Fractal Activity Time Risky Asset Models with Dependence

Thesis submitted to
Cardiff University
for the degree of
Doctor of Philosophy

by Stuart Gary Petherick

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Acknowledgements

Firstly, I would like to thank my PhD supervisor, Professor Nikolai Leonenko, for his unwavering support throughout my research program. Over the last four years, your endless suggestions and ideas have helped shape this thesis. Your continued commitment and passion for research never fails to inspire, and I will always treasure our time together.

I must also mention my fellow collaborators, Alla Sikorskii from Michagan State Univerity and Emmanuelle Taufer from Trento University. Thanks to your efforts and valuable insights, we have managed to submit four papers to journals. I will be forever grateful for these contributions, and I wish you both the very best for the future.

A big thank you to all my family and friends. You are truly the driving force behind me, and none of this would be possible without your love and support.

My research was supported in part by the Commission of the European Communities grant PIRSES-GA-2008-230804 within the programme "Marie Curie Actions". The grant enable me to attend and present at the "On Long-Range Dependence" workshop and the "Modern Stochastics: Theory and Applications II" conference at Kiev University. Thank you in particular to Yuliya Mishura, Lyudmyla Sakhno, Georgiy Shevchenko and Kostiantyn Ralchenko for organising the events, and for making me feel welcome.

I would also like to acknowledge ESPRC for further provision of funding.

Stuart Petherick, September 2011

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Abstract

The paradigm Black-Scholes model for risky asset prices has occupied a central place in asset-liability management since its discovery in 1973. While the underlying geometric Brownian motion surely captured the essence of option pricing (helping spawn a multi-billion pound derivatives industry), three decades of statistical study has shown that the model departs significantly from the realities of returns (increments in the logarithm of risky asset price) data.

To remedy the shortcomings of the Black-Scholes model, we present the fractal activity time geometric Brownian motion model proposed by Chris Heyde in 1999. This model supports the desired empirical features of returns including; no correlation but dependence, and distributions with heavier tails and higher peaks than Gaussian. In particular, the model generalises geometric Brownian motion whereby the standard Brownian motion is evaluated at random activity time instead of calendar time.

There are also strong suggestions from literature that the activity time process here is approximately self-similar. Thus we require a way to accommodate both the desired distributional and dependence features as well as the property of asymptotic self-similarity. In this thesis, we describe the construction of this fractal activity time; based on chi-square type processes, through Ornstein-Uhlenbeck processes driven by Lévy noise, and via diffusion-type processes. Once we validate the model by fitting real data, we endeavour to state a new explicit formula for the price of a European option. This is made possible as Heyde's model remains within the Black-Scholes framework of option pricing, which allows us to use their engendered arbitrage-free methodology.

Finally, we introduce an alternative to the previously considered approach. The motivation for which comes from the understanding that activity time cannot be exactly self-similar. We provide evidence that multi-scaling occurs in financial data and outline another construction for the activity time process.

Presentations and Publications

1. University of Wales Intercollegiate Colloquium in Mathematics, Gregynog 2008. "Monofractal and Multifractal Models in Finance".

2. Workshop on Long-Range Dependence, Kiev 2009. "Fractal Activity Time Risky Asset Models".

3. Young Statisticians Meeting, Liverpool 2010. "Why Financial Data are Interesting to Statisticians".

4. Modern Stochastics: Theory and Applications II, Kiev 2010. "Fractal Activity Time Risky Asset Models with Dependence".

5. LEONENKO NN PETHERICK S AND SIKORSKII A. *The Student Subordinator Model with Dependence for Risky Asset Returns.* Communications in Statistics: Theory and Methods **40,** 3509-3523 (2011).

6. LEONENKO NN PETHERICK S AND SIKORSKII A. *Fractal activity time models for risky asset with dependence and generalized hyperbolic distributions.* em Stochastic Analysis and its Applications (in press).

7. LEONENKO NN PETHERICK S AND SIKORSKII A. *Normal Inverse Gaussian Model for Risky Asset with Dependence.* Statistics and Probability Letters **82,** 109-115 (2012).

8. LEONENKO NN PETHERICK S AND TAUFER E. Multifractal Scaling for Risky Asset Modelling (submitted).

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Chapter 1

Introduction

With many modern finance applications such as; derivative pricing, systematic trading and risk control, risky asset modelling has become an extremely popular task. The growing interest largely stems from the complexities found in real-life data, and is shared by both theorists in academia and practitioners in industry. Such price movements over time appeal to those who crave a sense of order in a financial world where there are a convolution of industrial, economic and political factors. In recent years, many elaborate models have been built to overcome these challenges. As a result, there now exists a vast and rapidly expanding literature, in the field of Mathematical Finance, dedicated to the topic of building, validating and developing suitable models for risky assets.

An asset is said to be risky if it has an uncertain future (unlike a bank account, for example, where a fixed interest is added each year). They can come in many different forms; a stock traded at some exchange, a quoted interest rate, a foreign exchange rate etc, and can be sampled at many different frequencies; monthly, daily, intra-daily etc. The main focus of Mathematical Finance, however, is not these prices themselves, but their random changes over time. If one denotes $P(t), t \geq 0$, as the price of a risky asset at time t, the so-called returns $X_\tau(t)$, at time t and scale τ is simply the relative variation of the price from $t - \tau$ to t,

$$X_\tau(t) = \frac{P(t) - P(t - \tau)}{\tau}.$$

If τ is small enough, one has approximately,

$$X_\tau(t) \simeq \log P(t) - \log P(t - \tau).$$

For consistency with the statistical studies of past and present papers in Mathematical Finance, we set $\tau = 1$ in this thesis, and use the (one-period) returns process $X(t) = X_1(t)$

exclusively. These fluctuations are convenient to model because the resulting time series has the attractive statistical properties of stationarity and ergodicity.

Most academics would agree that the field of Mathematical Finance dates back to the beginning of the 20th century. Indeed, it was Louis Bachelier's dissertation, "The Theory of Speculation" [10], first published in 1900, which provided us with; many concepts of stochastic analysis, the first model of the stochastic process known today as Brownian motion and a theory for the valuation of options. Unfortunately however, Bachelier was not recognised for his immense contribution, as he died in 1946 relatively unknown. In fact, it wasn't until the late 1950's, that his achievements were identified.

The first person to rediscover and promote Bachelier's pioneering work was M.F. Maury Osborne. The paper Osbourne (1959) [104] began to address some of the imperfections of the Brownian motion model in [10] i.e. negative stock prices, and a more appropriate model was suggested. Almost simultaneously with Paul Samuelson, the first economist to win the Nobel prize, Osborne then successfully employed a "geometric Brownian motion" (GBM) to model risky assets. First coined geometric or, in Samuelson (1965) [113], "economic" Brownian motion, it gave the underlying price process an exponential form. Their argument eventuating from the attractive property that the price of a risky asset appeared to follow a log-normal distribution.

A few years later, Fisher Black, Myron Scholes and Robert C. Merton demonstrated how to analytically obtain the price of a European option based on the GBM model. Most significantly, Black and Scholes (1973) [26] and Merton (1973) [100] were able to bring the GBM model to the attention of the finance community. Their option pricing formula and the elegant theory which had engendered, gave people a new and improved understanding of financial data plus an early statistical description of asset price changes. One part in particular, the arbitrage-free methodology, still occupies a central place in asset-liability management, theory and practice. Movement from a risky to a non-risky financial world meant we could approximate real market. It was an essential building block for Black, Scholes and Merton, and one that led to option prices being calculated using palatable statistical techniques.

