix of a (possibly disconnected) graph. By [10, Theorem 4.2] the connected component of $*_{\mathcal{A}}$ of this graph gives the principal graph of the A_2 Goodman-de la Harpe-Jones subfactor. For the graph $\mathcal{E}^{(8)}$ with vertex i_1 chosen as the distinguished vertex this is the graph illustrated in Figure 5.2, which was shown to be the principal graph for this subfactor in [113].

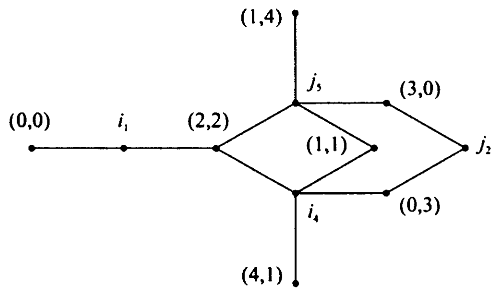


Figure 5.2: Principal graph for the A_2 Goodman-de la Harpe-Jones subfactor for $\mathcal{E}^{(8)}$

5.2 Modular invariants associated to the dual canonical endomorphisms.

Let $N \subset M$ be the $SU(3)$-GHJ subfactor for the finite $\mathcal{ADE}$ graph $\mathcal{G}$, where the distinguished vertex $*_{\mathcal{G}}$ is the vertex with lowest Perron-Frobenius weight. Then the dual canonical endomorphism θ for $N \subset M$ is given by (5.27) where V is now determined in Proposition 5.1.10. We list these θ's below for the $\mathcal{ADE}$ graphs. We must point out that as we have been unable to explicitly construct the Ocneanu cells W for $\mathcal{E}_4^{(12)}$, the existence of the A_2 Goodman-de la Harpe-Jones subfactor which realizes the candidate for the dual canonical endomorphism for $\mathcal{E}_4^{(12)}$ is not shown here.

$$\mathcal{A}^{(n)}: \quad [\theta] = [\lambda_{(0,0)}], \tag{5.31}$$

$$\mathcal{D}^{(n)}: \quad [\theta] = [\lambda_{(0,0)}] \oplus [\lambda_{A(0,0)}] \oplus [\lambda_{A^2(0,0)}], \tag{5.32}$$

$$\mathcal{A}^{(n)*}: \quad [\theta] = \bigoplus_{\mu \in \mathcal{A}^{(n)}} [\lambda_\mu], \tag{5.33}$$

$$\mathcal{D}^{(2k)*}: \quad [\theta] = \bigoplus_{\substack{\mu \in \mathcal{A}^{(2k)}: \\ \tau(\mu)=0}} [\lambda_\mu], \tag{5.34}$$

$$\mathcal{D}^{(2k+1)*}: \quad [\theta] = \bigoplus_{\substack{\mu=(2\mu_1,2\mu_2) \in \mathcal{A}^{(2k+1)}: \\ \tau(\mu)=0}} [\lambda_\mu], \tag{5.35}$$

$$\begin{aligned}
\mathcal{E}^{(8)}: &\quad [\theta] = [\lambda_{(0,0)}] \oplus [\lambda_{(2,2)}], && (5.36)\\
\mathcal{E}^{(8)*}: &\quad [\theta] = [\lambda_{(0,0)}] \oplus [\lambda_{(2,1)}] \oplus [\lambda_{(1,2)}] \oplus [\lambda_{(2,2)}] \oplus [\lambda_{(5,0)}] \oplus [\lambda_{(0,5)}], && (5.37)\\
\mathcal{E}_1^{(12)}: &\quad [\theta] = [\lambda_{(0,0)}] \oplus [\lambda_{(4,1)}] \oplus [\lambda_{(1,4)}] \oplus [\lambda_{(4,4)}] \oplus [\lambda_{(9,0)}] \oplus [\lambda_{(0,9)}], && (5.38)\\
\mathcal{E}_2^{(12)}: &\quad [\theta] = [\lambda_{(0,0)}] \oplus 2[\lambda_{(2,2)}] \oplus [\lambda_{(4,1)}] \oplus [\lambda_{(1,4)}] \oplus 2[\lambda_{(5,2)}] \oplus 2[\lambda_{(2,5)}] \\
&\qquad \oplus [\lambda_{(4,4)}] \oplus [\lambda_{(9,0)}] \oplus [\lambda_{(0,9)}], && (5.39)\\
\mathcal{E}_4^{(12)}: &\quad [\theta] = [\lambda_{(0,0)}] \oplus [\lambda_{(2,2)}] \oplus [\lambda_{(4,1)}] \oplus [\lambda_{(1,4)}] \oplus [\lambda_{(5,2)}] \oplus [\lambda_{(2,5)}] \oplus [\lambda_{(4,4)}] \\
&\qquad \oplus [\lambda_{(9,0)}] \oplus [\lambda_{(0,9)}], && (5.40)\\
\mathcal{E}_5^{(12)}: &\quad [\theta] = [\lambda_{(0,0)}] \oplus [\lambda_{(3,3)}] \oplus [\lambda_{(9,0)}] \oplus [\lambda_{(0,9)}], && (5.41)\\
\mathcal{E}^{(24)}: &\quad [\theta] = [\lambda_{(0,0)}] \oplus [\lambda_{(4,4)}] \oplus [\lambda_{(10,1)}] \oplus [\lambda_{(1,10)}] \oplus [\lambda_{(6,6)}] \oplus [\lambda_{(9,6)}] \oplus [\lambda_{(6,9)}] \\
&\qquad \oplus [\lambda_{(13,4)}] \oplus [\lambda_{(4,13)}] \oplus [\lambda_{(10,10)}] \oplus [\lambda_{(21,0)}] \oplus [\lambda_{(0,21)}]. && (5.42)
\end{aligned}$$

Note that these dual canonical endomorphisms depend only on the existence of a cell system W for each graph $\mathcal{G}$, but not on the choice of cell system since Lemma 5.1.9 and Proposition 5.1.10 did not depend on this choice. Where we have found two inequivalent solutions, the computations below show that either choice will give the same M-N graph, since the computations in these particular cases only depend on the dual canonical endomorphism θ. Similarly, even if there exists other solutions for the cells W for the $\mathcal{D}$, $\mathcal{D}^*$ and $\mathcal{E}_1^{(12)}$ graphs, these will not give any new M-N graphs either. It is conceivable however that in certain situations, for $SU(N)$, $N > 3$, the M-N graph will depend on the connection and not just on the GHJ graph.

Remark. For $SU(2)$ it was shown in [33] that the modular invariant Z can be realized from a subfactor with a dual canonical endomorphism of the form

$$[\theta] = \bigoplus_{\mu} Z_{\mu,\overline{\mu}}[\mu], \tag{5.43}$$

where the direct summation is over all μ even. At level k these are given by

$$\begin{aligned}
A_l: &\quad k = l-1 &&[\theta] = [\lambda_0] \oplus [\lambda_2] \oplus [\lambda_4] \oplus \cdots \oplus [\lambda_{2\lfloor k/2 \rfloor}],\\
D_{2l}: &\quad k = 4l-4 &&[\theta] = [\lambda_0] \oplus [\lambda_2] \oplus \cdots \oplus [\lambda_{2l-4}] \oplus 2[\lambda_{2l-2}] \oplus [\lambda_{2l}] \oplus \cdots \oplus [\lambda_k],\\
D_{2l+1}: &\quad k = 4l-2 &&[\theta] = [\lambda_0] \oplus [\lambda_2] \oplus [\lambda_4] \oplus \cdots \oplus [\lambda_k],\\
E_6: &\quad k = 10 &&[\theta] = [\lambda_0] \oplus [\lambda_4] \oplus [\lambda_6] \oplus [\lambda_{10}],\\
E_7: &\quad k = 16 &&[\theta] = [\lambda_0] \oplus [\lambda_4] \oplus [\lambda_6] \oplus [\lambda_8] \oplus [\lambda_{10}] \oplus [\lambda_{12}] \oplus [\lambda_{16}],\\
E_8: &\quad k = 28 &&[\theta] = [\lambda_0] \oplus [\lambda_6] \oplus [\lambda_{10}] \oplus [\lambda_{12}] \oplus [\lambda_{16}] \oplus [\lambda_{18}] \oplus [\lambda_{22}] \oplus [\lambda_{28}].
\end{aligned}$$

This raises the question of whether all the $SU(3)$ modular invariants be realized from some subfactor with dual canonical endomorphism θ of the form (5.43), where we now allow μ to be of any colour? For the $\mathcal{A}^{(n)*}$ graphs the θ given in (5.33) is automatically in the form (5.43), where Z is the conjugate modular invariant $Z_{\mathcal{A}^{(n)*}} = C$. For the $\mathcal{A}^{(n)}$ graphs, if we choose the M-N morphism $[\overline{a}]$ to be $[\iota\lambda_{(p,0)}]$, where $p = \lfloor (n-3)/2 \rfloor$, the sector $[a\overline{a}]$ gives $[\lambda_{(0,0)}] \oplus [\lambda_{(1,1)}] \oplus [\lambda_{(2,2)}] \oplus \cdots \oplus [\lambda_{(p,p)}]$, and we obtain a dual canonical endomorphism $[\theta] = [a\overline{a}]$ such that $[\theta] = \bigoplus_\mu Z_{\mu,\overline{\mu}}[\mu]$, where the direct summation is over all μ (of any colour) and Z is the identity modular invariant $Z_{\mathcal{A}^{(n)}} = I$.

For each of the $\mathcal{ADE}$ graphs (with the exception of $\mathcal{E}_4^{(12)}$) we have shown the existence of a braided subfactor $N \subset M$ with dual canonical endomorphisms θ given by (5.31)-(5.42). By the α-induction of [8, 9, 10], a matrix Z can be defined by $Z_{\lambda,\mu} = \langle \alpha_\lambda^+, \alpha_{\overline{\mu}}^- \rangle$, $\lambda, \mu \in {}_N\mathcal{X}_N$. If the braiding is non-degenerate, Z is a modular invariant mass matrix.

For the dual canonical endomorphisms θ in (5.31)-(5.42), what is the corresponding M-N system or Cappelli-Itzykson-Zuber graph which classifies the modular invariant? And what is the corresponding modular invariant? For $\mathcal{A}^{(n)}$ the M-M, M-N and N-N systems are all equal since $N = M$. Subfactors given by conformal inclusions were considered in [9, 10]. Those conformal inclusions which have $SU(3)$ invariants give identical dual canonical endomorphisms θ to those computed above. The M-N system was computed for conformal inclusions with corresponding modular invariants associated to the graphs $\mathcal{D}^{(6)}$ and $\mathcal{E}^{(8)}$ in [9], and to $\mathcal{E}_1^{(12)}$ and $\mathcal{E}^{(24)}$ in [10]. The M-N system was also computed in [9] for the inclusion with dual canonical endomorphism (5.32) for $n \equiv 0 \bmod 3$, which do not come from conformal inclusions. For each of these graphs, the graph of the M-N system and the α-graph can both be identified with the original graph itself, and the modular invariant is that associated with the original graph. We compute the M-N graph for the remaining θ's. Knowledge of the dual canonical endomorphism θ is not sufficient to determine the M-N graph, but we can utilize the fact that the list of $SU(3)$ modular invariants is complete. For a $\mathcal{ADE}$ graph $\mathcal{G}$ with Coxeter number n, the basic method is to compute $\langle \iota\lambda, \iota\mu \rangle$ for representations λ, μ on $\mathcal{A}^{(n)}$, and decompose into irreducibles. Sometimes there is an ambiguity about the decomposition, e.g. if $\langle \iota\lambda, \iota\lambda \rangle = 4$ then could have $\iota\lambda = 2\iota\lambda^{(1)}$ or $\iota\lambda = \iota\lambda^{(1)} + \iota\lambda^{(2)} + \iota\lambda^{(3)} + \iota\lambda^{(4)}$ where $\iota\lambda^{(i)}$, $i = 1, 2, 3, 4$ are irreducible sectors. By [12, Cor. 6.13], $\sharp_M\mathcal{X}_N = \mathrm{tr}(Z)$ for some modular invariant Z, and therefore, since we have a complete list of $SU(3)$ modular invariants, we can eliminate any particular decomposition if the total number of irreducible sectors obtained does not agree with the trace of any of the modular invariants (1.14)-(1.25). We compute the trace for the modular invariants at level k in the following lemma:

Lemma 5.2.1 *The traces of the level k modular invariants Z are*

$$\mathrm{tr}(Z_{\mathcal{A}^{(k+3)}}) = \frac{1}{2}(k+1)(k+2), \tag{5.44}$$

$$\mathrm{tr}(Z_{\mathcal{D}^{(k+3)}}) = \frac{1}{6}(k+1)(k+2) + c_k, \tag{5.45}$$

$$\mathrm{tr}(Z_{\mathcal{A}^{(k+3)*}}) = \lfloor\frac{k+2}{2}\rfloor, \tag{5.46}$$

$$\mathrm{tr}(Z_{\mathcal{D}^{(k+3)*}}) = 3\lfloor\frac{k+2}{2}\rfloor, \tag{5.47}$$

$$\mathrm{tr}(Z_{\mathcal{E}^{(8)}}) = 12, \tag{5.48}$$

$$\mathrm{tr}(Z_{\mathcal{E}^{(8)*}}) = 4, \tag{5.49}$$

$$\mathrm{tr}(Z_{\mathcal{E}_1^{(12)}}) = 12, \tag{5.50}$$

$$\mathrm{tr}(Z_{\mathcal{E}_4^{(12)}}) = 11, \tag{5.51}$$

$$\mathrm{tr}(Z_{\mathcal{E}_5^{(12)}}) = 17, \tag{5.52}$$

$$\mathrm{tr}(Z_{\mathcal{E}^{(24)}}) = 24, \tag{5.53}$$

where $c_k = 0$ if $k \not\equiv 0$ mod 3, $c_{3m} = 2/3$ for $m \in \mathbb{N}$ and $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.

Proof

For the $\mathcal{A}$ graphs, $\mathrm{tr}(Z_{\mathcal{A}^{(k+3)}})$ is given by the number of vertices of $\mathcal{A}^{(k+3)}$, which is $1+2+3+\cdots+k+1 = (k+1)(k+2)/2$. For $k \not\equiv 0$ mod 3, the diagonal terms in $Z_{\mathcal{D}^{(k+3)}}$ are given by the 0-coloured vertices of $\mathcal{A}^{(k+3)}$, so $\mathrm{tr}(Z_{\mathcal{D}^{(k+3)}})$ is $\mathrm{tr}(Z_{\mathcal{A}^{(k+3)}})/3$. For $k \equiv 0$ mod 3 the 0-coloured vertices of $\mathcal{A}^{(k+3)}$ again give the diagonal terms in $Z_{\mathcal{D}^{(k+3)}}$ but the number of 0-coloured vertices of $\mathcal{A}^{(k+3)}$ is now one greater than the number of 1,2-coloured vertices. The trace of $Z_{\mathcal{A}^{(k+3)*}}$ is given by the number of "diagonal" elements $\mu = \overline{\mu}$ of $\mathcal{A}^{(k+3)}$, which is $\lfloor k+2/2 \rfloor$. For the $\mathcal{D}^*$ graphs, when $k \not\equiv 0$ mod 3, the trace is given by the number of vertices $\mu = (\mu_1, \mu_2)$ of $\mathcal{A}^{(k+3)}$ such that $A^{(n-3)(\mu_1-\mu_2)}\mu = \overline{\mu}$. For the 0-coloured vertices this is the number of diagonal elements, whilst for the 1,2-coloured vertices this is where $A\mu = \overline{\mu}$ or $A^2\mu = \overline{\mu}$, depending on the parity of n. In each case the number of such vertices is $\lfloor k+2/2 \rfloor$. For $k \equiv 0$ mod 3 the trace is again given by a third of the number of vertices of $\mathcal{A}^{(k+3)}$ which satisfy each of the following $\mu = \overline{\mu}$, $A\mu = \overline{A^2\mu}$, $A^2 = \overline{A\mu}$, $\mu = \overline{A\mu}$, $A\mu = \overline{\mu}$, $A^2\mu = \overline{A^2\mu}$, $\mu = \overline{A^2\mu}$, $A\mu = \overline{A\mu}$ and $A^2\mu = \overline{\mu}$. The first three equalities are satisfied when $\mu = \overline{\mu}$, the second three when $A\mu = \overline{\mu}$ and the last three when $A^2\mu = \overline{\mu}$. So we have $\mathrm{tr}(Z_{\mathcal{D}^{(k+3)*}}) = 3\lfloor k+2/2 \rfloor$ also. The computations of $\mathrm{tr}(Z_{\mathcal{E}})$ for the exceptional invariants is clear from inspection of the modular invariant. $\square$

Lemma 5.2.2 *The trace of the modular invariants at level k are all different.*

Proof For level 5 we have $\mathrm{tr}(\mathcal{A}^{(8)}) = 21$, $\mathrm{tr}(\mathcal{D}^{(8)}) = 7$, $\mathrm{tr}(\mathcal{A}^{(8)*}) = 3$ and $\mathrm{tr}(\mathcal{D}^{(8)*}) = 9$, and compare these with (5.48) and (5.49). For level 9, $\mathrm{tr}(\mathcal{A}^{(12)}) = 55$, $\mathrm{tr}(\mathcal{D}^{(12)}) = 19$, $\mathrm{tr}(\mathcal{A}^{(12)*}) = 5$ and $\mathrm{tr}(\mathcal{D}^{(12)*}) = 15$, and compare these with (5.50)-(5.52). For level 21 we compare $\mathrm{tr}(\mathcal{A}^{(24)}) = 253$, $\mathrm{tr}(\mathcal{D}^{(24)}) = 85$, $\mathrm{tr}(\mathcal{A}^{(24)*}) = 11$ and $\mathrm{tr}(\mathcal{D}^{(24)*}) = 33$ with (5.53). For all other levels we need to compare the modular invariants for the $\mathcal{A}$, $\mathcal{D}$, $\mathcal{A}^*$ and $\mathcal{D}^*$ graphs.

Comparing the $\mathcal{A}$ and $\mathcal{D}$ modular invariants, the traces can only be equal if $3(k+1)(k+2) = (k+1)(k+2) + 6c_k$. For $k \equiv 0 \bmod 3$ this gives $k = 0, -3$, whilst if $k \not\equiv 0 \bmod 3$ we obtain $k = -1, -2$. So these traces cannot be equal except when $k = 0$, but the graphs $\mathcal{A}^{(3)}$ and $\mathcal{D}^{(3)}$ are both a single vertex. Comparing $\mathcal{A}$-$\mathcal{A}^*$, the traces are only equal if $(k+1)(k+2) = 2\lfloor (k+2)/2 \rfloor$. For even k this gives solutions $k = 0, -4$, but when $k = 0$ the graphs $\mathcal{A}^{(3)*}$ is also just a single vertex, so identical to the graph $\mathcal{A}^{(3)}$. For k odd we have $k = -1$. Next, comparing $\mathcal{A}$-$\mathcal{D}^*$, the traces are only equal if $(k+1)(k+2) = 6\lfloor (k+2)/2 \rfloor$. For k even this gives solutions $k = \pm 2$, but for $k = 2$ the graph $\mathcal{D}^{(5)*}$ is identical to $\mathcal{A}^{(5)}$. For k odd we obtain solutions $k = -3, 1$, but we again have for $k = 1$ that the graphs $\mathcal{D}^{(4)*}$ and $\mathcal{A}^{(4)}$ are the same. We now compare $\mathcal{D}$-$\mathcal{A}^*$. When $k \equiv 0 \bmod 3$, the traces are equal only if $(k+1)(k+2)+4 = 6\lfloor (k+2)/2 \rfloor = 6\lfloor k/2 \rfloor + 6$, so we have the quadratic $k^2 + 3(k - 2\lfloor k/2 \rfloor) = 0$. When k is even we have only the solution $k = 0$, whilst when k is odd this gives $k^2 = -3$. When $k \not\equiv 0 \bmod 3$, we obtain instead the quadratic $k^2 + 3(k - 2\lfloor k/2 \rfloor) - 4 = 0$. For even k this gives the solutions $k = \pm 2$, but we notice that the graphs $\mathcal{D}^{(5)}$ and $\mathcal{A}^{(5)*}$ are the same, whilst for odd k we have the solutions $k = \pm 1$, but we again see that the graphs $\mathcal{D}^{(4)}$ and $\mathcal{A}^{(4)*}$ are the same. Comparing $\mathcal{D}$-$\mathcal{D}^*$ we now obtain the quadratic equations $k^2 + 3(k - 6\lfloor k/2 \rfloor) - 14 = 0$, $k^2 + 3(k - 6\lfloor k/2 \rfloor) - 18 = 0$ for $k \equiv 0 \bmod 3$, $k \not\equiv 0 \bmod 3$ respectively. Neither of these equations has integer solutions for odd or even k. Finally, comparing the $\mathcal{A}^*$ and $\mathcal{D}^*$ modular invariants, the traces are only equal if $\lfloor (k+2)/2 \rfloor = 3\lfloor (k+2)/2 \rfloor$, giving $\lfloor (k+2)/2 \rfloor = 0$ which has solutions $k = -2, -3$. $\square$

Since the traces of the modular invariants at any level are all different, once we have found the number of irreducible sectors, we can identify the corresponding modular invariant. There may however still be an ambiguity with regard to the fusion rules that these irreducible sectors satisfy, with different seemingly possible fusion rules giving different fusion graphs for the M-N system. However, as in Section 1.5, the eigenvalues of the fusion graph must be $S_{1,\mu}/S_{0,\mu}$ with multiplicities given by the diagonal entry $Z_{\mu,\mu}$ of the associated modular invariant, where 0 is the irreducible sector $[\iota\lambda_{(0,0)}]$ and 1 the irreducible sector $[\iota\lambda_{(1,0)}]$. It turns out that the consideration of the trace and the eigenvalues is sufficient to compute the M-N graphs for $\mathcal{A}^{(12)*}$, $\mathcal{D}^{(12)*}$, $\mathcal{E}_2^{(12)}$, $\mathcal{E}_4^{(12)}$ and $\mathcal{E}_5^{(12)}$, and identify the corresponding modular invariant. The results are summarized in Table 5.1. We will say that an irreducible sector $[\iota\lambda_{(\mu_1,\mu_2)}]$ such that $\mu_1+\mu_2=m$ is at tier m.

5.2.1 $\mathcal{E}^{(8)*}$

For the graph $\mathcal{E}^{(8)*}$, we have $[\theta]=[\lambda_{(0,0)}]\oplus[\lambda_{(2,1)}]\oplus[\lambda_{(1,2)}]\oplus[\lambda_{(2,2)}]\oplus[\lambda_{(5,0)}]\oplus[\lambda_{(0,5)}]$. Then computing $\langle\iota\lambda,\iota\mu\rangle=\langle\lambda,\theta\mu\rangle$ (by Frobenius reciprocity) for λ, μ on $\mathcal{A}^{(8)}$, we find $\langle\iota\lambda,\iota\lambda\rangle=1$ and $\langle\iota\lambda,\iota\mu\rangle=0$ for $\lambda,\mu=\lambda_{(0,0)},\lambda_{(1,0)},\lambda_{(0,1)}$. At tier 2 we have $\langle\iota\lambda_{(2,0)},\iota\lambda_{(2,0)}\rangle=2$, $\langle\iota\lambda_{(2,0)},\iota\lambda_{(1,0)}\rangle=1$ and $\langle\iota\lambda_{(2,0)},\iota\mu\rangle=0$ for $\mu=\lambda_{(0,1)},\lambda_{(0,0)}$. So $[\iota\lambda_{(2,0)}]=[\iota\lambda_{(1,0)}]\oplus[\iota\lambda_{(2,0)}^{(1)}]$. Since $\langle\iota\lambda_{(0,2)},\iota\lambda_{(0,2)}\rangle=\langle\iota\lambda_{(0,2)},\iota\lambda_{(2,0)}\rangle=2$ we have $[\iota\lambda_{(0,2)}]=[\iota\lambda_{(2,0)}]$. Lastly at tier 2 we have $\langle\iota\lambda_{(1,1)},\iota\lambda_{(1,1)}\rangle=2$ and $\langle\iota\lambda_{(1,1)},\iota\lambda_{(1,0)}\rangle=\langle\iota\lambda_{(1,1)},\iota\lambda_{(0,1)}\rangle=1$, giving $[\iota\lambda_{(1,1)}]=[\iota\lambda_{(1,0)}]\oplus[\iota\lambda_{(0,1)}]$. At tier 3 we have $\langle\iota\lambda_{(3,0)},\iota\lambda_{(3,0)}\rangle=\langle\iota\lambda_{(3,0)},\iota\lambda_{(0,2)}\rangle=2$, so $[\iota\lambda_{(3,0)}]=[\iota\lambda_{(0,2)}]$. Similarly $[\iota\lambda_{(0,3)}]=[\iota\lambda_{(2,0)}]$. For $\iota\lambda_{(2,1)}$ we find $\langle\iota\lambda_{(2,1)},\iota\lambda_{(2,1)}\rangle=2$ and $\langle\iota\lambda_{(2,1)},\iota\lambda_{(0,0)}\rangle=\langle\iota\lambda_{(2,1)},\iota\lambda_{(1,0)}\rangle=1$, giving $[\iota\lambda_{(2,1)}]=[\iota\lambda_{(0,0)}]\oplus[\iota\lambda_{(1,0)}]$ and similarly $[\iota\lambda_{(1,2)}]=[\iota\lambda_{(0,0)}]\oplus[\iota\lambda_{(0,1)}]$. So no new irreducibles appear at tier 3. No new irreducible sectors appear at the other tiers either, so we have 4 irreducible sectors $[\iota\lambda_{(0,0)}]$, $[\iota\lambda_{(1,0)}]$, $[\iota\lambda_{(0,1)}]$ and $[\iota\lambda_{(2,0)}^{(1)}]$. We now compute the sector products of these irreducible sectors with the M-N sector $[\rho]=[\lambda_{(1,0)}]$. It is easy to compute $[\iota\lambda_{(0,0)}][\rho]=[\iota\lambda_{(1,0)}]$, $[\iota\lambda_{(1,0)}][\rho]=[\iota\lambda_{(0,1)}]\oplus[\iota\lambda_{(2,0)}]=[\iota\lambda_{(0,1)}]\oplus[\iota\lambda_{(1,0)}]\oplus[\iota\lambda_{(2,0)}^{(1)}]$ and $[\iota\lambda_{(0,1)}][\rho]=[\iota\lambda_{(0,0)}]\ominus[\iota\lambda_{(1,0)}]\ominus[\iota\lambda_{(0,1)}]$. We can invert these formula to obtain $[\iota\lambda_{(2,0)}^{(1)}]=[\iota\lambda_{(2,0)}]\ominus[\iota\lambda_{(1,0)}]$, and so $[\iota\lambda_{(2,0)}^{(1)}][\rho]=[\iota\lambda_{(1,1)}\oplus[\iota\lambda_{(3,0)}]\ominus([\iota\lambda_{(2,0)}]\oplus[\iota\lambda_{(0,1)}])=[\iota\lambda_{(0,1)}]$. Then we see that the multiplication graph for $[\rho]$ is the original graph $\mathcal{E}^{(8)*}$ itself, and the modular invariant associated to θ is $Z_{\mathcal{E}^{(8)*}}$.

5.2.2 $\mathcal{E}_2^{(12)}$

For the graph $\mathcal{E}_2^{(12)}$, we have $[\theta]=[\lambda_{(0,0)}]\oplus 2[\lambda_{(2,2)}]\oplus[\lambda_{(4,1)}]\oplus[\lambda_{(1,4)}]\oplus 2[\lambda_{(5,2)}]\oplus 2[\lambda_{(2,5)}]\oplus[\lambda_{(4,4)}]\oplus[\lambda_{(9,0)}]\oplus[\lambda_{(0,9)}]$. We have $\langle\iota\lambda,\iota\lambda\rangle=1$ and $\langle\iota\lambda,\iota\mu\rangle=0$ for all $\lambda,\mu\in\{\lambda_{(0,0)},\lambda_{(1,0)},\lambda_{(0,1)}\}$. At tier 2 we have $\langle\iota\lambda,\iota\lambda\rangle=3$ and $\langle\iota\lambda,\iota\mu\rangle=0$ for $\lambda=\lambda_{(2,0)},\lambda_{(1,1)},\lambda_{(0,2)}$, $\mu=\lambda_{(0,0)},\lambda_{(1,0)},\lambda_{(0,1)}$. Then $\lambda_{(2,0)}$, $\lambda_{(1,1)}$, $\lambda_{(0,2)}$ decompose into irreducibles as

$$[\iota\lambda_{(2,0)}] = [\iota\lambda_{(2,0)}^{(1)}]\oplus[\iota\lambda_{(2,0)}^{(2)}]+[\iota\lambda_{(2,0)}^{(3)}], \tag{5.54}$$

$$[\iota\lambda_{(1,1)}] = [\iota\lambda_{(1,1)}^{(1)}]\oplus[\iota\lambda_{(1,1)}^{(2)}]\oplus[\iota\lambda_{(1,1)}^{(3)}], \tag{5.55}$$

$$[\iota\lambda_{(0,2)}] = [\iota\lambda_{(0,2)}^{(1)}]\oplus[\iota\lambda_{(0,2)}^{(2)}]\oplus[\iota\lambda_{(0,2)}^{(3)}]. \tag{5.56}$$

At tier 3 we find $\langle \iota\lambda_{(3,0)}, \iota\lambda_{(3,0)}\rangle = \langle \iota\lambda_{(3,0)}, \iota\lambda_{(1,1)}\rangle = 3$ so that $[\iota\lambda_{(3,0)}] = [\iota\lambda_{(1,1)}]$, and similarly $[\iota\lambda_{(0,3)}] = [\iota\lambda_{(1,1)}]$. From $\langle \iota\lambda_{(2,1)}, \iota\lambda_{(2,1)}\rangle = 7$, $\langle \iota\lambda_{(2,1)}, \iota\lambda_{(1,0)}\rangle = 2$ and $\langle \iota\lambda_{(2,1)}, \iota\lambda_{(0,2)}\rangle = 3$, and similarly for $\iota\lambda_{(1,2)}$, we obtain

$$\begin{aligned} [\iota\lambda_{(2,1)}] &= 2[\iota\lambda_{(1,0)}] \oplus [\iota\lambda^{(1)}_{(0,2)}] \oplus [\iota\lambda^{(2)}_{(0,2)}] + [\iota\lambda^{(3)}_{(0,2)}], \\ [\iota\lambda_{(1,2)}] &= 2[\iota\lambda_{(0,1)}] \oplus [\iota\lambda^{(1)}_{(2,0)}] \oplus [\iota\lambda^{(2)}_{(2,0)}] \oplus [\iota\lambda^{(3)}_{(2,0)}], \end{aligned} \tag{5.57}$$

and no new irreducible sectors appear at tier 3. Then we have twelve irreducible sectors $[\iota\lambda_{(0,0)}]$, $[\iota\lambda_{(1,0)}]$, $[\iota\lambda_{(0,1)}]$, $[\iota\lambda^{(i)}_{(2,0)}]$, $[\iota\lambda^{(i)}_{(1,1)}]$, $[\iota\lambda^{(i)}_{(0,2)}]$ for $i = 1,2,3$, and the corresponding modular invariant must be $Z_{\mathcal{E}_1^{(12)}}$ since $\mathrm{tr}(Z_{\mathcal{E}_1^{(12)}}) = 12$.

We now look at the fusion rules that these irreducible sectors satisfy. With $\rho = \lambda_{(1,0)}$, we have $[\iota\lambda_{(0,0)}][\rho] = [\iota\lambda_{(1,0)}]$,

$$[\iota\lambda_{(1,0)}][\rho] = [\iota\lambda_{(0,1)}] \oplus [\iota\lambda_{(2,0)}] = [\iota\lambda_{(0,1)}] \oplus [\iota\lambda^{(1)}_{(2,0)}] \oplus [\iota\lambda^{(2)}_{(2,0)}] \oplus [\iota\lambda^{(3)}_{(2,0)}], \tag{5.58}$$

and similarly $[\iota\lambda_{(0,1)}][\rho] = [\iota\lambda_{(0,0)}] \oplus [\iota\lambda^{(1)}_{(1,1)}] \oplus [\iota\lambda^{(2)}_{(1,1)}] \oplus [\iota\lambda^{(3)}_{(1,1)}]$. Since $[\iota\lambda_{(2,0)}][\rho] = [\iota\lambda_{(1,1)}] \oplus [\iota\lambda_{(3,0)}] = 2[\iota\lambda^{(1)}_{(1,1)}] \oplus 2[\iota\lambda^{(2)}_{(1,1)}] \oplus 2[\iota\lambda^{(3)}_{(1,1)}]$, we obtain $([\iota\lambda^{(1)}_{(2,0)}][\rho]) \oplus ([\iota\lambda^{(2)}_{(2,0)}][\rho]) \oplus ([\iota\lambda^{(3)}_{(2,0)}][\rho]) = 2[\iota\lambda^{(1)}_{(1,1)}] \oplus 2[\iota\lambda^{(2)}_{(1,1)}] \oplus +2[\iota\lambda^{(3)}_{(1,1)}]$.

We now use a similar argument to that in [9, §2.4]. The statistical dimension of the positive energy representation (μ_1, μ_2) of $SU(3)_k$ is given by the Perron-Frobenius eigenvector (1.29): $d_{(\mu_1,\mu_2)} = [\mu_1+1][\mu_2+1][\mu_1+\mu_2+2]/[2]$. Then from (5.58) we obtain $d^{(1)}_{(2,0)} + d^{(2)}_{(2,0)} + d^{(3)}_{(2,0)} = d^2_{(1,0)} - d_{(1,0)} = [3]^3 - [3] = [3][4]/[2]$, where $d^{(i)}_{(2,0)} = d_{\iota\lambda^{(i)}_{(2,0)}}$. We may then assume without loss of generality that $d^{(1)}_{(2,0)} < [3][4]/(3[2]) = [2][3]/[4]$. Then since $([2][3]/[4])^2 \approx 2.488 < 3$, $[\overline{\iota\lambda^{(1)}_{(2,0)}}][\iota\lambda^{(1)}_{(2,0)}]$ decomposes into at most two irreducible N-N sectors. Then $\langle \iota\lambda^{(1)}_{(2,0)} \circ \rho, \iota\lambda^{(1)}_{(2,0)} \circ \rho\rangle = \langle \rho \circ \overline{\rho}, \overline{\iota\lambda^{(1)}_{(2,0)}} \circ \iota\lambda^{(1)}_{(2,0)}\rangle \leq 2$. So $[\iota\lambda^{(1)}_{(2,0)}][\rho]$ cannot contain an irreducible sector with multiplicity greater than one. Since, by (5.54) and (5.57), $\langle \iota\lambda^{(1)}_{(2,0)} \circ \rho, \iota\lambda_{(1,1)}\rangle = \langle \iota\lambda^{(1)}_{(2,0)}, \iota\lambda_{(1,1)} \circ \overline{\rho}\rangle = \langle \iota\lambda^{(1)}_{(2,0)}, \iota\lambda_{(0,1)} + \iota\lambda_{(2,0)} + \iota\lambda_{(1,2)}\rangle = 2$, using (5.55) we may assume, again without loss of generality, that

$$[\iota\lambda^{(1)}_{(2,0)}][\rho] = [\iota\lambda^{(1)}_{(1,1)}] \oplus [\iota\lambda^{(2)}_{(1,1)}].$$

Since $[\iota\lambda_{(1,0)}][\rho] \supset [\iota\lambda^{(1)}_{(2,0)}]$ and $\langle \iota\lambda_{(1,0)}, \iota\lambda^{(1)}_{(2,0)} \circ \overline{\rho}\rangle = \langle \iota\lambda_{(1,0)} \circ \rho, \iota\lambda^{(1)}_{(2,0)}\rangle > 0$, then $[\iota\lambda^{(1)}_{(2,0)}][\overline{\rho}] \supset [\iota\lambda_{(1,0)}]$. Then since $\langle \iota\lambda^{(1)}_{(2,0)} \circ \overline{\rho}, \iota\lambda^{(1)}_{(2,0)} \circ \overline{\rho}\rangle = \langle \iota\lambda^{(1)}_{(2,0)} \circ \rho, \iota\lambda^{(1)}_{(2,0)} \circ \rho\rangle = 2$, we have $[\iota\lambda^{(1)}_{(2,0)}][\overline{\rho}] = [\iota\lambda_{(1,0)}] \oplus [\iota\lambda^{(j)}_{(0,2)}]$, for $j \in \{1,2,3\}$. By a similar argument we may also assume that $[\iota\lambda^{(1)}_{(2,0)}]$ has statistical dimension $< [2][3]/[4]$, and using $[\overline{\rho}]$ instead of $[\rho]$, we find $[\iota\lambda^{(1)}_{(0,2)}][\rho] = [\iota\lambda_{(0,1)}] \oplus [\iota\lambda^{(j')}_{(2,0)}]$, and have the freedom to set $j' = 3$. Then we also have $[\iota\lambda^{(j)}_{(0,2)}][\rho] \supset [\iota\lambda_{(0,1)}]$ for $j = 2,3$ and $([\iota\lambda^{(2)}_{(0,2)}] \oplus [\iota\lambda^{(3)}_{(0,2)}])[\rho] = 2[\iota\lambda_{(0,1)}] \oplus [\iota\lambda^{(1)}_{(2,0)}] \oplus [\iota\lambda^{(2)}_{(2,0)}]$. From $[\iota\lambda_{(1,1)}][\rho]$ we obtain $([\iota\lambda^{(1)}_{(1,1)}] \oplus [\iota\lambda^{(2)}_{(1,1)}] \oplus [\iota\lambda^{(3)}_{(1,1)}])[\rho] = 3[\iota\lambda_{(1,0)}] \oplus 2[\iota\lambda^{(1)}_{(0,2)}] \oplus 2[\iota\lambda^{(2)}_{(0,2)}] \oplus 2[\iota\lambda^{(3)}_{(0,2)}]$

and since $[\iota\lambda_{(1,0)}][\overline{\rho}] = [\iota\lambda_{(2,0)}] \oplus [\iota\lambda_{(1,1)}] = [\iota\lambda_{(2,0)}] \oplus [\iota\lambda^{(1)}_{(1,1)}] \oplus [\iota\lambda^{(2)}_{(1,1)}] \oplus [\iota\lambda^{(3)}_{(1,1)}]$ then $\langle \iota\lambda^{(j)}_{(1,1)} \circ \rho, \iota\lambda_{(1,0)} \rangle = \langle \iota\lambda^{(j)}_{(1,1)}, \iota\lambda_{(1,0)} \circ \overline{\rho} \rangle = 1$ and $[\iota\lambda^{(j)}_{(1,1)}][\rho] \supset [\iota\lambda_{(1,0)}]$ for $j = 1,2,3$. There is still some ambiguity surrounding the decompositions of $[\iota\lambda^{(j)}_{(2,0)}][\rho]$, $[\iota\lambda^{(j)}_{(1,1)}][\rho]$ and $[\iota\lambda^{(j)}_{(0,2)}][\rho]$, for $j = 2,3$. Computing the eigenvalues of the fusion graphs for the different possibilities, we find that the only fusion graph which has eigenvalues $S_{1,\mu}/S_{0,\mu}$ with multiplicities given by the diagonal entry $Z_{\mu,\mu}$ of the modular invariant is that for: $[\iota\lambda^{(j)}_{(2,0)}][\rho] = [\iota\lambda^{(j)}_{(1,1)}] \oplus [\iota\lambda^{(j+1)}_{(1,1)}]$, $[\iota\lambda^{(j)}_{(1,1)}][\rho] = [\iota\lambda^{(l)}_{(0,2)}] \oplus [\iota\lambda^{(l+1)}_{(0,2)}]$ and $[\iota\lambda^{(j)}_{(0,2)}][\rho] = [\iota\lambda_{(0,1)}] \oplus [\iota\lambda^{(j+1)}_{(2,0)}]$ for $j = 1,2,3$, $l \in \{1,2,3\}$. The fusion graph is the same for any choice of $l = 1,2,3$, up to a relabeling of the irreducible representations $[\iota\lambda^{(j)}_{(2,0)}]$, $[\iota\lambda^{(j)}_{(1,1)}]$ and $[\iota\lambda^{(j)}_{(0,2)}]$, and the graph is just the graph $\mathcal{E}_2^{(12)}$ itself, illustrated in Figure 5.3. The associated modular invariant is $Z_{\mathcal{E}_2^{(12)}} = Z_{\mathcal{E}_1^{(12)}}$.

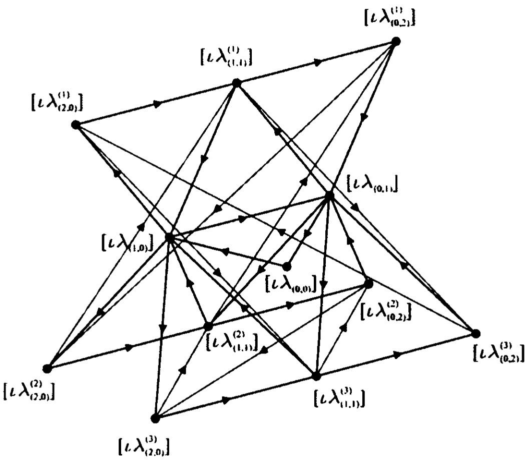


Figure 5.3: M-N graph for the $\mathcal{E}_2^{(12)}$ A_2-GHJ subfactor

5.2.3 $\mathcal{E}_4^{(12)}$

Warning: the existence of the A_2 Goodman-de la Harpe-Jones subfactor which gives the dual canonical endomorphism for $\mathcal{E}_4^{(12)}$ has not been shown yet by us.

For $\mathcal{E}_4^{(12)}$, we suppose $[\theta] = [\lambda_{(0,0)}] \oplus [\lambda_{(2,2)}] \oplus [\lambda_{(4,1)}] \oplus [\lambda_{(1,4)}] \oplus [\lambda_{(5,2)}] \oplus [\lambda_{(2,5)}] \oplus [\lambda_{(4,4)}] \oplus [\lambda_{(9,0)}] \oplus [\lambda_{(0,9)}]$. Then computing $\langle \iota\lambda, \iota\mu \rangle = \langle \lambda, \theta\mu \rangle$ for λ, μ on $\mathcal{A}^{(12)}$, we find $\langle \iota\lambda, \iota\lambda \rangle = 1$ for $\lambda = \lambda_{(0,0)}, \lambda_{(1,0)}, \lambda_{(0,1)}$. At tier 2 we have $\langle \iota\lambda, \iota\lambda \rangle = 2$ and $\langle \iota\lambda, \iota\mu \rangle = 0$ for $\lambda = \lambda_{(2,0)}, \lambda_{(1,1)}, \lambda_{(0,2)}$, $\mu = \lambda_{(0,0)}, \lambda_{(1,0)}, \lambda_{(0,1)}$. Then $[\lambda_{(2,0)}]$, $[\lambda_{(1,1)}]$, $[\lambda_{(0,2)}]$ decompose into irreducibles as

$$[\iota\lambda_{(2,0)}] = [\iota\lambda^{(1)}_{(2,0)}] \oplus [\iota\lambda^{(2)}_{(2,0)}], \tag{5.59}$$

$$[\iota\lambda_{(1,1)}] = [\iota\lambda^{(1)}_{(1,1)}] \oplus [\iota\lambda^{(2)}_{(1,1)}], \tag{5.60}$$
$$[\iota\lambda_{(0,2)}] = [\iota\lambda^{(1)}_{(0,2)}] \oplus [\iota\lambda^{(2)}_{(0,2)}]. \tag{5.61}$$

At tier 3, $\langle\iota\lambda_{(3,0)}, \iota\lambda_{(3,0)}\rangle = \langle\iota\lambda_{(3,0)}, \iota\lambda_{(1,1)}\rangle = 2$ and similarly for $\iota\lambda_{(0,3)}$, so that $[\iota\lambda_{(3,0)}] = [\iota\lambda_{(0,3)}] = [\iota\lambda_{(1,1)}]$. From $\langle\iota\lambda_{(2,1)}, \iota\lambda_{(2,1)}\rangle = 5$, $\langle\iota\lambda_{(2,1)}, \iota\lambda_{(1,0)}\rangle = 1$ and $\langle\iota\lambda_{(2,1)}, \iota\lambda_{(0,2)}\rangle = 2$, we have two possibilities for the decomposition of $[\iota\lambda_{(2,1)}]$:

$$[\iota\lambda_{(2,1)}] = \begin{cases} [\iota\lambda_{(1,0)}] \oplus 2[\iota\lambda^{(j)}_{(0,2)}] & \text{case } (i), \\ [\iota\lambda_{(1,0)}] \oplus [\iota\lambda^{(1)}_{(0,2)}] \oplus [\iota\lambda^{(2)}_{(0,2)}] \oplus [\iota\lambda^{(1)}_{(2,1)}] \oplus [\iota\lambda^{(2)}_{(2,1)}] & \text{case } (ii), \end{cases} \tag{5.62}$$

where we may assume $j = 1$ without loss of generality. Similarly,

$$[\iota\lambda_{(1,2)}] = \begin{cases} [\iota\lambda_{(0,1)}] \oplus 2[\iota\lambda^{(1)}_{(2,0)}] & \text{case } (i'), \\ [\iota\lambda_{(0,1)}] \oplus [\iota\lambda^{(1)}_{(2,0)}] \oplus [\iota\lambda^{(2)}_{(2,0)}] \oplus [\iota\lambda^{(1)}_{(1,2)}] \oplus [\iota\lambda^{(2)}_{(1,2)}] & \text{case } (ii'), \end{cases} \tag{5.63}$$

At tier 4 we have $\langle\iota\lambda_{(4,0)}, \iota\lambda_{(4,0)}\rangle = 3$, $\langle\iota\lambda_{(4,0)}, \iota\lambda_{(1,0)}\rangle = 1$ and $\langle\iota\lambda_{(4,0)}, \iota\lambda_{(0,2)}\rangle = 2$, and similarly for $\iota\lambda_{(0,4)}$, giving

$$\begin{aligned} [\iota\lambda_{(4,0)}] &= [\iota\lambda_{(1,0)}] \oplus [\iota\lambda^{(1)}_{(0,2)}] \oplus [\iota\lambda^{(2)}_{(0,2)}], \\ [\iota\lambda_{(0,4)}] &= [\iota\lambda_{(0,1)}] \oplus [\iota\lambda^{(1)}_{(2,0)}] \oplus [\iota\lambda^{(2)}_{(2,0)}]. \end{aligned}$$

From $\langle\iota\lambda_{(3,1)}, \iota\lambda_{(3,1)}\rangle = 8$, $\langle\iota\lambda_{(3,1)}, \iota\lambda_{(0,1)}\rangle = 2$, $\langle\iota\lambda_{(3,1)}, \iota\lambda_{(2,0)}\rangle = 2$ and $\langle\iota\lambda_{(3,1)}, \iota\lambda_{(1,2)}\rangle = 6$ we have

$$\begin{aligned} [\iota\lambda_{(3,1)}] &= \begin{cases} 2[\iota\lambda_{(0,1)}] \oplus 2[\iota\lambda^{(1)}_{(2,0)}] & \text{for case } (i'), \\ 2[\iota\lambda_{(0,1)}] \oplus [\iota\lambda^{(1)}_{(2,0)}] \oplus [\iota\lambda^{(2)}_{(2,0)}] \oplus [\iota\lambda^{(1)}_{(1,2)}] \oplus [\iota\lambda^{(2)}_{(1,2)}] & \text{for case } (ii'), \end{cases} \\ [\iota\lambda_{(1,3)}] &= \begin{cases} 2[\iota\lambda_{(1,0)}] \oplus 2[\iota\lambda^{(1)}_{(0,2)}] & \text{for case } (i), \\ 2[\iota\lambda_{(1,0)}] \oplus [\iota\lambda^{(1)}_{(0,2)}] \oplus [\iota\lambda^{(2)}_{(0,2)}] \oplus [\iota\lambda^{(1)}_{(2,1)}] \oplus [\iota\lambda^{(2)}_{(2,1)}] & \text{for case } (ii). \end{cases} \end{aligned}$$

We have $\langle\iota\lambda_{(2,2)}, \iota\lambda_{(2,2)}\rangle = 11$, $\langle\iota\lambda_{(2,2)}, \iota\lambda_{(0,0)}\rangle = 1$ and $\langle\iota\lambda_{(2,2)}, \iota\lambda_{(1,1)}\rangle = 4$, giving

$$[\iota\lambda_{(2,2)}] = \begin{cases} [\iota\lambda_{(0,0)}] \oplus 3[\iota\lambda^{(j)}_{(1,1)}] \oplus [\iota\lambda^{(3-j)}_{(1,1)}] & \text{case I}, \\ [\iota\lambda_{(0,0)}] \oplus 2[\iota\lambda^{(1)}_{(1,1)}] \oplus 2[\iota\lambda^{(2)}_{(1,1)}] \oplus [\iota\lambda^{(1)}_{(2,2)}] \oplus [\iota\lambda^{(2)}_{(2,2)}] & \text{case II}, \end{cases} \tag{5.64}$$

where $j \in \{1, 2\}$. Again, without loss of generality, we may assume that $j = 1$, and we see that for case I nothing new appears at tier 4. For case II, at tier 5 we find $[\iota\lambda_{(5,0)}] = [\iota\lambda_{(0,4)}]$, $[\iota\lambda_{(0,5)}] = [\iota\lambda_{(4,0)}]$, $[\iota\lambda_{(4,1)}] = [\iota\lambda_{(1,4)}] = [\iota\lambda_{(0,0)}] \oplus 2[\iota\lambda^{(1)}_{(1,1)}] \oplus 2[\iota\lambda^{(2)}_{(1,1)}]$ and

$$\begin{aligned} [\iota\lambda_{(3,2)}] &= \begin{cases} 2[\iota\lambda_{(1,0)}] \oplus 3[\iota\lambda^{(1)}_{(0,2)}] \oplus [\iota\lambda^{(2)}_{(0,2)}] & \text{for case } (i), \\ 2[\iota\lambda_{(1,0)}] \oplus 2[\iota\lambda^{(1)}_{(0,2)}] \oplus 2[\iota\lambda^{(2)}_{(0,2)}] \oplus [\iota\lambda^{(1)}_{(2,1)}] \oplus [\iota\lambda^{(2)}_{(2,1)}] & \text{for case } (ii), \end{cases} \\ [\iota\lambda_{(2,3)}] &= \begin{cases} 2[\iota\lambda_{(0,1)}] \oplus 3[\iota\lambda^{(1)}_{(2,0)}] \oplus [\iota\lambda^{(2)}_{(2,0)}] & \text{for case } (i'), \\ 2[\iota\lambda_{(0,1)}] \oplus 2[\iota\lambda^{(1)}_{(2,0)}] \oplus 2[\iota\lambda^{(2)}_{(2,0)}] \oplus [\iota\lambda^{(1)}_{(1,2)}] \oplus [\iota\lambda^{(2)}_{(1,2)}] & \text{for case } (ii'), \end{cases} \end{aligned}$$

and nothing new appears at tier 5. Then the total number of irreducible sectors for case I$(i)(i')$ is 9, for cases I$(i)(ii')$, I$(ii)(i')$, II$(i)(i')$ we have 11, for cases I$(ii)(ii')$, II$(i)(ii')$, II$(ii)(i')$ we have 13 and for case II$(ii)(ii')$ we have 15. The values of $\mathrm{tr}(Z)$ at level 12 are $\mathrm{tr}(Z_{\mathcal{A}^{(12)}}) = 55$, $\mathrm{tr}(Z_{\mathcal{D}^{(12)}}) = 19$, $\mathrm{tr}(Z_{\mathcal{A}^{(12)*}}) = 5$, $\mathrm{tr}(Z_{\mathcal{D}^{(12)*}}) = 15$, $\mathrm{tr}(Z_{\mathcal{E}_1^{(12)}}) = 12$, $\mathrm{tr}(Z_{\mathcal{E}_4^{(12)}}) = 11$ and $\mathrm{tr}(Z_{\mathcal{E}_5^{(12)}}) = 17$. So we see that the only possible cases are I$(i)(ii')$, I$(ii)(i')$, II$(i)(i')$ which have corresponding modular invariant $Z_{\mathcal{E}_4^{(12)}}$, and II$(ii)(ii')$ associated with the modular invariant $Z_{\mathcal{D}^{(12)*}}$. For case II$(i)(i')$, where we again use the notation $\rho = \lambda_{(1,0)}$, we have $[\iota\lambda_{(1,2)}][\rho] = [\iota\lambda_{(1,1)}] \oplus [\iota\lambda_{(0,3)}] \oplus [\iota\lambda_{(2,2)}]$ and $[\iota\lambda_{(1,2)}][\rho] = ([\iota\lambda_{(0,1)}] \oplus 2[\iota\lambda^{(1)}_{(2,0)}])[\rho] = [\iota\lambda_{(0,0)}] \oplus [\iota\lambda_{(1,1)}] \oplus 2([\iota\lambda^{(1)}_{(2,0)}][\rho])$, giving $2[\iota\lambda^{(1)}_{(2,0)}][\rho] = 3[\iota\lambda^{(1)}_{(1,1)}] \oplus 3[\iota\lambda^{(2)}_{(1,1)}] \oplus [\iota\lambda^{(1)}_{(2,2)}] \oplus [\iota\lambda^{(2)}_{(2,2)}]$, which is impossible since $[\iota\lambda^{(1)}_{(2,0)}][\rho]$ must have integer coefficients. Note that case II$(ii)(i')$ is the conjugate of case II$(i)(ii')$, where we replace $\iota\lambda_{(\mu_1,\mu_2)} \leftrightarrow \iota\lambda_{(\mu_2,\mu_1)}$. So we need to only consider cases I$(i)(ii')$ and II$(ii)(ii')$.

Consider first the case I$(i)(ii')$. From $[\iota\lambda_{(2,1)}][\rho] = [\iota\lambda_{(2,0)}] \oplus [\iota\lambda_{(1,2)}] \oplus [\iota\lambda_{(3,1)}]$ and (5.62) we find $[\iota\lambda^{(1)}_{(2,1)}][\rho] = [\iota\lambda_{(2,0)}] \oplus [\iota\lambda_{(1,2)}] \oplus [\iota\lambda_{(3,1)}] \ominus ([\iota\lambda_{(0,1)}] \oplus [\iota\lambda_{(2,0)}]) = [\iota\lambda_{(0,1)}] \oplus [\iota\lambda^{(1)}_{(2,1)}] \ominus [\iota\lambda^{(2)}_{(2,1)}] \oplus [\iota\lambda^{(1)}_{(1,2)}] \oplus [\iota\lambda^{(2)}_{(1,2)}]$. Then by $[\iota\lambda_{(0,2)}][\rho] = [\iota\lambda_{(0,1)}] \oplus [\iota\lambda_{(1,2)}]$ and (5.61), $[\iota\lambda^{(2)}_{(0,2)}][\rho] = [\iota\lambda_{(0,1)}]$. From $[\iota\lambda_{(1,1)}][\rho] = [\iota\lambda_{(1,0)}] \oplus [\iota\lambda_{(0,2)}] \oplus [\iota\lambda_{(2,1)}]$ and (5.60) we obtain

$$([\iota\lambda^{(1)}_{(1,1)}] \oplus [\iota\lambda^{(2)}_{(1,1)}])[\rho] = 2[\iota\lambda_{(1,0)}] \oplus 3[\iota\lambda^{(1)}_{(0,2)}] \oplus [\iota\lambda^{(2)}_{(0,2)}], \tag{5.65}$$

whilst from $[\iota\lambda_{(2,2)}][\rho] = [\iota\lambda_{(2,1)}] \ominus [\iota\lambda_{(1,3)}] \oplus [\iota\lambda_{(3,2)}]$ and (5.64) we have

$$(3[\iota\lambda^{(1)}_{(1,1)}] \oplus [\iota\lambda^{(2)}_{(1,1)}])[\rho] = 4[\iota\lambda_{(1,0)}] \oplus 7[\iota\lambda^{(1)}_{(0,2)}] \oplus [\iota\lambda^{(2)}_{(0,2)}]. \tag{5.66}$$

Then from (5.65) and (5.66) we find

$$[\iota\lambda^{(1)}_{(1,1)}][\rho] = [\iota\lambda_{(1,0)}] \oplus 2[\iota\lambda^{(1)}_{(0,2)}], \qquad [\iota\lambda^{(2)}_{(1,1)}][\rho] = [\iota\lambda_{(1,0)}] \oplus [\iota\lambda^{(1)}_{(0,2)}] \oplus [\iota\lambda^{(2)}_{(0,2)}].$$

In the same manner, by considering $[\iota\lambda_{(2,0)}][\rho] = [\iota\lambda_{(1,1)}] \oplus [\iota\lambda_{(3,0)}]$ and $[\iota\lambda_{(1,2)}][\rho] = [\iota\lambda_{(1,1)}] \ominus [\iota\lambda_{(0,3)}] \oplus [\iota\lambda_{(2,2)}]$, and using (5.59) and (5.63), we have

$$\begin{aligned} ([\iota\lambda^{(1)}_{(2,0)}] \oplus [\iota\lambda^{(2)}_{(2,0)}])[\rho] &= 2[\iota\lambda^{(1)}_{(1,1)}] \oplus 2[\iota\lambda^{(2)}_{(1,1)}], & (5.67)\\ ([\iota\lambda^{(1)}_{(2,0)}] \oplus [\iota\lambda^{(2)}_{(2,0)}] \oplus [\iota\lambda^{(1)}_{(1,2)}] \oplus [\iota\lambda^{(2)}_{(1,2)}])[\rho] &= [\iota\lambda_{(0,0)}] \oplus 5[\iota\lambda^{(1)}_{(1,1)}] \ominus 3[\iota\lambda^{(2)}_{(1,1)}] \\ &\quad \ominus([\iota\lambda_{(0,0)}] \oplus [\iota\lambda_{(1,1)}]). & (5.68) \end{aligned}$$

Then from (5.67), (5.65) and (5.60), we have $([\iota\lambda^{(1)}_{(1,2)}][\rho]) \oplus ([\iota\lambda^{(2)}_{(1,2)}][\rho]) = 2[\iota\lambda^{(1)}_{(1,1)}]$ giving $[\iota\lambda^{(j)}_{(1,2)}][\rho] = [\iota\lambda^{(1)}_{(1,1)}]$ for $j = 1, 2$. From $[\iota\lambda_{(2,2)}][\overline{\rho}] = [\iota\lambda_{(1,2)}] \oplus [\iota\lambda_{(3,1)}] \oplus [\iota\lambda_{(2,3)}]$ and (5.64) we have

$$([\iota\lambda_{(1,1)}] \oplus 2[\iota\lambda^{(1)}_{(1,1)}])[\overline{\rho}] = 4[\iota\lambda_{(0,1)}] \oplus 4[\iota\lambda^{(1)}_{(2,0)}] \oplus 4[\iota\lambda^{(2)}_{(2,0)}] \oplus 3[\iota\lambda^{(1)}_{(1,2)}] \oplus 3[\iota\lambda^{(2)}_{(1,2)}], \tag{5.69}$$

giving $2[\iota\lambda^{(1)}_{(1,1)}][\overline{\rho}] = 2[\iota\lambda_{(0,1)}] \oplus 2[\iota\lambda^{(1)}_{(2,0)}] \oplus 2[\iota\lambda^{(2)}_{(2,0)}] \oplus 2[\iota\lambda^{(1)}_{(1,2)}] \oplus 2[\iota\lambda^{(2)}_{(1,2)}]$. Then $\langle \iota\lambda^{(j)}_{(2,0)} \circ \rho, \lambda^{(1)}_{(1,1)} \rangle = \langle \iota\lambda^{(j)}_{(2,0)}, \lambda^{(1)}_{(1,1)} \circ \overline{\rho} \rangle = 1$ for $j = 1,2$, and the decompositions of $[\iota\lambda^{(1)}_{(2,0)}][\rho]$ and $[\iota\lambda^{(2)}_{(2,0)}][\rho]$ both contain the irreducible sector $[\iota\lambda^{(1)}_{(1,1)}]$. Then $[\iota\lambda^{(2)}_{(1,1)}][\overline{\rho}] = ([\iota\lambda_{(1,1)}][\overline{\rho}]) \ominus ([\iota\lambda^{(1)}_{(1,1)}][\overline{\rho}]) = [\iota\lambda_{(0,1)}] \oplus [\iota\lambda^{(1)}_{(2,0)}] \oplus [\iota\lambda^{(2)}_{(2,0)}]$ and $[\iota\lambda^{(1)}_{(2,0)}][\rho]$ and $[\iota\lambda^{(2)}_{(2,0)}][\rho]$ both also contain $[\iota\lambda^{(2)}_{(1,1)}]$. Then from (5.67) we have $[\iota\lambda^{(j)}_{(2,0)}][\rho] = [\iota\lambda^{(1)}_{(1,1)}] \oplus [\iota\lambda^{(2)}_{(1,1)}]$. The fusion graph with respect to $[\rho]$ for the case I$(i)(ii')$ is then seen to be just the graph $\mathcal{E}^{(12)}_4$.

Now consider the case II$(ii)(ii')$, which has corresponding modular invariant $Z_{\mathcal{D}^{(12)*}}$. We obtain the following sector products:

$$\begin{aligned}
([\iota\lambda^{(1)}_{(2,0)}] \oplus [\iota\lambda^{(2)}_{(2,0)}])[\rho] &= 2[\iota\lambda^{(1)}_{(1,1)}] \oplus 2[\iota\lambda^{(2)}_{(1,1)}], \\
([\iota\lambda^{(1)}_{(1,1)}] \oplus [\iota\lambda^{(2)}_{(1,1)}])[\rho] &= 2[\iota\lambda_{(1,0)}] \oplus 2[\iota\lambda^{(1)}_{(0,2)}] \oplus 2[\iota\lambda^{(2)}_{(0,2)}] \oplus [\iota\lambda^{(1)}_{(2,1)}] \oplus [\iota\lambda^{(2)}_{(2,1)}], \\
([\iota\lambda^{(1)}_{(0,2)}] \oplus [\iota\lambda^{(2)}_{(0,2)}])[\rho] &= 2[\iota\lambda_{(0,1)}] \oplus [\iota\lambda^{(1)}_{(2,0)}] \oplus [\iota\lambda^{(2)}_{(2,0)}] \oplus [\iota\lambda^{(1)}_{(1,2)}] \oplus [\iota\lambda^{(2)}_{(1,2)}], \\
([\iota\lambda^{(1)}_{(2,1)}] \oplus [\iota\lambda^{(2)}_{(2,1)}])[\rho] &= [\iota\lambda^{(1)}_{(2,0)}] \oplus [\iota\lambda^{(2)}_{(2,0)}] \oplus [\iota\lambda^{(1)}_{(1,2)}] \oplus [\iota\lambda^{(2)}_{(1,2)}], \\
([\iota\lambda^{(1)}_{(1,2)}] \oplus [\iota\lambda^{(2)}_{(1,2)}])[\rho] &= [\iota\lambda^{(1)}_{(1,1)}] \oplus [\iota\lambda^{(2)}_{(1,1)}] \oplus [\iota\lambda^{(1)}_{(2,2)}] \oplus [\iota\lambda^{(2)}_{(2,2)}],
\end{aligned}$$

and from $([\iota\lambda^{(1)}_{(2,2)}] \oplus [\iota\lambda^{(2)}_{(2,2)}])[\rho] = [\iota\lambda^{(1)}_{(2,1)}] \oplus [\iota\lambda^{(2)}_{(2,1)}]$ we may choose without loss of generality $[\iota\lambda^{(j)}_{(2,2)}][\rho] = [\iota\lambda^{(j)}_{(2,1)}]$ for $j = 1,2$. Then there are four different possibilities for $[\iota\lambda^{(j)}_{(1,1)}][\rho]$, three for $[\iota\lambda^{(j)}_{(2,0)}][\rho]$, six for $[\iota\lambda^{(j)}_{(0,2)}][\rho]$ and six for $[\iota\lambda^{(j)}_{(2,1)}][\rho]$, $j = 1,2$. From these, the only fusion graph which has eigenvalues $S_{1,\mu}/S_{0,\mu}$ with multiplicities given by the diagonal entry $Z_{\mu,\mu}$ of the modular invariant for $\mathcal{D}^{(12)*}$ is that for the following sector products:

$$\begin{aligned}
[\iota\lambda^{(j)}_{(2,0)}][\rho] &= 2[\iota\lambda^{(j)}_{(1,1)}], \\
[\iota\lambda^{(1)}_{(1,1)}][\rho] &= [\iota\lambda_{(1,0)}] \oplus 2[\iota\lambda^{(j)}_{(0,2)}] \oplus [\iota\lambda^{(j)}_{(2,1)}], \\
[\iota\lambda^{(j)}_{(0,2)}][\rho] &= [\iota\lambda_{(0,1)}] \oplus [\iota\lambda^{(j)}_{(2,0)}] \oplus [\iota\lambda^{(j)}_{(1,2)}], \\
[\iota\lambda^{(j)}_{(2,1)}][\rho] &= [\iota\lambda^{(j)}_{(2,0)}] \oplus [\iota\lambda^{(j)}_{(1,2)}], \\
[\iota\lambda^{(j)}_{(1,2)}][\rho] &= [\iota\lambda^{(j)}_{(1,1)}] \oplus [\iota\lambda^{(3-j)}_{(2,2)}],
\end{aligned}$$

for $j = 1,2$. For any $\lambda \in {}_M\mathcal{X}_N$, let $[\lambda][\rho] = \bigoplus_{\mu \in {}_M\mathcal{X}_N} a_\mu[\mu]$, $a_\mu \in \mathbb{C}$. Then $\langle \mu \circ \overline{\rho}, \lambda \rangle = \langle \mu, \lambda \circ \rho \rangle = a_\mu$ for all $\mu \in {}_M\mathcal{X}_N$, so $[\mu][\overline{\rho}] \supset a_\mu[\lambda]$. Then if G is the multiplication matrix for $[\rho]$, G^T is the multiplication matrix for $[\overline{\rho}]$. This graph cannot be the fusion graph since $GG^T \neq G^TG$, which means $[\iota\lambda][\rho][\overline{\rho}] \neq [\iota\lambda][\overline{\rho}][\rho]$. Then the only possibility for the fusion graph for the M-N system is the graph $\mathcal{E}^{(12)}_4$, and the associated modular invariant is $Z_{\mathcal{E}^{(12)}_4}$, assuming that θ is as expressed in (5.40).

5.2.4 $\mathcal{E}^{(12)}_5$

For the graph $\mathcal{E}^{(12)}_5$, we have $[\theta] = [\lambda_{(0,0)}] \oplus [\lambda_{(3,3)}] \oplus [\lambda_{(9,0)}] \oplus [\lambda_{(0,9)}]$. Then computing $\langle \iota\lambda, \iota\mu \rangle = \langle \lambda, \theta\mu \rangle$ for λ, μ on $\mathcal{A}^{(12)}$, we find $\langle \iota\lambda, \iota\lambda \rangle = 1$ for $\lambda = \lambda_{(\mu_1,\mu_2)}$ such that

$\mu_1+\mu_2 \le 2$. At tier 3 we have $\langle \iota\lambda, \iota\lambda\rangle = 2$ and $\langle \iota\lambda, \iota\mu\rangle = 0$ for $\lambda = \lambda_{(3,0)}, \lambda_{(2,1)}, \lambda_{(1,2)}, \lambda_{(0,3)}$, $\mu = \lambda_{(\mu_1,\mu_2)}$ such that $\mu_1 + \mu_2 \le 2$. We also have $\langle \iota\lambda_{(3,0)}, \iota\lambda_{(0,3)}\rangle = 0$. Then $\lambda_{(3,0)}$, $\lambda_{(2,1)}$, $\lambda_{(1,2)}$, $\lambda_{(0,3)}$ decompose into irreducibles as

$$\begin{aligned}
[\iota\lambda_{(3,0)}] &= [\iota\lambda^{(1)}_{(3,0)}] \oplus [\iota\lambda^{(2)}_{(3,0)}], && (5.70)\\
[\iota\lambda_{(2,1)}] &= [\iota\lambda^{(1)}_{(2,1)}] \oplus [\iota\lambda^{(2)}_{(2,1)}], && (5.71)\\
[\iota\lambda_{(1,2)}] &= [\iota\lambda^{(1)}_{(1,2)}] \oplus [\iota\lambda^{(2)}_{(1,2)}], && (5.72)\\
[\iota\lambda_{(0,3)}] &= [\iota\lambda^{(1)}_{(0,3)}] \oplus [\iota\lambda^{(2)}_{(0,3)}]. && (5.73)
\end{aligned}$$

At tier 4 we have $\langle \iota\lambda_{(4,0)}, \iota\lambda_{(4,0)}\rangle = 2$, $\langle \iota\lambda_{(4,0)}, \iota\lambda_{(2,1)}\rangle = 1$ and $\langle \iota\lambda_{(4,0)}, \iota\mu\rangle = 0$ for $\mu = \lambda_{(1,0)}, \lambda_{(0,2)}$. Then $[\iota\lambda_{(4,0)}] = [\iota\lambda^{(j)}_{(2,1)}] \ominus [\iota\lambda^{(1)}_{(4,0)}]$ for $j \in \{1,2\}$. We have the freedom to choose $j = 1$ without loss of generality. Similarly for $\iota\lambda_{(0,4)}$. Then

$$\begin{aligned}
[\iota\lambda_{(4,0)}] &= [\iota\lambda^{(1)}_{(2,1)}] \oplus [\iota\lambda^{(1)}_{(4,0)}], && (5.74)\\
[\iota\lambda_{(0,4)}] &= [\iota\lambda^{(1)}_{(1,2)}] \oplus [\iota\lambda^{(1)}_{(0,4)}]. && (5.75)
\end{aligned}$$

From $\langle \iota\lambda_{(3,1)}, \iota\lambda_{(3,1)}\rangle = 3$, $\langle \iota\lambda_{(3,1)}, \iota\lambda_{(2,0)}\rangle = 1$, $\langle \iota\lambda_{(3,1)}, \iota\lambda_{(1,2)}\rangle = 1$ and $\langle \iota\lambda_{(3,1)}, \iota\lambda_{(0,4)}\rangle = 1$, we have two possibilities for the decomposition of $[\iota\lambda_{(3,1)}]$:

$$[\iota\lambda_{(3,1)}] = \begin{cases} [\iota\lambda_{(2,0)}] \oplus [\iota\lambda^{(1)}_{(1,2)}] \oplus [\iota\lambda^{(1)}_{(3,1)}] & \text{case } (i), \\ [\iota\lambda_{(2,0)}] \oplus [\iota\lambda^{(2)}_{(1,2)}] \oplus [\iota\lambda^{(1)}_{(0,4)}] & \text{case } (ii). \end{cases} \qquad (5.76)$$

Similarly,

$$[\iota\lambda_{(1,3)}] = \begin{cases} [\iota\lambda_{(0,2)}] \oplus [\iota\lambda^{(1)}_{(2,1)}] \ominus [\iota\lambda^{(1)}_{(1,3)}] & \text{case } (i'), \\ [\iota\lambda_{(0,2)}] \oplus [\iota\lambda^{(2)}_{(2,1)}] \ominus [\iota\lambda^{(1)}_{(4,0)}] & \text{case } (ii'), \end{cases} \qquad (5.77)$$

Since $\langle \iota\lambda_{(2,2)}, \iota\lambda_{(2,2)}\rangle = 3$, $\langle \iota\lambda_{(2,2)}, \iota\lambda_{(1,1)}\rangle = 1$, $\langle \iota\lambda_{(2,2)}, \iota\lambda_{(3,0)}\rangle = 1$ and $\langle \iota\lambda_{(2,2)}, \iota\lambda_{(0,3)}\rangle = 1$, we have $[\iota\lambda_{(2,2)}] = [\iota\lambda_{(1,1)}] \oplus [\iota\lambda^{(j_1)}_{(3,0)}] \ominus [\iota\lambda^{(j_2)}_{(0,3)}]$ for $j_1, j_2 \in \{1,2\}$. We again have the freedom to choose, without loss of generality, $j_1 = j_2 = 1$, so that

$$[\iota\lambda_{(2,2)}] = [\iota\lambda_{(1,1)}] \ominus [\iota\lambda^{(1)}_{(3,0)}] \oplus [\iota\lambda^{(1)}_{(0,3)}]. \qquad (5.78)$$

At tier 5, $\langle \iota\lambda_{(5,0)}, \iota\lambda_{(5,0)}\rangle = \langle \iota\lambda_{(5,0)}, \iota\lambda_{(0,4)}\rangle = 2$ giving $[\iota\lambda_{(5,0)}] = [\iota\lambda_{(0,4)}]$, and similarly $[\iota\lambda_{(0,5)}] = [\iota\lambda_{(4,0)}]$. Since $\langle \iota\lambda_{(3,2)}, \iota\lambda_{(3,2)}\rangle = 4$, $\langle \iota\lambda_{(3,2)}, \iota\lambda_{(1,0)}\rangle = 1$, $\langle \iota\lambda_{(3,2)}, \iota\lambda_{(0,2)}\rangle = 1$ and $\langle \iota\lambda_{(3,2)}, \iota\lambda_{(2,1)}\rangle = 2$, we have $[\iota\lambda_{(3,2)}] = [\iota\lambda_{(1,0)}] \oplus [\iota\lambda_{(0,2)}] \oplus [\iota\lambda^{(1)}_{(2,1)}] \oplus [\iota\lambda^{(2)}_{(2,1)}]$, and similarly $[\iota\lambda_{(2,3)}] = [\iota\lambda_{(0,1)}] \oplus [\iota\lambda_{(2,0)}] \oplus [\iota\lambda^{(1)}_{(1,2)}] \oplus [\iota\lambda^{(2)}_{(1,2)}]$. We have $\langle \iota\lambda_{(4,1)}, \iota\lambda_{(4,1)}\rangle = \langle \iota\lambda_{(4,1)}, \iota\lambda_{(1,4)}\rangle = \langle \iota\lambda_{(1,4)}, \iota\lambda_{(1,4)}\rangle = 3$ so that $[\iota\lambda_{(4,1)}] = [\iota\lambda_{(1,4)}]$. Since $\langle \iota\lambda_{(4,1)}, \iota\lambda_{(1,1)}\rangle = 1$, $\langle \iota\lambda_{(4,1)}, \iota\lambda_{(2,2)}\rangle = 2$, $\langle \iota\lambda_{(4,1)}, \iota\lambda_{(3,0)}\rangle = 1$ and $\langle \iota\lambda_{(4,1)}, \iota\lambda_{(0,3)}\rangle = 1$, we have two possibilities for the decomposition of $[\iota\lambda_{(4,1)}]$:

$$[\iota\lambda_{(4,1)}] = \begin{cases} [\iota\lambda_{(1,1)}] \oplus [\iota\lambda^{(1)}_{(3,0)}] \oplus [\iota\lambda^{(2)}_{(0,3)}] & \text{case I}, \\ [\iota\lambda_{(1,1)}] \oplus [\iota\lambda^{(2)}_{(3,0)}] \oplus [\iota\lambda^{(1)}_{(0,3)}] & \text{case II}. \end{cases} \qquad (5.79)$$

Then we see that no new irreducible sectors appear at tier 5. We also have at tier 6, $\langle \iota\lambda_{(5,1)}, \iota\lambda_{(5,1)} \rangle = \langle \iota\lambda_{(5,1)}, \iota\lambda_{(1,3)} \rangle = 3$ giving $[\iota\lambda_{(5,1)}] = [\iota\lambda_{(1,3)}]$. Case $(i)(i')$ gives 16 irreducible sectors, whilst case $(ii)(ii')$ gives 18 irreducibles, and therefore by looking at $\mathrm{tr}(Z)$ for the level 12 modular invariants Z we see that neither of these cases is possible. Case $(ii)(i')$ is the 'conjugate' of case $(i)(ii')$, that is, we replace each irreducible sector $[\iota\lambda]$ in case $(i)(ii')$ by $[\overline{\iota\lambda}]$ in case $(ii)(i')$. We therefore only need to consider case $(i)(ii')$, which has seventeen irreducible sectors: $[\lambda_{(0,0)}]$, $[\lambda_{(1,0)}]$, $[\lambda_{(0,1)}]$, $[\lambda_{(2,0)}]$, $[\lambda_{(1,1)}]$, $[\lambda_{(0,2)}]$, $[\lambda^{(1)}_{(3,0)}]$, $[\lambda^{(2)}_{(3,0)}]$, $[\lambda^{(1)}_{(0,3)}]$, $[\lambda^{(2)}_{(0,3)}]$, $[\lambda^{(1)}_{(2,1)}]$, $[\lambda^{(2)}_{(2,1)}]$, $[\lambda^{(1)}_{(1,2)}]$, $[\lambda^{(2)}_{(1,2)}]$, $[\lambda^{(1)}_{(4,0)}]$, $[\lambda^{(1)}_{(0,4)}]$ and $[\lambda^{(1)}_{(3,1)}]$.

We now consider the sector products for these irreducible sectors, where we again denote by $[\rho]$ the irreducible N-N sector $[\lambda_{(1,0)}]$. The products $[\iota\lambda][\rho]$ are inherited from those for the N-N system for $\lambda = \lambda_{(\mu_1,\mu_2)}$ such that $\mu_1 + \mu_2 \le 2$, and we use (5.70)-(5.73) to decompose into irreducibles where necessary, e.g.

$$[\iota\lambda_{(0,2)}][\rho] = [\iota\lambda_{(0,1)}] \oplus [\iota\lambda_{(1,2)}] = [\iota\lambda_{(0,1)}] \oplus [\iota\lambda^{(1)}_{(1,2)}] \oplus [\iota\lambda^{(2)}_{(1,2)}]. \tag{5.80}$$

From $[\iota\lambda_{(2,1)}][\rho] = [\iota\lambda_{(2,0)}] \oplus [\iota\lambda_{(1,2)}] \oplus [\iota\lambda_{(3,1)}]$ and (5.71) we obtain

$$([\iota\lambda^{(1)}_{(2,1)}] \oplus [\iota\lambda^{(2)}_{(2,1)}])[\rho] = 2[\iota\lambda_{(2,0)}] \oplus 2[\iota\lambda^{(1)}_{(1,2)}] \oplus [\iota\lambda^{(2)}_{(1,2)}] \oplus [\iota\lambda^{(1)}_{(3,1)}]. \tag{5.81}$$

Similarly, by considering $[\iota\lambda_{(1,3)}][\rho]$ and $[\iota\lambda_{(4,0)}][\rho]$, and using (5.77) and (5.74) we have

$$\begin{aligned}
([\iota\lambda^{(2)}_{(2,1)}] \oplus [\iota\lambda^{(1)}_{(4,0)}])[\rho] &= [\iota\lambda_{(2,0)}] \oplus 2[\iota\lambda^{(1)}_{(1,2)}] \oplus [\iota\lambda^{(2)}_{(1,2)}] \oplus [\iota\lambda^{(1)}_{(0,4)}], & (5.82)\\
([\iota\lambda^{(1)}_{(2,1)}] \oplus [\iota\lambda^{(1)}_{(4,0)}])[\rho] &= [\iota\lambda_{(2,0)}] \oplus 2[\iota\lambda^{(1)}_{(1,2)}] \oplus [\iota\lambda^{(1)}_{(3,1)}] \oplus [\iota\lambda^{(1)}_{(0,4)}]. & (5.83)
\end{aligned}$$

Then from (5.81)-(5.83) we find

$$\begin{aligned}
[\iota\lambda^{(1)}_{(2,1)}][\rho] &= [\iota\lambda_{(2,0)}] \oplus [\iota\lambda^{(1)}_{(1,2)}] \oplus [\iota\lambda^{(1)}_{(3,1)}], & (5.84)\\
[\iota\lambda^{(2)}_{(2,1)}][\rho] &= [\iota\lambda_{(2,0)}] \oplus [\iota\lambda^{(1)}_{(1,2)}] \oplus [\iota\lambda^{(2)}_{(1,2)}], & (5.85)\\
[\iota\lambda^{(1)}_{(4,0)}][\rho] &= [\iota\lambda^{(1)}_{(1,2)}] \oplus [\iota\lambda^{(1)}_{(0,4)}]. & (5.86)
\end{aligned}$$

Now we focus on case I. From $[\iota\lambda_{(3,0)}][\rho] = [\iota\lambda_{(4,0)}] \oplus [\iota\lambda_{(2,1)}]$ and (5.70) we obtain

$$([\iota\lambda^{(1)}_{(3,0)}] \oplus [\iota\lambda^{(2)}_{(3,0)}])[\rho] = 2[\iota\lambda^{(1)}_{(2,1)}] \oplus [\iota\lambda^{(2)}_{(2,1)}] \oplus [\iota\lambda^{(1)}_{(4,0)}]. \tag{5.87}$$

Similarly by considering $[\iota\lambda_{(0,3)}][\rho]$ we have

$$([\iota\lambda^{(1)}_{(0,3)}] \oplus [\iota\lambda^{(2)}_{(0,3)}])[\rho] = 2[\iota\lambda_{(0,2)}] \oplus [\iota\lambda^{(2)}_{(2,1)}] \oplus [\iota\lambda^{(1)}_{(4,0)}]. \tag{5.88}$$

From $[\iota\lambda_{(2,2)}][\rho] = [\iota\lambda_{(2,1)}] \oplus [\iota\lambda_{(1,3)}] \oplus [\iota\lambda_{(3,2)}]$ and (5.78) we find

$$([\iota\lambda^{(1)}_{(3,0)}] \oplus [\iota\lambda^{(1)}_{(0,3)}])[\rho] = [\iota\lambda_{(0,2)}] \oplus [\iota\lambda^{(1)}_{(2,1)}] \oplus 2[\iota\lambda^{(2)}_{(2,1)}] \oplus [\iota\lambda^{(1)}_{(4,0)}], \tag{5.89}$$

whilst from $[\iota\lambda_{(4,1)}][\rho] = [\iota\lambda_{(4,0)}] \oplus [\iota\lambda_{(3,2)}] \oplus [\iota\lambda_{(5,1)}]$ and (5.79) we find

$$([\iota\lambda^{(1)}_{(3,0)}] \oplus [\iota\lambda^{(2)}_{(0,3)}])[\rho] = [\iota\lambda_{(1,0)}] \oplus 2[\iota\lambda_{(0,2)}] \oplus 2[\iota\lambda^{(1)}_{(2,1)}] \oplus 2[\iota\lambda^{(2)}_{(2,1)}] \oplus 2[\iota\lambda^{(1)}_{(4,0)}]. \tag{5.90}$$

Then from (5.87)-(5.90) we obtain

$$\begin{aligned}
[\iota\lambda^{(1)}_{(3,0)}][\rho] &= [\iota\lambda^{(1)}_{(2,1)}] \oplus [\iota\lambda^{(2)}_{(2,1)}] \oplus [\iota\lambda^{(1)}_{(4,0)}], && (5.91)\\
[\iota\lambda^{(2)}_{(3,0)}][\rho] &= [\iota\lambda^{(1)}_{(2,1)}], && (5.92)\\
[\iota\lambda^{(1)}_{(0,3)}][\rho] &= [\iota\lambda_{(0,2)}] \oplus [\iota\lambda^{(2)}_{(2,1)}], && (5.93)\\
[\iota\lambda^{(2)}_{(0,3)}][\rho] &= [\iota\lambda_{(0,2)}] \oplus [\iota\lambda^{(1)}_{(4,0)}]. && (5.94)
\end{aligned}$$

Next, by considering $[\iota\lambda][\rho]$ for $\lambda = \lambda_{(1,2)}, \lambda_{(3,1)}, \lambda_{(0,4)}$, and (5.72), (5.76) and (5.75) we obtain

$$\begin{aligned}
([\iota\lambda^{(1)}_{(1,2)}] \oplus [\iota\lambda^{(2)}_{(1,2)}])[\rho] &= 2[\iota\lambda_{(1,1)}] \oplus [\iota\lambda^{(1)}_{(3,0)}] \oplus 2[\iota\lambda^{(1)}_{(0,3)}] \oplus [\iota\lambda^{(2)}_{(0,3)}], && (5.95)\\
([\iota\lambda^{(1)}_{(1,2)}] \oplus [\iota\lambda^{(1)}_{(3,1)}])[\rho] &= [\iota\lambda_{(1,1)}] \oplus 2[\iota\lambda^{(1)}_{(3,0)}] \oplus [\iota\lambda^{(1)}_{(0,3)}] \oplus [\iota\lambda^{(2)}_{(0,3)}], && (5.96)\\
([\iota\lambda^{(1)}_{(1,2)}] \oplus [\iota\lambda^{(1)}_{(0,4)}])[\rho] &= [\iota\lambda_{(1,1)}] \oplus [\iota\lambda^{(1)}_{(3,0)}] \oplus [\iota\lambda^{(1)}_{(0,3)}] \oplus 2[\iota\lambda^{(2)}_{(0,3)}]. && (5.97)
\end{aligned}$$

We see from (5.95)-(5.97) that $[\iota\lambda^{(1)}_{(1,2)}][\rho] \subset [\iota\lambda_{(1,1)}] \oplus [\iota\lambda^{(1)}_{(3,0)}] \oplus [\iota\lambda^{(1)}_{(0,3)}] \oplus [\iota\lambda^{(2)}_{(0,3)}]$. From (5.80) and (5.84)-(5.86) we see that $[\iota\lambda^{(1)}_{(1,2)}][\overline{\rho}] = [\iota\lambda_{(0,2)}] \oplus [\iota\lambda^{(1)}_{(2,1)}] \oplus [\iota\lambda^{(2)}_{(2,1)}] \oplus [\iota\lambda^{(1)}_{(4,0)}]$, since $\langle \iota\lambda^{(1)}_{(1,2)} \circ \overline{\rho}, \iota\lambda \rangle = \langle \iota\lambda^{(1)}_{(1,2)}, \iota\lambda^{(1)}_{(1,2)} \circ \rho \rangle = 1$ for $\lambda = \lambda_{(0,2)}, \lambda^{(1)}_{(2,1)}, \lambda^{(2)}_{(2,1)}, \lambda^{(1)}_{(4,0)}$. Then $\langle \iota\lambda^{(1)}_{(1,2)} \circ \rho, \iota\lambda^{(1)}_{(1,2)} \circ \rho \rangle = \langle \iota\lambda^{(1)}_{(1,2)} \circ \overline{\rho}, \iota\lambda^{(1)}_{(1,2)} \circ \overline{\rho} \rangle = 4$ implies that we must have $[\iota\lambda^{(1)}_{(1,2)}][\rho] = [\iota\lambda_{(1,1)}] \oplus [\iota\lambda^{(1)}_{(3,0)}] \oplus [\iota\lambda^{(1)}_{(0,3)}] \oplus [\iota\lambda^{(2)}_{(0,3)}]$. Then from (5.95)-(5.97) we obtain

$$\begin{aligned}
[\iota\lambda^{(2)}_{(1,2)}][\rho] &= [\iota\lambda_{(1,1)}] \oplus [\iota\lambda^{(1)}_{(0,3)}],\\
[\iota\lambda^{(2)}_{(3,0)}][\rho] &= [\iota\lambda^{(2)}_{(0,3)}],\\
[\iota\lambda^{(1)}_{(0,3)}][\rho] &= [\iota\lambda^{(1)}_{(3,0)}].
\end{aligned}$$

It is easy to check that the fusion graph with respect to $[\rho]$ obtained in case I is just the graph $\mathcal{E}^{(12)}_5$.

For case II, we again have (5.95), and by considering $[\iota\lambda_{(3,1)}][\rho] = [\iota\lambda_{(3,0)}] \oplus [\iota\lambda_{(2,2)}] \oplus [\iota\lambda_{(4,1)}]$ and (5.70), (5.78) and (5.79) we obtain

$$([\iota\lambda^{(1)}_{(1,2)}] \oplus [\iota\lambda^{(1)}_{(3,1)}])[\rho] = [\iota\lambda_{(1,1)}] \oplus [\iota\lambda^{(1)}_{(3,0)}] \oplus 2[\iota\lambda^{(2)}_{(3,0)}] \oplus 2[\iota\lambda^{(1)}_{(0,3)}], \tag{5.98}$$

and similarly from $[\iota\lambda_{(0,4)}][\rho]$, (5.75), (5.73) and (5.79) we obtain

$$([\iota\lambda^{(1)}_{(1,2)}] \oplus [\iota\lambda^{(1)}_{(0,4)}])[\rho] = [\iota\lambda_{(1,1)}] \oplus [\iota\lambda^{(2)}_{(3,0)}] \oplus 2[\iota\lambda^{(1)}_{(0,3)}] \oplus [\iota\lambda^{(2)}_{(0,3)}]. \tag{5.99}$$

Then from (5.95), (5.98) and (5.99) we see that $[\iota\lambda^{(1)}_{(1,2)}][\rho] \subset [\iota\lambda_{(1,1)}] \oplus 2[\iota\lambda^{(1)}_{(0,3)}]$. Since $\langle \iota\lambda^{(1)}_{(1,2)} \circ \rho, \iota\lambda^{(1)}_{(1,2)} \circ \rho \rangle = \langle \iota\lambda^{(1)}_{(1,2)} \circ \overline{\rho}, \iota\lambda^{(1)}_{(1,2)} \circ \overline{\rho} \rangle = 4$, we must have $[\iota\lambda^{(1)}_{(1,2)}][\rho] = 2[\iota\lambda^{(1)}_{(0,3)}]$. Then from (5.95) we obtain $[\iota\lambda^{(2)}_{(1,2)}][\rho] = 2[\iota\lambda_{(1,1)}] \oplus [\iota\lambda^{(1)}_{(3,0)}] \oplus [\iota\lambda^{(2)}_{(0,3)}]$, and we have $\langle \iota\lambda^{(2)}_{(1,2)} \circ \overline{\rho}, \iota\lambda^{(2)}_{(1,2)} \circ \overline{\rho} \rangle = \langle \iota\lambda^{(1)}_{(1,2)} \circ \rho, \iota\lambda^{(1)}_{(1,2)} \circ \rho \rangle = 6$. From (5.80) and (5.84)-(5.86) we see that $[\iota\lambda^{(2)}_{(1,2)}][\overline{\rho}] = [\iota\lambda_{(0,2)}] \oplus [\iota\lambda^{(2)}_{(2,1)}]$, giving $\langle \iota\lambda^{(2)}_{(1,2)} \circ \overline{\rho}, \iota\lambda^{(2)}_{(1,2)} \circ \overline{\rho} \rangle = 2 \neq 6$, which is a contradiction. Then we reject case II.

Then the only possibility for the graph of the M-N system is $\mathcal{E}_5^{(12)}$, and the modular invariant for θ is $Z_{\mathcal{E}_5^{(12)}}$.