The timing of the "Black-Scholes formula" coincided with the advent of tradable listed options. The largest options exchange, with an annual trading volume now of around one billion contracts - the Chicago Board Options Exchange, was established just one month prior to the release of Black and Scholes (1973) [26]. Many people at that time suggested it was created so that the US Securities and Exchange Commission would sanction exchange-regulated options trading. The author's simplistic approach was perfectly in tune with something regulators could understand and accept.

In the decades that followed, investment bankers have had the idea that simple agreements such like derivatives would be a more efficient way of transferring risk than actually buying and selling assets. Thus, traders everywhere began using the Black-Scholes formula, along with its ensuing comprehensive theory. Considered then by most as the best model available to price an European option, it is still regarded as a powerful tool today. Investors will often use it as the starting point for their option valuations.

Undoubtedly, Black, Scholes and Merton captured the essence of option pricing with their formula, however recently, the suitability of the underlying stochastic process to model the price of risky assets has been under scrutiny. Ever since large sets of financial data became more widely available, intensive investigations of GBM (see for instance, Heyde and Liu (2001) [67], and Cont (2001) [36]) have been prompted. These have shown that the model departs from the realities of risky asset price and risky asset returns data in quite a number of important ways. The empirical characteristics of returns, which are now universally accepted, include; no correlation but some dependence, and a higher peak and heavier tails than the Gaussian distribution.

In order to find a model which incorporates all the empirical realities of risky assets, an alternative to the GBM model is required. One idea is to generalise GBM by replacing the standard Brownian motion with a Lévy process (Eberlain and Raible (1999) [44] or Schoutens (2003) [116]). In turn, this would allow for: flexible non-Gaussian marginals for returns, the use of discontinuous processes i.e. the hyperbolic Lévy motion, and the ability to derive a comparison to the Black-Scholes pricing formula. Whilst a Lévy process is a clear and sensible generalisation with rich theory and many attractive properties, their increments are independent by definition.

One may instead consider fractional Brownian motion as a replacement to standard Brownian motion in the GBM model. First introduced by Mandelbrot and Van Ness (1968) [97], fractional Brownian motion also has many attractive properties which give a good description of asset price movements. Such models can incorporate heavy tails and dependence. However, this could also lead to correlated returns which also violates one of the observations of the typical returns time series. In addition, it is important to note that fractional Brownian motion is not a martingale, nor a semi-martingale, so the standard stochastic calculus is lost and arbitrage opportunities would exist (Rogers (1997) [109]). This has particular implications when we look to price financial derivatives. Starting with the idea of Cheridito (2001) [34], Mishura (2008) [101] looks to address this problem by considering risky assets guided by a mixed model of both a fractional and a standard Brownian motion. This process is arbitrage-free without any restriction on dependence of components, so for this case, an competing form of the Black-Scholes pricing formula could possibly be deduced.

Another way to generalise Brownian motion is by changing the time variable at which Brownian motion is evaluated to a random process. Over the years, there have been various ways in which this idea has been used to model time series. Notably, Feller (1966) [48] introduced a construction using a Markov process and a "randomized operational time" process with independent increments i.e. Poisson process. Mandelbrot and Taylor (1967) [96] was another early study of the concept of changed time, or in their words, "trading time". However, it was the work of Clark (1973) [35] which established time-changed Brownian motion into a finance setting. Here the author wrote down a risky asset model driven by a standard Brownian motion evaluated at some "activity time". In this thesis, we will promote this concept of changed time as it has the feature of explicit modelling the unobserved but natural time-scale of the returns process.

Furthermore, Madan, Carr and Chang (1998) [88] provides us with the theory for using Lévy processes for the random time. Similarly with the generalisation of Brownian motion to a Lévy process, however, this will result in independent returns which we will later show is not reflected in risky asset returns data. Mandelbrot et al (1997) [95] argues instead that fractional Brownian motion evaluated at a random process should be used where the random time is a process with non-decreasing paths and stationary (but not independent) increments. Additionally, this process was thought to be a multifractal process defined on a set of restrictions of the process's moments as the timescale of observations changes. Calvet and Fisher (2002) [29] shows empirical evidence to support this idea. In this case though, we cannot escape the fact that returns may be correlated and arbitrage opportunities exist.

For some clarity, the majority of the models discussed will be continuous-time models. They are predominantly used by statisticians in Mathematical Finance, whereas discrete-time models such as the ARCH (Engle (1982) [47]) and the GARCH (Bollerslev (1986) [27]) and moving average models are predominantly used by econometricians. Using discrete-time is popular because most financial data are observed at fixed discrete time points and recorded at low frequency, making it a good candidate to be modelled in discrete-time. However, we say that although the data is discrete, the underlying process is continuous and a discrete approximation of a continuous model can deal appropriately with discrete data if the data set is large enough.

So far we have only suggested models which fail to capture the features of financial data significantly well enough. As a result, we have do not have the ability to derive pricing formulae for any financial derivatives. In this thesis, we propose that we can achieve this by using an approach first introduced in Heyde (1999) [63]. Further developed in Heyde and Leonenko (2005) [66], we will successfully generalise the GBM model, and remain in a simple enough form to provide an elegant alternative to the Black-Scholes pricing formula. Chris Heyde's model is distinctively different from previously mentioned models, in its recognition of approximate monofractal (self-similar) scaling and the associated dependence structure. The results of this thesis are quantitatively similar to those which are obtained for stochastic volatility models of Barndorff-Nielsen and Shephard (2001) [16] (for a discussion of pricing formulae see Nicolato and Venardos (2003) [102]), but with important quantitative differences. Both models provide a convenient basis for risky calculations when specific distributional assumptions are made. Carr et al (2003) [33] used subordinated processes to construct stochastic volatility for activity time, however, the activity time process is again assumed to be Lévy processes. One other recent model to mention is the paper Bender and Marquardt (2009) [18] which offers a construction where activity time is modelled as a convolution between a Lévy subordinator and a deterministic kernal. While initially this construction seems to have many desirable features, there is no specification of the marginal distribution of returns.

Our investigations to build a model for risky assets have been heavily influenced by the authors mentioned in this introduction, and their findings. A special mention must also go to Andrey Kolmogorov, Paul Lévy, Wolfgang Doeblin and Kiyoshi Itô, as their huge contributions, in particular; advances in stochastic processes, stochastic calculus, and measure theory, has equipped us with many of the tools which we need. The financial market has so many effects acting on it that we argue for probability theory as best suitable to assess the uncertainties. In Bachelier (1900) [10], he states, "the determination of these fluctuations depends on an infinite number of factors; it is, therefore, impossible to aspire to mathematical prediction of it".

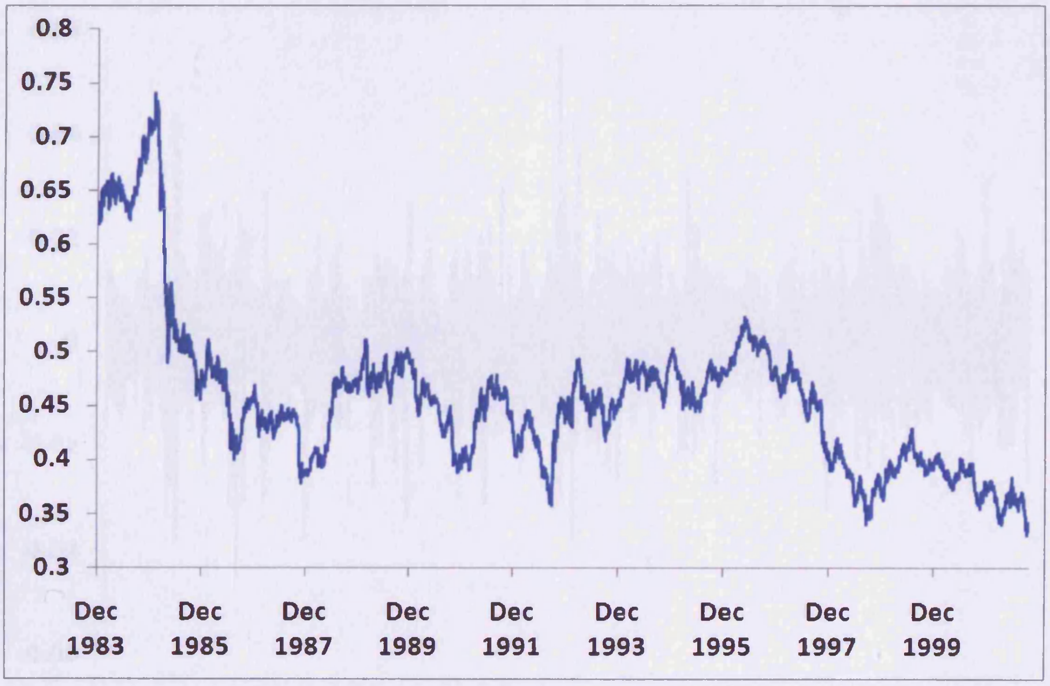
In the next chapter, we will illustrate some of the empirical findings that contradict GBM, plus some details on the features of risky asset returns. We will then introduce Heyde's alternative to GBM which incorporates dependence structure and desirable marginal distributions of the returns process as described in Heyde and Leonenko (2005) [66]. Notably, it is a subordinator model which gives asset prices as GBM driven by some nondecreasing stochastic "activity time" process based on fractal activity time, which endeavors to encompass all empirically found characteristics of real data. We will refer to this model from now as the "fractal activity time geometric Brownian motion" (FATGBM) model.

In chapter 3, we present three different constructions of the fractal activity time that leads to; a given marginal distribution, a flexible dependence structure and asymptotic self-similarity. Chapter 4 is devoted to parameter estimation and hypothesis testing, to validate the FATGBM model. In chapter 5, we state the competing formula for the price of a European call option which was first proposed by Heyde and Gay (2002) [58]. To do this we will follow the classical framework of Black, Scholes and Merton.

All work up to this point, uses the asymptotical self-similarity nature of fractal activity time found in Heyde and Liu (2001) [67]. In chapter 6 we consider an alternative multifractal approach and also a new construction for the activity time process. We first give a short description of the main features of multifractals in a finance setting, and then provide empirical evidence that multifractality exists for real financial data, along with the methodology to support. Section 7 concludes the thesis.

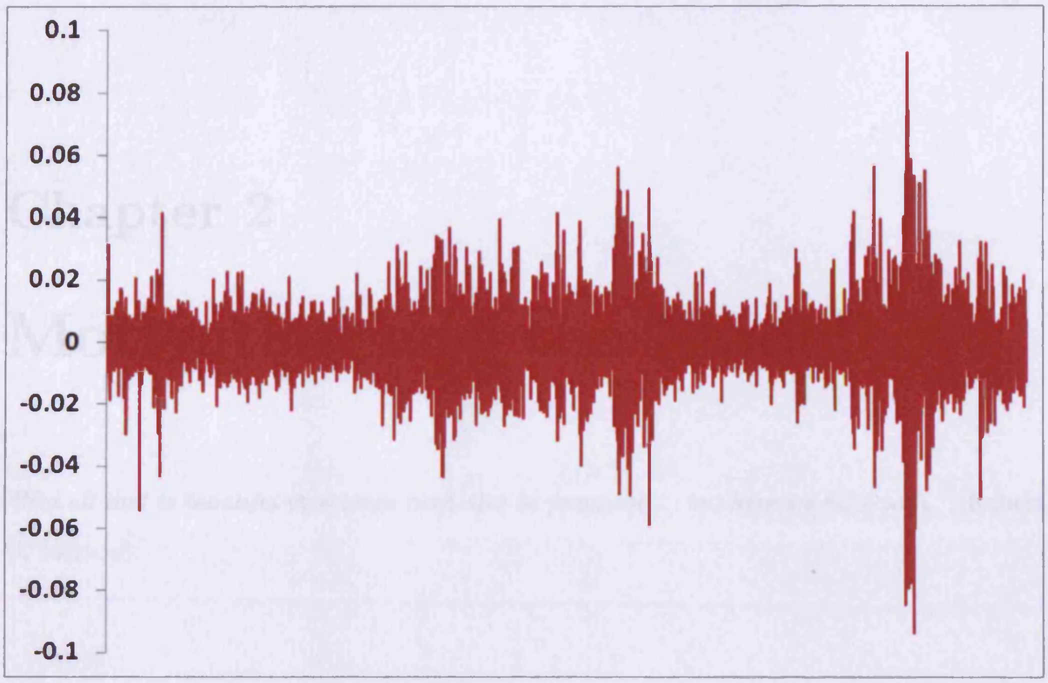
The real financial data sets, which we will employ for all statistical studies in this thesis, are exchange rates from United Sates Dollar to currencies: Australian Dollar (AD), Canadian Dollar (CD), Deutsche Mark (DM), Euro (EUR), French Franc (FF), Great British Pound (GBP), Japanese Yen (JY) and New Taiwan Dollar (NTD), and stock indices CAC40 and FTSE100. In total we have 10 data sets where an observation has been taken at the close of every trading day. *Figure 1.1* shows the asset price over time for each data set, with the corresponding returns process given in *Figure 1.2*. To keep the main body of the text clean throughout, only the empirical findings from GBP and FTSE100 data will be found in support of the theory. Please find the pictures generated using the remaining 8 data sets in Appendix A.