5.2.5 $\mathcal{A}^{(n)*}$

We compute the fusion graph for the case $n = 12$. It appears that the results will carry over to all other n, however we have not been able to show this in general. For the graph $\mathcal{A}^{(12)*}$, we have $[\theta] = \bigoplus_\mu [\lambda_\mu]$, where the direct sum is over all representations μ on $\mathcal{A}^{(12)}$. Then computing $\langle \iota\lambda, \iota\mu \rangle = \langle \lambda, \theta\mu \rangle$ for λ, μ on $\mathcal{A}^{(12)}$, we find that $\langle \iota\lambda_{(\mu_2,\mu_1)}, \iota\lambda_{(\mu_2,\mu_1)} \rangle = \langle \iota\lambda_{(\mu_2,\mu_1)}, \iota\lambda_{(\mu_1,\mu_2)} \rangle$ we have $[\iota\lambda_{(\mu_2,\mu_1)}] = [\iota\lambda_{(\mu_1,\mu_2)}]$ for all (μ_1, μ_2) on $\mathcal{A}^{(12)}$. At tier 0 we have $\langle \iota\lambda_{(0,0)}, \iota\lambda_{(0,0)} \rangle = 1$. At tier 1, $\langle \iota\lambda_{(1,0)}, \iota\lambda_{(1,0)} \rangle = 2$ and $\langle \iota\lambda_{(1,0)}, \iota\lambda_{(0,0)} \rangle = 1$, giving

$$[\iota\lambda_{(1,0)}] = [\iota\lambda_{(0,0)}] \oplus [\iota\lambda^{(1)}_{(1,0)}]. \tag{5.100}$$

At tier 2 we have $\langle \iota\lambda_{(2,0)}, \iota\lambda_{(2,0)} \rangle = 3$ and $\langle \iota\lambda_{(2,0)}, \iota\lambda_{(1,0)} \rangle = 2$, so $[\iota\lambda_{(2,0)}] = [\iota\lambda_{(0,0)}] \oplus [\iota\lambda^{(1)}_{(1,0)}] \oplus [\iota\lambda^{(1)}_{(2,0)}]$. We also have $\langle \iota\lambda_{(1,1)}, \iota\lambda_{(1,1)} \rangle = 6$, $\langle \iota\lambda_{(1,1)}, \iota\lambda_{(0,0)} \rangle = 1$, $\langle \iota\lambda_{(1,1)}, \iota\lambda_{(1,0)} \rangle = 3$ and $\langle \iota\lambda_{(1,1)}, \iota\lambda_{(2,0)} \rangle = 4$, giving $[\iota\lambda_{(1,1)}] = [\iota\lambda_{(0,0)}] \oplus 2[\iota\lambda^{(1)}_{(1,0)}] \oplus [\iota\lambda^{(1)}_{(2,0)}]$. At tier 3 we have $\langle \iota\lambda_{(3,0)}, \iota\lambda_{(3,0)} \rangle = 4$ and $\langle \iota\lambda_{(3,0)}, \iota\lambda_{(2,0)} \rangle = 3$, so $[\iota\lambda_{(3,0)}] = [\iota\lambda_{(0,0)}] \oplus [\iota\lambda^{(1)}_{(1,0)}] \oplus [\iota\lambda^{(1)}_{(2,0)}] \oplus [\iota\lambda^{(1)}_{(3,0)}]$. We also have $\langle \iota\lambda_{(2,1)}, \iota\lambda_{(2,1)} \rangle = 10$, $\langle \iota\lambda_{(2,1)}, \iota\lambda_{(0,0)} \rangle = 1$, $\langle \iota\lambda_{(2,1)}, \iota\lambda_{(1,0)} \rangle = 3$, $\langle \iota\lambda_{(2,1)}, \iota\lambda_{(2,0)} \rangle = 5$ and $\langle \iota\lambda_{(2,1)}, \iota\lambda_{(3,0)} \rangle = 6$, giving $[\iota\lambda_{(2,1)}] = [\iota\lambda_{(0,0)}] \oplus 2[\iota\lambda^{(1)}_{(1,0)}] \ominus 2[\iota\lambda^{(1)}_{(2,0)}] \oplus [\iota\lambda^{(1)}_{(3,0)}]$. Similarly, at tier 4 we find

$$\begin{aligned}
[\iota\lambda_{(4,0)}] &= [\iota\lambda_{(0,0)}] \oplus [\iota\lambda^{(1)}_{(1,0)}] \ominus [\iota\lambda^{(1)}_{(2,0)}] \oplus [\iota\lambda^{(1)}_{(3,0)}] \oplus [\iota\lambda^{(1)}_{(4,0)}], \\
[\iota\lambda_{(3,1)}] &= [\iota\lambda_{(0,0)}] \oplus 2[\iota\lambda^{(1)}_{(1,0)}] \ominus 2[\iota\lambda^{(1)}_{(2,0)}] \oplus 2[\iota\lambda^{(1)}_{(3,0)}] \oplus [\iota\lambda^{(1)}_{(4,0)}], \\
[\iota\lambda_{(2,2)}] &= [\iota\lambda_{(0,0)}] \oplus 2[\iota\lambda^{(1)}_{(1,0)}] \ominus 3[\iota\lambda^{(1)}_{(2,0)}] \oplus 2[\iota\lambda^{(1)}_{(3,0)}] \oplus [\iota\lambda^{(1)}_{(4,0)}],
\end{aligned}$$

and at tier 5:

$$\begin{aligned}
[\iota\lambda_{(5,0)}] &= [\iota\lambda_{(4,0)}], \\
[\iota\lambda_{(4,1)}] &= [\iota\lambda_{(0,0)}] \ominus 2[\iota\lambda^{(1)}_{(1,0)}] \oplus 2[\iota\lambda^{(1)}_{(2,0)}] \oplus 2[\iota\lambda^{(1)}_{(3,0)}] \oplus 2[\iota\lambda^{(1)}_{(4,0)}] \oplus [\iota\lambda^{(1)}_{(4,0)}], \\
[\iota\lambda_{(3,2)}] &= [\iota\lambda_{(0,0)}] \oplus 2[\iota\lambda^{(1)}_{(1,0)}] \oplus 3[\iota\lambda^{(1)}_{(2,0)}] \oplus 3[\iota\lambda^{(1)}_{(3,0)}] \oplus 2[\iota\lambda^{(1)}_{(4,0)}] \oplus [\iota\lambda^{(1)}_{(4,0)}].
\end{aligned}$$

Then we have six irreducible sectors $[\iota\lambda_{(0,0)}]$, $[\iota\lambda^{(1)}_{(1,0)}]$, $[\iota\lambda^{(1)}_{(2,0)}]$, $[\iota\lambda^{(1)}_{(3,0)}]$, $[\iota\lambda^{(1)}_{(4,0)}]$ and $[\iota\lambda^{(1)}_{(5,0)}]$.

We now compute the sector products. We have $[\iota\lambda_{(0,0)}][\rho] = [\iota\lambda_{(1,0)}] = [\iota\lambda_{(0,0)}]\oplus[\iota\lambda^{(1)}_{(1,0)}]$. From $[\iota\lambda_{(1,0)}][\rho] = [\iota\lambda_{(2,0)}] \oplus [\iota\lambda_{(0,1)}] = 2[\iota\lambda_{(0,0)}] \oplus 2[\iota\lambda^{(1)}_{(1,0)}] \oplus [\iota\lambda^{(1)}_{(2,0)}]$ and (5.100) we find $[\iota\lambda^{(1)}_{(1,0)}][\rho] = 2[\iota\lambda_{(0,0)}]\oplus 2[\iota\lambda^{(1)}_{(1,0)}]\oplus[\iota\lambda^{(1)}_{(2,0)}]\ominus([\iota\lambda_{(0,0)}]\oplus[\iota\lambda^{(1)}_{(1,0)}]) = [\iota\lambda_{(0,0)}]\oplus[\iota\lambda^{(1)}_{(1,0)}]\oplus[\iota\lambda^{(1)}_{(2,0)}]$. Similarly, we find

$$\begin{aligned}
[\iota\lambda^{(1)}_{(2,0)}][\rho] &= [\iota\lambda^{(1)}_{(1,0)}] \oplus [\iota\lambda^{(1)}_{(2,0)}] \oplus [\iota\lambda^{(1)}_{(3,0)}], \\
[\iota\lambda^{(1)}_{(3,0)}][\rho] &= [\iota\lambda^{(1)}_{(2,0)}] \oplus [\iota\lambda^{(1)}_{(3,0)}] \oplus [\iota\lambda^{(1)}_{(4,0)}], \\
[\iota\lambda^{(1)}_{(4,0)}][\rho] &= [\iota\lambda^{(1)}_{(3,0)}] \oplus [\iota\lambda^{(1)}_{(4,0)}],
\end{aligned}$$

and the fusion graph with respect to $[\rho]$ is $\mathcal{A}^{(12)*}$. The associated modular invariant is $Z_{\mathcal{A}^{(12)*}}$.

In the case above, since $n = 12$ is even, we have $[\iota\lambda_{(5,0)}] = [\iota\lambda_{(4,0)}]$ and so $[\iota\lambda_{(4,0)}][\rho] = [\iota\lambda_{(5,0)}] \oplus [\iota\lambda_{(3,1)}] = [\iota\lambda_{(4,0)}] \oplus [\iota\lambda_{(3,1)}]$. This leads to $[\iota\lambda^{(1)}_{(4,0)}][\rho] \supset [\iota\lambda^{(1)}_{(4,0)}]$, and there is a loop from $[\iota\lambda^{(1)}_{(4,0)}]$ to itself in the fusion graph. However, when n is odd, e.g. for $n = 11$, we have instead $[\iota\lambda_{(5,0)}] = [\iota\lambda_{(3,0)}]$ so $[\iota\lambda_{(4,0)}][\rho] = [\iota\lambda_{(5,0)}] \oplus [\iota\lambda_{(3,1)}] = [\iota\lambda_{(3,0)}] \oplus [\iota\lambda_{(3,1)}]$. This causes $[\iota\lambda^{(1)}_{(4,0)}][\rho] \not\supset [\iota\lambda^{(1)}_{(4,0)}]$, hence there is a loop from $[\iota\lambda^{(1)}_{(4,0)}]$ to itself in the fusion graph for the $n = 11$ case.

5.2.6 $\mathcal{D}^{(n)*}$

We compute the fusion graph for the case $n = 12$. For the graph $\mathcal{D}(12)*$, we have $[\theta] = \bigoplus_\mu [\lambda_\mu]$, where the direct sum is over all representations μ of colour 0 on $\mathcal{A}^{(12)}$. At tier 0 we have $\langle \iota\lambda_{(0,0)}, \iota\lambda_{(0,0)} \rangle = 1$. At tier 1, $\langle \iota\lambda_{(1,0)}, \iota\lambda_{(1,0)} \rangle = 2$ and $\langle \iota\lambda_{(1,0)}, \iota\lambda_{(0,0)} \rangle = 0$, and similarly for $\iota\lambda_{(0,1)}$, giving

$$[\iota\lambda_{(1,0)}] = [\iota\lambda^{(1)}_{(1,0)}] \oplus [\iota\lambda^{(2)}_{(1,0)}], \tag{5.101}$$
$$[\iota\lambda_{(0,1)}] = [\iota\lambda^{(1)}_{(0,1)}] \oplus [\iota\lambda^{(2)}_{(0,1)}]. \tag{5.102}$$

At tier 2 we have $\langle \iota\lambda_{(2,0)}, \iota\lambda_{(2,0)} \rangle = 3$ and $\langle \iota\lambda_{(2,0)}, \iota\lambda_{(0,1)} \rangle = 1$, and similarly for $\iota\lambda_{(0,2)}$, so we have

$$[\iota\lambda_{(2,0)}] = [\iota\lambda^{(1)}_{(0,1)}] \oplus [\iota\lambda^{(2)}_{(0,1)}] \oplus [\iota\lambda^{(1)}_{(2,0)}], \tag{5.103}$$
$$[\iota\lambda_{(0,2)}] = [\iota\lambda^{(1)}_{(1,0)}] \oplus [\iota\lambda^{(2)}_{(1,0)}] \oplus [\iota\lambda^{(1)}_{(0,2)}]. \tag{5.104}$$

For $\iota\lambda_{(1,1)}$ we have $\langle \iota\lambda_{(1,1)}, \iota\lambda_{(1,1)} \rangle = 6$ and $\langle \iota\lambda_{(1,1)}, \iota\lambda_{(0,0)} \rangle = 1$, so there are two possibilities for the decomposition of $[\iota\lambda_{(1,1)}]$ as irreducible sectors:

$$[\iota\lambda_{(1,1)}] = \begin{cases} [\iota\lambda_{(0,0)}] \oplus 2[\iota\lambda^{(1)}_{(1,1)}] \oplus [\iota\lambda^{(2)}_{(1,1)}] & \text{case I,} \\ [\iota\lambda_{(0,0)}] \oplus [\iota\lambda^{(1)}_{(1,1)}] \oplus [\iota\lambda^{(2)}_{(1,1)}] \oplus [\iota\lambda^{(3)}_{(1,1)}] \oplus [\iota\lambda^{(4)}_{(1,1)}] \oplus [\iota\lambda^{(5)}_{(1,1)}] & \text{case II.} \end{cases} \tag{5.105}$$

At tier 3 we have $\langle\iota\lambda_{(3,0)},\iota\lambda_{(3,0)}\rangle = 4$, $\langle\iota\lambda_{(3,0)},\iota\lambda_{(1,1)}\rangle = 4$ and $\langle\iota\lambda_{(3,0)},\iota\lambda_{(0,0)}\rangle = 1$, giving

$$[\iota\lambda_{(3,0)}] = \begin{cases} [\iota\lambda_{(0,0)}] \oplus [\iota\lambda_{(1,1)}^{(1)}] \oplus [\iota\lambda_{(1,1)}^{(2)}] \oplus [\iota\lambda_{(3,0)}^{(1)}] & \text{for case I,} \\ [\iota\lambda_{(0,0)}] \oplus [\iota\lambda_{(1,1)}^{(1)}] \oplus [\iota\lambda_{(1,1)}^{(2)}] \oplus [\iota\lambda_{(1,1)}^{(3)}] & \text{for case II.} \end{cases} \tag{5.106}$$

Then we see that for case II $[\iota\lambda_{(1,1)}] \supset [\iota\lambda_{(3,0)}]$. However, this contradicts the following values of the inner-products at tier 6, $\langle\iota\lambda_{(3,3)},\iota\lambda_{(1,1)}\rangle = 8$ and $\langle\iota\lambda_{(3,3)},\iota\lambda_{(3,0)}\rangle = 10$. So we reject case II.

Continuing at tier 3 we have $\langle\iota\lambda_{(0,3)},\iota\lambda_{(0,3)}\rangle = \langle\iota\lambda_{(0,3)},\iota\lambda_{(3,0)}\rangle = 4$, so that $[\iota\lambda_{(0,3)}] = [\iota\lambda_{(3,0)}]$. From $\langle\iota\lambda_{(2,1)},\iota\lambda_{(2,1)}\rangle = 10$, $\langle\iota\lambda_{(2,1)},\iota\lambda_{(1,0)}\rangle = 3$ and $\langle\iota\lambda_{(2,1)},\iota\lambda_{(0,2)}\rangle = 5$, and similarly for $\iota\lambda_{(1,2)}$, we have

$$[\iota\lambda_{(2,1)}] = 2[\iota\lambda_{(1,0)}^{(1)}] \oplus [\iota\lambda_{(1,0)}^{(2)}] \oplus 2[\iota\lambda_{(0,2)}^{(1)}] \oplus [\iota\lambda_{(2,1)}^{(1)}], \tag{5.107}$$

$$[\iota\lambda_{(1,2)}] = 2[\iota\lambda_{(0,1)}^{(1)}] \oplus [\iota\lambda_{(0,1)}^{(2)}] \oplus 2[\iota\lambda_{(2,0)}^{(1)}] \oplus [\iota\lambda_{(1,2)}^{(1)}]. \tag{5.108}$$

Next, at tier 4, we have $\langle\iota\lambda_{(4,0)},\iota\lambda_{(4,0)}\rangle = 5$, $\langle\iota\lambda_{(4,0)},\iota\lambda_{(1,0)}\rangle = 2$, $\langle\iota\lambda_{(4,0)},\iota\lambda_{(0,2)}\rangle = 3$ and $\langle\iota\lambda_{(4,0)},\iota\lambda_{(2,1)}\rangle = 6$, so there are two possibilities for the decomposition of $[\iota\lambda_{(4,0)}]$, and similarly for $[\iota\lambda_{(0,4)}]$:

$$[\iota\lambda_{(4,0)}] = \begin{cases} [\iota\lambda_{(1,0)}^{(1)}] \oplus [\iota\lambda_{(1,0)}^{(2)}] \oplus [\iota\lambda_{(0,2)}^{(1)}] \oplus [\iota\lambda_{(0,2)}^{(2)}] \oplus [\iota\lambda_{(4,0)}^{(1)}] & \text{case } (i), \\ 2[\iota\lambda_{(1,0)}^{(1)}] \oplus [\iota\lambda_{(0,2)}^{(1)}] & \text{case } (ii), \end{cases} \tag{5.109}$$

$$[\iota\lambda_{(0,4)}] = \begin{cases} [\iota\lambda_{(0,1)}^{(1)}] \oplus [\iota\lambda_{(0,1)}^{(2)}] \oplus [\iota\lambda_{(2,0)}^{(1)}] \oplus [\iota\lambda_{(2,0)}^{(2)}] \oplus [\iota\lambda_{(0,4)}^{(1)}] & \text{case } (i'), \\ 2[\iota\lambda_{(0,1)}^{(1)}] \oplus [\iota\lambda_{(2,0)}^{(1)}] & \text{case } (ii'). \end{cases} \tag{5.110}$$

Since $\langle\iota\lambda_{(3,1)},\iota\lambda_{(3,1)}\rangle = 14$, $\langle\iota\lambda_{(3,1)},\iota\lambda_{(0,1)}\rangle = 3$, $\langle\iota\lambda_{(3,1)},\iota\lambda_{(2,0)}\rangle = 5$, $\langle\iota\lambda_{(3,1)},\iota\lambda_{(1,2)}\rangle = 11$ and $\langle\iota\lambda_{(3,1)},\iota\lambda_{(0,4)}\rangle = 8$, then

$$[\iota\lambda_{(3,1)}] = \begin{cases} 2[\iota\lambda_{(0,1)}^{(1)}] \oplus [\iota\lambda_{(0,1)}^{(2)}] \oplus 2[\iota\lambda_{(2,0)}^{(1)}] \oplus 2[\iota\lambda_{(1,2)}^{(1)}] \oplus [\iota\lambda_{(0,4)}^{(1)}] & \text{for case } (i'), \\ 3[\iota\lambda_{(0,1)}^{(1)}] \oplus 2[\iota\lambda_{(2,0)}^{(1)}] \oplus [\iota\lambda_{(1,2)}^{(1)}] & \text{for case } (ii'). \end{cases} \tag{5.111}$$

Similarly, for $[\iota\lambda_{(1,3)}]$,

$$[\iota\lambda_{(1,3)}] = \begin{cases} 2[\iota\lambda_{(1,0)}^{(1)}] \oplus [\iota\lambda_{(1,0)}^{(2)}] \oplus 2[\iota\lambda_{(0,2)}^{(1)}] \oplus 2[\iota\lambda_{(2,1)}^{(1)}] \oplus [\iota\lambda_{(4,0)}^{(1)}] & \text{for case } (i), \\ 3[\iota\lambda_{(1,0)}^{(1)}] \oplus 2[\iota\lambda_{(0,2)}^{(1)}] \oplus [\iota\lambda_{(2,1)}^{(1)}] & \text{for case } (ii). \end{cases} \tag{5.112}$$

From $\langle\iota\lambda_{(2,2)},\iota\lambda_{(2,2)}\rangle = 19$, $\langle\iota\lambda_{(2,2)},\iota\lambda_{(0,0)}\rangle = 1$, $\langle\iota\lambda_{(2,2)},\iota\lambda_{(1,1)}\rangle = 8$ and $\langle\iota\lambda_{(2,2)},\iota\lambda_{(3,0)}\rangle = 8$, we must have

$$[\iota\lambda_{(1,3)}] = [\iota\lambda_{(0,0)}] \oplus 2[\iota\lambda_{(1,1)}^{(1)}] \oplus 3[\iota\lambda_{(1,1)}^{(2)}] \oplus 2[\iota\lambda_{(3,0)}^{(1)}] \oplus [\iota\lambda_{(2,2)}^{(1)}]. \tag{5.113}$$

At tier 5 we have $\langle\iota\lambda_{(5,0)},\iota\lambda_{(5,0)}\rangle = \langle\iota\lambda_{(5,0)},\iota\lambda_{(0,4)}\rangle 5$, giving $[\iota\lambda_{(5,0)}] = [\iota\lambda_{(0,4)}]$, and similarly $[\iota\lambda_{(0,5)}] = [\iota\lambda_{(4,0)}]$. From $\langle\iota\lambda_{(3,2)},\iota\lambda_{(3,2)}\rangle = 27$, $\langle\iota\lambda_{(3,2)},\iota\lambda_{(1,0)}\rangle = 3$, $\langle\iota\lambda_{(3,2)},\iota\lambda_{(0,2)}\rangle = 6$, $\langle\iota\lambda_{(3,2)},\iota\lambda_{(2,1)}\rangle = 14$ and $\langle\iota\lambda_{(3,2)},\iota\lambda_{(1,3)}\rangle = 19$ we must have

$$[\iota\lambda_{(3,2)}] = \begin{cases} 2[\iota\lambda^{(1)}_{(1,0)}] \oplus [\iota\lambda^{(2)}_{(1,0)}] \oplus 3[\iota\lambda^{(1)}_{(0,2)}] \oplus 3[\iota\lambda^{(1)}_{(2,1)}] \oplus 2[\iota\lambda^{(1)}_{(4,0)}] & \text{for case } (i), \\ 3[\iota\lambda^{(1)}_{(1,0)}] \oplus 3[\iota\lambda^{(1)}_{(0,2)}] \oplus 2[\iota\lambda^{(1)}_{(2,1)}] \oplus 2[\iota\lambda^{(1)}_{(4,0)}] \ominus [\iota\lambda^{(1)}_{(3,2)}] & \text{for case } (ii). \end{cases} \tag{5.114}$$

However, case (ii) does not satisfy $\langle\iota\lambda_{(3,2)},\iota\lambda_{(4,0)}\rangle = 11$, and hence we discard it. Similarly we discard case (ii') since no possible decomposition of $[\iota\lambda_{(2,3)}]$ exists for that case. Then we are left with only the one case $(i)(i')$. We have

$$[\iota\lambda_{(2,3)}] = 2[\iota\lambda^{(1)}_{(0,1)}] \oplus [\iota\lambda^{(2)}_{(0,1)}] \oplus 3[\iota\lambda^{(1)}_{(2,0)}] \oplus 3[\iota\lambda^{(1)}_{(1,2)}] \oplus 2[\iota\lambda^{(1)}_{(0,4)}]. \tag{5.115}$$

From $\langle\iota\lambda_{(4,1)},\iota\lambda_{(4,1)}\rangle = 17$, $\langle\iota\lambda_{(4,1)},\iota\lambda_{(0,0)}\rangle = 1$, $\langle\iota\lambda_{(4,1)},\iota\lambda_{(1,1)}\rangle = 7$, $\langle\iota\lambda_{(4,1)},\iota\lambda_{(3,0)}\rangle = 7$ and $\langle\iota\lambda_{(4,1)},\iota\lambda_{(2,2)}\rangle = 17$, we have

$$[\iota\lambda_{(4,1)}] = [\iota\lambda_{(0,0)}] \oplus 2[\iota\lambda^{(1)}_{(1,1)}] \oplus 2[\iota\lambda^{(2)}_{(1,1)}] \oplus 2[\iota\lambda^{(1)}_{(3,0)}] \oplus 2[\iota\lambda^{(1)}_{(2,2)}], \tag{5.116}$$

and since $\langle\iota\lambda_{(1,4)},\iota\lambda_{(1,4)}\rangle = \langle\iota\lambda_{(1,4)},\iota\lambda_{(4,1)}\rangle = 17$, $[\iota\lambda_{(1,4)}] = [\iota\lambda_{(4,1)}]$. We see that no new irreducible sectors appear at tier 5, so the M-N system contains 15 irreducible sectors. We also have the following decompositions at tier 6:

$$\begin{aligned}
[\iota\lambda_{(6,0)}] &= [\iota\lambda_{(0,6)}] \;=\; [\iota\lambda_{(3,0)}], & (5.117)\\
[\iota\lambda_{(5,1)}] &= 2[\iota\lambda^{(1)}_{(1,0)}] \oplus [\iota\lambda^{(2)}_{(1,0)}] \ominus 2[\iota\lambda^{(1)}_{(0,2)}] \oplus 2[\iota\lambda^{(1)}_{(2,1)}] \oplus [\iota\lambda^{(1)}_{(4,0)}], & (5.118)\\
[\iota\lambda_{(4,2)}] &= 2[\iota\lambda^{(1)}_{(0,1)}] \oplus [\iota\lambda^{(2)}_{(0,1)}] \ominus 3[\iota\lambda^{(1)}_{(2,0)}] \oplus 3[\iota\lambda^{(1)}_{(1,2)}] \oplus 2[\iota\lambda^{(1)}_{(0,4)}], & (5.119)\\
[\iota\lambda_{(1,5)}] &= 2[\iota\lambda^{(1)}_{(0,1)}] \oplus [\iota\lambda^{(2)}_{(0,1)}] \ominus 2[\iota\lambda^{(1)}_{(2,0)}] \oplus 2[\iota\lambda^{(1)}_{(1,2)}] \oplus [\iota\lambda^{(1)}_{(0,4)}]. & (5.120)
\end{aligned}$$

We now find the sector products of the irreducible sectors with the N-N sector $[\rho] = [\lambda_{(1,0)}]$. We have $[\iota\lambda_{(0,0)}][\rho] = [\iota\lambda_{(1,0)}] = [\iota\lambda^{(1)}_{(1,0)}] \oplus [\iota\lambda^{(2)}_{(1,0)}]$. From $[\iota\lambda_{(1,1)}][\rho] = [\iota\lambda_{(1,0)}] \oplus [\iota\lambda_{(0,2)}] \ominus [\iota\lambda_{(2,1)}]$ and (5.105) we have

$$\begin{aligned}
(2[\iota\lambda^{(1)}_{(1,1)}] \oplus [\iota\lambda^{(2)}_{(1,1)}])[\rho] &= 4[\iota\lambda^{(1)}_{(1,0)}] \oplus 3[\iota\lambda^{(2)}_{(1,0)}] \oplus 3[\iota\lambda^{(1)}_{(0,2)}] \oplus [\iota\lambda^{(1)}_{(2,1)}] \ominus ([\iota\lambda_{(0,0)}][\rho]) \\
&= 3[\iota\lambda^{(1)}_{(1,0)}] \oplus 2[\iota\lambda^{(2)}_{(1,0)}] \oplus 3[\iota\lambda^{(1)}_{(0,2)}] \oplus [\iota\lambda^{(1)}_{(2,1)}]. & (5.121)
\end{aligned}$$

Similarly, by considering $[\iota\lambda_{(3,0)}][\rho]$, $[\iota\lambda_{(2,2)}][\rho]$ and $[\iota\lambda_{(4,1)}][\rho]$, and using (5.106), (5.113) and (5.116), we have the following:

$$\begin{aligned}
([\iota\lambda^{(1)}_{(1,1)}] \oplus [\iota\lambda^{(2)}_{(1,1)}] \oplus [\iota\lambda^{(1)}_{(3,0)}])[\rho] &= 2[\iota\lambda^{(1)}_{(1,0)}] \oplus [\iota\lambda^{(2)}_{(1,0)}] \ominus 3[\iota\lambda^{(1)}_{(0,2)}] \\
&\quad \oplus 2[\iota\lambda^{(1)}_{(2,1)}] \oplus [\iota\lambda^{(1)}_{(4,0)}], & (5.122)
\end{aligned}$$

$$
\begin{aligned}
(2[\iota\lambda^{(1)}_{(1,1)}] \oplus 3[\iota\lambda^{(2)}_{(1,1)}] \oplus 2[\iota\lambda^{(1)}_{(3,0)}] \oplus [\iota\lambda^{(1)}_{(2,2)}])[\rho] &= 5[\iota\lambda^{(1)}_{(1,0)}] \oplus 2[\iota\lambda^{(2)}_{(1,0)}] \oplus 7[\iota\lambda^{(1)}_{(0,2)}] \\
&\quad \oplus 6[\iota\lambda^{(1)}_{(2,1)}] \oplus 3[\iota\lambda^{(1)}_{(4,0)}], \qquad (5.123) \\
(2[\iota\lambda^{(1)}_{(1,1)}] \oplus 2[\iota\lambda^{(2)}_{(1,1)}] \oplus 2[\iota\lambda^{(1)}_{(3,0)}] \oplus 2[\iota\lambda^{(1)}_{(2,2)}])[\rho] &= 4[\iota\lambda^{(1)}_{(1,0)}] \oplus 2[\iota\lambda^{(2)}_{(1,0)}] \oplus 6[\iota\lambda^{(1)}_{(0,2)}] \\
&\quad \oplus 6[\iota\lambda^{(1)}_{(2,1)}] \oplus 4[\iota\lambda^{(1)}_{(4,0)}]. \qquad (5.124)
\end{aligned}
$$

Then from (5.121)-(5.124) we obtain the following sector products:

$$
\begin{aligned}
[\iota\lambda^{(1)}_{(1,1)}][\rho] &= [\iota\lambda^{(1)}_{(1,0)}] \oplus [\iota\lambda^{(2)}_{(1,0)}] \oplus [\iota\lambda^{(1)}_{(0,2)}], \\
[\iota\lambda^{(2)}_{(1,1)}][\rho] &= [\iota\lambda^{(1)}_{(1,0)}] \oplus [\iota\lambda^{(1)}_{(0,2)}] \oplus [\iota\lambda^{(1)}_{(2,1)}], \\
[\iota\lambda^{(1)}_{(3,0)}][\rho] &= [\iota\lambda^{(1)}_{(0,2)}] \oplus [\iota\lambda^{(1)}_{(2,1)}] \oplus [\iota\lambda^{(1)}_{(4,0)}], \\
[\iota\lambda^{(1)}_{(2,2)}][\rho] &= [\iota\lambda^{(1)}_{(2,1)}] \oplus [\iota\lambda^{(1)}_{(4,0)}].
\end{aligned}
$$

Next, from $[\iota\lambda_{(1,0)}][\rho] = [\iota\lambda_{(0,1)}] \oplus [\iota\lambda_{(2,0)}]$ and (5.101) we have

$$
([\iota\lambda^{(1)}_{(1,0)}] \oplus [\iota\lambda^{(2)}_{(1,0)}])[\rho] = 2[\iota\lambda^{(1)}_{(0,1)}] \oplus 2[\iota\lambda^{(2)}_{(0,1)}] \oplus [\iota\lambda^{(1)}_{(2,0)}]. \qquad (5.125)
$$

By considering $[\iota\lambda_{(0,2)}][\rho] = [\iota\lambda_{(0,1)}] \oplus [\iota\lambda_{(1,2)}]$ and (5.104) we obtain $([\iota\lambda^{(1)}_{(1,0)}] \oplus [\iota\lambda^{(2)}_{(1,0)}] \oplus [\iota\lambda^{(1)}_{(0,2)}])[\rho] = 3[\iota\lambda^{(1)}_{(0,1)}] \oplus 2[\iota\lambda^{(2)}_{(0,1)}] \oplus 2[\iota\lambda^{(1)}_{(2,0)}] \oplus [\iota\lambda^{(1)}_{(1,2)}]$. Then from (5.125) we see that

$$
[\iota\lambda^{(1)}_{(0,2)}][\rho] = [\iota\lambda^{(1)}_{(0,1)}] \oplus [\iota\lambda^{(1)}_{(2,0)}] \oplus [\iota\lambda^{(1)}_{(1,2)}]. \qquad (5.126)
$$

From $[\iota\lambda_{(2,1)}][\rho]$, (5.107) and (5.126) we find

$$
(2[\iota\lambda^{(1)}_{(1,0)}]\oplus[\iota\lambda^{(2)}_{(1,0)}]\oplus[\iota\lambda^{(1)}_{(2,1)}])[\rho] = 3[\iota\lambda^{(1)}_{(0,1)}]\oplus 3[\iota\lambda^{(2)}_{(0,1)}]\oplus 3[\iota\lambda^{(1)}_{(2,0)}]\oplus[\iota\lambda^{(1)}_{(1,2)}]\oplus[\iota\lambda^{(1)}_{(0,4)}]. \qquad (5.127)
$$

Similarly, by considering $[\iota\lambda_{(1,3)}][\rho]$ and $[\iota\lambda_{(0,5)}][\rho]$, and using using (5.109), (5.112) and $[\iota\lambda_{(0,5)}] = [\iota\lambda_{(4,0)}]$, we have the following:

$$
\begin{aligned}
(2[\iota\lambda^{(1)}_{(1,0)}] \oplus [\iota\lambda^{(2)}_{(1,0)}] \oplus 2[\iota\lambda^{(1)}_{(2,1)}] \oplus [\iota\lambda^{(1)}_{(4,0)}])[\rho] &= 3[\iota\lambda^{(1)}_{(0,1)}] \oplus 3[\iota\lambda^{(2)}_{(0,1)}] \oplus 4[\iota\lambda^{(1)}_{(2,0)}] \\
&\quad \oplus 3[\iota\lambda^{(1)}_{(1,2)}] \oplus 2[\iota\lambda^{(1)}_{(0,4)}], \qquad (5.128) \\
([\iota\lambda^{(1)}_{(1,0)}] \oplus [\iota\lambda^{(2)}_{(1,0)}] \oplus [\iota\lambda^{(1)}_{(2,1)}] \oplus [\iota\lambda^{(1)}_{(4,0)}])[\rho] &= 2[\iota\lambda^{(1)}_{(0,1)}] \oplus 2[\iota\lambda^{(2)}_{(0,1)}] \oplus 2[\iota\lambda^{(1)}_{(2,0)}] \\
&\quad \oplus 2[\iota\lambda^{(1)}_{(1,2)}] \oplus 2[\iota\lambda^{(1)}_{(0,4)}]. \qquad (5.129)
\end{aligned}
$$

Then from (5.125), (5.127)-(5.129) we obtain the following sector products:

$$
\begin{aligned}
[\iota\lambda^{(1)}_{(1,0)}][\rho] &= [\iota\lambda^{(1)}_{(0,1)}] \oplus [\iota\lambda^{(2)}_{(0,1)}] \oplus [\iota\lambda^{(1)}_{(2,0)}], \\
[\iota\lambda^{(2)}_{(1,0)}][\rho] &= [\iota\lambda^{(1)}_{(0,1)}] \oplus [\iota\lambda^{(2)}_{(0,1)}], \\
[\iota\lambda^{(1)}_{(2,1)}][\rho] &= [\iota\lambda^{(1)}_{(2,0)}] \oplus [\iota\lambda^{(1)}_{(1,2)}] \oplus [\iota\lambda^{(1)}_{(0,4)}], \\
[\iota\lambda^{(1)}_{(4,0)}][\rho] &= [\iota\lambda^{(1)}_{(1,2)}] \oplus [\iota\lambda^{(1)}_{(0,4)}].
\end{aligned}
$$

Next, since $[\iota\lambda_{(0,1)}][\rho] = [\iota\lambda_{(0,0)}] \oplus [\iota\lambda_{(1,1)}]$, from (5.102) we have

$$([\iota\lambda^{(1)}_{(0,1)}] \oplus [\iota\lambda^{(2)}_{(0,1)}])[\rho] = 2[\iota\lambda_{(0,0)}] \oplus 2[\iota\lambda^{(1)}_{(1,1)}] \oplus [\iota\lambda^{(2)}_{(1,1)}]. \tag{5.130}$$

By considering $[\iota\lambda_{(2,0)}][\rho] = [\iota\lambda_{(1,1)}] \oplus [\iota\lambda_{(3,0)}]$ and (5.103) we obtain $([\iota\lambda^{(1)}_{(0,1)}] \oplus [\iota\lambda^{(2)}_{(0,1)}] \oplus [\iota\lambda^{(1)}_{(2,0)}])[\rho] = 2[\iota\lambda_{(0,0)}] \oplus 3[\iota\lambda^{(1)}_{(1,1)}] \oplus 2[\iota\lambda^{(2)}_{(1,1)}] \oplus [\iota\lambda^{(1)}_{(3,0)}]$. Then from (5.130) we see that

$$[\iota\lambda^{(1)}_{(2,0)}][\rho] = [\iota\lambda^{(1)}_{(1,1)}] \oplus [\iota\lambda^{(2)}_{(1,1)}] \oplus [\iota\lambda^{(1)}_{(3,0)}]. \tag{5.131}$$

From $[\iota\lambda_{(1,2)}][\rho]$, (5.108) and (5.131) we obtain

$$(2[\iota\lambda^{(1)}_{(0,1)}] \oplus [\iota\lambda^{(2)}_{(0,1)}] \oplus [\iota\lambda^{(1)}_{(1,2)}])[\rho] = 3[\iota\lambda_{(0,0)}] \oplus 3[\iota\lambda^{(1)}_{(1,1)}] \oplus 3[\iota\lambda^{(2)}_{(1,1)}] \oplus [\iota\lambda^{(1)}_{(3,0)}] \oplus [\iota\lambda^{(1)}_{(2,2)}]. \tag{5.132}$$

Similarly, by considering $[\iota\lambda_{(3,1)}][\rho]$ and $[\iota\lambda_{(0,4)}][\rho]$, and using (5.111) and (5.110), we have the following:

$$\begin{aligned}(2[\iota\lambda^{(1)}_{(0,1)}] \oplus [\iota\lambda^{(2)}_{(0,1)}] \oplus 2[\iota\lambda^{(1)}_{(1,2)}] \oplus [\iota\lambda^{(1)}_{(0,4)}])[\rho] &= 3[\iota\lambda_{(0,0)}] \oplus 3[\iota\lambda^{(1)}_{(1,1)}] \oplus 4[\iota\lambda^{(2)}_{(1,1)}] \\ &\quad \oplus 3[\iota\lambda^{(1)}_{(3,0)}] \oplus 3[\iota\lambda^{(1)}_{(2,2)}], \qquad (5.133)\\ ([\iota\lambda^{(1)}_{(0,1)}] \oplus [\iota\lambda^{(2)}_{(0,1)}] \oplus [\iota\lambda^{(1)}_{(1,2)}] \oplus [\iota\lambda^{(1)}_{(0,4)}])[\rho] &= 2[\iota\lambda_{(0,0)}] \oplus 2[\iota\lambda^{(1)}_{(1,1)}] \oplus 2[\iota\lambda^{(2)}_{(1,1)}] \\ &\quad \oplus 2[\iota\lambda^{(1)}_{(3,0)}] \oplus 2[\iota\lambda^{(1)}_{(2,2)}]. \qquad (5.134)\end{aligned}$$

Then from (5.130), (5.132)-(5.134) we obtain the following sector products:

$$\begin{aligned}[\iota\lambda^{(1)}_{(0,1)}][\rho] &= [\iota\lambda_{(0,0)}] \oplus [\iota\lambda^{(1)}_{(1,1)}] \oplus [\iota\lambda^{(2)}_{(1,1)}],\\ [\iota\lambda^{(2)}_{(0,1)}][\rho] &= [\iota\lambda_{(0,0)}] \oplus [\iota\lambda^{(1)}_{(1,1)}],\\ [\iota\lambda^{(1)}_{(1,2)}][\rho] &= [\iota\lambda^{(2)}_{(1,1)}] \oplus [\iota\lambda^{(1)}_{(3,0)}] \oplus [\iota\lambda^{(1)}_{(2,2)}],\\ [\iota\lambda^{(1)}_{(0,4)}][\rho] &= [\iota\lambda^{(1)}_{(3,0)}] \oplus [\iota\lambda^{(1)}_{(2,2)}].\end{aligned}$$

We thus obtain the graph $\mathcal{D}^{(12)*}$ as the fusion graph for the M-N system, and the associated modular invariant is $Z_{\mathcal{D}^{(12)*}}$.

5.2.7 The type I parent

Thus we have constructed subfactors which realize all of the $SU(3)$ modular invariants, except for the $\mathcal{E}_4^{(12)}$ case, since the existence of this subfactor is not yet shown. However, for the modular invariant associated to the graph $\mathcal{E}_4^{(12)}$, we have $Z_{\mathcal{E}_4^{(12)}} = Z_{\mathcal{E}_5^{(12)}} C$, where C is the modular invariant associated to the graph $\mathcal{A}^{(12)*}$. Since C is symmetric, and both $Z_{\mathcal{E}_5^{(12)}}$, $C^T = C$ are shown to be realised by subfactors, the result of [40, Theorem 3.6] shows that the modular invariant $Z_{\mathcal{E}_4^{(12)}}$ is also realised by a subfactor.

The M-N graph $\mathcal{G}$ of a subfactor $N \subset M$ is defined by the matrix Δ_ρ which gives the decomposition of the M-N sectors with respect to multiplication by the fundamental representation ρ. Similarly, multiplication by the conjugate representation defines the matrix $\Delta_{\overline{\rho}} = \Delta_\rho^T$ which is the adjacency matrix of the conjugate graph $\widetilde{\mathcal{G}}$. Then since ${}_N\mathcal{X}_N$ is commutate, the matrices Δ_ρ and Δ_ρ^T commute, i.e. Δ_ρ is normal. This provides a proof that the adjacency matrices of the $\mathcal{ADE}$ graphs are all normal, since each of the $\mathcal{ADE}$ graphs appears as the M-N graph for a subfactor $N \subset M$.

The zero-column of the modular invariant Z associated with the subfactor $N \subset M$ determines $\langle \alpha_j^+, \alpha_{j'}^+ \rangle$ since α preserves the sector product

$$\begin{aligned}
\langle \alpha_j^+, \alpha_{j'}^+ \rangle &= \langle \alpha_j^+ \alpha_{j'}^+, \mathrm{id} \rangle = \sum_{j''} N_{j,j'}^{j''} \langle \alpha_{j''}^+, \mathrm{id} \rangle \\
&= \sum_{j''} N_{j,j'}^{j''} Z_{j'',0},
\end{aligned} \tag{5.135}$$

and similarly the zero-row determines $\langle \alpha_j^-, \alpha_{j'}^- \rangle$. Then for all modular invariants with the same zero-column, the sectors $[\alpha_1^\pm]$ satisfy the same equation (5.135) and hence have the same fusion graphs. Let v be an isometry which intertwines the identity and the canonical endomorphism $\gamma = \iota\bar{\iota}$. Proposition 3.2 in [11] states that the following conditions are equivalent:

1. $Z_{\lambda,0} = \langle \theta, \lambda \rangle$ for all $\lambda \in {}_N\mathcal{X}_N$.

2. $Z_{0,\lambda} = \langle \theta, \lambda \rangle$ for all $\lambda \in {}_N\mathcal{X}_N$.

3. Chiral locality holds: $\varepsilon^+(\theta, \theta) v^2 = v^2$.

The chiral locality condition, which can be expressed in terms of the single inclusion $N \subset M$ and the braiding, expresses local commutativity (locality) of the extended net, if $N \subset M$ arises from a net of subfactors [81]. Chiral locality holds if and only if the dual canonical endomorphism is visible in the vacuum row, $[\theta] = \bigoplus_\lambda Z_{0,\lambda}[\lambda]$ (and hence also in the vacuum column also).

We will call the inclusion $N \subset M$ **type I** if and only if one of the above equivalent conditions 1-3 hold. Otherwise we will call the inclusion type II. Note that the inclusions obtained for the $\mathcal{E}_1^{(12)}$ and $\mathcal{E}_2^{(12)}$ graphs realize the same modular invariant $Z_{\mathcal{E}_1^{(12)}}$, but the inclusion for $\mathcal{E}_1^{(12)}$ is type I whilst the inclusion for $\mathcal{E}_2^{(12)}$ is type II. This shows that it is possible for a type I modular invariant to be realized by a type II inclusion, and suggests that care needs to be taken with the type I, II labelling of modular invariants. The fusion graph of $[\alpha_1^\pm]$ for the identity modular invariant is the fusion graph of the original N-N

system, whilst the fusion graph of $[\alpha_1^\pm]$ for the modular invariants associated to $\mathcal{D}^{(3k+3)}$ and $\mathcal{E}^{(8)}$ were computed in [9], and for $\mathcal{E}_1^{(12)}$ and $\mathcal{E}^{(24)}$ in [10]. In these cases we have $Z_{\lambda,0} = \langle \theta, \lambda \rangle$ for all $\lambda \in {}_N\mathcal{X}_N$, for θ given in (5.31)-(5.42). The principal graph of the inclusion $\alpha_1^\pm(N) \subset N$ is then the fusion graph of $[\alpha_1^\pm]$. The other modular invariants all have the same zero-column as one of these modular invariants, and hence the fusion graph of $[\alpha_1^\pm]$ for these modular invariants must be the graph given by the type I parent of Z, that is, the type I modular invariant which has the same first column as Z. The results are summarized in the table below, where "Type" refers to the type of the inclusion $N \subset M$ given by the A_2-GHJ construction, where the distinguished vertex $*_{\mathcal{G}}$ is the vertex with lowest Perron-Frobenius weight.[1] We again warn that the existence of the A_2-GHJ subfactor which gives the dual canonical endomorphism for $\mathcal{E}_4^{(12)}$ has not been shown yet by us.

GHJ graph	Modular invariant	Type	M-N graph	Type I parent
$\mathcal{A}^{(n)}$	$Z_{\mathcal{A}^{(n)}}$	I	$\mathcal{A}^{(n)}$	$\mathcal{A}^{(n)}$
$\mathcal{A}^{(n)*}$	$Z_{\mathcal{A}^{(n)*}}$	II	$\mathcal{A}^{(n)*}$	$\mathcal{A}^{(n)}$
$\mathcal{D}^{(3k)}$	$Z_{\mathcal{D}^{(3k)}}$	I	$\mathcal{D}^{(3k)}$	$\mathcal{D}^{(3k)}$
$\mathcal{D}^{(n)}$ $(n \not\equiv 0 \bmod 3)$	$Z_{\mathcal{D}^{(n)}}$	II	?	$\mathcal{A}^{(n)}$
$\mathcal{D}^{(3k)*}$	$Z_{\mathcal{D}^{(3k)*}}$	II	$\mathcal{D}^{(3k)*}$	$\mathcal{D}^{(3k)}$
$\mathcal{D}^{(n)*}$ $(n \not\equiv 0 \bmod 3)$	$Z_{\mathcal{D}^{(n)*}}$	II	$\mathcal{D}^{(n)*}$	$\mathcal{A}^{(n)}$
$\mathcal{E}^{(8)}$	$Z_{\mathcal{E}^{(8)}}$	I	$\mathcal{E}^{(8)}$	$\mathcal{E}^{(8)}$
$\mathcal{E}^{(8)*}$	$Z_{\mathcal{E}^{(8)*}}$	II	$\mathcal{E}^{(8)*}$	$\mathcal{E}^{(8)}$
$\mathcal{E}_1^{(12)} = \mathcal{E}_1^{(12)*}$	$Z_{\mathcal{E}_1^{(12)}}$	I	$\mathcal{E}_1^{(12)}$	$\mathcal{E}_1^{(12)}$
$\mathcal{E}_2^{(12)} = \mathcal{E}_2^{(12)*}$	$Z_{\mathcal{E}_1^{(12)}}$	II	$\mathcal{E}_2^{(12)}$	$\mathcal{E}_1^{(12)}$
$\mathcal{E}_3^{(12)}$	-	-	-	-
$\mathcal{E}_4^{(12)} = \mathcal{E}_5^{(12)*}$	$Z_{\mathcal{E}_4^{(12)}}$	II	$\mathcal{E}_4^{(12)}$	$\mathcal{D}^{(12)}$
$\mathcal{E}_5^{(12)}$	$Z_{\mathcal{E}_5^{(12)}}$	II	$\mathcal{E}_5^{(12)}$	$\mathcal{D}^{(12)}$
$\mathcal{E}^{(24)} = \mathcal{E}^{(24)*}$	$Z_{\mathcal{E}^{(24)}}$	I	$\mathcal{E}^{(24)}$	$\mathcal{E}^{(24)}$

Table 5.1: The $SU(3)$ modular invariants realized by A_2-GHJ subfactors

[1]Note, we have only showed the $\mathcal{A}^*$ and $\mathcal{D}^*$ case for $n = 12$. We have not done any computations for the $D^{(n)}$ graphs, $n \not\equiv 0 \bmod 3$.

Chapter 6

A_2-Planar Algebras

In this chapter we give a diagrammatic representation of the A_2-Temperley-Lieb algebra, and show that it is isomorphic to Wenzl's representation of a Hecke algebra. Generalizing Jones's notion of a planar algebra, we construct an A_2-planar algebra which will capture the structure contained in the $SU(3)$ $\mathcal{ADE}$ subfactors. We show that the subfactor for an $\mathcal{ADE}$ graph with a flat connection has a description as a flat A_2-planar algebra, and give the A_2-planar algebra description of the dual subfactor. We introduce the notion of modules over an A_2-planar algebra, and describe certain irreducible Hilbert A_2-TL-modules. A partial decomposition of the A_2-planar algebras for the $\mathcal{ADE}$ graphs is achieved.

6.1 A_2-tangles

6.1.1 Basis diagrams

In [78], Kuperberg defined the notion of a spider, which is a way of depicting the operations of the representation theory of groups and other group-like objects with certain planar graphs. These graphs are called webs, hence the term "spider". In [78] certain spiders were defined in terms of generators and relations, isomorphic to the representation theories of rank two Lie algebras and the quantum deformations of these representation theories. This formulation generalized a well-known construction for $A_1 = \mathrm{su}(2)$ by Kauffman, see Section 1.2.1.

For the $A_2 = \mathrm{su}(3)$ case, we have the A_2 webs, which we will call incoming and outgoing trivalent vertices, illustrated in Figure 6.1. We call the oriented lines **strings**. We may join the A_2 webs together by attaching free ends of outgoing trivalent vertices to free ends of incoming trivalent vertices, and isotoping the strings if needed so that they are smooth.

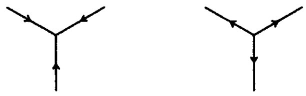
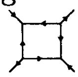
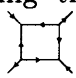
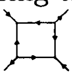


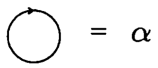
Figure 6.1: A_2 webs

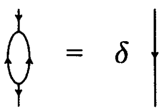
We define a **diagram** D to be any oriented planar graph embedded in a disc, formed by joining incoming and outgoing trivalent vertices together as described above. The diagram D may have free ends, that is, strings whose endpoints are attached to the boundary of the disc. We identify isotopic diagrams, i.e. diagrams which can be transformed into each other by moving the strings and trivalent vertices in a planar fashion. The following local pictures (which Kuperberg calls elliptic faces) may appear in D:  which we call an embedded circle, and  which we call an embedded square. We will call the diagrams without embedded circles or squares **basis diagrams**. Let D be a diagram which contains embedded circles and squares. If we choose one of the embedded circles we can obtain a new diagram by 'removing' this embedded circle, i.e. we replace the local picture  by . If we choose one of the embedded squares, we can obtain two new diagrams by 'splicing' the embedded square, that is, we form a new diagram by replacing the local picture  by  , and form a second new diagram by replacing the  by  . Repeating these steps as required for each new diagram, we eventually obtain a family of diagrams which do not contain any embedded circles or squares. We call this family of diagrams the **states** of the original diagram D. We attach a weight to each diagram in the above procedure, where if w is the weight of one diagram, the weight for the new diagram obtained from it by removing an embedded circle is δw, where $\delta = [2]_q$ for some variable q, whilst the weights of the two new diagrams obtained by removing an embedded square are both just w. The weight attached to the original diagram D is set to be 1. A state σ_i of D is not necessarily a basis diagram as it may now contain closed loops  or  . Let w_i be the weight attached to the state σ_i, and suppose σ_i contains k such closed loops. We remove these closed loops to obtain a basis diagram $\widetilde{\sigma_i}$, and define weight associated to $\widetilde{\sigma_i}$ to be $w_i\alpha^n$ where $\alpha = [3]_q$. If the diagram D has no free ends then its states σ_i will consist of only closed loops. Then by Proposition 6.1.11, the $\widetilde{\sigma_i}$ will be polynomials in $\mathbb{N}[q, q^{-1}]$ (since $\delta = q + q^{-1}$ and $\alpha = q^2 + 1 + q^{-2}$).

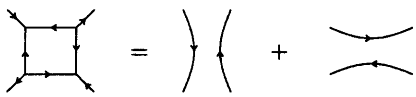
A sign string is a string of symbols '+' and '−'. There is a sign string associated to any basis diagram (up to any cyclic permutation of the symbols) given by the orientation at each free end. Moving in an anticlockwise direction from one free end to the next, we insert a '+' at the end of our string if the orientation of the string at the free end is towards the endpoint, and a '−' if the orientation is away from the endpoint. Let s be a sign string. The A_2 basis web set $B(s)$ is defined to be the set of all basis diagrams which have sign string s, up to cyclic permutation.

6.1.2 The A_2 web space $W(s)$

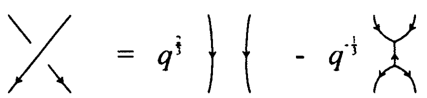
The A_2 web space $W(s)$ is defined to be the free vector space over $\mathbb{C}$ generated by diagrams in $B(s)$. Let D be a diagram which contains embedded squares or circles. We can write D as a linear combination of basis diagrams by $D = \sum_i w_i \alpha^{k_i} \widetilde{\sigma}_i$, where state σ_l contains k_l closed loops. We will call this procedure "reducing" the diagram D. Then the diagrams in $W(s)$ can be said to satisfy the Kuperberg relations, which are relations on local parts of the diagrams:

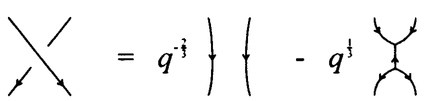
K1: 

K2: 

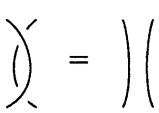
K3: 

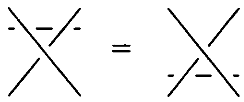
There is also a braiding on an A_2 web space $W(s)$, defined locally by the following linear combinations of local diagrams in $W(s)$ (see [78, 109]):


(6.1)

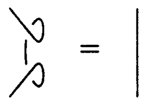

(6.2)

The braiding satisfies the following properties locally:


(6.3)



(6.4)



(6.5)

(6.6)

where we also have relation (6.6) with the crossings all reversed.

We call the local picture illustrated on the left hand sides of relation (6.1), (6.2) respectively a negative, positive crossing respectively. With this braiding, 'kinks' contribute a scalar factor of $q^{8/3}$ for those involving a positive crossing, and $q^{-8/3}$ for those involving a negative crossing, as shown in Figure 6.2.

6.1.3 A_2-Tangles

We are now going to systematically define an algebra of web tangles, and express this in terms of generators and relations.

Definition 6.1.1 *An A_2-**tangle** will be a connected collection of strings joined together at incoming or outgoing trivalent vertices (see Figure 6.1), possibly with some free ends, such that the orientations of the individual strings are consistent with the orientations of the trivalent vertices.*

Definition 6.1.2 *We call a vertex a **source** vertex if the string attached to it has orientation away from the vertex. Similarly, a **sink** vertex will be a vertex where the string attached has orientation towards the vertex.*

Definition 6.1.3 *For $m, n \geq 0$, an A_2-(m,n)-**tangle** will be an A_2-tangle T on a rectangle, where T has $m+n$ free ends attached to m source vertices along the top of the*

Figure 6.2: Removing kinks

rectangle and n sink vertices along the bottom such that the orientation of the strings is respected. If $m = n$ we call T simply an A_2-m-tangle, and we position the vertices so that for every vertex along the top there is a corresponding vertex directly beneath it along the bottom.

More generally, for $m_1, m_2, n_1, n_2 \geq 0$, define an $((m_1, m_2), (n_1, n_2))$-rectangle to have $m_1 + m_2$ vertices along the top such that the first m_1 are sources and the next m_2 are sinks, and $n_1 + n_2$ vertices along the bottom such that the first n_1 are sinks and the next n_2 are sources. Then an A_2-$((m_1, m_2), (n_1, n_2))$-**tangle** T' *will be an A_2-tangle on an $((m_1, m_2), (n_1, n_2))$-rectangle such that every free end of T' is attached to a vertex in a way that respects the orientation of the strings, and every vertex has a string attached to it.*

Two A_2-$((m_1, m_2), (n_1, n_2))$-tangles are equivalent if one can be obtained from the other by an isotopy which moves the strings and trivalent vertices, but leaves the boundary vertices unchanged. We define $\mathcal{T}^{A_2}_{(m_1,m_2),(n_1,n_2)}$ to be the set of all (equivalence classes of) A_2-$((m_1, m_2), (n_1, n_2))$-tangles

Note, an A_2-(m, n)-tangle is just an A_2-$((m, 0), (n, 0))$-tangle.

The composition $TS \in \mathcal{T}^{A_2}_{(m_1,m_2),(k_1,k_2)}$ of an A_2-$((m_1, m_2), (n_1, n_2))$-tangle T and an A_2-$((n_1, n_2), (k_1, k_2))$-tangle S is given by gluing S vertically below T such that the vertices at the bottom of T and the top of S coincide, removing these vertices, and isotoping the glued strings if necessary to make them smooth. The composition is clearly associative.

Definition 6.1.4 *We define the vector space $\mathcal{V}^{A_2}_{(m_1,m_2),(n_1,n_2)}$ to be the free vector space over $\mathbb{C}$ with basis $\mathcal{T}^{A_2}_{(m_1,m_2),(n_1,n_2)}$. Then $\mathcal{V}^{A_2}_{(m_1,m_2),(n_1,n_2)}$ has an algebraic structure with multiplication given by composition of tangles. In particular, we will write $\mathcal{V}^{A_2}_m$ for $\mathcal{V}^{A_2}_{(m,0),(m,0)}$, and $\mathcal{V}^{A_2} = \bigcup_{m\geq 0} \mathcal{V}^{A_2}_m$. For $n < m$ we have $\mathcal{V}^{A_2}_n \subset \mathcal{V}^{A_2}_m$, with the inclusion of an n-tangle $T \in \mathcal{T}^{A_2}_n$ in $\mathcal{T}^{A_2}_m$ given by adding $m - n$ vertices along the top and bottom of the rectangle after the rightmost vertex, with $m - n$ downwards oriented vertical strings connecting the extra vertices along the top to those along the bottom. The inclusion for $\mathcal{V}^{A_2}_n$ in $\mathcal{V}^{A_2}_m$ is the linear extension of this map.*

Note that $\mathcal{T}^{A_2}_{(m_1,m_2),(n_1,n_2)}$ is infinite , and thus the vector space $\mathcal{V}^{A_2}_{(m_1,m_2),(n_1,n_2)}$ is infinite dimensional. However, we will take a quotient of $\mathcal{V}^{A_2}_{(m_1,m_2),(n_1,n_2)}$ which will turn out to be finite dimensional.

Definition 6.1.5 *We define $I_m \subset \mathcal{V}^{A_2}_m$ to be the ideal of $\mathcal{V}^{A_2}_m$ which is the linear span of the relations K1-K3.*

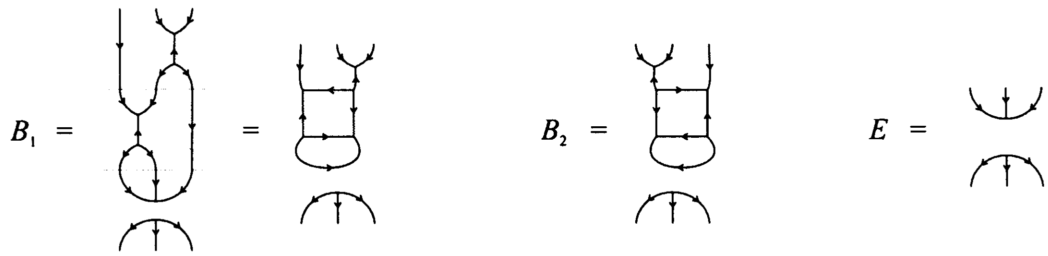


Figure 6.3: 3-tangles B_1, B_2, E

By the linear span of the relations K1-K3 is meant the linear span of the differences of the left hand side and the right hand side of each of the relations as local parts of the diagrams, where the rest of the diagram is identical in each term in the difference. Note that $I_m \subset I_{m+1}$.

Definition 6.1.6 *The algebra $V_m^{A_2}$ is defined to be the quotient of the space $\mathcal{V}_m^{A_2}$ by the ideal I_m, and $V^{A_2} = \bigcup_{m \geq 0} V_m^{A_2}$.*

The algebra $V_m^{A_2}$ is an A_2 web space $W(s_m)$, in the sense of Section 6.1.1, where s_m is the string of length $2m$ given by $++\cdots+--\cdots-$, with '+' and '−' both appearing m times. The multiplication on $W(s_m)$, for basis diagrams D_1 and D_2, is given by joining the m free ends of D_1 labelled '+' to the m free ends of D_2 labelled '−'. The new diagram may now contain embedded circles or squares, so we may write $D_1 D_2$ as a linear combination of basis diagrams in $V_m^{A_2}$, as described in Section 6.1.1.

We could replace the Kuperberg relation K1 by the more general relations:

K1':

Although it now appears that we have three independent parameters $\alpha_1, \alpha_1, \delta$, we actually have only one, as shown in the following Lemma:

Lemma 6.1.7 *For a fixed complex number $\delta \neq 0$ we must have either $\alpha_1 = \alpha_2 = \delta^2 - 1$ or $\alpha_1 = \alpha_2 = 0$ in V^{A_2}.*

Proof Let B_1 be the 3-tangle illustrated in Figure 6.3, which is the composition of three basis tangles in $V_3^{A_2}$. Let B_2 be a 3-tangle which comes from a similar composition, and E a basis tangle in $V_3^{A_2}$, both also illustrated in Figure 6.3. Reducing B_1 using K2 twice, we get $B_1 = \delta^2 E$. On the other hand, if we reduce B_1 using K3, we get an anticlockwise

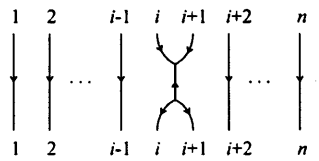
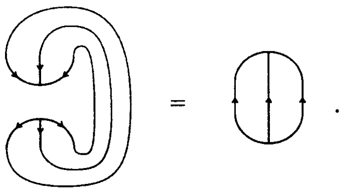


Figure 6.4: The m-tangle W_i, $i = 1, \dots, m-1$.

oriented closed loop, which by K1' contributes a scalar factor α_1. Then we also have $B_1 = E + \alpha_1 E$. If $E \neq 0$, then $\delta^2 = 1 + \alpha_1$, and by the same argument on B_2 we also obtain $\delta^2 = 1 + \alpha_2$. Suppose now that $E = 0$. Let $\sigma_{a,0} : W(aa^*) \to W(\emptyset)$ be the 'stitch' operation of Kuperberg [78], where $a = +++$ and $a^* = ---$. Then $\sigma_{a,0}(E) = 0$ since $E = 0$. Pictorially, $\sigma_{a,0}(E)$ gives



If we use K2 to remove the left embedded circle, we obtain an anticlockwise oriented loop, and so the diagram gives the scalar $\alpha_1 \delta$. If instead we used K2 to remove the right embedded circle we would obtain the scalar $\alpha_2 \delta$. Since $\sigma_{a,0}(E) = 0$ we have either $\alpha_1 = \alpha_2 = 0$ or $\delta = 0$. $\square$

We now define a $*$-operation on $\mathcal{V}_m^{A_2}$, which is an involutive conjugate linear map. For an m-tangle $T \in \mathcal{T}_m^{A_2}$, T^* is the m-tangle obtained by reflecting T about a horizontal line halfway between the top and bottom vertices of the tangle, and reversing the orientations on every string. Then $*$ on $\mathcal{V}_m^{A_2}$ is the conjugate linear extension of $*$ on $\mathcal{T}_m^{A_2}$. Note that the $*$-operation leaves the relation K2 invariant if and only if $\delta \in \mathbb{R}$. For $\delta \in \mathbb{R}$, the $*$-operation leaves the ideal I_m invariant due to the symmetry of the relations K1-K3. Then $*$ passes to $V_m^{A_2}$, and is an involutive conjugate linear anti-automorphism.

From now on let δ be real. Then δ can be written as $\delta = [2]_q$ for some q, and by Lemma 6.1.7 $\alpha = [3]_q$.

We define the tangle $\mathbf{1}_n$ to be the m-tangle with all strings vertical through strings. Then $\mathbf{1}_m$ is the identity of the algebra $\mathcal{V}_m^{A_2}$, $\mathbf{1}_m a = a = a \mathbf{1}_m$ for all $a \in \mathcal{V}_m^{A_2}$. We also define W_i to be the m-tangle with all vertices along the top connected to the vertices along the bottom by vertical lines, except for the i^{th} and $(i+1)^{\text{th}}$ vertices. The strings attached to the i^{th} and $(i+1)^{\text{th}}$ vertices along the top are connected at an incoming trivalent vertex, with the third string coming from an outgoing trivalent vertex connected to the strings

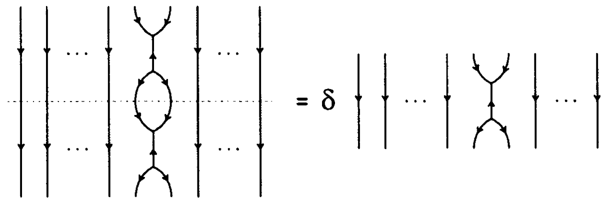


Figure 6.5: $w_i^2 = \delta w_i$

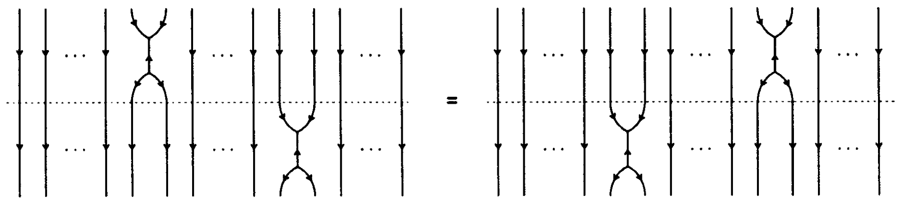


Figure 6.6: $w_i w_j = w_j w_i$ for $|i-j| > 1$.

attached to the i^{th} and $(i+1)^{\text{th}}$ vertices along the bottom. The tangle W_i is illustrated in Figure 6.4.

For $m \in \mathbb{N} \cup \{0\}$ we define the algebra $SU(3)\text{-}TL_m$ to be $\text{alg}(\mathbf{1}_m, w_i | i = 1, \ldots, m-1)$, where $w_i = W_i + I_m$. The w_i's in $SU(3)\text{-}TL_m$ are clearly self-adjoint, and satisfy the relations H1-H3, as illustrated in Figures 6.5, 6.6 and 6.7.

Let F_i be the m-tangle illustrated in Figure 6.8, and define $f_i = F_i + I_m$ so that $f_i = w_i w_{i+1} w_i - w_i = w_{i+1} w_i w_{i+1} - w_{i+1}$. By drawing pictures, it is easy to see that

$$\begin{aligned} f_i f_{i\pm1} f_i &= \delta^2 f_i, \\ f_i f_{i+2} f_i &= \delta f_i w_{i+3}, \end{aligned} \tag{6.7}$$

and

$$f_i f_{i-2} f_i = \delta f_i w_{i-2}. \tag{6.8}$$

We also find that the w_i satisfy the $SU(3)$ relation (1.7):

$$(w_i - w_{i+2} w_{i+1} w_i + w_{i+1}) f_{i+1} = 0.$$

The following lemma is found in [95, Lemma 3.3, p.385]:

Lemma 6.1.8 *Let T be a basis A_2-(m,n)-tangle. Then T must satisfy one of the following three conditions:*

(1) *There are two consecutive vertices along the top which are connected by a cup or whose strings are joined at an (incoming) trivalent vertex,*

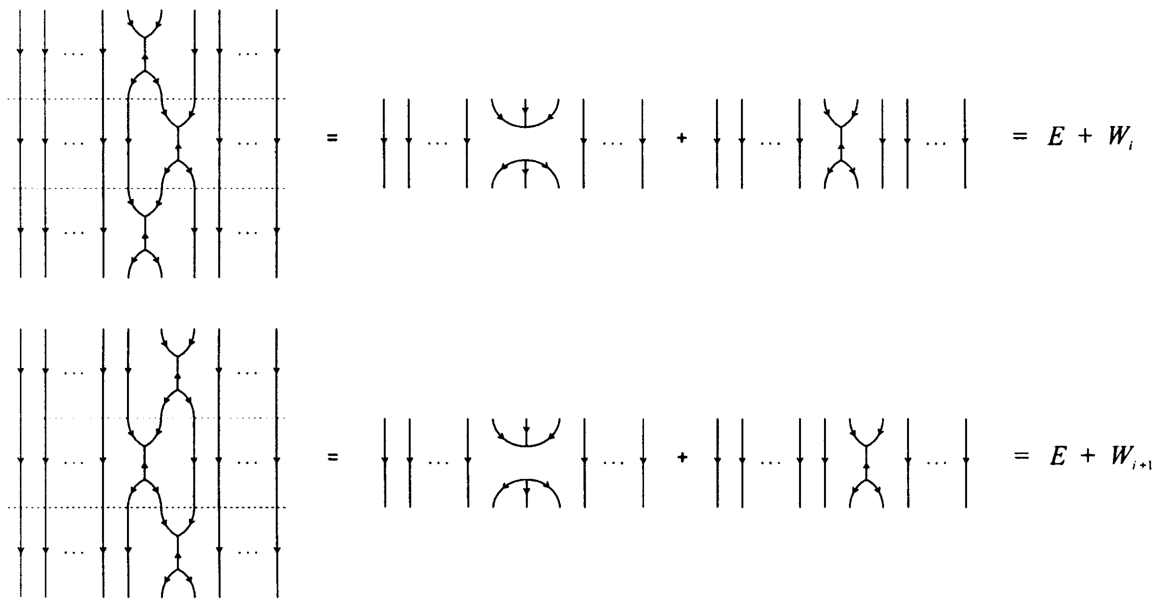


Figure 6.7: $w_i w_{i+1} w_i - w_i = w_{i+1} w_i w_{i+1} - w_{i+1}$

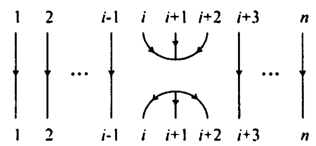


Figure 6.8: The m-tangle F_i, $i = 1, \ldots, m-2$.

(2) *There are two consecutive vertices along the bottom which are connected by a cap or whose strings are joined at an (outgoing) trivalent vertex,*

(3) T *is the identity tangle.*

Corollary 6.1.9 *For any basis A_2-m-tangle which is not the identity tangle, there must be two consecutive vertices along the top or bottom whose strings are joined at an (incoming) trivalent vertex.*

Proof

This follows from Lemma 6.1.8 with $m = 0$. □

Then we have the following lemma which says that the $SU(3)$-Temperley-Lieb algebra $SU(3)$-TL is equal to the algebra V^{A_2} of all A_2-tangles subject to the relations K1-K3. This is the A_2 analogue of the fact that the Temperley-Lieb algebra $TL_n = \mathrm{alg}(1, e_1, e_2, \ldots, e_{n-1})$ is isomorphic to Kauffman's diagram algebra [69], which is the algebra generated by the elements $E_1, E_2, \ldots, E_{n-1}$ on n strings, illustrated in Figure 1.6,

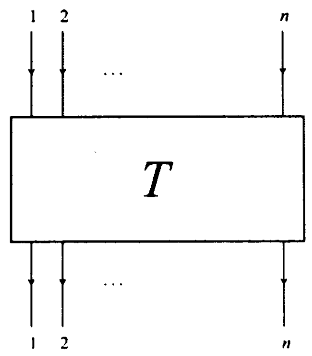


Figure 6.9: m-tangle T

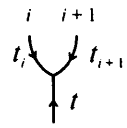


Figure 6.10:

along with the identity tangle $\mathbf{1}_n$ where every vertex along the top is connected to a vertex along the bottom by a vertical through string.

Lemma 6.1.10 *The algebra $V_m^{A_2}$ is generated by $\mathbf{1}_m$ and $W_i \in V_m^{A_2}$, $i = 1, \dots, m-1$. So $V_m^{A_2} \cong SU(3)\text{-}TL_m$.*

Proof

The proof that the algebra $V_m^{A_2}$ is generated by $\mathbf{1}_m$ and W_i, $i = 1, \dots, m-1$, is by induction on m. For $m = 1$ there is only one basis tangle, $\mathbf{1}_1$, whilst for $m = 2$ there are only two basis tangles, $\mathbf{1}_2$ and W_1. Assume the claim is true for $(m-1)$-tangles, $m \geq 3$. Let T be a basis m-tangle in $V_m^{A_2}$. We draw T as in Figure 6.9.

If T is the identity tangle then the $T = \mathbf{1}_m$, which is trivial. In what follows, vertex i along the top, bottom respectively, will mean the i^{th} vertex along the top, bottom respectively, counting from the left. By Corollary 6.1.9, for any other T there exists $i \in \{1, \dots, m-1\}$ such that strings t_i and t_{i+1}, which have vertices i and $i+1$ along the top or bottom as endpoints, are joined at a trivalent vertex. Let us suppose that this is the case for vertices along the top, as in Figure 6.10. If this isn't the case there must be vertices along the bottom for which it is true and we proceed similarly.

For a tangle T, let I_T be the set of all vertices i along the top of T such that strings t_i and t_{i+1} are joined at a trivalent vertex, let $I_T^1 \subset I_T$ be the subset consisting of the vertices $i \in I_T$ such that the endpoint of the string t isn't one of the other vertices along the top, and let I_T^2 be the subset of I_T such that string t in Fig 6.10 is attached to vertex $i+2$ along the top. Note that $I_T^1 \cap I_T^2 = \emptyset$. Suppose T contains d trivalent vertices.

Step 1: For any $i \in I_T^1$, the string t must have an outgoing trivalent vertex as its endpoint (it cannot have one of the vertices along the bottom as its endpoint due to its orientation). Choose the smallest $i \in I_T^1$ and isotope the strings so that we pull out these two trivalent vertices from the rest of the tangle as shown in Fig 6.11. Then $T = W_i T_1$,

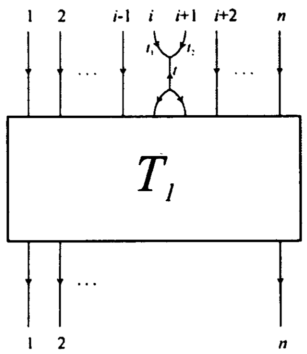


Figure 6.11:

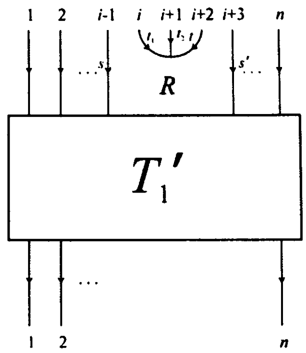


Figure 6.12:

and the number of trivalent vertices in T_1 is $d-2$, so that Step 1 reduces the complexity of the resulting tangle T_1.

If $I^1_{T_1} \neq \emptyset$, we choose the smallest $i \in I^1_{T_1}$, and repeat Step 1 for the tangle T_1 to get $T = W_{i_1}W_{i_2}T_2$. We continue in this way until we have $T = W_{i_1}W_{i_2}\cdots W_{i_l}T'$ for some T' and $I^1_{T'} = \emptyset$. If T' is the identity tangle we are done. Otherwise T' is a tangle with $d' = d - 2l$ trivalent vertices, for some $l \in \mathbb{N}$.

Step 2: Now let T' be an n-tangle such that $I^1_{T'} = \emptyset$, choose the smallest $i \in I^2_{T'}$ such that R has outgoing trivalent vertices on its boundary, where R is the region in T' bounded by strings s and s', as in Fig 6.12, where T_1' is an $(m-3,m)$-tangle.

We choose the first outgoing trivalent vertex we meet as we move along the boundary in an anti-clockwise direction, starting from the vertex $i-1$ along the top, and isotope the strings to pull this vertex out from the rest of the tangle as shown in Figure 6.13. Then $T' = (W_iW_{i+1}W_i - W_i)T_2'$, for some tangle T_2' which contains $d'-2$ trivalent vertices. If T_2' is the identity tangle, we are done. Otherwise we repeat Step 1 for the tangle T_2'.

Continuing in this way reduces the number of trivalent vertices contained in each new tangle T' by two each time. However, suppose now that for every $i \in I^2_{T'}$, the region R in Fig 6.12 does not have any outgoing trivalent vertices along its boundary. Let j be the smallest such vertex.

Case (i): If $j = 1$ then there must be a through string from a vertex $k \geq 4$ along the top to vertex 1 along the bottom, since otherwise the string which has vertex 1 along the bottom as its endpoint must have an outgoing trivalent vertex as its other endpoint, contradicting the fact that the region R does not have any outgoing trivalent vertices along its boundary. We insert an embedded circle on this string, by replacing a part of the string by , and multiply by a scalar factor δ^{-1}. Then isotoping the strings to pull the new outgoing vertex up out of the rest of the tangle as in Step 2, we obtain $T_1' = \delta^{-1}(W_iW_{i+1}W_i - W_i)T_2'$, where T_2' has the same number of trivalent vertices

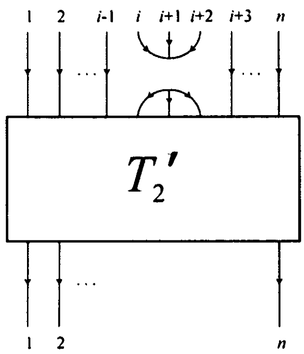


Figure 6.13:

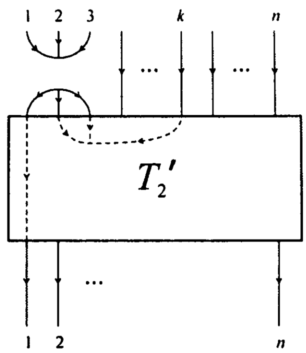


Figure 6.14:

as T_1'. The tangle T_2' has a through string from vertex 1 along the top to vertex 1 along the bottom (see Figure 6.14), and hence the sub-tangle to the right of this string is an $(m-1)$-tangle, which we know to be generated by W_i, $i = 1, \ldots, m-2$. Hence in Case (i) we have a tangle generated by W_i, $i = 1, \ldots, m-1$.

If $j > 1$, then there is a string s which has vertex $j-1$ as an endpoint.

Case (ii): Suppose first that the string s is a through string which has vertex k along the bottom as its other endpoint. We insert an embedded circle on the string s (and multiply by a scalar factor δ^{-1}), and isotope the strings to pull the outgoing vertex up out of the rest of the tangle as in Step 2 to obtain $T' = \delta^{-1}(W_i W_{i+1} W_i - W_i)T_2'$. We have now added and removed two trivalent vertices, hence the resulting tangle T_2' has the same number of trivalent vertices as T'. We now have $(j-1) \in I^2_{T_2'}$, and the string from vertex $(j+2)$ along the top is a through string which has vertex k along the bottom as its endpoint, as in Figure 6.15. If the region R' has an outgoing trivalent vertex along its boundary, we proceed as in Step 2, pulling the outgoing vertex out. Otherwise the string coming from vertex $(j-2)$ along the top must be a through string with endpoint vertex $(k-1)$ along the bottom. As before we insert an embedded circle on this string and isotope the strings to pull the outgoing vertex up out of the rest of the tangle as in Step 2. Continuing in this way will result in a tangle for which we can perform Step 2 without needing to insert an embedded circle. To see this, notice first that each new tangle T_l' now has a vertex $j_l \in I^2_{T_l'}$ such that $j_l = j_{l-1} - 1$, where j_{l-1} was the least integer in $I^2_{T_{l-1}'}$ for the previous tangle T_{l-1}'. Suppose we have the vertex $1 \in I^2_{T_l'}$, for some $l \in \mathbb{Z}$. Then T_l' will have a through string from the vertex 4 along the top to a vertex $k > 1$ ($k \equiv 0 \bmod 3$) along the bottom. The sub-tangle to the left of this string is a $(3,k)$-tangle, where the strings from the 3 vertices along the top meet at an incoming trivalent vertex, so the strings which have the vertices along the bottom as endpoints must each have an outgoing trivalent vertex as their other endpoint, hence there will be an outgoing trivalent vertex along the

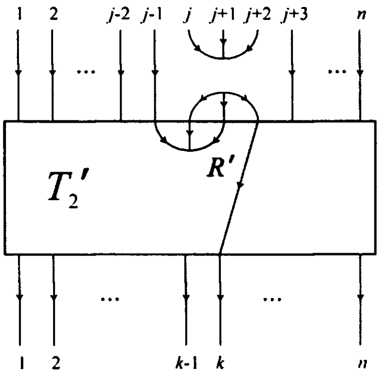


Figure 6.15:

boundary of R'. Then we perform the procedure of Step 2, which removes two trivalent vertices without first inserting any, and thus the resulting tangle will be less complex than T'.

Case (iii): Finally, suppose the string s has an incoming trivalent vertex as its endpoint. Then we insert an embedded circle along the string as before, and isotope the strings to pull the outgoing vertex up out of the rest of the tangle as in Step 2. If the resulting tangle T_1' is as in Case (i) or (ii), we follow the procedure described for those cases. Otherwise we are in Case (iii) again and we repeat the procedure described for Case (iii). Repeating this procedure enough times will also result in a situation where we can proceed as in Step 2 without needing to insert an embedded circle, since the smallest vertex $i \in I^2_{T_l'}$ is reduced by one each time. If we have $1 \in I^2_{T_p'}$ for some $p \in \mathbb{N}$, and we cannot use Step 2, then we must be in the situation of Case (i) considered above. In each case, we are able to reduce the number of trivalent vertices.

We now return to Step 1, and continue as above. Each use of Step 1, and each use of Step 2 without first inserting an embedded circle causes a tangle T containing d trivalent vertices to be written as $T = L_1 T'$, where L_1 is an element generated by W_i, $i = 1, \ldots, m-1$, and the number of trivalent vertices contained in T' is $d-2$. Cases (i)-(iii) each also result in a situation where we may use Step 1 or Step 2 without first needing to insert an embedded circle, and thus we can write $T = L_2 T''$, where L_2 is an element generated by W_i, $i = 1, \ldots, m-1$, and T'' is an m-tangle which does not contain any trivalent vertices, and hence must be the identity. Hence any m-tangle can be written as a linear combination of products of W_i, $i = 1, \ldots, m-1$.

Then the ideal I_m is contained in $SU(3)\text{-}TL_m$ and there is an isomorphism ψ : $SU(3)\text{-}TL_m \to V_m^{A_2}$ given by $\psi(w_i) = W_i$. □

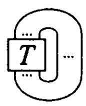
$$\mathrm{Tr}(T) \;=\; $$ 

Figure 6.16: $\mathrm{Tr}(T)$

6.1.4 Trace on $\mathcal{V}_n^{A_2}$

The following proposition is from [95, Prop. 1.2, p.375]:

Proposition 6.1.11 *The quotient $V_0^{A_2} = SU(3)\text{-}TL_0$ of the free vector space of all planar 0-tangles by the Kuperberg relations K1-K3 is isomorphic to $\mathbb{C}$.*

Proof

For an oriented surface F define $S(F)$ to be the quotient of the space of linear combinations of 0-tangles on F by the relations K1-K3. Let S^2 be the two-sphere, obtained from $\mathbb{R}^2$ by adding a point at infinity. Any 0-tangle on S^2 is a polyhedron which consists of polygons. These polygons must have an even number of edges since all the edges are oriented and the orientation changes at each vertex. Then $S = S(\mathbb{R}^2)$ is isomorphic to $S(S^2)$, where the isomorphism from S^2 to $\mathbb{R}^2$ is given by removing a point in the interior of one of the polygons. The polyhedron necessarily contain embedded circles or squares, because if it consisted of polygons which all have six or more edges, then we have $6f \leq 2e$ where f, e are the numbers of faces, edges respectively. Since the Euler number of S^2 is two and the tangle is trivalent, we have $v - e + f = 2$ and $2e = 3v$, where v is the number of vertices. Hence we have $6f = 12 + 2e$ which contradicts the above inequality. Therefore we can reduce the 0-tangle to obtain a scalar multiple of the empty tangle. $\square$

We define a trace Tr on $\mathcal{V}_m^{A_2}$ as follows. For an A_2-m-tangle $T \in \mathcal{V}_m^{A_2}$, we form the 0-tangle $\mathrm{Tr}(T)$ as in Figure 6.16 by joining the last vertex along the top of T to the last vertex along the bottom by a string which passes round the tangle on the right hand side, and joining the other vertices along the top to those on the bottom similarly. Then $\mathrm{Tr}(T)$ gives a value in $\mathbb{C}$ by Proposition 6.1.11. We could define the above trace as a right trace, and define a left trace similarly where the strings pass round the tangle on the left hand side. However, by the comments after Proposition 6.2.12 the right and left traces are equal. The trace of a linear combination of tangles is given by linearity. Clearly $\mathrm{Tr}(ab) = \mathrm{Tr}(ba)$ for any $a, b \in \mathcal{V}_m^{A_2}$, as in Figure 6.17. For any $x \in I_m$ we have $\mathrm{Tr}(x) = 0$, which follows trivially from the definition of Tr. Then Tr is well defined on $V_m^{A_2}$. We define a normalized trace tr on $\mathcal{V}_m^{A_2}$ by $\mathrm{tr} = \alpha^{-m}\mathrm{Tr}$, so that $\mathrm{tr}(\mathbf{1}_m) = 1$. Then tr is a Markov trace on V^{A_2} since for $x \in V_k^{A_2}$, $\mathrm{tr}(W_k x) = \delta\alpha^{-1}\mathrm{tr}(x)$, as illustrated in Figure

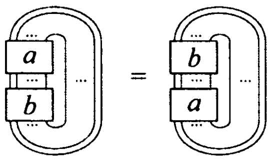


Figure 6.17: $\mathrm{Tr}(ab) = \mathrm{Tr}(ba)$

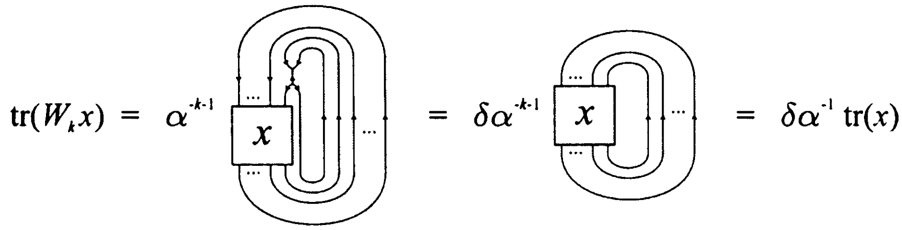


Figure 6.18: Markov trace on V^{A_2}

6.18, and in particular $\mathrm{tr}(W_i) = \delta\alpha^{-1}$. The Markov trace tr is positive by Lemma 6.1.12 and Theorem 1.2.1(b).

For each non-negative integer m we define an inner-product on $\mathcal{V}_m^{A_2}$ by

$$\langle S, T \rangle = \mathrm{tr}(T^* S), \tag{6.9}$$

which is well defined on $V_m^{A_2}$ since tr is.

For $\delta \geq 2$, there is an $x \geq 0$ such that $q = e^{2x}$ and $\delta = [2]_q$. For $\delta < 2$ such that $\delta = [2]_q = [2]$ where $q = e^{\pi i/n}$, $n \in \mathbb{N}$, we define $\widehat{V}_m^{A_2}$ to be the quotient of $V_m^{A_2}$ by the zero-length vectors in $V_m^{A_2}$ with respect to the inner-product defined in (6.9). Then the following lemma gives an identification between (a subalgebra of) the algebra of A_2-tangles and Wenzl's Hecke representations for $SU(3)$ (see Section 1.2.2).

Lemma 6.1.12 *For $\delta \geq 2$, the algebra $V_m^{A_2}$ is equivalent to Wenzl's representation π of the Hecke algebra, and consequently V^{A_2} gives a representation of the path algebra for $\mathcal{A}^{(\infty)}$. For $\delta = [2]_q$, $q = e^{\pi i/n}$, the algebra $\widehat{V}_m^{A_2}$ is equivalent to Wenzl's representation $\pi^{(3,n)}$ of the Hecke algebra, and consequently V^{A_2} gives a representation of the path algebra for $\mathcal{A}^{(n)}$.*

Proof

Clearly $\delta^{-1}W_i$, $i = 1, \ldots, m-1$, is a self-adjoint projection in $V_m^{A_2}$, and hence $V_m^{A_2}$ is a C^*-representation of $H_m(q^2)$ for any real $q \geq 1$ or $q = e^{\pi i/n}$. When $q = e^x$, $x \geq 0$, we have $\eta = (1-q^{2(-k+1)})/(1+q^2)(1-q^{-2k}) = \sinh((k-1)x)/2\cosh(x)\sinh(kx) = [k+1]_q/[2]_q[k]_q$, whilst for $q = e^{\pi i/n}$, $\eta = \sin((k-1)\pi/n)/2\cos(\pi/n)\sin(k\pi/n) = [k-1]/[2][k]$. Then for $k=3$, $\eta = [3]_q^{-1}$ so that the Markov trace on $V_m^{A_2}$ satisfies the condition in Theorem 1.2.1.

$\square$

Then the algebra $V_m^{A_2}$ is finite-dimensional for all finite m since the m^{th} level of the path algebra for $\mathcal{A}^{(n)}$ is finite-dimensional.

6.2 General A_2-planar algebras

6.2.1 Jones' planar algebras

Jones introduced the notion of a planar algebra in [64] to study subfactors. These planar algebras gave a geometric reformulation of the standard invariant. Since then there has been much interest in planar algebras; see for example [7], [6], [17], [48], [47], [52], [53], [62], [63], [65], [66], [74], [73], [90], [105].

Let us briefly review the basic construction of Jones' planar algebras. A planar k-tangle consists of a disc D in the plane with $2k$ vertices on its boundary, $k \geq 0$, and $n \geq 0$ internal discs D_j, $j = 1, \ldots, n$, where the disc D_j has $2k_j$ vertices on its boundary, $k_j \geq 0$. One vertex on the boundary of each disc (including the outer disc D) is chosen as a marked vertex, and the segment of the boundary of each disc between the marked vertex and the vertex immediately adjacent to it as we move around the boundary in an anti-clockwise direction is labelled $*$. Inside D we have a collection of disjoint smooth curves, called strings, where any string is either a closed loop, or else has as its endpoints the vertices on the discs, and such that every vertex is the endpoint of exactly one string. Any tangle must also allow a checkerboard coulouring of the regions inside D, which are bounded by the strings and the boundaries of the discs, where every region is coloured black or white such that any two regions which share a common boundary are not coloured the same, and any region which meets the boundary of a disc at the segment marked $*$ is coloured white. When the outer disc has no vertices on its boundary, we replace 0 by $\pm$, where the region which meets the outer boundary is coloured black for a $+$-tangle and white for a $-$-tangle.

A planar k-tangle with an internal disc D_j with $2k_j$ vertices on its boundary can be composed with a k_j-tangle S, giving a new k-tangle $T \circ_j S$, by inserting the tangle S inside the inner disc D_j of T such that the vertices on the outer disc of S coincide with those on the disc D_j, and in particular the two marked vertices must coincide. The boundary of the disc D_j is then removed, and the strings are smoothed if necessary. The collection of all diffeomorphism classes of such planar tangles, with composition defined as above, is called the planar operad.

A planar algebra P is then defined to be an algebra over this operad, i.e. a family $P = (P_0^+, P_0^-, P_k, k > 0)$ of vector spaces with $P_0^\pm \subset P_k \subset P_{k'}$ for $0 < k < k'$, and with the following property. For every k-tangle T with n internal discs D_j labelled by elements $x_j \in P_{k_j}$, $j = 1, \dots, n$, there is an associated linear map $Z(T) : \otimes_{j=1}^n P_{k_j} \to P_k$, which is compatible with the composition of tangles and re-ordering of internal discs.

6.2.2 General A_2-planar algebras

We will now define an A_2-version of Jones' planar algebra, using tangles generated by Kuperberg's A_2-spiders rather than genuinely planar tangles. Under certain assumptions, these A_2-planar algebras will correspond to certain subfactors of $SU(3)$ $\mathcal{ADE}$ graphs which have flat connections. The best way to describe planar algebras is in terms of operads (see May [85]).

Definition 6.2.1 *An* ***operad*** *consists of a sequence $(\mathcal{C}(n))_{n\in\mathbb{N}}$ of sets. There is a unit element 1 in $\mathcal{C}(1)$, and a function $\mathcal{C}(n) \otimes \mathcal{C}(j_1) \otimes \cdots \otimes \mathcal{C}(j_n) \to \mathcal{C}(j_1 + \cdots + j_n)$ called composition, given by $(y \otimes x_1 \otimes \cdots \otimes x_n) \to y \circ (x_1 \otimes \cdots \otimes x_n)$, satisfying the following properties*

- *associativity:* $y \circ (x_1 \circ (x_{1,1} \otimes \cdots \otimes x_{1,k_1}) \otimes \cdots \otimes x_n \circ (x_{n,1} \otimes \cdots \otimes x_{n,k_n}))$
$= (y \circ (x_1 \otimes \cdots \otimes x_n)) \circ (x_{1,1} \otimes \cdots \otimes x_{1,k_1} \otimes \cdots \otimes x_{n,1} \otimes \cdots \otimes x_{n,k_n})$,
- *identity:* $y \circ (1 \otimes \cdot \otimes 1) = y = 1 \circ y$.