(a) FTSE100

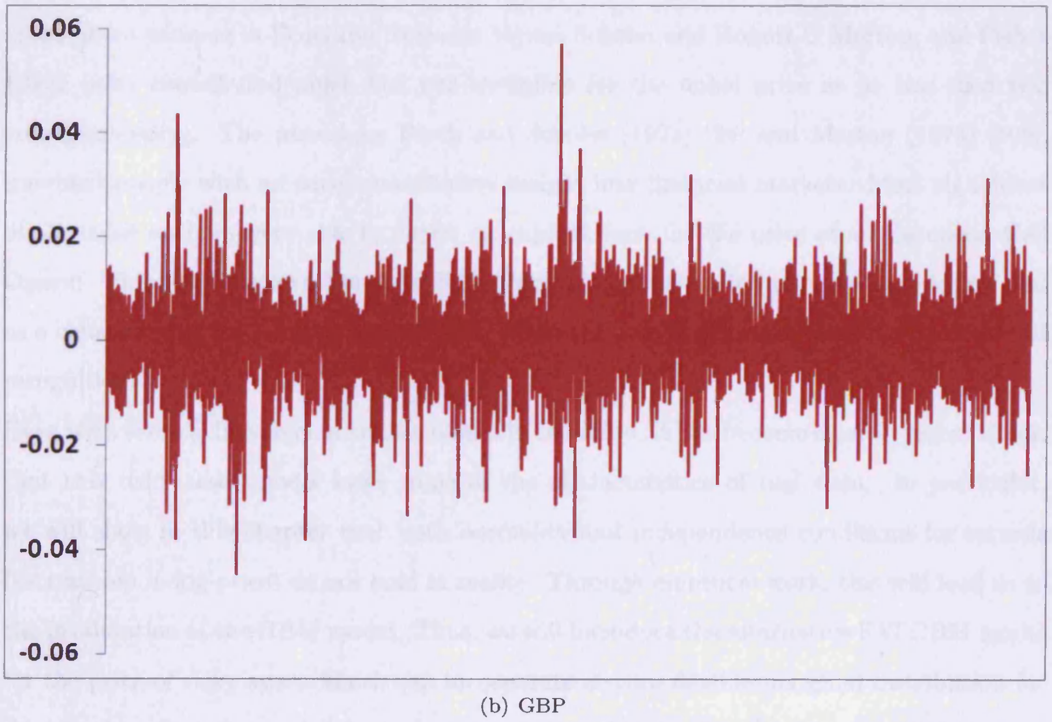


(b) GBP

Figure 1.1: Risky asset price $P(t)$



(a) FTSE100



(b) GBP

Figure 1.2: Risky asset returns $X(t)$

Chapter 2

Motivation and the Model

"Not all that is beautiful in science need also be practical ... but here we have both." (Robert C. Merton).

The paradigm GBM model for the price of a risky asset was first introduced to us by 1997 Nobel prize winners in Economic Sciences Myron Scholes and Robert C Merton, and Fisher Black (who contributed much but was ineligible for the nobel prize as he had died two year previously). The papers by Black and Scholes (1973) [26] and Merton (1973) [100], provided people with an early quantitative insight into financial markets. Most significant of all, these authors were able to derive an explicit form for the price of an European Call Option. This helped spawn the multi-billion pound derivatives industry we have today, and as a consequence, the subject area of Mathematical Finance gained substantial prestige and recognition.

Even with these achievements and accolades in the mind, it has become clear in recent times, that this risky asset model lacks some of the characteristics of real data. In particular, we will show in this chapter that both normality and independence conditions for returns (increments in log-price) do not hold in reality. Through empirical work, this will lead us to the invalidation of the GBM model. Thus, we will introduce the alternative FATGBM model for the price of risky assets which can incorporate a more flexible marginal distribution for returns than Gaussian and a dependence structure. Also in this chapter, we will determine whether the efficient market hypothesis holds for the proposed model, and investigate the scaling properties of the fractal activity time process.

2.1 Invalidation of the GBM model

Under the GBM model, the price P_t of a risky asset at time t is,

$$P_t = P_0 e^{\mu t + \sigma W(t)}, \ t \geq 0, \ P_0 > 0, \tag{2.1}$$

where drift parameter $\mu \in \mathbb{R}$ and volatility parameter $\sigma > 0$ are fixed constants, and $W(t), \ t \geq 0$ is a standard Brownian motion (Wiener process). The corresponding returns (or log returns) process X_t is thus given by,

$$X_t = \mu + \sigma(W(t) - W(t-1)), \ t = 1, 2, \ldots \tag{2.2}$$

From (2.1)-(2.2), we have the following hypothesis which we would like to test:

H_0 - The returns $X_t, t = 1, 2, \ldots$ are i.i.d Gaussian random variables with mean μ and σ^2.

All features now discussed are commonly found across all types of risky asset and all sampling frequencies.

2.1.1 Testing normality of the marginal distribution

Under the null hypothesis H_0, we should expect;

- the coefficient of skewness $\gamma_1 = \frac{\mu_3}{\sigma^3}$ to be 0,
- the coefficient of kurtosis $\gamma_2 = \frac{\mu_4}{\sigma^4} - 3$ to be 0,

where $\mu_k = E(X-\mu)^k, \ k = 2, 3, \ldots$ and $\mu_1 = \mu$.
To investigate the equality to zero for γ_1 and γ_2, we will assume μ and σ^2 are unknown and employ the method of moments to estimate the skewness and excess kurtosis of the marginal distribution of the returns,

$$\hat{\gamma_1}^{(N)} = \frac{\hat{\mu_3}^{(N)}}{\left(\hat{\mu_2}^{(N)}\right)^{\frac{3}{2}}}, \quad \hat{\gamma_2}^{(N)} = \frac{\hat{\mu_4}^{(N)}}{\left(\hat{\mu_2}^{(N)}\right)^{2}} - 3,$$

where $\hat{\mu_k}^{(N)} = \frac{1}{N}\sum_{i=1}^{N}\left(X_i - \bar{X}^{(N)}\right)^k, \ k = 2, 3, 4$ and $\bar{X}^{(N)} = \frac{1}{N}\sum_{i=1}^{N} X_i$. *Table 2.1* contains the number of observations N, and estimates of the mean, variance, coefficient of skewness, and coefficient of excess kurtosis for our 10 risky assets.
From Cramer (1946) [38], and under H_0 for $k = 2, 3, 4$, we can use,

$$\hat{\mu_k}^{(N)} \xrightarrow{p} \mu_k, \ N \to \infty, \quad (consistency)$$

$$\sqrt{N}\left(\hat{\mu_k}^{(N)} - \mu_k\right) \xrightarrow{d} \mathrm{N}(0, b_k^2), \quad (asymptotic\ normality)$$

where $b_k^2 = \mu_{2k} - \mu_k^2 - 2k\mu_{k-1}\mu_{k+1} + k^2\mu_2\mu_{k-1}^2$, to formally test the following:

1. For the estimate of skewness,

$$E\hat{\gamma_1}^{(N)} = \gamma_1 + O\left(\frac{1}{N}\right), \quad Var\hat{\gamma_1}^{(N)} = \frac{c_1}{N} + O\left(\frac{1}{N^{\frac{3}{2}}}\right),$$

where $c_1 = (4\mu_2^2\mu_6 - 12\mu_2\mu_3\mu_5 - 24\mu_2^3\mu_4 + 9\mu_3^2\mu_4 + 35\mu_2^2\mu_3^2 + 36\mu_2^5)/45\mu_2^5$ assuming moments exist, so

$$\sqrt{N}\left(\hat{\gamma_1}^{(N)} - \gamma_1\right) \xrightarrow{d} \mathrm{N}(0, 6), \ N \to \infty. \tag{2.3}$$

2. For the estimate of kurtosis,

$$E\hat{\gamma_2}^{(N)} = \gamma_2 + O\left(\frac{1}{N}\right), \quad Var\hat{\gamma_2}^{(N)} = \frac{c_2}{N} + O\left(\frac{1}{N^{\frac{3}{2}}}\right),$$

where $c_2 = (\mu_2^2\mu_8 - 4\mu_2\mu_4\mu_6 - 8\mu_2^2\mu_3\mu_5 + 4\mu_4^3 - \mu_2^2\mu_4^2 + 16\mu_2\mu_3^2\mu_4 + 16\mu_2^3\mu_3^2)/\mu_2^6$ assuming moments exist, so

$$\sqrt{N}\left(\hat{\gamma_2}^{(N)} - \gamma_2\right) \xrightarrow{d} \mathrm{N}(0, 24), \ N \to \infty. \tag{2.4}$$

Both these statements are derived from asymptotic normality and consistency of the empirical moments.