We will define two types of A_2-planar tangles, which we will call $(+)i, j$-tangles and $(-)i, j$-tangles. An A_2-planar $(\pm)i, j$-tangle (note, this is different from an (i, j)-tangle) will be the unit disc $D = D_0$ in $\mathbb{C}$ together with a finite (possibly empty) set of disjoint sub-discs $D_1, D_2, \dots, D_n$ in the interior of D. Each disc D_k, $k \geq 0$, will have an even number $2(i_k + j_k) \geq 0$ of vertices on its boundary ∂D_k ($i_0 = i, j_0 = j$). The first j_k vertices are restricted to be sources, the next $2i_k$ vertices alternate between sources and sinks, and finally the last j_k vertices are all sinks. For a $(+)$-tangle, vertex $j_k + 1$ is restricted to be a source for all k, whilst for a $(-)$-tangle it is a sink. We will position the vertices so that the first $i_k + j_k$ are along the boundary for the upper half of the disc, which we will call the top edge, and the next $i_k + j_k$ vertices are along the boundary for the bottom half of the disc, which we will call the bottom edge. We will use the convention of numbering the vertices along the bottom edge in reverse order, so that the $2(i_k + j_k)$-th vertex is called the first vertex along the bottom edge. For a $(+)$-tangle, the total number of source vertices along the top edge is $\lfloor j_k + (i_k+1)/2 \rfloor$, and the number of sink vertices is $\lfloor i_k/2 \rfloor$, whilst for a $(-)$-tangle the corresponding numbers are $\lfloor j_k + (i_k/2) \rfloor$ and $\lfloor (i_k+1)/2 \rfloor$. Inside D we have a tangle where the endpoint of any string is either a trivalent vertex (see Figure 6.1) or one of the vertices on the boundary of a disc D_k, $k = 0, \ldots, n$, or else the string forms a closed loop. Each vertex on the boundaries of the D_k is the endpoint of exactly one string, which meets ∂D_k transversally. An example of an A_2-planar $(+)0,4$-tangle is illustrated in Figure 6.19.

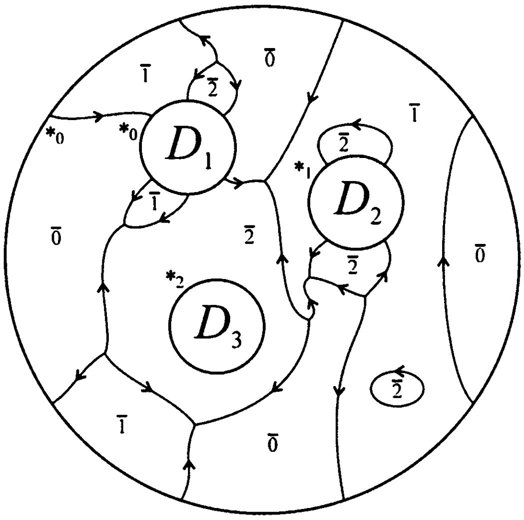


Figure 6.19: A_2-planar tangle

The regions inside D have as boundaries segments of the ∂D_k or the strings. These regions are labelled $\bar{0}$, $\bar{1}$ or $\bar{2}$- called the colouring- such that if we pass from a region R of colour $\bar{a}$ to an adjacent region R' by crossing a vertical string with downwards orientation, the R' has colour $\overline{a+1}$ (mod 3). We mark the segment of each ∂D_k between the last and first vertices with $*_{b_k}$, $b_k \in \{0,1,2\}$, so that the region inside D which meets ∂D_k at this segment is of colour $\overline{b_k}$, and the choice of these $*_{b_k}$ must give a consistent colouring of the regions. For the outer boundary ∂D we impose the restriction $b_0 = 0$. For $i,j = 0,0$ we have three types of tangle, depending on the colour $\bar{b}$ of the region near ∂D.

Let σ be + or -. We define $\widetilde{\mathcal{P}}_{(\sigma)i,j}(L_\sigma)$ to be the free vector space generated by orientation-preserving diffeomorphism classes of A_2-planar $(\sigma)i,j$-tangles with labelling sets L_σ. The diffeomorphisms preserve the boundary of D, but may move the D_k's, $k \geq 1$. Let $\mathcal{P}_{(\sigma)i,j}(L_\sigma)$ be the quotient of $\widetilde{\mathcal{P}}_{(\sigma)i,j}(L_\sigma)$ by the Kuperberg relations K1-K3. The A_2-(σ)-**planar operad** $\mathcal{P}_{(\sigma)}(L_\sigma)$ is defined to be $\mathcal{P}_{(\sigma)}(L_\sigma) = \bigcup_{i,j} \mathcal{P}_{(\sigma)i,j}(L_\sigma)$. We will usually simply write $\mathcal{P}_{(\sigma)}$ for $\mathcal{P}_{(\sigma)}(L_\sigma)$ when the labelling set is obvious. For $\mathcal{P}_{(\sigma)i,\infty}$ we use the

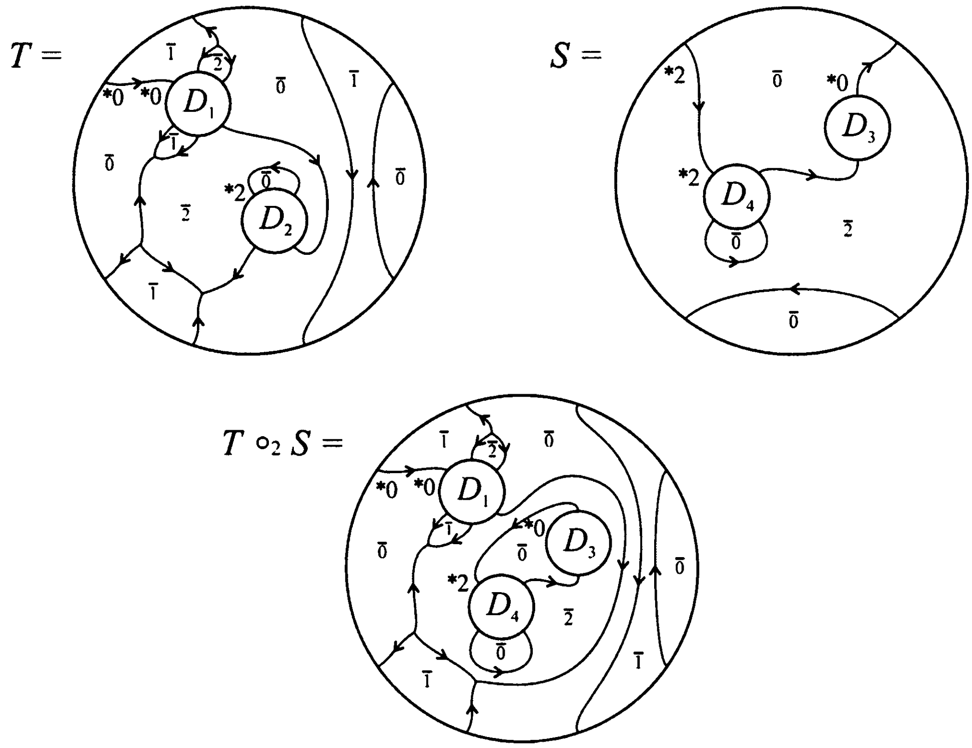


Figure 6.20: Composition of planar tangles

convention that the region of any tangle in $\mathcal{P}_{(\sigma)i,\infty}$ which meets the segment of the outer boundary between vertices v_0 and v_1 has colour $\overline{0}$.

We define composition in $\mathcal{P}_{(\sigma)}$ as follows. Given an A_2-planar $(\sigma)i,j$-tangle T with an internal disc D_l with $i_l, j_l = i', j'$ vertices on its boundary, and an A_2-planar $(\sigma)i',j'$-tangle S with external disc D', such that the orientations of the vertices on its boundary are consistent with those of D_l and $*_{D'} = *_{D_l}$. We define the $(\sigma)i,j$-tangle $T \circ_l S$ by isotoping S so that its boundary and vertices coincide with those of D_l, join the strings at ∂D_l and smooth if necessary. We the remove ∂D_l to obtain the tangle $T \circ_l S$ whose diffeomorphism class clearly depends only on those of T and S. This gives $\mathcal{P}_{(\sigma)}$ the structure of a **coloured operad**, where each D_k, $k > 0$, is assigned the pattern i_k, j_k, and composition is only allowed when the colouring of the regions match (which forces the orientations of the vertices to agree). There are three distinct patterns for $i,j = 0,0$, corresponding to the colouring of the region near the boundary. The D_k's, $k \geq 1$ are to be thought of as inputs, and $D = D_0$ is the output.

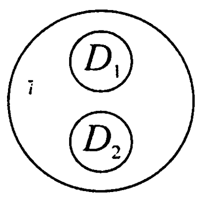
The most general notion of an A_2-planar algebra will be an algebra over the operad $\mathcal{P}_{(\sigma)}$, i.e. a **general A_2-(σ)-planar algebra** $P_{(\sigma)}$ is a family

$$P_{(\sigma)} = \left(P_{(\sigma)i,j}, i,j > 0, i,j \neq 0,0, P^{\overline{a}}_{(\sigma)0,0}, a \in \{0,1,2\}\right)$$

of vector spaces with $P^{\overline{a}}_{(\sigma)0,0} \subset P_{(\sigma)i,j} \subset P_{(\sigma)i',j'}$ for $0 < i \leq i'$, $0 < j \leq j'$, $a \in \{0,1,2\}$, and with the following property: for every labelled $(\sigma)i,j$-tangle $T \in \mathcal{P}_{(\sigma)i,j}$ with internal discs $D_1, D_2, \ldots, D_n$, where D_k has pattern i_k, j_k, there is associated a linear map $Z(T) : \otimes_{k=1}^{n} P_{(\sigma)i_k,j_k} \longrightarrow P_{(\pm)i,j}$ which is compatible with the composition of tangles in the following way. If S is a $(\sigma)i_k, j_k$-tangle with internal discs $D_{n+1}, \ldots, D_{n+m}$, where D_k has pattern i_k, j_k, then the composite tangle $T \circ_l S$ is a $(\sigma)i,j$-tangle with $n+m-1$ internal discs D_k, $k = 1,2,\ldots l-1, l+1, l+2, \ldots, n+m$. From the definition of an operad, associativity means that the following diagram commutes:

$$\begin{tikzcd}
\left(\bigotimes_{\substack{k=1\\k\neq l}}^{n} P_{(\sigma)i_k,j_k}\right) \otimes \left(\bigotimes_{k=n+1}^{n+m} P_{(\sigma)i_k,j_k}\right) \arrow[d, "\mathrm{id}\otimes Z(S)"'] \arrow[dr, "Z(T\circ_l S)"] & \\
\bigotimes_{k=1}^{n} P_{(\sigma)i_k,j_k} \arrow[r, "Z(T)"'] & P_{(\sigma)i,j},
\end{tikzcd} \tag{6.10}$$

so that $Z(T \circ_l S) = Z(T')$, where T' is the tangle T with $Z(S)$ used as the label for disc D_l. We also require $Z(T)$ to be independent of the ordering of the internal discs, that is, independent of the order in which we insert the labels into the discs. If $i = j = 0$, we adopt the convention that the empty tensor product is the complex numbers $\mathbb{C}$. By using the tangle



we see that each $P^{\overline{a}}_{(\sigma)0,0}$ (or simply $P^{\overline{a}}_{(\sigma)0}$) is a commutative associative algebra, $a \in \{0,1,2\}$. So each $P_{(\sigma)i,j}$ has a distinguished subset $\{Z(T)\}$ for every $(\sigma)i,j$-tangle T without any internal discs. This is the unital operad (see [85]). Following Jones's terminology, we call the linear map Z the **presenting map** for $P_{(\sigma)}$.

The usual A_2-planar algebra will be the $(+)$ one. However, in the rest of this subsection we will mean by a general A_2-planar algebra P either $(+)$ or $(-)$ versions. In the figures we omit the orientation on the strings from $(j+1)$-th vertices along the top and bottom of an i,j-tangle- these will be determined by whether the tangle is an $(+)$ or a $(-)$-tangle.

6.2.3 Partial Braiding

We now introduce the notion of a partial braiding in our A_2-planar operad. We will allow over and under crossings in our diagrams, which are interpreted as follows. For a tangle T with n crossings $c_1, \ldots, c_n$, choose one of the crossings c_i and, isotoping any strings if necessary, we enclose c_i in a disc b, as shown in Figure 6.21 for c_i a (i) negative crossing and (ii) positive crossing (up to some rotation of the disc).

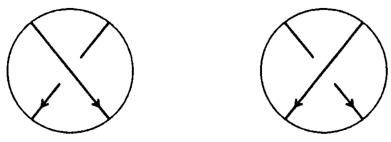


Figure 6.21: Disc b for (i) negative crossing, (ii) positive crossing

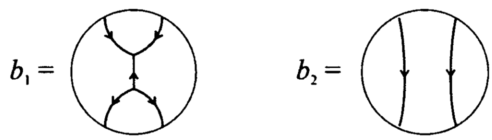


Figure 6.22: Discs b_1 and b_2

Let b_1, b_2 be the discs illustrated in Figure 6.22. We form two new tangles $S_1^{(1)}$ and $T_1^{(1)}$ which are identical to T except that we replace the disc b by b_1 for $S_1^{(1)}$ and by b_2 for $T_1^{(1)}$. If c_i is a negative crossing then T is equal to the linear combination of tangles $q^{-2/3}S_1^{(1)} - q^{1/3}T_1^{(1)}$, and if c_i is a positive crossing $T = q^{2/3}S_1^{(1)} - q^{-1/3}T_1^{(1)}$, where $q > 0$ satisfies $q + q^{-1} = \delta$ (cf. (6.1) and (6.2)). Then for both $S_1^{(1)}$ and $T_1^{(1)}$ we consider another crossing c_j and repeat the above process to obtain $S_1^{(1)} = r_1 S_1^{(2)} - r_1' T_1^{(2)}$, $T_1^{(1)} = r_2 S_2^{(2)} - r_2' T_2^{(2)}$, where $r_1, r_2 \in \{q^{\pm 2}\}$ and $r_1', r_2' \in \{q^{\pm 1}\}$ depending on whether c_j is a positive or negative crossing. Since this 'expansion' of the crossings is independent of the order in which the crossings are selected, repeating this procedure we obtain a linear combination $T = \sum_{i=1}^{2(n-1)} (s_i S_i^{(n)} + s_i' T_i^{(n)})$, where the s_i, s_i' are powers of $q^{\pm 1/3}$.

With this definition of a partial braiding, two tangles give identical elements of the planar algebra if one can be deformed into the other using relations (6.3)-(6.6). It is not a genuine braiding as we cannot in general pull strings over or under labelled inner discs D_k.

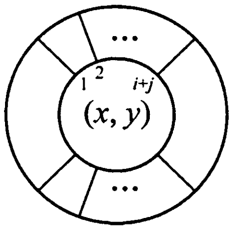


Figure 6.23: Tangle $I_{i,j}$

6.2.4 Finite-Dimensionality and General A_2-Planar Subalgebras

The tangles $I_{i,j}$ illustrated in Figure 6.23 have $2i, 2j$ vertices on the inner and outer discs and all strings are through strings from vertex k on the outer boundary along the top, bottom respectively to vertex k on the inner boundary along the top, bottom respectively. These tangles satisfy $I_{i,j} \circ T = T$ and also inserting I_{i_k,j_k} inside every inner disc D_k with pattern i_k, j_k also gives the original tangle T. Then $I_{i,j}$ is the unit element (see Definition 6.2.1). We let $I_{i,j}(x)$ denote the tangle $I_{i,j}$ with $x \in P_{i,j}$ as the label for the inner disc. Then, since we must have $Z(I_{i,j}(x)) = x$ we require $Z(I_{i,j}) = \mathrm{id}_{P_{i,j}}$. This means that the range of Z spans $P_{i,j}$, by using any element of $P_{i,j}$ as the insertion in the inner disc of $I_{i,j}$.

The condition $\dim(P_0^{\overline{a}}) = 1$, $a = 0,1,2$, implies that there is a unique way to identify each $P_0^{\overline{a}}$ with $\mathbb{C}$ as algebras, and $Z(\bigcirc_a) = 1$, $a = 0,1,2$, where $\bigcirc_a$ is the empty tangle with no vertices or strings at all, with the interior coloured a. By Lemma 6.1.7 there is thus also one scalar, or parameter, associated to a general A_2-planar algebra:

$$Z(\circledcirc) = \alpha, \tag{6.11}$$

where the inner circle is a closed loop not an internal disc.

It follows from the compatability condition (6.10) that Z is multiplicative on connected components, i.e. if a part of a tangle Y can be surrounded by a disc so that $T = T' \circ_l S$ for a tangle T' and 0-tangle S, then $Z(T) = Z(S)Z(T')$ where $Z(S)$ is a multilinear map into the field $\mathbb{C}$.

Every general A_2-planar algebra contains the A_2-planar subalgebra STL, where for $a \in \{0,1,2\}$, $STL_{\overline{a}} \cong \mathbb{C}$, and $STL_{i,j} = \mathcal{P}_{i,j}(\varnothing)$. Here the presenting map Z is just the identity map. Note that the partial braiding defined above is a genuine braiding in STL. The $SU(3)$-Temperley-Lieb algebra introduced in section 6.1.3 is a subalgebra of STL, given by $SU(3)\text{-}TL_n = STL_{0,n}$. The action of an A_2-planar i,j-tangle T on STL is given by filling the internal discs of T with basis elements of STL. The resulting tangle may then contain embedded circles and squares, which are removed using K2 and K3, and

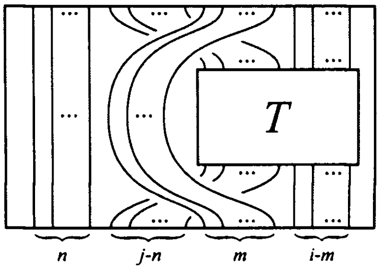


Figure 6.24: A tangle in $\mathcal{P}_{i,j}^{(m,n)}$

closed curves are removed using 6.11. The result is a linear combination of elements of STL.

Definition 6.2.2 *A general A_2-planar algebra P will be called* ***finite-dimensional*** *if* $\dim P_{i,j} < \infty$ *for all i, j.*

Remark. The algebras $SU(3)$-TL_n are finite dimensional, since from section 6.1.3 we know that they are isomorphic to the path algebra for the $SU(3)$ graph $\mathcal{A}^{(\infty)}$. Then by Lemma 6.2.17 it can be shown that $STL_{i,j}$ is finite dimensional for all $i, j \geq 0$. This result also follows from [78], since $STL_{i,j}$ and $STL_{0,i+j}$ have the same number of source and sink vertices along the outer boundary, and by Theorem 6.3 in [78] the dimensions must be the same.

We will now define A_2-planar subalgebras $P_{i,j}^{(m,n)} \subset P_{i,j}$.

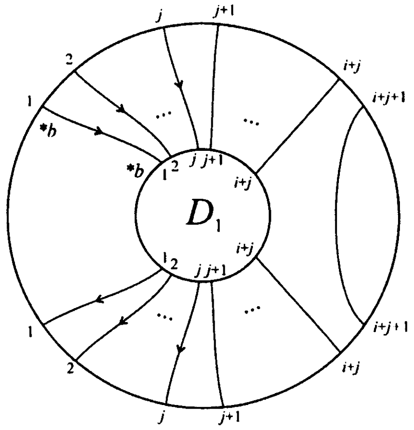
Definition 6.2.3 *For $0 \leq m \leq j$, let $\mathcal{P}_{i,j}^{(m,n)}$ denote the subset of $\mathcal{P}_{i,j}$ spanned by all tangles with the first n vertices along the top and bottom connected by vertical straight lines, or through strings. The vertices $n+j+1, \ldots, n+j+m$ are connected by through strings which pass over every string they cross such that there are no internal discs in the region between the strings and the outer boundary of the tangle to the left of them. If P is a general A_2-planar algebra with presenting map Z, we define $P_{i,j}^{(m,n)} = Z(\mathcal{P}_{i,j}^{(m,n)}) \subset P_{i,j}$.*

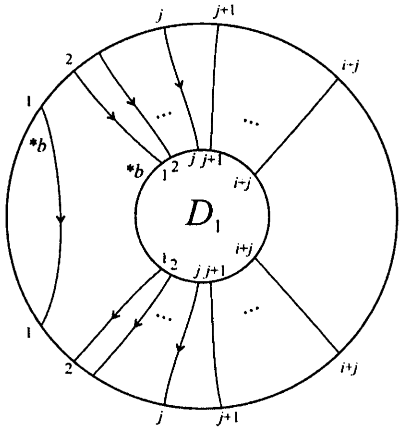
A general tangle in $\mathcal{P}_{i,j}^{(m,n)}$ is illustrated in Figure 6.24, where we have replaced the outer disc by a rectangle, and T is any tangle in $\mathcal{P}_{i-m,j-n}$. Note that $\mathcal{P}_{i,j}^{(0,0)} = \mathcal{P}_{i,j}$. We also have $P_{0,1}^{(0,1)} \cong P_0^{\bar{1}}$ since $P_{0,1}^{(0,1)}$ is just $P_0^{\bar{1}}$ with a vertical line added to the left. Similarly $P_{0,2}^{(0,2)} \cong P_0^{\bar{2}}$.

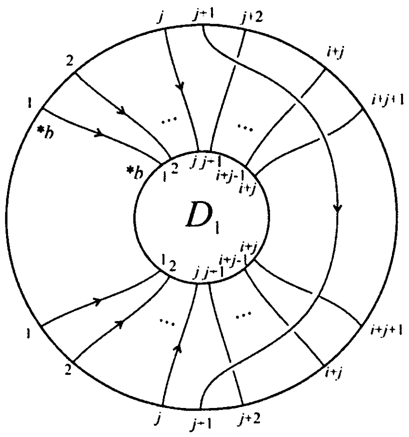
6.2.5 Basic Tangles in a General A_2-Planar Algebra

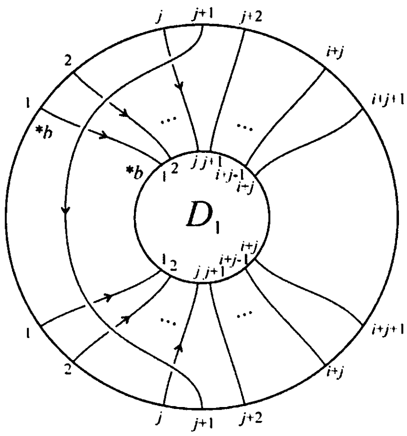
We have the following basic tangles:

- *Inclusion tangles* $IR^{i,j}_{i+1,j}$, $IL^{i,j}_{i,j+1}$, $IR^{i,j}_{i,j+1}$ and $\widetilde{IR}^{i,j}_{i,j+1}$:

$IR^{i,j}_{i+1,j} =$ 

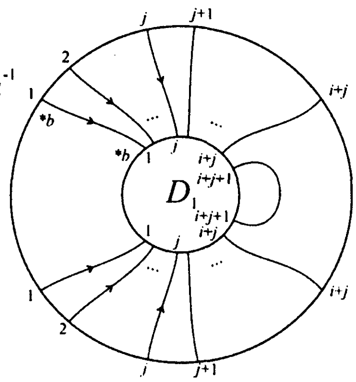
$IL^{i,j}_{i,j+1} =$ 

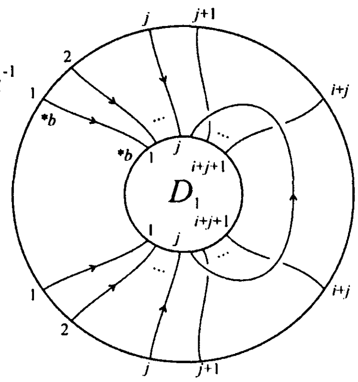
$IR^{i,j}_{i,j+1} =$ 

$\widetilde{IR}^{i,j}_{i,j+1} =$ 

where the orientation of the rightmost string in $I^{i,j}_{i+1,j}$ is downwards for i even and upwards for i odd. Both $IR^{i,j}_{i,j+1}$ and $\widetilde{IR}^{i,j}_{i,j+1}$ add a new source vertex along the top which immediately to the right of the first j source vertices, and a sink vertex along the bottom immediately to the right of the first j sink vertices along the bottom. These new vertices are regarded as being among the downwards oriented vertices rather than the alternating vertices. They are connected by a through string, and differ only in that the through string passes to the right of the inner disc in $IR^{i,j}_{i,j+1}$ and to the left of the inner disc in $\widetilde{IR}^{i,j}_{i,j+1}$. We have $Z(I^{i+1,j}_{i,j}) : P_{i,j} \to P_{i+1,j}$, and $Z(IL^{i,j}_{i,j+1}), Z(IR^{i,j}_{i,j+1}), Z(\widetilde{IR}^{i,j}_{i,j+1}), : P_{i,j} \to P_{i,j+1}$.

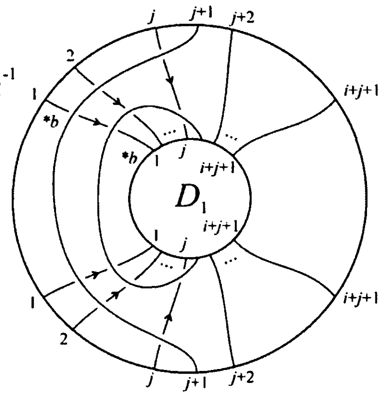
- *Conditional expectation tangles* $ER^{i+1,j}_{i,j}$ and $ER^{i,j+1}_{i,j}$:

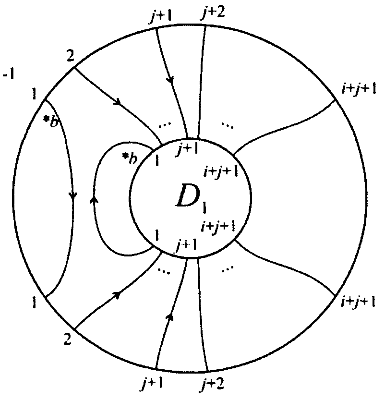
$ER^{i+1,j}_{i,j} = \alpha^{-1}$ 

$ER^{i,j+1}_{i,j} = \alpha^{-1}$ 

(6.12)

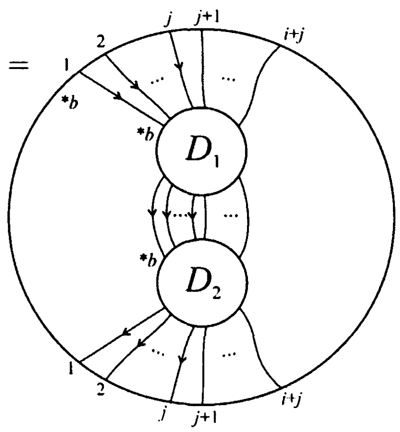
The orientation of the string from vertex $i+j+1$ on the inner disc of $ER^{i+1,j}_{i,j}$ is clockwise for i odd and anticlockwise for i even. We have $Z(ER^{i+1,j}_{i,j}) : P_{i+1,j} \to P_{i,j}$ and $Z(ER^{i,j+1}_{i,j}) : P_{i,j+1} \to P_{i,j}$. We also have the conditional expectation tangles $EL^{i+1,j}_{i+1,j}$ and $EL^{i,j+1}_{i,j+1}$:

$EL^{i+1,j}_{i+1,j} = \alpha^{-1}$ 

$EL^{i,j+1}_{i,j+1} = \alpha^{-1}$ 

where $Z(EL^{i+1,j}_{i+1,j}) : P_{i+1,j} \to P^{(1,0)}_{i+1,j}$ and $Z(EL^{i,j+1}_{i,j+1}) : P_{i,j+1} \to P^{(0,1)}_{i,j+1}$.

- *Multiplication tangles* $M_{i,j} : \mathcal{P}_{i,j} \times \mathcal{P}_{i,j} \to \mathcal{P}_{i,j}$:

$M_{i,j} =$ 

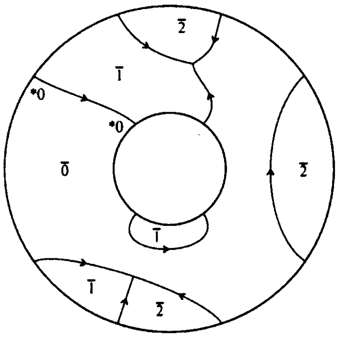


Figure 6.25: Annular tangle

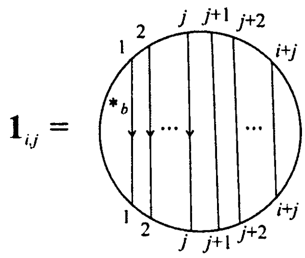


Figure 6.26: Identity Tangle $\mathbf{1}_{i,j}$

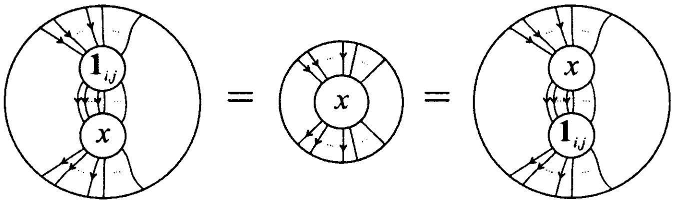


Figure 6.27: $Z(\mathbf{1}_{i,j})x = x = xZ(\mathbf{1}_{i,j})$

Each $P_{i,j}$ is then an associative algebra, with multiplication being defined by $x_1x_2 = Z(M_{i,j}(x_1, x_2))$, where $M_{i,j}(x_1, x_2)$ has $x_k \in P_{i,j}$ as the insertion in disc D_k. The multiplication is also clearly compatible with the inclusion tangles, as can be seen by drawing pictures.

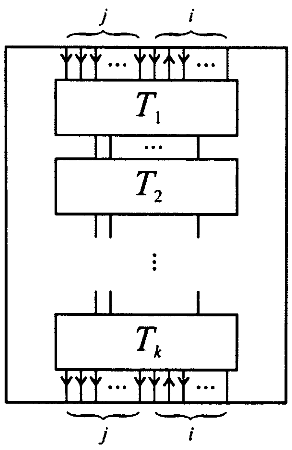
An annular tangle with outer disc with pattern i, j and inner disc with pattern i', j' will be called an annular $(i, j : i', j')$-tangle. An example of an annular $(2, 2 : 0, 2)$-tangle is illustrated in Figure 6.25.

The tangle $\mathbf{1}_{i,j}$ illustrated in Figure 6.26 is called the identity tangle. By inserting $\mathbf{1}_{i,j}$ and $x \in P_{i,j}$ into the discs of the multiplication tangle $M_{i,j}$ as in Figure 6.27 we see that $Z(\mathbf{1}_{i,j})x = x = xZ(\mathbf{1}_{i,j})$, hence $Z(\mathbf{1}_{i,j})$ is the left and right identity for $P_{i,j}$.

Proposition 6.2.4 *The A_2-planar operad $\mathcal{P}$ is generated by the algebra STL, multiplication tangles M, and annular tangles, which are tangles with only one internal disc.*

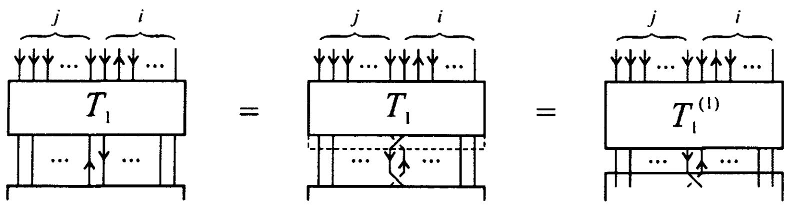
Proof

Consider an arbitrary tangle $T \in \mathcal{P}_{i,j}$ with k inner discs D_l with labels x_l, $l = 1, \ldots, k$. By an isotopy of the tangle, we may move all the inner discs so that in any horizontal strip there is only one disc. Then we may draw T as

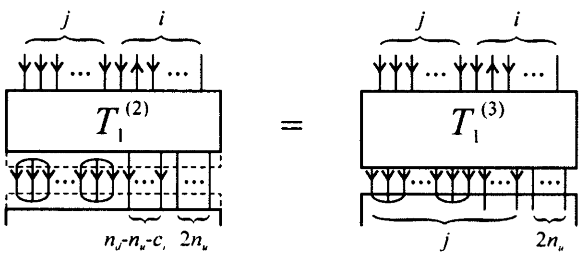


where the T_l are annular tangles with inner disc labelled by x_l, $l = 1, \ldots, k$.

Consider first the tangle T_1, which has pattern i, j along the top. Let n_d, n_u be the number of strings along the bottom of T_1 with downwards, upwards orientation respectively. Using the partial braiding we may switch any pair of strings along the bottom of T_1, and we replace T_1 by the annular tangle $T_1^{(1)}$:

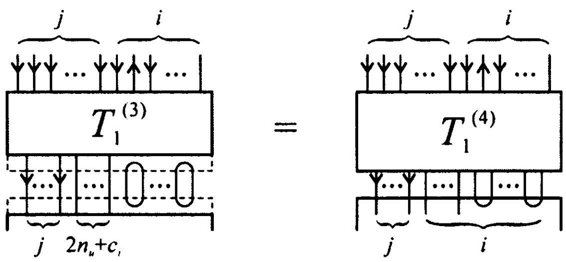


In this way we may permute all the strings along the bottom of T_1 to obtain an annular tangle $T_1^{(2)}$, where the first $n_d - n_u - c_i$ strings along the bottom all have downwards orientation, and the next $2n_u + c_i$ have alternating orientations (with the $(n_d - n_u - c_i + 1)$-th string oriented downwards), where c_i is 0 if i is even and 1 if i is odd. Then if $j > n_d - n_u - c_i$ we have $j = n_d - n_u - c_i + 3p$ for some $p \in \mathbb{N}$, so we add p "double loops"  at the bottom of $T_1^{(2)}$ (and multiply the tangle T by a scalar factor $\alpha^{-p}\delta^{-p}$):



On the other hand, if $j < n_d - n_u - c_i$ we have $j = n_d - n_u - c_i - 3p$ for some $p \in \mathbb{N}$, so we add p "double loops" at the top of $T_1^{(2)}$ instead. Similarly, if $i > 2n_u + c_i$ (respectively, $i < 2n_u + c_i$) then $i = 2n_u + c_i + 2p'$ for some $p' \in \mathbb{N}$ (respectively, $-p' \in \mathbb{N}$), and we add

p' closed loops at the bottom (respectively, top) of $T_1^{(3)}$ (and multiply by a scalar factor $\alpha^{-p'}$), and replace $T_1^{(3)}$ by the annular tangle $T_1^{(4)}$:



where the orientation of the closed loops is anticlockwise for i odd, clockwise for i even. Let $i' = \max(i, 2n_u + c_i)$, $j' = \max(j, n_d - n_u - c_i)$. Then we have a multiplication tangle $M_{i',j'}$ surrounded by an annular $(i,j : i',j')$-tangle A, with $T_1^{(4)}$ as the insertion for the first disc of $M_{i',j'}$, and the rest of the tangle, which we will call T', as the insertion for the second disc. If $i' = i$ and $j' = j$ then the annular tangle A is just $I_{i,j}$. Then T' is an i',j'-tangle with $k-1$ inner discs, and by the above procedure we can write T' as a multiplication tangle (possibly surrounded by an annular tangle), where the insertion for the second disc now only has $k-2$ inner discs. Continuing in this way we see that T is generated by multiplication tangles and annular tangles. Finally, tangles with no inner discs are elements of STL. $\square$

6.2.6 A_2-Planar Algebras

We now define an A_2-planar algebra P and two traces on P, as well as notions of non-degeneracy, sphericity and flatness.

Definition 6.2.5 *An A_2-**planar algebra** will be a general A_2-planar algebra P with* $\dim(P_0) = \dim(P_{0,1}^{(0,1)}) = \dim(P_{0,2}^{(0,2)}) = 1$, *and* $Z(\circledcirc) = \alpha$ *non-zero.*

Definition 6.2.6 *We call the presenting map Z the **partition function** when it is applied to a closed $0,0$-tangle T with internal discs D_k of pattern i_k, j_k. We identify $P_0^{\bar{\alpha}}$ with $\mathbb{C}$, so that $Z(T) : \otimes_k P_{i_k,j_k} \longrightarrow \mathbb{C}$.*

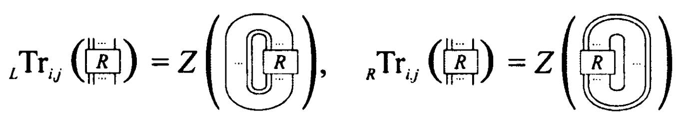
We define non-degeneracy and sphericity in the same way as Jones (see Definition 1.27 of [64]):

Definition 6.2.7 *An A_2-planar algebra will be called **non-degenerate** if, for $x \in P_{i,j}$, $x = 0$ if and only if $Z(A(x)) = 0$ for all annular $(0 : i,j)$-tangles A.*

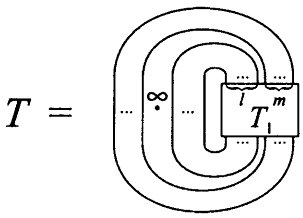
Definition 6.2.8 *An A_2-planar algebra will be called* **spherical** *if its partition function is an invariant of tangles on the two-sphere S^2 (obtained from $\mathbb{R}^2$ by adding a point at infinity).*

Definition 6.2.7 of non-degeneracy of an A_2-planar algebra involves all ways of closing a tangle. For a spherical algebra it is enough only to consider the following:

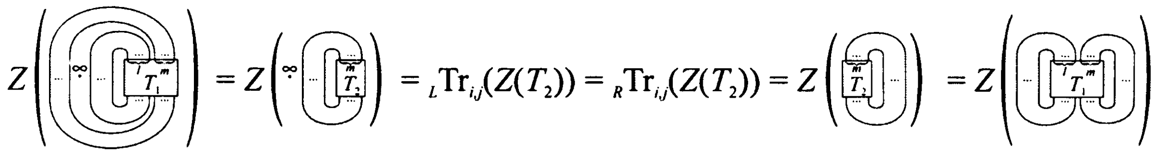
Definition 6.2.9 *Let P be an A_2-planar algebra. Define two traces ${}_L\mathrm{Tr}_{i,j}$ and ${}_R\mathrm{Tr}_{i,j}$ on $P_{i,j}$ by*



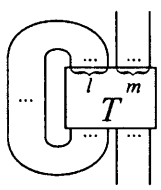
For a spherical A_2-planar algebra ${}_L\mathrm{Tr}_{i,j} = {}_R\mathrm{Tr}_{i,j} =: \mathrm{Tr}_{i,j}$. The converse is also true-that is, if ${}_L\mathrm{Tr}_{i,j} = {}_R\mathrm{Tr}_{i,j}$ on $P_{i,j}$ for all $i, j \geq 0$ then P is spherical. Let T be any labelled 0-tangle. For sphericity we require $Z(T)$ to be invariant no matter where we choose the "point at infinity" to be. We make a choice of a point at infinity and isotope T so that it looks as follows



where ∞ indicates the chosen point. Then



for any integers $l, m \geq 0$, where T_2 is the tangle:

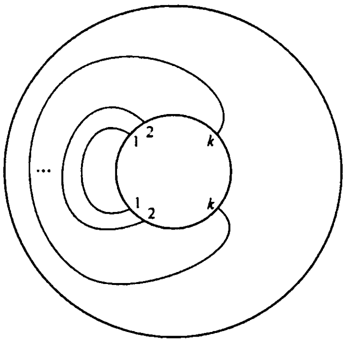


Let P be a spherical algebra. If we define $\mathrm{tr}(x) = \alpha^{-i-j}\,\mathrm{Tr}_{i,j}(x)$ for $x \in P_{i,j}$, then tr is compatible with the inclusions $P_{i,j} \subset P_{i,j+1}$ and $P_{i,j} \subset P_{i+1,j}$, given by $IR^{i,j}_{i,j+1}$, $IR^{i,j}_{i+1,j}$ respectively, and $\mathrm{tr}(1) = 1$, and so defines a trace on P itself. The following proposition is given in [64] in the setting of his A_1-planar algebras. The proof for A_2-planar algebras is identical, the only difference being in the orientation of the strings.

Proposition 6.2.10 *A spherical A_2-planar algebra P is non-degenerate if and only if $\mathrm{Tr}_{i,j}$ defines a non-degenerate bilinear form on $P_{i,j}$ for each i, j.*

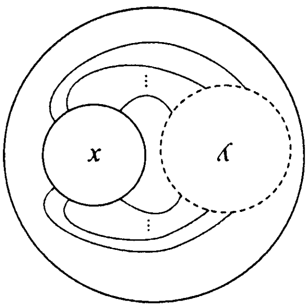
Proof

($\Rightarrow$) The picture defining Tr is $Z(A(x))$ where A is the annular $(0 : i, j)$-tangle

$A =$ 

and therefore $\mathrm{Tr}(x) = Z(A(x)) = 0 \Leftrightarrow x = 0$.

($\Leftarrow$) It is enough to show that for any annular $(0 : i, j)$-tangle A, there is a $y \in P_{i,j}$ such that $\mathrm{Tr}(xy) = Z(A(x))$. By spherical invariance one may arrange $A(x)$ so that it has no strings crossing the radius from a point between the last and first vertices on the outer boundary to a point between the last and first vertices on the inner boundary of A. Then by isotoping the strings we may contain the part of $A(x)$ outside x in a disc with pattern i, j:



$= \mathrm{Tr}(xy)$

□

Then for non-degenerate P, if we have a finite set of labelled tangles which linearly span $P_{i,j}$ (under the presenting map Z), the calculation of $\dim P_{i,j}$ is reduced to the finite problem of calculating the rank of the bilinear form defined on $P_{i,j}$ by Tr as for the A_1-case.

Definition 6.2.11 *Let T be any tangle with internal discs D_k, $k = 1, \ldots, n$. We call an A_2-planar algebra* ***flat*** *if $Z(T) = Z(T')$ where T' is any tangle obtained from T by pulling strings over an internal disc D_k, for any $k = 1, \ldots, n$. This is illustrated in Figure 6.28, where we only show a local part of the tangle.*

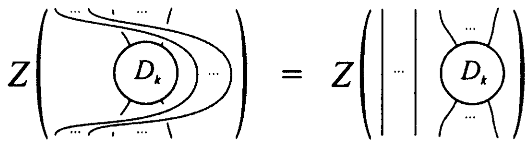


Figure 6.28: Flatness

Note that we could have defined a flat A_2-planar algebra to be one where strings can be pulled under internal discs instead of over. Such an A_2-planar algebra is isomorphic to the one defined above, with the isomorphism given by replacing q by q^{-1}- this is equivalent to reversing all the crossings in any tangle. For a flat A_2-planar algebra, the two 'right' inclusion tangles $IR^{i,j}_{i,j+1}$ and $\widetilde{IR}^{i,j}_{i,j+1}$ are equal, and we will simply write $IR^{i,j}_{i,j+1}$. For a flat A_2-planar algebra the partial braiding is a genuine braiding, as inner discs may now be pulled through crossings.

Proposition 6.2.12 *A flat A_2-planar algebra is spherical.*

Proof

Given a 0-tangle, we isotope the strings so that we have an n-tangle T, for $n \in \mathbb{N}$, with n vertices along the top and bottom of T connected be closed strings which pass to the left of T. Then the string from the n^{th} vertex along the top and bottom of T can be pulled over all the other strings and all internal discs of T, introducing two opposite kinks, which contribute a scalar factor $q^{8/3}q^{-8/3} = 1$ (see Figure 6.29). We may similarly pull the other strings which pass to the left of T over T. $\square$

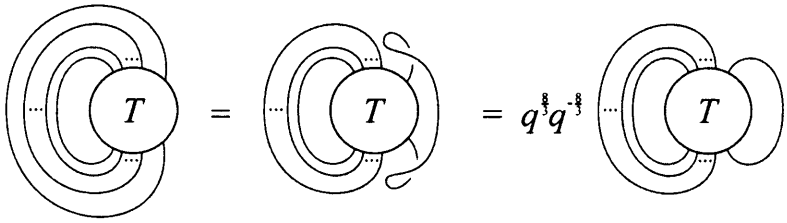


Figure 6.29: Flatness gives sphericity

6.2.7 The involution on P

We can define the adjoint T^* of a tangle $T \in \mathcal{P}_{i,j}(L)$, where L has a $*$ operation defined on it, by reflecting the whole tangle about the horizontal line that passes through its centre and reversing all orientations. The labels $x_k \in L$ of T are replaced by labels x_k^* in T^*. If φ is the map which sends $T \to T^*$, then every region $\varphi(R)$ of T^* has the same colour as the region R of T. For any linear combination of tangles in $\mathcal{P}_{i,j}(L)$ we extend $*$ by conjugate linearity. Then P is an A_2-planar $*$-algebra if each $P_{i,j}$ is a $*$-algebra, and for a i,j-tangle T with internal discs D_k with pattern i_k, j_k, labelled by $x_k \in P_{i_k,j_k}$, $k = 1, \dots, n$, then

$$Z(T)^* = Z(T^*),$$

where the labels of the discs in T^* are x_k^*. We extend the definition of $Z(T)$ to linear combinations of i,j-tangles by conjugate linearity. The partition function on an A_2-planar algebra will be called **positive** if ${}_R\mathrm{Tr}_{i,j}(x^*x) \geq 0$, for all $x \in P_{i,j}$, $i,j \geq 0$, and **positive definite** if ${}_R\mathrm{Tr}_{i,j}(x^*x) > 0$, for all non-zero $x \in P_{i,j}$. It is not clear whether ${}_R\mathrm{Tr}_{i,j}$ positive implies that ${}_L\mathrm{Tr}_{i,j}(x^*x) \geq 0$ for all $x \in P_{i,j}$ also, however we will only be interested in non-degenerate A_2-planar algebras, and hence we will have both ${}_R\mathrm{Tr}_{i,j}(x^*x) > 0$ and ${}_L\mathrm{Tr}_{i,j}(x^*x) > 0$ for all $x \in P_{i,j}$, $i,j \geq 0$, by the following proposition which is contained in [64, Prop. 1.33] for Jones's A_1-planar algebras. The proof carries over to A_2-planar algebras where the only modification is that we allow slightly different orientations on the strings.

Proposition 6.2.13 *Let P be an A_2-planar $*$-algebra with positive partition function Z. The following are equivalent:*

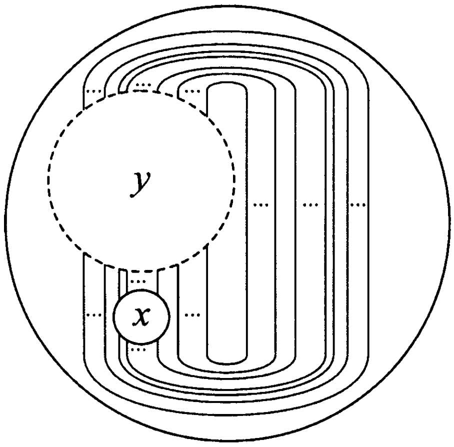
(i) P is non-degenerate,

(ii) ${}_R\mathrm{Tr}_{i,j}$ is positive definite,

(ii) ${}_L\mathrm{Tr}_{i,j}$ is positive definite.

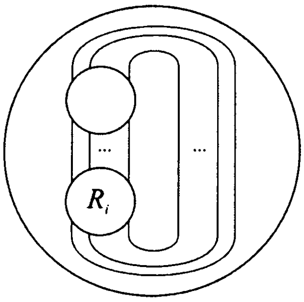
Proof

(i) $\Rightarrow$ (ii): Suppose $x \in P_{i,j}$ satisfies $\mathrm{Tr}_R(x^*x) = 0$. Then if A is any annular $(0 : i,j)$-tangle, we may isotope $A(x)$ so it looks like



where $y \in P_{i',j'}$, $i' \geq i, j' \geq j$. Thus $Z(A(x)) = {}_R\mathrm{Tr}_{i,j}(\widetilde{x}y)$, where $\widetilde{x}$ is x with $j'-j$ vertical straight lines added to the left and $i'-i$ vertical straight lines added to the right. By the Cauchy-Schwarz inequality, $|{}_R\mathrm{Tr}_{i,j}(\widetilde{x}y)| \leq \sqrt{{}_R\mathrm{Tr}_{i,j}(\widetilde{x}^*\widetilde{x})}\sqrt{{}_R\mathrm{Tr}_{i,j}(y^*y)}$, so $Z(A(x)) = 0$. (So ${}_R\mathrm{Tr}_{i,j}(x^*x) = 0 \Rightarrow x = 0$ which is statement (ii).) Similarly (i) $\Rightarrow$ (iii).

(ii) $\Rightarrow$ (i): For $x = \sum_l \lambda_l R_l$, for basis elements R_l of $P_{i,j}$, we can write ${}_R\mathrm{Tr}_{i,j}(x^*x) = \sum_l \lambda_l Z(A_l(x^*))$ where A_l is the annular $(0:i,j)$-tangle



Then for non-zero x, ${}_R\mathrm{Tr}_{i,j}(x^*x) > 0 \Rightarrow Z(A_n(x^*)) \neq 0$ for at least one n. Alternatively, ${}_R\mathrm{Tr}_{i,j}(x^*x) = 0 \Rightarrow Z(A_l(x^*)) = 0$ for all A_l. Similarly (iii) $\Rightarrow$ (i). $\square$

Then we have the following Corollary, as in [64, Cor. 1.36]

Corollary 6.2.14 *If P is a non-degenerate finite-dimensional A_2-planar $*$-algebra with positive partition function then $P_{i,j}$ is semisimple for all i,j, so there is a unique norm on $P_{i,j}$ making it into a C^*-algebra.*

Definition 6.2.15 *We call an A_2-planar algebra over $\mathbb{R}$ or $\mathbb{C}$ an **A_2-C^*-planar algebra** if it is a non-degenerate finite-dimensional A_2-planar $*$-algebra with positive definite partition function.*

If P is a spherical A_2-C^*-planar algebra we can define an inner-product on $P_{i,j}$ by $\langle x, y \rangle = \mathrm{tr}(x^* y)$ for $x, y \in P_{i,j}$, which is consistent with the inclusions $P_{i,j} \subset P_{i,j+1}$ and $P_{i,j} \subset P_{i+1,j}$, given by $IR^{i,j}_{i,j+1}$, $IR^{i,j}_{i+1,j}$ respectively, since tr is.

6.2.8 The Conditional Expectation

The justification for calling the tangles in (6.12) "conditional expectation" tangles is seen in the following Lemma:

Lemma 6.2.16 *Let P be an A_2-C^*-planar algebra. For the tangles $ER^{i+1,j}_{i,j}$ and $ER^{i,j+1}_{i,j}$ defined in (6.12), $E_1(x) = Z(ER^{i+1,j}_{i,j}(x))$ is the conditional expectation of $x \in P_{i+1,j}$ onto $P_{i,j}$ with respect to the trace, and $E_2(y) = Z(ER^{i,j+1}_{i,j}(y))$ is the conditional expectation of $y \in P_{i,j+1}$ onto $P_{i,j}$ with respect to the trace.*

Proof

We first check positivity of $E_1(x)$ for positive $x \in P_{i+1,j}$. As P is an A_2-C^*-planar algebra, the inner-product defined above is positive definite. We need to show that $\langle E_1(x)y, y \rangle \geq 0$ for all $y \in P_{i,j}$. From Figure 6.30 we see that $\mathrm{tr}(y^* ER^{i+1,j}_{i,j}(x)^* y) = \mathrm{tr}(y'^* x^* y') = \langle xy', y' \rangle \geq 0$ for all $y \in P_{i,j}$, where $y' = \boxed{y}\,| \in P_{i+1,j}$.

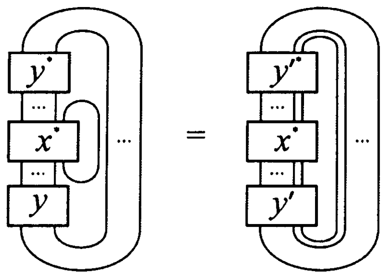
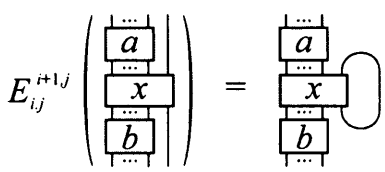


Figure 6.30:

From

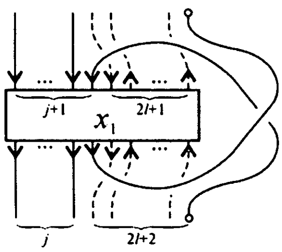
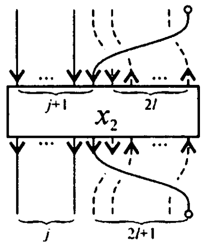


we see that $E_1(axb) = aE_1(x)b$, for $x \in P_{i+1,j}$, $a, b \in P_{i,j}$. Since also $\langle E_1(x), y \rangle = \langle x, y' \rangle$, E_1 is the trace-preserving conditional expectation from $P_{i+1,j}$ onto $P_{i,j}$. The proof for E_2 is similar. $\square$

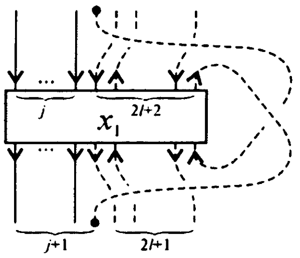
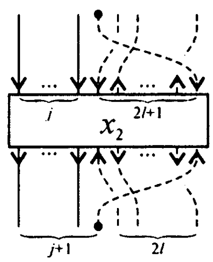
Similarly, $Z(EL^{i+1,j}_{i+1,j}(x))$ is the conditional expectation of $x \in P_{i+1,j}$ onto $P^{(1,0)}_{i+1,j}$, and $Z(EL^{i,j+1}_{i,j+1}(y))$ is the conditional expectation of $y \in P_{i,j+1}$ onto $P^{(0,1)}_{i,j+1}$ with respect to the trace.

6.2.9 Dimensions of A_2-planar algebras and A_2-STL.

We now present some other general results for A_2-planar algebras. We define maps $\varphi : \mathcal{P}_{2l+1,j+1}(L) \to \mathcal{P}_{2l+2,j}(L)$, $\omega : \mathcal{P}_{2l,j+1}(L) \to \mathcal{P}_{2l+1,j}(L)$ by

$\varphi(x_1) =$  $\qquad \omega(x_2) =$ 

for $x_1 \in \mathcal{P}_{2l+1,j+1}(L)$, $x_2 \in \mathcal{P}_{2l,j+1}(L)$, where the white circle at the end of a string indicates that this vertex is now regarded as one of the i vertices of $\mathcal{P}_{i,j}$ with alternating orientation ($i = 2l+2, 2l+1$ for φ, ω respectively). The maps φ, ω are invertible, with φ^{-1}, ω^{-1} given by

$\varphi^{-1}(x_1) =$  $\qquad \omega^{-1}(x_2) =$ 

for $x_1 \in \mathcal{P}_{2l+2,j}(L)$, $x_2 \in \mathcal{P}_{2l+1,j}(L)$, where the solid black circle at the end of a string indicates that this vertex is now regarded as one of the $j+1$ vertices of $\mathcal{P}_{i,j+1}$ with alternating orientation ($i = 2l+1, 2l$ for φ^{-1}, ω^{-1} respectively). Clearly $\varphi(\mathcal{P}_{2l+1,j+1}(L)) \subset \mathcal{P}_{2l+2,j}(L)$ and $\varphi(\mathcal{P}_{2l+1,j+1}(L)) \supset \mathcal{P}_{2l+2,j}(L)$ since $\mathcal{P}_{2l+1,j+1}(L) \supset \varphi^{-1}(\mathcal{P}_{2l+2,j}(L))$. So $\varphi(\mathcal{P}_{2l+1,j+1}(L)) = \mathcal{P}_{2l+2,j}(L)$ and φ a bijection. Similarly, we see that ω is a bijection and $\omega(\mathcal{P}_{2l,j+1}(L)) = \mathcal{P}_{2l+1,j}(L)$. Let $Z : \mathcal{P}_{i,j}(L) \to P_{i,j}$ be the presenting map for an A_2-C^*-planar algebra P. We define maps $\widetilde{\varphi} : P_{2l+1,j+1}(L) \to P_{2l+2,j}(L)$, $\widetilde{\omega} : P_{2l,j+1}(L) \to P_{2l+1,j}(L)$ by $\widetilde{\varphi}(x_1) = Z(\varphi(x_1))$ and $\widetilde{\omega}(x_2) = Z(\omega(x_2))$. The inverse $\widetilde{\varphi}^{-1}$ of $\widetilde{\varphi}$ is $\widetilde{\varphi}^{-1}(x) = Z(\varphi^{-1}(x))$ for $x \in P_{2l+2,j}$, since $\widetilde{\varphi}^{-1}\widetilde{\varphi}(x) = Z(\varphi^{-1}(Z(\varphi(x)))) = Z(\varphi^{-1}\varphi(x)) = Z(I_{2l+2,j}(x)) = x$. Similarly $\widetilde{\omega}^{-1}(x) = Z(\omega^{-1}(x))$ for $x \in P_{2l+1,j}$. Then by a similar argument as for φ, we have $\widetilde{\varphi}(P_{2l+1,j+1}) = P_{2l+2,j}$, and $\widetilde{\varphi}$ is a bijection since for every $y \in P_{2l+2,j}$ there is an $x = \widetilde{\varphi}^{-1}(y) \in P_{2l+1,j+1}$ such that $\widetilde{\varphi}(x) = y$. Similarly $\widetilde{\omega}$ is a bijection and $\widetilde{\omega}(P_{2l,j+1}) = P_{2l+1,j}$. Then we have the following lemma:

Lemma 6.2.17 *Let P be an A_2-C^*-planar algebra, with presenting map $Z : \mathcal{P}_{i,j}(L) \to P_{i,j}$, for some labelling set L. Then for all integers k such that $-i \leq k \leq j$:*

(i) $\dim(\mathcal{P}_{i,j}(L)) = \dim(\mathcal{P}_{i+k,j-k}(L))$,

(ii) $\dim(P_{i,j}) = \dim(P_{i+k,j-k})$.

Proof

For (i) there is a map $\varrho_k : \mathcal{P}_{i,j} \to \mathcal{P}_{i+k,j-k}$ for each $-i \leq k \leq j$ which is a composition of the maps φ and ω. The result follows from the fact that ϱ_k is a bijection and $\varrho_k(\mathcal{P}_{i,j}(L)) = \mathcal{P}_{i+k,j-k}(L)$. For (ii) we define maps $\widetilde{\varrho}_k : P_{i,j} \to P_{i+k,j-k}$ by $\widetilde{\varrho}_k(x) = Z(\varrho_k(x))$, and the result follows by the same argument as for (i). $\square$

For $L = \emptyset$, we define $\mathcal{STL}_{i,j}$ to be the quotient of $STL_{i,j} = \mathcal{P}_{i,j}(\emptyset)$ by the subspace of zero-length vectors with respect to the inner-product on $STL_{i,j}$ defined by $\langle x, y \rangle = \widehat{x^* y}$, for $x, y \in STL_{i,j}$, where $\widehat{T}$ is the tangle defined as in Figure 6.16.

Then we have the following result:

Lemma 6.2.18 *The element $\varphi(x)$ is a zero-length vector in $STL_{2l+2,j}$ if and only if x is a zero-length vector in $STL_{2l+1,j+1}$. Similarly, $\omega(x)$ is zero-length vector in $STL_{2l+1,j}$ if and only if x is a zero-length vector in $STL_{2l,j+1}$.*

Proof

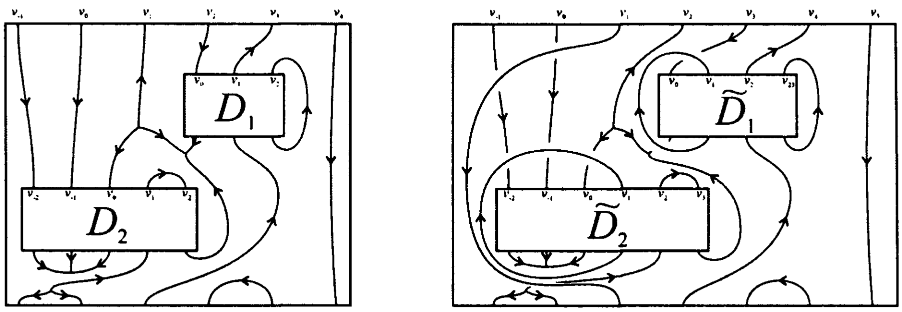
For φ, if x is a zero-length vector in $STL_{2l+1,j+1}$ then $\langle x, y \rangle = 0$ for all $y \in STL_{2l+1,j+1}$. Then for all $y_1 \in STL_{2l+2,j}$, we see by drawing tangles $\langle \varphi(x), y_1 \rangle = \widehat{\varphi(x)^* y_1} = \widehat{x^* \varphi^{-1}(y_1)} = \langle x, y_2 \rangle = 0$, where $y_2 = \varphi^{-1}(y_1) \in STL_{2l+1,j+1}$. The only if part follows by a similar argument on φ^{-1}. The result for ω is similar. $\square$

Corollary 6.2.19 *For all integers k with $-i \leq k \leq j$,* $\dim(\mathcal{STL}_{i,j}) = \dim(\mathcal{STL}_{i+k,j-k})$.

Proof

By Lemma 6.2.17 with $L = \emptyset$, we have $\dim(STL_{i,j}) = \dim(STL_{i+k,j-k})$. The result follows by Lemma 6.2.18 since φ and ω are bijections. $\square$

If we consider the sub-operad $\mathcal{Q} = \bigcup \mathcal{Q}_i$ where $\mathcal{Q}_i$ is the subset of $\mathcal{P}_{i,0}$ generated by tangles with no trivalent vertices (and hence no crossings) and where each internal disc D_k only has pattern $i_k, 0$, then $\mathcal{Q}$ is the coloured planar operad of Jones in [64], and $Q = Z(\mathcal{Q})$ is his planar algebra.



(-)4,2-tangle T (+)5,2-tangle $\widetilde{T}$

Figure 6.31: $T \to \widetilde{T}$ for duality

6.2.10 Duality

Let $P_{(\pm)}$ be an A_2-$(\pm)$-planar algebra with presenting map $Z : \mathcal{P}_{(\pm)}(P_{(\pm)}) \to P_{(\pm)}$, and let $\overline{P} = \bigcup_{i,j} \overline{P}_{i,j}$, where $\overline{P}_{i,j}$ is isomorphic to $P^{(1,0)}_{(\pm)i+1,j}$ via λ. Let $\mathcal{P}_{(\mp)} = \mathcal{P}_{(\mp)}(\overline{P})$ be the A_2-$(\mp)$-planar operad. Given any $(\mp)i,j$-tangle $T \in \mathcal{P}_{(\mp)i,j}$ we form the $(\pm)i+1,j$-tangle $\widetilde{T}$ in the following way. First, add a vertical line to the left of the tangle T, with downwards (upwards) orientation for T an $(-)i,j$-tangle ($(+)i,j$-tangle), and relabel the vertices along the top edge (and similarly along the bottom edge) by $v_1, v_{-j+1}, v_{-j+2}, \ldots, v_0, v_2, v_3, \ldots, v_{i+1}$. Then using the braiding the tangle is put in standard form, i.e. so that the vertices are ordered $v_{-j+1}, v_{-j+2}, \ldots, v_{i+1}$.

Each internal disc D_k with pattern i_k, j_k and vertices labelled $v_{-j_k+1}, v_{-j_k+2}, \ldots, v_{i_k}$ along the top and bottom, is replaced by a disc $\widetilde{D}_k$, with pattern i_k+1, j_k, where along the top and bottom an extra vertex is added between vertices v_0 and v_1, and the vertices along both top and bottom are relabeled $v_{-j_k+1}, v_{-j_k+2}, \ldots, v_{i_k+1}$. The new vertex v_1 along the top is a source, sink if T an $(+)i,j$-tangle, $(-)i,j$-tangle respectively, and is connected to vertex v_1 along the bottom by a string which goes around the disc to the left, passing over the strings coming from vertices $v_{-j_k+1}, \ldots, v_0$ along the top and bottom of the disc.

The labels $\widetilde{x}_k$ for the tangle $\widetilde{T}$ are given by $\lambda(x_k)$, where the x_k are the labels of the original tangle T. An example of $\widetilde{T}$ is shown in Figure 6.31, for a $(-)4,2$-tangle T. We set $\lambda(T) = \widetilde{T}$.

Proposition 6.2.20 *Let $P_{(\pm)} = \bigcup_{i,j} P_{(\pm)i,j}$ be an A_2-$(\pm)$-planar algebra with parameter α and presenting map $Z : \mathcal{P}_{(\pm)}(P_{(\pm)}) \to P_{(\pm)}$. Then the dual A_2-planar algebra $\overline{P}$ defined above is an A_2-$(\mp)$-planar algebra with parameter α and presenting map $\overline{Z} : \mathcal{P}_{(\mp)}(\overline{P}) \to \overline{P}$ defined by $\overline{Z}(T) = \alpha_1^{-p}\lambda^{-1}(Z(\widetilde{T}))$, where p is the number of internal discs in T.*

Proof

Since $\lambda(\overline{P}_{i,j}) \subset P_{(\pm)i+1,j}$, the labels in a tangle $T \in \mathcal{P}_{(\mp)}(\overline{P})$ give valid labels for $\widetilde{T} \in \mathcal{P}_{(\pm)}(P_{(\pm)})$ and $\overline{Z}$ satisfies the compatability condition (6.10) since Z does. Thus $\overline{P}$ is a general A_2-$(\mp)$-planar algebra. It clearly has the same parameter as $P_{(\pm)}$, and $\dim(\overline{P}_0) = \dim(P^{(1,0)}_{(\pm)1,0}) = 1$, $\dim(\overline{P}^{(0,1)}_{0,1}) = \dim(P^{(1,1)}_{(\pm)1,1}) = 1$ and $\dim(\overline{P}^{(0,2)}_{0,2}) = \dim(P^{(1,2)}_{(\pm)1,2}) = 1$, so $\overline{P}$ is an A_2-$(\mp)$-planar algebra. $\square$

Note that in our A_2 situation there is a distinction between $(+)$ and $(-)$ planar algebras, and for an A_2-$(+)$-planar algebra P the dual A_2-planar algebra $\overline{P}$ is an A_2-$(-)$-planar algebra which is isomorphic to the subalgebra $P^{(1,0)}$ of P. For Jones's planar algebras [64] no such distinction is necessary, and every planar algebra could be regarded as a $(+)$-planar algebra. The dual (A_1-)planar algebra $\overline{P}$ of a $(+)$-planar algebra P is then also a $(+)$-planar algebra, identified with a subalgebra of P given by the tangles where the string from the first vertex on the outer boundary of any tangle is a vertical through string whose endpoint is the last vertex on the outer boundary. The reason for this distinction is that since in Jones's planar algebras the orientation of each vertex alternates, he can embed any tangle in the operad $\mathcal{P}_k$ (of tangles with $2k$ vertices on the outer disc) in $\mathcal{P}_{k+1}$ by adding a vertical through string to the left of the tangle and reversing all the orientations. However, in our A_2 situation, in $\mathcal{P}_{i,j}$ the first j vertices along the top of any tangle all have downwards orientation whilst the next i have alternating orientations, and we would first add a vertex along the top and bottom between the j^{th} and $(j+1)^{\text{th}}$ vertices and connect them by a vertical through string. But then reversing all the orientations causes the first j vertices to all have upwards rather than downwards orientation, which is not an allowed tangle in the operad $\mathcal{P}_{i+1,j}$. So we wanted to define a notion of duality which did not involve reversing all orientations, which led us to define both A_2-$(+)$-planar algebras and A_2-$(-)$-planar algebras.

6.2.11 Tensor Product

Let $P^1 = \bigcup_{i,j} P^1_{i,j}$ and $P^2 = \bigcup_{i,j} P^2_{i,j}$ be general A_2-planar algebras with presenting maps $Z_1 : \mathcal{P}(P^1) \to P^1$ and $Z_2 : \mathcal{P}(P^2) \to P^2$ respectively. We will define the tensor product $P^1 \otimes P^2$ in a very similar way to the how Jones does in [64]. Let $L = \bigsqcup_{i,j} P^1_{i,j} \times P^2_{i,j}$ be the labelling set for $P^1 \otimes P^2$. We define a linear map $\mathcal{L} : \mathcal{P}(L) \to \mathcal{P}(P^1) \otimes \mathcal{P}(P^2)$ by $\mathcal{L}(T) = T_1 \otimes T_2$, where T is an i,j-tangle with internal discs D_k labelled by (x^1_k, x^2_k), $x^1_k \in P^1$, $x^2_k \in P^2$, and T_1, T_2 have the same unlabelled tangle as T, but with discs labelled by x^1_k, x^2_k respectively. We define $Z^{P^1 \otimes P^2} : \mathcal{P}(L) \to P^1 \otimes P^2$ by $Z^{P^1 \otimes P^2}(T) = (Z_1 \otimes Z_2) \circ \mathcal{L}(T) = Z_1(T_1) \otimes Z_2(T_2)$. This map is surjective since an arbitrary label

$(x, y) \in L$ will go to $x \otimes y \in P^1 \otimes P^2$ if inserted as the label of the tangle in $\mathcal{P}(L)$ illustrated in Figure 6.26.

Let A be an annular tangle labelled by L and A_1, A_2 have the same unlabelled tangle as A but are labelled by the first, second component respectively of the labels of A. Then $\mathcal{L} \circ A = A_1 \otimes A_2 \circ \mathcal{L}$, and if $T \in \ker\left(Z^{P^1 \otimes P^2}\right)$, then $T_i \in \ker(Z_i)$, $i = 1, 2$, with T_i the tangles as described above, and $Z^{P^1 \otimes P^2}(A(T)) = Z_1(A_1(T_1)) \otimes Z_2(A_2(T_2)) = 0$, since P^1 and P^2 are both A_2-planar algebras. It is clear that $Z^{P^1 \otimes P^2}(T) = Z_1(T_1)Z_2(T_2)$ for a 0-tangle T (with T_1, T_2 defined as above), in the sense that a 0-tangle labelled with $P^1 \times P^2$ is the same as two 0-tangles labelled with P^1, P^2 respectively. So $P^1 \otimes P^2$ is an A_2-planar algebra if both P^1 and P^2 are. Non-degeneracy, $*$-structure, positivity, sphericity and irreducibility are all inherited by $P^1 \otimes P^2$ from P^1 and P^2. Then the tensor product of two A_2-C^*-planar algebras is also an A_2-C^*-planar algebra. Clearly $P^1 \otimes P^2 \cong P^2 \otimes P^1$ as (general) A_2-planar algebras.

6.3 A_2-Planar algebra description of subfactors

We are now going to associate flat A_2-planar C^*-algebras to subfactors associated to $\mathcal{ADE}$ graphs with flat connections.

Let $\mathcal{G}$ be any finite $SU(3)$ $\mathcal{ADE}$ graph with Coxeter number n. Let $\alpha = [3]_q$, $q = e^{i\pi/n}$, be the Perron-Frobenius eigenvalue of $\mathcal{G}$ and let (ϕ_v) be the corresponding eigenvector. With any choice of distinguished vertex $*$, we define the double sequence $(B_{i,j})$ of finite dimensional algebras by:

$$\begin{array}{ccccccccc}
B_{0,0} & \subset & B_{0,1} & \subset & B_{0,2} & \subset & \cdots & \longrightarrow & B_{0,\infty} \\
\cap & & \cap & & \cap & & & & \cap \\
B_{1,0} & \subset & B_{1,1} & \subset & B_{1,2} & \subset & \cdots & \longrightarrow & B_{1,\infty} \\
\cap & & \cap & & \cap & & & & \cap \\
B_{2,0} & \subset & B_{2,1} & \subset & B_{2,2} & \subset & \cdots & \longrightarrow & B_{2,\infty} \\
\cap & & \cap & & \cap & & & & \cap \\
\vdots & & \vdots & & \vdots & & & & \vdots
\end{array}$$

The Bratteli diagrams for horizontal inclusions $B_{i,j} \subset B_{i,j+1}$ given by $\mathcal{G}$. If $\mathcal{G}$ is three-colourable, the vertical inclusions $B_{i,j} \subset B_{i+1,j}$ given by its $\overline{j}, \overline{j+1}$-part $\mathcal{G}_{\overline{j},\overline{j+1}}$, where $\overline{p} = \tau(p)$ is the colour of p for $p = j, j+1$, whilst if $\mathcal{G}$ is not three-colourable we use the graph $\mathcal{G}$ for all the vertical inclusions $B_{i,j} \subset B_{i+1,j}$. We identify $B_{0,0} = \mathbb{C}$ with the distinguished vertex $*$ of $\mathcal{G}$.

For i even we define a connection by

$$\begin{array}{ccc} & \xrightarrow{\sigma_1} & \\ \sigma_3\downarrow & & \downarrow\sigma_2 \\ & \xrightarrow[\sigma_4]{} & \end{array} = q^{\frac{2}{3}}\delta_{\sigma_1,\sigma_3}\delta_{\sigma_2,\sigma_4} - q^{-\frac{1}{3}}\mathcal{U}^{\sigma_1,\sigma_2}_{\sigma_3,\sigma_4}, \tag{6.13}$$

which satisfies the unitarity axiom (1.31) as shown in Lemma 4.1.2. We denote by $\widetilde{\mathcal{G}}$ the reverse graph of $\mathcal{G}$, which is the graph obtained by reversing the direction of every edge of $\mathcal{G}$. For i odd, let σ_1, σ_4 be edges on $\mathcal{G}$ and let $\widetilde{\sigma}_2$, $\widetilde{\sigma}_3$ be edges on the reverse graph $\widetilde{\mathcal{G}}$ (so that σ_2, σ_3 are edges on $\mathcal{G}$). We define the connection by

$$\begin{array}{ccc} & \xrightarrow{\sigma_1} & \\ \widetilde{\sigma}_3\downarrow & & \downarrow\widetilde{\sigma}_2 \\ & \xrightarrow[\sigma_4]{} & \end{array} = \sqrt{\frac{\phi_{s(\sigma_3)}\phi_{r(\sigma_2)}}{\phi_{r(\sigma_3)}\phi_{s(\sigma_2)}}}\ \overline{\begin{array}{ccc} & \xrightarrow{\sigma_4} & \\ \sigma_3\downarrow & & \downarrow\sigma_2 \\ & \xrightarrow[\sigma_1]{} & \end{array}}\ . \tag{6.14}$$

Then for the inclusions

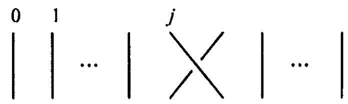
$$\begin{array}{ccc} B_{i,j} & \subset & B_{i,j+1} \\ \cap & & \cap \\ B_{i+1,j} & \subset & B_{i+1,j+1} \end{array} \tag{6.15}$$

an element indexed by paths in the basis $\downarrow\!\rightarrow$ can be transformed to an element indexed by paths in the basis $\rightarrow\!\downarrow$ using the above connection: Let $(\sigma\cdot\sigma'\cdot\alpha_1\cdot\alpha_2, \sigma\cdot\sigma'\cdot\alpha_1'\cdot\alpha_2')$ be an element in $B_{i+1,j+1}$ in the basis $\downarrow\!\rightarrow$, where σ is a horizontal path of length j, σ' is a vertical path of length i, α_1, α_1' are vertical paths of length 1, α_2, α_2' are horizontal paths of length 1, and $r(\alpha_2) = r(\alpha_2')$. We transform this to an element in the basis $\rightarrow\!\downarrow$ by

$$(\sigma\cdot\sigma'\cdot\alpha_1\cdot\alpha_2, \sigma\cdot\sigma'\cdot\alpha_1'\cdot\alpha_2') = \sum_{\beta_i,\beta_i'} \begin{array}{ccc} & \xrightarrow{\beta_1} & \\ \alpha_1\downarrow & & \downarrow\beta_2 \\ & \xrightarrow[\alpha_2]{} & \end{array}\ \overline{\begin{array}{ccc} & \xrightarrow{\beta_1'} & \\ \alpha_1'\downarrow & & \downarrow\beta_2' \\ & \xrightarrow[\alpha_2']{} & \end{array}}\ (\sigma\cdot\sigma'\cdot\beta_1\cdot\beta_2, \sigma\cdot\sigma'\cdot\beta_1'\cdot\beta_2'),$$

where the summation is over all horizontal paths β_1, β_1' of length 1, and vertical paths β_2, β_2' of length 1.

The Markov trace on $B_{i,j}$ is given in (5.6), for $(\sigma_1,\sigma_2) \in B_{i,j}$, where $k = i+j$, $\alpha = [3]_q$ as usual, $q = \exp(i\pi/n)$. We define $B_{i,\infty}$ to be the GNS-completion of $\bigcup_{k\geq 0} B_{i,k}$ with respect to the trace. As in [38], the braid elements

$\sigma_j =$ 

appear as the connection.

If $\mathcal{G}$ is three-colourable then its adjacency matrix $\Delta_{\mathcal{G}}$ which may be written in the form

$$\Delta_{\mathcal{G}} = \begin{pmatrix} 0 & \Delta_{01} & 0 \\ 0 & 0 & \Delta_{12} \\ \Delta_{20} & 0 & 0 \end{pmatrix},$$

where Δ_{01}, Δ_{12} and Δ_{20} are matrices which give the number of edges between each 0,1,2-coloured vertex respectively of $\mathcal{G}$ to each 1,2,0-coloured vertex respectively. By a suitable ordering of the vertices the matrix Δ_{12} may be chosen to be symmetric. These matrices satisfy the conditions

$$\Delta_{01}^T \Delta_{01} = \Delta_{20}\Delta_{20}^T = \Delta_{12}^2 \tag{6.16}$$

$$\Delta_{01}\Delta_{01}^T = \Delta_{20}^T\Delta_{20}, \tag{6.17}$$

which follow from the fact that $\Delta_{\mathcal{G}}$ is normal.

Lemma 6.3.1 *For the double sequence $(B_{i,j})$ defined above, $dim(B_{i,j}) = dim(B_{i+k,j-k})$ for all integers k such that $-i \leq k \leq j$.*

Proof

If $\mathcal{G}$ is not three-colourable, then $B_{i,j}$ is the space of all pairs of paths of length $i+j$ on $\mathcal{G}$, hence the result is trivial. If $\mathcal{G}$ is three-colourable, let $\Lambda_{i,j}^1$ be the product of j matrices $\Lambda_{i,j}^1 = \Delta_{01}\Delta_{12}\Delta_{20}\Delta_{01}\cdots\Delta_{\overline{j-1},\overline{j}}$, and $\Lambda_{i,j}^2$ the product of i matrices $\Lambda_{i,j}^2 = \Delta_{\overline{j},\overline{j+1}}\Delta_{\overline{j},\overline{j+1}}^T\Delta_{\overline{j},\overline{j+1}}\Delta_{\overline{j},\overline{j+1}}^T\cdots\Delta'$, where Δ' is $\Delta_{\overline{j},\overline{j+1}}$ if i is odd, $\Delta_{\overline{j},\overline{j+1}}^T$ if i is even, and $\overline{p}$ is the colour of p. Then if $\Lambda_{i,j} = \Lambda_{i,j}^1\Lambda_{i,j}^2$, the dimension of $B_{i,j}$ is given by $\left(\Lambda_{i,j}\Lambda_{i,j}^T\right)_{0,0}$. Using (6.16), (6.17) it is easy to show by induction that $\Lambda_{i,j}\Lambda_{i,j}^T = (\Delta_{01}\Delta_{01}^T)^{i+j}$. So $\dim(B_{i+k,j-k}) = \left(\Lambda_{i+k,j-k}\Lambda_{i+k,j-k}^T\right)_{0,0} = \left((\Delta_{01}\Delta_{01}^T)^{i+j}\right)_{0,0} = \dim(B_{i,j})$. □

For all $i, j \geq 0$ we define operators $U_{-k} \in B_{i,j}$, $k = 0, 1, \ldots, j-1$, which satisfy the Hecke relations H1-H3, by

$$U_{-k} = \sum_{\substack{|\zeta_1|=j-2-k, |\zeta'|=i \\ |\gamma_i|=|\eta_i|=1, |\zeta_2|=k}} \mathcal{U}_{\gamma_1,\eta_1}^{\gamma_2,\eta_2} (\zeta_1\cdot\gamma_1\cdot\eta_1\cdot\zeta_2\cdot\zeta', \zeta_1\cdot\gamma_2\cdot\eta_2\cdot\zeta_2\cdot\zeta'), \qquad 0 \leq k \leq j-2,$$

$$U_{-j+1} = \sum_{\substack{|\zeta|=j-1, |\zeta'|=i-1 \\ |\gamma_i|=|\eta_i'|=1}} \mathcal{U}_{\gamma_1,\eta_1}^{\gamma_2,\eta_2} (\zeta\cdot\gamma_1\cdot\eta_1'\cdot\zeta', \zeta\cdot\gamma_2\cdot\eta_2'\cdot\zeta'),$$

where ξ, ξ' are horizontal, vertical paths respectively, and $\mathcal{U}_{\gamma_1,\eta_1}^{\gamma_2,\eta_2}$ are the Boltzmann weights for $\mathcal{A}^{(n)}$. The embedding of $U_{-k} \in B_{i,j}$ into $B_{i+1,j}$ is U_{-k}, whilst the embedding of $U_{-k} \in B_{i,j}$ into $B_{i,j+1}$ is U_{-k-1}. We have $B_{i,j} \supset \mathrm{alg}(U_{-j+1}, U_{-j+2}, \ldots, U_{-1}, U_0)$. It was noted in Section 1.2.2 that when $\mathcal{G} = \mathcal{A}^{(n)}$, the algebra $B_{l,j} = \mathrm{alg}(U_{-j+1}, U_{-j+2}, \ldots, U_{-l-1})$ for $l = 0, 1$.

Lemma 6.3.2 *The square 6.15 is a commuting square.*

Proof

Note that for the $\mathcal{A}$ graphs, the result follows by [112, Prop. 3.2]. However, we prove the case for a general $SU(3)$ $\mathcal{ADE}$ graph $\mathcal{G}$. By [39, Theorem 11.2], the square 6.15 is a commuting square if and only if the corresponding connection satisfies

$$\sum_{\sigma_2,\sigma_4} \frac{\phi_{r(\sigma_2)}\sqrt{\phi_{s(\sigma_3)}\phi_{s(\sigma_3')}}}{\phi_{s(\sigma_2)}\phi_{s(\sigma_4)}} \begin{array}{ccc} & \xrightarrow{\sigma_1} & \\ \sigma_3\downarrow & & \downarrow\sigma_2 \\ & \xrightarrow[\sigma_4]{} & \end{array} \overline{\begin{array}{ccc} & \xrightarrow{\sigma_1'} & \\ \sigma_3'\downarrow & & \downarrow\sigma_2 \\ & \xrightarrow[\sigma_4]{} & \end{array}} = \delta_{\sigma_1,\sigma_1'}\delta_{\sigma_3,\sigma_3'}, \tag{6.18}$$

where σ_1, σ_1' are any edges on the graph of the Bratteli diagram for $B_{i,j} \subset B_{i,j+1}$, σ_3, σ_3' are any edges on the graph of the Bratteli diagram for $B_{i,j} \subset B_{i+1,j}$, σ_1, and σ_2, σ_4 are any edges on the graphs of the Bratteli diagrams for $B_{i,j+1} \subset B_{i+1,j+1}$, $B_{i+1,j} \subset B_{i+1,j+1}$ respectively, such that $s(\sigma_2) = r(\sigma_1) = r(\sigma_1')$ and $s(\sigma_4) = r(\sigma_3) = r(\sigma_3')$.

For i even, the connection on $\mathcal{G}$ is given by (6.13). Then the left hand side of (6.18) becomes

$$\begin{aligned}
&\sum_{\sigma_2,\sigma_4} \frac{\phi_{r(\sigma_2)}\sqrt{\phi_{s(\sigma_3)}\phi_{s(\sigma_3')}}}{\phi_{s(\sigma_2)}\phi_{r(\sigma_3)}} \left(\delta_{\sigma_1,\sigma_3}\delta_{\sigma_1',\sigma_3'}\delta_{\sigma_2,\sigma_4} - q^{-1}\frac{\delta_{\sigma_1',\sigma_3'}\delta_{\sigma_2,\sigma_4}}{\phi_{s(\sigma_1)}\phi_{r(\sigma_2)}} \sum_{\gamma} W(\triangle^{(\gamma,\sigma_3,\sigma_4)})\overline{W(\triangle^{(\gamma,\sigma_1,\sigma_2)})} \right. \\
&\qquad -q\frac{\delta_{\sigma_1,\sigma_3}\delta_{\sigma_2,\sigma_4}}{\phi_{s(\sigma_1')}\phi_{r(\sigma_2)}} \sum_{\gamma'} W(\triangle^{(\gamma',\sigma_1',\sigma_2)})\overline{W(\triangle^{(\gamma',\sigma_3',\sigma_4)})} \\
&\qquad \left. +\frac{1}{\phi_{s(\sigma_1)}\phi_{s(\sigma_1')}\phi_{r(\sigma_2)}^2} \sum_{\gamma,\gamma'} W(\triangle^{(\gamma,\sigma_3,\sigma_4)})\overline{W(\triangle^{(\gamma,\sigma_1,\sigma_2)})}W(\triangle^{(\gamma',\sigma_1',\sigma_2)})\overline{W(\triangle^{(\gamma,\sigma_3',\sigma_4)})} \right) \\
&= \frac{\sqrt{\phi_{s(\sigma_3)}\phi_{s(\sigma_3')}}}{\phi_{r(\sigma_3)}} \left(\sum_{\sigma_2} \frac{\phi_{r(\sigma_2)}}{\phi_{s(\sigma_2)}} \delta_{\sigma_1,\sigma_3}\delta_{\sigma_1',\sigma_3'} \right. \\
&\qquad -q^{-1}\frac{\delta_{\sigma_1',\sigma_3'}\delta_{\sigma_2,\sigma_4}}{\phi_{s(\sigma_1)}\phi_{s(\sigma_2)}}[2]\phi_{s(\sigma_1)}\phi_{s(\sigma_2)}\delta_{\sigma_1,\sigma_3} - q\frac{\delta_{\sigma_1,\sigma_3}\delta_{\sigma_2,\sigma_4}}{\phi_{s(\sigma_1')}\phi_{s(\sigma_2)}}[2]\phi_{s(\sigma_1')}\phi_{s(\sigma_2)}\delta_{\sigma_1',\sigma_3'} \\
&\qquad \left. +\frac{1}{\phi_{s(\sigma_1)}\phi_{s(\sigma_2)}\phi_{s(\sigma_1')}} (\phi_{s(\sigma_3)}\phi_{r(\sigma_3)}\phi_{s(\sigma_3')}\delta_{\sigma_1,\sigma_3}\delta_{\sigma_1',\sigma_3'} + \phi_{r(\sigma_1)}\phi_{s(\sigma_3)}\phi_{r(\sigma_3)}\delta_{\sigma_1,\sigma_1'}\delta_{\sigma_3,\sigma_3'}) \right) \\
&= \frac{\sqrt{\phi_{s(\sigma_3)}\phi_{s(\sigma_3')}}}{\phi_{r(\sigma_3)}} \left(\sum_{\sigma_2} \frac{\phi_{r(\sigma_2)}}{\phi_{s(\sigma_2)}} - [2]^2 + 1 \right) \delta_{\sigma_1,\sigma_3}\delta_{\sigma_1',\sigma_3'} + \delta_{\sigma_1,\sigma_1'}\delta_{\sigma_3,\sigma_3'} \quad = \quad \delta_{\sigma_1,\sigma_1'}\delta_{\sigma_3,\sigma_3'},
\end{aligned}$$

where $q+q^{-1} = [2]$, $\sum_{\sigma_2} \frac{\phi_{r(\sigma_2)}}{\phi_{s(\sigma_2)}} - [2]^2+1 = [3]-[2]^2+1 = 0$ since $[3]$ is the Perron-Frobenius eigenvalue for $\mathcal{G}$, and where we have used equations (4.1) and (4.2) for the first equality.

For i odd, the connection on $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ is given by (6.14). Then the left hand side of

(6.18) becomes

$$
\begin{aligned}
& \sum_{\sigma_2,\sigma_4} \overline{\begin{array}{ccc} & \xrightarrow{\sigma_4} & \\ \widetilde{\sigma}_3\downarrow & & \downarrow\widetilde{\sigma}_2 \\ & \xrightarrow[\sigma_1]{} & \end{array}} \begin{array}{ccc} & \xrightarrow{\sigma_4} & \\ \widetilde{\sigma}'_3\downarrow & & \downarrow\widetilde{\sigma}_2 \\ & \xrightarrow[\sigma'_1]{} & \end{array} \\
= & \sum_{\sigma_2,\sigma_4} \Bigg(\delta_{\widetilde{\sigma}_3,\sigma_4}\delta_{\sigma_1,\widetilde{\sigma}_2}\delta_{\widetilde{\sigma}'_3,\sigma_4}\delta_{\sigma'_1,\widetilde{\sigma}_2} - q\frac{\delta_{\widetilde{\sigma}_3,\sigma_4}\delta_{\sigma_1,\widetilde{\sigma}_2}}{\phi_{s(\sigma_4)}\phi_{r(\widetilde{\sigma}_2)}} \sum_{\gamma} W(\triangle^{(\gamma,\sigma_4,\widetilde{\sigma}_2)})\overline{W(\triangle^{(\gamma,\widetilde{\sigma}_3,\sigma_1)})} \\
& -q^{-1}\frac{\delta_{\widetilde{\sigma}'_3,\sigma_4}\delta_{\sigma'_1,\widetilde{\sigma}_2}}{\phi_{s(\sigma_4)}\phi_{r(\widetilde{\sigma}_2)}} \sum_{\gamma'} W(\triangle^{(\gamma',\widetilde{\sigma}'_3,\sigma'_1)})\overline{W(\triangle^{(\gamma',\sigma_4,\widetilde{\sigma}_2)})} \\
& +\frac{1}{\phi^2_{s(\sigma_4)}\phi^2_{r(\widetilde{\sigma}_2)}} \sum_{\gamma,\gamma'} W(\triangle^{(\gamma,\sigma_4,\widetilde{\sigma}_2)})\overline{W(\triangle^{(\gamma',\sigma_4,\widetilde{\sigma}_2)})} W(\triangle^{(\gamma',\widetilde{\sigma}'_3,\sigma'_1)})\overline{W(\triangle^{(\gamma,\widetilde{\sigma}_3,\sigma_1)})} \Bigg) \\
= & \delta_{\sigma_1,\sigma'_1}\delta_{\sigma_3,\sigma'_3} - \frac{[2]}{\phi_{r(\sigma_1)}\phi_{r(\sigma_3)}} \sum_{\gamma} W(\triangle^{(\gamma,\sigma_4,\widetilde{\sigma}_2)})\overline{W(\triangle^{(\gamma,\widetilde{\sigma}_3,\sigma_1)})} \\
& +\frac{1}{\phi^2_{s(\sigma_4)}\phi^2_{r(\widetilde{\sigma}_2)}} \sum_{\gamma,\gamma'} [2]\delta_{\gamma,\gamma'}\phi_{s(\sigma_4)}\phi_{s(\widetilde{\sigma}_2)} W(\triangle^{(\gamma',\widetilde{\sigma}'_3,\sigma'_1)})\overline{W(\triangle^{(\gamma,\widetilde{\sigma}_3,\sigma_1)})} \Bigg) \\
= & \delta_{\sigma_1,\sigma'_1}\delta_{\sigma_3,\sigma'_3},
\end{aligned}
$$

since $s(\sigma_4) = r(\sigma_3)$, $r(\widetilde{\sigma}_2) = r(\sigma_1)$, where we have used (4.1) for the last term of the second equality.