Using (2.3) and (2.4), and for large N,

$$P\left\{\left|\sqrt{\frac{N}{6}}\left(\hat{\gamma_1}^{(N)} - \gamma_1\right)\right| \le u_\epsilon\right\} \approx P\{|\mathrm{N}(0,1)| < u_\epsilon\} = 1 - \epsilon,$$

$$P\left\{\left|\sqrt{\frac{N}{24}}\left(\hat{\gamma_2}^{(N)} - \gamma_2\right)\right| \le u_\epsilon\right\} \approx P\{|\mathrm{N}(0,1)| < u_\epsilon\} = 1 - \epsilon.$$

where ϵ (decimal) is the significance level and u_ϵ can be obtained from statistical tables. Here we fix $\epsilon = 0.05$, $u_\epsilon = 1.96$. From *Table 2.2* we retain the hypothesis that $\gamma_1 = 0$ at a 5% significance level for data sets AD, CAC40, CD, DM and GBP, but reject $\gamma_1 = 0$ at a 5% significance level for data sets EUR, FF, FTSE100, JY and NTD. From *Table 2.3* we reject the hypothesis that $\gamma_2 = 0$ at a 5% significance level for all 10 data sets. Thus, we reject the null hypothesis H_0 for all 10 data sets, since either skewness or excess kurtosis (or both) are significantly different from zero.

Under an independence assumption, there are several statistical methods to test the normality of data i.e. the D'Agostino's K-squared test, the Anderson-Darling test, the Cramer-von Mises criterion etc. In addition, the popular Chi-Square goodness-of-fit test is available to us. *Table 2.4* shows that for each of our data sets, the Chi-Square statistics are very much higher than corresponding critical values (given in statistical tables). Thus, we can reject H_0 for all 10 cases, which again further supports the idea that returns data are not well-modelled by a Gaussian distribution.

Furthermore, in *Table 2.2* there is some evidence to suggest that any distribution used to explain risky asset returns must allow for occasional skewness. Specifically, when $\gamma_1 \neq 0$ (or not sufficiently close enough to zero) then $\gamma_1 < 0$ for the majority of our data sets, indicating that the returns are slightly negatively skewed. One explanation for this is that traders tend to react more strongly to negative information rather than positive information (Rydberg (2000) [112]).

From *Table 2.3*, we can also see that $\gamma_2 > 0$ for all data sets so any distribution we use to fit the data must be leptokurtic with (high peaks and) heavy tails. The light tails of the Gaussian distribution has been proven by many to be insufficient at covering the number of large variations of risky asset returns. The actual tailweight of risky asset returns has been a strongly debated subject in recent years. From Eugene Fama and Benoit Mandelbrot in the 1960s up to the present day, many authors have considered this issue (see Heyde and Kou (1994) [65] for a detailed review).

Mandelbrot suggested to use symmetric stable distributions which contain both Gaussian and a continuum of distributions with attractive properties such as Paretian tails. This family of distributions could sacrifice intermediate range for higher peaks and heavier peaks, with tails decaying by power law in comparison to the rapid exponential decay of Gaussian. However, the argument against this idea rests on the fact that the second moment of risky asset returns ceases to exist by definition. Even though the number of moments is too a keenly debated topic, it is widely accepted in modern times that the variance is finite. One example of a distribution which has heavy tails of Pareto-type that decays by power law, but can also operate fine with a finite second moment, is the Student's t-distribution. The suitability of Student's t to model risky asset returns was first advocated by Praetz (1972) [107].

For this thesis, we state that tail behaviour of returns is a power function at least asymptotically. However, to claim that all returns decay exact by exactly by power law would be too strong a statement. This is because estimating the subsequent power index has proven a historically problematic task (see Fung and Seneta (2007) [56]). We propose, therefore, to extend the study to the class of hyperbolic distributions. This class can be thought of having semi-heavy tails, in the sense they have an exponential term as well as a power term. In summary, the three distributions which we have chosen to analyse and believe fit returns data sufficiently well are the Variance Gamma, the Normal Inverse Gaussian and the Student's t-distribution. As it is hard in reality to distinguish between them, we will avoid any controversy of selecting a preference by investigating the suitability of all three distributions going forward.

2.1.2 Testing the independence assumption

Another important testable property ensues from the formulation of Black, Scholes and Merton. Under H_0, we should also expect:

- $\{X_t\}$, $t = 1, 2, \ldots$ are uncorrelated (and so independent because of Gaussianity).

To calculate the sample autocovariance based on the returns data $X_t, t = 0, 1, \ldots, N - k$, we will use,

$$\hat{R}^{(N)}(k) = \frac{1}{N} \sum_{t=1}^{N-k} \left(X_t - \bar{X}^{(N)}\right) \left(X_{t+k} - \bar{X}^{(N)}\right),$$

where the corresponding sample autocorrelations,

$$\hat{\rho}^{(N)}(k) = \frac{\hat{R}^{(N)}(k)}{\hat{R}^{(N)}(0)},$$

are calculated by normalizing at zero.

From *Figure 2.1* the sample autocorrelations of real returns diminish rapidly and are statistically insignificant. In fact, there is little or no autocorrelation present in returns past one or two lags (nearly white noise). We see that the returns of all kinds of risky assets show hardly any serial correlation but, importantly, this is not enough to say that they are independent. To study the correlation structure, it is necessary not to just study the returns themselves, but also an appropriate function of the returns. In this context, "appropriate" is related to the variance, i.e. an appropriate function is a function which reveals information about the variance of the asset prices.

The most widely used function is the quadratic function, which does reveal more information about the serial correlation of the variance, since it enlarges large returns and diminishes small ones. Another appropriate function for returns is the absolute value. The absolute value will compare the returns size-wise, and a strong correlation between first lags will then, like the use of the squared returns, reveal any accumulation of big movements of risky asset price.