Then (6.18) is satisfied, and 6.15 is a commuting square. □

Then as in [38], we define the Jones projections in $B_{i,j}$, for $i = 1, 2, \ldots$, by:

$$
e_{i-1} = \sum_{\substack{|\zeta|=j,|\zeta'|=i-2 \\ |\gamma'|=|\eta'|=1}} \frac{1}{[3]} \frac{\sqrt{\phi_{r(\gamma')}\phi_{r(\eta')}}}{\phi_{r(\zeta')}} (\zeta\cdot\zeta'\cdot\gamma'\cdot\widetilde{\gamma}', \zeta\cdot\zeta'\cdot\eta'\cdot\widetilde{\eta}')
$$

where $\widetilde{\xi}$ denotes the reverse edge of ξ. Let $E_{M_{i-1}}$ be the conditional expectation from $B_{i+1,\infty}$ onto $B_{i,\infty}$ with respect to the trace. For $x \in B_{i+1,j}$, $E_{M_{i-1}}(x)$ is given by the conditional expectation of x onto $B_{i,j}$, because of Lemma 6.3.2. Clearly $e_l x = x e_l$, for $x \in B_{l-1,\infty}$, since x and e_l live on distinct parts of the Bratteli diagram (cf. (1.34)). Let $x = (\alpha_1\cdot\alpha'_1\cdot\beta'_1, \alpha_2\cdot\alpha'_2\cdot\beta'_2) \in B_{l,j}$, where α_i are horizontal paths of length j, α'_i are vertical paths of length $l-1$ and β'_i are vertical paths of length 1, $i = 1, 2$. Then

$$
\begin{aligned}
e_l x e_l = & \sum_{\substack{|\zeta_i|=j,|\zeta'_i|=l-1 \\ |\gamma'_i|=|\eta'_i|=|\mu'|=1}} \frac{1}{[3]^2} \frac{\sqrt{\phi_{r(\gamma'_1)}\phi_{r(\eta'_1)}}}{\phi_{r(\zeta'_1)}} \frac{\sqrt{\phi_{r(\gamma'_2)}\phi_{r(\eta'_2)}}}{\phi_{r(\zeta'_2)}} (\zeta_1\cdot\zeta'_1\cdot\gamma'_1\cdot\widetilde{\gamma}'_1, \zeta_1\cdot\zeta'_1\cdot\eta'_1\cdot\widetilde{\eta}'_1) \\
& \times(\alpha_1\cdot\alpha'_1\cdot\beta'_1\cdot\mu', \alpha_2\cdot\alpha'_2\cdot\beta'_2\cdot\mu')(\zeta_2\cdot\zeta'_2\cdot\gamma'_2\cdot\widetilde{\gamma}'_2, \zeta_2\cdot\zeta'_2\cdot\eta'_2\cdot\widetilde{\eta}'_2) \\
= & \sum_{|\gamma'_1|=|\eta'_2|=1} \frac{1}{[3]^2} \frac{\sqrt{\phi_{r(\gamma'_1)}\phi_{r(\eta'_2)}}}{\phi_{r(\alpha'_2)}} \frac{\sqrt{\phi_{r(\beta'_1)}\phi_{r(\beta'_2)}}}{\phi_{r(\alpha'_1)}} \delta_{\beta'_1,\beta'_2} (\alpha_1\cdot\alpha'_1\cdot\gamma'_1\cdot\widetilde{\gamma}'_1, \alpha_2\cdot\alpha'_2\cdot\eta'_2\cdot\widetilde{\eta}'_2),
\end{aligned}
$$

and

$$E_{l-1}(x)e_l$$

$$= \sum_{\substack{|\zeta|=j,|\zeta'|=l-1 \\ |\gamma'|=|\eta'|=|\mu'|=|\nu'|=1}} \frac{1}{[3]^2} \frac{\sqrt{\phi_{r(\gamma')}\phi_{r(\eta')}}}{\phi_{r(\zeta')}} \frac{\phi_{r(\beta_1')}}{\phi_{r(\alpha_1')}} \delta_{\beta_1',\beta_2'} \, (\alpha_1 \cdot \alpha_1' \cdot \mu' \cdot \nu', \alpha_2 \cdot \alpha_2' \cdot \mu' \cdot \nu')$$
$$\times(\zeta \cdot \zeta' \cdot \gamma' \cdot \widetilde{\gamma'}, \zeta \cdot \zeta' \cdot \eta_1' \cdot \widetilde{\eta'})$$

$$= \sum_{|\gamma'|=|\eta'|=1} \frac{1}{[3]^2} \frac{\sqrt{\phi_{r(\gamma')}\phi_{r(\eta')}}}{\phi_{r(\alpha_2')}} \frac{\sqrt{\phi_{r(\beta_1')}\phi_{r(\beta_2')}}}{\phi_{r(\alpha_1')}} \delta_{\beta_1',\beta_2'} \, (\alpha_1 \cdot \alpha_1' \cdot \gamma' \cdot \widetilde{\gamma'}, \alpha_2 \cdot \alpha_2' \cdot \eta' \cdot \widetilde{\eta'}).$$

So $e_l x e_l = E_{l-1}(x)e_l$ for all $x \in B_{l,\infty}$. Let $y = (\alpha_1 \cdot \alpha_1' \cdot \alpha_2' \cdot \alpha_3', \beta_1 \cdot \beta_1' \cdot \beta_2' \cdot \beta_3') \in B_{l+1,j}$, where α_1, β_1 are horizontal paths of length j, α_1', β_1' are vertical paths of length $l-1$ and α_i', β_i' ($i = 2,3$) are vertical paths of length 1. It can be easily checked that y can be written (up to some scalar factor) as $x_1 e_l x_2$ for $x_1, x_2 \in B_{l,j}$, by choosing $x_1 = (\alpha_1 \cdot \alpha_1' \cdot \alpha_2', \sigma_1 \cdot \sigma_1' \cdot \widetilde{\alpha_3'})$, $x_2 = (\sigma_2 \cdot \sigma_2' \cdot \widetilde{\beta_3'}, \beta_1 \cdot \beta_1' \cdot \beta_2')$, where σ_i, σ_i', $i = 1,2$, are any paths such that $r(\sigma_1') = r(\sigma_2') = r(\alpha_3')$. So $B_{l+1,\infty}$ is generated by $B_{l,\infty}$ and e_l. Then e_l is the Jones projection for the basic construction $B_{l-1,\infty} \subset B_{l,\infty} \subset B_{l+1,\infty}$, $l = 1,2,\dots$. By [101, Prop. 1.2] if we set $N = B_{0,\infty}$ and $M = B_{1,\infty}$, the sequence $B_{0,\infty} \subset B_{1,\infty} \subset B_{2,\infty} \subset B_{3,\infty} \subset \cdots$ can be identified with the Jones tower $N \subset M \subset M_1 \subset M_2 \subset \cdots$. It was shown in [38] that for $\mathcal{G} = \mathcal{A}^{(n)}$, $n < \infty$, if $*$ is now the apex vertex $(0,0)$ of $\mathcal{A}^{(n)}$, then this subfactor is the same as Wenzl's subfactor in [112] for $SU(3)$, and we have the following theorem from Theorems 3.3, 5.8 and Corollary 3.4 in [38]:

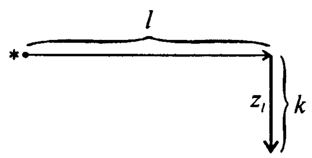
Theorem 6.3.3 *In the double sequence $(B_{i,j})$ above for $\mathcal{G} = \mathcal{A}^{(n)}$ or $\mathcal{D}^{(n)}$, $n < \infty$, with $*$ the vertex with lowest Perron-Frobenius weight, we have $B_{0,\infty}' \cap B_{i,\infty} = B_{i,0}$, i.e. $N' \cap M_{i-1} = B_{i,0}$. The principal graph for the above subfactors is given by the 01-part $\mathcal{G}_{01}$ of $\mathcal{G}$.*

The connection will be called flat if any two elements $x \in B_{k,0}$ and $y \in B_{0,l}$ commute. This is equivalent to the relation

$$\begin{tikzcd} * \arrow[r, "\sigma"] \arrow[d, "\rho"'] & * \arrow[d, "\rho"] \\ * \arrow[r, "\sigma"'] & * \end{tikzcd} = 1 \qquad (6.19)$$

where σ is a path of length $2l$, $l \in \mathbb{N}$, with the first l edges on $\mathcal{G}$ and the last l edges on $\widetilde{\mathcal{G}}$

Suppose we have an arbitrary element z^0 in $B_{0,\infty}' \cap B_{k,\infty}$. If we set $z_l = E_{B_{k,l}}(z^0)$ then $z_l \in B_{0,l}' \cap B_{k,l}$:



Let d be the smallest integer such that the Bratteli diagram for the inclusion $B_{0,d} \subset B_{0,d+1}$ is the same as the Bratteli diagram for the inclusion $B_{0,d+3} \subset B_{0,d+4}$. Then for all $3l \geq d$, the algebras $B'_{0,3l} \cap B_{k,3l}$ are isomorphic, since they have as basis elements indexed by pairs of paths on the 01-part $\mathcal{G}_{01}$ of $\mathcal{G}$ which start from any 0-coloured vertex of $\mathcal{G}$. Let A be a finite dimensional C^*-algebra isomorphic to these algebras, with natural isomorphism $\phi_{3l} : B'_{0,3l} \cap B_{k,3l} \rightarrow A$, so that A has as basis elements indexed by pairs of paths on the 01-part $\mathcal{G}_{01}$ of $\mathcal{G}$ which start from any 0-coloured vertex of $\mathcal{G}$. Let $||\cdot||$ be the operator norm. Since $||\phi_{3l}(z_{3l})|| = ||z_{3l}|| = ||E_{B_{k,3l}}(z^0)|| \leq ||z^0||$, the sequence $\{\phi_{3l}(z_{3l})\}_l$ is bounded in the finite dimensional algebra A. Then by compactness of any bounded set in A, every sequence has a convergent subsequence, and hence there is a subsequence $\{l_j\}_j$ such that $\phi_{3l_j}(z_{3l_j}) \rightarrow z$ and $\phi_{3l_j+3}(z_{3l_j+3}) \rightarrow z'$ as $j \rightarrow \infty$, for some $z, z' \in A$. Let $\widetilde{A}$ be the algebra illustrated in Figure 6.32, where we have two embeddings of A into $\widetilde{A}$, and along the top and bottom we have elements indexed by paths of length 3 on $\mathcal{G}$. Since $||z_{3l} - z_{3l+3}||_2 \rightarrow 0$ as $l \rightarrow \infty$, we have $z \cdot \mathrm{id}^{(3)} = \mathrm{id}^{(3)} \cdot z'$, as in Figure 6.33. Here the equality means that $z \cdot \mathrm{id}^{(3)}$ is identified with $\mathrm{id}^{(3)} \cdot z'$ using the connection, $\mathrm{id}^{(3)} = \sum_{|\sigma|=3}(\sigma, \sigma)$ on $\mathcal{G}$, and by $a \cdot b$ we mean the concatenation of the paths that index the elements a, b, e.g. if $a = (\rho_1, \rho_2)$, $b = (\sigma_1, \sigma_2)$ then $a \cdot b = (\rho_1 \cdot \sigma_1, \rho_2 \cdot \sigma_2)$.

If the connection is flat we have $z' = z$, i.e. $z \cdot \mathrm{id}^{(3)} = \mathrm{id}^{(3)} \cdot z$. Let $z(v)$ be the component of z which has initial vertex v, and regard $z(*)$ as an element of $B_{k,0}$. Then $\lim_{j\rightarrow\infty} ||z_{3l_j} - z(*)||_2 = 0$, and $z^0 = z(*) \in B_{k,0}$. Similarly, and $z(*) \in B_{k,0}$ is in $B'_{0,\infty} \cap B_{k,\infty}$. Then for graphs where the connection (6.13) is flat, the higher relative

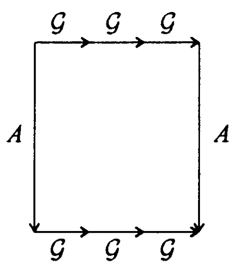


Figure 6.32: Algebra $\widetilde{A}$

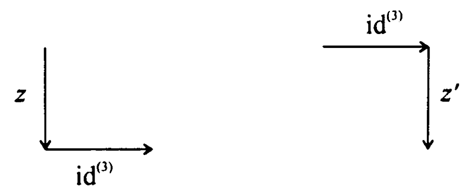


Figure 6.33: $z \cdot \mathrm{id}^{(3)}$ and $\mathrm{id}^{(3)} \cdot z'$

commutants are given by the $B_{k,0}$, that is, $B'_{0,\infty} \cap B_{k,\infty} = B_{k,0}$. The above is Ocneanu's compactness argument (which first appeared in [92]) in the setting of our $SU(3)$ subfactors. If $\mathcal{G}$ is a graph with flat connection, then the principal graph of the subfactor $B_{0,\infty} \subset B_{1,\infty}$ will be the 01-part $\mathcal{G}_{01}$ of $\mathcal{G}$.

Flatness of the connection for the $\mathcal{A}$, $\mathcal{D}$ graphs was shown in Theorem 6.3.3, where the distinguished vertex $*$ was chosen to be the vertex with lowest Perron-Frobenius weight. The flatness of the connection for the exceptional $\mathcal{E}$ graphs in not decided here. The determination of whether the connection is flat in these cases is a finite problem, involving checking the identity (6.19) for diagrams of size $2d_{\mathcal{G}_{01}} \times 2(d_{\mathcal{G}} + 3)$, where $d_{\mathcal{G}}$ is the depth of $\mathcal{G}$ and $d_{\mathcal{G}_{01}}$ is the depth of its 01-part $\mathcal{G}_{01}$. This is because for the vertical paths, the algebras $B_{l+1,j}$ are generated by $B_{l,j}$ and the Jones projection e_l for all $l \geq d_{\mathcal{G}_{01}}$, and e_l does not change its form under the change of basis using the connection. For the horizontal paths, by Lemma 5.1.7 we see that the algebras $B_{i,l+1}$ are generated by $B_{i,l}$ and U_{-l} for $l \geq d_{\mathcal{G}} + 3$, and the Hecke operators U_{-l} do not change their form under the change of basis, as is shown in the proof of Theorem 6.3.4 below.

We have not yet been able to determine whether or not the connection defined by (6.13), (6.14) is flat for the $\mathcal{E}$ cases, where the vertex $*$ is chosen to be the vertex with lowest Perron-Frobenius weight, since the number of computations involved, though finite, is extremely large. We expect that this connection will be flat for the exceptional graphs $\mathcal{E}^{(8)}$, $\mathcal{E}_1^{(12)}$ and $\mathcal{E}^{(24)}$, since these graphs appear as the M-N graphs for type I inclusions $N \subset M$, as in Table 5.1. We expect that this connection will not be flat for the remaining exceptional graphs $\mathcal{E}_2^{(12)}$, $\mathcal{E}_4^{(12)}$ and $\mathcal{E}_5^{(12)}$ for any choice of distinguished vertex $*$. We also expect that the connection will not be flat for the $\mathcal{A}^*$, $\mathcal{D}^*$ graphs, for any choice of distinguished vertex $*$. The principal graph for the graphs with a non-flat connection is given by its flat part, which should be the type I parents of Table 5.1.

6.3.1 Flat A_2-C^*-planar algebra from $SU(3)$ $\mathcal{ADE}$ subfactors

We will now associate a flat A_2-C^*-planar algebra P to a double sequence $(B_{i,j})$ of finite dimensional algebras with a flat connection.

We define the tangles W_{-k}, $k = 0, \ldots, j-1$, and f_l, $l = 1, \ldots, i$, in $\mathcal{P}_{i,j}(\emptyset)$ as in Figure 6.34, where the orientations of the strings without arrows depends on the parity of i and l.

Let $P_{i,j} = B_{i,j}$. We will define a presenting map $Z : \mathcal{P}_{i,j}(P) \to P_{i,j}$. Let T be a labelled tangle in $\mathcal{P}_{i,j}$ with m internal discs D_k with pattern i_k, j_k and labels $x_k \in B_{i_k,j_k}$, $k = 1, \ldots, m$. We define $Z(T)$ as follows. First, convert all the discs D_k to rectangles,

Figure 6.34: Tangles W_{-k} and f_l

with the first $i_k + j_k$ vertices along one edge, and the next $i_k + j_k$ vertices along the opposite edge, and rotate each rectangle so that those edges are horizontal with the first vertex on the top edge. Next, isotope the strings of T so that each horizontal strip only contains one of the following elements: a rectangle with label x_k, a cup, a cap, a Y-fork, or an inverted Y-fork. Let C be the set of all strips containing one of these elements except for a labelled rectangle. We will use the following notation for elements of C, as shown in Figures 6.35, 6.36 and 6.37: A strip containing a cup, cap will be $\cup^{(i)}$, $\cap^{(i)}$ respectively, where there are $i-1$ vertical strings to the left of the cup or cap. Strips containing an incoming Y-fork, inverted Y-fork will be $\curlyvee^{(i)}$, $\curlywedge^{(i)}$ respectively, where there are $i-1$ vertical strings to the left of the (inverted) Y-fork. A bar will denote that it is an outgoing (inverted) Y-fork.

Figure 6.35: Cup $\cup^{(i)}$ and cap $\cap^{(i)}$

Figure 6.36: Y-forks $\curlyvee^{(i)}$ and $\overline{\curlyvee}^{(i)}$

Figure 6.37: Inverted Y-forks $\curlywedge^{(i)}$ and $\overline{\curlywedge}^{(i)}$

For an element $c \in C$ with n_1, n_2 strings having endpoints (we will call these endpoints vertices) along the top, bottom edge respectively of the strip, let the orientations of these vertices along the top, bottom edge respectively of the strip be given by the sequences $\mathbf{v}^{(1)}$, $\mathbf{v}^{(2)}$ respectively, where for $i = 1,2$, $\mathbf{v}^{(i)} = (v_0^{(i)}, v_1^{(i)}, v_2^{(i)}, \ldots, v_{l_i}^{(i)})$, where $v_0^{(i)} \in \mathbb{N} \cup \{0\}$ and $v_k^{(i)} \in \mathbb{N}$ for $k \geq 1$, with $\sum_{k=1}^{l_i} v_k^{(i)} = n_i$. The numbers $v_k^{(i)}$ denote the number of consecutive vertices with downwards, upwards orientation for k even, odd respectively. Note that if the first vertex along the top, bottom of the strip has upwards orientation, then $v_0^{(i)} = 0$ for $i = 1,2$ respectively. The leftmost region of the strip c corresponds to the vertex $*$ of $\mathcal{G}$, and each vertex along the top (or bottom) with downwards, upwards orientation respectively, corresponds to an edge on $\mathcal{G}$, $\widetilde{\mathcal{G}}$ respectively ($\widetilde{\mathcal{G}}$ is the graph $\mathcal{G}$ with all orientations reversed). Then the top, bottom edge of the strip corresponds is labelled by all paths on $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ which start at $*$ and have the form given by $\mathbf{v}^{(i)}$. These paths are uniquely described by the sequence of edges they pass along. Let H_1, H_2 be the Hilbert spaces corresponding to all paths of the form $\mathbf{v}^{(1)}$, $\mathbf{v}^{(2)}$ respectively. Then $Z(c)$ defines an operator $M_c \in \mathrm{End}(H_1, H_2)$ as follows.

For a cup $\cup^{(i)}$,

$$(M_{\cup^{(i)}})_{\alpha,\beta} = \delta_{\alpha_1,\beta_1}\delta_{\alpha_2,\beta_2}\cdots\delta_{\alpha_{i-1},\beta_{i-1}}\delta_{\alpha_i,\beta_{i+2}}\delta_{\alpha_{i+1},\beta_{i+3}}\cdots\delta_{\alpha_m,\beta_{m+2}}\delta_{\widetilde{\beta}_i,\beta_{i+1}}\frac{\sqrt{\phi_{r(\beta_i)}}}{\sqrt{\phi_{s(\beta_i)}}}, \tag{6.20}$$

for paths $\alpha = \alpha_1 \cdot \alpha_2 \cdots \alpha_j$, $\beta = \beta_1 \cdots \beta_{j+2}$.

For a cap $\cap^{(i)}$,

$$M_{\cap^{(i)}} = M^*_{\cup^{(i)}}. \tag{6.21}$$

For an incoming (inverted) Y-fork $\curlyvee^{(i)}$ or $\curlywedge^{(i)}$,

$$(M_{\curlyvee^{(i)}})_{\alpha,\beta} = \delta_{\alpha_1,\beta_1}\cdots\delta_{\alpha_{i-1},\beta_{i-1}}\delta_{\alpha_{i+1},\beta_{i+2}}\cdots\delta_{\alpha_m,\beta_{m+1}}\frac{1}{\sqrt{\phi_{s(\alpha_i)}\phi_{r(\alpha_i)}}}W(\triangle_{(\widetilde{\alpha}_i\cdot\beta_i\cdot\beta_{i+1})}), \tag{6.22}$$

$$(M_{\curlywedge^{(i)}})_{\alpha,\beta} = \delta_{\alpha_1,\beta_1}\cdots\delta_{\alpha_{i-1},\beta_{i-1}}\delta_{\alpha_{i+2},\beta_{i+1}}\cdots\delta_{\alpha_{m+1},\beta_m}\frac{1}{\sqrt{\phi_{s(\beta_i)}\phi_{r(\beta_i)}}}\overline{W(\triangle_{(\beta_i\cdot\widetilde{\alpha}_{i+1}\cdot\widetilde{\alpha}_i)})}, \tag{6.23}$$

where W is a cell system on $\mathcal{G}$ satisfying (4.1) and (4.2).

For an outgoing (inverted) Y-fork $\overline{\curlyvee}^{(i)}$ or $\overline{\curlywedge}^{(i)}$,

$$M_{\overline{\curlyvee}^{(i)}} = M^*_{\curlywedge^{(i)}}, \tag{6.24}$$

$$M_{\overline{\curlywedge}^{(i)}} = M^*_{\curlyvee^{(i)}}. \tag{6.25}$$

For a strip b_k containing a rectangle with label $x_k = \sum_{\gamma,\gamma'} \lambda_{\gamma,\gamma'}(\gamma,\gamma')$ where $\lambda_{\gamma,\gamma'} \in \mathbb{C}$ and $(\gamma,\gamma') \in P_{i_k,j_k}$ are matrix units indexed by paths γ, γ', we define the operator

$M_{b_k} = Z(b_k)$ as follows. Let p_k, p'_k be the number of vertical strings to the left, right of the rectangle in strip b_k respectively, with orientations given by the sequences $\mathbf{v}^{(p_k)} = (v_0^{(p_k)}, v_1^{(p_k)}, \ldots, v_{p_k}^{(p_k)})$, $\mathbf{v}^{(p'_k)} = (v_0^{(p'_k)}, v_1^{(p'_k)}, \ldots, v_{p'_k}^{(p'_k)})$ respectively. We attach trivial tails of length p_k of the form $\mathbf{v}^{(p_k)}$ (on $\mathcal{G}$, $\widetilde{\mathcal{G}}$) to x_k and use the connection to transform this to an element in the basis which has the first p_k edges of the form $\mathbf{v}^{(p_k)}$, followed by j_k edges on $\mathcal{G}$ and lastly i_k edges on $\mathcal{G}$, $\widetilde{\mathcal{G}}$ alternately (with the $(p_k + j_k + 1)$-th edge on $\mathcal{G}$). By flatness of the connection on $\mathcal{G}$, this will be an element of the form $\sum_{\gamma,\gamma',\zeta,\zeta',\mu} \lambda_{\gamma,\gamma'} p_{\zeta,\zeta'}(\mu \cdot \zeta, \mu \cdot \zeta')$, where $p_{\zeta,\zeta'} \in \mathbb{C}$ are given by the connection, and ν are paths of the form $\mathbf{v}^{(p_k)}$. Adding trivial tails of length p'_k and of the form $\mathbf{v}^{(p'_k)}$ gives an element $\sum_{\gamma,\gamma',\zeta,\zeta',\mu,\nu} \lambda_{\gamma,\gamma'} p_{\zeta,\zeta'}(\mu \cdot \zeta \cdot \nu, \mu \cdot \zeta' \cdot \nu)$ which defines the matrix M_{b_k} (M_{b_k} is indexed by all paths of length $p_k + j_k + i_k + p'_k$ on $\mathcal{G}$, $\widetilde{\mathcal{G}}$ of the form $(\mathbf{v}^{(p_k)}, j_k + 1, 1, 1, 1, \ldots, \mathbf{v}^{(p'_k)})$).

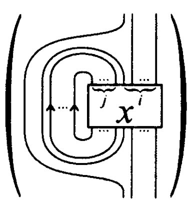
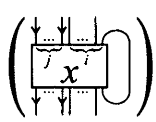
For a tangle $T \in \mathcal{P}_{i,j}$ with l horizontal strips s_l, where s_1 is the lowest strip, s_2 the strip immediately above it, and so on, we define $Z(T) = Z(s_1)Z(s_2) \cdots Z(s_l)$, which will be an element of $P_{i,j}$. This algebra is normalized in the sense that for the empty tangle $\bigcirc$, $Z(\bigcirc) = 1$. We need to show that this only depends on T, and not on the decomposition of T into horizontal strips.

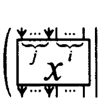
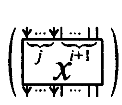
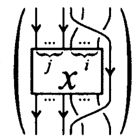
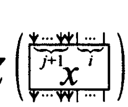
The following theorem shows that the double sequence $(B_{i,j})$ for an $\mathcal{ADE}$ graph with a flat connection gives a flat A_2-C^*-planar algebra. However, unlike [64, Theorem 4.2.1] for Jones's planar algebras, we have been unable to prove the uniqueness of this A_2-planar algebra, due to the existence of the tangles $f_m^{(p)}$ of Figure 6.47. These tangles and the corresponding elements in $(B_{i,j})$ are not understood very well.

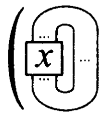
Theorem 6.3.4 *Let $\mathcal{G}$ be an $\mathcal{ADE}$ graph such that the connections (6.13), (6.14) are flat. The above definition of $Z(T)$ for any A_2-planar tangle T makes the above double sequence $(B_{i,j})$ for $\mathcal{G}$ into a flat A_2-C^*-planar algebra P ($= P_{(+)}$) with $\dim(P_0) = \dim(P_{0,1}^{(0,1)}) = \dim(P_{0,2}^{(0,2)}) = 1$. This A_2-C^*-planar algebra has parameter $\alpha = [3]$ (the Perron-Frobenius eigenvalue for $\mathcal{G}$), $Z(I_{i,j}(x)) = x$, where $I_{i,j}(x)$ is the tangle $I_{i,j}$ with $x \in P_{i,j}$ as the insertion in its inner disc, and*

(*i*) $Z(W_{-k}) = U_{-k}, \qquad k \geq 0,$

(*ii*) $Z(f_l) = \alpha e_l, \qquad l \geq 1,$

(*iii*) Z  $= \alpha^{j+1} E_{M' \cap M_{i-1}}(x)$, $\quad Z$  $= \alpha E_{M_{i-2}}(x)$,

(iv) $Z\Big($  $\Big) = Z\Big($  $\Big)$, $\quad Z\Big($  $\Big) = Z\Big($  $\Big)$,

(v) $\alpha^{-i-j} Z\Big($  $\Big) = \mathrm{tr}(x)$,

for $x \in P_{i,j}$, $i, j \geq 0$. In the first equation of (iii) the first $j+1$ vertices along the top and bottom of the rectangle are joined by loops, and the second equation only holds for $i \neq 0$. In the first, second equation of (iv) respectively, the x on the right hand side is considered as an element of $P_{i+1,j}$, $P_{i,j+1}$ respectively.

Proof

First we show that $Z(T)$ does not change if the labelled tangle is changed by isotopy of the strings. We use the following notation $\partial_{\alpha_i,\beta_j}^{\alpha_{i+k},\beta_{j+k}} := \delta_{\alpha_i,\beta_j}\delta_{\alpha_{i+1},\beta_{j+1}}\cdots\delta_{\alpha_{i+k},\beta_{j+k}}$.

Case (1)- Topological moves.

We consider the cup-cap simplifications (which Kauffman calls Move Zero in [70]) shown in Figure 6.38.

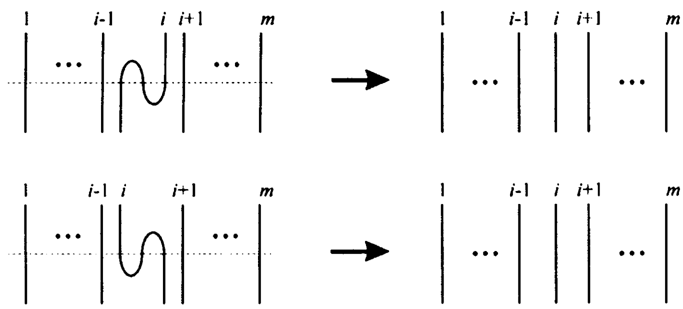


Figure 6.38: Two cup-cap simplifications

For the first cup-cap simplification of Figure 6.38 we have

$$\begin{aligned}
(M_{\cup(i+1)}M_{\cap(i)})_{\alpha,\beta} &= \sum_{\gamma} (M_{\cup(i+1)})_{\alpha,\gamma}(M_{\cup(i)})_{\beta,\gamma} \\
&= \sum_{\gamma} \partial_{\alpha_1,\gamma_1}^{\alpha_i,\gamma_i}\partial_{\alpha_{i+1},\gamma_{i+3}}^{\alpha_m,\gamma_{m+2}}\delta_{\widetilde{\gamma_{i+1}},\gamma_{i+2}}\frac{\sqrt{\phi_{r(\gamma_{i+1})}}}{\sqrt{\phi_{s(\gamma_{i+1})}}}\partial_{\beta_1,\gamma_1}^{\beta_{i-1},\gamma_{i-1}}\partial_{\beta_i,\gamma_{i+2}}^{\beta_m,\gamma_{m+2}}\delta_{\tilde{\gamma}_i,\gamma_{i+1}}\frac{\sqrt{\phi_{r(\gamma_i)}}}{\sqrt{\phi_{s(\gamma_i)}}} \\
&= \partial_{\alpha_1,\beta_1}^{\alpha_{i-1},\beta_{i-1}}\partial_{\alpha_{i+1},\beta_{i+1}}^{\alpha_m,\beta_m}\delta_{\alpha_i,\beta_i}\frac{\sqrt{\phi_{r(\gamma_{i+1})}}}{\sqrt{\phi_{r(\gamma_i)}}}\frac{\sqrt{\phi_{r(\gamma_i)}}}{\sqrt{\phi_{r(\tilde{\gamma}_i)}}} \quad = \quad \delta_{\alpha,\beta}.
\end{aligned} \tag{6.26}$$

The second simplification in Figure 6.38 follows from the above, since

$$M_{\cup(i)} M_{\cap(i+1)} = (M_{\cup(i+1)} M_{\cap(i)})^T = \mathbf{1}. \tag{6.27}$$

Case (2)- Isotopies involving incoming trivalent vertices.

We require the identities of Figure 6.39.

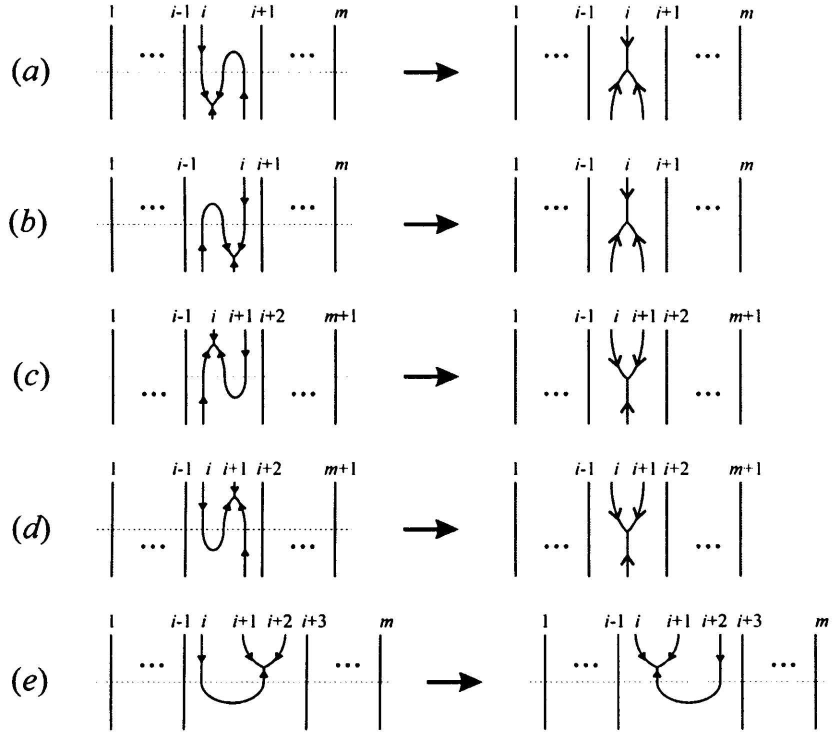


Figure 6.39: Isotopies involving an incoming trivalent vertex

For (a) : $\quad (M_{\curlyvee(i)} M_{\cap(i+1)})_{\alpha,\beta} = \sum_{\gamma} (M_{\curlyvee(i)})_{\alpha,\gamma} (M_{\cup(i+1)})_{\beta,\gamma}$

$$
\begin{aligned}
&= \sum_{\gamma} \partial^{\alpha_{i-1},\gamma_{i-1}}_{\alpha_1,\gamma_1} \partial^{\alpha_{m+1},\gamma_{m+2}}_{\alpha_{i+1},\gamma_{i+2}} \frac{1}{\sqrt{\phi_{s(\alpha_i)}\phi_{r(\alpha_i)}}} W_{(\widetilde{\alpha_i}\cdot\gamma_i\cdot\gamma_{i+1})} \\
&\qquad \cdot \partial^{\beta_i,\gamma_i}_{\beta_1,\gamma_1} \partial^{\beta_m,\gamma_{m+2}}_{\beta_{i+1},\gamma_{i+3}} \delta_{\widetilde{\gamma_{i+1}},\gamma_{i+2}} \frac{\sqrt{\phi_{r(\gamma_{i+1})}}}{\sqrt{\phi_{s(\gamma_{i+1})}}} \\
&= \partial^{\alpha_{i-1},\beta_{i-1}}_{\alpha_1,\beta_1} \partial^{\alpha_{m+1},\beta_m}_{\alpha_{i+2},\beta_{i+1}} \frac{1}{\sqrt{\phi_{s(\alpha_i)}\phi_{r(\alpha_i)}}} \delta_{\alpha_{i+1},\gamma_{i+2}} \sum_{\gamma_{i+1}} \delta_{\widetilde{\gamma_{i+1}},\gamma_{i+2}} W_{(\widetilde{\alpha_i}\cdot\beta_i\cdot\gamma_{i+1})} \frac{\sqrt{\phi_{s(\alpha_{i+1})}}}{\sqrt{\phi_{r(\beta_i)}}} \\
&= \partial^{\alpha_{i-1},\beta_{i-1}}_{\alpha_1,\beta_1} \partial^{\alpha_{m+1},\beta_m}_{\alpha_{i+2},\beta_{i+1}} \frac{1}{\sqrt{\phi_{s(\beta_i)}\phi_{r(\beta_i)}}} W_{(\widetilde{\alpha_i}\cdot\beta_i\cdot\widetilde{\alpha_{i+1}})} \quad = \quad (M_{\curlywedge(i)})_{\alpha,\beta} .
\end{aligned}
$$

The identities (b), (c) and (d) follow similarly. For (e):

$$
\begin{aligned}
(M_{\cup(i-1)}M_{\curlyvee(i)})_{\alpha,\beta} &= \sum_{\gamma}(M_{\cup(i-1)})_{\alpha,\gamma}(M_{\curlyvee(i)})_{\gamma,\beta} \\
&= \sum_{\gamma}\partial^{\alpha_{i-2},\gamma_{i-2}}_{\alpha_1,\gamma_1}\partial^{\alpha_m,\gamma_{m+2}}_{\alpha_{i-1},\gamma_{i+1}}\delta_{\widetilde{\gamma_{i-1}},\gamma_i}\frac{\sqrt{\phi_{r(\gamma_{i-1})}}}{\sqrt{\phi_{s(\gamma_{i-1})}}} \\
&\qquad\cdot\partial^{\gamma_{i-1},\beta_{i-1}}_{\gamma_1,\beta_1}\partial^{\gamma_{m+2},\beta_{m+3}}_{\gamma_{i+1},\beta_{i+2}}\frac{1}{\sqrt{\phi_{s(\gamma_i)}\phi_{r(\gamma_i)}}}W_{(\tilde{\gamma}_i\cdot\beta_i\cdot\beta_{i+1})} \\
&= \partial^{\alpha_{i-2},\beta_{i-2}}_{\alpha_1,\beta_1}\partial^{\alpha_m,\beta_{m+3}}_{\alpha_{i-1},\beta_{i+2}}\frac{\sqrt{\phi_{r(\beta_{i-1})}}}{\sqrt{\phi_{s(\beta_{i-1})}}}\frac{1}{\sqrt{\phi_{s(\beta_{i-1})}\phi_{r(\beta_{i-1})}}}W_{(\widetilde{\beta_{i-1}}\cdot\beta_i\cdot\beta_{i+1})}
\end{aligned}
$$

and

$$
\begin{aligned}
(M_{\cup(i-1)}M_{\curlyvee(i-1)})_{\alpha,\beta} &= \sum_{\gamma}(M_{\cup(i-1)})_{\alpha,\gamma}(M_{\curlyvee(i)})_{\gamma,\beta} \\
&= \sum_{\gamma}\partial^{\alpha_{i-2},\gamma_{i-2}}_{\alpha_1,\gamma_1}\partial^{\alpha_m,\gamma_{m+2}}_{\alpha_{i-1},\gamma_{i+1}}\delta_{\widetilde{\gamma_{i-1}},\gamma_i}\frac{\sqrt{\phi_{r(\gamma_{i-1})}}}{\sqrt{\phi_{s(\gamma_{i-1})}}} \\
&\qquad\cdot\partial^{\gamma_{i-2},\beta_{i-2}}_{\gamma_1,\beta_1}\partial^{\gamma_{m+2},\beta_{m+3}}_{\gamma_i,\beta_{i+1}}\frac{1}{\sqrt{\phi_{s(\gamma_{i-1})}\phi_{r(\gamma_{i-1})}}}W_{(\widetilde{\gamma_{i-1}}\cdot\beta_{i-1}\cdot\beta_i)} \\
&= \partial^{\alpha_{i-2},\beta_{i-2}}_{\alpha_1,\beta_1}\partial^{\alpha_m,\beta_{m+3}}_{\alpha_{i-1},\beta_{i+2}}\frac{1}{\phi_{s(\widetilde{\beta_{i+1}})}}W_{(\widetilde{\beta_{i-1}}\cdot\beta_i\cdot\beta_{i+1})} = (M_{\cup(i-1)}M_{\curlyvee(i)})_{\alpha,\beta}
\end{aligned}
$$

The corresponding identities for outgoing trivalent vertices hold in the same way. Then the identity in Figure 6.40 follows from the cup-cap simplifications and identities (a)-(e) for incoming and outgoing trivalent vertices.

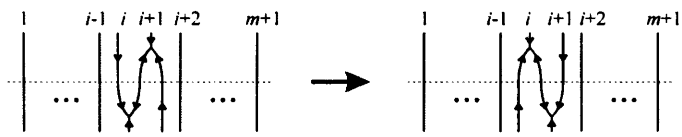


Figure 6.40: An isotopy involving an incoming and outgoing trivalent vertex

Kuperberg relations. Before checking isotopies that involve rectangles, we will show that the Kuperberg relations K1-K3 are satisfied. For K1, a closed loop gives

$$
\begin{aligned}
(M_{\cup(i)}M_{\cap(i)})_{\alpha,\beta} &= \sum_{\gamma}(M_{\cup(i)})_{\alpha,\gamma}(M_{\cup(i)})_{\beta,\gamma} = \delta_{\alpha,\beta}\sum_{\substack{\gamma_i:\\ s(\gamma_i)=r(\alpha_{i-1})}}\frac{\phi_{r(\gamma_i)}}{\phi_{s(\gamma_i)}} \\
&= \delta_{\alpha,\beta}\sum_{\gamma_i}\Lambda(r(\alpha_{i-1}),r(\gamma_i))\frac{\phi_{r(\gamma_i)}}{\phi_{r(\alpha_{i-1})}} = \delta_{\alpha,\beta}[3],
\end{aligned} \tag{6.28}
$$

by the Perron-Frobenius eigenvalue equation $\Lambda x = [3]x$, $x = (\phi_v)_v$, where Λ is $\Delta_{\mathcal{G}}$ or $\Delta_{\mathcal{G}}^T$ depending on whether the loop has anticlockwise, clockwise orientation respectively. Next consider K2. For the first diagram in Figure 6.41 we have

$$\begin{aligned}
\left(M_{\curlyvee^{(i)}} M_{\overline{\curlywedge}^{(i)}}\right)_{\alpha,\beta} &= \sum_{\gamma} \left(M_{\curlyvee^{(i)}}\right)_{\alpha,\gamma} \overline{\left(M_{\curlyvee^{(i)}}\right)_{\beta,\gamma}} \\
&= \sum_{\gamma} \partial_{\alpha_1,\gamma_1}^{\alpha_{i-1},\gamma_{i-1}} \partial_{\alpha_{i+1},\gamma_{i+2}}^{\alpha_m,\gamma_{m+1}} \frac{1}{\sqrt{\phi_{s(\alpha_i)}\phi_{r(\alpha_i)}}} W_{(\widetilde{\alpha_i}\cdot\gamma_i\cdot\gamma_{i+1})} \\
&\qquad \partial_{\beta_1,\gamma_1}^{\beta_{i-1},\gamma_{i-1}} \partial_{\beta_{i+1},\gamma_{i+2}}^{\beta_m,\gamma_{m+1}} \frac{1}{\sqrt{\phi_{s(\beta_i)}\phi_{r(\beta_i)}}} \overline{W_{(\tilde{\beta}_i\cdot\gamma_i\cdot\gamma_{i+1})}} \\
&= \partial_{\alpha_1,\beta_1}^{\alpha_{i-1},\beta_{i-1}} \partial_{\alpha_{i+1},\beta_{i+1}}^{\alpha_m,\beta_m} \sum_{\gamma_i,\gamma_{i+1}} \frac{1}{\phi_{s(\alpha_i)}\phi_{r(\alpha_i)}} W_{(\widetilde{\alpha_i}\cdot\gamma_i\cdot\gamma_{i+1})} \overline{W_{(\tilde{\beta}_i\cdot\gamma_i\cdot\gamma_{i+1})}} \\
&= \partial_{\alpha_1,\beta_1}^{\alpha_{i-1},\beta_{i-1}} \partial_{\alpha_{i+1},\beta_{i+1}}^{\alpha_m,\beta_m} \delta_{\alpha_i,\beta_i}[2] \qquad (6.29) \\
&= \delta_{\alpha,\beta}[2],
\end{aligned}$$

where (6.29) follows from Ocneanu's type I formula (4.1).

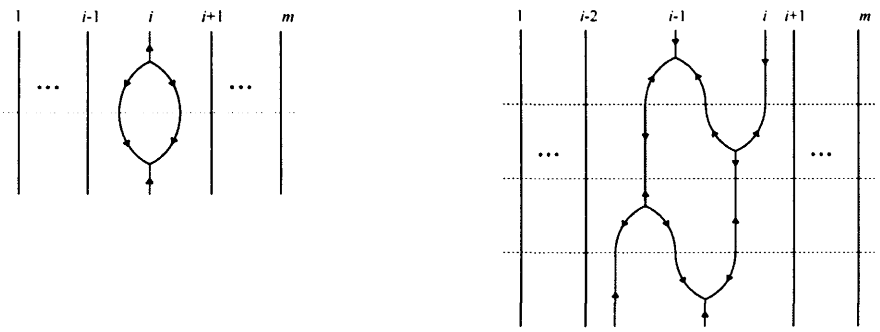


Figure 6.41: Tangles for checking K2 and K3

Finally, for K3 we have the second diagram in Figure 6.41, which gives

$$\begin{aligned}
&\left(M_{\curlyvee^{(i)}} M_{\overline{\curlywedge}^{(i-1)}} M_{\overline{\curlyvee}^{(i)}} M_{\curlywedge^{(i-1)}}\right)_{\alpha,\beta} = \sum_{\gamma,\zeta,\eta} \left(M_{\curlyvee^{(i)}}\right)_{\alpha,\gamma} \overline{\left(M_{\curlyvee^{(i-1)}}\right)_{\zeta,\gamma} \left(M_{\curlywedge^{(i)}}\right)_{\eta,\zeta}} \left(M_{\curlywedge^{(i-1)}}\right)_{\eta,\beta} \\
&= \sum_{\gamma,\zeta,\eta} \partial_{\alpha_1,\gamma_1}^{\alpha_{i-1},\gamma_{i-1}} \partial_{\alpha_{i+1},\gamma_{i+2}}^{\alpha_m,\gamma_{m+1}} \frac{1}{\sqrt{\phi_{s(\alpha_i)}\phi_{r(\alpha_i)}}} W_{(\widetilde{\alpha_i}\cdot\gamma_i\cdot\gamma_{i+1})} \\
&\qquad \cdot\partial_{\zeta_1,\gamma_1}^{\zeta_{i-2},\gamma_{i-2}} \partial_{\zeta_i,\gamma_{i+1}}^{\zeta_m,\gamma_{m+1}} \frac{1}{\sqrt{\phi_{s(\zeta_{i-1})}\phi_{r(\zeta_{i-1})}}} \overline{W_{(\widetilde{\zeta_{i-1}}\cdot\gamma_{i-1}\cdot\gamma_i)}} \\
&\qquad \cdot\partial_{\eta_1,\zeta_1}^{\eta_{i-1},\zeta_{i-1}} \partial_{\eta_{i+2},\zeta_{i+1}}^{\eta_{m+1},\zeta_m} \frac{1}{\sqrt{\phi_{s(\zeta_i)}\phi_{r(\zeta_i)}}} \overline{W_{(\zeta_i\cdot\widetilde{\eta_{i+1}}\cdot\tilde{\eta}_i)}} \\
&\qquad \cdot\partial_{\eta_1,\beta_1}^{\eta_{i-2},\beta_{i-2}} \partial_{\eta_{i+1},\beta_i}^{\eta_{m+1},\beta_m} \frac{1}{\sqrt{\phi_{s(\beta_{i-1})}\phi_{r(\beta_{i-1})}}} W_{(\beta_{i-1}\cdot\tilde{\eta}_i\cdot\widetilde{\eta_{i-1}})}
\end{aligned}$$

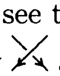
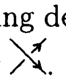
$$
\begin{aligned}
&= \partial_{\alpha_1,\beta_1}^{\alpha_{i-2},\beta_{i-2}} \partial_{\alpha_{i+1},\beta_{i+1}}^{\alpha_m,\beta_m} \frac{1}{\phi_{s(\alpha_{i-1})}\phi_{r(\alpha_i)}\sqrt{\phi_{r(\alpha_{i-1})}\phi_{r(\beta_{i-1})}}} \\
&\qquad \cdot \sum_{\substack{\gamma_i,\gamma_{i+1},\zeta_{i-1} \\ \zeta_i,\eta_{i-1},\eta_i}} \delta_{\zeta_i,\gamma_{i+1}}\delta_{\zeta_{i-1},\eta_{i-1}} \frac{1}{\phi_{r(\eta_{i-1})}} W_{(\widetilde{\alpha_i}\cdot\gamma_i\cdot\gamma_{i+1})} \overline{W_{(\widetilde{\zeta_{i-1}}\cdot\alpha_{i-1}\cdot\gamma_i)}}\, \overline{W_{(\zeta_i\cdot\widetilde{\beta}_i\cdot\widetilde{\eta_i})}} W_{(\beta_{i-1}\cdot\widetilde{\eta_i}\cdot\widetilde{\eta_{i-1}})} \\
&= \partial_{\alpha_1,\beta_1}^{\alpha_{i-2},\beta_{i-2}} \partial_{\alpha_{i+1},\beta_{i+1}}^{\alpha_m,\beta_m} \frac{1}{\phi_{s(\alpha_{i-1})}\phi_{r(\alpha_i)}\sqrt{\phi_{r(\alpha_{i-1})}\phi_{r(\beta_{i-1})}}} \\
&\qquad \cdot \sum_{\xi_1,\xi_2,\xi_3,\xi_4} \frac{1}{\phi_{r(\xi_1)}} W_{(\widetilde{\alpha_i}\cdot\xi_1\cdot\xi_2)} \overline{W_{(\alpha_{i-1}\cdot\xi_1\cdot\xi_3)}}\, \overline{W_{(\widetilde{\beta}_i\cdot\xi_4\cdot\xi_2)}} W_{(\beta_{i-1}\cdot\xi_4\cdot\xi_3)} \\
&= \partial_{\alpha_1,\beta_1}^{\alpha_{i-2},\beta_{i-2}} \partial_{\alpha_{i+1},\beta_{i+1}}^{\alpha_m,\beta_m} \frac{1}{\phi_{s(\alpha_{i-1})}\phi_{r(\alpha_i)}\sqrt{\phi_{r(\alpha_{i-1})}\phi_{r(\beta_{i-1})}}} \\
&\qquad \cdot \Big(\delta_{\alpha_{i-1},\beta_{i-1}}\delta_{\alpha_i,\beta_i}\phi_{s(\alpha_{i-1})}\phi_{r(\alpha_{i-1})}\phi_{s(\widetilde{\alpha_i})} \\
&\qquad\qquad + \delta_{\alpha_{i-1},\widetilde{\alpha_i}}\delta_{\beta_{i-1},\widetilde{\beta}_i}\phi_{r(\beta_{i-1})}\phi_{s(\alpha_{i-1})}\phi_{r(\widetilde{\alpha_{i-1}})} \Big) \\
&= \delta_{\alpha,\beta}\frac{\phi_{s(\widetilde{\alpha_i})}}{\phi_{r(\alpha_i)}} + \partial_{\alpha_1,\beta_1}^{\alpha_{i-2},\beta_{i-2}} \partial_{\alpha_{i+1},\beta_{i+1}}^{\alpha_m,\beta_m} \delta_{\alpha_{i-1},\widetilde{\alpha_i}}\delta_{\beta_{i-1},\widetilde{\beta}_i} \frac{\sqrt{\phi_{r(\beta_{i-1})}\phi_{r(\alpha_{i-1})}}}{\phi_{r(\alpha_i)}} \qquad (6.30) \\
&= \mathbf{1}_{\alpha,\beta} + \alpha(e_i)_{\alpha,\beta},
\end{aligned}
$$

where (6.30) follows from Ocneanu's formula for Type II frames (4.2). This is the reason why the weights W were used in the definitions of $M_{\curlyvee}$, $M_{\overline{\curlyvee}}$, $M_{\curlywedge}$ and $M_{\overline{\curlywedge}}$.

Property (ii) and the connection.

We will next show property (ii) in the statement of the theorem:

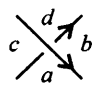
$$
\begin{aligned}
\left(M_{\overline{\curlywedge}(j-k)} M_{\curlyvee(j-k)}\right)_{\alpha,\beta} &= \sum_{\gamma} \overline{\left(M_{\curlyvee(j-k)}\right)_{\gamma,\alpha}} \left(M_{\curlyvee(j-k)}\right)_{\gamma,\beta} \\
&= \sum_{\gamma} \partial_{\gamma_1,\alpha_1}^{\gamma_{j-k-1},\alpha_{j-k-1}} \partial_{\gamma_{j-k+1},\alpha_{j-k+2}}^{\gamma_{i+j-1},\alpha_{i+j}} \frac{1}{\sqrt{\phi_{s(\gamma_{j-k})}\phi_{r(\gamma_{j-k})}}} \overline{W_{(\widetilde{\gamma_{j-k}}\cdot\alpha_{j-k}\cdot\alpha_{j-k+1})}} \\
&\qquad \cdot \partial_{\gamma_1,\beta_1}^{\gamma_{j-k-1},\beta_{j-k-1}} \partial_{\gamma_{j-k+1},\beta_{j-k+2}}^{\gamma_{i+j-1},\beta_{i+j}} \frac{1}{\sqrt{\phi_{s(\gamma_{j-k})}\phi_{r(\gamma_{j-k})}}} W_{(\widetilde{\gamma_{j-k}}\cdot\beta_{j-k}\cdot\beta_{j-k+1})} \\
&= \partial_{\alpha_1,\beta_1}^{\alpha_{j-k-1},\beta_{j-k-1}} \partial_{\alpha_{j-k+2},\beta_{j-k+2}}^{\alpha_{i+j},\beta_{i+j}} \sum_{\lambda} \frac{1}{\phi_{s(\lambda)}\phi_{r(\lambda)}} W_{(\lambda\cdot\beta_{j-k}\cdot\beta_{j-k+1})} \overline{W_{(\lambda\cdot\alpha_{j-k}\cdot\alpha_{j-k+1})}} \\
&= \partial_{\alpha_1,\beta_1}^{\alpha_{j-k-1},\beta_{j-k-1}} \partial_{\alpha_{j-k+2},\beta_{j-k+2}}^{\alpha_{i+j},\beta_{i+j}} \mathcal{U}_{\beta_{j-k},\beta_{j-k+1}}^{\alpha_{j-k},\alpha_{j-k+1}} = (U_{-k})_{\alpha,\beta}\,.
\end{aligned}
$$

Since U_{-k} is given by the tangle W_{-k}, we see that the partial braiding defined in (6.2) gives the connection, where (6.13) is given by  and (6.14) is given by . For the latter connection, which involves the reverse graph $\widetilde{\mathcal{G}}$, if

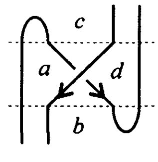
$$
\begin{array}{ccc}
a & \rightarrow & b \\
\downarrow & & \downarrow \\
c & \rightarrow & d
\end{array}
$$

is a connection on the graph $\mathcal{G}$, then

$$\begin{array}{ccc} c & \longrightarrow & d \\ \downarrow & & \downarrow \\ a & \longrightarrow & b \end{array} =$$

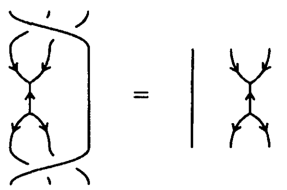


$$=$$



$$= \sqrt{\frac{\phi_a \phi_d}{\phi_b \phi_c}} \; \overline{\begin{array}{ccc} a & \longrightarrow & b \\ \downarrow & & \downarrow \\ c & \longrightarrow & d \end{array}} \, .$$

So we have that $Z(T)$ is invariant under all isotopies that only involve strings (and the partial braiding). This shows that the operators U_{-k} do not change their form under the change of basis using the connection, since



Note that we have not used the fact that the connection is flat yet, so the operators U_{-k} do not change their form under the change of basis for any of the $SU(3)$ $\mathcal{ADE}$ graphs.

Case (3)- Isotopies that involve rectangles.

We need to check invariance as in Figure 6.42.

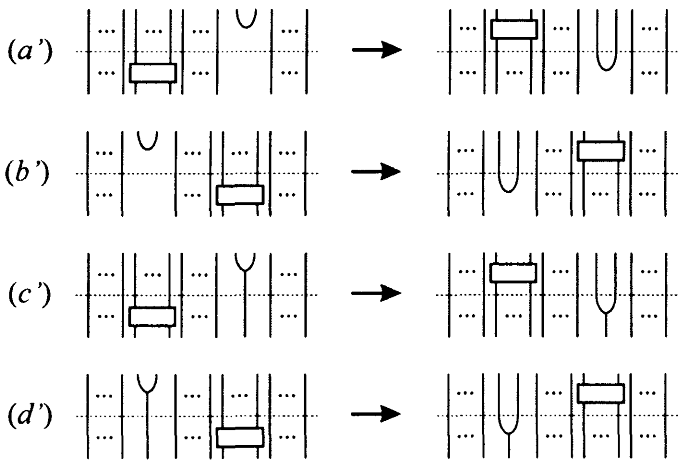
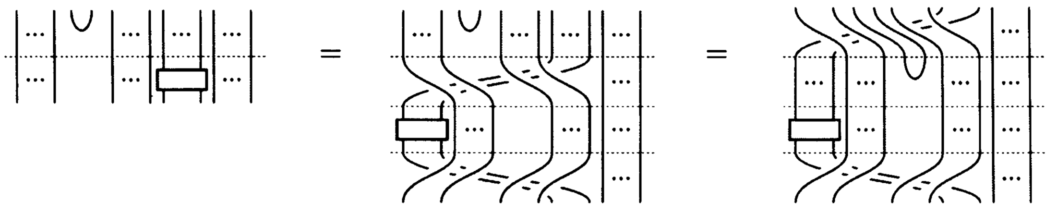
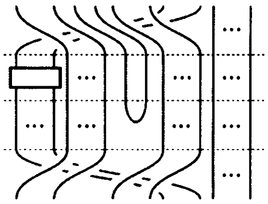


Figure 6.42: Isotopies involving rectangles

For (a'), pulling a cup down to the right of a rectangle b is trivial since $M_{\cup}$ commutes with M_b (since b, $\cup$ are localized on separate parts of the Bratteli diagram). Now consider (b'). By definition of $Z(b)$ for a horizontal strip b containing a rectangle labelled x, we have for the left hand side



where the second equality follows since Z is invariant under all isotopies that only involve strings and the partial braiding. Similarly, for the right hand side we obtain



and the result follows from (a'). The situations for (c'), (d') are similar to (a'), (b'). We also have the isotopy in Figure 6.43. Let $x = (\alpha_1, \alpha_2) \in P_{i_1,j_1}$, $y = (\beta_1, \beta_2) \in P_{i_2,j_2}$ such that $|\alpha_l| = k_2 = i_1+j_1$, $|\beta_l| = k_4 = i_2+j_2$, $l = 1, 2$. The case for general elements $x \in P_{i_1,j_1}$, $y \in P_{i_2,j_2}$ follows by linearity. We have $Z(s_1) = \sum_{\mu_i,\alpha'_j} p_{\alpha'_1,\alpha'_2}(\mu_1 \cdot \alpha'_1 \cdot \mu_3 \cdot \mu_4 \cdot \mu_5, \mu_1 \cdot \alpha'_2 \cdot \mu_3 \cdot \mu_4 \cdot \mu_5)$ and $Z(s_2) = \sum_{\nu_i,\beta'_j} q_{\beta'_1,\beta'_2}(\nu_1 \cdot \nu_2 \cdot \nu_3 \cdot \beta'_1 \cdot \nu_5, \nu_1 \cdot \nu_2 \cdot \nu_3 \cdot \beta'_2 \cdot \nu_5)$, where $|\mu_i| = |\nu_i| = k_i$, $i = 1, \ldots, 5$, $|\alpha'_j| = k_2$, $|\beta'_j| = k_4$, $j = 1, 2$, and $p_{\alpha'_1,\alpha'_2}, q_{\beta'_1,\beta'_2} \in \mathbb{C}$ are given by the connection. Then

$$
\begin{aligned}
& Z(s_1)Z(s_2) \\
& \quad = \sum_{\substack{\mu_i,\nu_i \\ \alpha'_j,\beta'_j}} p_{\alpha'_1,\alpha'_2} q_{\beta'_1,\beta'_2}\, \delta_{\mu_1,\nu_1}\delta_{\alpha'_2,\nu_2}\delta_{\mu_3,\nu_3}\delta_{\mu_4,\beta'_2}\delta_{\mu_5,\nu_5}\, (\mu_1 \cdot \alpha'_1 \cdot \mu_3 \cdot \mu_4 \cdot \mu_5, \nu_1 \cdot \nu_2 \cdot \nu_3 \cdot \beta'_2 \cdot \nu_5) \\
& \quad = \sum_{\mu_i,\alpha'_j,\beta'_j} p_{\alpha'_1,\alpha'_2} q_{\beta'_1,\beta'_2}\, (\mu_1 \cdot \alpha'_1 \cdot \mu_3 \cdot \beta'_1 \cdot \mu_5, \mu_1 \cdot \alpha'_2 \cdot \mu_3 \cdot \beta'_2 \cdot \mu_5) \;=\; Z(s_1)Z(s_2).
\end{aligned}
$$

Case (4)- Rotational invariance.

The other isotopy that needs to be checked is the rotation of internal rectangles by 2π. We illustrate the case where rectangle b has $k_b = 2$ vertices along its top and bottom edges in Figure 6.44.

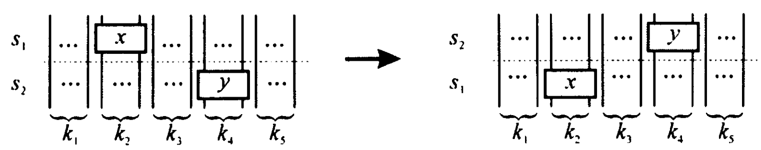


Figure 6.43: An isotopy involving two rectangles

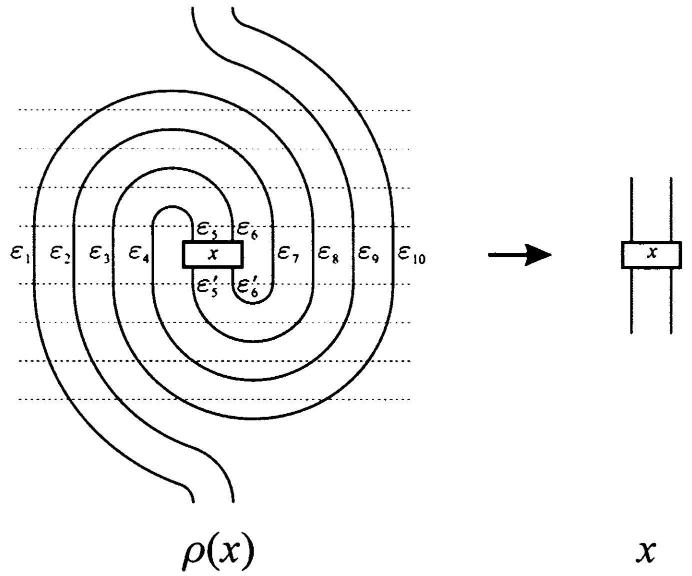


Figure 6.44: Rotation of internal rectangles by 2π

Let x be the label of the rectangle b, where x is the element (ν, ν') given by the pair of paths ν, ν' on the graphs $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ according to the orientations of the vertices along the top and bottom of the rectangle b. We add $4k_b$ vertical strings to the right of the rectangle b such that the first $2k_b$ have orientations corresponding to the first $2k_b$ strings in the strip containing the rectangle b in $\rho(x)$, and the next $2k_b$ have orientations corresponding to the last $2k_b$ strings in the strip. Then we have $x \longrightarrow x' = \sum_{\mu_1,\mu_2} (\nu \cdot \mu_1 \cdot \mu_2, \nu' \cdot \mu_1 \cdot \mu_2)$, where the sum is over all paths μ_1, μ_2 of length $2k_b$ on the graphs $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ according to the orientations of the vertical strings described above. Using the connection, we can write $\mathrm{Ad}(u)(x') = \sum_{\mu_1,\mu_2} \sum_{\zeta,\zeta'} a_{\zeta,\zeta'} (\mu_1 \cdot \zeta \cdot \mu_2, \mu_1 \cdot \zeta' \cdot \mu_2) = Y$, where u is the unitary given by the connections which change the basis of the paths which index x' so that it is indexed by paths on $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ according to the orientations of the strings in the strip containing the rectangle b in $\rho(x)$, the second sum is over all paths ζ, ζ' which have the same form as ν, ν', and the numbers $a_{\zeta,\zeta'} \in \mathbb{C}$ are given by the connections. Then $Y = Z(b')$ where b' is the strip in $\rho(x)$ which contains the rectangle b.

For a horizontal strip s_1 and strip s_2 immediately above it, an entry in the operator $Z(s') = Z(s_1)Z(s_2)$ is only defined when the path corresponding to the bottom edge of the strip s_2 is equal to the path given by the top edge of s_1. So for example, for the two strips s_1, s_2 in Figure 6.45, even though there are non-zero entries in $Z(s_1)$ for any path $\alpha = \alpha_1 \cdot \alpha_2 \cdots$, the entries in $Z(s')$ will be zero unless edge α_i is the reverse edge $\widetilde{\alpha_{i+1}}$ of α_{i+1} since the entries in $Z(s_2)$ are only non-zero for the paths $\gamma = \gamma_1 \cdot \gamma_2 \cdots$ such that

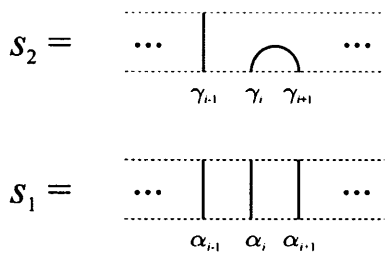


Figure 6.45: Horizontal strips s_1, s_2

$\gamma_i = \widetilde{\gamma_{i+1}}$.

Let $\varepsilon = \varepsilon_1 \cdots \varepsilon_{5k_b}$, $\varepsilon' = \varepsilon'_1 \cdots \varepsilon'_{5k_b}$ be two paths which label the indices for Y. For simplicity we consider the case $k_b = 2$ as in Figure (6.44). By considering the horizontal strip containing the rectangle, we see that $Y_{\varepsilon,\varepsilon'} = 0$ unless $\varepsilon_i = \varepsilon'_i$ for $i =$ 1,2,3,4,7,8,9,10. We see that in $\rho(x)$, ε_1 is the same string as ε_8 and ε'_5, but that ε_8 has the opposite orientation to ε_1 and ε'_5. We define the operator $\widehat{Y}$ by $\widehat{Y}_{\varepsilon,\varepsilon'} = 0$ unless $\varepsilon_1 = \widetilde{\varepsilon_8} = \varepsilon'_5$, $\varepsilon_2 = \widetilde{\varepsilon_7} = \varepsilon'_6$ $\varepsilon_3 = \widetilde{\varepsilon_6} = \widetilde{\varepsilon_{10}}$ and $\varepsilon_4 = \widetilde{\varepsilon_5} = \widetilde{\varepsilon_9}$, and $\widehat{Y}_{\varepsilon,\varepsilon'} = Y_{\varepsilon,\varepsilon'}$ otherwise. Then $\rho(x) = M_{\cup(k_b+1)} M_{\cup(k_b+2)} \cdots M_{\cup(3k_b)} Y M_{\cap(2k_b)} M_{\cap(2k_b-1)} \cdots M_{\cap(1)} = M_{\cup(k_b+1)} M_{\cup(k_b+2)} \cdots M_{\cup(3k_b)} \widehat{Y} M_{\cap(2k_b)} M_{\cap(2k_b-1)} \cdots M_{\cap(1)}$. For any ε and ε' such that $\widehat{Y}_{\varepsilon,\varepsilon'}$ is non-zero, the caps contribute a scalar factor $\sqrt{\phi_{r(\varepsilon_4)}}/\sqrt{\phi_{s(\varepsilon_1)}} = \sqrt{\phi_{s(\varepsilon'_5)}}/\sqrt{\phi_{s(\varepsilon'_5)}} = 1$, and similarly we have a scalar factor of 1 from the cups. Now ε_1 is an edge on $\mathcal{G}$ (or $\widetilde{\mathcal{G}}$) with $s(\varepsilon_1) = *$, and hence $\rho(x)$ is only non-zero for paths $\varepsilon_5 \cdot \varepsilon_6$ and $\varepsilon'_5 \cdot \varepsilon'_6$ such that $s(\varepsilon_5) = s(\varepsilon'_5) = *$. By the flatness of the connection on $\mathcal{G}$, the paths ζ, ζ' starting from $*$ in $\mathrm{Ad}(u)(x')$ are ν, ν'- the paths which indexed the original element x. Then the resulting operator given by $\rho(x)$ will have all entries 0 except for that for the pair ν, ν'. Then $\rho(x) = (\nu, \nu') = x$.

Then $Z(T)$ is invariant under all isotopies of the tangle T.

Properties (i)-(v).

For (i), we have $i + j$ vertices along the top and bottom:

$$(M_{\cap(l+j)} M_{\cup(l+j)})_{\alpha,\beta} = \sum_{\gamma} (M_{\cup(l+j)})_{\gamma,\alpha} (M_{\cup(l+j)})_{\gamma,\beta}$$

$$= \sum_{\gamma} \partial_{\gamma_1,\alpha_1}^{\gamma_{l+j-1},\alpha_{l+j-1}} \partial_{\gamma_{l+j},\alpha_{l+j+2}}^{\gamma_{i+j},\alpha_{i+j+2}} \delta_{\widetilde{\alpha_{l+j}},\alpha_{l+j+1}} \frac{\sqrt{\phi_{r(\alpha_{l+j})}}}{\sqrt{\phi_{s(\alpha_{l+j})}}}$$

$$\cdot \partial_{\gamma_1,\beta_1}^{\gamma_{l+j-1},\beta_{l+j-1}} \partial_{\gamma_{l+j},\beta_{l+j+2}}^{\gamma_{i+j},\beta_{i+j+2}} \delta_{\widetilde{\beta_{l+j}},\beta_{l+j+1}} \frac{\sqrt{\phi_{r(\beta_{l+j})}}}{\sqrt{\phi_{s(\beta_{l+j})}}}$$

$$
\begin{aligned}
&= \partial_{\alpha_1,\beta_1}^{\alpha_{l+j-1},\beta_{l+j-1}} \partial_{\alpha_{l+j+2},\beta_{l+j+2}}^{\alpha_{i+j+2},\beta_{i+j+2}} \delta_{\widetilde{\alpha_{l+j}},\alpha_{l+j+1}} \delta_{\widetilde{\beta_{l+j}},\beta_{l+j+1}} \frac{\sqrt{\phi_{r(\alpha_{l+j})}\phi_{r(\beta_{l+j})}}}{\phi_{s(\alpha_{l+j})}} \\
&= \alpha\left(e_l\right)_{\alpha,\beta} .
\end{aligned}
$$

Now consider property (iii). We start with the first equation. For any $x \in P_{i,j}$, $E_{M'\cap M_{i-1}}(x) = Z(EL_{i,0}^{i,0} ER_{i,0}^{i,1} ER_{i,1}^{i,2} \cdots ER_{i,j-1}^{i,j}(x))$, and so $E_{M'\cap M_{i-1}}$ is the conditional expectation onto $P_{i,0}^{(1,0)}$. We now show that $P_{i,0}^{(1,0)} = M' \cap M_{i-1}$. Embedding the subalgebra $P_{i,0}^{(1,0)}$ of $P_{i,0}$ in $P_{i,\infty}$ we see that it lives on the last $i-1$ strings, with the rest all vertical through strings. Then $P_{i,0}^{(1,0)}$ clearly commutes with M, since the embedding of $M = P_{1,\infty}$ in $P_{i,\infty}$ has the last $i-1$ strings all vertical through strings, so we have $M' \cap M_{i-1} \supset P_{i,0}^{(1,0)}$. For the opposite inclusion, we extend the double sequence $(B_{i,j})$ to the left to get

$$
\begin{array}{ccccccccccc}
 & & B_{0,0} & \subset & B_{0,1} & \subset & B_{0,2} & \subset & \cdots & \longrightarrow & B_{0,\infty} \\
\cap & & \cap & & \cap & & \cap & & & & \cap \\
B_{1,-1} & \subset & B_{1,0} & \subset & B_{1,1} & \subset & B_{1,2} & \subset & \cdots & \longrightarrow & B_{1,\infty} \\
\cap & & \cap & & \cap & & \cap & & & & \cap \\
B_{2,-1} & \subset & B_{2,0} & \subset & B_{2,1} & \subset & B_{2,2} & \subset & \cdots & \longrightarrow & B_{2,\infty} \\
\cap & & \cap & & \cap & & \cap & & & & \cap \\
\vdots & & \vdots & & \vdots & & \vdots & & & & \vdots
\end{array}
$$

Note that $B_{1,-1} = B_{0,0} = \mathbb{C}$. Since the connection is flat, by Ocneanu's compactness argument we have $B_{1,\infty}' \cap B_{i,\infty} = B_{i,-1}$. Let $x = (\alpha_1, \alpha_2)$ be an element of $B_{i,-1}$. We embed x in $B_{i,0}$ by adding trivial horizontal tails of length one, and using the connection we can write x as $x' = \sum_\mu p_{\beta_1,\beta_2}(\mu\cdot\beta_1, \mu\cdot\beta_2)$, where $p_{\beta_1,\beta_2} \in \mathbb{C}$. We see that $x' \in B_{i,0} = P_{i,0}$ is summed over all trivial edges μ of length 1 starting at $*$, and hence is given by $Z(T)$ for some $T \in \mathcal{P}_{i,0}$ which has a vertical through string from the first vertex along the top to the first vertex along the bottom, i.e $x \in P_{i,0}^{(1,0)}$. So $M' \cap M_{i-1} = B_{i,-1} \subset P_{i,0}^{(1,0)}$. Similarly we find that $M_k' \cap M_{i-1} = P_{i,0}^{(k+1,0)}$, for $-1 \leq k \leq i$, where $M_{-1} = N$, $M_0 = M$.

For the second equation of (iii), if $x \in P_{i,\infty}$ then $x \to Z(E_{i-1,\infty}^{i,\infty}(x))$ is the conditional expectation onto $P_{i-1,\infty} = M_{i-2}$, and the result for $x \in P_{i,j}$ follows by Lemma 6.3.2.

Property (iv) is clear. Finally, for (v) let x be a matrix unit $(\alpha, \beta) \in B_{i,j}$, where $i + j = k$. Then

$$
[3]^{-k} Z(\widehat{x}) = [3]^{-k} \delta_{\alpha,\beta} \frac{\phi_{r(\alpha)}}{\phi_*} \phi_*^2 = [3]^{-k} \delta_{\alpha,\beta} \phi_{r(\alpha)} = \mathrm{tr}((\alpha, \beta)),
$$

since $\phi_* = 1$, where $\widehat{x}$ is the tangle defined by joining the last vertex along the top of T to the last vertex along the bottom by a string which passes round the tangle on the right hand side, and joining the other vertices along the top to those on the bottom similarly. For $b_l = (\alpha_l, \beta_l) \in B_{i,j}$ such that $i + j = k$, $l = 1, 2$, we have $\mathrm{tr}(b_1^* b_2) = \delta_{\alpha_1,\alpha_2} \mathrm{tr}((\beta_1, \beta_2)) = [3]^{-k} \delta_{\alpha_1,\alpha_2} \delta_{\beta_1,\beta_2} \phi_{r(\alpha_1)} = [3]^{-k} \delta_{b_1,b_2} \phi_{r(\alpha_1)}$. Then the trace is positive definite since the matrix units b_l are mutually orthogonal elements of positive length.

The A_2-planar algebra is clearly flat, since by the definition of $Z(b)$ for a horizontal strip b containing a disc with label x with n vertical strings to the left of the disc, if Y is the operator defined by the horizontal strip containing the disc with label x and n vertical strings to the right of the disc which have the same orientations as those in the strip b to the left of the disc, then $Z(b) = \mathrm{Ad}(u)Y$, where the unitary u is given by the connection, which is just the definition of flatness.

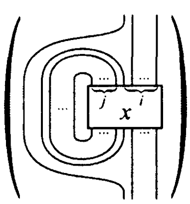
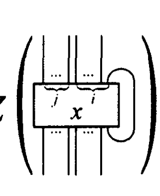
To see the $*$-structure, note that under $*$, the order of the strips is reversed so that $(Z(s_1)Z(s_2)\cdots Z(s_l))^* = Z(s_l)^* Z(s_{l-1})^* \cdots Z(s_1)^*$. For $M_{\cup^{(i)}}$, $\sqrt{\phi_{r(\beta_i)}}/\sqrt{\phi_{s(\beta_i)}}$ does not change under reflection of the tangle and reversing the orientation, so that $(M_{\cup^{(i)}})^*$ is the conjugate transpose of $M_{\cup^{(i)}}$ as required, and similarly for $M_{\cap^{(i)}}$. Since the involution of the strip $\Upsilon^{(i)}$ containing an incoming trivalent vertex is $\overline{\curlywedge}^{(i)}$, whilst the involution of the strip $\curlywedge^{(i)}$ containing an incoming trivalent vertex is $\overline{\Upsilon}^{(i)}$, so by (6.24), $(M_{\overline{\Upsilon}^{(i)}})^*$ is the conjugate transpose of $M_{\curlywedge^{(i)}}$ and by (6.25), $(M_{\overline{\curlywedge}^{(i)}})^*$ is the conjugate transpose of $M_{\Upsilon^{(i)}}$ as required. To show that P is an A_2-C^*-planar algebra we need to show that P is non-degenerate, which is immediate from property (v) in the statement of the theorem, Proposition 6.2.13 and the fact that tr is positive definite. $\square$

Definition 6.3.5 *We will say that an A_2-planar algebra P is an A_2-planar algebra for the subfactor $N \subset M$ if $P_{0,\infty} = N$, $P_{1,\infty} = M$, $P_{n,\infty} = M_{n-1}$, the sequence $P_{0,0} \subset P_{1,0} \subset P_{2,0} \subset \cdots$ is the tower of relative commutants, and if conditions (i)-(v) of Theorem 6.3.4 are satisfied.*

We now give the subfactor interpretation of duality:

Corollary 6.3.6 *If $P^{N\subset M} = P^{N\subset M}_{(+)}$ is the A_2-planar algebra for an $SU(3)$-$\mathcal{ADE}$ subfactor $N \subset M$ then, with the notation of Section 6.2.10, $\overline{P}$ is an A_2-$(-)$-planar algebra for the subfactor $M \subset M_1$, which satisfies conditions (i), (ii), (iv) and (v) of Theorem 6.3.4, and where condition (iii) becomes $(iii)'$:*

$(iii)'$

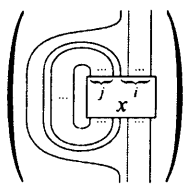
Z  $= \alpha^{j+1} E_{M_i' \cap M_{i-1}}(x)$, Z  $= \alpha E_{M_{i-1}}(x)$,

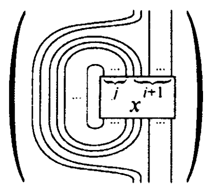
for $i, j \geq 0$.

Proof

Let $\mathcal{G}$ be the $SU(3)$ $\mathcal{ADE}$ graph for the subfactor $N \subset M$, $*$ its distinguished vertex, and $*_M$ the (unique) vertex given by $r(\zeta)$ where ζ is the edge such that $s(\zeta) = *$. The $(-)$-planar algebra $\overline{P}$ is the path algebra on the double sequence $(B_{i,j})$ where $B_{0,0}$ is identified with $*_M$, the Bratteli diagrams for inclusions $B_{i,j} \subset B_{i,j+1}$ are given by the graph $\mathcal{G}$, and the inclusions $B_{i,j} \subset B_{i+1,j}$ are given by its $\overline{j-1},\overline{j}$-part $\mathcal{G}_{\overline{j-1},\overline{j}}$, where $\overline{p} \in \{0,1,2\}$, $p \equiv \overline{p} \bmod 3$ for $p = j-1, j$. For $x = \sum_{\gamma_1,\gamma_2} \lambda_{\gamma_1,\gamma_2}(\gamma_1,\gamma_2)$, $\lambda_{\gamma_1,\gamma_2} \in \mathbb{C}$, the isomorphism $\lambda : \overline{P}_{i,j} \longrightarrow P^{(1,0)}_{i+1,j}$ is given by sending (γ_1,γ_2) to $(\zeta \cdot \gamma_1, \zeta \cdot \gamma_1)$ and using the connection to transform the paths $\zeta \cdot \gamma_i$ to the basis for paths which index $P_{i+1,j}$, which can be represented graphically as adding a string to the left of the disc containing x and conjugating by the connection. With $I^-_{i,j}$ the $(-)i,j$-identity tangle and $x \in \overline{P}_{i,j}$, we have $\overline{Z}(I^-_{i,j}(x)) = \alpha^{-1}\lambda^{-1}(Z(\widetilde{I^-_{i,j}}(\lambda(x))))$. In $\widetilde{I^-_{i,j}}(\lambda(x))$ the added string forms a closed loop, which can be removed to contribute a factor α, giving $Z(\widetilde{I^-_{i,j}}(\lambda(x))) = \lambda(x)$. Then $\overline{Z}(I^-_{i,j}(x)) = \lambda^{-1}\lambda(x) = x$.

Property (i) follows from $\overline{Z}(f_i) = Z(f_{i+1}) = \alpha e_{i+1}$, whilst (ii) is unchanged. Conditions (iv) and (v) are obvious, as is the second equality of $(iii)'$. For the first equality of condition $(iii)'$ we have

$\overline{Z}$ 

$= \alpha^{-1} Z$ 

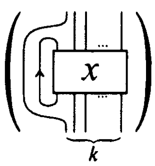
$= \alpha^{j+1} E_{M_i' \cap M_{i-1}}(x).$

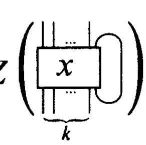
□

For the subalgebra Q introduced in §6.2, we give an alternative proof of Jones's theorem that extremal subfactors give planar algebras [64, Theorem 4.2.1] in the finite depth case.

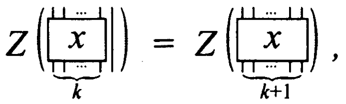
Corollary 6.3.7 *Let $N \subset M$ be a finite depth type II_1 subfactor. For each k let $Q_k = N' \cap M_{k-1}$. Then $Q = \bigcup_k Q_k$ has a spherical $(A_1\text{-})C^*$-planar algebra structure (in the sense of Jones), with labelling set Q, for which $Z(I_k(x)) = x$, where $I_k(x)$ is the tangle I_k with $x \in Q_k$ as the insertion in its inner disc, and*

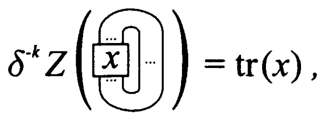
(i) $\quad Z(f_l) = \alpha e_l, \qquad l \geq 1,$

(ii) $\quad Z$ 

$= \delta E_{M' \cap M_{k-1}}(x), \qquad Z$ 

$= \delta E_{M_{k-1}}(x),$

(iii) 

(iv) 

for $x \in Q_k$, $k \geq 0$. *In condition (iii), the* x *on the right hand side is considered as an element of* Q_{k+1}. *Moreover, any other spherical planar algebra structure* Z' *with* $Z'(I_k(x)) = x$ *and (i), (ii), (iv) for* Z' *is equal to* Z.

Proof

We define Z in the same way as above, by converting all the discs of a tangle T to horizontal rectangles and isotoping the tangle so that in each horizontal strip there is either a labelled rectangle, a cup or a cap. Then we define $M_{\cup(i)}$ and $M_{\cap(i)}$ as in (6.20), (6.21). For strip b_l containing a rectangle with label x_l, we define M_{b_l} as in Theorem 6.3.4, using the connection on the principal graph $\mathcal{G}$ and its reverse graph $\widetilde{\mathcal{G}}$. The cup-cap simplification of Figure 6.38 follows from (6.26) and (6.27). The invariance of Z under isotopies involving rectangles as in Figures 6.42, 6.44 follows as in the proof of Theorem 6.3.4. That closed loops give a scalar factor of δ follows from (6.28), where the Perron-Frobenius eigenvalue now is δ.

Properties (*i*)-(*iv*) are proved in the same way as properties (*i*), (*iii*), (*iv*), (*v*) of Theorem 6.3.4, and uniqueness is proved as in [64]. □

6.3.2 Representation of Path Algebras as STL Algebras

We now show that each $B_{i,j}$ for the double sequence $(B_{i,j})$ defined above for $\mathcal{G} = \mathcal{A}^{(n)}$ also has a representation as $\mathcal{STL}_{i,j}$, where $\mathcal{STL}_{i,j}(\emptyset)$ is the quotient of $STL_{i,j}$ by the subspace of zero-length vectors, as in Section 6.2. Now $B_{1,j} \cong \mathcal{STL}_{1,j}$ by Lemma 6.1.12. Let $\psi : B_{1,j} \rightarrow \mathcal{STL}_{1,j}$ be the isomorphism given by $\psi(U_{-k}) = W_{-k}$, $k = 0, \ldots, j-1$. We define maps ϱ_i for $i \geq 2$ by $\varrho_2 = \varphi$, $\varrho_3 = \omega\varphi$, $\varrho_4 = \varphi\omega\varphi$, $\varrho_5 = \omega\varphi\omega\varphi$, ...

Let $x = \sum_{\gamma,\gamma'} \lambda_{\gamma,\gamma'}(\gamma,\gamma')$, $\lambda_{\gamma,\gamma'} \in \mathbb{C}$, be an element of $B_{i,j}$. Then $Z(\varrho_i^{-1}(x)) \in B_{1,i+j-1}$. We set $x_W \in \mathcal{STL}_{1,i+j-1}$ to be the element $\psi(Z(\varrho_i^{-1}(x)))$, and since $Z(W_{-k}) = U_{-k}$ we have $Z(x_W) = Z(\varrho_i^{-1}(x))$. For any $x \in B_{i,j}$, $\varrho_i(x_W) \in STL_{i,j}$ and $Z(\varrho_i(x_W)) = Z(\varrho_i(Z(x_W))) = Z(\varrho_i(Z(\varrho_i^{-1}(x)))) = Z(\varrho_i\varrho_i^{-1}(x)) = Z(I_{i,j}(x)) = x$. In fact, $\varrho_i(x_W) \in \mathcal{STL}_{i,j}$, since if $\langle \varrho_i(x_W), \varrho_i(x_W)\rangle = 0$, then $\langle x,x\rangle = \langle \varrho_i^{-1}(x), \varrho_i^{-1}(x)\rangle = \langle x_W, x_W\rangle = \langle \varrho_i(x_W), \varrho_i(x_W)\rangle = 0$ by Lemma 6.2.18, so that $\varrho_i(x_W)$ is a zero-length vector only if

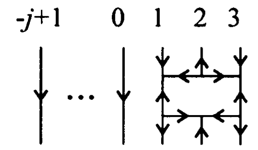


Figure 6.46: Element $f_1^{(3)}$

x is. Then for every $x \in B_{i,j}$ there exists a unique $y = \varrho_i(x_W) \in \mathcal{STL}_{i,j}$ such that $Z(y) = x$, so that Z is onto. Since, by Lemma 6.2.17 $\dim(\mathcal{STL}_{i,j}) = \dim(\mathcal{STL}_{1,i+j-1}) = \dim(B^{1,i+j-1}) = \dim(B^{i,j})$, Z is a bijection. By its definition, Z is a homomorphism since it is linear and preserves multiplication. Then $Z : \mathcal{STL}_{i,j} \to B_{i,j}$ is an isomorphism, and we have shown the following:

Lemma 6.3.8 *In the double sequence $(B_{i,j})$ defined above for $\mathcal{G} = \mathcal{A}^{(n)}$, each $B_{i,j}$ is isomorphic to $\mathcal{STL}_{i,j}$*

In particular, there is a representation of the path algebra for the 01-part $\mathcal{A}_{01}^{(n)}$ of $\mathcal{A}^{(n)}$ given by vectors of non-zero length, which linear combinations of tangles generated by Kuperberg's A_2 spiders, where $A(\mathcal{A}_{01}^{(n)})_k$ is the space of all such tangles on a rectangle with k vertices along the top and bottom, with the orientations of the vertices alternating.

Since $\mathcal{STL}_{1,2} = \mathrm{alg}(\mathbf{1}_{1,2}, W_{-1}, W_0)$, we have $\varphi(W_{-1}) = q^{8/3}W_{-1}$ and $\varphi(W_0) = q^{5/3}\mathbf{1}_{2,1} - q^{-1/3}f_1$ so that $\mathcal{STL}_{2,1} = \mathrm{alg}(\mathbf{1}_{2,1}, W_0, f_1)$. The action of ω on $\mathcal{STL}_{2,1}$ is given by $\omega(f_1) = f_1$, $\omega(W_0) = f_1^{(3)} - q\alpha^2 f_1 f_2 - q^{-1}\alpha^2 f_2 f_1$ and $\omega(f_1 - qW_0 f_1 - q^{-1} f_1 W_0 + W_0 f_1 W_0) = f_2$, where $f_1^{(3)}$ is the tangle illustrated in Figure 6.46. We see that $\mathcal{STL}_{3,0}$ is generated by $\mathbf{1}, f_1, f_2$ and $f_1^{(3)}$. This new element $f_1^{(3)}$ cannot be written as a linear combination of products of $\mathbf{1}, f_1$ and f_2. The following hold for $f_1^{(3)}$ (they can be easily checked by drawing pictures):

(i) $\left(f_1^{(3)}\right)^2 = \delta f_1^{(3)} + \alpha(f_1 + f_2) + \alpha^2(f_1 f_2 + f_2 f_1)$,

(ii) $f_1 f_1^{(3)} = \delta f_1 + \delta\alpha f_1 f_2, \qquad f_2 f_1^{(3)} = \delta f_2 + \delta\alpha f_2 f_1$,

(iii) $f_i f_1^{(3)} f_i = \delta^3 \alpha^{-1} f_i, \quad i = 1, 2$,

(iv) $f_1^{(3)} f_i f_1^{(3)} = \delta^2(f_1 + f_2) + \delta^2\alpha(f_1 f_2 + f_2 f_1), \quad i = 1, 2.$

Define $g_1^{(3)} = Z(f_1^{(3)})$. Then $A(\mathcal{A}_{01}^{(n)})_3 = \mathrm{alg}(\mathbf{1}, e_1, e_2, g_1^{(3)})$.

For $n \geq 6$, with the rows and columns indexed by the paths of length 3 on $\mathcal{A}_{01}^{(n)}$ which start at vertex $(0,0)$, $g_1^{(3)}$ can be written explicitly as the matrix

$$g_1^{(3)} = \begin{pmatrix} [2]^3/[3] & \sqrt{[2]^3[4]}/[3] & 0 & 0 \\ \sqrt{[2]^3[4]}/[3] & [4]/[3] & 0 & 0 \\ 0 & 0 & [2] & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

For $n = 5$, $g_1^{(3)} = \alpha\mathbf{1} - e_1 - e_2 + \alpha e_1 e_2 + \alpha e_2 e_1$, so is a linear combination of $\mathbf{1}, e_1$ and e_2. This is not a surprise since $\mathcal{A}_{01}^{(5)}$ is just the Dynkin diagram A_4, and we know that $A(A_4)_3$ is generated by $\mathbf{1}$, e_1 and e_2. Note also that in this case we have $\alpha = \delta = \sin(2\pi i/5)$.

It appears that $\mathcal{STL}_{i,j} = \mathrm{alg}(\mathbf{1}_{i,j}, W_l, f_l, f_m^{(p)} | k = 0, \ldots, j-1; l = 1, \ldots, i-1; p = 3, \ldots, i; m = 1, \ldots, i-p+1)$, where $f_m^{(p)}$ is the tangle illustrated in Figure 6.47 (with m odd).

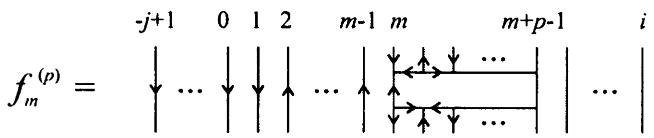


Figure 6.47: Element $f_m^{(p)}$

6.4 Planar Modules and A_2-ATL

We now return to the abstract setting, where our A_2-planar algebras are not necessarily flat, and extend Jones's notion of planar algebra modules and the annular Temperley algebra to our A_2-planar algebras (cf. Section 2 of [63]).

Definition 6.4.1 *(cf. [63, Def. 2.1]) An A_2-**annular tangle** T will be a tangle in $\mathcal{P}$ with the choice of a distinguished internal disc, which we will call the inner disc. T will be called an A_2-**annular** $(m_1, m_2 : k_1, k_2)$-**tangle** if it is an A_2-annular tangle with pattern m_1, m_2 on its outer disc and pattern k_1, k_2 on its inner disc. If $m_1 = m_2 = 0$ or $k_1 = k_2 = 0$, we replace the $0,0$ with $\overline{a}$, $a \in \{0,1,2\}$, corresponding to the colour of the region which meets the outer or inner disc respectively. When $m_1 = k_1$ and $m_2 = k_2$ we will call T an A_2-annular m_1, m_2-tangle.*

Note, this annular tangle is different to the one defined in Section 6.2.5- here more than one internal disc is allowed, but one of those is chosen to be the distinguished disc.