We compute the sample autocorrelations for the absolute and squared values of the returns $(|X_t|, t = 0, 1, \ldots, N - k)$ and $(X_t^2, t = 0, 1, \ldots, N - k)$, respectively, in the same way as for the returns themselves. Interestingly, however, the figure does show some strong persistence in autocorrelations for the absolute and squared values of returns. The upper and lower bounds (black dotted lines) in *Figure 2.1* are included for illustrative purposes to provide a reference for the magnitude of sample autocorrelations. These bounds correspond to the levels of $\pm 2/\sqrt{N}$ and are based on the asymptotics of the sample autocorrelations if independence assumption were satisfied. With this in mind, there is strong evidence to suggest the autocorrelations of $|X_t|^d$, where d is an integer, declines slower as lag increases, with the slowest decline for $d = 1$ (an effect described in the literature as Taylor effect.) This empirical evidence matches the observations of Heyde and Liu (2001) [67] and Cont (2001) [36], and supports the "stylized facts" outlined in Granger (2005) [60]. Most authors now view the empirical features of risky asset returns data to be a critical step for setting the foundations on which reliable models can be built. Also included in *Figure 2.1* are the autocorrelations for the square root value of returns (or $d = \frac{1}{2}$). From our empirical investigation, we find that there is even more persistence in this case for significantly large lags. This additional $|X_t|^{\frac{1}{2}}$ case requires further discussion.

Many authors also suggest that returns are long-range dependent. By definition, a stationary process displays long-range dependence (long memory, strong memory or strong dependence) if its autocorrelation function decreases so that,

$$\rho(t) \propto \frac{1}{|t|^a}, \; t \in \mathbb{R},$$

for $0 < a < 1$. Therefore the autocorrelation is not integrable,

$$\int_{-\infty}^{\infty} |\rho(t)| \, dt = \infty, \; t \in \mathbb{R}.$$

In contrast, a series whose autocorrelation is integrable in $\mathbb{R}$ may display short-range dependence (short memory, weak memory or weak dependence).

Equivalently, for the discrete case, the process displays long-range dependence if the auto-correlation series is non-summable,

$$\sum_{-\infty}^{\infty} |\rho(k)| = \infty.$$

Recent models have claimed that long-range dependence exists for risky asset data (see Heyde and Yang (1997) [69] and references therein), however, this is not an universally accepted view. One way to distinguish between whether we have strong or weak dependence for our data sets, is to estimate Hurst parameter. For now, Hurst parameter can be considered as a direct link between the intensity of dependence of a given process and its self-similar scaling nature. In Appendix B, we will describe two methods of estimating Hurst parameter. However, for this thesis, we will avoid stating the strength of dependence for risky asset returns going forward and refrain from using either short-range or long-range dependence exclusively.

2.2 The FATGBM model

The FATGBM model for risky assets was first proposed by Heyde (1999) [63] and further developed upon by Heyde and Liu (2001) [67], Heyde and Leonenko (2005) [66], and Finlay and Seneta (2006) [50]. This model generalizes classical GBM model by using a random subordinator, as opposed to time, to evaluate the standard Brownian motion. For a discussion of subordinator models see Rachev and Mittnik (2000) [108].

Under the FATGBM model, the price P_t of a risky asset at time t is,

$$P_t = P_0 e^{\mu t + \theta T_t + \sigma W(T_t)},\ t \geq 0,\ P_0 > 0, \tag{2.5}$$

where drift parameter $\mu \in \mathbb{R}$, asymmetry parameter $\theta \in \mathbb{R}$ and volatility parameter $\sigma > 0$ are fixed constants, and $W(t), t \geq 0$ is a standard Brownian motion (Wiener process) independent of "activity time" $\{T_t\}$. $\{T_t, t = 0, 1, 2, \ldots, T_0 = 0\}$ is a positive, non-decreasing stochastic process with stationary but not necessarily independent increments. The increments over the unit time are,

$$\tau_t = T_t - T_{t-1},\ t = 1, 2, \ldots$$

In a financial context, this "activity time" process $\{T_t\}$ can be interpreted as time over which market prices evolve, and is often associated with trading volume or the flow of new price-sensitive information (Howison and Lamper (2001) [68]). The more frenzied trading

becomes, or the more information is released to the market, the faster the activity time flows. The calender time t plays a secondary role. Note that if $T_t = t$, then equation describes classical Black-Scholes model.
The corresponding returns are given by,

$$X_t = \mu + \theta\tau_t + \sigma\tau_t^{\frac{1}{2}} W(1), \; t = 1, 2, \ldots \tag{2.6}$$

We will now show that $\{T_t\}$ plays a crucial role in our model, determining both the distribution of $\{X_t\}$ and their correlation structure.

2.2.1 Distribution theory for the model

From (2.6), the conditional distribution of X_t, given $\tau_t = V$, is Normal with mean $\mu + \theta V$ and variance $\sigma^2 V$. Therefore the conditional distributions of X_t are normal mixed or the so-called generalized hyperbolic distributions.
Furthermore,

$$EX_t = \mu + \theta,$$

$$E(X_t - EX_t)^2 = \sigma^2 + \theta^2 M_2,$$

$$E(X_t - EX_t)^3 = 3\theta\sigma^2 M_2 + \theta^3 M_3,$$

$$E(X_t - EX_t)^4 = 3\sigma^4(1 + M_2) + 6\sigma^2\theta^2(M_2 + M_3) + \theta^4 M_4,$$

for any t, where $M_i = E(\tau_t - E\tau_t)^i$, $i = 2, 3, 4$, assuming these moments exist.
So the skewness coefficient of the distribution of X_t is,

$$\gamma_1 = \frac{3\theta\sigma^2 M_2 + \theta^3 M_3}{(\sigma^2 + \theta^2 M_2)^{\frac{3}{2}}},$$

and it's kurtosis coefficient becomes,

$$\gamma_2 = \frac{3\sigma^4(1 + M_2) + 6\sigma^2\theta^2(M_2 + M_3) + \theta^4 M_4}{(\sigma^2 + \theta^2 M_2)^2}.$$

Note that if $\theta = 0$, we have a symmetric model, otherwise we have a skewed model ($\theta \neq 0$).

<u>Case I: The Student model</u>

If τ_t follows an inverse Gamma distribution, the distribution of X_t will be Student. Specifically, if τ_t is distributed as $R\Gamma(\nu, \delta)$, $\nu, \delta > 0$, its density is,

$$f_{R\Gamma}(x) = \frac{\delta^\nu}{\Gamma(\nu)} x^{-\nu-1} e^{-\delta/x},\ x > 0. \tag{2.7}$$

The characteristic function of τ_t is,

$$\phi_{R\Gamma}(u) = E\left[e^{iu\tau_t}\right] = \frac{2(-i\delta u)^{\frac{\nu}{2}}}{\Gamma(\nu)} K_\nu(\sqrt{-4i\delta u}),\ u \in \mathbb{R},$$

where K_ν is modified Bessel function of the third kind, or McDonalds function (see Appendix C at the end of the thesis or Kotz, Kozubowski and Podgorski (2001) [80]).