Definition 6.4.2 *(cf. [63, Def. 2.2]) If P is an A_2-planar algebra, a* **module** *over P, or* **P-module**, *will be a graded vector space $V = (V_{i,j}, i,j \geq 0, i,j \neq 0,0, V^{\overline{a}}_{0,0})$ with an action of P. Given an A_2-annular $(i,j : i',j')$-tangle T in $\mathcal{P}$ with distinguished ("V input") internal disc D_1 with pattern i',j' and other ("P input") internal discs D_p, $p = 2,\dots,n$, with patterns i_p, j_p, there is a linear map $Z(T) : V_{i',j'} \otimes (\otimes_{p=2}^{n} P_{i_p,j_p}) \to V_{i,j}$. $Z(T)$ satisfies the same compatability condition (6.10) for the gluing of tangles as P itself.*

An A_2-planar algebra is always a module over itself- we will call it the **trivial module**. Any relation (i.e. linear combination of labelled A_2-planar tangles) that holds in P will hold in V, e.g. K1-K3 hold in V where α, δ have the same values as in P.

A module over an A_2-planar algebra P can be understood as a module over the A_2-annular algebra A_2-AP, defined as follows. We define the associated annular category A_2-$AnnP$ to have three objects $\overline{a}$ for $i = j = 0$, $a \in \{0,1,2\}$, and one object for each $i,j \geq 0$ with i, j not both equal to zero, and whose morphisms are A_2-annular labelled tangles with labelling set all of P. Let A_2-FAP be the linearization of A_2-$AnnP$- it has the same objects, but the set of morphisms from object i,j to object i',j' is the vector space having as basis the morphisms in A_2-$AnnP$ from i,j to i',j'. Composition of morphisms is A_2-FAP is by linear extension of composition in A_2-$AnnP$. The **A_2-annular algebra** A_2-$AP = \{A_2\text{-}AP(i,j : i',j')\}$ is the quotient of A_2-FAP by all A_2-planar relations.

Definition 6.4.3 *(cf. [63, Def. 2.6]) We define A_2-$AP_{i,j}$ to be the algebra A_2-$AP(i,j : i,j)$ for $i,j \geq 0$ with i and j not both zero, and A_2-$AP_{\overline{a}}$, $a \in \{0,1,2\}$, to be the algebras spanned by A_2-annular tangles with no vertices on the outer and inner boundaries, and with the regions which meet the boundaries coloured $\overline{a}$.*

Let us apply this procedure to the $SU(3)$-Temperley-Lieb algebra STL defined in Section 6.1.3, for fixed $\delta \in \mathbb{C}$. The labels for the internal discs are now just A_2-annular tangles. For $m_1, m_2, n_1, n_2 \geq 0$ let A_2-$AnnTL(m_1,m_2 : n_1,n_2)$ be the set of all basis A_2-annular $(m_1,m_2 : n_1,n_2)$-tangles. Elements of A_2-$AnnTL(m_1,m_2 : n_1,n_2)$ define elements of A_2-$ATL(m_1,m_2 : n_1,n_2)$ by passing to the quotient of A_2-$FATL$ by relations K1-K3. The objects of A_2-ATL are $\overline{0}$, $\overline{1}$ and $\overline{2}$ for $m_1 = m_2 = 0$. When m_1 and m_2 are not both equal to zero, the objects are the sets of $2(m_1 + m_2)$ points with pattern m_1, m_2. A_2-$ATL_{m_1,m_2}(\delta)$ has as basis the set of A_2-annular m_1, m_2-tangles with no contractible circles, or embedded circles or squares. However, non-contractible circles are allowed, which make each algebra A_2-ATL_{m_1,m_2} infinite dimensional. Multiplication in A_2-$ATL_{m_1,m_2}(\delta)$ is by composition of tangles, then reducing the resulting tangle using relations K1-K3.

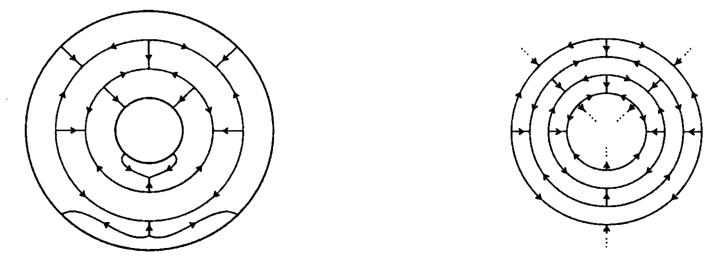


Figure 6.48: A basis A_2-annular $(2,0:2,0)$-tangle containing hexagons, and the possibility of an infinite number of hexagons

For all $m_1, m_2 \geq 0$ such that $m_1 + m_2 \geq 2$, the algebras $A_2\text{-}ATL_{m_1,m_2}$ are also infinite dimensional due to the possibility of an infinite number of embedded hexagons in basis tangles in the annular picture, as illustrated in Figure 6.48.

We have a notion of the rank of a tangle. A minimal cut loop γ in an annular $(i,j : i',j')$-tangle T will be a clockwise closed path which encloses the distinguished internal disc and crosses the least number of strings. We associate a weight $w_\gamma = (t_1, t_2)$ to a minimal cut loop γ, where t_1 is the number of strings of T that cross γ with orientation from left to right, and t_2 the number of strings that have orientation from right to left, as we move along γ in a complete clockwise loop. For a weight (t_1, t_2), let $t_{\text{max}} = \max\{t_1, t_2\}$ and $t_{\text{min}} = \min\{t_1, t_2\}$. We will say (t_1', t_2') is less than (t_1, t_2), and write $(t_1', t_2') < (t_1, t_2)$, if $t_1' + t_2' < t_1 + t_2$, and if $t_1' + t_2' = t_1 + t_2$ then $(t_1', t_2') < (t_1, t_2)$ if $2t_{max}' + t_{min}' < 2t_{max} + t_{min}$. The **rank** of T is then given by $w_\gamma = (t_1, t_2)$ such that $(t_1, t_2) \leq w_{\gamma'}$ for all minimal cut loops γ'.

Let $A_2\text{-}AnnTL(m_1, m_2 : m_1, m_2)_{(t_1,t_2)}$ denote the set of tangles in $A_2\text{-}AnnTL(m_1, m_2 : m_1, m_2)$ with rank (t_1, t_2). Since the rank cannot increase under composition of tangles, the linear span of $A_2\text{-}AnnTL(m_1, m_2 : m_1, m_2)_{(t_1,t_2)}$ for all $(t_1, t_2) < (t_1', t_2')$ for any fixed t_1', t_2' is an ideal in $A_2\text{-}ATL_{m_1,m_2}$.

Lemma 6.4.4 *(cf. [63, Lemma 2.10]) Let P be an A_2-planar algebra and let t_1, t_2 satisfy $2t_{max} + t_{min} = 3m$. For any t_1', t_2' such that $2t_{max}' + t_{min}' \leq 3m$, denote by $A_2\text{-}AP_{t_1,t_2}^{(t_1',t_2')}$ the linear span in the algebra $A_2\text{-}AP_{t_1,t_2}$ of all labelled A_2-annular t_1, t_2-tangles with rank $(s_1, s_2) < (t_1', t_2')$. Then $A_2\text{-}AP_{(t_1,t_2)}^{(t_1',t_2')}$ is a two-sided ideal.*

Remark For $A_2\text{-}ATL$ the quotient of $A_2\text{-}ATL_{t_1,t_2}$ by the ideal $A_2\text{-}AP_{t_1,t_2}^{(t_1',t_2')}$ is not in general finite dimensional, for $2t_{\text{max}}' + t_{\text{min}}' \leq 2t_{\text{max}} + t_{\text{min}}$. For example, consider the quotient of $A_2\text{-}ATL_{t_1,t_2}$ by $A_2\text{-}AP_{t_1,t_2}^{(3k,0)}$ (or $A_2\text{-}AP_{t_1,t_2}^{(0,3k)}$), for $3 \leq 3k \leq 2t_{\text{max}} + t_{\text{min}}$. The elements $\varphi_{(3k,0)}$ and $\varphi_{(0,3k)}$ (see Figure 6.49) have ranks $(3k, 0)$ and $(0, 3k)$ respectively, and can be composed an infinite number of times without being able to reduce the resulting tangle.

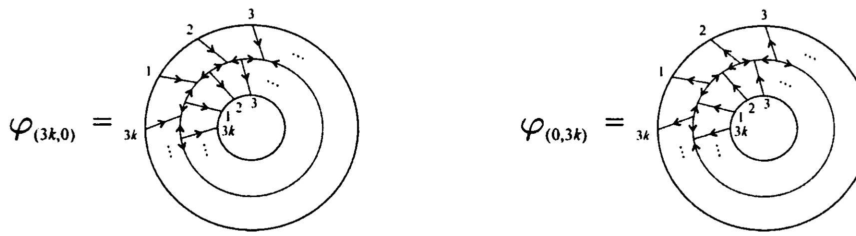


Figure 6.49: $\varphi_{(3k,0)}$ and $\varphi_{(0,3k)}$

Lemma 6.4.5 *(cf. [63, Lemma 2.11]) Let $V = (V_{i,j})$ be a P-module. Then V is indecomposable if and only if $V_{i,j}$ is an indecomposable A_2-$AP_{i,j}$-module for each $i, j \geq 0$.*

Proof

Suppose $V_{i,j} = \bigoplus_k V_{i,j}^{(k)}$ as an A_2-AP-module, for a collection of proper submodules $V_{i,j}^{(l)}$, $l = 1, \ldots, m$ for some integer m. Then applying A_2-AP to $\bigoplus_k V_{i,j}^{(k)}$ we obtain A_2-$AP(\bigoplus_k V_{i',j'}^{(k)}) = \bigoplus_k A_2$-$AP(V_{i',j'}^{(k)})$ as P-modules, so V is decomposable. Otherwise, if

$$A_2\text{-}AP(i',j' : i,j)(V_{i',j'}^{(k_1)} \oplus V_{i',j'}^{(k_2)}) \neq A_2\text{-}AP(i',j' : i,j)(V_{i',j'}^{(k_1)}) \oplus A_2\text{-}AP(i',j' : i,j)(V_{i',j'}^{(k_2)})$$

as an A_2-$AP_{i',j'}$-module for some $k_1, k_2 \in 1, \ldots, m$, $k_1 \neq k_2$, and some i', j', then for $v_l \in V_{i',j'}^{(k_l)}$, $l = 1, 2$, and some $(i', j' : i, j)$-tangle T there exists an annular i', j'-tangle S such that $STv_1 = Tv_2$. Then applying any $(i, j : i', j')$-tangle T' to Tv_1 and Tv_2 we find that $T'Tv_l \in V_{i',j'}^{(k_l)}$, $l = 1, 2$, since $T'T \in A_2$-$AP_{i,j}$ and $V_{i',j'}^{(k_l)}$ is an A_2-$AP_{i,j}$-module. Now $T'ST$ is also an element of A_2-$AP_{i,j}$, so we obtain $T'STv_1 \in V_{i',j'}^{(k_1)}$. But, $T'STv_1 = T'Tv_2 \in V_{i',j'}^{(k_2)}$, which contradicts $V_{i,j} = \bigoplus_k V_{i,j}^{(k)}$ as an A_2-AP-module. $\square$

Definition 6.4.6 *(cf. [63, Def. 2.12]) The* ***weight*** *$wt(V)$ of a P-module V is the smallest integer $i + j$ for which $V_{i,j}$ is non-zero. If $V_{\overline{a}}$ is non-zero for $a \in \{0, 1, 2\}$ we say V has weight zero. Elements of $V_{i',j'}$ for $i' + j' = wt(V)$ will be called lowest weight vectors in V, and $V_{i',j'}$ is an A_2-$AP_{i',j'}$-module which we call a* ***lowest weight module****.*

Note that for $i' + j' = wt(V)$, all $V_{i'+k,j'-k}$, $-i' \leq k \leq j'$, are lowest weight modules for V.

Definition 6.4.7 *(cf. [63, Def. 2.13]) The Hilbert series (called the dimension in [63]) of a P-module V is the formal power series*

$$\Phi_V(z_1, z_2) = \frac{1}{3} dim(V_{\overline{0}} \oplus V_{\overline{1}} \oplus V_{\overline{2}}) + \sum_{\substack{i,j=0 \\ i,j \text{ not both } =0}}^{\infty} dim(V_{i,j}) z_1^i z_2^j.$$

Again, as with $SU(2)$, the Hilbert series is additive under the direct sum of two P-modules.

6.4.1 Hilbert P-modules

If P is a C^*-A_2-planar algebra, the $*$-algebra structure on P induces a $*$-structure on A_2-AP, where the involution $*$ is defined by reflecting an A_2-annular $(m_1, m_2 : k_1, k_2)$-tangle T about a circle halfway between the inner and outer disc, and reversing the orientation. T^* will be an A_2-annular $(k_1, k_2 : m_1, m_2)$-tangle. If P is a C^*-A_2-planar algebra this defines an antilinear involution $*$ on A_2-FAP by taking the $*$ of the underlying unlabelled tangle for a labelled tangle T, replacing the labels of T by their $*$'s, and extending by antilinearity. Since P is an A_2-planar $*$-algebra, all the A_2-planar relations are preserved under $*$ on A_2-FAP, so $*$ passes to an antilinear involution on the algebra $SU(3)$-AP. In particular, all the A_2-$AP_{i,j}$ are $*$-algebras.

Definition 6.4.8 *(cf. [63, Def. 3.1]) Let P be a C^*-A_2-planar algebra. A P-module H will be called a* ***Hilbert P-module*** *if each $H_{i,j}$ is a finite dimensional Hilbert space with inner-product $\langle \cdot, \cdot \rangle$ satisfying*

$$\langle av, w \rangle = \langle v, a^* w \rangle, \tag{6.31}$$

for all $v, w, \in H$ and $a \in A_2$-AP.

As in the $SU(2)$ situation, a P-submodule of a Hilbert P-module is a Hilbert P-module. Also, the orthogonal complement of a P-submodule is a P-module, so that indecomposability and irreducibility are the same for Hilbert P-modules.

Lemma 6.4.9 *(cf. [63, Lemma 3.4]) Let P be an A_2-C^*-planar algebra and H a Hilbert P-module. If $W \subseteq H_{i,j}$ is an irreducible A_2-$AP_{i,j}$-submodule of $H_{i,j}$ for some i, j, then A_2-$AP(W)$ is an irreducible P-submodule of H.*

Proof

Suppose v, w are non-zero elements of A_2-$AP(W)_{i',j'}$ such that A_2-$AP_{i',j'}(v)$ is orthogonal to A_2-$AP_{i',j'}(w)$. Since $v, w \in A_2$-$AP(i', j' : i, j)(W)$, we have $v = av'$, $w = bw'$, for some $a, b \in A_2$-$AP(i', j' : i, j)$ and $v', w' \in W$. Then $a^*v = a^*av'$ and $b^*w = b^*bw'$ are non-zero elements of W, and A_2-$AP_{i,j}(a^*v)$ is orthogonal to A_2-$AP_{i,j}(b^*w)$, which contradicts W being an irreducible A_2-$AP_{i,j}$-module. Then by Lemma 6.4.5, A_2-$AP(W)$ is an irreducible P-submodule of H. $\square$

Lemma 6.4.10 *(cf. [63, Lemma 3.5]) Let P be an A_2-C^*-planar algebra and H a Hilbert P-module. Let V and W be orthogonal A_2-$AP_{i,j}$ invariant subspaces of $H_{i,j}$ for some i, j. Then A_2-$AP(V)$ is orthogonal to A_2-$AP(W)$.*

Proof

For any i', j', let $v \in V$, $w \in W$ and $a, b \in A_2\text{-}AP(i', j' : i, j)$. Since $a^*b \in A_2\text{-}AP_{i,j}$ and W is invariant under $A_2\text{-}AP_{i,j}$, we have $a^*bw = w' \in W$. Then $\langle av, bw \rangle = \langle v, a^*bw \rangle = \langle v, w' \rangle = 0$. □

As in the proof of Lemma 6.2.17, there is a bijection $\varrho_k : V_{i,j} \to V_{i+k,j-k}$, $-i \leq k \leq j$. Then $\dim(V_{i,j}) = \dim(V_{i+k,j-k})$, since if $v \neq 0$ in $V_{i,j}$ but $\varrho_k(v) = 0$ in $V_{i+k,j-k}$ then $v = \varrho_k^{-1}\varrho_k(v) = \varrho_k^{-1}(0) = 0$ which is a contradiction. The following Lemma (which is virtually identical to Lemma 3.7 in [63]) shows that an irreducible Hilbert P-module H is determined by its lowest weight modules, and in particular H is determined by its lowest weight module $H_{0,\mathrm{wt}(H)}$, since for all other $i + j = \mathrm{wt}(H)$, $H_{i,j} = \varrho_i(H_{0,\mathrm{wt}(H)})$.

Lemma 6.4.11 *(cf. [63, Lemma 3.7]) Let P be an A_2-C^*-planar algebra and let $H^{(1)}$, $H^{(2)}$ be Hilbert P-modules with $H^{(1)}$ irreducible. Suppose there is a non-zero A_2-$AP_{i,j}$ homomorphism $\theta : H^{(1)}_{i,j} \to H^{(2)}_{i,j}$. Then θ extends to an injective homomorphism Θ of P-modules.*

Proof

Since $H^{(1)}_{i',j'}$ is irreducible for all i', j', we can write any element $v \in H^{(1)}_{i',j'}$ as av' for some $a \in A_2\text{-}AP(i', j' : i, j)$ and $v' \in H^{(1)}_{i,j}$. We set $\Theta(v) = a\theta(v')$. Now suppose $av' = bv'$ for some $b \in A_2\text{-}AP(i', j' : i, j)$. Then for any $w' \in H^{(2)}_{i,j}$ and $c \in A_2\text{-}AP(i', j' : i, j)$, we have

$$\begin{aligned}
\langle a\theta(v'), cw' \rangle &= \langle c^*a\theta(v'), w' \rangle = \langle \theta(c^*av'), w' \rangle = \langle \theta(c^*bv'), w' \rangle \\
&= \langle c^*b\theta(v'), w' \rangle = \langle b\theta(v'), cw' \rangle,
\end{aligned}$$

since $c^*a \in A_2\text{-}AP_{i,j}$ and θ is an $A_2\text{-}AP_{i,j}$ homomorphism. Then $\Theta(av') = \Theta(bv')$ so that Θ is well defined. Now suppose $\Theta(v) = \Theta(w)$ for $v, w \in H^{(1)}_{i',j'}$ for some i', j'. Let $v = av'$, $w = bw'$ for some $a, b \in A_2\text{-}AP(i', j' : i, j)$ and $v', w' \in H^{(1)}_{i,j}$. Since $\Theta(v - w) = 0$, for all $c \in A_2\text{-}AP(i, j : i', j')$, we have $c\Theta(v - w) = ca\theta(v') - cb\theta(w') = \theta(cav' - cbw') = 0$. Now $cav' - cbw' \in H^{(1)}_{i,j}$ and θ is a non-zero homomorphism, so we have $cav' = cbw'$ for all $c \in A_2\text{-}AP(i, j : i', j')$. Hence $v = av' = bw' = w$. So Θ is injective. □

We will now determine which A_2-$AP_{i,j}$-modules can be lowest weight modules.

Lemma 6.4.12 *Let P be an A_2-C^*-planar algebra and H a Hilbert P-module of lowest weight k and rank (t_1, t_2). With $i + j = k$, any element $w \in H_{i,j}$ can be written, up to a scalar, as aw for some $a \in A_2$-$AP_{i,j}$ with* $\mathrm{rank}(a) = \mathrm{rank}(w)$.

Proof

First form $ww^* \in A_2$-$AP_{i,j}$. Then dividing out by the relations K1-K3 we obtain a linear combination of elements in A_2-$AP_{i,j}$, and we remove those elements that have rank $< (t_1, t_2)$. Ignoring the scalar factor we are left with a single element $a \in A_2$-$AP_{i,j}$ with $\mathrm{rank}(a) = (t_1, t_2)$. If we form aw, then dividing out by K1-K3 we obtain $aw = lw + \sum_i l_i w_i$, where $l, l_i \in \mathbb{C}$ and $w_i \in H_{i,j}$ with $\mathrm{rank}(w_i) < (t_1, t_2)$ for each i. Then in H the w_i are all zero, so that $l^{-1}aw = w$. □

Lemma 6.4.13 *(cf. [63, Lemma 3.8]) Let P be an A_2-C^*-planar algebra and H a Hilbert P-module. For $3(i+j-1) < 2t'_{max} + t'_{min} \leq 3(i+j)$, let $H_{i,j}^{(t_1,t_2)}$ be the A_2-$AP_{i,j}$-submodule of $H_{i,j}$ spanned by the i,j-graded pieces of all P-submodules with rank $< (t_1, t_2)$. Then*

$$(H_{i,j}^{(t_1,t_2)})^\perp = \bigcap_{a \in A_2\text{-}AP_{i,j}^{(t_1,t_2)}} \ker(a).$$

Proof

(i) We will first show $(H_{i,j}^{(t_1,t_2)})^\perp \supset \bigcap \ker(a)$. Choose any element $w \in H_{i,j}^{(t_1,t_2)}$. Then by Lemma 6.4.12 and the definition of $H_{i,j}^{(t_1,t_2)}$, w is a linear combination of elements of the form aw', where $a \in A_2$-$AP(i, j : i', j')$ with $i' + j' \leq i + j$, with $\mathrm{rank}(a) = (t'_1, t'_2) < (t_1, t_2)$, and w' is a lowest weight vector. Then for $v \in H_{i,j}$, $\langle aw', v \rangle = \langle w', a^* v \rangle$, and a^{ast} can be written up to some scalar as $t^* t a^*$ for some $(i, j : i', j')$-tangle t. Then ta^* has rank at most (t'_1, t'_2), so $ta^* \in A_2$-$AP_{i,j}^{(t_1,t_2)}$. So if $v \in \ker(ta^*)$ then $t^* t a^* v = 0$, and $a^* v = 0$ since a^* is just $t^* t a^*$ up to some scalar. Then $\langle w, v \rangle = 0$ since $\langle w', a^* v \rangle = \langle w', 0 \rangle$, and w is orthogonal to $\bigcap_{a \in A_2\text{-}AP_{i,j}^{(t_1,t_2)}} \ker(a)$.

(ii) For the opposite inclusion, $(H_{i,j}^{(t_1,t_2)})^\perp \subset \bigcap \ker(a)$, suppose $v \perp H_{i,j}^{(t_1,t_2)}$ and $a \in A_2$-$AP_{i,j}^{(t_1,t_2)}$. Then a is a linear combination of elements $a_i \in H_{i,j}$, with $\mathrm{rank}(a_i) = (t_1^{(i)}, t_2^{(i)}) < (t_1, t_2)$ for each i. For any i and any $w \in H_{i,j}^{(t_1,t_2)}$ we have $\langle a_i v, w \rangle = \langle v, a_i^* w \rangle$. Now $a_i^* w$ has rank at most $(t_1^{(i)}, t_2^{(i)}) < (t_1, t_2)$, so $a_i^* w \in H_{i,j}^{(t_1,t_2)}$. Since v is orthogonal to $H_{i,j}^{(t_1,t_2)}$, $\langle a_i v, w \rangle = \langle v, a_i^* w \rangle = 0$. Then $\langle av, w \rangle = 0$ for any $w \in H_{i,j}^{(t_1,t_2)}$, and since $av \in H_{i,j}^{(t_1,t_2)}$, $\langle av, av \rangle = 0$. This gives $av = 0$, so $v \in \ker(a)$. □

Corollary 6.4.14 *(cf. [63, Cor. 3.10]) The lowest weight modules of an irreducible P-module of rank (t_1, t_2) are A_2-$AP_{i,j}/A_2$-$AP_{i,j}^{(t_1,t_2)}$-modules, where $2t_{max} + t_{min} = 3(i + j)$.*

Then for an A_2-C^*-planar algebra P, we can determine all Hilbert P-modules by first determining the algebras $A_2\text{-}AP_{0,j}/A_2\text{-}AP_{0,j}^{(t_1,t_2)}$ and their irreducible modules, for $2t_{\text{max}} + t_{\text{min}} = 3j$, and then determining which of these modules extend to P-modules.

6.4.2 Irreducible A_2-STL-modules

We can easily determine certain irreducible A_2-STL-modules. We will describe some zero-weight modules. However we have not determined all irreducible A_2-STL-modules, even for the zero-weight case, since it is not clear that elements of the from $\sigma^{(i)}_{(k_1,k_2)}$ defined below must necessarily a contribute scalar factor, as A_2-$ATL_{\overline{a}}$ is not one-dimensional (and hence not isomorphic to $\mathbb{C}$).

Proposition 6.4.15 *(cf. [63, Prop. 5.9]) The algebra A_2-$ATL_{\overline{a}}$, $a \in \{0,1,2\}$, is generated by the 0-tangles $\sigma_{j,j\pm1}$ illustrated in Figure 6.50, $j \in \{0,1,2\}$.*

Proof

Given any $(\overline{a},\overline{a})$-tangle, removing all contractible circles, and embedded circles and squares using K1-K3 we obtain a tangle consisting only of non-contractible circles about the inner disc such that the regions that meet the inner and outer boundaries are coloured $\overline{a}$. Clearly, such a tangle must be a product of elements $\sigma_{j,j\pm1}$, $j \in \{0,1,2\}$. $\square$

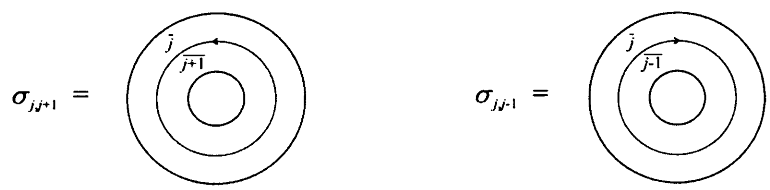


Figure 6.50: $\sigma_{j,j+1}$ and $\sigma_{j,j-1}$

Let H be an irreducible Hilbert A_2-STL-module of lowest weight zero. For $i \in \{0,1,2\}$ and integers $k_1 \equiv k_2 \bmod 3$, let

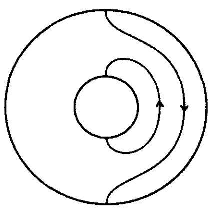
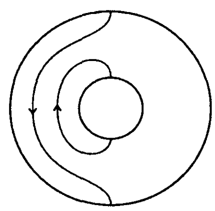
$$\sigma^{(i)}_{(k_1,k_2)} = (\sigma_{i,i+1}\sigma_{i+1,i+2}\cdots\sigma_{i+k_1-1,i+k_1})(\sigma_{i+k_1,i+k_1-1}\sigma_{i+k_1-1,i+k_1-2}\cdots\sigma_{i+k_1-k_2+1,i+k_1-k_2}).$$

If the maps $\sigma^{(i)}_{(k_1,k_2)}$ just give the complex number $\beta^{k_1}\overline{\beta}^{k_2}$ for some fixed $\beta \in \mathbb{C}$, i.e. $\sigma^{(i)}_{(k_1,k_2)} = \beta^{k_1}\overline{\beta}^{k_2}\mathbf{1}_{\overline{i}}$, then the dimensions of $H_{\overline{a}}$ are at most 5, for $a \in \{0,1,2\}$. To see this consider an arbitrary element given by a product of elements $\sigma_{j,j\pm1}$. Whenever the product $\sigma_{l,l+1}\sigma_{l+1,l}$ appears, for some $l \in \{0,1,2\}$, we get a factor of $|\beta|^2$. Removing all such products we will be left with an element which contains only non-contractible circles with the same orientation. Any three consecutive such circles contribute a factor of β^3 or $\overline{\beta}^3$. Then up to some scalar, any element will have at most two non-contractible circles, with each circle having the same orientation.

Proposition 6.4.16 *(cf. [63, Theorem 5.12]) An irreducible Hilbert A_2-STL-module H of weight zero in which the maps $\sigma^{(i)}_{(k_1,k_2)}$, $i \in \{0,1,2\}$, are given by the complex number $\beta^{k_1}\overline{\beta}^{k_2}$ for some fixed $\beta \in \mathbb{C}$ is determined up to isomorphism by the dimensions of $H_{\overline{a}}$, $a \in \{0,1,2\}$, and the number β, where we require $|\beta| \leq \alpha$.*

Proof

The uniqueness of the A_2-STL-module is a consequence of Lemma 6.4.11 since at least one of $H_{\overline{0}}$, $H_{\overline{1}}$ and $H_{\overline{2}}$ is non-zero. Let E_1, E_2 be the tangles

$E_1 =$  $E_2 =$ 

so that $\alpha^{-1}E_1$, $\alpha^{-1}E_2$ are projections. Then since $E_1E_2E_1 = |\beta|^2E_1$ we have $||\alpha^{-1}E_1 \cdot \alpha^{-1}E_2 \cdot \alpha^{-1}E_1|| = |\beta|^2\alpha^{-2}||\alpha^{-1}E_1||$ so that $1 \geq |\beta|^2\alpha^{-2}$. Hence $|\beta| \leq \alpha$. $\square$

For $\beta = \alpha$ let $V^{\alpha}_{i,j} = STL_{i,j}$ (since when $\beta = \alpha$ there is no distinction between contractible and non-contractible circles). For $\alpha > 3$ (which corresponds to $\delta > 2$), the inner product is positive definite by Lemma 6.1.12 and Theorem 1.2.1, and $H^{\alpha}_{i,j} = V^{\alpha}_{i,j}$ is a Hilbert A_2-STL-module. For $0 < \alpha \leq 3$, if the inner product is positive semi-definite on $V^{\alpha}_{i,j}$ we let $H^{\alpha}_{i,j}$ be the quotient of $V^{\alpha}_{i,j}$ by the subspace of vectors of length zero; otherwise $H^{\alpha}_{i,j}$ does not exist.

Now consider the case when $0 < |\beta| < \alpha$. We define for each $i,j \geq 0$ (with $0,0$ replaced by $\overline{a}$, $a \in \{0,1,2\}$, as usual), the set $Th_{i,j}$ to be the set of all $(i,j:\overline{0})$-tangles with no contractible circles and at most two non-contractible circles. Now for each β we form the graded vector space V^{β}, where $V^{\beta}_{i,j}$ has basis $Th_{i,j}$, and we equip it with an A_2-STL-module structure as follows. Let $T \in A_2$-$ATL(i',j':i,j)$ and $R \in A_2$-$ATL_{i,j}$. We from the tangle TR and reduce it using K1-K3, so that $TR = \sum_j \delta^{b_j}\alpha^{c_j}TR_j$, for some basis A_2-annular $(i',j':\overline{0})$-tangles TR_j. Let $\natural^a_j$, $\natural^c_j$ denote the number of non-contractible circles in the tangle TR_j which have anti-clockwise, clockwise orientation respectively. We define integers d_j, f_j and g_j as follows: $d_j = \min(\natural^a_j, \natural^c_j)$, $f_j = \natural^a_j - \natural^c_j - \gamma_{f_j}$ if $\natural^a_j \geq \natural^c_j$ and $f_j = 0$ otherwise, and $g_j = \natural^c_j - \natural^a_j - \gamma_{g_j}$ if $\natural^a_j \leq \natural^c_j$ and $g_j = 0$ otherwise, where $\gamma_{f_j}, \gamma_{g_j} \in \{0,1,2\}$ such that $f_j, g_j \equiv 0 \bmod 3$. Then we set $T(R) = \sum_j \delta^{b_j}\alpha^{c_j}\beta^{d_j+f_j}\overline{\beta}^{d_j+g_j}\widehat{TR_j}$, where $\widehat{TR_j}$

is the tangle TR_j with $d_j + f_j$ anti-clockwise non-contractible circles removed, and $d_j + g_j$ clockwise ones removed.

Proposition 6.4.17 *The above definition make V^β into an A_2-STL-module of weight zero in which $\sigma^{(a)}_{(k_1,k_2)} = \beta^{k_1}\overline{\beta}^{k_2}$ for $a = 0, 1, 2$.*

As with the $SU(2)$ situation, the choice of $(i, j : \overline{0})$-tangles rather than $(i, j : \overline{1})$- or $(i, j : \overline{2})$-tangles to define V^β was arbitrary. For these other two choices, the maps $T \to \beta^{-1}T\sigma_{01}$, $T \to \overline{\beta}^{-1}T\sigma_{02}$ respectively would have defined isomorphisms from those modules to the one defined above.

Definition 6.4.18 *(cf. [63, Def. 5.17]) Given $S, T \in Th_{i,j}$, we reduce T^*S using K1-K3 so that $T^*S = \sum_j \delta^{b_j}\alpha^{c_j}(T^*S)_j$ for basis $(\overline{0} : \overline{0})$-tangles $(T^*S)_j$. Define d_j, f_j and g_j for each $(T^*S)_j$ as above. We define an inner-product by $\langle S, T\rangle = \sum_j \delta^{b_j}\alpha^{c_j}\beta^{d_j+f_j}\overline{\beta}^{d_j+g_j}$.*

Invariance of this inner-product follows from the fact that $T^*S = \langle S, T\rangle T_0$ where T_0 is the annular $(\overline{0} : \overline{0})$-tangle with no strings at all. When the above inner-product is positive semi-definite, we define the Hilbert A_2-STL-module H^β of weight zero to be the quotient of V^β by the subspace of vectors of length zero. Otherwise H^β does not exist.

Proposition 6.4.19 *For the above Hilbert A_2-STL-module H^β of weight zero, the dimension of $H^\beta_{\overline{a}}$ is either 0 or 1 for any $\beta \in \mathbb{C} \setminus \{0\}$.*

Proof

For $a = 0$ the result is trivial since $V^\beta_{\overline{0}}$ is the linear span of the empty tangle T_0 given in Definition 6.4.18. For $a = 1$, $V^\beta_{\overline{1}} = \mathrm{span}(\sigma_{10}, \sigma_{12}\sigma_{20})$. Let $w = |\beta|^2\sigma_{12}\sigma_{20} - \beta^3\sigma_{10}$. Then

$$\begin{aligned}\langle w, w\rangle &= |\beta|^4\langle\sigma_{12}\sigma_{20}, \sigma_{12}\sigma_{20}\rangle - |\beta|^2\overline{\beta}^3\langle\sigma_{12}\sigma_{20}, \sigma_{10}\rangle - \beta^3|\beta|^2\langle\sigma_{10}, \sigma_{12}\sigma_{20}\rangle + |\beta|^6\langle\sigma_{10}, \sigma_{10}\rangle \\ &= |\beta|^4(|\beta|^4) - |\beta|^2\overline{\beta}^3\beta^3 - \beta^3|\beta|^2\overline{\beta}^3 + |\beta|^6|\beta|^2 \;=\; 0.\end{aligned}$$

Then $\sigma_{10} = |\beta|^2\beta^{-3}\sigma_{12}\sigma_{20} = \overline{\beta}\beta^{-2}\sigma_{12}\sigma_{20}$ in $H^\beta_{\overline{1}}$. Similarly when $a = 2$, $\sigma_{21}\sigma 10 = \overline{\beta}^2\beta^{-1}\sigma_{20}$.

□

So we may define H^β so that it does not contain any clockwise non-contractible circles, where we replace every σ_{10} by $\overline{\beta}\beta^{-2}\sigma_{12}\sigma_{20}$ and every $\sigma_{21}\sigma_{10}$ by $\overline{\beta}^2\beta^{-1}\sigma_{20}$.

Proposition 6.4.20 *(cf. [63, Cor. 5.8]) The Hilbert A_2-STL-module H^β, $|\beta| < \alpha$, is irreducible (when it exists).*

Proof

Since $H^{\beta}_{\overline{a}}$ is at most one-dimensional it must be irreducible, for each $a \in \{0,1,2\}$. The maps $\sigma_{j,j+1}$ moves a non-zero element in $H^{\beta}_{\overline{j}}$ to an element in $H^{\beta}_{\overline{j+1}}$, and hence the lowest weight module $H^{\beta}_{0} = H^{\beta}_{\overline{0}} \oplus H^{\beta}_{\overline{1}} \oplus H^{\beta}_{\overline{2}}$ is irreducible as an A_2-ATL_0-module. Since $H^{\beta} = A_2\text{-}ATL(H^{\beta}_{0})$, the result follows from Lemma 6.4.9. □

Now we consider the case when $\beta = 0$. For each $i, j \geq 0$ (with $0,0$ replaced by $\overline{a}$, $a \in \{0,1,2\}$, as usual), the set $Th^{\overline{a}}_{i,j}$ is defined to be the set of all $(i,j : \overline{a})$-tangles with no contractible or non-contractible circles at all. The cardinality of $Th^{\overline{a}}_{\overline{b}}$ is $\delta_{a,b}$. We form the graded vector space $V^{0,\overline{a}}$, where $V^{0,\overline{a}}_{i,j}$ has basis $Th^{\overline{a}}_{i,j}$. We equip it with an A_2-STL-module structure of lowest weight zero as follows. Let $T \in A_2\text{-}ATL(i',j' : i,j)$ and $R \in Th^{\overline{a}}_{i,j}$. We form TR and reduce it using K1-K3, so that $TR = \sum_j \delta^{b_j}\alpha^{c_j}$ as in the case $0 < |\beta| < \alpha$. We define $T(R)_j$ to be zero if there are any non-contractible circles in TR_j, and TR_j otherwise. Then $T(R) = \sum_j \delta^{b_j}\alpha^{c_j}T(R)_j$.

Proposition 6.4.21 *The above definition make $V^{0,\overline{a}}$ into an A_2-STL-module of weight zero in which $\sigma_{j,j\pm1} = 0$ for $j \in \{0,1,2\}$.*

Definition 6.4.22 *(cf. [63, Def. 5.22]) Given $S, T \in Th^{\overline{a}}_{i,j}$, we reduce T^*S using K1-K3 so that $T^*S = \sum_j \delta^{b_j}\alpha^{c_j}(T^*S)_j$ for basis $(\overline{a} : \overline{a})$-tangles $(T^*S)_j$. We define $\langle S,T\rangle_j$ to be 0 if there are any non-contractible circles in $(T^*S)_j$, and 1 otherwise. Then we define an inner-product by $\langle S,T\rangle = \sum_j \delta^{b_j}\alpha^{c_j}\langle S,T\rangle_j$.*

This inner-product is invariant as in the case $0 < |\beta| < \alpha$. Again, if the inner product is positive semi-definite we define $H^{0,\overline{a}}$ to be the quotient of $V^{0,\overline{a}}$ by the subspace of vectors with length zero; otherwise $H^{0,\overline{a}}$ does not exist.

Proposition 6.4.23 *The Hilbert A_2-STL-module $H^{0,\overline{a}}$, $a \in \{0,1,2\}$, is irreducible (when it exists).*

Proof is as for H^{β}.

6.5 The A_2-planar algebra of an $\mathcal{ADE}$ graph

Let $\mathcal{G}$ be any finite $SU(3)$ $\mathcal{ADE}$ graph (not necessarily one for which there exists a flat connection) with vertex set $\mathfrak{V}^{\mathcal{G}} = \mathfrak{V}^{\mathcal{G}}_{0} \cup \mathfrak{V}^{\mathcal{G}}_{1} \cup \mathfrak{V}^{\mathcal{G}}_{2}$, where $\mathfrak{V}^{\mathcal{G}}_{a}$ is the set of a-coloured vertices of $\mathcal{G}$, $a = 0,1,2$. Let $n_a = |\mathfrak{V}^{\mathcal{G}}_{a}|$ denote the number of a-coloured vertices and $n = |\mathfrak{V}^{\mathcal{G}}| = n_0 + n_1 + n_2$ the total number of vertices of $\mathcal{G}$. Note that $n_1 = n_2$ due to *double sequence* $(C_{i,j})$ *into an* A_2*-*C^**-planar algebra* $P^{\mathcal{G}}$, *with* $\dim(P^{\mathcal{G}}_{\overline{a}}) = n_a$, $a = 0, 1, 2$, *and parameter* $[3]$.

Proof

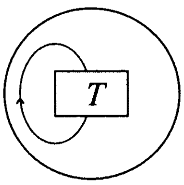
This follows as in the proof of Theorem 6.3.4, where the only small difference occurs for isotopies of the tangle which involve rectangles. However the invariance is simpler here as the connection is not used. □

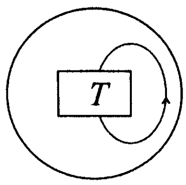
The partition functions $Z : \mathcal{P}_{\overline{a}} \longrightarrow \mathbb{C}$ are defined as the linear extensions of the function which takes the basis path v to ϕ_v^2. There is an extra multiplicative factor of ϕ_v^2 for the external region. This is required for spherical isotopy.

Proposition 6.5.2 *(cf. [62, Prop. 3.4]) The partition function of a closed labelled tangle* T *depends only on* T *up to isotopies of the 2-sphere.*

Proof

It is enough to show spherical invariance for T a 1-tangle. Let T_1 and T_2 be the 0-tangles

$T_1 =$ 

$T_2 =$ 

If $Z(T) = (\gamma, \gamma)$ for an edge γ on $\mathcal{G}$, $Z(T_1) = \phi^2_{r(\gamma)}\phi_{s(\gamma)}/\phi_{r(\gamma)} = \phi_{s(\gamma)}\phi_{r(\gamma)}$ and $Z(T_2) = \phi^2_{s(\gamma)}\phi_{r(\gamma)}/\phi_{s(\gamma)} = \phi_{s(\gamma)}\phi_{r(\gamma)} = Z(T_1)$. □

We normalize (ϕ_v) so that the partition function of an empty closed tangle is equal to one. We will say that the SU(3)-planar algebra of a graph $\mathcal{G}$ is **normalized** if

$$\sum_{v \in \mathfrak{V}_0^{\mathcal{G}}} \phi_v^2 = 1.$$

Theorem 6.5.3 *(cf. [62, Theorem 3.6]) Let* $P^{\mathcal{G}}$ *be the normalized* A_2*-planar algebra of an* $\mathcal{ADE}$ *graph* $\mathcal{G}$, *with (normalized) Perron-Frobenius eigenvector* (ϕ_v). *Then for* $x \in P^{\mathcal{G}}_{i,j}$,

$$\mathrm{tr}(x) = [3]^{-i-j} Z(\widehat{x})$$

defines a normalized trace on the union of the P*'s, where* $\widehat{x}$ *is any 0-tangle obtained from* x *by connecting the first* $i+j$ *boundary points to the last* $i+j$. *The scalar product* $\langle x, y \rangle = \mathrm{tr}(x^* y)$ *is positive definite.*

Proof

The normalization makes the definition of the trace consistent with the inclusions. The property $\mathrm{tr}(ab) = \mathrm{tr}(ba)$ is a consequence of planar isotopy when all the strings added to x to get $\widehat{x}$ go round x in the same direction, as in Figure 6.17. Spherical isotopy reduces the general case to the one above. Positive definiteness follows from the fact that the matrix units $e = (\gamma, \gamma') \in P^{\mathcal{G}}_{i,j}$ are mutually orthogonal elements of positive length: $\langle e, e \rangle = [3]^{-i-j}\phi_{v_1}\phi_{v_2} > 0$, where $e \in P^{\mathcal{G}}_{i,j}$ is a pair of paths of length $i+j$ starting at vertex v_1 and ending at vertex v_2, and $\phi_v > 0$ for all v since ϕ is a Perron-Frobenius eigenvector. $\square$

6.5.1 $P^{\mathcal{G}}$ as a TL-module for ADE Dynkin diagrams $\mathcal{G}$

In the case of $SU(2)$, Jones [63] determined all Hilbert modules $H^{k,\omega}$ of lowest weight $k > 0$ and H^{μ} of lowest weight 0. We will give a brief overview of these modules. For $1 \leq k \leq m$, $m \in \mathbb{N}$, let $\mathcal{T}^k_m$ be the set of all annular (m,k)-tangles (having $2m$ vertices on the outer disc and $2k$ vertices on the (distinguished) inner disc, where the vertices have alternating orientations) with no internal discs and $2k$ through strings. If $\widetilde{ATL}_{m,k}$ denotes the quotient of $ATL_{m,k}$ by the ideal generated by all annular (m,k)-tangles with no internal discs and strictly less than $2k$ through strings, then the equivalence classes of the elements of $\mathcal{T}^k_m$ form a basis for $\widetilde{ATL}_{m,k}$. The group $\mathbb{Z}_k$ acts by an internal rotation, which permutes the basis elements. The action of ATL on $\widetilde{ATL}_{m,k}$ is given as follows. Let T be an annular (p,m)-tangle in $ATL_{p,m}$ and $R \in \mathcal{T}^k_m$. Define $T(R)$ to be $\delta^r \widehat{TR}$ if the (p,k)-tangle TR has $2k$ through strings and 0 otherwise, where TR contains r contractible circles and $\widehat{TR}$ is the tangle TR with all the contractible circles removed. Since the action of ATL commutes with the action of $\mathbb{Z}_k$, as a TL-module $\widetilde{ATL}_{m,k}$ splits as a direct sum, over the k^{th} roots of unity ω, of TL-modules $V^{k,\omega}_m$ which are the eigenspaces for the action of $\mathbb{Z}_k$ with eigenvalue ω. For each k one can choose a faithful trace tr on the abelian C^*-algebra $\widetilde{ATL}_{k,k}$, which extends to $ATL_{k,k}$ by composition with the quotient map. The inner-product on $\widetilde{ATL}_{m,k}$ is then defined to be $\langle S, T \rangle = \mathrm{tr}(T^*S)$ for $S, T \in \widetilde{ATL}_{m,k}$.

We now turn to the zero-weight case ($k = 0$). The algebras $ATL_{\pm}$, which have the regions adjacent to both inner and outer boundaries shaded $\pm$, are generated by elements $\sigma_{\pm}\sigma_{\mp}$, where $\sigma_{\pm}$ is the $(\pm, \mp)$-tangle which is just a single non-contractible circle, with the region which meets the outer boundary shaded $\pm$ and the region which meets the inner boundary shaded $\mp$. Then the dimensions on V_+ and V_- must be 1 or 0 for any TL-module V. Then in V, the maps $\sigma_{\pm}\sigma_{\mp}$ must contribute a scalar factor μ^2, where $0 \leq \mu \leq \delta$. If $\mu = \delta$, V^{δ} is simply the ordinary Temperley-Lieb algebra described in Section 1.2.1. When

$0 < \mu < \delta$, V^μ is the TL-module such that V^μ_m $m \geq 0$, has as basis the set of $(m,+)$-tangles with no internal discs and at most one non-contractible circle. The action of ATL on V^μ, $0 \leq \mu \leq \delta$, is given as follows. Let T be an annular (p,m)-tangle in $ATL_{p,m}$ and R be a basis element of V^μ. Define $T(R)$ to be $\delta^r \mu^{2d} \widehat{TR}$, where TR contains r contractible circles and $2d+i$ non-contractible circles, where $i \in \{0,1\}$, and $\widehat{TR}$ is the tangle TR with all the contractible circles removed and $2d$ of the non-contractible circles removed. The inner product on V^μ is defined by $\langle S,T \rangle = \delta^r \mu^{2d}$, where T^*S contains r contractible circles and $2d$ non-contractible circles. When $\mu = 0$, we have TL-modules $V^{0,+}$ and $V^{0,-}$, where $V^{0,\pm}_m$ has as basis the set of $(m,\pm)$-tangles with no internal discs and no contractible circles. The action of ATL on $V^{0,\pm}$ is given as follows. Let T be an annular (p,m)-tangle in $ATL_{p,m}$ and R be a basis element of $V^{0,\pm}$. Define $T(R)$ to be $\delta^r \widehat{TR}$, where TR contains r contractible circles. Now $\widehat{TR}$ is zero if TR contains any non-contractible circles, and is the tangle TR with all the contractible circles removed otherwise. The inner product on $V^{0,\pm}$ is defined by $\langle S,T \rangle = 0$ if T^*S contains any non-contractible circles, and $\langle S,T \rangle = \delta^r$ otherwise, where r is the number of contractible circles in T^*S.

In the generic case, $\delta > 2$, it was shown that the inner-product is always positive definite, so that $H = V$ is a Hilbert TL-module, for the irreducible lowest weight k TL-module V. In the non-generic case, if the inner product is positive semi-definite, H is defined to be the quotient of V by the vectors of zero-length with respect to the inner product.

Let $\mathcal{G}$ be a bipartite graph. Then the vertex set of $\mathcal{G}$ is given by $\mathfrak{V} = \mathfrak{V}_+ \cup \mathfrak{V}_-$, where there are no connecting a vertex in $\mathfrak{V}_+$ to another, and similarly for $\mathfrak{V}_-$. We call the vertices in $\mathfrak{V}_+$, $\mathfrak{V}_-$ the even, odd respectively vertices of $\mathcal{G}$, and the distinguished vertex $*$ of $\mathcal{G}$, which has the highest Perron-Frobenius weight, is an even vertex. The adjacency matrix of $\mathcal{G}$ can thus be written in the form $\left(\begin{array}{cc} 0 & \Lambda_{\mathcal{G}} \\ \Lambda_{\mathcal{G}}^T & 0 \end{array} \right)$. We let $r_\pm = |\mathfrak{V}_\pm|$. The planar algebra $P^{\mathcal{G}}$ of a bipartite graph $\mathcal{G}$ was constructed in [62], which is the path algebra on $\mathcal{G}$ where paths may start at any of the even vertices of $\mathcal{G}$, and where the m^{th} graded part $P^{\mathcal{G}}_m$ is given by all pairs of paths of length m on $\mathcal{G}$ which start at the same even vertex and have the same end vertex. Let μ_j, $j = 1, \ldots, r_+$, denote the eigenvalues of $\Lambda_{\mathcal{G}} \Lambda_{\mathcal{G}}^T$. Then the following result is given in [105, Prop. 13], which motivated Proposition 6.5.4: The irreducible weight-zero submodules of $P^{\mathcal{G}}$ are H^{μ_j}, $j = 1, \ldots, r_-$, and $r_+ - r_-$ copies of H^0, and these can be assumed to be mutually orthogonal.

Reznikoff [105] computed the decomposition of $P^{\mathcal{G}}$ as a TL-module into irreducible

TL-modules for the ADE Dynkin diagrams. For the graphs A_m, $m \geq 3$,

$$P^{A_m} = \bigoplus_{j=1}^{s} H^{\mu_j}, \tag{6.32}$$

where $s = \lfloor (m+1)/2 \rfloor$ is the number of even vertices of A_m and $\mu_j = 2\cos(j\pi/(m+1))$, $j = 1, \ldots, s$. For D_m, $m \geq 3$,

$$P^{D_m} = \bigoplus_{j=1}^{t} H^{\mu_j} \oplus (s-t) H^{0,\pm} \oplus \bigoplus_{j=1}^{s-2} H^{2j,-1}, \tag{6.33}$$

where $s = \lfloor (m+2)/2 \rfloor$, $t = \lfloor (m-1)/2 \rfloor$ are the number of even, odd vertices respectively of D_m, and $\mu_j = 2\cos((2j-1)\pi/(2m-2))$, $j = 1, \ldots, t$. For the exceptional graphs the results are

$$\begin{aligned}
P^{E_6} &= H^{\mu_1} \oplus H^{\mu_4} \oplus H^{\mu_5} \oplus H^{2,-1} \oplus H^{3,\omega} \oplus H^{3,\omega^{-1}}, && (6.34)\\
P^{E_7} &= H^{0,\pm} \oplus H^{\mu_1} \oplus H^{\mu_5} \oplus H^{\mu_7} \oplus H^{2,-1} \oplus H^{3,\omega} \oplus H^{3,\omega^{-1}} \oplus H^{4,-1} \oplus H^{8,-1}, && (6.35)\\
P^{E_8} &= H^{\mu_1} \oplus H^{\mu_7} \oplus H^{\mu_{11}} \oplus H^{\mu_{13}} \oplus H^{2,-1} \oplus H^{3,\omega} \oplus H^{3,\omega^{-1}} \oplus H^{4,-1} && (6.36)\\
&\quad \oplus H^{5,\zeta} \oplus H^{5,\zeta^{-1}} \oplus H^{5,\zeta^{2}} \oplus H^{5,\zeta^{-2}}, && (6.37)
\end{aligned}$$

where $\omega = e^{2\pi i/3}$, $\zeta = e^{2\pi i/5}$, and $\mu_j = 2\cos(\pi j/h)$ where h is the Coxeter number.

6.5.2 $P^{\mathcal{G}}$ as an A_2-STL-module

We now return to the A_2-case for an $SU(3)$ graph $\mathcal{G}$. As in the proof of Lemma 6.3.1, if $\mathcal{G}$ is three-colourable let $\Lambda_{i,j}^1$, $\Lambda_{i,j}^2$ be the product of j, i matrices respectively, given by

$$\Lambda_{i,j}^1 = \Delta_{01}\Delta_{12}\Delta_{20}\Delta_{01}\cdots\Delta_{\overline{j-1},\overline{j}}, \qquad \Lambda_{i,j}^2 = \Delta_{\overline{j},\overline{j+1}}\Delta_{\overline{j},\overline{j+1}}^T\Delta_{\overline{j},\overline{j+1}}\Delta_{\overline{j},\overline{j+1}}^T\cdots\Delta',$$

where Δ' is $\Delta_{\overline{j},\overline{j+1}}$ if i is odd, $\Delta_{\overline{j},\overline{j+1}}^T$ if i is even, and $\overline{p}$ is the colour of p. If $\mathcal{G}$ is not three-colourable we let $\Lambda_{i,j}^1 = \Delta^i$ and $\Lambda_{i,j}^2 = \Delta^j$. Note that $\Lambda_{i,j}^1$ is a normal operator since $\Lambda_{i,j}^1(\Lambda_{i,j}^1)^* = \Lambda_{i,j}^1(\Lambda_{i,j}^1)^T = (\Delta_{01}\Delta_{01}^T)^j$ by the proof of Lemma 6.3.1, and similarly $(\Lambda_{i,j}^1)^*\Lambda_{i,j}^1 = (\Lambda_{i,j}^1)^T\Lambda_{i,j}^1 = (\Delta_{01}\Delta_{01}^T)^j$. Similarly $\Lambda_{i,j}^2$ is a normal operator.

Let β_l^3, $l \in \mathfrak{V}_0^{\mathcal{G}}$, be the eigenvalues of $\Lambda_{i,3}^1$, and $v^{(l)}$ their corresponding eigenvectors. Then $(\Lambda_{i,3}^1)^T v^{(l)} = \overline{\beta}_l^3 v^{(l)}$ and $(\Delta_{01}\Delta_{01}^T)^3 v^{(l)} = \Lambda_{i,j}^1(\Lambda_{i,j}^1)^T v^{(l)} = |\beta_j|^6 v^{(l)}$. Then if λ_l, $l \in \mathfrak{V}_0^{\mathcal{G}}$, are the eigenvalues of $\Delta_{01}\Delta_{01}^T$, with corresponding eigenvectors $v^{(l)\prime}$, we have $(\Delta_{01}\Delta_{01}^T)^3 v^{(l)\prime} = \lambda_l^3 v^{(l)\prime}$ so that $v^{(l)\prime} = v^{(l)}$ and $\lambda_l = |\beta_l|^2$.

Let $n' = \min\{n_0, n_1\}$. The dimension of $P_{i,j}^{\mathcal{G}}$ is given by the trace of $\Lambda\Lambda^T$ where $\Lambda = (\Lambda_{i,j}^1)^i(\Lambda_{i,j}^2)^j$, which counts the number of pairs of paths on $\mathcal{G}$, $\widetilde{\mathcal{G}}$. Since $\Lambda\Lambda^T = (\Delta\Delta^T)^{i+j}$, the trace of $\Lambda\Lambda^T$ is given by the sum $\sum_l \nu_{i,j}^{(l)}$ of its eigenvalues, $l = 1, 2, \ldots, n'$. The eigenvalues $\nu_{i,j}^{(l)}$ are given by $|\beta_l|^{2(i+j)}$, where β_l^3 are the eigenvalues of $\Lambda_{i,3}^1$. The Hilbert series for $P^{\mathcal{G}}$ is then given by

$$\Phi_{P^{\mathcal{G}}}(z_1, z_2) = \frac{1}{3}(n_0 + 2n_1 - 3n') + \sum_{l=1}^{n'} \frac{1}{(1 - |\beta_l|^2 z_1)(1 - |\beta_l|^2 z_2)}.$$

Proposition 6.5.4 *(cf. [105, Prop. 13]) Let $\mathcal{G}$ be one of the finite $SU(3)$ $\mathcal{ADE}$ graphs, let ζ_l, $l = 1, 2, \ldots, n'$, be the non-zero eigenvalues of $\Lambda_{0,3}^1$, counting multiplicity, and let β_l be any cubic root of ζ_l, $l = 1, 2, \ldots, n'$. For all the three-colourable graphs except $\mathcal{E}_5^{(12)}$, we have $n_0 \geq n_1$, and all the irreducible weight-zero A_2-ATL-submodules of $P^{\mathcal{G}}$ are H^{β_l}, $l = 1, 2, \ldots, n_1$, and $(n_0 - n_1)$ copies of H^0, and these can be assumed to be mutually orthogonal. For $\mathcal{E}_5^{(12)}$ we have $n_1 > n_0$, and all the irreducible weight-zero A_2-ATL-submodules of $P^{\mathcal{E}_5^{(12)}}$ are H^{β_l}, $l = 1, 2, \ldots, n_0$, and $2(n_1 - n_0)$ copies of H^0, which can again be assumed to be mutually orthogonal. If $\mathcal{G}$ is not three-colourable, all the irreducible weight-zero A_2-ATL-submodules of $P^{\mathcal{G}}$ are H^{β_l}, $l = 1, 2, \ldots, n_0$, where n_0 is the total number of vertices of $\mathcal{G}$.*

Proof

Consider the case where $n_0 > n_1$ (the case for $\mathcal{E}_5^{(12)}$ where $n_1 > n_0$ is similar). Each β_l-eigenvector $v^{(l)} = (v_w^{(l)})$, $w \in \mathfrak{V}_0^{\mathcal{G}}$ of $\Delta_{01}\Delta_{01}^T$ spans a one-dimensional subspace of $P_{\overline{0}}^{\mathcal{G}}$ that is invariant under A_2-$ATL_{\overline{0}}$. To see this, first consider the element $\sigma_{01}\sigma_{12}\sigma_{20}$:

$$\sigma_{01}\sigma_{12}\sigma_{20}v^{(l)} = \sigma_{01}\sigma_{12}\sigma_{20} \sum_{w \in \mathfrak{V}_0^{\mathcal{G}}} v_w^{(l)} = \sum_{w',w} (\Delta_{01}\Delta_{12}\Delta_{20})_{w',w} v_w^{(l)},$$

which, by the β_l eigenequation gives

$$\sigma_{01}\sigma_{12}\sigma_{20}v^{(l)} = \sum_{w'} \beta_l^3 v_{w'}^{(l)} = \beta_l^3 v^{(l)}. \tag{6.38}$$

Similarly for $\sigma_{20}^*\sigma_{12}^*\sigma_{01}^*$. Next consider the general element σ given by the composition of $2k$ elements $\sigma = \sigma_{01}\sigma_{12}\sigma_{20}\sigma_{01}\cdots\sigma_{\overline{k-1},\overline{k}}\sigma_{\overline{k-1},\overline{k}}^*\cdots\sigma_{12}^*\sigma_{01}^*$:

$$\begin{aligned}
\sigma v^{(l)} &= \sum_{w',w} (\Delta_{01}\Delta_{12}\cdots\Delta_{\overline{k-1},\overline{k}}\Delta_{\overline{k-1},\overline{k}}^T\cdots\Delta_{01}^T)_{w',w} v_w^{(l)} \\
&= \sum_{w',w} ((\Delta_{01}\Delta_{01}^T)^k)_{w',w} v_w^{(l)} = \sum_{w'} |\beta_l|^{2k} v_{w'}^{(l)} = |\beta_l|^{2k} v^{(l)}.
\end{aligned} \tag{6.39}$$

Any element of A_2-$ATL_{\overline{0}}$ is a linear combination of products of elements $\sigma_{j,j\pm 1}$ such that the regions which meet the outer and inner boundaries have colour 0. Let σ be such an element. Then the action of σ on the β_l-eigenvector $v^{(l)}$ is given by $\sigma v^{(l)} =$

$\sum_{w',w} M(w',w)v_w^{(l)}$, where M is the product of matrices Δ, Δ^T given by replacing every $\sigma_{j,j+1}$, $\sigma_{j',j'-1}$ in σ by Δ, Δ^T respectively. Then by (6.38) and (6.39), this gives some scalar multiple of $v^{(l)}$. Then each β_l-eigenvector $v^{(l)}$ generates the submodule H^{β_l} by Proposition 6.4.16. The inner product on H^{β_l} coincides with the inner product on $P^{\mathcal{G}}$. Because of (6.31) we only need to check its restriction to the zero-weight part. For any element $A \in A_2\text{-}ATL_{\bar{0}}$, $\langle Av, v\rangle_{H^{\beta_l}} = c\langle v, v\rangle_{H^{\beta_l}}$ whilst $\langle Av^{(l)}, v^{(l)}\rangle_{P^{\mathcal{G}}} = d\langle v^{(l)}, v^{(l)}\rangle_{P^{\mathcal{G}}}$. The element A is necessarily a combination of non-contractible circles, which gives the same contribution in $P^{\mathcal{G}}$ as in H^{β_l} by (6.38), (6.39). So $c = d$. This shows that the inner product on the H^{β_l} is positive definite.

Similarly, a 0-eigenvector generates the submodule H^0, where for $n_0 > n_1$, $\dim(H_{\bar{0}}^{0,\bar{0}}) = 1$ and $\dim(H_{\bar{1}}^{0,\bar{1}}) = \dim(H_{\bar{2}}^{0,\bar{2}}) = 0$, whilst for $\mathcal{E}_5^{(12)}$ we have $\dim(H_{\bar{1}}^{0,\bar{1}}) = \dim(H_{\bar{2}}^{0,\bar{2}}) = 1$ and $\dim(H_{\bar{0}}^{0,\bar{0}}) = 0$. As in the $SU(2)$ case, in order to make the resulting submodules orthogonal we take an orthogonal set of eigenvectors. □

For an $\mathcal{ADE}$ graph $\mathcal{G}$ with Coxeter number n, let $\beta_{(l_1,l_2)}$ be the eigenvalue of $\mathcal{G}$ given by (1.26) for exponent (l_1,l_2). Then for the graphs $\mathcal{A}^{(n)}$, we have for $n \not\equiv 0 \bmod 3$,

$$P^{\mathcal{A}^{(n)}} \supset \bigoplus_{(l_1,l_2)} H^{\beta_{(l_1,l_2)}}, \tag{6.40}$$

whilst for $n = 3k$, $k \geq 2$,

$$P^{\mathcal{A}^{(3k)}} \supset \bigoplus_{(l_1,l_2)} H^{\beta_{(l_1,l_2)}} \oplus H^{0,\bar{0}}, \tag{6.41}$$

where in both cases the summation is over all $(l_1,l_2) \in \{(m_1,m_2)|\ 3m_2 \leq n-3, 3m_1 + 3m_2 \leq 2n-6\}$, i.e. each $\beta_{(l_1,l_2)}$ is a cubic root of an eigenvalue of $\Lambda_{0,3}^1$. We believe that we in fact have equality here, so that $P^{\mathcal{A}^{(n)}} = \bigoplus_{(l_1,l_2)} H^{\beta_{(l_1,l_2)}}$. In the $SU(2)$ case this was achieved by a dimension count of the left and right hand sides [105, Theorem 15]. However, we have not yet been able to determine a similar result for the $SU(3)$ $\mathcal{A}$ graphs.

For the other $\mathcal{ADE}$ graphs, Proposition 6.5.4 gives the following results for the zero-weight part of $P^{\mathcal{G}}$. For the $\mathcal{D}$ graphs, we have

$$P^{\mathcal{D}^{(3k)}} \supset \bigoplus_{(l_1,l_2)} H^{\beta_{(l_1,l_2)}} \oplus 3H^{0,\bar{0}}, \tag{6.42}$$

for $k \geq 2$, where the summation is over all $(l_1,l_2) \in \{(m_1,m_2)|m_2 \leq k-1, m_1+m_2 \leq 2k-2, m_1-m_2 \equiv 0 \bmod 3\}$, whilst for $n \not\equiv 0 \bmod 3$,

$$P^{\mathcal{D}^{(n)}} \supset \bigoplus_{(l_1,l_2)} H^{\beta_{(l_1,l_2)}}, \tag{6.43}$$

where the summation is over all $(l_1,l_2) \in \{(m_1,m_2)|3m_2 \leq n-3, 3m_1+3m_2 \leq 2n-6\}$. The path algebras for $\mathcal{A}^{(n)*}$ and $\mathcal{D}^{(n)*}$ are identified under the map which send the vertices

i_l, j_l and k_l of $\mathcal{D}^{(n)*}$ with the vertex l of $\mathcal{A}^{(n)*}$, $l = 1, 2, \ldots, \lfloor l/2 \rfloor$. We have

$$P^{\mathcal{A}^{(n)*}} = P^{\mathcal{D}^{(n)*}} \supset \bigoplus_{(l_1,l_2)} H^{\beta_{(l_1,l_2)}}, \tag{6.44}$$

where the summation is over all $(l_1, l_2) \in \{(m,m) | m = 0, 1, \ldots, \lfloor (n-3)/2 \rfloor\}$. Similarly, the path algebras for $\mathcal{E}^{(8)}$ and $\mathcal{E}^{(8)*}$ are identified, and

$$P^{\mathcal{E}^{(8)}} = P^{\mathcal{E}^{(8)*}} \supset H^{\beta_{(0,0)}} \oplus H^{\beta_{(3,0)}} \oplus H^{\beta_{(0,3)}} \oplus H^{\beta_{(2,2)}}. \tag{6.45}$$

For the graphs $\mathcal{E}_i^{(12)}$, $i = 1, 2, 3$, we have

$$P^{\mathcal{E}_i^{(12)}} \supset H^{\beta_{(0,0)}} \oplus 2H^{\beta_{(2,2)}} \oplus H^{\beta_{(4,4)}}. \tag{6.46}$$

For the remaining exceptional graphs we have

$$P^{\mathcal{E}_4^{(12)}} \supset H^{\beta_{(0,0)}} \oplus H^{\beta_{(2,2)}} \oplus H^{\beta_{(4,4)}} \oplus 2H^{0,\overline{0}}, \tag{6.47}$$

$$P^{\mathcal{E}_5^{(12)}} \supset H^{\beta_{(0,0)}} \oplus H^{\beta_{(3,0)}} \oplus H^{\beta_{(0,3)}} \oplus H^{\beta_{(2,2)}} \oplus H^{\beta_{(4,4)}} \oplus H^{0,\overline{1}} \oplus H^{0,\overline{2}}, \tag{6.48}$$

$$P^{\mathcal{E}^{(24)}} \supset H^{\beta_{(0,0)}} \oplus H^{\beta_{(6,0)}} \oplus H^{\beta_{(0,6)}} \oplus H^{\beta_{(4,4)}} \oplus H^{\beta_{(7,4)}} \oplus H^{\beta_{(4,7)}} \oplus H^{\beta_{(6,6)}} \oplus H^{\beta_{(10,10)}}. \tag{6.49}$$

The A_2-planar algebra P of Section 6.3 for the graphs $\mathcal{A}^{(n)}$ clearly have decomposition $P = H^{\alpha}$ as an A_2-ATL-module, since $P \cong STL$ which is equal to the A_2-ATL-module H^{α} (see Section 6.4.2). Since every A_2-planar algebra contains STL, the A_2-planar algebra for all the $\mathcal{ADE}$ graphs with a flat connection will contain the zero-weight module H^{α}.

6.5.3 Irreducible modules with non-zero weight

We will now present some conjectured irreducible A_2-ATL-modules with non-zero weight. It is not known whether the inner-products we define on these modules are positive definite. Our construction of these modules is also based on the following assumption. Let $\varphi_{(t_1,t_2)}$, $\widetilde{\varphi}_{(t_1,t_2)}$ be the tangles illustrated in Figure 6.51. Note that $\widetilde{\varphi}_{(t_2,t_1)}$ is the rotation of $\varphi_{(t_1,t_2)}$ by π. These tangles can be viewed as some sort of "rotation by one". They have rank (t_1, t_2). It appears that the infinite dimensional algebra $\widehat{A}_k = A_2\text{-}ATL_{0,k}/A_2\text{-}ATL_{0,k}^{(k,k)}$ is generated by the two tangles $\varphi_{(k,k)}$ and $\widetilde{\varphi}_{(k,k)}$, $k \geq 1$. From now on will assume that this is true.

Let $\rho_{0,k}$ be the $0,k$-tangle given by the image of $\varphi_{(k,k)}\widetilde{\varphi}_{(k,k)}$ in $\widehat{A}_k$, illustrated in Figure 6.52, and let $\rho_{i,k-i}$ be the image of $\rho_{0,k}$ under the map $\varrho_k : A_2\text{-}ATL_{0,k} \to A_2\text{-}ATL_{i,k-i}$ as in Section 6.4.1. Then $\rho_{i,j}$ is some sort of "rotation by two". Indeed, it can be shown

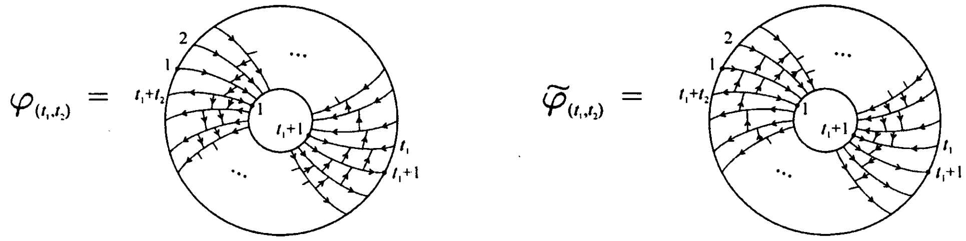


Figure 6.51: $\varphi_{(t_1,t_2)}$ and $\widetilde{\varphi}_{(t_1,t_2)}$

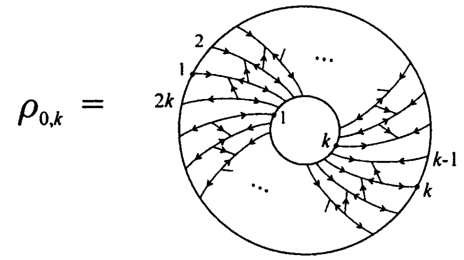


Figure 6.52: "Rotation" tangle $\rho_{0,k}$

for $2t_{\max} + t_{\min} = 3(i+j)$, for any $i,j \geq 0$, that $\rho_{i,j}$ is a rotation of order $i+j$ in $A_2\text{-}ATL_{i,j}/A_2\text{-}ATL_{i,j}^{(t_1,t_2)}$, i.e. $(\rho_{i,j})^{i+j} = \mathbf{1}_{i,j}$.

By drawing pictures it is easy to see that $\widetilde{\varphi}_{(k,k)}\varphi_{(k,k)} = \rho_{0,k}$ in $\widehat{A}_k$, and hence that $\varphi_{(k,k)}$, $\widetilde{\varphi}_{(k,k)}$ commute in $\widehat{A}_k$. It is also easy to check that

$$\varphi_{(k,k)}\varphi^*_{(k,k)} = \varphi^*_{(k,k)}\varphi_{(k,k)} = \widetilde{\varphi}_{(k,k)}\widetilde{\varphi}^*_{(k,k)} = \widetilde{\varphi}^*_{(k,k)}\widetilde{\varphi}_{(k,k)} = \mathbf{1}_{0,k},$$

so that $\varphi^*_{(k,k)} = \varphi^{-1}_{(k,k)}$ and $\widetilde{\varphi}^*_{(k,k)} = \widetilde{\varphi}^{-1}_{(k,k)}$ are inverse elements in $\widehat{A}_k$. Again, by drawing pictures it is clear that $(\varphi_{(k,k)}\widetilde{\varphi}^*_{(k,k)})^k = \varphi^{2k}_{(k,k)}$. Then we have

$$\varphi^k_{(k,k)} = (\varphi^*_{(k,k)}\varphi_{(k,k)})^k\varphi^k_{(k,k)} = (\varphi^*_{(k,k)})^k\varphi^{2k}_{(k,k)} = (\varphi^*_{(k,k)})^k(\varphi_{(k,k)}\widetilde{\varphi}^*_{(k,k)})^k = (\widetilde{\varphi}^*_{(k,k)})^k, \quad (6.50)$$

and so $\widetilde{\varphi}^*_{(k,k)} = \varphi^k_{(k,k)}\widetilde{\varphi}^{k-1}_{(k,k)}$ and $\varphi^*_{(k,k)} = \varphi^{k-1}_{(k,k)}\widetilde{\varphi}^k_{(k,k)}$.

The algebras $\widehat{A}_k$ are infinite dimensional, since $\varphi^l_{(k,k)}$, $l = 1,2,\dots$, are all distinct and non-zero in $\widehat{A}_k$, as are $\widetilde{\varphi}^l_{(k,k)}$, $l = 1,2,\dots$. One way to obtain a finite-dimensional $A_2\text{-}ATL$-module $V^{(k,k),\gamma}$ is to consider the element $\varphi_{(k,k)}\widetilde{\varphi}^*_{(k,k)}$ as acting as a scalar γ^2 in the lowest weight module $V^{(k,k),\gamma}_{0,k}$, i.e. $\varphi_{(k,k)}\widetilde{\varphi}^*_{(k,k)} = \gamma^2\mathbf{1}_{0,k}$ in $V^{(k,k),\gamma}_{0,k}$, for some $\gamma \in \mathbb{C}$. By drawing the element $\varphi^{2k}_{(k,k)}$ we see that $\varphi^{2k}_{(k,k)} = \gamma^{2k}\mathbf{1}_{0,k}$. Then we have $\varphi^*_{(k,k)} = \gamma^{-2k}\varphi^{2k-1}_{(k,k)}$, and by (6.50),

$$\widetilde{\varphi}^k_{(k,k)} = (\varphi^*_{(k,k)})^k = (\gamma^{-2k}\varphi^{2k-1}_{(k,k)})^k = \gamma^{-2k^2}\varphi^{2k^2-k}_{(k,k)} = \gamma^{-2k^2}\gamma^{2k(k-1)}\varphi^k_{(k,k)} = \gamma^{-2k}\varphi^k_{(k,k)},$$

so that $\widetilde{\varphi}_{(k,k)}^{2k} = \gamma^{-4k}\varphi_{(k,k)}^{2k} = \gamma^{-2k}\mathbf{1}_{0,k}$. Now $\varphi_{(k,k)}^{k+1}\widetilde{\varphi}_{(k,k)}^{k-1} = \varphi_{(k,k)}\widetilde{\varphi}_{(k,k)}^{*} = \gamma^2\mathbf{1}_{0,k}$, so that

$$\widetilde{\varphi}_{(k,k)}^{k+1}\varphi_{(k,k)}^{k+1}\widetilde{\varphi}_{(k,k)}^{k-1} = \gamma^2\widetilde{\varphi}_{(k,k)}^{k+1}. \tag{6.51}$$

But we also have

$$\widetilde{\varphi}_{(k,k)}^{k+1}\varphi_{(k,k)}^{k+1}\widetilde{\varphi}_{(k,k)}^{k-1} = \varphi_{(k,k)}^{k+1}\widetilde{\varphi}_{(k,k)}^{2k} = \gamma^{-2k}\varphi_{(k,k)}^{k+1}. \tag{6.52}$$

Comparing (6.51) and (6.52) we find that $\gamma^2\widetilde{\varphi}_{(k,k)}^{k+1} = \gamma^{-2k}\varphi_{(k,k)}^{k+1}$, which gives

$$\widetilde{\varphi}_{(k,k)}^{k+1} = \gamma^{-2(k+1)}\varphi_{(k,k)}^{k+1}. \tag{6.53}$$

Then by (6.50), (6.53) we have

$$\widetilde{\varphi}_{(k,k)} = \widetilde{\varphi}_{(k,k)}\widetilde{\varphi}_{(k,k)}^{k}(\widetilde{\varphi}_{(k,k)}^{*})^{k} = \widetilde{\varphi}_{(k,k)}^{k+1}(\widetilde{\varphi}_{(k,k)}^{*})^{k} = \gamma^{-2(k+1)}\varphi_{(k,k)}^{k+1}(\gamma^{2k}(\varphi_{(k,k)}^{*})^{k}) = \gamma^{-2}\varphi_{(k,k)}.$$

Then it appears that

$$V_{0,k}^{(k,k),\gamma} = \mathrm{span}(\varphi_{(k,k)}^{l} \mid l = 0,1,\dots,2k-1),$$

where $\varphi_{(k,k)}^{2k} = \gamma^{2k}\mathbf{1}_{0,k}$. We see that $\varphi_{(k,k)}$ acts on $V_{0,k}^{(k,k),\gamma}$ as $\mathbb{Z}_{2k}$, by permuting the $2k$ basis elements $\varphi_{(k,k)}^{l}$, and so the A_2-ATL-module $V^{(k,k),\gamma}$ decomposes as a direct sum over the $2k^{\text{th}}$ roots of unity ω of A_2-ATL-modules $V^{(k,k),\gamma,\omega}$, where $V^{(k,k),\gamma,\omega}$ is the ω-eigenspace for the action of $\mathbb{Z}_{2k}$ with eigenvalue ω.

For each k, we can choose a faithful trace tr$'$ on $\widehat{A}_k$, which we extend to a trace tr on A_2-$ATL_{0,k}$ by $\mathrm{tr} = \mathrm{tr}'\circ\pi$, where π is the quotient map $\pi : A_2\text{-}ATL_{0,k} \to \widehat{A}_k$. We can define an inner product on A_2-$ATL(i,j:0,k)$ by $\langle S,T\rangle = \mathrm{tr}(T^*S)$ for any $S,T \in A_2\text{-}ATL(i,j:0,k)$. Since $\varphi_{(k,k)}^{*}\varphi_{(k,k)} = \mathbf{1}_{0,k}$, the decomposition into $V^{(k,k),\gamma,\omega}$ is orthogonal. If we let $\psi_{0,k}^{\gamma,\omega}$ be the vector in $V_{0,k}^{(k,k),\gamma,\omega}$ which is proportional to $\sum_{j=0}^{2k-1}(\omega\gamma)^{-j}\varphi_{(k,k)}^{j}$ such that $\langle\psi_{0,k}^{\gamma,\omega},\psi_{0,k}^{\gamma,\omega}\rangle = 1$, then $\varphi_{(k,k)}\psi_{0,k}^{\gamma,\omega} = \omega\gamma\psi_{0,k}^{\gamma,\omega}$. We see that $\dim(V_{0,k}^{(k,k),\gamma,\omega}) = 1$, and $V_{0,k}^{(k,k),\gamma,\omega}$ is the span of $\psi_{0,k}^{\gamma,\omega}$. We define the Hilbert A_2-ATL-module $H^{(k,k),\gamma,\omega}$ to be the quotient of $V^{(k,k),\gamma,\omega}$ by the zero-length vectors with respect to this inner product.

We can also construct a finite-dimensional A_2-ATL-module $V^{(3,0)}$ with lowest weight 2 and minimum rank $(3,0)$ as follows. Let $W_{i,j}^{(3,0)}$ be the vector space of all linear combinations of tangles with one inner disc, where the outer disc has pattern i,j, the inner disc has 3 sink vertices, with one of these vertices chosen as a distinguished vertex, and such that as we pass along the string that has this distinguished vertex as its endpoint, the region to its right must be coloured $\overline{0}$. Let $V_{i,j}^{(3,0)}$ be the quotient of $W_{i,j}^{(3,0)}$ by the ideal generated by the Kuperberg relations K1-K3. The vector space $V_{i,j}^{(3,0)}$ is infinite dimensional due to the possibility of composing the elements $\varphi_{(3,0)}$ an infinite number of times, where each $\varphi_{(3,0)}^{l}$, $l = 1,2,\dots$, is a tangle which has rank $(3,0)$ and does not

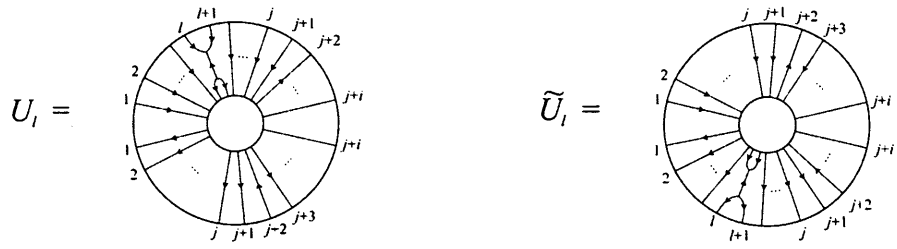
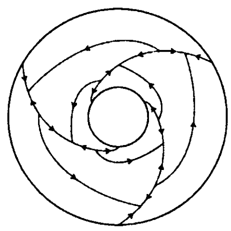


Figure 6.53: Annular i,j-tangles U_l, $\widetilde{U}_l$, $l = 1, \dots, j$

contain any closed circles, or embedded circles or squares. If however, we let $\varphi_{(3,0)}\widetilde{\varphi}^*_{(3,0)}$ count as some scalar in $V^{(3,0)}$, $\varphi_{(3,0)}\widetilde{\varphi}^*_{(3,0)} = \gamma^3 \in \mathbb{C}$, then $V^{(3,0)}_{i,j}$ is finite-dimensional since

$$\widetilde{\varphi}^*_{(3,0)} =$$



$$= \varphi^2_{(3,0)}$$

and hence $\varphi^3_{(3,0)} = \gamma^3 \in \mathbb{C}$ (and similarly $\widetilde{\varphi}^3_{(3,0)}$ is also a scalar). Since the elements $\varphi^*_{(3,0)}$ and $\varphi_{(3,0)}$ are the same, $\gamma^3 = \varphi^3_{(3,0)} = \widetilde{\varphi}^3_{(3,0)} = \overline{\gamma}^3$, so $\gamma^3 \in \mathbb{R}$. We will denote the module $V^{(3,0)}$ where $\varphi_{(3,0)}\widetilde{\varphi}^*_{(3,0)} = \gamma^3 \in \mathbb{R}$ by $V^{(3,0),\gamma}$, where $\gamma \in \mathbb{R}$.

Let $U_l, \widetilde{U}_l \in A_2\text{-}ATL_{i,j}$, $l = 1, \dots, j$, be the annular i,j-tangles illustrated in Figure 6.53. We claim that the lowest weight module $V^{(3,0),\gamma}_{0,2}$ is the span of v_l, $l = 1, \dots, 6$, where v_1 is the tangle in Figure 6.54, $v_{2l} = \varphi_{(2,2)}v_{2l-1}$, $l = 1,2,3$, and $v_{2l+1} = \widetilde{U}_1 v_{2l}$, $l = 1,2$. These are the only tangles we can find that have rank no smaller than $(3,0)$, do not contain any closed circles or embedded circles or squares, and which cannot be written as a linear combination of tangles of the form $v'\varphi^{3p}_{(3,0)}$ for some $p \in \mathbb{N}$, where v' is one of the elements v_l above, and the tangle $\varphi^{3p}_{(3,0)}$ is inserted in the inner disc of v'.

The action of $A_2\text{-}ATL_{0,2}$ on $V^{(3,0),\gamma}_{0,2}$ is given as follows. For a tangle $T \in A_2\text{-}ATL(i,j : 0,2)$ and one of the elements v_l above, we form Tv_l and divide out by the relations K1-K3 to obtain a linear combination of tangles with pattern i,j on the outer disc and three sink vertices on the inner disc. Any tangle that has rank $< (3,0)$ is equal to zero. For the remaining tangles, any tangle that is of the form $v'\varphi^{p}_{(3,0)}$ (p must necessarily by some integer multiple of 3 to respect the colouring at the inner disc) becomes $\gamma^p v' \in V^{(3,0),\gamma}_{i,j}$.

For any two elements $S, T \in V^{(3,0),\gamma}_{i,j}$, the tangle T^*S will have three (source) vertices on its outer disc and three (sink) vertices on its inner disc. We use relations K1-K3 on T^*S to obtain a linear combination $\sum_j c_j(T^*S)_j$ of tangles $(T^*S)_l$ which do not contain any closed circles, or embedded circles or squares, where $c_j \in \mathbb{C}$. We let $\langle S,T \rangle_l$ be zero if

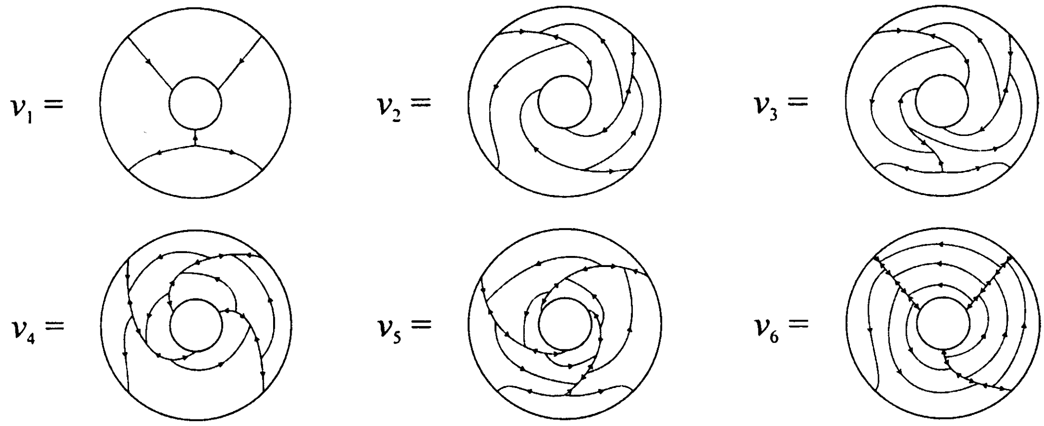


Figure 6.54: The basis elements v_i, $i = 1, \ldots, 6$, of $V_{0,2}^{(3,0),\gamma}$

rank$((T^*S)_l < (3,0)$. Otherwise, $(T^*S)_l$ will be equal to $\varphi_{(3,0)}^p$ for some $p = 0, 1, 2, \ldots$, and we let $\langle S, T\rangle_l$ be γ^p. We then define an inner product on $V^{(3,0),\gamma}$ by $\langle S, T\rangle = \sum_j c_j \langle S, T\rangle_j$. We define the Hilbert A_2-ATL-module $H^{(3,0),\gamma}$ to be the quotient of $V^{(3,0),\gamma}$ by the zero-length vectors with respect to this inner product.

For $\gamma \neq \pm 1$, $H_{0,2}^{(3,0),\gamma}$ has dimension 6, and the action of A_2-$ATL_{0,2}$ on $H_{0,2}^{(3,0),\gamma}$ is given explicitly by

$$\begin{array}{rclrcll rcl}
\varphi_{(2,2)} v_{2l-1} & = & v_{2l}, & \varphi_{(2,2)} v_{2l} & = & v_{2l-1}, & l = 1,2,3, & & & \\
\widetilde{U}_1 v_{2l-1} & = & \delta v_{2l-1}, \quad l = 1,2,3, & \widetilde{U}_1 v_{2l} & = & v_{2l+1}, & l = 1,2, & \widetilde{U}_1 v_6 & = & \gamma^3 v_1, \\
\widetilde{\varphi}_{(2,2)} v_l & = & v_l, & U_1 v_l & = & 0, & \text{for all } l. & & &
\end{array}$$

For $\gamma = \pm 1$, the dimension of $H_{0,2}^{(3,0),\pm 1}$ is 2, and $H_{0,2}^{(3,0),\pm 1}$ is the span of the elements v_1, v_2 above. The action of A_2-$ATL_{0,2}$ on $H_{0,2}^{(3,0),\pm 1}$ is given by

$$\begin{array}{rclrcll}
\varphi_{(2,2)} v_1 & = & v_2, & \varphi_{(2,2)} v_2 & = & v_1, & \\
\widetilde{U}_1 v_1 & = & \delta v_1, & \widetilde{U}_1 v_2 & = & \gamma v_1, & \\
\widetilde{\varphi}_{(2,2)} v_l & = & v_l, & U_1 v_l & = & 0, & l = 1,2.
\end{array}$$

There is a similar description of modules $H^{(0,3),\gamma}$ of minimum rank $(0,3)$, where there are now three source vertices on the inner disc. The roles of U_l and $\widetilde{U}_l$ are interchanged for $H^{(0,3),\gamma}$.

We were able to conjecture certain irreducible modules of non-zero weight that the A_2-planar algebra $P^{\mathcal{G}}$ for the graphs $\mathcal{E}^{(8)}$ and $\mathcal{D}^{(6)}$ should contain, since the action of the rotation $\rho_{0,2}$ on the A_2-planar algebras for these graphs was much easier to write down than for the other graphs.

For the graph $\mathcal{E}^{(8)}$, its zero-weight irreducible modules are $H^{\beta_{(0,0)}}$, $H^{\beta_{(3,0)}}$, $H^{\beta_{(0,3)}}$ and $H^{\beta_{(2,2)}}$. By computing the inner-products $\langle v_i, v_j \rangle$ of the elements $v_l \in H^{\beta}_{0,1}$ explicitly, and using Mathematica to compute the rank of the matrix $(\langle v_i, v_j \rangle)_{i,j}$, we computed the dimension of $H^{\beta_{(0,0)}}_{0,1}$, $H^{\beta_{(3,0)}}_{0,1}$, $H^{\beta_{(0,3)}}_{0,1}$ and $H^{\beta_{(2,2)}}_{0,1}$ and found that $P^{\mathcal{E}^{(8)}}$ did not contain any irreducible modules of lowest weight 1. It should be noted that Mathematica is not an open-source software, and the users have no way of knowing the reliability of results obtained using it. Similarly, by computing the dimensions of $W = H^{\beta_{(0,0)}}_{0,2} \oplus H^{\beta_{(3,0)}}_{0,2} \oplus H^{\beta_{(0,3)}}_{0,2} \oplus H^{\beta_{(2,2)}}_{0,2}$, we find that $\dim(W) = 30$ whilst $\dim(P^{\mathcal{E}^{(8)}}_{0,2}) = 36$, so that the dimension of $W^{\perp} \cap P^{\mathcal{E}^{(8)}}_{0,2}$ is 6. Then for modules of lowest weight 2, we conjecture

$$P^{\mathcal{E}^{(8)}}_{0,2} = H^{\beta_{(0,0)}}_{0,2} \oplus H^{\beta_{(3,0)}}_{0,2} \oplus H^{\beta_{(0,3)}}_{0,2} \oplus H^{\beta_{(2,2)}}_{0,2} \oplus H^{(3,0),\varepsilon_1}_{0,2} \oplus H^{(0,3),\varepsilon_1}_{0,2} \oplus H^{(2,2),\gamma_1,\varepsilon_2 i}_{0,2} \oplus H^{(2,2),\gamma_2,\varepsilon_3 i}_{0,2},$$

where $\varepsilon_i \in \{\pm 1\}$, $i = 1, 2, 3$, and $\gamma_1, \gamma_2 \in \mathbb{T}$, where the exact values of these six parameters has not yet been determined. This conjecture arises from computing the eigenvalues of the actions of $\rho_{0,2}$, U_1 and $\widetilde{U}_1$ on $W^{\perp} \cap P^{\mathcal{E}^{(8)}}_{0,2}$. Each action is a linear transformation, which we computed by hand, and then computed using Mathematica the eigenvalues of the matrix which gives this linear transformation. These eigenvalues are

$$\rho_{0,2}: \quad 1 \text{ twice}, \; -1 \text{ four times}, \tag{6.54}$$

$$U_1, \widetilde{U}_1: \quad [4]\alpha\delta^{-2}, \text{ once}, \; 0 \text{ five times}. \tag{6.55}$$

The eigenvalues of the actions of these elements on $H^{(2,2),\gamma,\omega}_{0,2}$, $H^{(3,0),\gamma}_{0,2}$ and $H^{(0,3),\gamma}_{0,2}$ are given in the Table 6.1.