The moments of order k of τ_k exist when $\nu > k$, (for example, when $\nu \leq 2$, $Var(\tau_t) = \infty$):

$$E\tau_t = \frac{\delta}{\nu - 1},\ \nu > 1,$$

$$M_2 = \frac{\delta^2}{(\nu-1)^2(\nu-2)},\ \nu > 2,$$

$$M_3 = \frac{4\delta^3}{(\nu-1)^3(\nu-2)(\nu-3)},\ \nu > 3,$$

$$M_4 = \frac{3\delta^4(\nu+5)}{(\nu-1)^4(\nu-3)(\nu-4)},\ \nu > 4.$$

The density of the marginal distribution of X_t is Student. When $\theta = 0$, the density of the distribution of X_t is,

$$f_S(x) = \frac{\Gamma(\nu + \frac{1}{2})}{\sqrt{2\sigma^2\delta}\sqrt{\pi}\Gamma(\nu)} \frac{1}{\left(1 + \left(\frac{x-\mu}{\sqrt{2\sigma^2\delta}}\right)^2\right)^{\nu+\frac{1}{2}}},\ x \in \mathbb{R}.$$

If $\theta \neq 0$,

$$f_S(x) = \sqrt{\frac{2}{\pi}} \frac{(\nu-1)^\nu e^{\frac{(x-\mu)\theta}{\sigma^2}}}{\sigma\Gamma(\nu)} \left(\frac{\theta^2}{2\delta\sigma^2 + (x-\mu)^2}\right)^{\frac{\nu+\frac{1}{2}}{2}} K_{\nu+1/2}\left(\frac{|\theta|\sqrt{2\delta\sigma^2 + (x-\mu)^2}}{\sigma^2}\right),\ x \in \mathbb{R}.$$

These expressions of densities above were given by Sørensen and Bibby (2003) [24].

If X_t has a symmetric Student distribution, then,

$$P(|X_t| > x) \sim const(\delta, \nu, \sigma) x^{-2\nu},$$

where $f(x) \sim g(x)$ means that $\lim_{x\to\infty} f(x)/g(x) = 1$.

The characteristic function of the distribution of X_t is,

$$\phi_S(u) = \frac{2^{1-\nu/2} e^{i\mu u}}{\Gamma(\nu)} \left(\delta(\sigma^2 u^2 - 2i\theta u)\right)^{\frac{\nu}{2}} K_\nu(\sqrt{2\delta(\sigma^2 u^2 - 2i\theta u)}).$$

We will use the notation $S(\mu, \theta, \sigma^2, \nu, \delta)$ for the Student model.

Case II: The Variance Gamma (VG) model

If instead τ_t follows a Gamma distribution, the conditional distribution of X_t is VG. Specifically, if τ_t is distributed as $\Gamma(\alpha, \beta)$, where $\alpha, \beta > 0$, its density is,

$$f_\Gamma(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \ x > 0. \tag{2.8}$$

The characteristic function is,

$$\phi_\Gamma(u) = \left(1 - \frac{iu}{\beta}\right)^{-\alpha}, \ u \in \mathbb{R},$$

and the moments of the process τ_t in this case are,

$$E\tau_t = \frac{\alpha}{\beta}, \quad M_2 = \frac{\alpha}{\beta^2}, \quad M_3 = \frac{2\alpha}{\beta^3}, \quad M_4 = \frac{3\alpha(\alpha+2)}{\beta^4}.$$

The density of the marginal distribution of X_t is then given by,

$$f_{VG}(x) = \sqrt{\frac{2}{\pi}} \frac{e^{-\frac{(x-\mu)\theta}{\sigma^2}}}{\sigma\Gamma(\alpha)} \left(\frac{|x-\mu|}{\sqrt{\theta^2 + 2\beta\sigma^2}}\right)^{\alpha-\frac{1}{2}} K_{\alpha-\frac{1}{2}}\left(\frac{|x-\mu|\sqrt{\theta^2+2\beta\sigma^2}}{\sigma^2}\right),$$

where $K_{\alpha-\frac{1}{2}}$ is the modified Bessel function of the third kind with index $\alpha - \frac{1}{2}$ (Appendix C).

If X_t has VG distribution, then as $x \to \infty$,

$$P(|X_t| > x) \sim const(\alpha, \beta, \sigma) x^{\alpha-1} e^{-x\sqrt{2\beta/\sigma^2}}.$$

The characteristic function of X_t is,

$$\phi_{VG}(u) = e^{i\mu u}\left(1 - \frac{i\theta u}{\beta} + \frac{1}{2\beta}\sigma^2 u^2\right)^{-\alpha}, \ u \in \mathbb{R}. \tag{2.9}$$

We will use the notation $VG(\mu, \theta, \sigma^2, \alpha, \beta)$ for the VG model.

Case III: The Normal Inverse Gaussian (NIG) model

Alternatively, we may consider τ_t that follows an inverse Gaussian distribution, where the resulting distribution of X_t will be NIG. Specifically, if τ_t is distributed as $IG(\delta, \gamma)$, $\delta >$

$0, \gamma \geq 0$, its density is,

$$f_{IG}(x) = \frac{\gamma e^{\gamma\delta}}{\sqrt{2\pi x^3}} e^{-\frac{1}{2}\left(\frac{\gamma^2}{x}+\delta^2 x\right)}, \; x > 0. \tag{2.10}$$

The characteristic function of $IG(\delta, \gamma)$ is,

$$\phi_{IG}(u) = e^{\frac{\gamma}{\delta}\left(1-\sqrt{1-\frac{2iu}{\delta^2}}\right)}, \; u \in \mathbb{R},$$

and the moments are

$$E\tau_t = \frac{\gamma}{\delta}, \quad M_2 = \frac{\gamma}{\delta^3}, \quad M_3 = \frac{3\gamma}{\delta^5}, \quad M_4 = \frac{3(5+\gamma\delta)\gamma}{\delta^7}.$$

The density of the marginal distribution of X_t is then given by,

$$f_{NIG}(x) = \frac{\sqrt{\theta^2+\delta^2\gamma^2}}{\sigma^2\pi} e^{\gamma\delta+\frac{\theta^2}{\sigma^2}(x-\mu)} \frac{\sigma\gamma}{\sqrt{\sigma^2\gamma^2+(x-\mu)^2}} K_1\left(\frac{\sqrt{(\theta^2+\delta^2\gamma^2)(\sigma^2\gamma^2+(x-\mu)^2)}}{\sigma^2}\right), x \in \mathbb{R},$$

where K_1 is the modified Bessel function of the third kind with index 1 (see Appendix C). If X_t has NIG distribution, then as $x \to \infty$,

$$P(|X_t| > x) \sim const(\delta, \gamma, \sigma) x^{-3/2} e^{-\alpha x}.$$

Note that the behaviour of the tails of the VG and NIG distributions is different from that of the Student distribution. When $\mu = \theta = 0$, Student distribution has heavy tails which decay by power law, while VG and NIG have semi-heavy tails which contain both a power and an exponential factor.

The characteristic function of the distribution of X_t is,

$$\phi_{NIG}(u) = e^{i\mu u+\gamma\delta-\gamma\sqrt{\sigma^2u^2-2\theta iu+\delta^2}}, \; u \in \mathbb{R}.$$

We will use the notation $NIG(\mu, \theta, \sigma^2, \delta, \gamma)$ for the NIG model.