	Eigenvalues of the action of		
A_2-ATL-module	$\rho_{0,2}$	U_1	$\widetilde{U}_1$
$H^{(2,2),\gamma,\omega}_{0,2}$	ω^2	0	0
$H^{(3,0),\pm 1}_{0,2}$	$1, \; -1$	$0 \; (\times 2)$	$[4]\alpha\delta^{-2}, \; 0$
$H^{(0,3),\pm 1}_{0,2}$	$1, \; -1$	$[4]\alpha\delta^{-2}, \; 0$	$0 \; (\times 2)$
$H^{(0,3),\pm 1}_{0,2}$, $\gamma \neq \pm 1$	$1 \; (\times 3), \; -1 \; (\times 3)$	$0 \; (\times 6)$	$[4]\alpha\delta^{-2} \; (\times 3), \; 0 \; (\times 3)$
$H^{(3,0),\pm 1}_{0,2}$, $\gamma \neq \pm 1$	$1 \; (\times 3), \; -1 \; (\times 3)$	$[4]\alpha\delta^{-2} \; (\times 3), \; 0 \; (\times 3)$	$0 \; (\times 6)$

Table 6.1: The eigenvalues of the actions of $\rho_{0,2}$, U_1, $\widetilde{U}_1$ on $H^{(2,2),\gamma,\omega}_{0,2}$, $H^{(3,0),\gamma}_{0,2}$, $H^{(0,3),\gamma}_{0,2}$.

Then we see that $W^{\perp} \cap P^{\mathcal{E}^{(8)}}_{0,2}$ should contain one copy of both of $H^{(3,0),\varepsilon_1}_{0,2}$ and $H^{(0,3),\varepsilon'_1}_{0,2}$, $\varepsilon_1, \varepsilon'_1 \in \{\pm 1\}$, and since $P^{\mathcal{E}^{(8)}}$ is invariant under conjugation of the graph $\mathcal{E}^{(8)}$, we should have $\varepsilon_1 = \varepsilon'_1$. Then we need to rank $(2,2)$ modules of $H^{(2,2),\gamma_1,\omega}_{0,2}$, $H^{(2,2),\gamma_2,\omega}_{0,2}$ such that the action of $\rho_{0,2}$ on both has an eigenvalue $\omega^2 = -1$, i.e. $\omega = \pm i$. Since $P^{\mathcal{E}^{(8)}}$ is invariant under complex conjugation, we would either have $\gamma_1, \gamma_2 \in \mathbb{R}$ or else $\gamma_1 = \overline{\gamma}_2$. However, to determine the exact values of ε_i, $i = 1,2,3$, and γ_1, γ_2, we would need to consider the action of $\varphi_{(2,2)}$ on $W^\perp \cap P_{0,2}^{\mathcal{E}^{(8)}}$, the computation of which would take many weeks to write down. So we conjecture that

$$P^{\mathcal{E}^{(8)}} \supset H^{\beta_{(0,0)}} \oplus H^{\beta_{(3,0)}} \oplus H^{\beta_{(0,3)}} \oplus H^{\beta_{(2,2)}} \oplus H^{(3,0),\varepsilon_1} \oplus H^{(0,3),\varepsilon_1} \oplus H^{(2,2),\gamma_1,\varepsilon_2 i} \oplus H^{(2,2),\gamma_2,\varepsilon_3 i}.$$

Similarly for the graph $\mathcal{D}^{(6)}$, we found that $P^{\mathcal{D}^{(6)}}$ also contains no irreducible modules of lowest weight 1. Computing the dimensions of $P_{0,2}^{\mathcal{D}^{(6)}}$ and $W = H_{0,2}^{\beta_{(0,0)}} \oplus H_{0,2}^{0,\overline{0}}$ as for the $\mathcal{E}^{(8)}$ case, we find $\dim(P_{0,2}^{\mathcal{D}^{(6)}}) = 16$ and $\dim(W) = 14$. Then the dimension of $W^\perp \cap P_{0,2}^{\mathcal{D}^{(6)}}$ is 2, and hence $P_{0,2}^{\mathcal{D}^{(6)}}$ must either contain one copy of $H_{0,2}^{(3,0),\gamma}$ or else $H_{0,2}^{(2,2),\gamma_1,\omega_1} \oplus H_{0,2}^{(2,2),\gamma_2,\omega_2}$. By considering the action of $\rho_{0,2}$ on $W^\perp \cap P_{0,2}^{\mathcal{D}^{(6)}}$, we have the eigenvalue 1 twice. Then $W = H_{0,2}^{(2,2),\gamma_1,\omega_1} \oplus H_{0,2}^{(2,2),\gamma_2,\omega_2}$, where $\omega_i^2 = 1$, $i = 1,2$. Then we conjecture that

$$P^{\mathcal{D}^{(6)}} \supset H^{\beta_{(0,0)}} \oplus H^{0,\overline{0}} \oplus H^{(2,2),\gamma_1,\varepsilon_1} \oplus H^{(2,2),\gamma_2,\varepsilon_2},$$

where $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$, and either $\gamma_1, \gamma_2 \in \mathbb{R}$ or else $\gamma_1 = \overline{\gamma}_2$. Again, to determine the values of ε_i, γ_i, $i = 1,2$, explicitly requires considering the eigenvalues of the action of $\varphi_{(2,2)}$ on $W^\perp \cap P_{0,2}^{\mathcal{D}^{(6)}}$.

Chapter 7

Spectral Measures and Generating Series for Nimrep Graphs in Subfactor Theory

The spectral measures for the ADE graphs were computed in terms of probability measures on the circle $\mathbb{T}$ in [3]. We reproduce their results via a different method, which depends on the fact that the ADE graphs are nimrep graphs. This method can then be generalized to $SU(3)$ much easier, which we do, and in particular obtain spectral measures for the infinite graphs $\mathcal{A}^{(\infty)}$ and $\mathcal{A}^{(\infty)_6}$ corresponding to the representation graphs of the fixed point algebra of $\bigotimes_{\mathbb{N}} M_3$ under the action of $SU(3)$ and $\mathbb{T}^2$ respectively. We also obtain the spectral measure for the finite graphs $\mathcal{A}^{(n)}$, $\mathcal{A}^{(n)*}$, $n \geq 4$, and $\mathcal{D}^{(3k)}$, $k \geq 2$, and the subgroups $\mathbb{Z}_n \times \mathbb{Z}_n$, $n \geq 2$, $\Delta(3n^2)$, $n \equiv 0 \bmod 3$, and $(G) = \Sigma(216 \times 3)$ of $SU(3)$.

We are also going to compute various Hilbert series of dimensions associated to ADE models. In the $SU(2)$ case this corresponds to the study of the McKay correspondence [104], Kostant polynomials of [75], the T-series of [3], and the study of pre-projective algebras [16, 83]. The corresponding $SU(3)$ theory will be more complex, related to the AdS-CFT correspondence and the Calabi-Yau algebras arise in the geometry of Calabi-Yau (CY) manifolds.

7.1.1 Spectral Measures

A non-commutative C^*-probability space (A, φ) is defined to be a unital C^*-algebra A over $\mathbb{C}$ together with a state $\varphi : A \to \mathbb{C}$ such that $\varphi(\mathbf{1}_A) = 1$, where $\mathbf{1}_A$ is the unit of A. A random variable is then an element $a \in A$.

If $b \in A$ is a normal bounded operator then there exists a compactly supported probability measure μ_b on the spectrum $\sigma(b) \subset \mathbb{C}$ of b, which is uniquely determined by its moments

$$\varphi(b^m b^{*n}) = \int_{\sigma(b)} z^m \overline{z}^n d\mu_b(z), \tag{7.1}$$

for any non-negative integers m, n.

Suppose a be self-adjoint. Then (7.1) reduces to

$$\varphi(a^m) = \int_{\sigma(a)} x^m d\mu_a(x), \tag{7.2}$$

with $\sigma(a) \subset \mathbb{R}$, for any non-negative integer m. The generating series of the moments of a is the Stieltjes transform $\sigma(z)$ of μ_a, given by

$$\sigma(z) = \sum_{m=0}^{\infty} \varphi(a^m) z^m = \sum_{m=0}^{\infty} \int_{\sigma(a)} x^m z^m d\mu_a(x) = \int_{\sigma(a)} \frac{1}{1-xz} d\mu_a(x). \tag{7.3}$$

7.1.2 General Polynomials

The classical McKay correspondence relates finite subgroups Γ of $SU(2)$ with the algebraic geometry of the quotient Kleinian singularities $\mathbb{C}^2/\Gamma$ but also with the classification of $SU(2)$ modular invariants, classification of subfactors of index less than 4, and quantum subgroups of $SU(2)$. The study of quotient singularities and their resolution has been assisted with the study of structure of certain noncommutative algebras. Minimal resolutions of Kleinian singularities can be described via the moduli space of representations of the pre-projective algebra associated to the action of Γ. This leads to general programme to understand singularities via a non commutative algebra A, often called a noncommutative resolution, whose centre corresponds to the coordinate ring of the singularity. The algebras should be finitely generated over its centre, and the desired favourable resolutions is the modular space of representations of A, whose category of finitely generated modules is derived equivalent to the category of coherent sheaves of the resolution.

In three dimensions, as part of the AdS-CFT correspondence or programme, the work of Hanany, He and coworkers has identified intriguing connections and puzzles related to the following concepts

- Toric geometry of singular CY-threefolds in a supergravity AdS model
- Brane tilings in the two dimension plane
- Quiver diagrams describing bifundamental fields in an effective gauge theory of a conformal field theory

This can probably be regarded as a higher rank analogue of the McKay correspondence in dimension two.

Some of these links are simply explained but on the whole, these relations are mainly not obvious. The AdS-CFT correspondence seeks to relate a supersymmetric gauge theory on a 10 dimensional space with a conformal field theory (CFT) described by a quiver gauge theory on a lower dimensional factor. This is encapsulated by the geometry of the toric singularity modeled on $\mathbb{C}^3/\Gamma$ where Γ is a finite (usually abelian) subgroup of $SU(3)$. A brane tiling is a tiling of the plane with a planar graph where each face has an even number of edges or the graph is bipartite. The dual graph is the (periodic) quiver describing the gauge theory.

A dimer model has allowed configurations of non overlapping edges on a graph so that each edge of the dimer picks up exactly one vertex of the underlying graph once only. Taking the determinant of the Kasteleyn matrix leads to the toric geometry described via the toric diagram of the singularity.

Evans and Gannon [36] have noted that there are similarities between this circle of ideas and those arising in the bundle structures in twisted equivariant K-theory analysis of conformal embeddings.

We take the superpotentials built on the $\mathcal{ADE}$ cells and corresponding associated algebraic structures and proceed in another direction and try to compute the Hilbert series of dimensions of the corresponding algebras.

We can go from a toric diagram to a noncommutative geometry or a noncommutative algebra given by a superpotential. The fundamental ideas can be gleaned from the following example.

Take the quiver $\mathcal{G}$ on two vertices $\{1,2\}$ with four oriented edges $\{a_1, a_2 : 0 \to 1; b_1, b_2 : 1 \to 0\}$ and quartic superpotential (of Klebanov-Witten [72]) $W = a_1 b_1 a_2 b_2 - a_1 b_2 a_2 b_1$. Thus we form the noncommutative algebra A by dividing the path algebra $\mathbb{C}\mathcal{G}$ by the relations given by the derivatives of W. Here the relations $x_1 = a_1 b_1 + b_1 a_1$, $x_2 = a_2 b_2 + b_2 a_2$, $x_3 = a_1 b_2 + b_2 a_1$, $x_4 = a_2 b_1 + b_1 a_2$ span the centre $R = Z(A) = \mathbb{C}[x_1, x_2, x_3, x_4]/(x_1 x_2 - x_3 x_4)$. This is the ring of functions on Z a conifold singularity or threefold double point. Then A can be regarded as a noncommutative resolution of the toric singularity Z the spectrum of R.

If H_n is the matrix of dimensions of paths of length n in a graph $\mathcal{G}$ in the pre-projective algebra (see Section 7.5.4), with indices labeled by the vertices, then the matrix Hilbert series H of the pre-projective algebra is defined as $H(t) = \sum H_n t^n$. Let Δ denote the adjacency matrix of $\mathcal{G}$. Then if $\mathcal{G}$ is a finite (unoriented) graph which is not an $ADET$

graph (where T denotes the tadpole graph Tad_n), then $H(t) = (1 - \Delta t + t^2)^{-1}$, whilst if $\mathcal{G}$ is an $ADET$ graph, then $H(t) = (1+Pt^h)(1-\Delta t+t^2)^{-1}$, where h is the Coxeter number of $\mathcal{G}$ and P is the permutation matrix corresponding to an involution of the vertices of $\mathcal{G}$ [83]. In the $ADET$ case, one can go further and compute the dimension of Π as $h(h+1)r/6$, where r is the number of vertices of $\mathcal{G}$.

The dual $\Pi^* = \mathrm{Hom}(\Pi, \mathbb{C})$ is a Π-Π bimodule, not usually identified with ${}_\Pi\Pi_\Pi$ or ${}_1\Pi_1$ with trivial right and left actions but with ${}_1\Pi_\nu$ with trivial left action and the right action twisted by an automorphism, the Nakayama automorphism ν. The Nakayama automorphism measures how far away Π is from being symmetric. In the case of the pre-projective algebra of a Dynkin quiver, this Nakayama automorphism is identified with an involution on the underlying Dynkin diagram. More precisely it is trivial in all cases, except for the Dynkin diagrams A_n, D_{2n+1}, E_6 where it is the unique non-trivial involution.

The examples coming from finite subgroups of $SU(3)$ give CY algebras of rank three. We are mainly interested not in the fusion graphs of $SU(3)$, whose adjacency matrices have norm 3, but in the fusion ADE graphs arising in our subfactor setting as describing the $SU(3)$ modular invariants through M-N systems which have norm less than 3.

An abelian category is a category with addition of objects and morphisms where kernels and cokernels exist and are well behaved. An abelian category $\mathcal{A}$ gives rise to a derived category $\mathcal{D}(\mathcal{A})$. We first pass to the chain complexes $\mathrm{Kom}(\mathcal{A})$. These are sequences of objects with connecting morphisms where the product of any two connecting maps are zero. The quotient of a kernel (the cycles) by the preceding image (boundaries) gives homology. The next step is to take $K(\mathcal{A})$, the homotopy category of chain complexes by identifying chain homotopic morphisms, i.e. those morphisms which yield identical results on homology. Quasi-isomorphism are the morphisms which identify the homology elements of the chain complexes. The derived category $\mathcal{D}(\mathcal{A})$, is obtained by localizing at the quasi-isomorphisms - adding to the category an inverse morphism for each quasi-isomorphism, constraining them to become isomorphisms. For the bounded derived category $\mathcal{D}^b(\mathcal{A})$, one only considers chain complexes of finite support. The derived category is a triangulated category. A triangulated category is a category $\mathcal{D}$ with a translation functor T, moving objects and morphisms, written say $X[n] = T^n X$ on objects. A triangle is $X \to Y \to Z \to Z[1]$. If $\mathcal{A}$ is an abelian category, then $K(\mathcal{A})$ is a triangulated category with objects the chain complexes, morphisms are homotopy classes of morphisms and the distinguished triangles are morphisms with their mapping cones. If $f : A \to B$ is a map of complexes, then the cone of f, $\mathrm{Cone}(f) = C(f) = C$ is the complex $A[1] \oplus B$, with triangles $A \to B \to C(f)$, where the maps $B \to C(f) \to A[1]$ are natural inclusions and projections .

A triangulated category $\mathcal{D}$ yields a cohomology and cohomological functors, say $F : \mathcal{D} \to \mathcal{A}$ into an abelian category $\mathcal{A}$, where each distinguished triangle is mapped to a long exact sequence in $\mathcal{A}$, where $\to FX[i] \to FY[i] \to FZ[i] \to FX[i+1] \to$. In particular $\mathrm{Hom}(A, -)$ is a cohomological functor, and $\mathrm{Ext}^i(A, X) = \mathrm{Hom}(A, X[i])$ are the extension groups.

A bounded derived category $\mathcal{D}^b(\mathcal{A})$ is Calabi-Yau of dimension n if there are natural isomorphisms

$$\mathrm{Hom}_{\mathcal{D}^b\mathcal{A}}(A, B) \to \mathrm{Hom}_{\mathcal{D}^b\mathcal{A}}(B, A[n])^* .$$

That is the n^{th}-shift is a Serre functor. An algebra A will be said to be Calabi-Yau of dimension n if the bounded derived category of the abelian category $\mathcal{A} = \mathrm{Rep}(A)$ of finite dimensional A-modules is a Calabi-Yau category of dimension n. In this case [14] one has the global dimension of A is n. That is for all X, Y in $\mathrm{Rep}(A)$ that $\mathrm{Ext}^i_A(X, Y) = 0$, unless $0 \leq i \leq 3$. Moreover, if X, Y are in $\mathrm{Rep}(A)$, then there are natural isomorphisms

$$\mathrm{Ext}^k_A(X, Y) \simeq \mathrm{Ext}^{n-k}_A(Y, X)^*$$

and natural pairings $\mathrm{Ext}^k_A(X, Y) \times \mathrm{Ext}^{n-k}_A(Y, X) \to \mathbb{C}$. The derived category of coherent sheaves $\mathcal{D}^b(CohX)$ over an n-dimensional Calabi-Yau manifold is Calabi-Yau category of dimension n and they appear naturally in the study of boundary conditions of the B-model in superstring theory over X. For more on Calabi-Yau algebras, see e.g. [14, 49].

Bocklandt [14] has studied the types of quivers and relations (superpotentials) that appear in graded Calabi-Yau algebras of dimension 3. Indeed he also points out that the zero-dimensional case consists of semi-simple algebras, i.e. quivers without arrows, the one dimensional case consists of direct sums of one-vertex-one-loop quivers. Moreover, a Calabi-Yau algebra of dimension 2 is the pre-projective algebra of a non-Dynkin quiver. The pre-projective algebra of a Dynkin quiver has global dimension 2.

7.2 $SU(2)$ Case

In this section we will compute the spectral measures for the ADE Dynkin diagrams and their affine counterparts. We will present a method for computing these spectral measures using the fact that the graphs are nimrep graphs. This method recovers the measures given in [3] and will allow for an easy generalization to the case of $SU(3)$ and associated nimrep graphs.

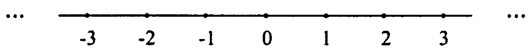


Figure 7.1: Doubly infinite graph $A_{\infty,\infty}$.

A graph is called locally finite if each vertex is the start or endpoint for a finite number of edges. Let $\mathcal{G}$ be any locally finite bipartite graph, with a distinguished vertex labelled $*$ and adjacency matrix Δ. Defining a state φ by the $*, *^{\text{th}}$ element, $\varphi(\,\cdot\,) = [\,\cdot\,]_{*,*}$, by (7.2) the spectral measure of $\mathcal{G}$ is the probability measure μ_Δ on $\mathbb{R}$ given by $\int_{\mathbb{R}} \psi(x) d\mu_\Delta(x) = [\psi(\Delta)]_{*,*}$, for any continuous function $\psi : \mathbb{R} \to \mathbb{C}$, as in [3].

7.2.1 Spectral measure for $A_{\infty,\infty}$

We begin by looking at the fixed point algebra of $\bigotimes_{\mathbb{N}} M_2$ under the action of the group $\mathbb{T}$. Let ρ be the fundamental representation of $SU(2)$, so that its restriction to $\mathbb{T}$ is given by 1.8.

Let $\{\chi_i\}_{i\in\mathbb{N}}$, $\{\sigma_i\}_{i\in\mathbb{Z}}$ be the irreducible characters of $SU(2)$, $\mathbb{T}$ respectively, where χ_0 is the trivial character of $SU(2)$, χ_1 is the character of ρ, and $\sigma_i(z) = z^i$, $i \in \mathbb{Z}$. If σ is the restriction of χ_1 to $\mathbb{T}$, we have $\sigma = \sigma_1 + \sigma_{-1}$ (by (1.8)), and $\sigma\sigma_i = \sigma_{i-1} + \sigma_{i+1}$, for any $i \in \mathbb{Z}$. Then the representation graph of $\mathbb{T}$ is identified with the doubly infinite graph $A_{\infty,\infty}$, illustrated in Figure 7.1, whose vertices are labelled by the integers $\mathbb{Z}$ which correspond to the irreducible representations of $\mathbb{T}$, where we choose the distinguished vertex to be $* = 0$. The Bratteli diagram for the path algebra of the graph $A_{\infty,\infty}$ with initial vertex $*$ is given by Pascal's triangle. The dimension of the 0^{th} level of the path algebra is 1, and we compute the dimensions of the matrix algebras corresponding to minimal central projections at the other levels according to the rule that for a vertex (v, n) at level n we take the sum of the dimensions at level $n-1$ corresponding to vertices $(v', n-1)$ for which there is an edge in the Bratteli diagram from $(v', n-1)$ to (v, n). It is well-known that these numbers give the binomial coefficients, with the j^{th} vertex along level m giving C^m_j, and we see that $\sigma^m = \sum_{j=0}^m C^m_j \sigma_{m-2j}$, where $C^m_j = \begin{pmatrix} m \\ j \end{pmatrix}$ are the binomial coefficients.

Recall that if $\{\pi_i\}$ denote irreducible representations of a group G, and if $\pi = n_1\pi_1 \oplus n_2\pi_2 \oplus \cdots$ on a full matrix algebra, then the fixed point algebra of the action $\mathrm{Ad}(\pi)$ is isomorphic to $M = M_{n_1} \oplus M_{n_2} \oplus \cdots$, and the dimension of M is given by the sum of the squares of the n_i. Then we see that $(\otimes^k M_2)^{\mathbb{T}} \cong \bigoplus_{j=0}^k M_{C^k_j}$, and $(\bigotimes_{\mathbb{N}} M_2)^{\mathbb{T}} \cong A(A_{\infty,\infty})$. Hence $\dim\left(\left(\otimes^k M_2\right)^{\mathbb{T}}\right) = \sum_{j=0}^k (C^k_j)^2$. By comparing the coefficient of x^k in the binomial expansions of $(1+x)^k(1+x)^k$ and $(1+x)^{2k}$, we have

$$\sum_{j=0}^{k} C_j^k C_{k-j}^k = C_k^{2k}, \tag{7.4}$$

and we obtain $\dim(A(A_{\infty,\infty})_k) = C_k^{2k}$. Counting the number p_j of pairs of paths in $A(A_{\infty,\infty})_k$ which end at a vertex $k-2j$ of $A_{\infty,\infty}$ is the same as the dimension of the subalgebra of $A(A_{\infty,\infty})_k$ which corresponds to the vertex $k-2j$ at level k of the Bratteli diagram for $A(A_{\infty,\infty})$, and hence p_j is given by the binomial coefficient $p_j = C_j^k$.

We define an operator w_Z on $\ell^2(\mathbb{Z})$ by $w_Z = s + s^{-1}$, where s is the bilateral shift on $\ell^2(\mathbb{Z})$. Let Ω be the vector $(\delta_{i,0})_i$. Then w_Z is identified with the adjacency matrix $\Delta_{\infty,\infty}$ of $A_{\infty,\infty}$, where we regard the vector Ω as corresponding to the vertex 0 of $A_{\infty,\infty}$, and the shifts s, s^{-1} correspond to moving along an edge to the right, left respectively on $A_{\infty,\infty}$. Then $s^k\Omega$ corresponds to the vertex k of A_∞, $k \in \mathbb{Z}$, the identity $s^{-1}s = ss^{-1} = 1$ correspond to moving along an edge of $A_{\infty,\infty}$ and then back along the reverse edge, arriving back at the original vertex we started at. Applying w_Z^n, $n \geq 0$, to Ω gives a vector $v = (v_i)_{i\in\mathbf{Z}}$ in $\ell^2(A_{\infty,\infty})$, where v_i gives the number of paths of length n from the vertex 0 to the vertex i of $A_{\infty,\infty}$.

The binomial coefficient C_n^{2n} counts the number of 'balanced' paths of length $2n$ on the integer lattice $\mathbb{Z}^2$ [29], that is, paths of length $2n$ starting from the point $(0,0)$ and ending at the point $(2n,0)$ where each edge is a vector equal to a translation of the vectors $(0,0) \to (1,1)$ or $(0,0) \to (1,-1)$.

We define a state φ on $C^*(w_Z)$ by $\varphi(\cdot) = \langle \Omega, \cdot\, \Omega\rangle$. The odd moments are all zero. For the even moments we have

$$\varphi(w_Z^{2k}) = \varphi((s+s^{-1})^{2k}) = \sum_{j=0}^{2k} C_j^{2k} \varphi(s^{2k-2j}) = \sum_{j=0}^{2k} C_j^{2k} \delta_{j,k} = C_k^{2k}.$$

Suppose the operator Δ has norm ≤ 2, so that the support of the spectral measure μ of Δ is contained in $[-2,2]$. There is a map $\Phi : \mathbb{T} \to [-2,2]$ given by

$$\Phi(u) = u + u^{-1}, \tag{7.5}$$

for $u \in \mathbb{T}$. Then any probability measure ε on $\mathbb{T}$ produces a probability measure μ on $[-2,2]$ by

$$\int_{-2}^{2} \psi(x) d\mu(x) = \int_{\mathbb{T}} \psi(u+u^{-1}) d\varepsilon(u),$$

for any continuous function $\psi : [-2,2] \to \mathbb{C}$.

The operator $\Delta_{\infty,\infty}$ has norm 2. Consider the measure $\epsilon(u)$ given by $d\epsilon(u) = du$, where du is the uniform Lebesgue measure du on $\mathbb{T}$. Now $\int_{\mathbb{T}} u^m du = \delta_{m,0}$, hence $\int_{\mathbb{T}}(u + u^{-1})^m du = 0$ for m odd, and

$$\int_{\mathbb{T}}(u+u^{-1})^{2k}du = \sum_{j=0}^{2k} C_j^{2k} \int_{\mathbb{T}} u^{2k-2j}du = C_k^{2k} = \varphi(w_Z^{2k}),$$

for $k \geq 0$ [3, Theorem 2.2]. Now, we can write

$$\int_{\mathbb{T}}(u+u^{-1})^m du = \int_0^1 (e^{2\pi i\theta} + e^{-2\pi i\theta})^m d\theta = 2\int_0^{1/2} (e^{2\pi i\theta} + e^{-2\pi i\theta})^m d\theta.$$

If we let $x = e^{2\pi i\theta} + e^{-2\pi i\theta} = 2\cos(2\pi\theta)$, then $dx/d\theta = 2\pi i(e^{2\pi i\theta} - e^{-2\pi i\theta}) = -4\pi\sin(2\pi\theta) = -2\pi\sqrt{4-x^2}$. Here the square root is always taken to be positive, since $\sin(2\pi\theta) \geq 0$ in the range $0 \leq \theta \leq 1/2$. So

$$\begin{aligned}\int_{\mathbb{T}}(u+u^{-1})^m du &= 2\int_0^{1/2} (e^{2\pi i\theta} + e^{-2\pi i\theta})^m d\theta &= -\frac{1}{\pi}\int_2^{-2} x^n \frac{1}{\sqrt{4-x^2}}\, dx \\ &= \frac{1}{\pi}\int_{-2}^{2} x^n \frac{1}{\sqrt{4-x^2}}\, dx.\end{aligned}$$

Thus the spectral measure μ_{w_Z} of w_Z (over $[-2,2]$) is $d\mu_{w_Z}(x) = (\pi\sqrt{4-x^2})^{-1}\, dx$.

Summarizing, we have the identifications

$$\dim(A(A_{\infty,\infty})_k) = \dim\left(\left(\otimes^k M_2\right)^{\mathbb{T}}\right) = C_k^{2k} = \varphi(w_Z^{2k}) = \frac{1}{\pi}\int_{-2}^{2} x^{2k}\frac{1}{\sqrt{4-x^2}}\, dx.$$

7.2.2 Spectral measure for A_∞

We now consider the fixed point algebra of $\bigotimes_{\mathbb{N}} M_2$ under the action of $SU(2)$. We have $\chi_1\chi_i = \chi_{i-1} + \chi_{i+1}$, for $i = 0, 1, 2, \ldots$, where $\chi_{-1} = 0$. Then the representation graph of $SU(2)$ is identified with the infinite Dynkin diagram A_∞ of Figure 1.1, with distinguished vertex $* = 1$. Then $(\bigotimes_{\mathbb{N}} M_2)^{SU(2)} \cong A(A_\infty)$.

We define an operator w_N on $\ell^2(\mathbb{N})$ by $w_N = l + l^*$, where l is the unilateral shift to the right on $\ell^2(\mathbb{N})$, and Ω by the vector $(\delta_{i,1})_i$. The operators l, l^* satisfy $l^*l = 1$ and $l^*\Omega = 0$. Then w_N is identified with the adjacency matrix Δ_∞ of A_∞, where we regard the vector Ω as corresponding to the vertex $* = 1$ of A_∞, the creation operator l as an edge to the right on A_∞ and the annihilation operator l^* as an edge to the left. As for the graph $A_{\infty,\infty}$, applying w_N^n, $n \geq 0$, to Ω gives a vector $v = (v_i)_{i\in\mathbb{N}}$ in $\ell^2(A_\infty)$, where v_i gives the number of paths of length n from the vertex 1 to the vertex i of A_∞. The relation $l^*\Omega = 0$ corresponds to the fact that there is no edge to the left from the vertex 1 on A_∞.

Let c_n be the n^{th} Catalan number which counts the number of Catalan (or Dyck) paths of length $2n$ in the sublattice L of $\mathbb{Z}^2$ given by all points with non-negative coordinates. A Catalan path begins at the point $(0,0)$ and must end at the point $(2n,0)$, and is constructed from edges which are translations of the vectors $(0,0) \to (1,1)$ or $(0,0) \to (1,-1)$. The Catalan numbers c_n are given explicitly by

$$c_n = \frac{1}{n+1}\begin{pmatrix} 2n \\ n \end{pmatrix}.$$

We define a state φ on $C^*(w_N)$ by $\varphi(\,\cdot\,) = \langle \Omega, \cdot\,\Omega\rangle$. Once again, the odd moments are all zero. For the even moments we have $\varphi(w_N^{2k}) = c_k$, since the sequences in l, l^* which contribute to the calculation of $\varphi(w_N^{2k})$ can be identified with the Catalan paths of length $2k$. As an example, the case for $k = 3$ is illustrated in Figure 7.2.

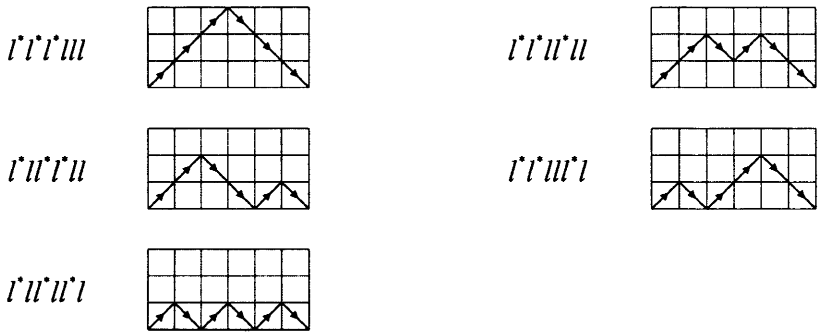


Figure 7.2: Catalan paths of length 6

By [61, Aside 5.1.1], the dimension of the k^{th} level of the path algebra for the infinite graph A_∞ is given by $\dim(A(A_\infty)_k) = c_k$. A connection with Catalan paths was also shown in [61, Aside 4.1.4], since any ordered reduced word in the Temperley-Lieb algebra $\mathrm{alg}(1, e_1, \ldots, e_{k-1})$ is of the form

$$(e_{j_1}e_{j_1-1}\cdots e_{l_1})(e_{j_2}e_{j_2-1}\cdots e_{l_2})\cdots(e_{j_p}e_{j_p-1}\cdots e_{l_p}),$$

where j_p is the maximum index, $j_i \geq l_i$, $i = 1, \ldots, p$, and $j_{i+1} > j_i$, $l_{i+1} > l_i$, $i = 1, \ldots, p-1$. In the generic case, when the Temperley-Lieb parameter $\delta \geq 2$, these words are linearly independent. Such an ordered reduced word corresponds to an increasing path on the integer lattice from $(0,0)$ to (k,k) which does not go below the diagonal. Rotating any such path on the lattice by $\pi/4$, we obtain a path of length $2k$ corresponding to a Catalan path. For example, consider the element $(e_3e_2e_1)(e_4e_3)(e_5e_4) \in \mathrm{alg}(1, e_1, \ldots, e_6)$. Then the increasing path and the corresponding Catalan path are shown in Figure 7.3. For $\delta < 2$, the ordered reduced words are linearly dependent, and we only have $\dim(A(A_\infty)_k) \leq c_k$.

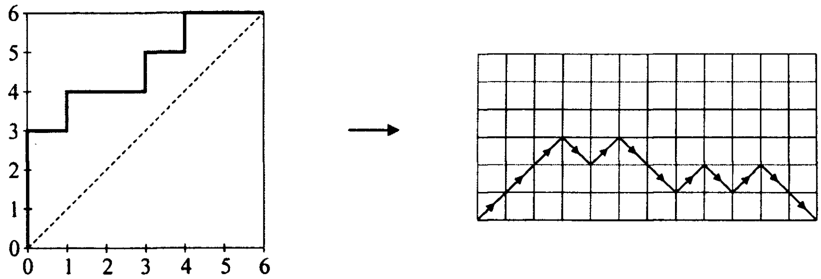


Figure 7.3: Increasing path and corresponding Catalan path for $(e_3e_2e_1)(e_4e_3)(e_5e_4)$.

A self-adjoint bounded operator a is called a semi-circular element with mean $\kappa \in \mathbb{R}$ and variance $r^2/4$ if its moments equal those of the semi-circular distribution centered at κ and of radius $r > 0$, i.e. a has the probability measure μ_a on $[\kappa - r, \kappa + r]$ given by

$$\mu_a(t) = \frac{2}{\pi r^2}\sqrt{r^2 - (x-\kappa)^2}dx. \tag{7.6}$$

When $\kappa = 0$, $r = 2$, this is equivalent to a being an even variable with even moments given by the Catalan numbers:

$$\varphi(a^m) = \begin{cases} c_k, & \text{if } m = 2k, \\ 0, & \text{if } m \text{ odd}, \end{cases}$$

Thus the operator w_N above is a semi-circular element. We will reproduce a proof that the probability measure μ_{w_N} on $[-2,2]$ is given by $d\mu_{w_N}(x) = (2\pi)^{-1}\sqrt{4-x^2}dx$ in the next section. This is the spectral measure for A_∞ given in [111].

Summarizing, we have the identifications

$$\begin{aligned} \dim(A(A_\infty)_k) &= \dim\left(\left(\otimes^k M_2\right)^{SU(2)}\right) = c_k = \frac{1}{k+1}C_k^{2k} \\ &= \varphi(w_N^{2k}) = \frac{1}{2\pi}\int_{-2}^{2} x^{2k}\sqrt{4-x^2}\,dx. \end{aligned}$$

7.3 Spectral measures for ADE Dynkin diagrams via nimreps

Let $\Delta_\mathcal{G}$ be the adjacency matrix of the finite (possibly affine) Dynkin diagram $\mathcal{G}$ with s vertices. The m^{th} moment $\int x^m d\mu(x)$ is given by $\langle \Delta_\mathcal{G}^m e_1, e_1\rangle$, where e_1 is the basis vector in $\ell^2(\mathcal{G})$ corresponding to the distinguished vertex $*$ of $\mathcal{G}$.

Let β^j be the eigenvalues of $\mathcal{G}$, with corresponding eigenvectors x^j, $j = 1, \ldots, s$. Now $\Delta_\mathcal{G} = \mathcal{U}\Lambda_\mathcal{G}\mathcal{U}^*$, where $\Lambda_\mathcal{G} = \mathrm{diag}(\beta^1, \beta^2, \ldots, \beta^s)$ and $\mathcal{U} = (x^1, x^2, \ldots, x^s)$. Then

$\Delta_{\mathcal{G}}^m = \mathcal{U}\Lambda_{\mathcal{G}}^m\mathcal{U}^*$, so that

$$\begin{aligned}\int_{\mathbb{T}} \psi(u+u^{-1})d\varepsilon(u) &= \langle \mathcal{U}\Lambda_{\mathcal{G}}^m\mathcal{U}^* e_1, e_1\rangle = \langle \Lambda_{\mathcal{G}}^m\mathcal{U}^* e_1, \mathcal{U}^* e_1\rangle \\ &= \sum_{j=1}^{s} (\beta^j)^m |y_i|^2, \end{aligned} \tag{7.7}$$

where $y_i = x_1^i$ is the first entry of the eigenvector x^i.

For a Dynkin diagram $\mathcal{G}$, its eigenvalues λ^j are given in (1.13), with corresponding eigenvectors $(\psi_a^{m_j})_{a\in\mathfrak{V}(\mathcal{G})}$, for the exponents m_j of $\mathcal{G}$, $j = 1,\dots,s$. Then (7.7) becomes

$$\int_{\mathbb{T}} \psi(u+u^{-1})d\varepsilon(u) = \sum_{j=1}^{s} (\lambda^j)^m |\psi_*^{m_j}|^2, \tag{7.8}$$

where $*$ is the distinguished vertex of $\mathcal{G}$ with lowest Perron-Frobenius weight. Using (7.8) we can obtain the results for the spectral measures of the Dynkin diagrams given in [3]. The advantage of this method is that it can be extended to the case of $SU(3)$ $\mathcal{ADE}$ graphs and subgroups of $SU(3)$, which we will do in Sections 7.7, 7.8.

7.3.1 Dynkin diagrams A_n, A_∞

The eigenvalues λ_n^j of A_n are given in (1.12), with corresponding eigenvectors $\psi_a^j = S_{a,j} = \sqrt{2/(n+1)}\ \sin(ja\pi/(n+1))$. The distinguished vertex $*$ of A_n is the vertex 1 in Figure 1.1. With $\widetilde{u} = e^{\pi i/(n+1)}$, we have $2\cos(j\pi/(n+1)) = \widetilde{u}^j + \widetilde{u}^{-j}$ and $\sin(j\pi/(n+1)) = \mathrm{Im}(\widetilde{u}^j)$. Note that $\mathrm{Im}(\widetilde{u}^j) = 0$ for $j = 0, n+1$. Then

$$\begin{aligned}\int_{\mathbb{T}} \psi(u+u^{-1})d\varepsilon(u) &= \frac{2}{n+1}\sum_{j=1}^{n}\left(2\cos\left(\frac{j\pi}{n+1}\right)\right)^m \sin^2\left(\frac{j\pi}{n+1}\right) &(7.9)\\ &= \frac{2}{n+1}\sum_{j=1}^{n}(\widetilde{u}^j+\widetilde{u}^{-j})^m\ \mathrm{Im}(\widetilde{u}^j)^2 \\ &= \frac{2}{2(n+1)}\sum_{j=0}^{2(n+1)}(\widetilde{u}^j+\widetilde{u}^{-j})^m\ \mathrm{Im}(\widetilde{u}^j)^2 \\ &= 2\int_{\mathbb{T}}(u+u^{-1})^m\ \mathrm{Im}(u)^2\, d_{n+1}u &(7.10)\end{aligned}$$

where d_{n+1} is the uniform measure on the $2(n+1)^{\text{th}}$ roots of unity. Thus the spectral measure (over $\mathbb{T}$) for A_n is $d\varepsilon(u) = 2\mathrm{Im}(u)^2\, d_{n+1}u$. This is the result given in [3, Theorem 3.1]

We again consider the infinite graph A_∞, and note that the computation of the m^{th} moment is a finite problem, $\int x^m d\mu_{w_N}(x) = \langle \Delta_{A_n}^m e_1, e_1\rangle$, for $m < 2n$. Taking the limit in

(7.9) as $n \to \infty$ (cf. the second proof of Theorem 1.1.5 in [54]), we obtain a sum which is the approximation of an integral,

$$\int x^m d\mu_{w_N}(x) = \frac{2}{\pi}\int_0^{\pi} (2\cos t)^m \sin^2 t dt = \frac{1}{2\pi}\int_{-2}^{2} x^m\sqrt{4-x^2}dx,$$

so that $d\mu_{w_N}(x) = (2\pi)^{-1}\sqrt{4-x^2}dx$, and the operator w_N is a semi-circular element.

Alternatively, if we take the limit as $n \to \infty$ in (7.10), we obtain

$$\int_{\mathbb{T}} \psi(u+u^{-1})d\varepsilon(u) = 2\int_{\mathbb{T}} (u+u^{-1})^m \operatorname{Im}(u)^2 du,$$

where du is the uniform measure over $\mathbb{T}$, as claimed in the previous section.

7.3.2 Dynkin diagrams D_n

For finite n, the distinguished vertex of the graph D_n is the vertex n in Figure 1.2. The exponents Exp of D_n are $1,3,5,\dots,2n-3,n-1$. For $n=2l$, the exponent $2l-1$ has multiplicity two, and we denote these exponents by $(2l-1,\pm)$. The eigenvectors of D_{2l} are given by [5, (B.6)] as:

$$\begin{array}{rcll}
\psi_a^j & = & \sqrt{2}S_{2l+1-a,j}, & a\neq 1,2,\ j\neq 2l-1,\\
\psi_a^{(2l-1,\pm)} & = & S_{2l+1-a,2l-1}, & a\neq 1,2,\\
\psi_1^j = \psi_2^j & = & \frac{1}{\sqrt{2}}S_{2l-1,j}, & j\neq 2l-1,\\
\psi_{1+\epsilon}^{(2l-1,\pm)} & = & \frac{1}{2}(S_{2l-1,2l-1}\pm(1-2\epsilon)\sqrt{(-1)^{l+1}}), &
\end{array}$$

where $\epsilon = 0,1$ and $j \in \mathcal{E}$. Using (7.8) and with $\widetilde{u} = e^{\pi i/(4l-2)}$,

$$\begin{aligned}
&\int_{\mathbb{T}} \psi(u+u^{-1})d\varepsilon(u)\\
&= \sum_{j\neq 2l-1} (2\cos(j\pi/(4l-2)))^m|\sqrt{2}S_{1,j}|^2 + 2(2\cos(j\pi/(4l-2)))^m|S_{1,j}|^2\\
&= \frac{4}{4l-2}\sum_{j\in\mathrm{Exp}} (2\cos(j\pi/(4l-2)))^m \sin^2(j\pi/(4l-2))\\
&= \frac{4}{4l-2}\sum_{j\in\mathrm{Exp}} (\widetilde{u}^j+\widetilde{u}^{-j})^m \operatorname{Im}(\widetilde{u}^j)^2 \;=\; \frac{2}{4l-2}\sum_{j\in\{1,3,\dots,8l-5\}} (\widetilde{u}^j+\widetilde{u}^{-j})^m \operatorname{Im}(\widetilde{u}^j)^2\\
&= 2\int_{\mathbb{T}} (u+u^{-1})^m \operatorname{Im}(u)^2 d'_{4l-2}u,
\end{aligned}$$

where d'_{4l-2} is the uniform measure on the $(8l-4)^{\text{th}}$ roots of unity of odd order.

For D_{2l+1}, the eigenvectors are given by [5, (B.8)] as:

$$\begin{array}{rcll}
\psi_a^j & = & (-1)^{\frac{j-1}{2}}\sqrt{2}S_{2l+2-a,j}, & a\neq 1,2,\ j\neq 2l,\\
\psi_a^{2l} & = & 0, \quad a\neq 1,2, &\\
\psi_1^j = \psi_2^j & = & (-1)^{\frac{j-1}{2}}\frac{1}{\sqrt{2}}S_{2l,j}, \;=\; \frac{1}{2\sqrt{l}} & j\neq 2l,\\
\psi_1^{2l} & = & \frac{1}{\sqrt{2}}, \qquad \psi_2^{2l} \;=\; -\frac{1}{\sqrt{2}}, &
\end{array}$$

where $j \in \mathrm{Exp} = \{1, 3, 5, \ldots, 4l-1, 2l\}$. Then, using (7.8) and with $\widetilde{u} = e^{\pi i/(4l)}$,

$$\begin{aligned}
&\int_{\mathbb{T}} \psi(u+u^{-1}) d\varepsilon(u) \\
&= 2\sum_{j \neq 2l} (2\cos(j\pi/4l))^m |S_{1,j}|^2 + 0 = \frac{4}{4l} \sum_{j\in\{1,3,\ldots,4l-1\}} (2\cos(j\pi/4l))^m \sin^2(j\pi/4l) \\
&= \frac{2}{4l} \sum_{j\in\{1,3,\ldots,8l-1\}} (\widetilde{u}^j + \widetilde{u}^{-j})^m \, \mathrm{Im}(\widetilde{u}^j)^2 = 2\int_{\mathbb{T}} (u+u^{-1})^m \, \mathrm{Im}(u)^2 d'_{4l}u.
\end{aligned}$$

So the spectral measure $d\varepsilon(u)$ on $\mathbb{T}$ for D_n is given by $d\varepsilon(u) = \alpha(u) d'_{2n-2}u$, where

$$\alpha(u) = 2\mathrm{Im}(u)^2, \tag{7.11}$$

which recovers the spectral measure given in [3, Theorem 3.2].

Taking the limit of the graph D_n as $n \to \infty$ with the vertex n as the distinguished vertex, we just obtain the infinite graph A_∞. In order to obtain the infinite graph D_∞ we must set the distinguished vertex $*$ of D_n to be the vertex 1 in Figure 1.2. Then using (7.8), and taking the limit as $n \to \infty$, we obtain the spectral measure for D_∞.

7.3.3 Dynkin diagram E_6

For E_6 the exponents are 1, 4, 5, 7, 8, 11. The eigenvectors for E_6 are given in [5, (B.9)]. In particular,

$$\psi_1^1 = \psi_1^{11} = \frac{1}{2}\sqrt{\frac{3-\sqrt{3}}{6}}, \qquad \psi_1^4 = \psi_1^8 = \frac{1}{2}, \qquad \psi_1^5 = \psi_1^7 = \frac{1}{2}\sqrt{\frac{3+\sqrt{3}}{6}}.$$

Then, by (7.8),

$$\int_{\mathbb{T}} \psi(u+u^{-1}) d\varepsilon(u) = \sum_{j\in\mathrm{Exp}} |\psi_1^j|^2 (2\cos(j\pi/12))^m = \frac{1}{2}\sum_{p\in B_6} |\psi_1^p|^2 (2\cos(p\pi/12))^m,$$

where $B_6 = \{1, 4, 5, 7, 8, 11, 13, 16, 17, 19, 20, 23\}$, and for $j > 12$ we define ψ_1^j by $\psi_1^j = \psi_1^{24-j}$. Then with $\widetilde{u} = e^{\pi i/12}$,

$$\int_{\mathbb{T}} \psi(u+u^{-1}) d\varepsilon(u) = \frac{1}{24}\sum_{p\in B_6} 12|\psi_1^p|^2 (\widetilde{u}^p + \widetilde{u}^{-p})^m.$$

Now for any $p \in B_6$, $\widetilde{u}^p$ is a 24^{th} root of unity, but for $p = 4, 8, 16, 20$, $\widetilde{u}^p$ is also a 6^{th} root of unity. Since $|\psi_1^p|^2$ takes different values for different p, clearly we cannot write the above summation as an integral using the uniform measure over 24^{th} roots of unity.

However, with α as in (7.11), we have

$$\begin{aligned}
\alpha(\widetilde{u}^p) &= 12|\psi_1^p|^2 - \tfrac{1}{2}, && \text{for } p = 1, 11, 13, 23,\\
\alpha(\widetilde{u}^p) &= 12|\psi_1^p|^2 - \tfrac{1}{2}, && \text{for } p = 4, 8, 16, 20,\\
\alpha(\widetilde{u}^p) &= 12|\psi_1^p|^2 - \tfrac{1}{2}, && \text{for } p = 5, 7, 17, 19.
\end{aligned}$$

By considering $a_p = \alpha(\widetilde{u}^p) + 1/2$, we can write

$$\begin{aligned}
\int_{\mathbb{T}} \psi(u+u^{-1}) d\varepsilon(u) &= \frac{1}{24} \sum_{p \in B_6} a_p (\widetilde{u}^p + \widetilde{u}^{-p})^m \\
&\quad - \frac{1}{24} \left((\widetilde{u}^4 + \widetilde{u}^{-4})^m + (\widetilde{u}^8 + \widetilde{u}^{-8})^m + (\widetilde{u}^{16} + \widetilde{u}^{-16})^m + (\widetilde{u}^{20} + \widetilde{u}^{-20})^m \right).
\end{aligned}$$

Since $\widetilde{u}^p$ is also a 6^{th} root of unity for $p = 4, 8, 16, 20$, it may be possible to obtain the last four terms by considering an integral using the uniform measure on 6^{th} roots of unity. First, we consider the integral $\int_{\mathbb{T}} (u+u^{-1})^m (2\text{Im}(u)^2 + 1/2) d_{12} u$, where d_{12} is the uniform measure on the 24^{th} roots of unity, to obtain the terms in the summation above, giving

$$\begin{aligned}
&\int_{\mathbb{T}} \psi(u+u^{-1}) d\varepsilon(u) \\
&= \int_{\mathbb{T}} (u+u^{-1})^m (2\text{Im}(u)^2 + \tfrac{1}{2}) d_{12} u \; - \frac{1}{24} \sum_q a_q (\widetilde{u}^q + \widetilde{u}^{-q})^m \\
&\quad + \frac{1}{24} \left((\widetilde{u}^4 + \widetilde{u}^{-4})^m + (\widetilde{u}^8 + \widetilde{u}^{-8})^m + (\widetilde{u}^{16} + \widetilde{u}^{-16})^m + (\widetilde{u}^{20} + \widetilde{u}^{-20})^m \right),
\end{aligned}$$

where the summation is over $q \in \{2, 3, 6, 9, 10, 12, 14, 15, 18, 21, 22, 24\}$, that is, the integers $1 \leq q \leq 24$ such that $q \notin B_6$. For these values of q, we have $a_2 = a_{10} = a_{14} = a_{22} = 1$, $a_3 = a_9 = a_{15} = a_{21} = 3/2$, $a_6 = a_{18} = 5/2$, and $a_{12} = a_{24} = 1/2$. Using these values for a_q, we now isolate the terms involving the 12^{th} roots of unity, giving

$$\begin{aligned}
&\int_{\mathbb{T}} \psi(u+u^{-1}) d\varepsilon(u) \\
&= \int_{\mathbb{T}} (u+u^{-1})^m (2\text{Im}(u)^2 + \tfrac{1}{2}) d_{12} u - \frac{1}{24} \sum_{k=1}^{12} (\widetilde{u}^{2k} + \widetilde{u}^{-2k})^m \\
&\quad - \frac{1}{16} (\widetilde{u}^3 + \widetilde{u}^{-3})^m + \frac{1}{12} (\widetilde{u}^4 + \widetilde{u}^{-4})^m - \frac{1}{16} (\widetilde{u}^6 + \widetilde{u}^{-6})^m + \frac{1}{12} (\widetilde{u}^8 + \widetilde{u}^{-8})^m \\
&\quad - \frac{1}{16} (\widetilde{u}^9 + \widetilde{u}^{-9})^m + \frac{1}{48} (\widetilde{u}^{12} + \widetilde{u}^{-12})^m - \frac{1}{16} (\widetilde{u}^{15} + \widetilde{u}^{-15})^m + \frac{1}{12} (\widetilde{u}^{16} + \widetilde{u}^{-16})^m \\
&\quad - \frac{1}{16} (\widetilde{u}^{18} + \widetilde{u}^{-18})^m + \frac{1}{12} (\widetilde{u}^{20} + \widetilde{u}^{-20})^m - \frac{1}{16} (\widetilde{u}^{21} + \widetilde{u}^{-21})^m + \frac{1}{48} (\widetilde{u}^{24} + \widetilde{u}^{-24})^m.
\end{aligned}$$

Now $\sum_{k=1}^{12} (\widetilde{u}^{2k} + \widetilde{u}^{-2k})^m / 12 = \int_{\mathbb{T}} (u+u^{-1})^m d_6$. For the remaining terms, we notice that $\sum_{k=1}^{8} (\widetilde{u}^{3k} + \widetilde{u}^{-3k})^m / 8 = \int_{\mathbb{T}} (u+u^{-1})^m d_4$, giving

$$\int_{\mathbb{T}} \psi(u+u^{-1}) d\varepsilon(u)$$

$$= \int_{\mathbb{T}} (u+u^{-1})^m (2\mathrm{Im}(u)^2 + \tfrac{1}{2}) d_{12}u - \frac{1}{2}\int_{\mathbb{T}} (u+u^{-1})^m d_6 u - \frac{1}{2}\int_{\mathbb{T}} (u+u^{-1})^m d_4 u$$
$$+\frac{1}{12}(\widetilde{u}^4+\widetilde{u}^{-4})^m + \frac{1}{12}(\widetilde{u}^8+\widetilde{u}^{-8})^m + \frac{1}{12}(\widetilde{u}^{12}+\widetilde{u}^{-12})^m$$
$$+\frac{1}{12}(\widetilde{u}^{16}+\widetilde{u}^{-16})^m + \frac{1}{12}(\widetilde{u}^{20}+\widetilde{u}^{-20})^m + \frac{1}{12}(\widetilde{u}^{24}+\widetilde{u}^{-24})^m.$$

These last six terms are given by the integral $(\int_{\mathbb{T}}(u+u^{-1})^m d_3)/2$. Then the spectral measure $d\varepsilon(u)$ (over $\mathbb{T}$) for E_6 is

$$d\varepsilon = \alpha d_{12} + \tfrac{1}{2}(d_{12} - d_6 - d_4 + d_3),$$

which recovers the spectral measure given in [3, Theorem 6.2].

7.3.4 Dynkin diagrams E_7, E_8

The following definition is given by Banica and Bisch [3, Def. 7.1]:

Definition 7.3.1 *A discrete measure supported by roots of unity is called* ***cyclotomic*** *if it is a linear combination of measures of type d_n, $n \geq 1$, and αd_n, $n \geq 2$.*

Note that since $d_n' = 2d_{2n} - d_n$, all the measures for the A and D diagrams, as well as for E_6, have been cyclotomic. However, Banica and Bisch proved that the spectral measures for E_7, E_8 are not cyclotomic. This can also be seen by our method using (7.8).

For E_7 the exponents are 1, 5, 7, 9, 11, 13, 17. The eigenvectors for E_7 are given in [5, (B.10)]. In particular,

$$\psi_1^1 = \psi_1^{17} = (18+12\sqrt{3}\cos(\pi/18))^{-1/2}, \qquad \psi_1^5 = \psi_1^{13} = (18+12\sqrt{3}\cos(13\pi/18))^{-1/2},$$
$$\psi_1^7 = \psi_1^{11} = (18+12\sqrt{3}\cos(11\pi/18))^{-1/2}, \qquad \psi_1^9 = 1/\sqrt{3}.$$

Then

$$\int_{\mathbb{T}} \psi(u+u^{-1}) d\varepsilon(u) = \sum_{j\in\mathrm{Exp}} |\psi_1^j|^2 (2\cos(j\pi/18))^m = \frac{1}{2}\sum_{p\in B_7} |\psi_1^p|^2 (2\cos(p\pi/18))^m,$$

where $B_7 = \{1,5,7,9,11,13,17,19,23,25,27,29,31,35\}$, and for $j > 18$ we define ψ_1^j by $\psi_1^j = \psi_1^{36-j}$. Then with $\widetilde{u} = e^{\pi i/18}$,

$$\int_{\mathbb{T}} \psi(u+u^{-1}) d\varepsilon(u) = \frac{1}{36}\sum_{p\in B_7} 18|\psi_1^p|^2 (\widetilde{u}^p + \widetilde{u}^{-p})^m. \tag{7.12}$$

Now for any $p \in B_7$, $\widetilde{u}^p$ is a 36^{th} root of unity, but not a root of unity of lower order, except for $p = 9, 27$, in which case $\widetilde{u}^p$ is also a 4^{th} root of unity. Since $|\psi_1^1|^2 \neq |\psi_1^5|^2$, clearly we cannot write the summation in (7.12) as an integral using the uniform measure over 36^{th} roots of unity. With α as in (7.11), we have

$$\begin{aligned}
\alpha(\widetilde{u}^p) &= 18|\psi_1^p|^2 - 0.4076, && \text{for } p = 1, 17, 19, 35, \\
\alpha(\widetilde{u}^p) &= 18|\psi_1^p|^2 - 2.7057, && \text{for } p = 5, 13, 23, 31, \\
\alpha(\widetilde{u}^p) &= 18|\psi_1^p|^2 + 0.1133, && \text{for } p = 7, 11, 25, 29, \\
\alpha(\widetilde{u}^p) &= 18|\psi_1^p|^2 - 4, && \text{for } p = 9, 27.
\end{aligned}$$

Since $\alpha(\widetilde{u}^p) - 18|\psi_1^p|^2$ also takes different values for certain $p \in B_7$, and for any $p \in B_7$, $\widetilde{u}^p$ is a 36^{th} root of unity, but not a root of unity of lower order, the summation in (7.12) cannot be written as an integral using the measure αd_{18} either. So we see that the spectral measure for E_7 is not cyclotomic.

For E_8 the exponents are 1, 7, 11, 13, 17, 19, 23, 29. The eigenvectors for E_8 are given in [5, (B.12)]. In particular,

$$\begin{aligned}
\psi_1^1 &= \psi_1^{29} = \left(2/(15(3+\sqrt{5}) + \sqrt{15(130+58\sqrt{5})})\right)^{1/2}, \\
\psi_1^7 &= \psi_1^{23} = \left(2/(15(3-\sqrt{5}) + \sqrt{15(130-58\sqrt{5})})\right)^{1/2}, \\
\psi_1^{11} &= \psi_1^{19} = \left(2/(15(3+\sqrt{5}) - \sqrt{15(130+58\sqrt{5})})\right)^{1/2}, \\
\psi_1^{13} &= \psi_1^{17} = \left(2/(15(3-\sqrt{5}) - \sqrt{15(130-58\sqrt{5})})\right)^{1/2}.
\end{aligned}$$

Then

$$\int_{\mathbb{T}} \psi(u+u^{-1}) d\varepsilon(u) = \sum_{j \in \text{Exp}} |\psi_1^j|^2 (2\cos(j\pi/30))^m = \frac{1}{60} \sum_{p \in B_8} 30 |\psi_1^p|^2 (\widetilde{u}^p + \widetilde{u}^{-p})^m, \quad (7.13)$$

where $\widetilde{u} = e^{\pi i/30}$, $B_8 = \{1, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 49, 53, 59\}$, and for $j > 30$ we define ψ_1^j by $\psi_1^j = \psi_1^{60-j}$. We have

$$\begin{aligned}
\alpha(\widetilde{u}^p) &= 30|\psi_1^p|^2 - 0.4038, && \text{for } p = 1, 29, 31, 59, \\
\alpha(\widetilde{u}^p) &= 30|\psi_1^p|^2 - 3.5135, && \text{for } p = 7, 23, 37, 53, \\
\alpha(\widetilde{u}^p) &= 30|\psi_1^p|^2 - 2.0511, && \text{for } p = 11, 19, 41, 49, \\
\alpha(\widetilde{u}^p) &= 30|\psi_1^p|^2 - 4.5316, && \text{for } p = 13, 17, 43, 47.
\end{aligned}$$

Now for all $p \in B_8$, $\widetilde{u}^p$ is a 60^{th} root of unity, but not a root of unity of lower order. By similar considerations as in the case of E_7, we see that the summation in (7.13) cannot be written as an integral using the uniform measure d_{30} or the measure αd_{30} either. So we see that the spectral measure for E_8 is not cyclotomic.

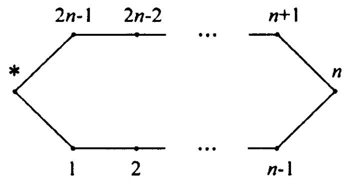


Figure 7.4: Affine Dynkin diagrams $A_{2n}^{(1)}$, $n = 2, 3, \ldots$

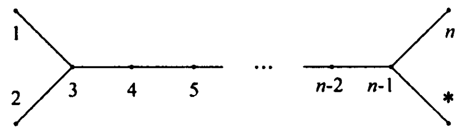


Figure 7.5: Affine Dynkin diagrams $D_n^{(1)}$, $n = 4, 5, \ldots$

7.4 Spectral measures for finite subgroups of $SU(2)$

The McKay correspondence [86] associates to every finite subgroup Γ of $SU(2)$ an affine Dynkin diagram $\mathcal{G}_\Gamma$ given by the fusion graph of the fundamental representation ρ acting on the irreducible representations of Γ. These affine Dynkin diagrams are illustrated in Figures 7.4-7.6, where $*$ denotes the identity representation. Hence there is associated to each finite subgroup of $SU(2)$ the corresponding (non-affine) ADE Dynkin diagram $\mathcal{G}$, which is obtained from the affine diagram by deleting the vertex $*$ and all edges attached to it. This correspondence is shown in the following table. The second column indicates the type of the associated modular invariant.

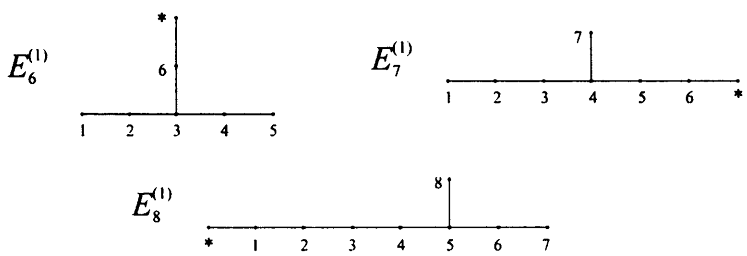


Figure 7.6: Affine Dynkin diagrams $E_6^{(1)}$, $E_7^{(1)}$ and $E_8^{(1)}$

Dynkin Diagram $\mathcal{G}$	Type	Subgroup $\Gamma \subset SU(2)$	$\lvert\Gamma\rvert$ = order of Γ
A_l	I	cyclic, $\mathbb{Z}_{l+1}$	$l+1$
D_{2k}	I	binary dihedral, $BD_{2k} = Q_{2k-2}$	$8k-8$
D_{2k+1}	II	binary dihedral, $BD_{2k+1} = Q_{2k-1}$	$8k-4$
E_6	I	binary tetrahedral, $BT = BA_4$	24
E_7	II	binary octahedral, $BO = BS_4$	48
E_8	I	binary icosahedral, $BI = BA_5$	120

It was shown in [71] that for any finite group Γ the S-matrix, which simultaneously diagonalizes the representations of Γ, can be written in terms of the characters $\chi_j(\Gamma_i)$ of Γ evaluated on the conjugacy classes Γ_i of Γ, $S_{ij} = \sqrt{|\Gamma_j|}\chi_i(\Gamma_j)/\sqrt{|\Gamma|}$. Let N_ρ be the fundamental representation matrix of the fusion rules of the irreducible characters of Γ. Then by the Verlinde formula (1.11), the eigenvalues of N_ρ are given by ratios of the S-matrix, $\sigma(N_\rho) = \{S_{\rho,j}/S_{\rho,0} | j = 1, \ldots, p\}$, where p is the number of conjugacy classes and ρ is the fundamental representation of G. Now

$$\frac{\sqrt{|\Gamma_j|}\chi_\rho(\Gamma_j)/\sqrt{|\Gamma|}}{\sqrt{|\Gamma_j|}\chi_\rho(\Gamma_0)/\sqrt{|\Gamma|}} = \chi_\rho(\Gamma_j),$$

since $\chi_\rho(\Gamma_0) = 1$. Then any eigenvalue of Γ can be written as $\chi_\rho(g) = \mathrm{Tr}(\rho(g))$, where g is any element of Γ_j.

The elements y_i in (7.7) are then given by $y_i = S_{0,j} = \sqrt{|\Gamma_j|}\chi_0(\Gamma_j)/\sqrt{|\Gamma|} = \sqrt{|\Gamma_j|}/\sqrt{|\Gamma|}$. Then the m^{th} moment ς_m is given by

$$\varsigma_m = \int x^m d\mu(x) = \sum_{j=1}^{n} \frac{|\Gamma_j|}{|\Gamma|}\chi_\rho(\Gamma_j)^m. \tag{7.14}$$

We define an inverse $\Phi^{-1} : [-2,2] \to \mathbb{T}$ of the map Φ given in (7.5) by

$$\Phi^{-1}(x) = (x + i\sqrt{4-x^2})/2, \tag{7.15}$$

for $x \in [-2,2]$. Then the spectral measure of Γ (over $\mathbb{T}$) is given by

$$\int_{\mathbb{T}} \psi(u+u^{-1})d\varepsilon(u) = \sum_{j=1}^{n} \frac{|\Gamma_j|}{|\Gamma|}(\Phi^{-1}(\chi_\rho(\Gamma_j)) + \overline{\Phi^{-1}(\chi_\rho(\Gamma_j))})^m. \tag{7.16}$$

The generating series of the moments $G(z) = \sum_{m=0}^{\infty} \varsigma_m z^m$, is

$$G(z) = \sum_{m=0}^{\infty}\sum_{j=1}^{n} \frac{|\Gamma_j|}{|\Gamma|}\chi_\rho(\Gamma_j)^m z^m = \sum_{j=1}^{n} \frac{|\Gamma_j|}{|\Gamma|}\frac{1}{1 - z\chi_\rho(\Gamma_j)}. \tag{7.17}$$

Γ_j	$\mathbf{1}$	$(\tau\sigma)^2$	$\sigma^j,\ j=1,\dots,n-3$	τ	$\tau\sigma$
$\lvert\Gamma_j\rvert$	1	1	2	$n-2$	$n-2$
$\chi_\rho(\Gamma_j)\in[-2,2]$	2	-2	$\xi^j+\xi^{-j}$	0	0
$e^{2\pi i\theta}=\Phi^{-1}(\chi_\rho(\Gamma_j))\in\mathbb{T}$	1	-1	ξ^j	i	$-i$
$\theta\in[0,1]$	0	$\frac{n-2}{2(n-2)}$	$\frac{j}{2(n-2)}$	$\frac{n-2}{4(n-2)}$	$\frac{3(n-2)}{4(n-2)}$

Table 7.1: Character table for BD_n. Here $\xi=e^{\pi i/(n-2)}$.

7.4.1 Cyclic Group $\mathbb{Z}_{2n}$

Suppose Γ is the cyclic subgroup $\mathbb{Z}_{2n}$ of $SU(2)$, which has McKay graph $A_{2n}^{(1)}$. Then $|\Gamma|=2n$, and each element of the group is a separate conjugacy class. Now $\chi_\rho(\Gamma_j)=\widetilde{u}^j+\widetilde{u}^{-j}\in[-2,2]$, where $\widetilde{u}=e^{\pi i/n}$, for each $j=1,\dots,2n$. Then by (7.14)

$$\int_{\mathbb{T}}\psi(u+u^{-1})d\varepsilon(u)=\sum_{j=1}^{2n}\frac{1}{2n}(\widetilde{u}^j+\widetilde{u}^{-j})^m=\int_{\mathbb{T}}(u+u^{-1})^m\,d_nu.$$

Hence the spectral measure for $A_{2n}^{(1)}$ (over $\mathbb{T}$) is $d\varepsilon(u)=d_nu$, as in [3, Theorem 2.1].

7.4.2 Binary Dihedral Group BD_n

Let Γ be the binary dihedral group $BD_n=\langle\sigma,\tau|\tau^2=\sigma^n=(\tau\sigma)^2\rangle$, which has McKay graph $D_n^{(1)}$. Then $|\Gamma|=4(n-2)$. The character table for BD_n is given in Table 7.1. Let $\widetilde{u}=e^{\pi i/2(n-2)}$ and $U(j)=(\widetilde{u}^j+\widetilde{u}^{-j})^m$. Then by (7.14)

$$\begin{aligned}
&\int_{\mathbb{T}}\psi(u+u^{-1})d\varepsilon(u)\\
=&\ \frac{1}{4(n-2)}U(0)+\frac{1}{4(n-2)}U(n-2)+\sum_{j=1}^{n-3}\frac{2}{4(n-2)}\left(\frac{U(j)+U(2n-2-j)}{2}\right)\\
&+\frac{n-2}{4(n-2)}U((n-2)/2)+\frac{n-2}{4(n-2)}U(3(n-2)/2)\\
=&\ \frac{1}{2}\sum_{j=0}^{2n-3}\frac{1}{2(n-2)}(\widetilde{u}^j+\widetilde{u}^{-j})^m+\frac{1}{4}\left((\widetilde{u}^{(n-2)/2}+\widetilde{u}^{-(n-2)/2})^m+(\widetilde{u}^{3(n-2)/2}+\widetilde{u}^{-3(n-2)/2})^m\right)\\
=&\ \frac{1}{2}\int_{\mathbb{T}}(u+u^{-1})^m\,d_{n-2}u+\frac{1}{4}\int_{\mathbb{T}}(u+u^{-1})^m\,(\delta_i+\delta_{-i}),
\end{aligned}$$

where δ_ω is the Dirac measure at $\omega\in\mathbb{T}$. Then the spectral measure for $D_n^{(1)}$ (over $\mathbb{T}$) is

$$d\varepsilon(u)=\frac{1}{2}d_{n-2}u+\frac{1}{4}(\delta_i+\delta_{-i}),$$

as given in [3, Theorem 4.1].

Γ_j	$\mathbf{1}$	$-\mathbf{1}$	τ	μ	μ^2	μ^4	μ^5
$\lvert\Gamma_j\rvert$	1	1	6	4	4	4	4
$\chi_\rho(\Gamma_j) \in [-2,2]$	2	-2	0	1	-1	-1	1
$e^{2\pi i\theta} = \Phi^{-1}(\chi_\rho(\Gamma_j)) \in \mathbb{T}$	1	-1	i	$\frac{1+\sqrt{3}i}{2}$	$\frac{-1+\sqrt{3}i}{2}$	$\frac{-1+\sqrt{3}i}{2}$	$\frac{1+\sqrt{3}i}{2}$
$\theta \in [0,1]$	0	$\frac{6}{12}$	$\frac{3}{12}$	$\frac{2}{12}$	$\frac{4}{12}$	$\frac{8}{12}$	$\frac{10}{12}$

Table 7.2: Character table for the binary tetrahedral group BT.

7.4.3 Binary Tetrahedral Group BT

Let Γ be the binary tetrahedral group BT, which has McKay graph $E_6^{(1)}$. It has order 24, and is generated by $BD_2 = \langle \sigma, \tau \rangle$ and μ:

$$\sigma = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad \tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \mu = \frac{1}{\sqrt{2}} \begin{pmatrix} \varepsilon^7 & \varepsilon^7 \\ \varepsilon^5 & \varepsilon \end{pmatrix},$$

where $\varepsilon = e^{2\pi i/8}$. The orders of the group elements σ, τ, μ are 2, 4, 6 respectively. The character table for BT is given in Table 7.2. Let $\widetilde{u} = e^{2\pi i/12}$ and $U(j) = (\widetilde{u}^j + \widetilde{u}^{-j})^m$. Then by (7.14),

$$\begin{aligned} &\int_{\mathbb{T}} \psi(u+u^{-1}) d\varepsilon(u) \\ &= \frac{1}{24}U(0) + \frac{1}{24}U(6) + \frac{6}{24}U(3) + \frac{4}{24}U(2) + \frac{4}{24}U(4) + \frac{4}{24}U(8) + \frac{4}{24}U(10). \end{aligned}$$

For the 6^{th} roots of unity we have $\alpha(e^{p\pi i/6}) - 1/2 = -1/2$, $p = 0, 6$, and $\alpha(e^{p\pi i/6}) - 1/2 = 1$, $p = 2, 4, 8, 10$, where α is given in (7.11). Then since $U(3) = U(9)$:

$$\begin{aligned} &\int_{\mathbb{T}} \psi(u+u^{-1}) d\varepsilon(u) \\ &= \frac{3}{24}(U(0) + U(3) + U(6) + U(9)) \\ &\quad + \frac{1}{24}(-2U(0) + 4U(2) + 4U(2) - 2U(6) + 4U(8) + 4U(10)) \\ &= \frac{1}{2}\sum_{j=0}^{3} \frac{1}{4}(\widetilde{u}^{3j} + \widetilde{u}^{-3j})^m + \sum_{j=0}^{5} (\alpha(\widetilde{u}^{2j}) - \tfrac{1}{2})(\widetilde{u}^{2j} + \widetilde{u}^{-2j})^m \\ &= \frac{1}{2}\int_{\mathbb{T}} (u+u^{-1})^m \, d_2 u + \int_{\mathbb{T}} (u+u^{-1})^m \, (\alpha(u) - \tfrac{1}{2}) d_3 u. \end{aligned}$$

Hence the spectral measure for $E_6^{(1)}$ (over $\mathbb{T}$) is

$$d\varepsilon = (\alpha - \tfrac{1}{2})d_3 + \tfrac{1}{2}d_2,$$

as given in [3, Theorem 6.1].

Γ_j	$\mathbf{1}$	$\mathbf{-1}$	μ	μ^2	τ	κ	$\tau\kappa$	κ^3
$\lvert\Gamma_j\rvert$	1	1	8	8	6	6	12	6
$\chi_\rho(\Gamma_j) \in [-2,2]$	2	-2	1	-1	0	$\sqrt{2}$	0	$-\sqrt{2}$
$e^{2\pi i\theta} = \Phi^{-1}(\chi_\rho(\Gamma_j)) \in \mathbb{T}$	1	-1	$\frac{1+\sqrt{3}i}{2}$	$\frac{-1+\sqrt{3}i}{2}$	i	$\frac{1+i}{\sqrt{2}}$	$-i$	$\frac{-1+i}{\sqrt{2}}$
$\theta \in [0,1]$	0	$\frac{12}{24}$	$\frac{4}{24}$	$\frac{8}{24}$	$\frac{6}{24}$	$\frac{3}{24}$	$\frac{18}{24}$	$\frac{9}{24}$

Table 7.3: Character table for the binary octahedral group BO.

7.4.4 Binary Octahedral Group BO

Let Γ be the binary octahedral group BO, which has McKay graph $E_7^{(1)}$. It has order 48 and is generated by the binary tetrahedral group BT and the element κ of order 8 given by

$$\kappa = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^7 \end{pmatrix},$$

where again $\varepsilon = e^{2\pi i/8}$. Its McKay graph is $E_7^{(1)}$. The character table for BO is given in Table 7.3. Let $\widetilde{u} = e^{2\pi i/24}$ and $U(j) = (\widetilde{u}^j + \widetilde{u}^{-j})^m$. Then by (7.14)

$$\begin{aligned}
&\int_{\mathbb{T}} \psi(u+u^{-1}) d\varepsilon(u) \\
&= \frac{1}{48}U(0) + \frac{1}{48}U(12) + \frac{8}{48}U(4) + \frac{8}{48}U(8) + \frac{6}{48}U(6) + \frac{6}{48}U(3) + \frac{12}{48}U(18) + \frac{6}{48}U(9).
\end{aligned}$$

For the 8^{th} roots of unity we have $\alpha(e^{p\pi i/8}) - 1/2 = -1/2$, for $p = 0, 12$, $\alpha(e^{p\pi i/6}) - 1/2 = 1/2$, for $p = 3, 9, 15, 21$, and $\alpha(e^{p\pi i/6}) - 1/2 = 3/2$, for $p = 6, 18$, where α is given in (7.11). Then since $U(j) = U(24-j)$, $j = 1, \ldots, 12$, we have

$$\begin{aligned}
&\int_{\mathbb{T}} \psi(u+u^{-1}) d\varepsilon(u) \\
&= \frac{4}{48}(U(0) + U(4) + U(8) + U(12) + U(16) + U(20)) \\
&\quad + \frac{1}{48}(-3U(0) + 3U(3) + 9U(6) + 3U(9) - 3U(12) + 3U(15) + 9U(18) + 3U(21)) \\
&= \frac{1}{2}\sum_{j=0}^{5} \frac{1}{6}(\widetilde{u}^{4j} + \widetilde{u}^{-4j})^m + \sum_{j=0}^{7}(\alpha(\widetilde{u}^{3j}) - \tfrac{1}{2})(\widetilde{u}^{3j} + \widetilde{u}^{-3j})^m \\
&= \frac{1}{2}\int_{\mathbb{T}} (u+u^{-1})^m \, d_3 u + \int_{\mathbb{T}} (u+u^{-1})^m \, (\alpha(u) - \tfrac{1}{2}) d_4 u.
\end{aligned}$$

Hence the spectral measure for $E_7^{(1)}$ (over $\mathbb{T}$) is

$$d\varepsilon = (\alpha - \tfrac{1}{2})d_4 + \tfrac{1}{2}d_3,$$

as given in [3, Theorem 6.1].

Γ_j	$\mathbf{1}$	$-\mathbf{1}$	σ	σ^2	σ^3	σ^4	τ	$\sigma^2\tau$	$\sigma^7\tau$
$\lvert\Gamma_j\rvert$	1	1	12	12	12	12	30	20	20
$\chi_\rho(\Gamma_j) \in [-2,2]$	2	-2	μ^+	$-\mu^-$	μ^-	$-\mu^+$	0	-1	1
$e^{2\pi i\theta} = \Phi^{-1}(\chi_\rho(\Gamma_j)) \in \mathbb{T}$	1	-1	ν^+	$-\overline{\nu^-}$	ν^-	$-\overline{\nu^+}$	i	$\frac{-1+\sqrt{3}i}{2}$	$\frac{1+\sqrt{3}i}{2}$
$\theta \in [0,1]$	0	$\frac{60}{120}$	$\frac{12}{120}$	$\frac{96}{120}$	$\frac{36}{120}$	$\frac{72}{120}$	$\frac{30}{120}$	$\frac{40}{120}$	$\frac{20}{120}$

Table 7.4: Character table for the binary icosahedral group BI. Here $\mu^\pm = (1 \pm \sqrt{5})/2$, $\nu^\pm = (1 \pm \sqrt{5} + i\sqrt{10 \mp 2\sqrt{5}})/4$.

7.4.5 Binary Icosahedral Group BI

Let Γ be the binary icosahedral group BI, which has McKay graph $E_8^{(1)}$. It has order 120, and is generated by σ, τ:

$$\sigma = \begin{pmatrix} -\varepsilon^3 & 0 \\ 0 & -\varepsilon^2 \end{pmatrix}, \qquad \tau = \frac{1}{\sqrt{5}} \begin{pmatrix} \varepsilon^4 - \varepsilon & \varepsilon^2 - \varepsilon^3 \\ \varepsilon^2 - \varepsilon^3 & \varepsilon - \varepsilon^4 \end{pmatrix},$$

where $\varepsilon = e^{2\pi i/5}$. The orders of the group elements σ, τ are 10, 4 respectively. The character table for BI is given in Table 7.4. Let $\widetilde{u} = e^{2\pi i/120}$ and $U(j) = (\widetilde{u}^j + \widetilde{u}^{-j})^m$. Then by (7.14)

$$\begin{aligned}\int_{\mathbb{T}} \psi(u+u^{-1}) d\varepsilon(u) &= \frac{1}{120}U(0) + \frac{1}{120}U(60) + \frac{12}{120}U(12) + \frac{12}{120}U(96) + \frac{12}{120}U(36) \\ &\quad + \frac{12}{120}U(72) + \frac{30}{120}U(30) + \frac{20}{120}U(40) + \frac{20}{120}U(20).\end{aligned}$$

For the 12^{th} roots of unity we have $\alpha(e^{p\pi i/6}) - 1/2 = -1/2$, for $p = 0, 6$, $\alpha(e^{p\pi i/6}) - 1/2 = 1$, for $p = 2, 4, 8, 10$, $\alpha(e^{p\pi i/6}) - 1/2 = 3/2$, for $p = 3, 9$, and $\alpha(e^{p\pi i/6}) - 1/2 = 0$, for $p = 1, 5, 7, 11$, where α is given in (7.11). Then since $U(j) = U(120 - j)$, $j = 1, \ldots, 60$, we have

$$\begin{aligned}\int_{\mathbb{T}} \psi(u+u^{-1}) d\varepsilon(u) &= \frac{6}{120}\Big(U(0) + U(12) + U(24) + U(36) + U(48) + U(60) \\ &\qquad + U(72) + U(84) + U(96) + U(108)\Big) \\ &\quad + \frac{1}{120}\Big(- 5U(0) + 10U(20) + 15U(30) + 10U(40) - 5U(60) \\ &\qquad + 10U(80) + 15U(90) + 10U(100)\Big) \\ &= \frac{1}{2}\sum_{j=0}^{9} \frac{1}{10}(\widetilde{u}^{12j} + \widetilde{u}^{-12j})^m + \sum_{j=0}^{11}(\alpha(\widetilde{u}^{10j}) - \tfrac{1}{2})(\widetilde{u}^{10j} + \widetilde{u}^{-10j})^m \\ &= \frac{1}{2}\int_{\mathbb{T}} (u+u^{-1})^m \, d_5 u + \int_{\mathbb{T}} (u+u^{-1})^m \, (\alpha(u) - \tfrac{1}{2}) d_6 u.\end{aligned}$$

Hence the spectral measure for $E_8^{(1)}$ (over $\mathbb{T}$) is

$$d\varepsilon = (\alpha - \tfrac{1}{2})d_6 + \tfrac{1}{2}d_5,$$

as given in [3, Theorem 6.1].

7.5 Hilbert Series of dimensions of ADE models.

We now compare various polynomials related to ADE models.

7.5.1 T-Series

We begin first with the T-series of Banica and Bisch [3]. Let $\mathcal{G}$ now be any bipartite graph with norm ≤ 2, that is, its adjacency matrix Δ has norm ≤ 2. These are the subgroups of $SU(2)$, with McKay graphs given by the affine Dynkin diagrams, and the modules and subgroups of $SU(2)_k$, which have McKay graphs given by the ADE Dynkin diagrams. The generating series $S(q)$ of the moments of the spectral measure ε_Δ (over $\mathbb{T}$) for Δ is given by

$$S(q) = \int_{\mathbb{T}} \frac{1}{1-qu} d\varepsilon_\Delta(u).$$

Let $A(\mathcal{G})$ be the path algebra for $\mathcal{G}$, with initial vertex the distinguished vertex $*$ which has lowest Perron-Frobenius weight. The Hilbert series (also called the Poincaré series in some literature)

$$f(z) = \sum_{k=0}^{\infty} \dim(A(\mathcal{G})_k) z^k \tag{7.18}$$

of $\mathcal{G}$ is the generating function counting the numbers l_{2k} of loops of length $2k$ on $\mathcal{G}$, from the vertex $*$ to itself, $f(z) = \sum_{k=0}^{\infty} l_{2k} z^k$. The Hilbert series f measures the dimension of the algebra at level k in the Bratteli diagram. If $\mathcal{G}$ is the principal graph of a subfactor $N \subset M$, the series f measures the dimensions of the higher relative commutants, giving an invariant of the subfactor $N \subset M$. We define another function $\widehat{f}$ by

$$\widehat{f}(z) = \varphi\left(\left(\mathbf{1} - z^{\frac{1}{2}}\Delta\right)^{-1}\right). \tag{7.19}$$

Then $\widehat{f}(z) = \varphi(\mathbf{1} + z^{1/2}\Delta + z\Delta^2 + z^{3/2}\Delta^3 + \dots) = \sum_{n=0}^{\infty} [\Delta^n]_{*,*} z^{n/2}$. Since $\mathcal{G}$ is bipartite, there are no paths of odd length from $*$ to $*$, and so $[\Delta^{2k+1}]_{*,*} = 0$ for $k = 0, 1, \dots$. Then $\widehat{f}(z) = \sum_{k=0}^{\infty} [\Delta^{2k}]_{*,*} z^k = f(z)$. Then it is easily seen from (7.3) and (7.19) that $f(z^2)$ is equal to the Stieltjes transform $\sigma(z)$ of μ_Δ.

Suppose P is the (A_1-)planar algebra for a subfactor $N \subset M$ with Jones index $[M : N] < 4$ and principal graph $\mathcal{G}$. If $\dim(P_0^{\pm}) = 1$, the Hilbert series $f(z)$ is identical to the Hilbert series $\Phi_P(z)$ which gives the dimension of the planar algebra P:

$$\Phi_P(z) = \frac{1}{2}(\dim(P_0^+) + \dim(P_0^-)) + \sum_{j=1}^{\infty} \dim(P_j) z^j.$$

As a Temperley-Lieb module, P decomposes into a sum of irreducible Temperley-Lieb modules, with the multiplicity of the irreducible module of lowest weight k given by the non-negative integer a_k. Jones [63] then defined the series Θ by

$$\Theta_P(q) = \sum_{j=0}^{\infty} a_j q^j.$$

It was shown in [3, Prop. 1.2] that $\Theta(q^2) = 2S(q) + q^2 - 1$. The series $\Theta(q)$ is essentially obtained from the Hilbert series $f(z)$ in (7.18) by the change of variables. More explicitly, in [3], $\Theta(q)$ is given in terms of $f(z)$ by:

$$\Theta(q) = q + \frac{1-q}{1+q} f\left(\frac{q}{(1+q)^2}\right).$$

Banica and Bisch then introduced their T series, which is defined for any Dynkin diagram (and affine Dynkin diagram) by

$$T(q) = \frac{2S(q^{1/2}) - 1}{1-q},$$

in order to compute the spectral measures for the Dynkin diagrams (and affine Dynkin diagrams) of type E. In terms of the Hilbert series f, we have

$$T(q) = \frac{\Theta(q) - q}{1-q} = \frac{1}{1+q} f\left(\frac{q}{(1+q)^2}\right).$$

We can define a generalized T series $\widetilde{T}_{ij}$ by

$$\widetilde{T}(q) = \frac{1}{1+q} \widetilde{f}\left(\frac{q}{(1+q)^2}\right), \tag{7.20}$$

where the matrix $\widetilde{f}(z) = \left(1 - z^{\frac{1}{2}}\Delta_X\right)^{-1}$, and $[\widetilde{f}(z)]_{ij}$ counts paths from i to j. Then $f(z) = \varphi(\widetilde{f}(z))$ and

$$T(q) = \varphi(\widetilde{T}(q)). \tag{7.21}$$

Since $(1+q^2)^2/q^2 = (q+q^{-1})^2 = [2]_q^2$, we can write $\widetilde{T}(q^2)$ as $\widetilde{T}(q^2) = \widetilde{f}([2]_q^{-2})/(1+q^2)$.

The T series for the exceptional graphs E_6, E_7 and E_8 and their affine versions are computed in [3]. Let T_j, $T_j^{(1)}$ denote the T series for the Dynkin diagrams E_j, affine Dynkin diagram $E_j^{(1)}$ respectively. The T series are given by:

$$
\begin{aligned}
T_6 &= \frac{(1-q^6)(1-q^8)}{(1-q^3)(1-q^{12})}, & T_6^{(1)} &= \frac{1+q^6}{(1-q^3)(1-q^4)},\\
T_7 &= \frac{(1-q^9)(1-q^{12})}{(1-q^4)(1-q^{18})}, & T_7^{(1)} &= \frac{1+q^9}{(1-q^4)(1-q^6)},\\
T_8 &= \frac{(1-q^{10})(1-q^{15})(1-q^{18})}{(1-q^5)(1-q^9)(1-q^{30})}, & T_8^{(1)} &= \frac{1+q^{15}}{(1-q^6)(1-q^{10})}.
\end{aligned}
$$

7.5.2 Kostant Polynomials

We now introduce a polynomial for subgroups of $SU(2)$ which is related to the T-series defined in Section 7.5.1. The precise relation between the two polynomials will be given later in Theorem 7.5.1. For a subgroup $\Gamma \subset SU(2)$ and an irreducible representation γ of Γ, the Kostant polynomial F_γ counts the multiplicity of γ in (j), the $j+1$-dimensional irreducible representation of $SU(2)$ restricted to Γ. The Kostant polynomial F_γ is given by

$$
F_\gamma(t) = \sum_{j=0}^{\infty} \langle (j), \gamma, \rangle_\Gamma \, t^j,
$$

where $\langle (j), \gamma \rangle_\Gamma$ is the multiplicity of γ in (j). Let $F(t) = \sum_{j=0}^{\infty} t^j (j) = \sum_\gamma F_\gamma(t)\gamma$. Then we obtain the recursion formulae

$$
\begin{aligned}
F(t) \otimes (1) &= \sum_\gamma F_\gamma(t)\gamma \otimes (1) &&= \sum_{j=0}^{\infty} t^j (j) \otimes (1)\\
&= \sum_{j=0}^{\infty} t^j ((j-1) \oplus (j+1)) &&= (t^{-1}+t)F(t) - \frac{\mathrm{id}}{t},
\end{aligned}
$$

where id is the identity representation of Γ. Evaluating this polynomial by taking its character on conjugation classes Γ_i of Γ we obtain [56]:

$$
F_\gamma(t) = \sum_i \frac{|\Gamma_i|}{|\Gamma|} \frac{\chi_\gamma^*(\Gamma_i)}{1 - t\chi_\rho(\Gamma_i) + t^2}. \tag{7.22}
$$

The explicit result was worked out by Kostant in [75], where he showed that the polynomials $F_\gamma(t)$ have the simple form

$$
F_\gamma(t) = \frac{z_\gamma(t)}{(1-t^a)(1-t^b)}, \tag{7.23}
$$

where a, b are positive integers which satisfy $a + b = h + 2$ and $ab = 2|\Gamma|$, where h is the Coxeter number of the Dynkin diagram $\mathcal{G}$, and $z_\gamma(t)$ is now a finite polynomial which we will reproduce below. The values of a, b are:

Dynkin Diagram $\mathcal{G}$	h	a, b
A_l	$l+1$	$2,\ l+1$
D_l	$2l-2$	$4,\ 2l-4$
E_6	12	6, 8
E_7	18	8, 12
E_8	30	12, 20

The Kostant polynomial is related to subfactors realizing the ADE modular invariants in [34, §3.3]. Let $*$ label the trivial representation of Γ. By the argument of changing the ι-vertex [33] it may be assumed that the subfactor $N \subset M$ realizing the ADE modular invariant has the ι-vertex on the vertex which would join the extended vertex $*$ of the affine Dynkin diagram $\mathcal{G}_\Gamma$. For all DE cases there is a natural bijection between (equivalence classes of) non-trivial irreducible representations of Γ and M-N sectors $[\iota\lambda_l]$, since the irreducible representations label the vertices of the DE graph, as do the sectors $[\iota\lambda_l]$. Let ρ denote the fundamental representation of Γ. Denoting the M-N morphism associated to the irreducible representation $\gamma \neq *$ by $\overline{a}_\gamma$ (so $\iota = \overline{a}_\rho$), it was shown in [34] that the polynomials p_γ defined by

$$\begin{aligned} p_*(t) &= 1 + q^{k+2}, \\ p_\gamma(t) &= \sum_{i=0}^{k} \langle \overline{a}_\gamma, \iota\lambda_j \rangle t^{j+1}, \end{aligned}$$

are equal to the numerators $z_\gamma(t)$ of the Kostant polynomial $F_\gamma(t)$, and consequently $F_\gamma(t) = p_\gamma(t)/\Omega(t)$, where $\Omega(t) = (1+t^2)p_*(t) - tp_\rho(t)$.

It was shown by McKay in [87] that the finite polynomials $z_\gamma(t)$ also arise by calculating weights associated to the vertices of so-called semi-affine Dynkin diagrams. The semi-affine Dynkin diagrams are given by the affine Dynkin diagrams where the edges attached to the affine vertex $*$ are now oriented edges, directed towards the affine vertex. Let $n_* = 1$ be a weight attached to vertex $*$. Then to each other vertex i of the semi-affine diagram we attach a weight n_i satisfying $sn_i = \sum n_j$ where the summation is over all vertices j such that there is an edge from i to j. The weights are quotients of polynomials, so we re-normalize the weights to remove the denominator. Making the change of variable $s = t + t^{-1}$, and re-normalizing so that $n_* = 1 + t^h$, the weights n_i give the numerators $z_i(t)$.

For the distinguished vertex $*$, $z_*(t) = 1 + t^h$ for each of the affine Dynkin diagrams. For A_l, the numerators $z_i(t)$ are given by $z_i(t) = t^i + t^{h-i}, \quad i = 1, \dots, l$. From Kostant [75], for the exceptional graphs, the numerators $z_i(t)$ are

E_6 :

$$
\begin{aligned}
z_1(t) &= t^4 + t^8, \\
z_2(t) &= t^3 + t^5 + t^7 + t^9, \\
z_3(t) &= t^2 + t^4 + 2t^6 + t^8 + t^{10}, \\
z_4(t) &= t^3 + t^5 + t^7 + t^9, \\
z_5(t) &= t^4 + t^8, \\
z_6(t) &= t + t^5 + t^7 + t^{11},
\end{aligned}
$$

E_7 :

$$
\begin{aligned}
z_1(t) &= t^6 + t^{12}, \\
z_2(t) &= t^5 + t^7 + t^{11} + t^{13}, \\
z_3(t) &= t^4 + t^6 + t^8 + t^{10} + t^{12} + t^{14}, \\
z_4(t) &= t^3 + t^5 + t^7 + 2t^9 + t^{11} + t^{13} + t^{15}, \\
z_5(t) &= t^2 + t^6 + t^8 + t^{10} + t^{12} + t^{16}, \\
z_6(t) &= t + t^7 + t^{11} + t^{17}, \\
z_7(t) &= t^4 + t^8 + t^{10} + t^{14},
\end{aligned}
$$

E_8 :

$$
\begin{aligned}
z_1(t) &= t + t^{11} + t^{19} + t^{29}, \\
z_2(t) &= t^2 + t^{10} + t^{12} + t^{18} + t^{20} + t^{28}, \\
z_3(t) &= t^3 + t^9 + t^{11} + t^{13} + t^{17} + t^{19} + t^{21} + t^{27}, \\
z_4(t) &= t^4 + t^8 + t^{10} + t^{12} + t^{14} + t^{16} + t^{18} + t^{20} + t^{22} + t^{26}, \\
z_5(t) &= t^5 + t^7 + t^9 + t^{11} + t^{13} + 2t^{15} + t^{17} + t^{19} + t^{21} + t^{23} + t^{25}, \\
z_6(t) &= t^6 + t^8 + t^{12} + t^{14} + t^{16} + t^{18} + t^{22} + t^{24}, \\
z_7(t) &= t^7 + t^{13} + t^{17} + t^{23}, \\
z_8(t) &= t^6 + t^{10} + t^{14} + t^{16} + t^{20} + t^{24}.
\end{aligned}
$$

Notice that the Kostant polynomial $F_*(t)$ for the graphs E_n, $n = 6, 7, 8$, is just the T-series $T_n^{(1)}(t^2)$ of Section 7.5.1 for the affine graphs $E_n^{(1)}$, $n = 6, 7, 8$ (see Theorem 7.5.1 (iii)).

7.5.3 Molien Series

Another related polynomial is the Molien series. Let Γ be a finite subgroup of $SU(N)$ as above. For $i = 0, 1, \dots$, let M_i be a representation of Γ with dim $M_i < \infty$, and let $M = \bigoplus_{i=0}^{\infty} M_i$. With γ an irreducible representation of Γ, the Molien series $P_{M,\gamma}$ of M is defined in [50] by

$$P_{M,\gamma}(t) = \sum_{i=0}^{\infty} \langle M_i, \gamma \rangle_\Gamma \, t^i,$$

and counts the multiplicity $\langle M_i, \gamma \rangle_\Gamma$ of γ in M_i.

Let $\overline{\mathbb{C}^N}$ denote the dual vector space of $\mathbb{C}^N$, and denote by $S = \bigoplus_k S^k(\overline{\mathbb{C}^N})$ the symmetric algebra of $\overline{\mathbb{C}^N}$ over $\mathbb{C}$, where $S^k(\overline{\mathbb{C}^N})$ is the k^{th} symmetric product of $\overline{\mathbb{C}^N}$. Let ρ be the fundamental representation of Γ and $\overline{\rho}$ its conjugate representation, let $\{\rho_0 = \mathrm{id}, \rho_1 = \rho, \rho_2, \ldots, \rho_s\}$ be the irreducible representations of Γ and χ_j be the character of ρ_j for $j = 0, 1, \ldots, s$. Then we have Molien's formula for $P_{S,\gamma_j}(t)$ given as [50]:

$$P_{S,\rho_j}(t) = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} \frac{\chi_j^*(g)}{\det(1 - \overline{\rho}(g)t)}.$$

Let R_k be the sum of the representations of $SU(N)$ with Dynkin labels $\lambda_1, \lambda_2, \ldots, \lambda_{(N-1)}$ such that $\lambda_1 + \cdots + \lambda_{(N-1)} = k$, and $R = \bigoplus_{k=0}^{\infty} R_k$. Then in this notation, $P_{R,\gamma}$ recovers the Kostant polynomial F_γ, where γ is an irreducible representation of Γ:

$$P_{R,\gamma}(t) = \sum_{i=0}^{\infty} \langle R_i, \gamma \rangle_\Gamma \, t^i = F_\gamma(t, t, \ldots, t). \tag{7.24}$$

Since there is only one Dynkin label λ for any representation of $SU(2)$, $R_k = (k)$, the $(k+1)$-dimensional representation of $SU(2)$, for each k. Then by (7.24) the Molien series $P_{R,\gamma}(t)$ for a subgroup $\Gamma \subset SU(2)$ is equal to the Kostant polynomial $F_\gamma(t)$. From Section 1.3.1, the k^{th} symmetric product of $\overline{\mathbb{C}^2}$ gives the irreducible level k representation, so that $R = S$ for $SU(2)$, and $P_{S,\gamma}(t) = F_\gamma(t)$.

7.5.4 Hilbert Series of Pre-projective Algebras

Finally, we introduce another related polynomial, the Hilbert series $H(t)$, which counts the dimensions of pre-projective algebras for the ADE and affine Dynkin diagrams. Let $\mathcal{G}$ be any (oriented or unoriented) graph, and let $\mathbb{C}\mathcal{G}$ be the algebra with basis given by the paths in $\mathcal{G}$, where paths may begin at any vertex of $\mathcal{G}$. Multiplication of two paths a, b is given by concatenation of paths $a \cdot b$ (or simply ab), where ab is defined to be zero if $r(a) \neq s(b)$. Note that the algebra $\mathbb{C}\mathcal{G}$ is not the path algebra $A(\mathcal{G})$ for $\mathcal{G}$ in the usual operator algebraic meaning. Let $[\mathbb{C}\mathcal{G}, \mathbb{C}\mathcal{G}]$ denote the subspace of $\mathbb{C}\mathcal{G}$ spanned by all commutators of the form $ab-ba$, for $a, b \in \mathbb{C}\mathcal{G}$. If a, b are paths in $\mathbb{C}\mathcal{G}$ such that $r(a) = s(b)$ but $r(b) \neq s(b)$, then $ab - ba = ab$, so in the quotient $\mathbb{C}\mathcal{G}/[\mathbb{C}\mathcal{G}, \mathbb{C}\mathcal{G}]$ the path ab will be zero. Then any non-cyclic path, i.e. any path a such that $r(a) \neq s(a)$, will be zero in $\mathbb{C}\mathcal{G}/[\mathbb{C}\mathcal{G}, \mathbb{C}\mathcal{G}]$. If $a = a_1 a_2 \cdots a_k$ is a cyclic path in $\mathbb{C}\mathcal{G}$, then $a_1 a_2 \cdots a_k - a_k a_1 \cdots a_{k-1} = 0$

in $\mathbb{C}\mathcal{G}/[\mathbb{C}\mathcal{G},\mathbb{C}\mathcal{G}]$, so $a_1a_2\cdots a_k$ is identified with $a_ka_1\cdots a_{k-1}$. Similarly, $a = a_1a_2\cdots a_k$ is identified with every cyclic permutation of the edges a_j, $j = 1,\dots,k$. So the commutator quotient $\mathbb{C}\mathcal{G}/[\mathbb{C}\mathcal{G},\mathbb{C}\mathcal{G}]$ may be identified, up to cyclic permutation of the arrows, with the vector space spanned by cyclic paths in $\mathcal{G}$.

The pre-projective algebra Π of a finite unoriented graph $\mathcal{G}$ is defined as the quotient of $\mathbb{C}\mathcal{G}$ by the two-sided ideal generated by

$$\theta = \sum_{i,\sigma} \theta_i^\sigma,$$

where the summation is over all vertices i and edges σ of $\mathcal{G}$ such that i is an endpoint for σ, and $\theta_i^\sigma \in \mathbb{C}\mathcal{G}$ is defined to be the loop of length two starting and ending at vertex i formed by going along the edge σ and back again. So the pre-projective algebra is the quotient algebra under relations θ, and any closed loop of length 2 on $\mathcal{G}$ is identified with a linear combination of all the other closed loops of length 2 on $\mathcal{G}$ which have the same initial vertex. In the language of planar algebras for bipartite graphs (see Section 6.5.1), this is closely related to taking the (complement of the) kernel of the insertion operators given by the cups and caps.

For a graph $\mathcal{G}$ without any closed loops of length one, i.e. edges from a vertex to itself, the pre-projective algebra Π has the following description as a quotient of a path algebra by a two-sided ideal generated by derivatives of a potential Φ. We fix an orientation for the edges of $\mathcal{G}$, and form the double $\overline{\mathcal{G}}$ of $\mathcal{G}$, where for each (oriented) edge γ we add the reverse edge $\widetilde{\gamma}$ which has $s(\widetilde{\gamma}) = r(\gamma)$, $r(\widetilde{\gamma}) = s(\gamma)$. We define a potential Φ by $\Phi = \sum_\gamma \gamma\widetilde{\gamma}$, where the summation is over all edges of $\mathcal{G}$. Let $\gamma_1\gamma_2\cdots\gamma_k$ be any closed loop of length k in $\mathbb{C}\overline{\mathcal{G}}/[\mathbb{C}\overline{\mathcal{G}},\mathbb{C}\overline{\mathcal{G}}]$, $k > 1$. We define derivatives $\partial_i : \mathbb{C}\overline{\mathcal{G}}/[\mathbb{C}\overline{\mathcal{G}},\mathbb{C}\overline{\mathcal{G}}] \to \mathbb{C}\overline{\mathcal{G}}$ for each vertex $i \in \mathfrak{V}_\mathcal{G}$ of $\mathcal{G}$ by $\partial_i(\gamma_1\gamma_2\cdots\gamma_k) = \sum_j \gamma_j\gamma_{j+1}\cdots\gamma_k\gamma_1\cdots\gamma_{j-1}$, where the summation is over all $1 \leq j \leq k$ such that $s(\gamma_j) = i$. Then on paths $\gamma\widetilde{\gamma} \in \mathbb{C}\overline{\mathcal{G}}/[\mathbb{C}\overline{\mathcal{G}},\mathbb{C}\overline{\mathcal{G}}]$, we have

$$\partial_i(\gamma\widetilde{\gamma}) = \begin{cases} \gamma\widetilde{\gamma} & \text{if } s(\gamma) = i, \\ \widetilde{\gamma}\gamma & \text{if } r(\gamma) = i, \\ 0 & \text{otherwise.} \end{cases},$$

and $\Pi \cong \mathbb{C}\overline{\mathcal{G}}/(\partial_i\Phi : i \in \mathfrak{V}_\mathcal{G})$. For any graph $\mathcal{G}$ and potential Φ, Bocklandt [14, Theorem 3.2] showed that if $A(\mathbb{C}\mathcal{G},\Phi)$ is Calabi-Yau of dimension 2 then $A(\mathbb{C}\mathcal{G},\Phi)$ is the pre-projective algebra of a non-Dynkin quiver.

We can define the Hilbert series for $A(\mathbb{C}\mathcal{G},\Phi)$ as

$$H_A(t) = \sum_{k=0}^{\infty} H_{ji}^k t^k,$$

where the H_{ji}^k are matrices which count the dimension of the subspace $\{i \cdot a \cdot j | a \in A(\mathbb{C}\mathcal{G}, \Phi)_k\}$, where $A(\mathbb{C}\mathcal{G}, \Phi)_k$ is the subspace of $A(\mathbb{C}\mathcal{G}, \Phi)$ of all paths of length k, and i, j are paths in $A(\mathbb{C}\mathcal{G}, \Phi)_0$, corresponding to vertices of $\mathcal{G}$.

Let $q \in \mathbb{C} \setminus \{0\}$. If $q = \pm 1$ or q not a root of unity, the tensor category $\mathcal{C}_q$ of representations of the quantum group $SU(2)_q$ has a complete set $\{L_s\}_{s=0}^{\infty}$ of simple objects of $\mathcal{C}_q$, where L_s is the deformation of the $(s+1)$-dimensional representation of $SU(2)$, which satisfy the tensor product decomposition

$$L_r \otimes L_s \simeq \bigoplus_{\substack{t=|r-s| \\ t \equiv r+s \bmod 2}}^{r+s} L_t. \tag{7.25}$$

If q is an n^{th} root of unity, $\mathcal{C}_q$ is the semisimple subquotient of the category of representations of $SU(2)_q$. In this case, the set $\{L_s\}_{s=0}^{h(q)-2}$ is the complete set of simple objects of $\mathcal{C}_q$, where L_s is again the deformation of the $(s+1)$-dimensional representation of $SU(2)$, and $h(q)$ is n when n is odd and $n/2$ when n is even. These simple objects satisfy the tensor product decomposition

$$L_r \otimes L_s \simeq \bigoplus_{\substack{t=|r-s| \\ t \equiv r+s \bmod 2}}^{k} L_t, \tag{7.26}$$

where

$$k = \begin{cases} r+s & \text{if } r+s < h(q)-1, \\ 2h(q)-4-r-s & \text{if } r+s \geq h(q)-1. \end{cases}$$

Semisimple module categories over $\mathcal{C}_q$ where classified in [32]. A semisimple $\mathcal{C}_q$-module category $\mathcal{D}$ is abelian, and is equivalent to the category of I-graded vector spaces $\mathcal{M}_I$, where I is the set of isomorphism classes of simple objects of $\mathcal{D}$. The structure of a $\mathcal{C}_q$ category on $\mathcal{M}_I$ is the same as a tensor functor $F : \mathcal{C}_q \to \mathrm{Fun}(\mathcal{M}_I, \mathcal{M}_I)$, where $\mathrm{Fun}(\mathcal{M}_I, \mathcal{M}_I) \cong \mathcal{M}_{I \times I}$ is the category of additive functors from $\mathcal{M}_I$ to itself. When $q = \pm 1$ or q is not a root of unity, by [32, Theorem 2.5], such functors are classified by the following data:

- a collection of finite dimensional vector spaces V_{ij}, $i, j \in I$,
- a collection of non-degenerate bilinear forms $E_{ij} : V_{ij} \otimes V_{ji} \to \mathbb{C}$, subject to the condition, $\sum_j \mathrm{Tr}(E_{ij}(E_{ji}^T)^{-1}) = -q - q^{-1}$, for each $i \in I$.

When q is a root of unity there is an extra condition given in [32], due to the fact that $\mathcal{C}_q$ is now a quotient of the tensor category whose objects are $V^{\otimes m}$, $m \in \mathbb{N}$.

Let Δ be the matrix given by $\Delta_{i,j} = \dim V_{ij}$ ($= \dim V_{ji}$ since E_{ij} is non-degenerate). The quantum McKay correspondence gives a graph with adjacency matrix Δ and vertex set I. The free algebra T in $\mathcal{C}_q$ generated by the self-dual object $V = L_1$ maps to the path algebra of the McKay graph under the functor $F : \mathcal{C}_q \to \mathcal{M}_{I\times I}$. Let S be the quotient of T by the two-sided ideal J generated by the image of $\mathbf{1} = L_0$ under the map $\mathbf{1} \stackrel{\mathrm{coev}_V}{\longrightarrow} V \otimes \overline{V} \stackrel{\mathrm{id}_V \otimes \phi^{-1}}{\longrightarrow} V \otimes V$, where ϕ is any choice of isomorphism from V to its conjugate representation $\overline{V}$. In the classical situation, $q = 1$, S is the algebra of polynomials in two commuting variables. More generally, S is called the q-symmetric algebra, or the algebra of functions on the quantum plane. The structure of these algebras is well known, see for example [67]. Applying the functor F to S gives an algebra $\widetilde{\Pi} = F(S)$ which is the quotient of the path algebra with respect to the two-sided ideal $F(J)$. Then given any arbitrary connected graph $\mathcal{G}$, there exists a particular value of q and choice of $\mathcal{C}_q$-module category $\mathcal{D}$ such that $\widetilde{\Pi}$ is equal to the pre-projective algebra Π of $\mathcal{G}$ [83, Lemma 2.2].

When q is not a root of unity, the m^{th} graded component of the q-symmetric algebra S is given by $S(m) = L_m$, for $m \in \mathbb{N}$. Then by the fusion rules given in (7.25),

$$L_1 \otimes L_m \simeq L_{m-1} \oplus L_{m+1}. \tag{7.27}$$

Then summing (7.27) over all $m \in \mathbb{N}$, with a grading t^m, gives $tL_1 \otimes S = t^2 S \oplus S \ominus L_0$. Applying the functor F one obtains a recursion $t\Delta H(t) = H(t) + t^2 H(t) - 1$, where Δ is the adjacency matrix of the (quantum) McKay graph $\mathcal{G}$. Then we obtain the following result [83, Theorem 2.3a]:

$$H(t) = \frac{1}{1 - \Delta t + t^2}. \tag{7.28}$$

For an $ADET$ graph $\mathcal{G}$, q is an n^{th} root of unity, and $h(q) = h$ is the Coxeter number of $\mathcal{G}$. The m^{th} graded component is given by $S(m) = L_m$ for $0 \leq m \leq h-2$, and $S(m) = 0$ for $m \geq h-1$. Defining $\widehat{S} = S \ominus t^h(L_{h-2} \otimes S) \oplus t^{2h}(L_{h-2} \otimes L_{h-2} \otimes S) \ominus \cdots$, the fusion rules (7.26) give the recursion $L_1 \otimes \widehat{S}(m) \simeq \widehat{S}(m-1) \oplus \widehat{S}(m+1)$. Applying the functor F gives $1 + t^h F(L_{h-2}) + t\Delta H(t) = H(t) + t^2 H(t)$, where the matrix $P = F(L_{h-2})$. Then for the Dynkin diagrams (and the graph Tad_n), there is a 'correction' term in the numerator, so that [83, Theorem 2.3b]:

$$H(t) = \frac{1 + Pt^h}{1 - \Delta t + t^2},$$

where P is a permutation corresponding to some involution of the vertices of the graph. Since $L_{h-2} \otimes L_{h-2} \simeq L_0$, $P^2 = F(L_{h-2} \otimes L_{h-2}) = F(\mathbf{1})$ so P^2 is the identity matrix. The matrix P is an automorphism of the underlying graph [83]; for A_n, D_{2n+1}, E_6 it is the unique nontrivial involution, while for D_{2n}, E_7, E_8 (and Tad_n) it is the identity matrix, i.e. the matrix P corresponds to the Nakayama permutation π for the ADE graph [31]. A Nakayama automorphism of Π is an automorphism ν of edges for which there exists an element $\widehat{b}$ of the dual Π^* of Π such that $\widehat{b}a = \nu(a)\widehat{b}$ for all $a \in \Pi$. The Nakayama automorphism is related to the Nakayama permutation by $\nu(a) = \epsilon(a)\pi(a)$ for all edges a of the Dynkin quiver, where $\epsilon(a) \in \{\pm 1\}$. For these graphs the pre-projective algebra is finite dimensional, and we have [83, Cor. 2.4]

$$\dim \Pi = \frac{h(h+1)r}{6},$$

where r is the rank, i.e. the number of vertices of the Dynkin diagram.

For a class of graphs called star graphs, which consist of n A-tails (or rays) of lengths $p_1, \dots, p_n$ attached to a central vertex i_*, one can define the spherical subalgebra $\Pi_{i_* i_*}$ of Π to be $\Pi_{i_* i_*} = i_* \cdot \Pi \cdot i_*$, the algebra of all paths in Π beginning and ending at the central vertex i_*. One can define a grading of $\Pi_{i_* i_*}$ by setting the degree of the loops given by any edge from i_* and its reverse edge to be one. If $h(t)$ is the Hilbert series of the subalgebra $\Pi_{i_* i_*}$ with respect to the above grading and the star graph is not of ADE type, then $h(t) = (1 + t - \sum_{s=1}^{n} (t - t^{p_s})/(1 - t^{p_s}))^{-1}$. If the star graph is of DE type then $h(t) = (1 + t - \sum_{s=1}^{n} (t - t^{p_s})/(1 - t^{p_s}))^{-1}(1 + t^{h/2})$.

In the case where the star graph is the McKay graph for a subgroup $\Gamma \subset SU(2)$, the algebra $\Pi_{i_* i_*}$ has the presentation $\Pi_{i_* i_*} = \mathrm{alg}(x, y, z | x^a = y^b = z^c = x + y + z = 0)$, where a, b, c denote the lengths of the rays on the McKay graph. Here x is identified with the loop of length one (due to the grading) from i_* to the adjacent vertex along the ray of length a, and back again. Similarly y and z are identified with loops of length one from i_*. The dimension of $\Pi_{i_* i_*}$ is given by $\dim \Pi_{i_* i_*} = |\Gamma|/2$.

We now present the following result which relates these various polynomials:

Theorem 7.5.1 *Let Γ be a finite subgroup of $SU(2)$ so that $\mathcal{G}_\Gamma$ is one of the affine Dynkin diagrams, with the vertices of $\mathcal{G}_\Gamma$ labelled by the irreducible representations γ of Γ, with the distinguished vertex $*$ labelled by* id. *Let $G(z)$ be the generating series of the moments for finite subgroups of $SU(2)$ in (7.17), $\widetilde{T}$ be the generalized T series defined in Section 7.5.1, and let P_γ, F_γ be the Molien series, Kostant polynomial respectively of Γ. Then for the Hilbert series H of $\mathcal{G}_\Gamma$ as in (7.28), the following hold:*

(i) $\widetilde{T}(t^2) = H(t)$,

(ii) $H_{\gamma,\mathrm{id}}(t) = P_\gamma(t) = F_\gamma(t)$,

(iii) $T(t^2) = H_{\mathrm{id},\mathrm{id}}(t) = P_{\mathrm{id}}(t) = F_{\mathrm{id}}(t) = \frac{1}{1+t^2} G\left(\frac{t}{1+t^2}\right)$.

Proof.

(i) From (7.20) we have

$$\begin{aligned}\widetilde{T}(t^2) &= \frac{1}{1+t^2}\widetilde{f}\left(\frac{t^2}{(1+t^2)^2}\right) = \frac{1}{1+t^2}\cdot\frac{1}{1-t(1+t^2)^{-1}\Delta} = \frac{1}{1+t^2-t\Delta}\\ &= H(t).\end{aligned}$$

(ii) By [50, Cor. 2.4 (ii)], for the symmetric algebra $S = S(\overline{\mathbb{C}^2})$, $P_{\gamma_j} = P_{S,\gamma_j}$ satisfies

$$\sum_{j=0}^{s}[\Delta_\Gamma]_{ij}P_{\gamma_j}(t) = (t+t^{-1})P_{\gamma_i}(t) - t^{-1}\delta_{i,0},$$

where $\gamma_1,\ldots,\gamma_s$ are the irreducible representations associated with the vertices $1,\ldots,s$ of $\mathcal{G}_\Gamma$. Then multiplying through by t we obtain

$$\sum_{j=0}^{s}\left[\mathbf{1}-\Delta_\Gamma t+\mathbf{1}t^2\right]_{ij}P_{S,\gamma_j}(t) = \delta_{i,0}.$$

From (7.28) we see that the matrix $(\mathbf{1}-\Delta_\Gamma t+\mathbf{1}t^2)$ is invertible, and hence by the definition of matrix multiplication, we see that

$$P_\gamma(t) = \left[\left(\mathbf{1}-\Delta_\Gamma t+\mathbf{1}t^2\right)^{-1}\right]_{\gamma,\mathrm{id}},$$

which is the first equality. The second was shown in Section 7.5.3.

(iii) The first equality follows from (7.21), and the next two are immediate from (ii). For the last equality, using (7.22) we have

$$\begin{aligned}F_{\mathrm{id}}(t) &= \sum_{j=1}^{n}\frac{|\Gamma_j|}{|\Gamma|}\frac{\chi_0^*(\Gamma_j)}{1-t\chi_\rho(\Gamma_j)+t^2} &= \frac{1}{1+t^2}\sum_{j=1}^{n}\frac{|\Gamma_j|}{|\Gamma|}\frac{1}{1-\left(\frac{t}{1+t^2}\right)\chi_\rho(\Gamma_j)}\\ &= \frac{1}{1+t^2}G\left(\frac{t}{1+t^2}\right).\end{aligned}$$

□

7.6 $SU(3)$ Case

We will now consider the case of $SU(3)$. We no longer have self-adjoint operators, but are in the more general setting of normal operators, whose moments are given by (7.1). We will first consider the fixed point algebra of $\bigotimes_{\mathbb{N}} M_3$ under the action of the group $\mathbb{T}^2$ to obtain the spectral measure for the infinite graph which we call $\mathcal{A}^{(\infty)_6}$. We will then generalize the method presented in Section 7.3 to the case of $SU(3)$ graphs.

7.6.1 Spectral measure for $\mathcal{A}^{(\infty)_6}$

We first consider the fixed point algebra of $\bigotimes_{\mathbb{N}} M_3$ under the action of the group $\mathbb{T}^2$. Let ρ be the fundamental representation of $SU(3)$, so that the restriction of ρ to $\mathbb{T}^2$ is given by

$$(\rho|_{\mathbb{T}^2})(\omega_1,\omega_2) = \begin{pmatrix} \omega_1 & 0 & 0 \\ 0 & \omega_2^{-1} & 0 \\ 0 & 0 & \omega_1^{-1}\omega_2 \end{pmatrix}, \tag{7.29}$$

for $(\omega_1,\omega_2) \in \mathbb{T}^2$.

Let $\{\chi_{(\lambda_1,\lambda_2)}\}_{\lambda_1,\lambda_2\in\mathbb{N}}$, $\{\sigma_{(\lambda_1,\lambda_2)}\}_{\lambda_1,\lambda_2\in\mathbb{Z}}$ be the irreducible characters of $SU(3)$, $\mathbb{T}^2$ respectively, where if $\chi_{(\lambda_1,\lambda_2)}$ is the character of a representation π then $\chi_{(\lambda_2,\lambda_1)}$ is the character of the conjugate representation $\overline{\pi}$ of π. The trivial character of $SU(3)$ is $\chi_{(0,0)}$, $\chi_{(1,0)}$ is the character of ρ, and $\sigma_{(\lambda_1,\lambda_2)}(p,q) = (p^{\lambda_1}, q^{\lambda_2})$, for $\lambda_1,\lambda_2 \in \mathbb{Z}$. If σ is the restriction of $\chi_{(1,0)}$ to $\mathbb{T}^2$, we have $\sigma = \sigma_{(1,0)} + \sigma_{(0,-1)} + \sigma_{(-1,1)}$ (by (7.29)), and $\sigma\sigma_{(\lambda_1,\lambda_2)} = \sigma_{(\lambda_1+1,\lambda_2)} + \sigma_{(\lambda_1,\lambda_2-1)} + \sigma_{(\lambda_1-1,\lambda_2+1)}$, for any $\lambda_1,\lambda_2\in\mathbb{Z}$. So the representation graph of $\mathbb{T}^2$ is identified with the infinite graph $\mathcal{A}^{(\infty)_6}$, illustrated in Figure 7.7, whose vertices are labelled by pairs $(\lambda_1,\lambda_2)\in\mathbb{Z}^2$, and which has an edge from vertex (λ_1,λ_2) to the vertices (λ_1+1,λ_2), (λ_1,λ_2-1) and $(\lambda_1-1,\lambda_2+1)$. The 6 in the notation $\mathcal{A}^{(\infty)_6}$ is to indicate that for this graph we are taking six infinities, one in each of the directions given by $\pm e_i$, $i = 1,2,3$, where the vectors e_i are as in Section 1.3.1. We choose the distinguished vertex to be $* = (0,0)$. Hence $\left(\bigotimes_{\mathbb{N}} M_3\right)^{\mathbb{T}^2} \cong A(\mathcal{A}^{(\infty)_6})$.

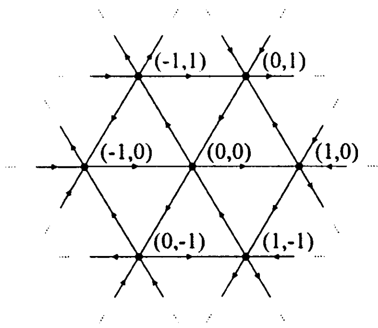


Figure 7.7: The infinite graph $\mathcal{A}^{(\infty)_6}$.

We define a normal operator v_Z in $\ell^2(\mathbb{Z})\otimes\ell^2(\mathbb{Z})$ by $v_Z = s\otimes 1 + 1\otimes s^{-1} + s^{-1}\otimes s$, where s is again the bilateral shift on $\ell^2(\mathbb{Z})$. Let $\Omega\otimes\Omega$ be the vector $(\delta_{i,0})_i\otimes(\delta_{i,0})_i$. Then v_Z is identified with the adjacency matrix Δ of $\mathcal{A}^{(\infty)_6}$, where we regard the vector $\Omega\otimes\Omega$ as corresponding to the vertex $(0,0)$ of $\mathcal{A}^{(\infty)_6}$, and the operators $s\otimes 1$, $s^{-1}\otimes s$, $1\otimes s^{-1}$ as corresponding to an edge on $\mathcal{A}^{(\infty)_6}$, in the direction of the vectors e_1, e_2, e_3

respectively. Then $(s^{\lambda_1} \otimes s^{-\lambda_2})(\Omega \otimes \Omega)$ corresponds to the vertex (λ_1, λ_2) of $\mathcal{A}^{(\infty)_6}$, for any $\lambda_1, \lambda_2 \in \mathbb{Z}$, and applying $v_Z^m v_Z^{*n}(\Omega \otimes \Omega)$ gives a vector $y = (y_{(\lambda_1,\lambda_2)})$ in $\ell^2(\mathcal{A}^{(\infty)_6})$, where $y_{(\lambda_1,\lambda_2)}$ gives the number of paths of length $m+n$ from $(0,0)$ to the vertex (λ_1, λ_2), where m edges are on $\mathcal{A}^{(\infty)_6}$ and n edges are on the reverse graph $\widetilde{\mathcal{A}^{(\infty)_6}}$. The relation $(1 \otimes s^{-1})(s^{-1} \otimes s)(s \otimes 1) = s^{-1}s \otimes s^{-1}s = 1 \otimes 1$ corresponds to the fact that traveling along edges in directions e_1 followed by e_2 and then e_3 forms a closed loop, and similarly for any permutations of $1 \otimes s^{-1}$, $s^{-1} \otimes s$, $s \otimes 1$.

Define a state φ on $C^*(v_Z)$ by $\varphi(\,\cdot\,) = \langle \Omega \otimes \Omega, \,\cdot\,(\Omega \otimes \Omega)\rangle$. When $m \not\equiv n \bmod 3$ it is impossible for there to be a closed loop of length $m+n$ beginning and ending at the vertex $(0,0)$, with the first m edges are on $\mathcal{A}^{(\infty)_6}$ and the next n edges are on the reverse graph $\widetilde{\mathcal{A}^{(\infty)_6}}$. Hence $\varphi(v_Z^m v_Z^{*n}) = 0$ for $m \not\equiv n \bmod 3$. We use the notation $(a,b,c)!$ to denote the multinomial coefficient $(a+b+c)!/(a!b!c!)$. For $m \equiv n \bmod 3$, we have

$$\begin{aligned}
\varphi(v_Z^m v_Z^{*n}) &= \sum_{\substack{0 \le k_1+k_2 \le m \\ 0 \le l_1+l_2 \le n}} (k_1, k_2, m-k_1-k_2)!(l_1, l_2, n-l_1-l_2)!\, \varphi(s^{r_1} \otimes s^{r_2}) \\
&= \sum_{\substack{0 \le k_1+k_2 \le m \\ 0 \le l_1+l_2 \le n}} (k_1, k_2, m-k_1-k_2)!(l_1, l_2, n-l_1-l_2)!\, \delta_{r_1,0}\, \delta_{r_2,0},
\end{aligned}$$

where

$$r_1 = 2k_1 + k_2 - 2l_1 - l_2 + n - m, \qquad r_2 = 2l_2 + l_1 - 2k_2 - k_1 + m - n. \tag{7.30}$$

Then we get a non-zero contribution when $l_1 = k_1 + r$, $l_2 = k_2 + r$, where $n = m + 3r$, $r \in \mathbb{Z}$. So we obtain

$$\varphi(v_Z^m v_Z^{*n}) = \sum_{k_1,k_2} (k_1, k_2, m-k_1-k_2)!(k_1+r, k_2+r, m+r-k_1-k_2)! \tag{7.31}$$

where the summation is over all non-negative integers k_1, k_2 such that $\max(0,-r) \le k_1, k_2 \le \min(m, m+2r)$ and $k_1 + k_2 \le \min(m, m+r)$.

Proposition 7.6.1 *The dimension of the m^{th} level of the path algebra for the infinite graph $\mathcal{A}^{(\infty)_6}$ is given by*

$$\dim\left((\otimes^m M_3)^{\mathbb{T}^2}\right) = \dim(A(\mathcal{A}^{(\infty)_6})_m) = \sum_{j=0}^{m} C_j^{2j} (C_j^m)^2.$$

Proof

When $m = n$ we have

$$\varphi(v_Z^m v_Z^{*m}) = \sum_{0 \le k_1+k_2 \le m} ((k_1, k_2, m-k_1-k_2)!)^2$$

$$= \sum_{k_1=0}^{m} \sum_{k_2=0}^{m-k_1} \left(\frac{m!}{k_1!k_2!(m-k_1-k_2)!} \right)^2$$

$$= \sum_{k_1=0}^{m} \left(\frac{m!}{k_1!(m-k_1)!} \right)^2 \sum_{k_2=0}^{m-k_1} \left(\frac{(m-k_1)!}{k_2!(m-k_1-k_2)!} \right)^2$$

$$= \sum_{k_1=0}^{m} (C_{k_1}^{m})^2 \sum_{k_2=0}^{m-k_1} (C_{k_2}^{m-k_1})^2 = \sum_{k_1=0}^{m} (C_{k_1}^{m})^2 C_{m-k_1}^{2(m-k_1)},$$

where we have used (7.4) in the last equality. □

Since the spectrum $\sigma(s)$ of s is $\mathbb{T}$, the spectrum $\sigma(v_Z)$ of v_Z is the set $\mathfrak{D} = \{\omega_1 + \omega_2^{-1} + \omega_1^{-1}\omega_2 | \, \omega_1, \omega_2 \in \mathbb{T}\}$, the closure of the interior of a three-cusp hypocycloid called a deltoid; see Figure 7.8. Any point in $\mathfrak{D}$ can be parameterized by

$$x = r(2\cos(2\pi t) + cos(4\pi t)), \qquad y = r(2\sin(2\pi t) - sin(4\pi t)), \tag{7.32}$$

where $0 \leq r \leq 1$, $0 \leq t < 1$, with $r = 1$ corresponding to the boundary of $\mathfrak{D}$.

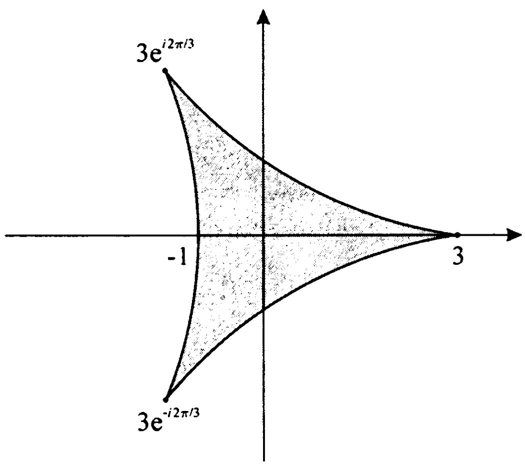


Figure 7.8: The set $\mathfrak{D}$, the closure of the interior of a deltoid.

Thus the support of the probability measure μ_{v_Z} is contained in $\mathfrak{D}$. There is a map $\Phi : \mathbb{T}^2 \to \mathfrak{D}$ from the torus to $\mathfrak{D}$ given by

$$\Phi(\omega_1, \omega_2) = \omega_1 + \omega_2^{-1} + \omega_1^{-1}\omega_2, \tag{7.33}$$

where $\omega_1, \omega_2 \in \mathbb{T}$.

Let G denote the subgroup of $GL(2, \mathbb{Z})$ generated by the matrices T_2, T_3, of orders 2, 3 respectively, given by

$$T_3 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \qquad T_2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \tag{7.34}$$

which is isomorphic to the permutation group S_3. Then $\Phi(\omega_1,\omega_2)$ is invariant under the action of G given by $T(\omega_1,\omega_2) = (\omega_1^{a_{11}}\omega_2^{a_{12}}, \omega_1^{a_{21}}\omega_2^{a_{22}})$, for $T = (a_{ij}) \in G$, i.e.

$$\Phi(\omega_1,\omega_2) = \Phi(\omega_1,\omega_1\omega_2^{-1}) = \Phi(\omega_2^{-1},\omega_1\omega_2^{-1}) = \Phi(\omega_2^{-1},\omega_1^{-1}) = \Phi(\omega_1^{-1}\omega_2,\omega_1^{-1}) = \Phi(\omega_1^{-1}\omega_2,\omega_2).$$

Any G-invariant probability measure ε on $\mathbb{T}^2$ produces a probability measure μ on $\mathfrak{D}$ by means of the map Φ:

$$\int_{\mathfrak{D}} \psi(z)d\mu(z) = \int_{\mathbb{T}^2} \psi(\omega_1 + \omega_2^{-1} + \omega_1^{-1}\omega_2)d\varepsilon(\omega_1,\omega_2),$$

for any continuous function $\psi : \mathfrak{D} \to \mathbb{C}$, where $d\varepsilon(\omega_1,\omega_2) = d\varepsilon(g(\omega_1,\omega_2))$ for all $g \in G$.

Theorem 7.6.2 *The spectral measure $d\varepsilon(\omega_1,\omega_2)$ (on $\mathbb{T}^2$) for the graph $\mathcal{A}^{(\infty)6}$ is given by the uniform Lebesgue measure*

$$d\varepsilon(\omega_1,\omega_2) = d\omega_1\, d\omega_2. \tag{7.35}$$

Proof

With this measure we have

$$\begin{aligned}
&\int_{\mathbb{T}^2} (\omega_1 + \omega_2^{-1} + \omega_1^{-1}\omega_2)^m(\omega_1^{-1} + \omega_2 + \omega_1\omega_2^{-1})^n d\omega_1\, d\omega_2 \\
&= \sum_{\substack{0\leq k_1+k_2\leq m \\ 0\leq l_1+l_2\leq n}} \left((k_1,k_2,m-k_1-k_2)!(l_1,l_2,n-l_1-l_2)! \int_{\mathbb{T}^2} \omega_1^{r_1}\omega_2^{r_2} d\omega_1\, d\omega_2\right) \\
&= \sum_{\substack{0\leq k_1+k_2\leq m \\ 0\leq l_1+l_2\leq n}} (k_1,k_2,m-k_1-k_2)!(l_1,l_2,n-l_1-l_2)!\, \delta_{r_1,0}\, \delta_{r_2,0},
\end{aligned}$$

where r_1, r_2 are as in (7.30). This is equal to $\varphi(v_Z^m v_Z^{*n})$ given in (7.31). □

The quotient $\mathbb{T}^2/\mathbb{Z}_3$, where the $\mathbb{Z}_3$ action is given by left multiplication by T_3 is a two-sphere $\mathbb{S}^2$ with three singular points corresponding to the points $(1,1)$, $(e^{2\pi i/3}, e^{4\pi i/3})$, $(e^{4\pi i/3}, e^{2\pi i/3})$ in $\mathbb{T}^2$ [42]. Under the $\mathbb{Z}^2$ action given by left multiplication by T_2 on this two-sphere, we obtain a disc with three singular points, which is topologically equal to the deltoid $\mathfrak{D}$. The boundaries of the deltoid $\mathfrak{D}$ are given by the lines $\theta_1 = 1 - \theta_2$, $\theta_1 = 2\theta_2$ and $2\theta_1 = \theta_2$. The diagonal $\theta_1 = \theta_2$ in $\mathbb{T}^2$ is mapped to the real interval $[-1,3] \subset \mathfrak{D}$. The mapping of the 'horizontal' lines on $\mathbb{T}^2$ between points $(e^{2\pi i m/12}, e^{2\pi i n/12})$ and $(e^{2\pi i(m+1)/12}, e^{2\pi i n/12})$, and the 'vertical' lines on $\mathbb{T}^2$ between points $(e^{2\pi i m/12}, e^{2\pi i n/12})$ and $(e^{2\pi i m/12}, e^{2\pi i(n+1)/12})$, onto $\mathfrak{D}$, for $0 \leq m,n \leq 11$, is illustrated in Figure 7.9.

Thus the quotient $\mathbb{T}^2/G$ is topologically equal to the set $\mathfrak{D}$. A fundamental domain C of $\mathbb{T}^2$ under the action of the group G is illustrated in Figure 7.10, where the axes are

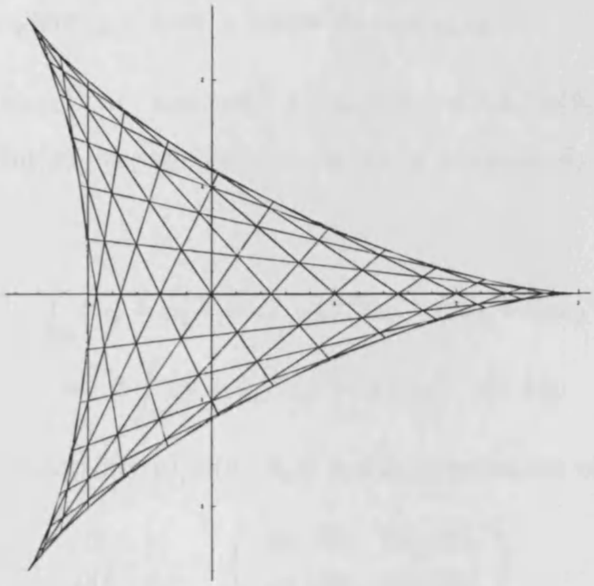


Figure 7.9: Mapping $\mathbb{T}^2$ onto the deltoid $\mathfrak{D}$.

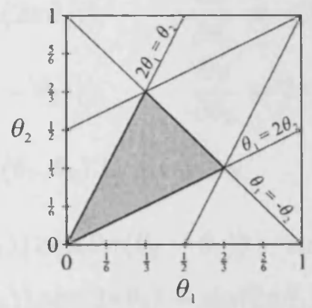


Figure 7.10: A fundamental domain C of $\mathbb{T}^2/G$.

labelled by the parameters θ_1, θ_2 in $(e^{2\pi i\theta_1}, e^{2\pi i\theta_2}) \in \mathbb{T}^2$. The boundaries of C map to the boundaries of the deltoid $\mathfrak{D}$. The torus $\mathbb{T}^2$ contains six copies of C.

We will now determine the spectral measure μ_{v_Z} over $\mathfrak{D}$. Now

$$\begin{aligned}
&\int_{\mathbb{T}^2} (\omega_1 + \omega_2^{-1} + \omega_1^{-1}\omega_2)^m (\omega_1^{-1} + \omega_2 + \omega_1\omega_2^{-1})^n d\omega_1\, d\omega_2 \\
&\quad = 6 \int_C (\omega_1 + \omega_2^{-1} + \omega_1^{-1}\omega_2)^m (\omega_1^{-1} + \omega_2 + \omega_1\omega_2^{-1})^n d\omega_1\, d\omega_2 \\
&\quad = 6 \int (e^{2\pi i\theta_1} + e^{-2\pi i\theta_2} + e^{2\pi i(\theta_2-\theta_1)})^m (e^{-2\pi i\theta_1} + e^{2\pi i\theta_2} + e^{2\pi i(\theta_1-\theta_2)})^n d\theta_1\, d\theta_2,
\end{aligned}$$

where the last integral is over the values of θ_1, θ_2 such that $(e^{2\pi i\theta_1}, e^{2\pi i\theta_2}) \in C$. Under the change of variable $z = e^{2\pi i\theta_1} + e^{-2\pi i\theta_2} + e^{2\pi i(\theta_2-\theta_1)}$, we have

$$\begin{aligned} x &:= \mathrm{Re}(z) &=& \cos(2\pi\theta_1) + \cos(2\pi\theta_2) + \cos(2\pi(\theta_2 - \theta_1)), \\ y &:= \mathrm{Im}(z) &=& \sin(2\pi\theta_1) - \sin(2\pi\theta_2) + \sin(2\pi(\theta_2 - \theta_1)). \end{aligned}$$

Then

$$\begin{aligned} &\int_{\mathbb{T}^2} (\omega_1 + \omega_2^{-1} + \omega_1^{-1}\omega_2)^m (\omega_1^{-1} + \omega_2 + \omega_1\omega_2^{-1})^n d\omega_1\, d\omega_2 \\ &= 6\int_{\mathfrak{D}} (x+iy)^m (x+iy)^n |J^{-1}| dx\, dy, \end{aligned} \tag{7.36}$$

where the Jacobian $J = \det(\partial(x,y)/\partial(\theta_1,\theta_2))$ is the determinant of the Jacobian matrix

$$\frac{\partial(x,y)}{\partial(\theta_1,\theta_2)} = \begin{pmatrix} \partial x/\partial\theta_1 & \partial x/\partial\theta_2 \\ \partial y/\partial\theta_1 & \partial y/\partial\theta_2 \end{pmatrix}.$$

Computing the partial derivatives:

$$\frac{\partial x}{\partial\theta_1} = 2\pi(\sin(2\pi(\theta_2-\theta_1)) - \sin(2\pi\theta_1)), \qquad \frac{\partial x}{\partial\theta_2} = -2\pi(\sin(2\pi\theta_2) + \sin(2\pi(\theta_2-\theta_1))),$$

$$\frac{\partial y}{\partial\theta_1} = 2\pi(\cos(2\pi\theta_1) + \cos(2\pi(\theta_2-\theta_1))), \qquad \frac{\partial y}{\partial\theta_2} = 2\pi(\cos(2\pi(\theta_2-\theta_1)) - \cos(2\pi\theta_2)),$$

we find that the Jacobian $J = J(\theta_1,\theta_2)$ is given by

$$\begin{aligned} J(\theta_1,\theta_2) &= 4\pi^2(\sin(2\pi(\theta_2-\theta_1))\cos(2\pi(\theta_2-\theta_1)) - \sin(2\pi\theta_1)\cos(2\pi(\theta_2-\theta_1)) \\ &\quad - \sin(2\pi(\theta_2-\theta_1))\cos(2\pi\theta_2) + \sin(2\pi\theta_1)\cos(2\pi\theta_2) + \sin(2\pi\theta_2)\cos(2\pi\theta_1) \\ &\quad - \sin(2\pi\theta_2)\cos(2\pi(\theta_2-\theta_1)) + \sin(2\pi(\theta_2-\theta_1))\cos(2\pi\theta_1) \\ &\quad - \sin(2\pi(\theta_2-\theta_1))\cos(2\pi(\theta_2-\theta_1))) \\ &= 4\pi^2(\sin(2\pi(\theta_1+\theta_2)) - \sin(2\pi(2\theta_1-\theta_2)) - \sin(2\pi(2\theta_2-\theta_1))). \end{aligned} \tag{7.37}$$

The Jacobian is real and vanishes on the boundary of the deltoid $\mathfrak{D}$. For the values of θ_1, θ_2 such that $(e^{2\pi i\theta_1}, e^{2\pi i\theta_2})$ are in the interior of the fundamental domain C illustrated in Figure 7.10, the value of J is always negative. In fact, restricting to any one of the fundamental domains shown in Figure 7.10, the sign of J is constant. It is negative over three of the fundamental domains, and positive over the remaining three. The Jacobian $J(\theta_1,\theta_2)$ is illustrated in Figure 7.11. When evaluating J at a point in $z \in \mathfrak{D}$, we pull back z to $\mathbb{T}^2$. However, there are six possibilities for $(\omega_1,\omega_2) \in \mathbb{T}^2$ such that $\Phi(\omega_1,\omega_2) = z$, one in each of the fundamental domains of $\mathbb{T}^2$ in Figure 7.10. Thus over $\mathfrak{D}$, J is only determined up to a sign. To obtain a positive measure over $\mathfrak{D}$ we take the absolute value $|J|$ of the Jacobian in the integral (7.36).

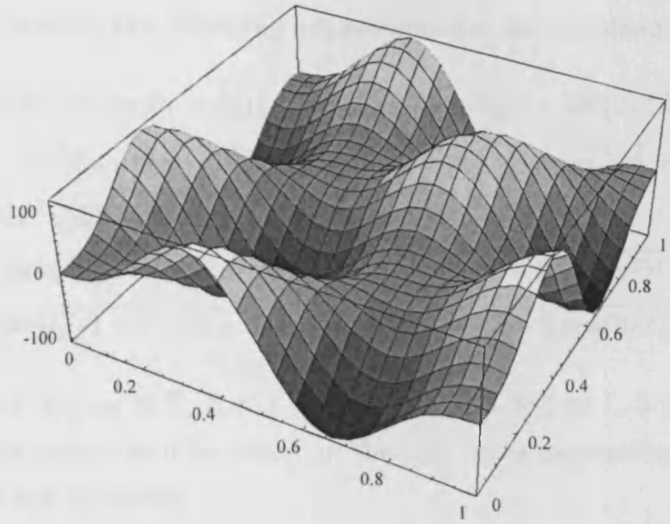


Figure 7.11: The Jacobian J.

Writing $\omega_j = e^{2\pi i\theta_j}$, $j = 1, 2$, J is given in terms of $\omega_1, \omega_2 \in \mathbb{T}$ by,

$$\begin{aligned} J(\omega_1, \omega_2) &= 4\pi^2 \mathrm{Im}(\omega_1\omega_2 - \omega_1^2\omega_2^{-1} - \omega_1^{-1}\omega_2^2) \\ &= -2\pi^2 i(\omega_1\omega_2 - \omega_1^{-1}\omega_2^{-1} - \omega_1^2\omega_2^{-1} + \omega_1^{-2}\omega_2 - \omega_1^{-1}\omega_2^2 + \omega_1\omega_2^{-2}). \end{aligned} \tag{7.38}$$

Since

$$\begin{aligned} &(\omega_1\omega_2 - \omega_1^{-1}\omega_2^{-1} - \omega_1^2\omega_2^{-1} + \omega_1^{-2}\omega_2 - \omega_1^{-1}\omega_2^2 + \omega_1\omega_2^{-2})^2 \\ &\quad = -6 + 2(\omega_1\omega_2 + \omega_1^{-1}\omega_2^{-1} + \omega_1\omega_2^{-2} + \omega_1^2\omega_2^{-1} + \omega_1^{-1}\omega_2^2 + \omega_1^{-2}\omega_2) \\ &\qquad -2(\omega_1^3 + \omega_1^{-3} + \omega_2^3 + \omega_2^{-3} + \omega_1^3\omega_2^{-3} + \omega_1^{-3}\omega_2^3) \\ &\qquad +(\omega_1^2\omega_2^2 + \omega_1^{-2}\omega_2^{-2} + \omega_1^2\omega_2^{-4} + \omega_1^4\omega_2^{-2} + \omega_1^{-2}\omega_2^4 + \omega_1^{-4}\omega_2^2), \end{aligned}$$

the square of the Jacobian is invariant under the action of G. Since z, $\overline{z}$ are also invariant under G, J^2 can be written in terms of z, $\overline{z}$, and we obtain $J(z,\overline{z})^2 = 4\pi^4(27 - 18z\overline{z} + 4z^3 + 4\overline{z}^3 - z^2\overline{z}^2)$ for $z \in \mathfrak{D}$. Since J is real, $J^2 \geq 0$. Then

$$\begin{aligned} |J| &= 2\pi^2\sqrt{27 - 18z\overline{z} + 4z^3 + 4\overline{z}^3 - z^2\overline{z}^2} \\ &= 2\pi^2\sqrt{27 - 18(x^2+y^2) + 8x(x^2 - 3y^2) - (x^2+y^2)^2}, \end{aligned} \tag{7.39}$$

where the expression under the square root is always real and non-negative since J^2 is, and by $\sqrt{\cdot}$ we mean the positive square root. Alternatively, in terms of the parameters r, t given in (7.32) we can write $|J|$ as

$$|J(r,t)| = 2\pi^2\sqrt{(1-r)((5 + 4\cos(6\pi t))^2 r^3 - 9(7 + 8\cos(6\pi t))r^2 + 27r + 27)}. \tag{7.40}$$

We have thus obtained the following expressions for the Jacobian J:

$$\begin{aligned}
J(\theta_1,\theta_2) &= 4\pi^2(\sin(2\pi(\theta_1+\theta_2)) - \sin(2\pi(2\theta_1-\theta_2)) - \sin(2\pi(2\theta_2-\theta_1))), \\
J(\omega_1,\omega_2) &= -2\pi^2 i(\omega_1\omega_2 - \omega_1^{-1}\omega_2^{-1} - \omega_1^2\omega_2^{-1} + \omega_1^{-2}\omega_2 - \omega_1^{-1}\omega_2^2 + \omega_1\omega_2^{-2}), \\
|J(z,\overline{z})| &= 2\pi^2\sqrt{27 - 18z\overline{z} + 4z^3 + 4\overline{z}^3 - z^2\overline{z}^2}, \\
|J(x,y)| &= 2\pi^2\sqrt{27 - 18(x^2+y^2) + 8x(x^2-3y^2) - (x^2+y^2)^2}, \\
|J(r,t)| &= 2\pi^2\sqrt{(1-r)((5+4\cos(6\pi t))^2r^3 - 9(7+8\cos(6\pi t))r^2 + 27r + 27)},
\end{aligned}$$

where $0 \leq \theta_1, \theta_2 < 1$, $\omega_1, \omega_2 \in \mathbb{T}$, $z = x + iy \in \mathfrak{D}$ and $0 \leq r \leq 1$, $0 \leq t < 1$. Again, the positive value of the square root is meant in the last three expressions.

We have shown the following:

Theorem 7.6.3 *The spectral measure μ_{v_Z} (over $\mathfrak{D}$) for the graph $\mathcal{A}^{(\infty)_6}$ is*

$$d\mu_{v_Z}(z) = \frac{6}{|J|}\,dz = \frac{3}{\pi^2\sqrt{27 - 18z\overline{z} + 4z^3 + 4\overline{z}^3 - z^2\overline{z}^2}}\,dz, \tag{7.41}$$

where $dz := d\mathrm{Re}z\, d\mathrm{Im}z$ denotes the Lebesgue measure on $\mathbb{C}$.

To summarize the situation for the fixed point algebra under the action of $\mathbb{T}^2$, we have obtained the following identifications

$$\begin{aligned}
\dim(A(\mathcal{A}^{(\infty)_6})_k) &= \dim\left(\left(\otimes^k M_3\right)^{\mathbb{T}^2}\right) = \sum_{j=0}^{k} C_j^{2j}(C_j^k)^2 = \varphi(|v_Z|^{2k}) \\
&= \frac{3}{\pi^2}\int_{\mathfrak{D}} |z|^{2k}\frac{1}{\sqrt{27 - 18z\overline{z} + 4z^3 + 4\overline{z}^3 - z^2\overline{z}^2}}\,dz.
\end{aligned}$$

7.6.2 Spectral measure for $\mathcal{A}^{(\infty)}$

We now consider the fixed point algebra under the action of the group $SU(3)$. The characters of $SU(3)$ satisfy $\chi_{(1,0)}\chi_{(\lambda_1,\lambda_2)} = \chi_{(\lambda_1+1,\lambda_2)} + \chi_{(\lambda_1,\lambda_2-1)} + \chi_{(\lambda_1-1,\lambda_2+1)}$, for any $\lambda_1, \lambda_2 \geq 0$, where $\chi_{(\lambda,-1)} = 0$ for all $\lambda \geq 0$. So the representation graph of $SU(3)$ is identified with the infinite graph $\mathcal{A}^{(\infty)}$ illustrated in Figure 1.7, with distinguished vertex $* = (0,0)$. Hence $(\bigotimes_{\mathbb{Z}} M_3)^{SU(3)} \cong A(\mathcal{A}^{(\infty)})$.

Let us define a normal operator v_N on $\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$ by

$$v_N = l \otimes 1 + 1 \otimes l^* + l^* \otimes l, \tag{7.42}$$

where l is again the unilateral shift on $\ell^2(\mathbb{N})$. If we regard the element $\Omega \otimes \Omega$ as corresponding to the apex vertex $(0,0)$, and the operators $l \otimes 1$, $l^* \otimes l$, $1 \otimes l^*$ as corresponding to the vectors e_1, e_2, e_3 on $\mathcal{A}^{(\infty)}$, then $(l^{\lambda_1} \otimes (l^*)^{\lambda_2})(\Omega \otimes \Omega)$ corresponds to the vertex (λ_1, λ_2) of $\mathcal{A}^{(\infty)}$, for $\lambda_1, \lambda_2 \geq 0$. We see that v_N is identified with the adjacency matrix $\Delta_{\mathcal{A}}$ of $\mathcal{A}^{(\infty)}$, and $v_N^m v_N^{*n}(\Omega \otimes \Omega)$ gives a vector $y = (y_{(\lambda_1,\lambda_2)})$ in $\ell^2(\mathcal{A}^{(\infty)})$, where $y_{(\lambda_1,\lambda_2)}$ gives the number of paths of length $m+n$ from $(0,0)$ to the vertex (λ_1, λ_2), where m edges are on $\mathcal{A}^{(\infty)}$ and n edges are on the reverse graph $\widetilde{\mathcal{A}}^{(\infty)}$. The relation $(l^* \otimes \cdot\,)(\Omega \otimes \cdot\,) = 0$ corresponds to the fact that there are no edges in the direction $-e_1$ from a vertex $(0, \lambda_2)$ on the boundary of $\mathcal{A}^{(\infty)}$, $\lambda_2 \geq 0$, and similarly $(\,\cdot \otimes l^*)(\,\cdot \otimes \Omega) = 0$ corresponds to there being no edges in the direction e_3 from a vertex $(\lambda_1, 0)$, $\lambda_1 \geq 0$. The relation $(1 \otimes l^*)(l^* \otimes l)(l \otimes 1) = l^*l \otimes l^*l = 1 \otimes 1$ again corresponds to the fact that travelling along edges in directions e_1 followed by e_2 and then e_3 forms a closed loop, and similarly for any permutations of $1 \otimes l^*$, $l^* \otimes l$, $l \otimes 1$, but now the product will be 0 along one of the boundaries $\lambda_1 = 0$ or $\lambda_2 = 0$ for certain of the permutations, but 1 everywhere else.

The vector $\Omega \otimes \Omega$ is cyclic in $\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$. We can show this by induction. Suppose any vector $l^{k_1}\Omega \otimes l^{k_2}\Omega \in \ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$, such that $k_1 + k_2 \leq p$, can be written as a linear combination of elements of the form $v_N^m v_N^{*n}(\Omega \otimes \Omega)$ where $m+n \leq p$. This is certainly true when $p = 1$ since $v_N(\Omega \otimes \Omega) = (l \otimes 1 + 1 \otimes l^* + l^* \otimes l)(\Omega \otimes \Omega) = l\Omega \otimes \Omega$ and $v_N^*(\Omega \otimes \Omega) = \Omega \otimes l\Omega$. For $j = 0, 1, \ldots, p$, we have $v_N(l^{p-j}\Omega \otimes l^j\Omega) = l^{p-j+1}\Omega \otimes l^j\Omega + l^{p-j}\Omega \otimes l^{j-1}\Omega + l^{p-j-1}\Omega \otimes l^{j+1}\Omega$. Then $l^{p-j+1}\Omega \otimes l^j\Omega = v_N(l^{p-j}\Omega \otimes l^j\Omega) - l^{p-j}\Omega \otimes l^{j-1}\Omega - l^{p-j-1}\Omega \otimes l^{j+1}\Omega$, and $l^{p-j+1}\Omega \otimes l^j\Omega$, for $j = 0, 1, \ldots, p$, can be written as a linear combination of elements of the form $v_N^m v_N^{*n}(\Omega \otimes \Omega)$ where $m+n \leq p+1$. Since also $\Omega \otimes l^{p+1}\Omega = v_N^*(\Omega \otimes l^p\Omega) - l\Omega \otimes l^{p-1}\Omega$, then every $l^{k_1}\Omega \otimes l^{k_2}\Omega$, such that $k_1 + k_2 \leq p+1$, can be written as a linear combination of elements of the form $v_N^m v_N^{*n}(\Omega \otimes \Omega)$ where $m+n \leq p+1$. Then $\overline{C^*(v_N)(\Omega \otimes \Omega)} = \ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$.

We define a state φ on $C^*(v_N)$ by $\varphi(\,\cdot\,) = \langle \Omega \otimes \Omega, \cdot\,(\Omega \otimes \Omega) \rangle$. Suppose $\varphi(x^*x) = 0$ for some $x \in C^*(v_N)$. Then $x(\Omega \otimes \Omega) = 0$. Since v_N is normal, $C^*(v_N)$ is abelian, and hence $xC^*(v_N)(\Omega \otimes \Omega) = C^*(v_N)x(\Omega \otimes \Omega) = 0$, so that $xy(\Omega \otimes \Omega) = 0$ for all $y \in C^*(v_N)$. Then $x(\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})) = 0$ since $\Omega \otimes \Omega$ is cyclic, giving $x = 0$. Then φ is faithful.

The moments $\varphi(v_N^m v_N^{*n})$ are all zero if $m - n \not\equiv 0 \bmod 3$, and for $m \equiv n \bmod 3$, the first few moments are given by

$$\begin{aligned}
\varphi(1) &= 1, \\
\varphi(v_N v_N^*) &= \varphi(ll^* \otimes 1) + \varphi(l \otimes l) + \varphi(l^2 \otimes l^*) + \varphi(l^* \otimes l^*) + \varphi(1 \otimes l^*l) + \varphi(l \otimes (l^*)^2) \\
&\quad + \varphi((l^*)^2 \otimes l) + \varphi(l^* \otimes l^2) + \varphi(l^*l \otimes ll^*) \;=\; \varphi(1 \otimes l^*l) \;=\; 1, \\
\varphi(v_N^3) &= \varphi(l^*l \otimes l^*l) \;=\; 1, \\
\varphi(v_N^{*3}) &= \varphi(l^*l \otimes l^*l) \;=\; 1,
\end{aligned}$$

$$
\begin{aligned}
\varphi(v_N^2 v_N^{*2}) &= \varphi(1\otimes l^*l^*ll)+\varphi(l^*l\otimes 1) = 2,\\
\varphi(v_N^4 v_N^{*}) &= \varphi(v_N v_N^{*4}) = 3,\\
\varphi(v_N^6) &= \varphi(v_N^{*6}) = 5,\\
\varphi(v_N^3 v_N^{*3}) &= 6.
\end{aligned}
$$

The moments $\varphi(v_N^m v_N^{*n})$ count the number of paths of length $m+n$ on the $SU(3)$ graph $\mathcal{A}^{(\infty)}$, starting from the apex vertex $(0,0)$, with the first m edges on $\mathcal{A}^{(\infty)}$ and the other n edges on the reverse graph $\widetilde{\mathcal{A}}^{(\infty)}$. Let $A'(\mathcal{A}^{(\infty)})_{m,n}$ be the algebra of all pairs (η_1,η_2) of paths from $(0,0)$ such that $r(\eta_1)=r(\eta_2)$, $|\eta_1|=m$ and $|\eta_2|=n$. Then we define the general path algebra $A'(\mathcal{A}^{(\infty)})$ for the graph $\mathcal{A}^{(\infty)}$ to be $A'(\mathcal{A}^{(\infty)}) = \bigoplus_{m,n} A'(\mathcal{A}^{(\infty)})_{m,n}$. Then $\varphi(v_N^m v_N^{*n})$ gives the dimension of the m,n^{th} level $A'(\mathcal{A}^{(\infty)})_{m,n}$ of the general path algebra $A'(\mathcal{A}^{(\infty)})$. In particular, $\varphi(v_N^m v_N^{*m})$ for $m=n$ gives the dimension of the m^{th} level of the path algebra for graph $\mathcal{A}^{(\infty)}$, i.e. $\varphi(v_N^m v_N^{*m}) = \dim(A(\mathcal{A}^{(\infty)})_m)$.

The moments $\varphi(v_N^m v_N^{*n})$ have a realization in terms of a higher dimensional analogue of Catalan paths: Let $E=\{f_1,f_2,f_3\}$ be the set of vectors $f_1=(1,1,0)$, $f_2=(1,-1,1)$, $f_3=(1,0,-1)\in\mathbb{Z}^3$, which are illustrated in Figure 7.12. These vectors correspond to the vectors e_i above, $i=1,2,3$.

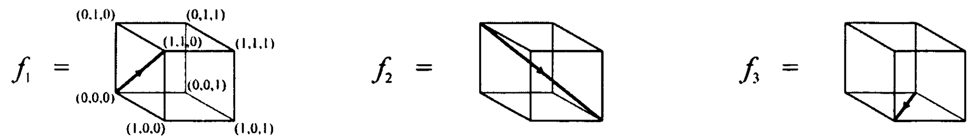


Figure 7.12: The vectors $f_i\in\mathbb{Z}^3$, $i=1,2,3$.

We define the conjugate $\overline{f}$ of a vector $f\in E$ by $\overline{(1,y,z)}=(1,-y,-z)$, and let $\overline{E}=\{\overline{f}_1,\overline{f}_2,\overline{f}_3\}$. Let L be the sublattice of $\mathbb{Z}^3$ given by all points with non-negative co-ordinates. Then define $c_{m,n}$ to be the number of paths of length $m+n$ in L, starting from $(0,0,0)$ and ending at $(m+n,0,0)$, where m edges are of the form of a vector from E and n edges are of the form of a vector from $\overline{E}$. Then $\varphi(v_N^m v_N^{*n})=c_{m,n}$, and for $m=n$, $\varphi(v_N^m v_N^{*m}) = c_{m,m} = \dim(A(\mathcal{A}^{(\infty)})_m)$.

We now consider the probability measure μ_{v_N} on $\mathfrak{D}$ for the normal element v_N. Since φ is a faithful state, by [111, Remark 2.3.2] the support of μ_{v_N} is equal to the spectrum $\sigma(v_N)$ of v_N. Consider the exact sequence $0\to\mathcal{K}\to C^*(v_N)\to C^*(v_N)/\mathcal{K}\to 0$, where $\mathcal{K}=\mathcal{K}(\ell^2(\mathbb{N})\otimes\ell^2(\mathbb{N}))\subset B(\ell^2(\mathbb{N})\otimes\ell^2(\mathbb{N}))$ are the compact operators. Let $\pi : B(\ell^2(\mathbb{N})\otimes\ell^2(\mathbb{N}))\to B(\ell^2(\mathbb{N})\otimes\ell^2(\mathbb{N}))/\mathcal{K}$ be the quotient map. The resolvent $\rho(v_N)$ of v_N is a subset of $\rho(\pi(v_N))$ since for any $\lambda\in\rho(\Delta)$, applying π to $(v_N-\lambda)b=1$, for some

$b \in B(\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N}))$, gives $(\pi(v_N) - \lambda)\pi(b) = 1$. Then $\sigma(v_N) \supset \sigma(\pi(v_N))$. Now $\pi(v_N) = u \otimes 1 + 1 \otimes u^* + u^* \otimes u$ where u is a unitary which has spectrum $\mathbb{T}$, so that the spectrum of $\pi(v_N)$ is given by $\sigma(\pi(v_N)) = \{\omega_1 + \omega_2^{-1} + \omega_1^{-1}\omega_2 | \, \omega_1, \omega_2 \in \mathbb{T}\} = \mathfrak{D}$. Then $\sigma(v_N) \subset \mathfrak{D}$.

Consider the measure $d\varepsilon(\omega_1, \omega_2)$ on $\mathbb{T}^2$ given by

$$\begin{aligned} d\varepsilon(\omega_1, \omega_2) &= \frac{1}{24\pi^4} J(\omega_1, \omega_2)^2 d\omega_1 \, d\omega_2 \\ &= -\frac{1}{6}(\omega_1\omega_2 + \omega_1\omega_2^{-2} + \omega_1^{-2}\omega_2 - \omega_1^{-1}\omega_2^{-1} - \omega_1^2\omega_2^{-1} - \omega_1^{-1}\omega_2^2)^2 \, d\omega_1 \, d\omega_2 \quad (7.43) \end{aligned}$$

on $\mathbb{T}^2$, where $d\omega_j$ is the uniform Lebesgue measure on $\mathbb{T}$, $j = 1, 2$. We will prove in the next section that this is the spectral measure (over $\mathbb{T}^2$) of v_N, so that $\sigma(v_N) = \mathfrak{D}$. With this measure we have

$$\begin{aligned} &-\frac{1}{6}\int_{\mathbb{T}^2} (\omega_1 + \omega_2^{-1} + \omega_1^{-1}\omega_2)^m (\omega_1^{-1} + \omega_2 + \omega_1\omega_2^{-1})^n \\ &\qquad \times (\omega_1\omega_2 + \omega_1\omega_2^{-2} + \omega_1^{-2}\omega_2 - \omega_1^{-1}\omega_2^{-1} - \omega_1^2\omega_2^{-1} - \omega_1^{-1}\omega_2^2)^2 \, d\omega_1 \, d\omega_2 \\ &= -\frac{1}{6} \sum_{\substack{0 \le k_1 + k_2 \le m \\ 0 \le l_1 + l_2 \le n}} \Big((k_1, k_2, m - k_1 - k_2)!(l_1, l_2, n - l_1 - l_2)! \\ &\qquad \times \int_{\mathbb{T}^2} \omega_1^{r_1}\omega_2^{r_2} (\omega_1\omega_2 + \omega_1\omega_2^{-2} + \omega_1^{-2}\omega_2 - \omega_1^{-1}\omega_2^{-1} - \omega_1^2\omega_2^{-1} - \omega_1^{-1}\omega_2^2)^2 \, d\omega_1 \, d\omega_2 \Big) \\ &= -\frac{1}{6} \sum_{a_1, a_2} \sum_{\substack{0 \le k_1 + k_2 \le m \\ 0 \le l_1 + l_2 \le n}} \Big((k_1, k_2, m - k_1 - k_2)!(l_1, l_2, n - l_1 - l_2)! \, \gamma_{a_1, a_2} \\ &\qquad \times \int_{\mathbb{T}^2} \omega_1^{r_1 + a_1} \omega_1^{r_2 + a_2} \, d\omega_1 \, d\omega_2 \Big), \end{aligned}$$

where r_1, r_2 are as in (7.30), and the summation is over all integers a_1, a_2 such that $(a_1, a_2) \in \Upsilon = \{(\lambda_1, \lambda_2) | \; \lambda_1 \equiv \lambda_2 \bmod 3, |\lambda_1 + \lambda_2| \le 4, |\lambda_1| + |\lambda_2| \le 6\}$. The set Υ is the set of all pairs (a_1, a_2) of exponents of $\omega_1^{a_1}\omega_2^{a_2}$ that appear in the expansion of $(\omega_1\omega_2 + \omega_1\omega_2^{-2} + \omega_1^{-2}\omega_2 - \omega_1^{-1}\omega_2^{-1} - \omega_1^2\omega_2^{-1} - \omega_1^{-1}\omega_2^2)^2$, and the integers γ_{a_1, a_2} are the corresponding coefficients. Let $b_1 = (2a_1 + a_2)/3$ and $b_2 = (a_1 + 2a_2)/3$. The m, n^{th} moment for the measure $d\varepsilon(\omega_1, \omega_2)$ is zero if $m \not\equiv 0 \bmod 3$, and for $n = m + 3r$, $r \in \mathbb{Z}$, the m, n^{th} moment is given by

$$-\frac{1}{6} \sum_{\substack{k_1, k_2 \\ a_1, a_2}} \gamma_{a_1, a_2} (k_1, k_2, m - k_1 - k_2)! \; (k_1 + r + b_1, k_2 + r - b_2, m + r - b_1 + b_2 - k_1 - k_2)! \quad (7.44)$$

where the summation is over all $a_1, a_2 \in \mathbb{Z}$ such that $(a_1, a_2) \in \Upsilon$, and all non-negative integers k_1, k_2 such that

$$\begin{aligned} \max(0, -r - b_1) &\le k_1 \le \min(m, m + 2r - b_1) && (7.45) \\ \max(0, -r + b_2) &\le k_2 \le \min(m, m + 2r + b_2) && (7.46) \\ k_1 + k_2 &\le \min(m, m + r - b_1 + b_2). && (7.47) \end{aligned}$$

As in (7.36), under the change of variables $\omega_1+\omega_2^{-1}+\omega_1^{-1}\omega_2 = z$, the spectral measure $\mu_{v_N}(z)$ is given by

$$d\mu_{v_N}(z) = \frac{6}{|J|}\frac{1}{24\pi^4}J^2\,dz = \frac{1}{4\pi^4}|J|\,dz.$$

Then to summarize the situation for the fixed point algebra under the action of $SU(3)$, we have obtained the following identifications

$$\begin{aligned} \dim(A(\mathcal{A}^{(\infty)})_k) &= \dim\left((\otimes^k M_3)^{SU(3)}\right) = \varphi(|v_N|^{2k}) \\ &= \frac{1}{2\pi^2}\int_{\mathfrak{D}} |z|^{2k}\sqrt{27-18z\overline{z}+4z^3+4\overline{z}^3-z^2\overline{z}^2}\,dz. \end{aligned}$$

7.7 Spectral measures for $\mathcal{ADE}$ graphs via nimreps

Let $\Delta_{\mathcal{G}}$ be the adjacency matrix of a finite graph $\mathcal{G}$ with s vertices, such that $\Delta_{\mathcal{G}}$ is normal. The m,n^{th} moment $\int z^m\overline{z}^n d\mu(z)$ is given by $\langle \Delta_{\mathcal{G}}^m(\Delta_{\mathcal{G}}^*)^n e_1, e_1\rangle$, where e_1 is the basis vector in $\ell^2(\mathcal{G})$ corresponding to the distinguished vertex $*$ of $\mathcal{G}$. For convenience we will use the notation

$$R_{m,n}(\omega_1,\omega_2) := (\omega_1+\omega_2^{-1}+\omega_1^{-1}\omega_2)^m(\omega_1^{-1}+\omega_2+\omega_1\omega_2^{-1})^n, \tag{7.48}$$

so that $\int_{\mathbb{T}^2} R_{m,n}(\omega_1,\omega_2)d\varepsilon(\omega_1,\omega_2) = \int z^m\overline{z}^n d\mu(z) = \langle \Delta_{\mathcal{G}}^m(\Delta_{\mathcal{G}}^*)^n e_1, e_1\rangle$.

Let β^j be the eigenvalues of $\mathcal{G}$, with corresponding eigenvectors x^j, $j=1,\dots,s$. Then as for $SU(2)$, $\Delta_{\mathcal{G}}^m = \mathcal{U}\Lambda_{\mathcal{G}}^m(\Lambda_{\mathcal{G}}^*)^n\mathcal{U}^*$, where $\Lambda_{\mathcal{G}} = \mathrm{diag}(\beta^1,\beta^2,\dots,\beta^s)$ and $\mathcal{U} = (x^1,x^2,\dots,x^s)$, so that

$$\begin{aligned} \int_{\mathbb{T}^2} R_{m,n}(\omega_1,\omega_2)d\varepsilon(\omega_1,\omega_2) &= \langle \mathcal{U}\Lambda_{\mathcal{G}}^m(\Lambda_{\mathcal{G}}^*)^n\mathcal{U}^*e_1,e_1\rangle = \langle \Lambda_{\mathcal{G}}^m(\Lambda_{\mathcal{G}}^*)^n\mathcal{U}^*e_1,\mathcal{U}^*e_1\rangle \\ &= \sum_{j=1}^s (\beta^j)^m(\overline{\beta^j})^n|y_j|^2, \end{aligned}\tag{7.49}$$

where $y_j = x_1^j$ is the first entry of the eigenvector x^j.

For a finite $\mathcal{ADE}$ graph $\mathcal{G}$ with Coxeter exponents Exp, its eigenvalues $\beta^{(\lambda)}$ are ratios of the S-matrix given by (1.27) for $\lambda\in\mathrm{Exp}$, with corresponding eigenvectors $(\psi_a^\lambda)_{a\in\mathfrak{V}(\mathcal{G})}$. Then (7.49) becomes

$$\int_{\mathbb{T}^2} R_{m,n}(\omega_1,\omega_2)d\varepsilon(\omega_1,\omega_2) = \sum_{\lambda\in\mathrm{Exp}} (\beta^{(\lambda)})^m(\overline{\beta^{(\lambda)}})^n|\psi_*^\lambda|^2, \tag{7.50}$$

where $*$ is the distinguished vertex of $\mathcal{G}$ with lowest Perron-Frobenius weight.

Before we compute the spectral measure for the graphs $\mathcal{A}^{(l)}$, $\mathcal{A}^{(l)*}$, $l\geq 4$, and $\mathcal{D}^{(3k)}$, $k\geq 2$, we briefly remark about the relation between the generalized T-series defined in (7.20) and the $SU(3)$ $\mathcal{ADE}$ graphs. For the 01-part $\mathcal{G}_{01}$ of a finite three-colourable $SU(3)$ $\mathcal{ADE}$ graph $\mathcal{G}$, we have $\widetilde{T}(t^2) = H(t)$, where $H(t)$ is the Hilbert series for the pre-projective algebra of the bipartite graph $\mathcal{G}_{01}$ (this follows in the same way as the proof of part (i) of Theorem 7.5.1). If $\mathcal{G}$ has a flat connection, then by the comments at the end of Section 6.3 it is expected that the Hilbert series f for $\mathcal{G}_{01}$ counts the dimensions of the higher relative commutants for the subfactor with principal graph $\mathcal{G}_{01}$.

7.7.1 Graphs $\mathcal{A}^{(l)}$, $l \leq \infty$.

The distinguished vertex $*$ of the graph $\mathcal{A}^{(l)}$ is the apex vertex $(0,0)$. Its eigenvalues $\beta^{(\lambda)}$ are given in (1.27), with corresponding eigenvectors $\psi_\mu^\lambda = S_{\mu,\lambda}$, where the S-matrix for $SU(3)$ at level $k = l-3$ is [45]:

$$\begin{aligned} S_{\lambda,\mu} & \\ = \frac{-i}{l\sqrt{3}} \Big[& \exp(\xi(2\lambda_1'\mu_1' + \lambda_1'\mu_2' + \lambda_2'\mu_1' + 2\lambda_2'\mu_2')) + \exp(\xi(\lambda_2'\mu_1' - \lambda_1'\mu_1' + 2\lambda_1'\mu_2' - \lambda_2'\mu_2')) \\ & + \exp(\xi(\lambda_1'\mu_2' - \lambda_1'\mu_1' - 2\lambda_2'\mu_1' - \lambda_2'\mu_2')) - \exp(\xi(-2\lambda_1'\mu_2' - \lambda_1'\mu_1' - \lambda_2'\mu_2' - 2\lambda_2'\mu_1')) \\ & - \exp(\xi(2\lambda_1'\mu_1' + \lambda_1'\mu_2' + \lambda_2'\mu_1' - \lambda_2'\mu_2')) - \exp(\xi(\lambda_1'\mu_2' - \lambda_1'\mu_1' + \lambda_2'\mu_1' + 2\lambda_2'\mu_2')) \Big], \end{aligned} \tag{7.51}$$

where $\xi = -2\pi i/3l$, $\lambda = (\lambda_1, \lambda_2)$, $\mu = (\mu_1, \mu_2)$, and $\lambda_j' = \lambda_j + 1$, $\mu_j' = \mu_j + 1$, for $j = 1,2$.

Then ψ_*^λ is found by setting $\mu = (0,0)$ in (7.51), giving

$$\begin{aligned} S_{(0,0),\lambda} &= \frac{-i}{l\sqrt{3}} \left[\exp\left(-\frac{2\pi i}{3l}(3\lambda_1' + 3\lambda_2')\right) + \exp\left(-\frac{2\pi i}{3l}(-3\lambda_2')\right) + \exp\left(-\frac{2\pi i}{3l}(-3\lambda_1')\right) \right. \\ & \qquad \left. - \exp\left(-\frac{2\pi i}{3l}(-3\lambda_1' - 3\lambda_2')\right) - \exp\left(-\frac{2\pi i}{3l}(3\lambda_2')\right) - \exp\left(-\frac{2\pi i}{3l}(3\lambda_1')\right) \right] \\ &= \frac{2}{l\sqrt{3}} \left[\sin\left(\frac{2\pi}{l}(\lambda_1+1)\right) + \sin\left(\frac{2\pi}{l}(\lambda_2+1)\right) - \sin\left(\frac{2\pi}{l}(\lambda_1+\lambda_2+2)\right) \right] \qquad (7.52) \\ &= -\frac{1}{2\sqrt{3}\pi^2 l} J\left((\lambda_1 + 2\lambda_2 + 3)/3l, (2\lambda_1 + \lambda_2 + 3)/3l\right), \qquad (7.53) \end{aligned}$$

where in (7.53) $\theta_1 = (\lambda_1+2\lambda_2+3)/3l$ and $\theta_2 = (2\lambda_1+\lambda_2+3)/3l$, so that $(\lambda_1+1)/l = 2\theta_2 - \theta_1$ and $(\lambda_2+1)/l = 2\theta_1 - \theta_2$.

Since the S-matrix is symmetric, we also have $\psi_\mu^\lambda = S_{\lambda,\mu}$, so that the Perron-Frobenius eigenvector $\psi^{(0,0)}$ has entries $\psi_\lambda^{(0,0)}$ given by (7.52). Since the S-matrix is unitary, the eigenvector $\psi^{(0,0)}$ has norm 1. Recall that the Perron-Frobenius eigenvector for $\mathcal{A}^{(l)}$ can also be written as (1.28), with $n = l$, where $\phi^{(0,0)}$ has norm > 1. In fact, $\phi^{(0,0)}$ has norm $l\sqrt{3}(8\sin(2\pi/l)\sin^2(\pi/l))^{-1}$, so that $\psi^{(0,0)} = 8\sin(2\pi/l)\sin^2(\pi/l)\ \phi^{(0,0)}/l\sqrt{3}$. Then using the expression for $S_{(0,0),\lambda}$ given in (7.53),

$$\begin{aligned} J(\theta_1,\theta_2) &= -2\sqrt{3}\pi^2 l\, \psi^{(0,0)}_{(l(2\theta_2-\theta_1)-1,l(2\theta_1-\theta_2)-1)} \\ &= -2\sqrt{3}\pi^2 l\, \frac{8}{l\sqrt{3}} \sin(2\pi/l)\sin^2(\pi/l)\, \phi^{(0,0)}_{(l(2\theta_2-\theta_1)-1,l(2\theta_1-\theta_2)-1)} \\ &= -16\pi^2 \sin((2\theta_2-\theta_1)\pi)\sin((2\theta_1-\theta_2)\pi)\sin((\theta_1+\theta_2)\pi), \end{aligned}$$

so that the Jacobian $J(\theta_1,\theta_2)$ can also be written as a product of sine functions. From this form for J we see that the expression for $J(\omega_1,\omega_2)$ in (7.38) factorizes as

$$J(\omega_1,\omega_2) = -2\pi^2 i(u_1^{-1}u_2^2 - u_1u_2^{-2})(u_1^2u_2^{-1} - u_1^{-2}u_2)(u_1u_2 - u_1^{-1}u_2^{-1}),$$

where $u_1 = \omega_1^{1/2}$ and $u_2 = \omega_2^{1/2}$ take their values in $\{e^{i\theta} |\, 0 \le \theta < \pi\}$.

We now compute the spectral measure for $\mathcal{A}^{(l)}$. The exponents of $\mathcal{A}^{(l)}$ are all the vertices of $\mathcal{A}^{(l)}$, i.e. Exp $= \{(\lambda_1,\lambda_2)|\ \lambda_1,\lambda_2 \ge 0;\ \lambda_1+\lambda_2 \le l-3\}$. Then summing over all $(\lambda_1,\lambda_2) \in$ Exp corresponds to summing over all $(\theta_1,\theta_2) \in \{(q_1/3l, q_2/3l)|\ q_1,q_2 = 0,1,\ldots,3l-1\}$, such that $\theta_1+\theta_2 \equiv 0 \bmod 3$ and

$$\begin{aligned} 2\theta_2-\theta_1 &= (\lambda_1+1)/l \ \ge\ 1/l, & 2\theta_1-\theta_2 &= (\lambda_2+1)/l \ \ge\ 1/l, \\ \theta_1+\theta_2 &= (\lambda_1+\lambda_2+2)/l \ \le\ (l-1)/l \ =\ 1-1/l. \end{aligned}$$

Let $L_{(\theta_1,\theta_2)}$ be the set of all such (θ_1,θ_2), and let C_l be the set of all $(\omega_1,\omega_2) \in \mathbb{T}$, where $\omega_j = e^{2\pi i\theta_j}$, $j=1,2$, such that $(\theta_1,\theta_2) \in L_{(\theta_1,\theta_2)}$. It is easy to check that $\beta^{(\lambda)} = \omega_1 + \omega_2^{-1} + \omega_1^{-1}\omega_2$. Using (7.50),

$$\begin{aligned} &\int_{\mathbb{T}^2} R_{m,n}(\omega_1,\omega_2)d\varepsilon(\omega_1,\omega_2) \qquad (7.54) \\ &= \frac{1}{12\pi^4 l^2} \sum_{\lambda\in\mathrm{Exp}} (\beta^{(\lambda)})^m(\overline{\beta^{(\lambda)}})^n J\left((2\lambda_1+\lambda_2+3)/3l, (\lambda_1+2\lambda_2+3)/3l\right)^2 \\ &= -\frac{1}{3l^2} \sum_{(\omega_1,\omega_2)\in C_l} (\omega_1+\omega_2^{-1}+\omega_1^{-1}\omega_2)^m(\omega_1^{-1}+\omega_2+\omega_1\omega_2^{-1})^n \\ &\qquad \times(\omega_1\omega_2+\omega_1\omega_2^{-2}+\omega_1^{-2}\omega_2-\omega_1^{-1}\omega_2^{-1}-\omega_1^2\omega_2^{-1}-\omega_1^{-1}\omega_2^2)^2. \qquad (7.55) \end{aligned}$$

If we let C be the limit of C_l as $l \to \infty$, then C is a fundamental domain of $\mathbb{T}^2$ under the action of the group G, illustrated in Figure 7.10. Since $J=0$ along the boundary of C, which is mapped to the boundary of $\mathfrak{D}$ under the map $\Phi : \mathbb{T}^2 \to \mathfrak{D}$, we can take the summation in (7.55) to be over $(\omega_1,\omega_2) \in C$. Since J^2 is invariant under the action of G, we have

$$\int_{\mathbb{T}^2} R_{m,n}(\omega_1,\omega_2)d\varepsilon(\omega_1,\omega_2)$$

$$= -\frac{1}{6}\frac{1}{3l^2}\sum_{(\omega_1,\omega_2)\in D_l}(\omega_1+\omega_2^{-1}+\omega_1^{-1}\omega_2)^m(\omega_1^{-1}+\omega_2+\omega_1\omega_2^{-1})^n$$
$$\times(\omega_1\omega_2+\omega_1\omega_2^{-2}+\omega_1^{-2}\omega_2-\omega_1^{-1}\omega_2^{-1}-\omega_1^2\omega_2^{-1}-\omega_1^{-1}\omega_2^2)^2, \quad (7.56)$$

where

$$D_l=\{(e^{2\pi i q_1/3l},e^{2\pi i q_2/3l})\in\mathbb{T}^2|\ q_1,q_2=0,1,\ldots,3l-1;q_1+q_2\equiv 0 \bmod 3\} \quad (7.57)$$

is the image of C_l under the action of G. The number $\sharp_{\mathrm{int}}^{(l)}$ of such pairs in the interior of a fundamental domain C can be seen to be equal to $n^{(l)}=(l-2)(l-1)/2$, where $n^{(l)}$ is the number of vertices of $\mathcal{A}^{(l)}$, whilst the number $\sharp_{\partial}^{(l)}$ of such pairs along the boundary of C is $n^{(l+3)}-n^{(l)}=[(l+1)(l+2)-(l-2)(l-1)]/2=3l$. Then the total number of such pairs over the whole of $\mathbb{T}^2$ is $|D_l|=6\sharp_{\mathrm{int}}^{(l)}+3\sharp_{\partial}^{(l)}-6$ since we count the interior of C six times but only count its boundary three times. The vertices at the corners of the boundary of C are overcounted twice each, hence the term -6. So $|D_l|=3(l-2)(l-1)+9l-6=3l^2$, and we have

$$\int_{\mathbb{T}^2}R_{m,n}(\omega_1,\omega_2)d\varepsilon(\omega_1,\omega_2)$$
$$= -\frac{1}{6}\frac{1}{|D_l|}\sum_{(\omega_1,\omega_2)\in D_l}(\omega_1+\omega_2^{-1}+\omega_1^{-1}\omega_2)^m(\omega_1^{-1}+\omega_2+\omega_1\omega_2^{-1})^n$$
$$\times(\omega_1\omega_2+\omega_1\omega_2^{-2}+\omega_1^{-2}\omega_2-\omega_1^{-1}\omega_2^{-1}-\omega_1^2\omega_2^{-1}-\omega_1^{-1}\omega_2^2)^2,$$
$$= -\frac{1}{6}\int_{\mathbb{T}^2}(\omega_1+\omega_2^{-1}+\omega_1^{-1}\omega_2)^m(\omega_1^{-1}+\omega_2+\omega_1\omega_2^{-1})^n$$
$$\times(\omega_1\omega_2+\omega_1\omega_2^{-2}+\omega_1^{-2}\omega_2-\omega_1^{-1}\omega_2^{-1}-\omega_1^2\omega_2^{-1}-\omega_1^{-1}\omega_2^2)^2\ d^{(l)}(\omega_1,\omega_2),$$

where $d^{(l)}$ is the uniform measure over all pairs $(\omega_1,\omega_2)\in D_l$. Then we have proved the following:

Theorem 7.7.1 *The spectral measure of $\mathcal{A}^{(l)}$ (over $\mathbb{T}^2$) is given by*

$$d\epsilon(\omega_1,\omega_2)=\frac{1}{24\pi^4}J(\omega_1,\omega_2)^2d^{(l)}(\omega_1,\omega_2). \quad (7.58)$$

We can now easily deduce the spectral measure of $\mathcal{A}^{(\infty)}$ claimed in Section 7.6.2. Letting $l\longrightarrow\infty$, the measure $d^{(l)}(\omega_1,\omega_2)$ becomes the uniform Lebesgue measure $d\omega_1\,d\omega_2$ on $\mathbb{T}^2$. Thus we have:

Theorem 7.7.2 *The spectral measure of $\mathcal{A}^{(\infty)}$ (over $\mathbb{T}^2$) is*

$$d\epsilon(\omega_1,\omega_2)=\frac{1}{24\pi^4}J(\omega_1,\omega_2)^2d\omega_1\,d\omega_2, \quad (7.59)$$

where $d\omega$ is the uniform Lebesgue measure over $\mathbb{T}$. Over $\mathfrak{D}$, the spectral measure $\mu_{v_N}(z)$ of $\mathcal{A}^{(\infty)}$ is

$$d\mu_{v_N}(z) = \frac{1}{2\pi^2}\sqrt{27 - 18z\overline{z} + 4z^3 + 4\overline{z}^3 - z^2\overline{z}^2}\, dz. \tag{7.60}$$

Remark: Gepner [46] proved that this is the measure required to make the polynomials $S_\mu(z,\overline{z})$, defined in Section 5.1.1 for vertices μ of $\mathcal{A}^{(\infty)}$, orthogonal, i.e.

$$\frac{1}{2\pi^2}\int_{\mathbb{T}^2} S_\mu(z,\overline{z})\overline{S_\nu(z,\overline{z})}\sqrt{27 - 18z\overline{z} + 4z^3 + 4\overline{z}^3 - z^2\overline{z}^2}\, dz = \delta_{\mu,\nu}.$$

Then in particular, the dimension of the n^{th} level of the path algebra for $\mathcal{A}^{(\infty)}$ is given by (7.44) with $m = n$ (i.e. $r = 0$), or equivalently by the integral $\int_{\mathfrak{D}} |z|^{2m} d\mu_{v_N}(z)$ with measure given by (7.60).

The dimension of the irreducible representation π_λ of the Hecke algebra $H_n(q)$, labelled by a Young diagram $\lambda = (p_1, p_2, n - p_1 - p_2)$ with at most 3 rows, is given by the determinantal formula (see e.g. [107]):

$$\dim(\pi_\lambda) = n! \begin{vmatrix} 1/p_1! & 1/(p_1+1)! & 1/(p_1+2)! \\ 1/(p_2-1)! & 1/p_2! & 1/(p_2+1)! \\ 1/(n-p_1-p_2-2)! & 1/(n-p_1-p_2-1)! & 1/(n-p_1-p_2)! \end{vmatrix}, \tag{7.61}$$

where $1/q!$ is understood to be zero if q is negative. Computing the determinant in equation (7.61), we can rewrite the right hand side as a sum of multinomial coefficients:

$$\begin{aligned} \dim(\pi_\lambda) &= (p_1, p_2, n-p_1-p_2)! - (p_1, p_2+1, n-p_1-p_2-1)! \\ &\quad + (p_1+1, p_2+1, n-p_1-p_2-2)! - (p_1+1, p_2-1, n-p_1-p_2)! \\ &\quad + (p_1+2, p_2-1, n-p_1-p_2-1)! - (p_1+2, p_2, n-p_1-p_2-2)! \end{aligned} \tag{7.62}$$

We can also obtain another formula for the dimension of $A(\mathcal{A}^{(\infty)})_n$. The number $c^{(n)}_{(\lambda_1,\lambda_2)}$ of paths of length n on the graph $\mathcal{A}^{(\infty)}$ from the apex vertex $(0,0)$ to a vertex (λ_1,λ_2) is given in [25] as

$$c^{(n)}_{(\lambda_1,\lambda_2)} = \frac{(\lambda_1+1)(\lambda_2+1)(\lambda_1+\lambda_2+2)\, n!}{((n+2\lambda_1+\lambda_2+6)/3)!((n-\lambda_1+\lambda_2+3)/3)!((n-\lambda_1-2\lambda_2)/3)!}. \tag{7.63}$$

Then we have the following results:

Lemma 7.7.3 *Let $c^{(n)}_{(\lambda_1,\lambda_2)}$ be the number of paths of length n from $(0,0)$ to the vertex (λ_1,λ_2) on the graph $\mathcal{A}^{(\infty)}$, as given in (7.63), and let $A'(\mathcal{A}^{(\infty)})$ be the general path algebra defined in Section 7.6.2. Then, for fixed integers $m,n < \infty$, the following are all equal:*

(1) $\dim(A'(\mathcal{A}^{(\infty)})_{m,n})$,

(2) $\frac{1}{2\pi^2}\int_{\mathfrak{D}} z^m \overline{z}^n \sqrt{27-18z\overline{z}+4z^3+4\overline{z}^3-z^2\overline{z}^2}\,dz$,

(3) $\frac{1}{24\pi^4}\int_{\mathbb{T}^2}(\omega_1+\omega_2^{-1}+\omega_1^{-1}\omega_2)^m(\omega_1^{-1}+\omega_2+\omega_1\omega_2^{-1})^n J(\omega_1,\omega_2)^2 d\omega_1\, d\omega_2$,

(4) $-\frac{1}{6}\sum \gamma_{a_1,a_2}(k_1,k_2,n-k_1-k_2)!\,(k_1+r+b_1,k_2+r-b_2,m+r-b_1+b_2-k_1-k_2)!$,

(5) $\sum c^{(m)}_{(\lambda_1,\lambda_2)}c^{(n)}_{(\lambda_1,\lambda_2)}$,

where in (4), $n=m+3r$, $r\in\mathbb{Z}$, $b_1=(2a_1+a_2)/3$, $b_2=(a_1+2a_2)/3$ and the summation is over all $a_1,a_2\in\mathbb{Z}$ such that $(a_1,a_2)\in\Upsilon$, and all non-negative integers k_1, k_2 which satisfy (7.45)-(7.47). The summation in (5) is over all $0\le\lambda_1,\lambda_2\le\min(m,n)$ such that $\lambda_1+\lambda_2\le\min(m,n)$ and $m\equiv n\equiv\lambda_1+2\lambda_2$ mod 3.

Proof

The identities (1) = (2) = (3) = (4) were shown above. The other identity (1) = (5) is trivial since the dimension of $A'(\mathcal{A}^{(\infty)})_{m,n}$ is equal to the number of pairs of paths (with lengths m, n respectively) which begin at $(0,0)$ and end at the same vertex of $\mathcal{A}^{(\infty)}$. □

Corollary 7.7.4 *Let $f^{(n)}_{p_1,p_2}$ be the sum of multinomial coefficients given by (7.62). Then, in particular, for fixed $n<\infty$, the following are all equal:*

(1) $\dim\left((\otimes^n M_3)^{SU(3)}\right)$,

(2) $\frac{1}{2\pi^2}\int_{\mathfrak{D}}|z|^{2n}\sqrt{27-18z\overline{z}+4z^3+4\overline{z}^3-z^2\overline{z}^2}\,dz$,

(3) $\frac{1}{24\pi^4}\int_{\mathbb{T}^2}|\omega_1+\omega_2^{-1}+\omega_1^{-1}\omega_2|^{2n}\,J(\omega_1,\omega_2)^2\,d\omega_1\,d\omega_2$,

(4) $-\frac{1}{6}\sum\gamma_{a_1,a_2}(k_1,k_2,n-k_1-k_2)!\,(k_1+b_1,k_2-b_2,n-b_1+b_2-k_1-k_2)!$,

(5) $\sum f^{(n)}_{p_1,p_2}$,

(6) $\sum(c^{(n)}_{(\lambda_1,\lambda_2)})^2$,

where in (4), $b_1=(2a_1+a_2)/3$, $b_2=(a_1+2a_2)/3$ and the summation is over all $a_1,a_2\in\mathbb{Z}$ such that $(a_1,a_2)\in\Upsilon$, and all non-negative integers k_1, k_2 which satisfy (7.45)-(7.47). The summation in (5) is over all $0\le p_2\le p_1\le n$ such that $n-p_1\le 2p_2$, whilst the summation in (6) is over all $0\le\lambda_1,\lambda_2\le n$ such that $\lambda_1+\lambda_2\le n$ and $n\equiv\lambda_1+2\lambda_2$ mod 3.

Proof

The identities (1) = (2) = (3) = (4) = (6) follow from Lemma 7.7.3. The identity (1) = (5) follows from (7.62) and the fact that $(\otimes^n M_3)^{SU(3)}=A(\mathcal{A}^{(\infty)})_n=\bigoplus_\lambda\pi_\lambda(H_n(q))$, where the summation is again over all Young diagrams λ with n boxes. □

7.7.2 Graphs $\mathcal{D}^{(n)}$, $n \equiv 0 \bmod 3$.

The exponents of $\mathcal{D}^{(3k)}$, for integers $k \geq 2$, are the 0-coloured vertices of $\mathcal{A}^{(3k)}$, i.e. $\mathrm{Exp} = \{(\lambda_1, \lambda_2) | \ \lambda_1, \lambda_2 \geq 0; \ \lambda_1 + \lambda_2 \leq 3k-3; \ \lambda_1 - \lambda_2 \equiv 0 \bmod 3\}$, where the exponent $(k-1, k-1)$ has multiplicity three. For $\mathcal{D}^{(3k)}$ we have $|\psi_*^\lambda| = \sqrt{3} S_{(0,0),\lambda}$ for all $\lambda \in \mathrm{Exp}$ except for $\lambda = (k-1, k-1)$. For this exponent however the eigenvalue $\beta^{(k,k)} = 0$, so that this term does not contribute in (7.50). Then for $\lambda \neq (k-1, k-1)$, $|\psi_*^\lambda| = J\left((\lambda_1 + 2\lambda_2 + 3)/3l, (2\lambda_1 + \lambda_2 + 3)/3l\right)/6k\pi^2$.

Since the exponents for $\mathcal{D}^{(3k)}$ are all of colour zero, under the above identification between λ_1, λ_2 and θ_1, θ_2, the exponents λ correspond to all pairs (θ_1, θ_2) such that $\theta_1 - \theta_2 \equiv 0 \bmod 3$ and $(e^{2\pi i \theta_1}, e^{2\pi i \theta_2}) \in C$. These pairs (θ_1, θ_2) are thus in fact all of the form $(p_1/3k, p_2/3k)$, for $p_1, p_2 \in \{1, 2, \ldots, 3k-1\}$. Under the action of G, these pairs are mapped to the all the points $(q_1, q_2) \in [0,1]^2$ such that $e^{2\pi i q_j}$ is a $3k^{\text{th}}$ root of unity, for $j = 1, 2$, except for the points the points (q_1, q_2) which parameterize to the boundary of $\mathfrak{D}$. However, we can again use the fact that the Jacobian is zero at the points which parameterize to the boundary of $\mathfrak{D}$.

Then by (7.50) we have

$$\begin{aligned}
&\int_{\mathbb{T}^2} R_{m,n}(\omega_1, \omega_2) d\varepsilon(\omega_1, \omega_2) \\
&= \frac{1}{4\pi^4} \frac{1}{(3k)^2} \sum_{\lambda \in \mathrm{Exp}} (\beta^{(\lambda)})^m (\overline{\beta^{(\lambda)}})^n J\left((\lambda_1 + 2\lambda_2 + 3)/3l, (2\lambda_1 + \lambda_2 + 3)/3l\right)^2 \\
&= \frac{1}{24\pi^4} \frac{1}{(3k)^2} \sum_{\theta_1, \theta_2} (\beta^{(\lambda)})^m (\overline{\beta^{(\lambda)}})^n J(\theta_1, \theta_2)^2,
\end{aligned}$$

where the last summation is over all $(\theta_1, \theta_2) \in \{(p_1/3k, p_2/3k) | \ p_1, p_2 = 1, 2, \ldots, 3k-1\}$. Then we have obtained the following result:

Theorem 7.7.5 *The spectral measure of $\mathcal{D}^{(3k)}$, $k \geq 2$, (over $\mathbb{T}^2$) is*

$$d\epsilon(\omega_1, \omega_2) = \frac{1}{24\pi^4} J(\omega_1, \omega_2)^2 \ d_{3k/2}\omega_1 \ d_{3k/2}\omega_2, \tag{7.64}$$

where $d_{3k/2}\omega$ is the uniform measure over the $3k^{\text{th}}$ roots of unity.

For the limit as $k \to \infty$ we simply recover the measure (7.59) for $\mathcal{A}^{(\infty)}$. This is due to the fact that taking the limit of the graph $\mathcal{D}^{(3k)}$ as $k \to \infty$ with the vertex $* = (0,0)$ as the distinguished vertex, we just obtain the infinite graph $\mathcal{A}^{(\infty)}$. In order to obtain the infinite graph $\mathcal{D}^{(\infty)}$ we must set the distinguished vertex $*$ of $\mathcal{D}^{(3k)}$ to be one of the triplicated vertices $(k-1, k-1)_i$, $i = 1, 2, 3$ (see e.g. Figure 4.5). Then using (7.50), and taking the limit as $k \to \infty$, we would obtain the spectral measure for $\mathcal{D}^{(\infty)}$.

7.7.3 Graphs $\mathcal{A}^{(l)*}$, $l \leq \infty$.

The exponents of $\mathcal{A}^{(l)*}$ are Exp $= \{(j,j)|\ j = 0,1,\ldots,\lfloor(l-3)/2\rfloor\}$. From [44] its eigenvectors are $\psi_a^\lambda = 2\sqrt{l^{-1}}\sin(2\pi a(\lambda_1+1)/l)$, where $\lambda = (\lambda_1,\lambda_2) \in$ Exp and $a = 1,2,\ldots,\lfloor(l-1)/2\rfloor$, as in Figure 1.10. Then

$$\int_{\mathbb{T}^2} R_{m,n}(\omega_1,\omega_2)d\varepsilon(\omega_1,\omega_2) = \frac{4}{l}\sum_{j=0}^{\lfloor(l-3)/2\rfloor}(\beta^{(j,j)})^m(\overline{\beta^{(j,j)}})^n\sin^2(2\pi(j+1)/l).$$

Since all the eigenvalues $\beta^{(j,j)}$ of $\mathcal{A}^{(l)*}$ are real, there is a map $\Phi_1 : \mathbb{T} \to \mathfrak{D}$ given by $\Phi_1(u) = u + u^{-1} + 1$ so that the eigenvalues are given by $\Phi_1(e^{2\pi i(j+1)/l}) \in [-1,3]$ for $j = 0,1,\ldots,\lfloor(l-3)/2\rfloor\}$. Then the spectral measure of $\mathcal{A}^{(l)*}$ can be written as a measure over $\mathbb{T}$. Then with $\widetilde{u} = e^{2\pi i/l}$, we have

$$\int_{\mathbb{T}}(u+u^{-1})^{m+n}d\varepsilon(u) = \frac{4}{l}\sum_{j=1}^{\lfloor(l-1)/2\rfloor}(\widetilde{u}^j + \widetilde{u}^{-j}+1)^{m+n}\ \sin(\widetilde{u}^j)^2.$$

For all l, $\sin(\widetilde{u}^0) = 0$, and $\sin(\widetilde{u}^j) = \sin(\widetilde{u}^l - j)$, for $l = 1,2,\ldots,\lfloor(l-1)/2\rfloor$. If l is even, we also must consider when $j = l/2$. In this case $\sin(\widetilde{u}^{l/2}) = 0$. Then we can write

$$\begin{aligned}\int_{\mathbb{T}}(u+u^{-1})^{m+n}d\varepsilon(u) &= \frac{2}{l}\sum_{j=0}^{l}(\widetilde{u}^j+\widetilde{u}^{-j}+1)^{m+n}\ \sin(\widetilde{u}^j)^2 \\ &= 2\int_{\mathbb{T}}(u+u^{-1}+1)^{m+n}\ \sin(u)^2 d_{l/2}u,\end{aligned} \tag{7.65}$$

where d_p is the uniform measure over the $2p^{\text{th}}$ roots of unity. Then we have:

Theorem 7.7.6 *The spectral measure of $\mathcal{A}^{(l)*}$, $l < \infty$, (over $\mathbb{T}$) is*

$$d\epsilon(u) = \alpha(u)d_{l/2}u, \tag{7.66}$$

where $d_{l/2}u$ is the uniform measure over the l^{th} roots of unity, and α is given in (7.11).

Since $(u+u^{-1}+1)^l = \sum_{i=0}^{l} C_i^l(u+u^{-1})^i$, for even $l = 2k$ we can express the m,n^{th} moment as a linear combination of the moments of the Dynkin diagram A_{k-1}:

$$\int_{\mathbb{T}}(u+u^{-1})^{m+n}d\varepsilon(u) = \sum_{j=0}^{m+n}C_j^{m+n}\int_{\mathbb{T}}(u+u^{-1})^j\ 2\mathrm{Im}(u)^2 d_{l/2}u = \sum_{j=0}^{m+n}C_j^{m+n}\varsigma^j,$$

where ς^j is the j^{th} moment of A_{k-1}. When $l \to \infty$, the j^{th} moment ς^j of A_∞ is given by the Catalan number $c_{j/2}$ when j is even, and 0 when j is odd. Then for the infinite graph $\mathcal{A}^{(\infty)*}$,

$$\int_{\mathbb{T}}(u+u^{-1})^{m+n}d\varepsilon(u) = \sum_{k=0}^{\lfloor(m+n)/2\rfloor}C_{2k}^{m+n}c_k.$$

In fact, the spectral measure for $\mathcal{A}^{(\infty)*}$ has semicircle distribution: Letting $l \to \infty$ in (7.65), we have the approximation of an integral

$$\lim_{l\to\infty} \frac{2}{l}\sum_{j=0}^{l}(\widetilde{u}^j + \widetilde{u}^{-j} + 1)^{m+n} \; \sin(\widetilde{u}^j)^2 = 2\int_0^1 (e^{2\pi i\theta} + e^{-2\pi i\theta} + 1)^m \; \sin^2(2\pi\theta)d\theta.$$

Making the change of variable $x = e^{2\pi i\theta} + e^{-2\pi i\theta} + 1 = 2\cos(2\pi\theta) + 1$, we have $2\sin(2\pi\theta) = \sqrt{4-(x-1)^2}$, and $dx/d\theta = -4\pi\sin(2\pi\theta) = -2\pi\sqrt{4-(x-1)^2}$. Then

$$\begin{aligned}
\int x^m d\mu(x) &= 2\int_0^1 (e^{2\pi i\theta} + e^{-2\pi i\theta} + 1)^m \; \sin^2(2\pi\theta)d\theta \\
&= 4\int_0^{\frac{1}{2}} (e^{2\pi i\theta} + e^{-2\pi i\theta} + 1)^m \; \sin^2(2\pi\theta)d\theta \\
&= \frac{-4}{8\pi}\int_3^{-1} x^m\sqrt{4-(x-1)^2}dx \;=\; \frac{1}{2\pi}\int_{-1}^{3} x^m\sqrt{4-(x-1)^2}dx,
\end{aligned}$$

which is the semicircle law centered at 1 with radius 2. Then the spectral measure (over $[-1,3]$) for the infinite graph $\mathcal{A}^{(\infty)*}$ has semicircle distribution with mean 1 and variance 1, i.e. $d\mu(x) = \sqrt{4-(x-1)^2}dx$.

The graph $\mathcal{A}^{(2l)*}$ has adjacency matrix $\Delta^{(2l)*} = \Delta_{l-1} + \mathbf{1}$, where Δ_l is the adjacency matrix of the Dynkin diagram A_l. Hence the spectral measure for $\mathcal{A}^{(2l)*}$ is the spectral measure for A_{l-1} but with a shift by one.

7.7.4 Graph $\mathcal{E}^{(8)}$

The spectral measures for the graphs $\mathcal{A}^{(l)}$, $\mathcal{D}^{(3k)}$ are measures of type $d_p \times d_p$, $J^2 d_p \times d_p$, $d^{(p)}$ or $J^2 d^{(p)}$, for $2p \in \mathbb{N}$. We will show in Section 7.8 that the spectral measures for certain finite subgroups of $SU(3)$ are also linear combinations of measures of these types. However, we will now show that the spectral measure for $\mathcal{E}^{(8)}$ is not a linear combination of measures of these types. The exponents of $\mathcal{E}^{(8)}$ are

$$\mathrm{Exp} = \{(0,0),(5,0),(0,5),(2,2),(2,1),(1,2),(3,0),(2,3),(0,2),(0,3),(3,2),(2,0)\}.$$

Let $\omega = e^{2\pi i/3}$ and A be the automorphism of order 3 on the vertices of $\mathcal{A}^{(8)}$ given by $A(\mu_1,\mu_2) = (5-\mu_1-\mu_2,\mu_1)$. For the eigenvalues $\beta^{(\lambda)}$, $\beta^{(A(\lambda))} = \omega\beta^{(\lambda)}$ and $\beta^{(A^2(\lambda))} = \overline{\omega}\beta^{(\lambda)}$, the corresponding eigenvectors are $(v^\lambda, v^\lambda, v^\lambda)$, $(v^\lambda, \omega v^\lambda, \overline{\omega} v^\lambda)$ and $(v^\lambda, \overline{\omega} v^\lambda, \omega v^\lambda)$ respectively, where the row vectors v^λ are given in [26, Table 17.3] (We normalize the eigenvectors so that $||\psi^\lambda|| = 1$). Hence $\psi_*^\lambda = \psi_*^{A(\lambda)} = \psi_*^{A^2(\lambda)}$ for $\lambda \in \mathrm{Exp}$. With $\theta_1 = (\lambda_1 + 2\lambda_2 + 3)/24$, $\theta_2 = (2\lambda_1 + \lambda_2 + 3)/24$, we have

$\lambda \in \text{Exp}$	$(\theta_1, \theta_2) \in [0,1]^2$	$\lvert\psi_*^\lambda\rvert^2$	$\frac{1}{16\pi^4} J(\theta_1, \theta_2)^2$
$(0,0)$, $(5,0)$, $(0,5)$	$(\frac{3}{24}, \frac{3}{24})$, $(\frac{8}{24}, \frac{13}{24})$, $(\frac{13}{24}, \frac{8}{24})$	$\frac{2-\sqrt{2}}{24}$	$3 - 2\sqrt{2}$
$(2,2)$, $(2,1)$, $(1,2)$	$(\frac{9}{24}, \frac{9}{24})$, $(\frac{7}{24}, \frac{8}{24})$, $(\frac{8}{24}, \frac{7}{24})$	$\frac{2+\sqrt{2}}{24}$	$3 + 2\sqrt{2}$
$(3,0)$, $(2,3)$, $(0,2)$	$(\frac{6}{24}, \frac{9}{24})$, $(\frac{11}{24}, \frac{10}{24})$, $(\frac{7}{24}, \frac{5}{24})$	$\frac{1}{12}$	2
$(0,3)$, $(3,2)$, $(2,0)$	$(\frac{9}{24}, \frac{6}{24})$, $(\frac{10}{24}, \frac{11}{24})$, $(\frac{5}{24}, \frac{7}{24})$	$\frac{1}{12}$	2

From (7.50),

$$\int_{\mathbb{T}^2} R_{m,n}(\omega_1, \omega_2) d\varepsilon(\omega_1, \omega_2) = \frac{1}{6} \sum_{g \in G} \sum_{\lambda \in \text{Exp}} (\beta^{(g(\lambda))})^m (\overline{\beta^{(g(\lambda))}})^n |\psi_*^{g(\lambda)}|^2. \tag{7.67}$$

Now the pairs (θ_1, θ_2) given by $g(\lambda)$ for $\lambda \in \text{Exp}$, $g \in G$, are illustrated in Figure 7.13. Consider the pairs $(\theta_1, \theta_2) = (7/24, 8/24), (8/24, 13/24), (10/24, 11/24)$. For each of these, $(\omega_1, \omega_2) = (e^{2\pi i \theta_1}, e^{2\pi i \theta_2}) \in \mathbb{T}^2$ can only be obtained in the integral in (7.67) from either the product measure $d_{12} \times d_{12}$ on pairs of 24^{th} roots of unity, or the uniform measure $d^{(8)}$ on the elements of D_8 ($(7/24, 8/24)$, $(8/24, 13/24)$, $(10/24, 11/24)$ are each in D_8, but none are in D_k for any integer $k < 8$). Since these points (ω_1, ω_2) cannot be obtained independently of each other, we must find a linear combination $\varepsilon' = c_1 \varepsilon_1 + c_2 J^2 \varepsilon_2$ of measures, where ε_j must be either $d_{12} \times d_{12}$ or $d^{(8)}$ for $j = 1, 2$ (it doesn't matter at this stage which of the two measures we take ε_j to be), such that the weight $\varepsilon'(e^{2\pi i \theta_1}, e^{2\pi i \theta_2})$ is $(2+\sqrt{2})/24$, $(2-\sqrt{2})/24$, $1/12$ for $(\theta_1, \theta_2) = (7/24, 8/24), (8/24, 13/24), (10/24, 11/24)$ respectively. Suppose for now that $\varepsilon_1 = \varepsilon_2$. Then we must find solutions $c_1, c_2 \in \mathbb{C}$ such that

$$c_1 + (3 - 2\sqrt{2}) c_2 = \frac{2 - \sqrt{2}}{24}, \qquad c_1 + (3 + 2\sqrt{2}) c_2 = \frac{2 + \sqrt{2}}{24}, \qquad c_1 + 2c_2 = \frac{1}{12}. \tag{7.68}$$

Solving the first two equations we obtain $c_1 = c_2 = 1/48$. However, substituting for these values into the third equation we get $1/48 + 2/48 = 1/16 \neq 1/12$, hence no solution exists to the equations (7.68), and hence the spectral measure for $\mathcal{E}^{(8)}$ is not a linear combination of measures of type $d_p \times d_p$, $J^2 d_p \times d_p$, $d^{(p)}$ or $J^2 d^{(p)}$, for $2p \in \mathbb{N}$.

7.7.5 Graph $\mathcal{E}_1^{(12)}$

We will now show that the spectral measure for $\mathcal{E}_1^{(12)}$ is also not a linear combination of measures of type $d_p \times d_p$, $J^2 d_p \times d_p$, $d^{(p)}$ or $J^2 d^{(p)}$, for $2p \in \mathbb{N}$. The exponents of $\mathcal{E}_1^{(12)}$ are

$$\text{Exp} = \{(0,0), (9,0), (0,9), (4,4), (4,1), (1,4), \text{ and twice } (2,2), (5,2), (2,5)\}.$$

Computing the first entries of the eigenvectors, we have

$$|\psi_*^{(0,0)}|^2 = |\psi_*^{(9,0)}|^2 = |\psi_*^{(0,9)}|^2 = \frac{2 - \sqrt{3}}{36},$$

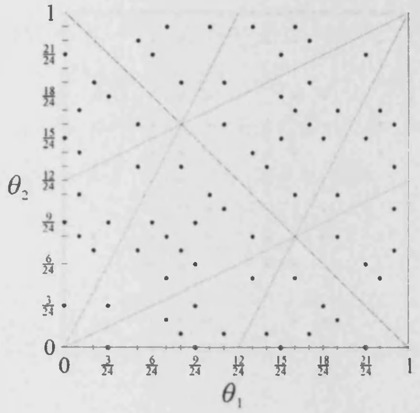


Figure 7.13: The points $(\theta_1, \theta_2) \in \{g(\lambda) | \lambda \in \text{Exp}, g \in G\}$ for $\mathcal{E}^{(8)}$.

$$|\psi_*^{(4,4)}|^2 = |\psi_*^{(4,1)}|^2 = |\psi_*^{(1,4)}|^2 = \frac{2+\sqrt{3}}{36},$$

whilst for the repeated eigenvalues, for the exponents with multiplicity two which we will label by $(\lambda_1, \lambda_2)_1$, $(\lambda_1, \lambda_2)_2$, we have

$$|\psi_*^{(2,2)_1}|^2 + |\psi_*^{(2,2)_2}|^2 = |\psi_*^{(5,2)_1}|^2 + |\psi_*^{(5,2)_2}|^2 = |\psi_*^{(2,5)_1}|^2 + |\psi_*^{(2,5)_2}|^2 = \frac{2}{9}.$$

With $\theta_1 = (\lambda_1 + 2\lambda_2 + 3)/24$, $\theta_2 = (2\lambda_1 + \lambda_2 + 3)/24$, we have

$\lambda \in \text{Exp}$	$(\theta_1, \theta_2) \in [0,1]^2$	$\frac{1}{16\pi^4} J(\theta_1, \theta_2)^2$
$(0,0), (9,0), (0,9)$	$(\frac{1}{12}, \frac{1}{12}), (\frac{7}{12}, \frac{4}{12}), (\frac{4}{12}, \frac{7}{12})$	$\frac{7-4\sqrt{3}}{4}$
$(4,4), (4,1), (1,4)$	$(\frac{5}{12}, \frac{5}{12}), (\frac{4}{12}, \frac{3}{12}), (\frac{3}{12}, \frac{4}{12})$	$\frac{7+4\sqrt{3}}{4}$
$(2,2), (5,2), (2,5)$	$(\frac{3}{12}, \frac{3}{12}), (\frac{5}{12}, \frac{4}{12}), (\frac{4}{12}, \frac{5}{12})$	4

Again, from (7.50),

$$\int_{\mathbb{T}^2} R_{m,n}(\omega_1, \omega_2) d\varepsilon(\omega_1, \omega_2) = \frac{1}{6} \sum_{g \in G} \sum_{\lambda \in \text{Exp}} (\beta^{(g(\lambda))})^m (\overline{\beta^{(g(\lambda))}})^n |\psi_*^{g(\lambda)}|^2. \qquad (7.69)$$

We illustrate the pairs (θ_1, θ_2) given by $g(\lambda)$ for $\lambda \in \text{Exp}$, $g \in G$, in Figure 7.14. Consider the pairs $(\theta_1, \theta_2) = (4/12, 7/12), (3/12, 5/12)$. For both of these, $(\omega_1, \omega_2) = (e^{2\pi i \theta_1}, e^{2\pi i \theta_2}) \in \mathbb{T}^2$ can only be obtained in the integral in (7.69) by using either the product measure $d_6 \times d_6$ or the measure $d^{(12)}$ ($(4/12, 7/12)$, $(3/12, 5/12)$ are both in D_{12}, but neither are in D_k for any integer $k < 12$). With either of these measures, we will also obtain the point $(e^{2\pi i 5/12}, e^{2\pi i 6/12})$ in the integral (7.69). The corresponding pair (θ_1, θ_2) is indicated by the white circle in Figure 7.14. The point $(e^{2\pi i 5/12}, e^{2\pi i 6/12})$ can also only obtained

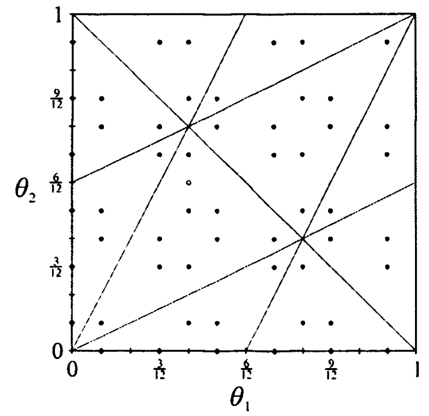


Figure 7.14: The points $(\theta_1, \theta_2) \in \{g(\lambda)|\ \lambda \in \mathrm{Exp}, g \in G\}$ for $\mathcal{E}_1^{(12)}$. The white circle indicates is the point $(5/12, 6/12)$.

by using the measures $d_6 \times d_6$ or $d^{(12)}$. Since these points (ω_1, ω_2) cannot be obtained independently of each other, we must find a linear combination $\varepsilon' = c_1\varepsilon_1 + c_2 J^2 \varepsilon_2$ of measures, where ε_j must be either $d_6 \times d_6$ or $d^{(12)}$ for $j = 1, 2$, such that the weight $\varepsilon'(e^{2\pi i\theta_1}, e^{2\pi i\theta_2})$ is $(2-\sqrt{3})/36$, $(2+\sqrt{3})/36$, 0 for $(\theta_1, \theta_2) = (4/12, 7/12), (3/12, 5/12), (5/12, 6/12)$ respectively. Suppose for now that $\varepsilon_1 = \varepsilon_2$ (again, it doesn't matter at this stage which of the two measures we take ε_1, ε_2 to be). Then since $J(5/12, 6/12)^2 = 3/4$, we must find solutions $c_1, c_2 \in \mathbb{C}$ such that

$$c_1 + \frac{7-4\sqrt{3}}{4}c_2 = \frac{2-\sqrt{3}}{36}, \qquad c_1 + \frac{7+4\sqrt{3}}{4}c_2 = \frac{2+\sqrt{3}}{36}, \qquad c_1 + \frac{3}{4}c_2 = 0. \tag{7.70}$$

Solving the first two equations we obtain $4c_1 = c_2 = 1/36$. However, substituting for these values into the third equation we get $1/144 + 3/144 = 1/36 \neq 0$, hence no solution exists to the equations (7.70), and hence the spectral measure for $\mathcal{E}_1^{(12)}$ is not a linear combination of measures of type $d_p \times d_p$, $J^2 d_p \times d_p$, $d^{(p)}$ or $J^2 d^{(p)}$, for $2p \in \mathbb{N}$.

7.8 Spectral measures for finite subgroups of $SU(3)$

The classification of finite subgroups of $SU(3)$ was begun by Miller, Blichfeldt and Dickson [88, Chapter XII] in 1916. Further work was done in [41, 15]. The classification was finally completed by Yau and Yu [115] in 1993. Clearly, any finite subgroup of $SU(2)$ is a finite subgroup of $SU(3)$, since we can embed $SU(2)$ in $SU(3)$ by sending $A \to \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \in SU(3)$, for any $A \in SU(2)$. These subgroups of $SU(3)$ are called type (B). There are three other infinite series of finite groups, called types (A), (C), (D). The groups of type

(A) are the diagonal abelian groups, which correspond to an embedding of the two torus $\mathbb{T}^2$ in $SU(3)$ given by (1.8). The groups of type (C), (D) are $\Delta(3n^2)$, $\Delta(6n^2)$ respectively, which are considered in [15]. They generalize the dihedral subgroups of $SU(2)$. There are also eight exceptional groups (E)-(L). The complete list of finite subgroups of $SU(3)$ is given in Table 7.5.

Subgroup $\Gamma \subset SU(3)$	$\lvert\Gamma\rvert$ = order of Γ
(A): $\mathbb{Z}_n \times \mathbb{Z}_n$	n^2
(B): Groups isomorphic to finite subgroups of $SU(2)$	-
(C): $\Delta(3n^2) = \mathbb{Z}_n \times \mathbb{Z}_n/\mathbb{Z}_3$	$3n^2$
(D): $\Delta(6n^2)$	$6n^2$
(E) = $\Sigma(36 \times 3)$	108
(F) = $\Sigma(72 \times 3)$	216
(G) = $\Sigma(216 \times 3)$	648
(H) = $\Sigma(60)$	60
(I) = $\Sigma(168)$	168
(J)	180
(K)	504
(L) = $\Sigma(360 \times 3)$	1080

Table 7.5: The finite subgroups Γ of $SU(3)$.

The fundamental representation ρ of $SU(3)$ corresponds to the vertex $(1,0)$ of the graph $\mathcal{A}^{(\infty)}$. The McKay graph $\mathcal{G}_\Gamma$ is the graph associated to the subgroup Γ, as for $SU(2)$ in Section 7.4. For most of the graphs $\mathcal{G}_\Gamma$ there is a corresponding $SU(3)$ $\mathcal{ADE}$ graph/quiver $\mathcal{G}$, which is obtained from $\mathcal{G}_\Gamma$ by now removing more than one vertex, and all the edges that start or end at those vertices, as well as possibly some other edges, as was noted in [27] (to obtain the graph $\mathcal{E}_5^{(12)}$ from the McKay graph for (K) an extra edge must also be inserted[1]). These graphs are illustrated in Figures 7.15 - 7.25. In the graph in Figure 7.16, the three vertices at the corners of the outer triangle are identified with the corresponding vertices of the innermost triangle. However, unlike with $SU(2)$, for $SU(3)$ there is a certain mismatch between the subgroups Γ, with their associated McKay graphs $\mathcal{G}_\Gamma$, and the $\mathcal{ADE}$ graphs. The correspondence is as in Table 7.6, where we use the same notation as Yau and Yu [115] for the subgroups (E)-(I). The notation $\lfloor x \rfloor$ denotes the integer part of x.

$\mathcal{ADE}$ graph	Type	Subgroup $\Gamma \subset SU(3)$
$\mathcal{A}^{(n)}$	I	(A): $\mathbb{Z}_{n-2} \times \mathbb{Z}_{n-2}$
$\mathcal{A}^{(n)*}$	II	-
$\mathcal{D}^{(n)}$ $(n \equiv 0 \bmod 3)$	I	(C): $\Delta(3(n-3)^2) = \mathbb{Z}_{n-3} \times \mathbb{Z}_{n-3}/\mathbb{Z}_3$
$\mathcal{D}^{(n)}$ $(n \not\equiv 0 \bmod 3)$	II	-
-	-	(C): $\Delta(3n^2)$, $(n \not\equiv 0 \bmod 3)$
-	-	(D): $\Delta(6n^2)$
$\mathcal{D}^{(n)*}$	II	$\mathbb{Z}_{\lfloor (n+2)/2 \rfloor} \times \mathbb{Z}_3$
$\mathcal{E}^{(8)}$	I	(E) $= \Sigma(36 \times 3)$
$\mathcal{E}^{(8)*}$	II	-
$\mathcal{E}_1^{(12)} = \mathcal{E}_1^{(12)*}$	I	(F) $= \Sigma(72 \times 3)$
$\mathcal{E}_2^{(12)} = \mathcal{E}_2^{(12)*}$	II	(G) $= \Sigma(216 \times 3)$
$\mathcal{E}_3^{(12)}$	-	$\widehat{\mathcal{D}_4} \otimes \sigma_{123}$
$\mathcal{E}_4^{(12)} = \mathcal{E}_5^{(12)*}$	II	(L) $= \Sigma(360 \times 3)$
$\mathcal{E}_5^{(12)}$	II	(K)
$\mathcal{E}^{(24)} = \mathcal{E}^{(24)*}$	I	-
-	-	(H) $= \Sigma(60)$
-	-	(I) $= \Sigma(168)$
-	-	(J)

Table 7.6: Relationship between $\mathcal{ADE}$ graphs and subgroups Γ of $SU(3)$.

We will now consider the spectral measure for the McKay graph $\mathcal{G}_\Gamma$ associated to a finite subgroup $\Gamma \subset SU(3)$. It was shown in Section 7.4 that any eigenvalue of Γ can be written in the form $\chi_\rho(g) = \mathrm{Tr}(\rho(g))$, where g is any element of the conjugacy class Γ_j.

The diagonal abelian groups $\mathbb{Z}_n \times \mathbb{Z}_n$ correspond to the torus, so clearly the trace of any element, which is the sum of its eigenvalues, will be of the form

$$e^{i\theta_1} + e^{i\theta_2} + e^{-i(\theta_1+\theta_2)}, \tag{7.71}$$

for $0 \le \theta_1, \theta_2 < 2\pi$, and hence the spectrum is contained in $\mathfrak{D}$. For any group isomorphic to a finite subgroup of $SU(2)$, the trace of any element is of the form $1+u+u^{-1}$ for some $u \in \mathbb{T}$, since the trace of any matrix in $SU(2)$ is given by $u+u^{-1}$, $u \in \mathbb{T}$. The generators of the subgroups (C)-(L) of $SU(3)$ are given in [115]. Using these, we computed the trace of any element in the groups (C)-(L), and found that they can all be written in the form (7.71), and hence are in $\mathfrak{D}$.

[1] I am grateful to Jean-Bernard Zuber for pointing this connection out to me.

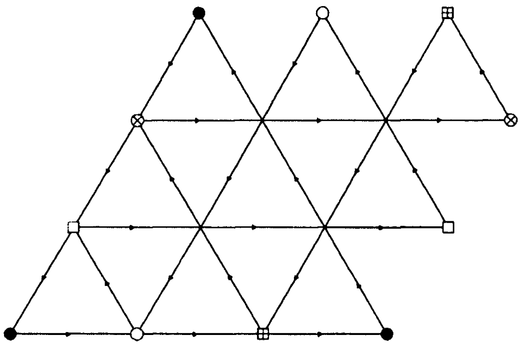


Figure 7.15: $\mathbb{Z}_{n-2} \times \mathbb{Z}_{n-2}$ for $n = 6$; vertices which have the same symbol are identified.

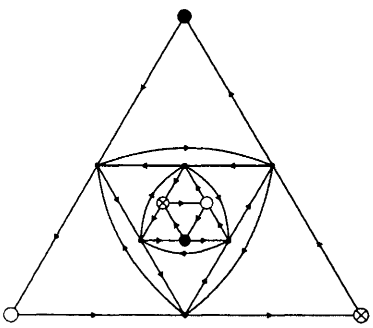


Figure 7.16: $\mathbb{Z}_p \times \mathbb{Z}_3$ for $p = 3$.

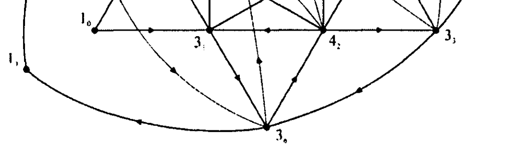


Figure 7.17: (E) $= \sum(36 \times 3)$

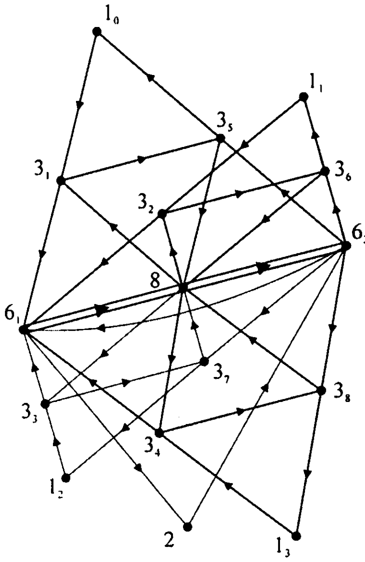


Figure 7.18: (F) $= \sum(72 \times 3)$

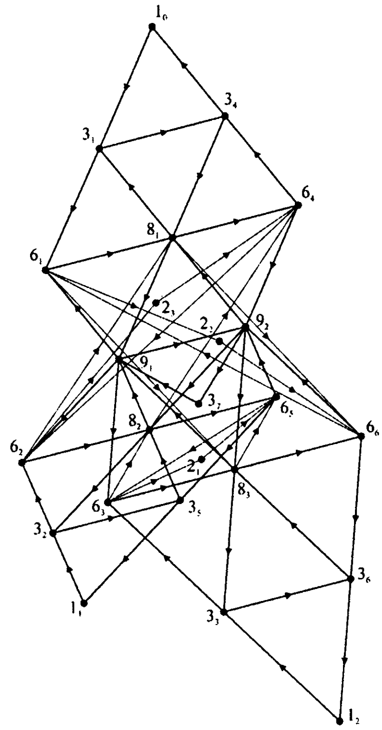


Figure 7.19: (G) $= \sum(216 \times 3)$

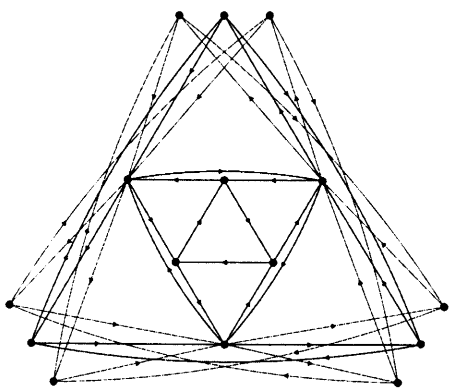


Figure 7.20: $\widehat{\mathcal{D}_4} \otimes \sigma_{123}$

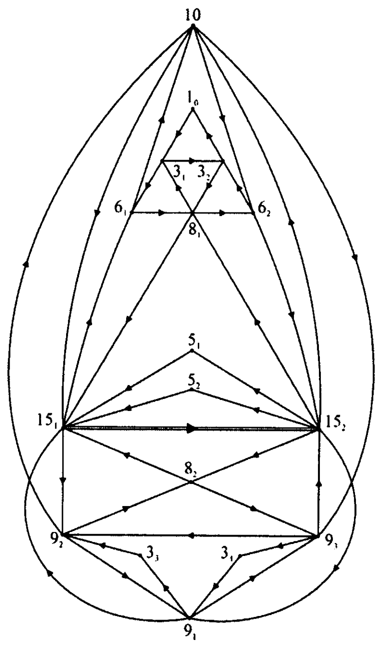


Figure 7.21: (L) $= \sum(360 \times 3)$

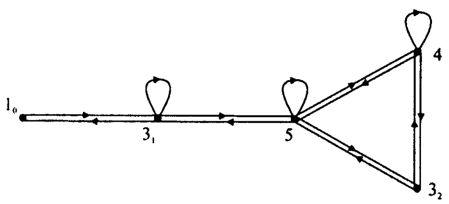


Figure 7.22: (H) = $\sum(60)$

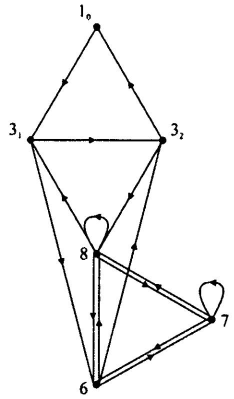


Figure 7.23: (I) = $\sum(168)$

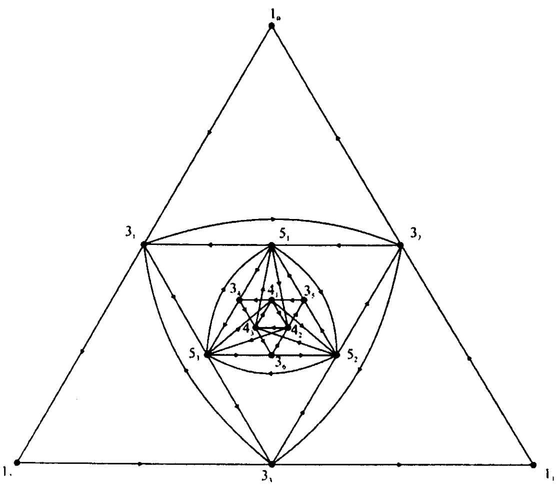


Figure 7.24: (J)

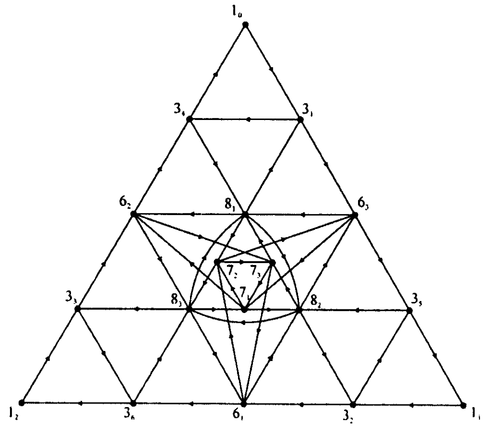


Figure 7.25: (K)

For the group $SU(3)$ itself, the adjacency matrix Δ of the fusion rules is given by the operator $\Delta = v_N \in B(\ell^2(\mathbb{N})\otimes\ell^2(\mathbb{N}))$, where v_N is as in (7.42). Consider the exact sequence $0 \to \mathcal{K} \to C^*(\Delta) \to C^*(\Delta)/\mathcal{K} \to 0$, where $\mathcal{K} = \mathcal{K}(\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})) \subset B(\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N}))$ are the compact operators. Let $\pi : B(\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})) \to B(\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N}))/\mathcal{K}$ be the quotient map. The resolvent $\rho(\Delta)$ of Δ is a subset of $\rho(\pi(\Delta))$ since for any $\lambda \in \rho(\Delta)$, applying π to $(\Delta - \lambda)b = 1$, for some $b \in B(\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N}))$, gives $(\pi(\Delta) - \lambda)\pi(b) = 1$. Then $\sigma(\Delta) \supset \sigma(\pi(\Delta)) = \mathfrak{D}$.

So if Γ is $SU(3)$ or one of its finite subgroups, the spectrum $\sigma(\Delta)$ of Δ is contained in $\mathfrak{D}$, illustrated in Figure 7.8. Thus the support of μ_Δ is contained in $\mathfrak{D}$, and $\int_{\mathbb{C}} \psi(z) d\mu_\Delta(z) = \int_{\mathfrak{D}} \psi'(z) d\mu_\Delta(z)$, where ψ' is the restriction of ψ to $\mathfrak{D}$. Since $\mathfrak{D}$ is bounded, the spectral measure μ_Δ is uniquely determined by its moments $\int_{\mathfrak{D}} z^m \overline{z}^n d\mu_\Delta(z)$.

Since the S-matrix simultaneously diagonalizes the representations of Γ, then as in Section 7.4 for $SU(2)$, the elements y_i in (7.49) are then given by $y_i = S_{0,j} = \sqrt{|\Gamma_j|}\chi_0(\Gamma_j)/\sqrt{|\Gamma|} = \sqrt{|\Gamma_j|}/\sqrt{|\Gamma|}$. Then the m, n^{th} moment $\varsigma_{m,n}$ is given by

$$\varsigma_{m,n} = \int z^m \overline{z}^n d\mu(z) = \sum_{j=1}^{n} \frac{|\Gamma_j|}{|\Gamma|} \chi_\rho(\Gamma_j)^m \overline{\chi_\rho(\Gamma_j)}^n. \tag{7.72}$$

Let $\Phi : \mathbb{T}^2 \to \mathfrak{D}$ be the map defined in (7.33). We wish to compute 'inverse' maps $\Phi^{-1} : \mathfrak{D} \to \mathbb{T}^2$ such that $\Phi \circ \Phi^{-1} = \mathrm{id}$. For $z \in \mathfrak{D}$, we can write $z = \omega_1 + \omega_2^{-1} + \omega_1^{-1}\omega_2$ and $\overline{z} = \omega_1^{-1} + \omega_2 + \omega_1\omega_2^{-1}$. Multiplying the first equation through by ω_1, we obtain $z\omega_1 = \omega_1^2 + \omega_1\omega_2^{-1} + \omega_2$. Then we need to find solutions ω_1 to the cubic equation

$$\omega_1^3 - z\omega_1^2 + \overline{z}\omega_1 - 1 = 0. \tag{7.73}$$

Similarly, we need to find solutions ω_2 to the cubic equation $\omega_2^3 - \overline{z}\omega_2^2 + z\omega_2 - 1 = 0$. We see that the three solutions for ω_2 are given by the complex conjugate of the three solutions for ω_1. Solving (7.73) we obtain solutions $\omega^{(k)}$, $k = 0, 1, 2$, given by

$$\omega^{(k)} = (z + 2^{-1/3}\epsilon_k P + 2^{1/3}\overline{\epsilon_k}(z^2 - 3\overline{z})P^{-1})/3,$$

where $\epsilon_k = e^{2\pi i k/3}$, $2^{1/3}$ takes a real value, and P is the cube root $P = (27 - 9z\overline{z} + 2z^3 + 3\sqrt{3}\sqrt{27 - 18z\overline{z} + 4z^2 + 4\overline{z}^3 - z^2\overline{z}^2})^{1/3}$ such that $P \in \{re^{i\theta} |\ 0 \leq \theta < 2\pi/3\}$. For the roots of a cubic equation that it does not matter whether the square root in P is taken to be positive or negative. We will take it to have positive value. We notice that the Jacobian J appears in the expression for P as the discriminant of the cubic equation (7.73). We can define maps $\Phi_{k,l}^{-1} : \mathfrak{D} \to \mathbb{T}^2$ by

$$\Phi_{k,l}^{-1}(z) = (\omega^{(k)}, \overline{\omega^{(l)}}), \qquad k, l \in \{0, 1, 2\}, \tag{7.74}$$

for $z \in \mathfrak{D}$. Now $\Phi(\Phi_{k,k}^{-1}(z)) \neq z$, for $k = 0,1,2$, however, for the other six cases ($k, l \in \{0,1,2\}$ such that $k \neq l$) we do indeed have $\Phi \circ \Phi_{k,l}^{-1} = \mathrm{id}$. These six $\Phi_{k,l}^{-1}(z)$ are the images of $\Phi_{0,1}^{-1}(z)$ under the action of the group $G \cong S_3$. The spectral measure of Γ (over $\mathbb{T}^2$) can then be taken as the average over these $\Phi_{k,l}^{-1}(z)$:

$$\begin{aligned}&\int_{\mathbb{T}^2} R_{m,n}(\omega_1,\omega_2) d\varepsilon(\omega_1,\omega_2)\\ &= \frac{1}{6}\sum_{j=1}^{n} \sum_{\substack{k,l\in\{0,1,2\}:\\ k\neq -l}} \frac{|\Gamma_j|}{|\Gamma|}(\omega^{(k,j)} + \omega^{(l,j)} + \overline{\omega^{(k,j)}\omega^{(l,j)}})^m(\overline{\omega^{(k,j)}} + \overline{\omega^{(l,j)}} + \omega^{(k,j)}\omega^{(l,j)})^n, \quad (7.75)\end{aligned}$$

where $\Phi^{-1}(\chi_\rho(\Gamma_j)) = (\omega^{(k,j)}, \overline{\omega^{(l,j)}})$, for $k,l \in \{0,1,2\}$, $k \neq l$.

7.8.1 Group $\mathbb{Z}_n \times \mathbb{Z}_n$

We will now compute the spectral measure for the graph $\mathcal{G}_\Gamma$ corresponding to the subgroup $\Gamma = \mathbb{Z}_n \times \mathbb{Z}_n$. This group has the $SU(3)$ McKay graph which is the "affine" version of the graph $\mathcal{A}^{(n+2)}$. The group contains $|\Gamma| = n^2$ elements, each of which is a separate conjugacy class $\Gamma_{k,l}$, $k,l \in \{0,1,2,\ldots,n-1\}$. Now $\chi_\rho(\Gamma_{k,l}) = \widetilde{\omega}_1^k + \widetilde{\omega}_2^{-l} + \widetilde{\omega}_1^{-k}\widetilde{\omega}_2^l \in \mathfrak{D}$, where $\widetilde{\omega}_j = e^{2\pi i/n}$, $j = 1,2$. Then by (7.75),

$$\int_{\mathbb{T}^2} R_{m,n}(\omega_1,\omega_2) d\varepsilon(\omega_1,\omega_2) = \sum_{j,k=0}^{n-1} \frac{1}{n^2}(\widetilde{\omega}_1^k + \widetilde{\omega}_2^{-l} + \widetilde{\omega}_1^{-k}\widetilde{\omega}_2^l)^m(\widetilde{\omega}_1^{-k} + \widetilde{\omega}_2^l + \widetilde{\omega}_1^k\widetilde{\omega}_2^{-l})^n.$$

Theorem 7.8.1 *For $\Gamma = \mathbb{Z}_n \times \mathbb{Z}_n$, the spectral measure of $\mathcal{G}_\Gamma$ on $\mathbb{T}^2$ is given by the product measure*

$$d\varepsilon(\omega_1,\omega_2) = d_{n/2}\omega_1 \; d_{n/2}\omega_2,$$

where d_m is the uniform measure on the $2m^{th}$ roots of unity.

7.8.2 Group $\Delta(3n^2)$, $n \equiv 0 \bmod 3$

This group has order $|\Gamma| = 3n^2$. Let $n = 3k$, $k \in \mathbb{N}$. Then the group has the $SU(3)$ McKay graph which is the "affine" version of the graph $\mathcal{D}^{(3k)}$. For $k,l \in \mathbb{Z}_n$, let $\widetilde{(k,l)}$ denote the set $\{(k,l),(l,-k-l),(-k-l,k)\}$. The character table for $\Delta(27k^2)$ is given in Table 7.7 ([82]). Here K_n is the set of all pairs $\{(k,l)|\ k,l \in \mathbb{Z}_n\} \setminus \{(0,0),(n/3,2n/3),(2n/3,n/3)\}$ such that K_n contains exactly one element from each $\widetilde{(k,l)}$ (except for $(0,0)$, $(n/3,2n/3)$, $(2n/3,n/3)$). There are two copies of the fundamental domain C_n (see Section 7.7) in K_n. The final row in the table denotes the pair (θ_1,θ_2) given by $(e^{2\pi i\theta_1}, e^{2\pi i\theta_2}) = \Phi^{-1}(\chi_\rho(\Gamma_j))$.

Γ_j	Γ_1	Γ_2	Γ_3	$\Gamma_{k,l}$, $(k,l) \in K_n$	Γ'_j, $j = 1,\ldots,6$
$\lvert\Gamma_j\rvert$	1	1	1	3	$n^2/3$
$\chi_\rho(\Gamma_j) \in \mathfrak{D}$	3	3ω	$3\overline{\omega}$	$e^{\frac{2\pi ik}{n}} + e^{-\frac{2\pi il}{n}} + e^{\frac{2\pi i(l-k)}{n}}$	0
$\Phi^{-1}(\chi_\rho(\Gamma_j)) \in \mathbb{T}^2$	(1,1)	$(\omega, \overline{\omega})$	$(\overline{\omega}, \omega)$	$(e^{2\pi ik}, e^{2\pi il})$	$(\omega, 1)$
$(\theta_1, \theta_2) \in [0,1]^2$	(0,0)	$(\frac{1}{3}, \frac{2}{3})$	$(\frac{2}{3}, \frac{1}{3})$	(k,l)	$(\frac{1}{3}, 0)$

Table 7.7: Character table for group $\Delta(3n^2)$, $n \equiv 0 \bmod 3$. Here $\omega = e^{2\pi i/3}$.

Let $\widetilde{\omega} = e^{2\pi i/n}$, and

$$\Omega(k,l) = (\widetilde{\omega}^k + \widetilde{\omega}^{-l} + \widetilde{\omega}^{l-k})^m (\widetilde{\omega}^{-k} + \widetilde{\omega}^l + \widetilde{\omega}_1^{k-l})^n. \tag{7.76}$$

Then by (7.75),

$$\begin{aligned}
&\int_{\mathbb{T}^2} R_{m,n}(\omega_1, \omega_2) d\varepsilon(\omega_1, \omega_2) \\
&= \frac{1}{3n^2}\Omega(0,0) + \frac{1}{3n^2}\Omega(\tfrac{1}{3}, \tfrac{2}{3}) + \frac{1}{3n^2}\Omega(\tfrac{2}{3}, \tfrac{1}{3}) + \frac{3}{3n^2}\sum_{k,l \in K_n} \Omega(k,l) + \frac{n^2/3}{3n^2}\sum_{j=1}^{6} \Omega(0, \tfrac{1}{3}) \\
&= \frac{1}{3n^2}\Omega(0,0) + \frac{1}{3n^2}\Omega(\tfrac{1}{3}, \tfrac{2}{3}) + \frac{1}{3n^2}\Omega(\tfrac{2}{3}, \tfrac{1}{3}) + \frac{1}{3n^2} \sum_{\substack{k,l \in \mathbb{Z}_n \\ (k,l) \neq (0,0), (\frac{1}{3}, \frac{2}{3}), (\frac{2}{3}, \frac{1}{3})}} \Omega(k,l) \\
&\quad + \frac{1}{9}\sum_{j=1}^{6} \frac{1}{6}(\Omega(0, \tfrac{1}{3}) + \Omega(0, \tfrac{2}{3}) + \Omega(\tfrac{1}{3}, 0) + \Omega(\tfrac{1}{3}, \tfrac{1}{3}) + \Omega(\tfrac{2}{3}, 0) + \Omega(\tfrac{2}{3}, \tfrac{2}{3})) \\
&= \frac{1}{3n^2} \sum_{k,l \in \mathbb{Z}_n} \Omega(k,l) + \frac{1}{9}(\Omega(0, \tfrac{1}{3}) + \Omega(0, \tfrac{2}{3}) + \Omega(\tfrac{1}{3}, 0) + \Omega(\tfrac{1}{3}, \tfrac{1}{3}) + \Omega(\tfrac{2}{3}, 0) + \Omega(\tfrac{2}{3}, \tfrac{2}{3})).
\end{aligned}$$

Then we have obtained:

Theorem 7.8.2 *The spectral measure (over $\mathbb{T}^2$) for the group $\Delta(3n^2)$, $n \equiv 0$ mod 3, is*

$$d\varepsilon(\omega_1, \omega_2) = \frac{1}{3}\, d_{n/2}\omega_1 \; d_{n/2}\omega_2 + \frac{\delta_{(1,\omega)} + \delta_{(1,\overline{\omega})} + \delta_{(\omega,1)} + \delta_{(\omega,\omega)} + \delta_{(\overline{\omega},1)} + \delta_{(\overline{\omega},\overline{\omega})}}{9}, \tag{7.77}$$

where d_n is the uniform measure over $2n^{\text{th}}$ roots of unity, $\omega = e^{2\pi i/3}$, and $\delta_{(u,u')}$ is the Dirac measure at $(u, u') \in \mathbb{T}^2$.

7.8.3 Group $(G) = \Sigma(216 \times 3)$

The subgroup (G) has order 648, and its McKay graph is the "affine" version of the graph $\mathcal{E}_2^{(12)}$. The character table for (G) is given in Table 7.8 ([23]).

j	1	2	3	4	5	6	7	8	9
$\lvert\Gamma_j\rvert$	1	1	1	12	12	12	12	12	12
$\chi_\rho(\Gamma_j) \in \mathfrak{D}$		3			$\sqrt{3}e^{\pi i/18}$			$\sqrt{3}e^{35\pi i/18}$	
$(\theta_1, \theta_2) \in [0,1]^2$	(0,0)	$(\frac{1}{3}, \frac{2}{3})$	$(\frac{2}{3}, \frac{1}{3})$	$(\frac{1}{9}, \frac{2}{9})$	$(\frac{1}{9}, \frac{5}{9})$	$(\frac{7}{9}, \frac{2}{9})$	$(\frac{2}{9}, \frac{1}{9})$	$(\frac{2}{9}, \frac{7}{9})$	$(\frac{5}{9}, \frac{1}{9})$

j	10	11	12	13	14	15
$\lvert\Gamma_j\rvert$	36	36	36	36	36	36
$\chi_\rho(\Gamma_j)$		$e^{17\pi i/9}$			$e^{\pi i/9}$	
(θ_1, θ_2)	$(\frac{5}{18}, \frac{1}{18})$	$(\frac{2}{18}, \frac{7}{18})$	$(\frac{8}{18}, \frac{1}{18})$	$(\frac{1}{18}, \frac{5}{18})$	$(\frac{1}{18}, \frac{8}{18})$	$(\frac{7}{18}, \frac{2}{18})$

j	16	17	18	19	20	21	22	23	24
$\lvert\Gamma_j\rvert$	54	54	54	9	9	9	24	72	72
$\chi_\rho(\Gamma_j)$		1			-1		0	0	0
(θ_1, θ_2)	$(0, \frac{1}{4})$	$(\frac{1}{12}, \frac{5}{12})$	$(\frac{5}{12}, \frac{1}{12})$	$(0, \frac{1}{2})$	$(\frac{2}{6}, \frac{1}{6})$	$(\frac{1}{6}, \frac{2}{6})$	$(0, \frac{1}{3})$	$(0, \frac{1}{3})$	$(0, \frac{1}{3})$

Table 7.8: Character table for group $(G) = \Sigma(216 \times 3)$. Here $\omega = e^{2\pi i/3}$.

Let us denote be $\Sigma_{j_1}^{j_2}$ the summation

$$\Sigma_{j_1}^{j_2} = \frac{1}{6} \sum_{j=j_1}^{j_2} \sum_{g \in G} (g(\widetilde{\omega}^{\theta_1} + \widetilde{\omega}^{-\theta_2} + \widetilde{\omega}^{\theta_2 - \theta_1}))^m (g(\widetilde{\omega}^{-\theta_1} + \widetilde{\omega}^{\theta_2} + \widetilde{\omega}^{\theta_1 - \theta_2}))^n,$$

where for each j, θ_1, θ_2 are given in Table 7.8, $\widetilde{\omega} = e^{2\pi i/18}$, and the action of $g \in G$ on $(\widetilde{\omega}^{\theta_1} + \widetilde{\omega}^{-\theta_2} + \widetilde{\omega}^{\theta_2 - \theta_1})$ is defined as follows. Suppose $g(\widetilde{\omega}^{\theta_1}, \widetilde{\omega}^{\theta_2}) = (\widetilde{\omega}^p, \widetilde{\omega}^q)$. Then $g(\widetilde{\omega}^{\theta_1} + \widetilde{\omega}^{-\theta_2} + \widetilde{\omega}^{\theta_2 - \theta_1}) = (\widetilde{\omega}^p + \widetilde{\omega}^{-q} + \widetilde{\omega}^{q-p})$. Then by (7.75),

$$\begin{aligned} &\int_{\mathbb{T}^2} R_{m,n}(\omega_1, \omega_2) d\varepsilon(\omega_1, \omega_2) \\ &= \frac{1}{648}\Sigma_1^3 + \frac{12}{648}\Sigma_4^9 + \frac{36}{648}\Sigma_{10}^{15} + \frac{54}{648}\Sigma_{16}^{18} + \frac{9}{648}\Sigma_{19}^{21} + \frac{24+72+72}{648}\Sigma_{22}^{24}. \end{aligned} \tag{7.78}$$

For $j = 1, 2, 3$, $\Sigma_1^3 = \sum \Omega(k, l)$, where $\Omega(k, l)$ is defined in (7.76) and the summation is over $(k, l) \in \{(0,0), (1/3, 2/3), (2/3, 1/3)\}$, which give the fixed points in $\mathbb{T}^2$ under the action of G. These are the points (k, l) such that $(e^{2\pi i k}, e^{2\pi i l}) \in D_{1/2}$. Let

$$I[d\varepsilon(\omega_1, \omega_2)] = \int_{\mathbb{T}^2} (\omega_1 + \omega_2^{-1} + \omega_1^{-1}\omega_2)^m (\omega_1^{-1} + \omega_2 + \omega_1\omega_2^{-1})^n d\varepsilon(\omega_1, \omega_2). \tag{7.79}$$

Then $\Sigma_1^3 = 3I[d^{(1/2)}(\omega_1, \omega_2)]$, where $d^{(m)}$ is again the uniform measure on the elements of D_m. For $j = 4, \ldots, 9$, the points $(\theta_1, \theta_2) \in [0,1]^2$ are $(1/9, 2/9)$, $(1/9, 5/9)$, $(7/9, 2/9)$,

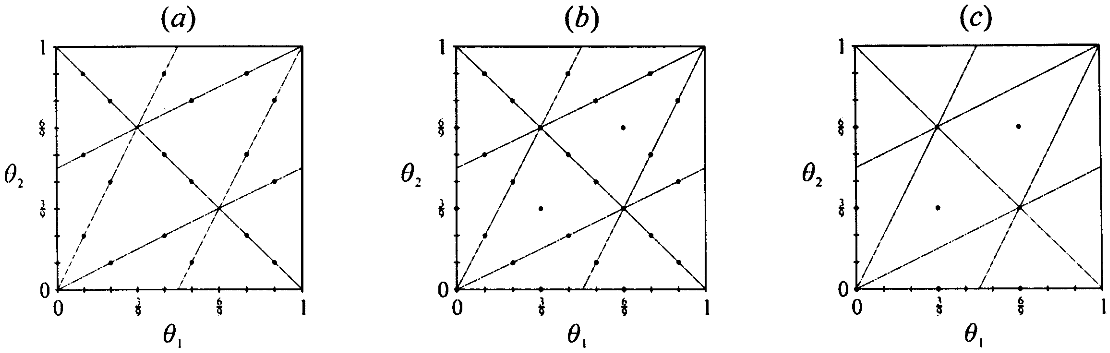


Figure 7.26: (a) the points $\{g(\theta_1, \theta_2)|g \in G\}$ for $j = 4, \ldots, 9$; (b) the points (θ_1, θ_2) such that $(e^{2\pi i\theta_1}, e^{2\pi i\theta_2}) \in D_{3/2}$; (c) the points (θ_1, θ_2) such that $e^{2\pi i\theta_k}$ is a 3rd root of unity, $k = 1, 2$.

$(2/9, 1/9)$, $(2/9, 7/9)$, $(5/9, 1/9)$. Under the action of G, each of these pairs has three images. These points are illustrated in Figure 7.26(a). We can obtain this distribution of points by taking the points (θ_1, θ_2) such that $(e^{2\pi i\theta_1}, e^{2\pi i\theta_2}) \in D_{3/2}$, illustrated in Figure 7.26(b), and then removing the points (θ_1, θ_2) such that $\theta_k \in \{0, 1/3, 2/3\}$, $k = 1, 2$, illustrated in Figure 7.26(c). Then we can write $3\Sigma_4^9 = \sum_{(e^{2\pi ik}, e^{2\pi il}) \in D_{3/2}} \Omega(k, l) - \sum_{k,l=0}^{2} \Omega(k/3, l/3)$, giving $3\Sigma_4^9 = 27I[d^{(3/2)}(\omega_1, \omega_2)] - 9I[d_{3/2}\omega_1 \; d_{3/2}\omega_2]$.

For $j = 10, \ldots, 15$, the points (θ_1, θ_2) are $(5/18, 1/18)$, $(2/18, 7/18)$, $(8/18, 1/18)$, $(1/18, 5/18)$, $(1/18, 8/18)$, $(7/18, 2/18)$. Under the action of G, each of these pairs has six images, which are illustrated in Figure 7.27(a). At these points, $J^2 = 48\pi^4$. We can obtain this distribution of points by taking the points (θ_1, θ_2) such that $(e^{2\pi i\theta_1}, e^{2\pi i\theta_2}) \in D_3$, illustrated in Figure 7.27(b), each with the weight J^2 evaluated at that point. Since the points indicated by white circles in Figure 7.27(b) map to the boundary of $\mathfrak{D}$, here $J^2 = 0$. We must then remove the points indicated by black circles in the interior of the triangular regions in Figure 7.27(b) which are not in $\{g(\theta_1, \theta_2)|g \in G\}$. This can be done by removing the points (θ_1, θ_2) such that $e^{2\pi i\theta_k}$ is a 6th root of unity, for $k = 1, 2$, illustrated in Figure 7.27(c), again with the weight J^2 evaluated at each point. The value of J^2 at the black circles near the corners of the triangular regions is $12\pi^4$. For the points in the centre of each triangular region, the eigenvalue is zero, therefore these points do not contribute to the summation in (7.75). Then $6\Sigma_{10}^{15} = \sum_{(e^{2\pi ik}, e^{2\pi il}) \in D_3} J^2\, \Omega(k, l)/48\pi^4 - \sum_{k,l=0}^{5} J^2\, \Omega(k/6, l/6)/12\pi^4$, and therefore for Σ_{10}^{15} we have obtained $\Sigma_{10}^{15} = 108I[J^2\, d^{(3)}(\omega_1, \omega_2)]/288\pi^4 - 36I[J^2\, d_3\omega_1 \; d_3\omega_2]/72\pi^4$.

For $j = 16, 17, 18$, the points (θ_1, θ_2) are $(0, 1/4)$, $(1/12, 5/12)$, $(5/12, 1/12)$. Under the action of G, each of these pairs has six images, which are given by the solid black circles

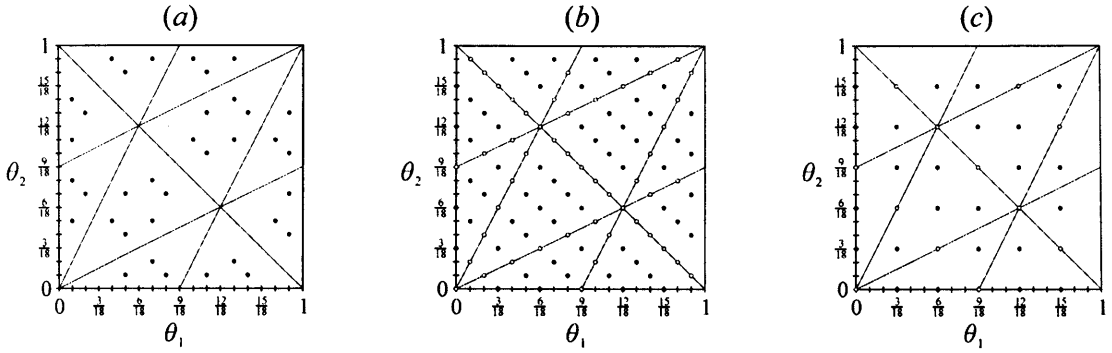


Figure 7.27: (a) the points $\{g(\theta_1, \theta_2)|g \in G\}$ for $j = 10, \ldots, 15$; (b) the points (θ_1, θ_2) such that $(e^{2\pi i\theta_1}, e^{2\pi i\theta_2}) \in D_3$; (c) the points (θ_1, θ_2) such that $e^{2\pi i\theta_k}$ is a 6^{th} root of unity, $k = 1, 2$.

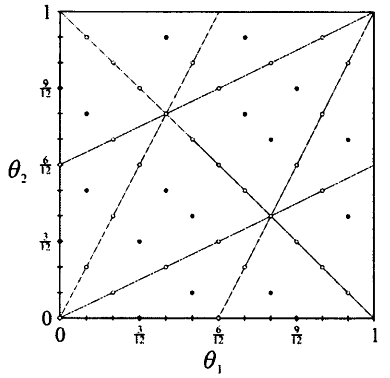


Figure 7.28: The points (θ_1, θ_2) such that $(e^{2\pi i\theta_1}, e^{2\pi i\theta_2}) \in D_2$.

in Figure 7.28. At these points, $J^2 = 64\pi^4$. Now the points indicated by white circles in Figure 7.28 all map to the boundary of $\mathfrak{D}$, and hence the value of J^2 at these points is zero. Therefore $6\Sigma_{16}^{18} = \sum_{(e^{2\pi ik}, e^{2\pi il}) \in D_2} J^2\, \Omega(k, l)/16\pi^4$, giving $\Sigma_{16}^{18} = 48I[J^2\, d^{(2)}(\omega_1, \omega_2)]/96\pi^4$.

For $j = 19, 20, 21$, the points (θ_1, θ_2) each have three images under the action of G. The points in $\{g(\theta_1, \theta_2)|g \in G\}$ are the points (k, l) such that $(e^{2\pi ik}, e^{2\pi il}) \in D_1$, apart from $(k, l) \in \{(0, 0), (1/3, 2/3), (2/3, 1/3)\}$, which give the fixed points of $\mathbb{T}^2$ under the action of G. So $3\Sigma_{19}^{21} = \sum_{(e^{2\pi ik}, e^{2\pi il}) \in D_1} \Omega(k, l) - \sum_{(e^{2\pi ik}, e^{2\pi il}) \in D_{1/2}} \Omega(k, l)$, i.e. $3\Sigma_{19}^{21} = 12I[d^{(1)}(\omega_1, \omega_2)] - 3I[d^{(1/2)}(\omega_1, \omega_2)]$. Finally, $(\theta_1, \theta_2) = (0, 1/3)$ for $j = 22, 23, 24$, which has six images under the action of G. These are the points in the interior of the triangular regions in Figure 7.26(c) (but not the fixed points $(\theta_1, \theta_2) \in \{(0, 0), (1/3, 2/3), (2/3, 1/3)\}$). These points can be obtained by taking the points (θ_1, θ_2) such that $e^{2\pi i\theta_k}$ is a 3^{rd} root of unity, for $k = 1, 2$, with the weight J^2 evaluated at each point. For the fixed points $J^2 = 0$, whilst for all the other points $J^2 = 12\pi^4$. Then $6\Sigma_{22}^{24} = \sum_{k,l=0}^{2} J^2\,\Omega(k/3,l/3)/12\pi^4$, giving $\Sigma_{22}^{24} = 9I[J^2\,d_{3/2}\omega_1\;d_{3/2}\omega_2]/72\pi^4$.

Then (7.78) becomes

$$\begin{aligned}
&\int_{\mathbb{T}^2} R_{m,n}(\omega_1,\omega_2)d\varepsilon(\omega_1,\omega_2)\\
&= \frac{1}{216}I[d^{(1/2)}(\omega_1,\omega_2)] + \frac{1}{6}I[d^{(3/2)}(\omega_1,\omega_2)] - \frac{1}{18}I[d_{3/2}\omega_1\;d_{3/2}\omega_2]\\
&\quad+\frac{1}{48\pi^4}I[J^2\,d^{(3)}(\omega_1,\omega_2)] - \frac{1}{36\pi^4}I[J^2\,d_3\omega_1\;d_3\omega_2] + \frac{1}{96\pi^4}I[J^2\,d^{(2)}(\omega_1,\omega_2)]\\
&\quad+\frac{1}{18}I[d^{(1)}(\omega_1,\omega_2)] - \frac{1}{72}I[d^{(1/2)}(\omega_1,\omega_2)] + \frac{7}{216\pi^4}I[J^2\,d_{3/2}\omega_1\;d_{3/2}\omega_2].
\end{aligned}$$

Thus we obtain the following result:

Theorem 7.8.3 *The spectral measure (over $\mathbb{T}^2$) for the group $(G) = \Sigma(216\times 3)$, is*

$$\begin{aligned}
d\varepsilon &= \frac{1}{48\pi^4}J^2\,d^{(3)} + \frac{1}{96\pi^4}J^2\,d^{(2)} + \frac{1}{6}d^{(3/2)} + \frac{1}{18}d^{(1)} - \frac{1}{108}d^{(1/2)}\\
&\quad-\frac{1}{36\pi^4}J^2\,d_3\,d_3 + \left(\frac{7}{216\pi^4}J^2 - \frac{1}{18}\right)d_{3/2}\,d_{3/2},
\end{aligned} \tag{7.80}$$

where d_m is the uniform measure over $2m^{\text{th}}$ roots of unity and $d^{(m)}$ is the uniform measure on the points in D_m.

7.8.4 Kostant Polynomial

We briefly mention the Kostant polynomial, which can also be defined for finite subgroups of $SU(3)$. For γ an irreducible representation of $\Gamma \subset SU(3)$, the Kostant polynomial is

$$F_\gamma(t_1,t_2) = \sum_{\lambda_1,\lambda_2}\langle\gamma,\rho_{(\lambda_1,\lambda_2)}\rangle_\Gamma\; t_1^{\lambda_1}t_2^{\lambda_2},$$

where the summation is over all $\lambda_1,\lambda_2 \geq 0$, and $\langle\gamma,\rho_{(\lambda_1,\lambda_2)}\rangle_\Gamma$ is the multiplicity of γ in the representation $\rho_{(\lambda_1,\lambda_2)}$ of $SU(3)$ restricted to Γ, where $\rho_{(\lambda_1,\lambda_2)}$ is the representation of $SU(3)$ with Dynkin labels (λ_1,λ_2).

For $SU(2)$ the Kostant polynomial F_γ has the simple form (7.23). Does a similarly simple form exist for the Kostant polynomial F_γ for $SU(3)$? Desmier, Sharp and Patera [23] compute this polynomial for the groups (I) $= \Sigma(168)$, (G) $= \Sigma(216\times 3)$, (L) $= \Sigma(360\times 3)$ and $G(13,3,3) = \mathbb{Z}_{12}\times\mathbb{Z}_3$, where they have the form

$$F_\gamma(t_1,t_2) = \frac{(1-t_1t_2)z_\gamma(t_1,t_2)}{(1-t_1^a)(1-t_1^b)(1-t_1^c)(1-t_2^a)(1-t_2^b)(1-t_2^c)},$$

where $z_\gamma(t_1,t_2)$ is a finite polynomial, and a, b, c are the integers given in Table 7.9

Subgroup $\Gamma \subset SU(3)$	a	b	c
(I) $= \Sigma(168)$	4	6	14
(G) $= \Sigma(216 \times 3)$	9	12	18
(L) $= \Sigma(360 \times 3)$	6	12	30
$\mathbb{Z}_{12} \times \mathbb{Z}_3$	3	13	16

Table 7.9: Integers a, b, c for Kostant polynomial F_γ

7.9 Hilbert Series of q-deformations of CY-Algebras of Dimension 3

We will now introduce the Calabi-Yau and q-deformed Calabi-Yau algebras of dimension 3, which are the $SU(3)$ generalizations of the pre-projective algebras of Section 7.5.4. For certain $\mathcal{ADE}$ graphs we will also compute the Hilbert series of the q-deformed CY-algebras of dimension 3.

Let $\mathcal{G}$ be an oriented graph, and $\mathbb{C}\mathcal{G}$, $[\mathbb{C}\mathcal{G}, \mathbb{C}\mathcal{G}]$ be as in Section 7.5.4. We define a derivation $\partial_a : \mathbb{C}\mathcal{G}/[\mathbb{C}\mathcal{G}, \mathbb{C}\mathcal{G}] \to \mathbb{C}\mathcal{G}$ by

$$\partial_a(a_1 \cdots a_n) = \sum_j a_{j+1} \cdots a_n a_1 \cdots a_{j-1},$$

where the summation is over all indices j such that $a_j = a$. Then for a potential $\Phi \in \mathbb{C}\mathcal{G}/[\mathbb{C}\mathcal{G}, \mathbb{C}\mathcal{G}]$, which is some linear combination of cyclic paths in $\mathcal{G}$, we define the algebra

$$A(\mathbb{C}\mathcal{G}, \Phi) \cong \mathbb{C}\mathcal{G}/\{\partial_a \Phi\},$$

which is the quotient of the path algebra by the two-sided ideal generated by the elements $\partial_a \Phi \in \mathbb{C}\mathcal{G}$, for all edges a of $\mathcal{G}$. We define the Hilbert series $H_A(t)$ as in Section 7.5.4.

If $A(\mathbb{C}\mathcal{G}, \Phi)$ is a Calabi-Yau algebra of dimensions $d \geq 3$ and $\deg \Phi = d$, then

$$H_A(t) = \frac{1}{1 - \Delta_{\mathcal{G}} t + \Delta_{\mathcal{G}}^T t^{d-1} - t^d}.$$

Note that since $H_A(t)$ is a formal power series in t, if the matrix $B(t) := (1 - \Delta_{\mathcal{G}} t + \Delta_{\mathcal{G}}^T t^{d-1} - t^d)$ is not invertible then it would not be invertible if t was any value in $\mathbb{C}$. However, when $t = 0$, $B(0)$ is just the identity matrix, which is trivially invertible. Hence $B(t)$ is invertible.

Let Γ be a subgroup of $SU(3)$. For the McKay graph $\mathcal{G}_\Gamma$ one can define a cell system W as in [49], where $W(\triangle_{ijk})$ is a complex number for every triangle $\triangle_{ijk}$ on $\mathcal{G}_\Gamma$ whose vertices are labelled by the irreducible representations i, j, k of Γ. We introduce the following potential

$$\Phi_\Gamma = \sum_{\triangle_{ijk} \in \mathcal{G}_\Gamma} W(\triangle_{ijk}) \cdot \triangle_{ijk} \quad \in \mathbb{C}\mathcal{G}_\Gamma / [\mathbb{C}\mathcal{G}_\Gamma, \mathbb{C}\mathcal{G}_\Gamma].$$

Then dividing out $\mathbb{C}\mathcal{G}_\Gamma$ by the ideal generated by $\delta_a \Phi_\Gamma$ for all edges a of $\mathcal{G}_\Gamma$, by [49, Theorem 4.4.6], $A(\mathbb{C}\mathcal{G}_\Gamma, \Phi_\Gamma)$ is a Calabi-Yau algebra of dimension 3, and the Hilbert series is [14, Theorem 4.6]

$$H_A(t) = \frac{1}{1 - \Delta_\Gamma t + \Delta_\Gamma^T t^2 - t^3}.$$

Theorem 7.9.1 *Let Γ be a finite subgroup of $SU(3)$, $\{\rho_0 = \mathrm{id}, \rho_1 = \rho, \rho_2, \dots, \rho_s\}$ its irreducible representations and $\mathcal{G}_\Gamma$ its McKay graph. Then if P_{S,ρ_j} is the Molien series of the symmetric algebra S of $\overline{\mathbb{C}^N}$, and $H(t)$ is the Hilbert series of $A(\mathbb{C}\mathcal{G}_\Gamma, \Phi_\Gamma)$,*

$$H_{\rho_j, 1_0}(t) = P_{S,\rho_j}(t).$$

Proof. Let Γ be a subgroup of $SU(3)$ with irreducible representations ρ_j, $j = 1, \dots, s$, where $\rho_0 = \mathrm{id}$ is the identity representation and $\rho_1 = \rho$ the fundamental representation. The fundamental matrices Δ_Γ, Δ_Γ^T defined by $\rho \otimes \rho_i = \sum_{j=0}^s (\Delta_\Gamma)_{i,j} \rho_j$, $\overline{\rho} \otimes \rho_i = \sum_{j=0}^s (\Delta_\Gamma^T)_{i,j} \rho_j$, satisfy, by [50, Cor. 2.4(i)],

$$\sum_{j=0}^{s} \left(-(\Delta_\Gamma)_{\rho_i,\rho_j} t + (\Delta_\Gamma^T)_{\rho_i,\rho_j} t^2\right) P_{S,\rho_j}(t) = -(1 - t^3) P_{S,\rho_i}(t) + \delta_{i,0},$$

so we have

$$\begin{aligned} \sum_{j=0}^{s} \left(\mathbf{1}_{\rho_i,\rho_j} - (\Delta_\Gamma)_{\rho_i,\rho_j} t + (\Delta_\Gamma^T)_{\rho_i,\rho_j} t^2 - \mathbf{1}_{\rho_i,\rho_j} t^3\right) P_{S,\rho_j}(t) &= \delta_{i,0} \\ \sum_{j=0}^{s} \left(\mathbf{1} - (\Delta_\Gamma) t + (\Delta_\Gamma^T) t^2 - \mathbf{1} t^3\right)_{\rho_i,\rho_j} P_{S,\rho_j}(t) &= \delta_{i,0}. \end{aligned}$$

Then $\left(P_{S,\rho_j}(t)\right)_{\rho_j}$ is the first column of the inverse of the matrix $\left(\mathbf{1} - (\Delta_\Gamma) t + (\Delta_\Gamma^T) t^2 - \mathbf{1} t^3\right)$, which is invertible since it is just the identity when $t = 0$, that is,

$$P_{S,\rho_j}(t) = \left(\left(\mathbf{1} - (\Delta_\Gamma) t + (\Delta_\Gamma^T) t^2 - \mathbf{1} t^3\right)^{-1}\right)_{\rho_j,\rho_0} = H_{\rho_j,\rho_0}.$$

□

For the $\mathcal{ADE}$ graphs, we define a potential Φ by

$$\Phi = \sum_{i,j,k} W(\triangle_{ijk}) \cdot \triangle_{ijk} \quad \in \mathbb{C}\mathcal{G} / [\mathbb{C}\mathcal{G}, \mathbb{C}\mathcal{G}],$$

where the Ocneanu cells $W(\triangle_{ijk})$ are computed in Chapter 4. If the $\mathcal{ADE}$ graph $\mathcal{G}$ has an associated subgroup Γ of $SU(3)$, as given in Table 7.6, then the quotient algebra $A(\mathbb{C}\mathcal{G},\Phi)$ provides a Calabi-Yau deformation of the algebra $A(\mathbb{C}\mathcal{G},\Phi_\Gamma)$. For certain $\mathcal{ADE}$ graphs, the Hilbert series for the q-deformed algebra $A(\mathbb{C}\mathcal{G},\Phi)$ is given by the following:

(i) Let $\mathcal{G}$ be one of the following $\mathcal{ADE}$ graphs, $\mathcal{A}^{(n)}$, $\mathcal{D}^{(n)}$. Then

$$H_A(t) = \frac{1-Pt^h}{1-\Delta t+\Delta^T t^2 - t^3},$$

where P is the permutation matrix corresponding to a $\mathbb{Z}/3\mathbb{Z}$ symmetry of the graph.

(ii) Let $\mathcal{G}$ be $\mathcal{A}^{(2m)*}$. Then

$$H_A(t) = \frac{1+Wt^m}{1-\Delta t+\Delta^T t^2 - t^3},$$

where W is the permutation matrix corresponding to the $\mathbb{Z}/2\mathbb{Z}$ involution of the graph $\mathcal{A}^{(2m)*}$.

(iii) Let $\mathcal{G}$ be $\mathcal{D}^{(2m)*}$. Then

$$H_A(t) = \frac{1+PWt^m}{1-\Delta t+\Delta^T t^2 - t^3},$$

where P, W respectively are permutation matrices corresponding to a $\mathbb{Z}/3\mathbb{Z}$ symmetry, $\mathbb{Z}/2\mathbb{Z}$ involution respectively of the graph $\mathcal{D}^{(2m)*}$.

(iv) Let $\mathcal{G}$ be $\mathcal{E}^{(8)}$. Then

$$H_A(t) = (1+B_5t^5 - B_6t^6 - B_7t^7 - B_8t^8)(1-\Delta t+\Delta^T t^2 - t^3)^{-1},$$

where B_k, $k=5,6,7,8$, are matrices which are zero almost everywhere.

In (i) and (iii) the involution P is an automorphism of the underlying graph, which is the identity for $\mathcal{D}^{(n)}$. For the graphs $\mathcal{A}^{(n)}$, $\mathcal{D}^{(2m)*}$, let V be the permutation matrix corresponding to the clockwise rotation of the graph by $2\pi/3$. Then

$$P = \begin{cases} V & \text{for } \mathcal{A}^{(n)}, \\ V^m & \text{for } \mathcal{D}^{(2m)*}. \end{cases}$$

For $\mathcal{A}^{(2m)*}$, W is the involution which sends vertices $p \longrightarrow l-p$, for $p=1,\dots,\lfloor (m-1)/2 \rfloor$, $l=\lfloor (m+1)/2 \rfloor$, whilst for $\mathcal{D}^{(2m)*}$, W is the involution which sends

$$\alpha_p \longrightarrow \alpha_{l-p}, \qquad \alpha \in \{i,j,k\},$$

for $p = 1, \dots, \lfloor (m-1)/2 \rfloor$, $l = \lfloor (m+1)/2 \rfloor$. Since the permutation matrices P and W commute, the matrix PW has order 6, and generates a group of permutations of the vertices of $\mathcal{D}^{(2m)*}$ which is isomorphic to $\mathbb{Z}_6$. The matrices B_k, $k = 5,6,7,8$, in (iv) are zero almost everywhere, apart from

$$B_5(j_l, j_{l+1}) = 1, \text{ for } l = 2,3,5,6, \text{ and } B_5(j_1, j_5) = B_5(j_4, j_2) = 1,$$

$$B_6(j_l, i_{l+2}) = 1, \text{ for } l = 2,3,5,6, \text{ and } B_6(j_1, i_6) = B_6(j_4, i_3) = 1,$$

$$B_7(j_l, i_{l-2}) = 1, \qquad B_8(i_l, i_{l-2}) = 1, \text{ for } l = 1, \dots, 6.$$

Since Q be a permutation matrix which corresponds to a symmetry of a graph with adjacency matrix Δ. Then $\Delta = Q^{-1} \Delta Q$, so that $Q\Delta = \Delta Q$. Similarly $Q\Delta^T = \Delta^T Q$. Then we see that the numerator in (i)-(iii) commutes with the denominator. In (iv), the matrices $(1 + B_5 t^5 - B_6 t^6 - B_7 t^7 - B_8 t^8)$ and $(1 - \Delta t + \Delta^T t^2 - t^3)^{-1}$ do not commute. Indeed, $H_A(t) = (1 - \Delta t + \Delta^T t^2 - t^3)^{-1}(1 + B_5 t^5 - B_6' t^6 - B_7' t^7 - B_8 t^8)$, where B_6', B_7' are unitarily equivalent to B_6, B_7 respectively, obtained by conjugating by the unitary U defined, for $i, j = 1, 2, \dots, 8$, by $U_{i,j} = 1$ if $j' \equiv i' + 1 \bmod 4$, where $i', j' \in \{1,2,3,4\}$ such that $j - j' = i - i' = 4k$ for $k \in \{0,1,2\}$, and $U_{i,j} = 0$ otherwise.

The formulas above have been checked "by hand" for the graphs $\mathcal{A}^{(n)}$ for $n = 4,5,6,7$, $\mathcal{D}^{(n)}$ for $n = 5,6,7$, $\mathcal{D}^{(n)*}$, $n = 6,8,10$, and $\mathcal{E}^{(8)}$, where we explicitly wrote out all the allowed paths in $A(\mathbb{C}\mathcal{G}, \Phi)$ and compared the dimensions that appeared with those given by the Hilbert series. The space of allowed paths for these graphs does not particularly depend on the values of the cells $W(\triangle_{ijk})$, except for whether the cells are zero or non-zero. In fact, were we to replace the non-zero cells $W(\triangle_{ijk})$ by an arbitrary choice of non-zero complex numbers $W'(\triangle_{ijk})$ (which would not be a solution for the cells of the graph), then $A(\mathbb{C}\mathcal{G}, \Phi)$ would most likely be isomorphic to $A(\mathbb{C}\mathcal{G}, \Phi')$, where $\Phi' = \sum_{i,j,k} W'(\triangle_{ijk}) \cdot \triangle_{ijk}$. However, suppose a path $\gamma \in \mathbb{C}\mathcal{G}$ is identified with a linear combination $\sum b_i \gamma_i$ of paths γ_i in $A(\mathbb{C}\mathcal{G}, \Phi)$, with $b_i \neq 0$ for all i. For certain choices of $W'(\triangle_{ijk})$ it is possible that now $b_i = 0$ for all i, and hence $\gamma = 0$ in $A(\mathbb{C}\mathcal{G}, \Phi')$. In this case $\dim(A(\mathbb{C}\mathcal{G}, \Phi')_k) < \dim(A(\mathbb{C}\mathcal{G}, \Phi)_k)$ for some $k \in \mathbb{N}$.

For $\mathcal{D}^{(n)*}$, $n = 6,8,10$, and $\mathcal{E}^{(8)}$ the polynomials are infinite, but there is only at most one allowed path of length $k > n$ from vertex i to j on the graph, found by adding a closed loop $j \to j' \to j'' \to j$ to the allowed path of length $k - 3$ from i to j, and hence the allowed paths in $A(\mathbb{C}\mathcal{G}, \Phi)$ could be written out explicitly here also. In the $SU(2)$ case, the permutation matrices P appearing in the numerator of $H_A(t)$ corresponded to the Nakayama permutation of the Dynkin diagram. The above claim then raises the question of the relation between the automorphisms which appear in the numerators of the expressions for $H_A(t)$ with Nakayama's automorphisms.

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