2.2.2 Dependence structure of the model

We will next express some properties of the covariance structure of the process $\{X_t\}$ in terms of properties of the process $\{\tau_t\}$, assuming finiteness of moments as necessary. For integer $k \geq 1$, we have,

$$\begin{aligned} Cov(X_t, X_{t+k}) &= Cov(\theta\tau_t + \sigma\tau_t^{\frac{1}{2}} W_1(1), \theta\tau_{t+k} + \sigma\tau_{t+k}^{\frac{1}{2}} W_2(1) \; , \\ &= \theta^2(E\tau_t\tau_{t+k} - E\tau_t E\tau_{t+k}), \\ &= \theta^2 Cov(\tau_t, \tau_{t+k}), \end{aligned}$$

where W_1 and W_2 are independent Brownian motions. When $\theta = 0$, $Cov(X_t, X_{t+k}) = 0$. In the general case $\mu = \theta \neq 0$, we have,

$$\begin{aligned} Cov(X_t^2, X_{t+k}^2) &= Cov((\mu + \theta\tau_t + \sigma\tau_t^{\frac{1}{2}}W_1(1))^2, (\mu + \theta\tau_{t+k} + \sigma\tau_{t+k}^{\frac{1}{2}}W_2(1)|)^2) , \\ &= (\sigma^4 + 4\theta^2\mu^2 + 4\theta\mu\sigma^2)Cov(\tau_t, \tau_{t+k}) + \theta^4 Cov(\tau_t^2, \tau_{t+k}^2) \\ &\quad +(\theta^2\sigma^2 + 2\theta^3\mu)(Cov(\tau_t^2, \tau_{t+k}) + Cov(\tau_t, \tau_{t+k}^2)) , \end{aligned} \tag{2.11}$$

which for $\theta = 0$ reduces to,

$$Cov(X_t^2, X_{t+k}^2) = \sigma^4 Cov(\tau_t, \tau_{t+k}), \tag{2.12}$$

irrespective of the value of μ. From (2.11) and (2.12), it is clear that structural dependence properties expressed by the autocovariances for $\{\tau_t\}$ imply those for the squared returns $\{X_t^2\}$.

For $\mu = \theta = 0$, we also have,

$$\begin{aligned} Cov(|X_t|, |X_{t+k}|) &= Cov(|\sigma\tau_t^{\frac{1}{2}}W_1(1)|, |\sigma\tau_{t+k}^{\frac{1}{2}}W_2(1)|) , \\ &= \sigma^2 E(|W_1(1)|)E(|W_2(1)|)Cov(\tau_t^{\frac{1}{2}}, \tau_{t+k}^{\frac{1}{2}}) , \\ &= \frac{2}{\pi}\sigma^2 Cov(\tau_t^{\frac{1}{2}}, \tau_{t+k}^{\frac{1}{2}}). \end{aligned}$$

For $\{\tau_t\}$ with dependence structure, $\{X_t\}$ also displays conditional heteroscedasticity, i.e. time dependent conditional variance. Let $\mathcal{F} = \sigma(\{W(u), u \leq T_t\}, \{T(u), u \leq t\})$, which can be thought of as information available up to time t. Then,

$$\begin{aligned} Var(X_t|\mathcal{F}_{t-1}) &= E(X_t^2|\mathcal{F}_{t-1}) - (E(X_t|\mathcal{F}_{t-1}))^2 , \\ &= \theta^2 E(\tau_t^2|\mathcal{F}_{t-1}) + (\sigma^2 + 2\mu\theta)E(\tau_t|\mathcal{F}_{t-1}) \\ &\quad -(2\mu\theta E(\tau_t|\mathcal{F}_{t-1}) + \theta^2(E(\tau_t|\mathcal{F}_{t-1}))^2) , \\ &= \theta^2 Var(\tau_t|\mathcal{F}_{t-1}) + \sigma^2 E(\tau_t|\mathcal{F}_{t-1}). \end{aligned}$$

Under the restricted model of $\theta = 0$, the above expression reduces to (see Heyde and Liu (2001) [67]),

$$Var(X_t|\mathcal{F}_{t-1}) = \sigma^2 E(\tau_t|\mathcal{F}_{t-1}).$$

2.2.3 The efficient market hypothesis and the subordinator model

The notion of efficient market was first introduced by Fama (1970) [49]. This developed into what is now a central idea in modern finance. The premise is a hypothesis that risky asset prices are always "right", and so therefore, no one can divine the market's future direction. For prices to be right, we assume that the people who set them must be both rational and well informed. They must be confident arbitrageurs who never pay more or less than the true value.

If we consider (2.5), the definition of weak efficiency by Campbell, Lo and MacKinlay (1997) [31] implies that $\{P_t\}$ is a martingale with respect to the σ-algebra of events reflecting information available up to time t. Since $\{B(T_u), u = 1, 2, \dots\}$ is a martingale with respect to $\mathcal{F}_u$ defined in the previous subsection, we have,

$$E[B(T_t)|\mathcal{F}_{t-1}] = B(T_{t-1}),$$

almost surely.

When $\theta = 0$, $Cov(X_t, X_{t+k}) = 0$, the efficient market hypothesis holds, but when $\theta \neq 0$, the covariance of X_t and X_{t+k} is not zero if $Cov(\tau_t, \tau_{t+k}) \neq 0$. However,

$$Corr(X_t, X_{t+k}) = \frac{\theta^2 Cov(\tau_t, \tau_{t+k})}{\theta^2 Var(\tau_t) + \sigma^2} \leq \theta^2 \frac{Corr(\tau_t, \tau_{t+k})}{\sigma^2} Var(\tau_t) \leq \theta^2 \frac{Var(\tau_t)}{\sigma^2},$$

when $corr(\tau_t, \tau_{t+k}) > 0$.

For the FATGBM model, it is natural to interpret $\sigma\sqrt{\tau_t}$ as the volatility at time t, and hence the process $\{\sigma\sqrt{\tau_t}\}$ is stochastic volatility process.

2.2.4 Asymptotic self-similarity

Another important feature of the FATGBM model is that of asymptotic self-similarity. The origin of this property comes from the paper of Heyde and Liu (2001) [67], in which, they investigate the fractal activity time process acting on the belief that it should exhibit dependence.

A flexible dependence structure is usually suggestive of self-similar scaling. This behaviour occurs when the structure of parts is the same as the structure of the whole time series. In fact, it has long been observed in the statistical analysis of financial time series that many series have this property (see Embrechts and Maejima (2002) [46] for a detailed review). By definition, a stochastic process $Y(t), t \geq 0$, is self-similar if for each $a > 0$ there exists b such that,

$$\{Y(ab)\} \stackrel{d}{=} \{bY(t)\},$$

where "$\stackrel{d}{=}$" denotes equality of finite dimensional distributions. Furthermore, Lamperti (1962) [82] showed that if $Y(t)$ is self-similar and stochastically continuous at $t = 0$ then there exists a unique $H > 0$ such that for all $a > 0$,

$$\{Y(at)\} \stackrel{d}{=} \{a^H Y(t)\}. \tag{2.13}$$

This so-called Hurst parameter H is named after the British engineer Harold Hurst *(1880-1978)* whose work on Nile river data played an important role in the development of self-similar processes.

From definition (2.13) we can show how self-similar scaling can be linked with a flexible dependence structure. In Appendix B, we illustrate the importance of Hurst parameter H. Indeed, if H can be estimated then it can be used to gauge the strength of dependence, and even determine whether or not long-range dependence exists.

If the process in (2.13) also has stationary increments, then it must have $0 < H \leq 1$ and $Y(0) = 0$ a.s. (almost surely). If it has finite mean and $H = 1$ then it is the degenerate process $Y(t) = tY$ for some random variable Y. If it has finite mean and $0 < H < 1$ then $EY(t) = 0$.

The most important and most basic self-similar processes are derived from the so-called fractional Brownian motion. When a process is H-self-similar and has a finite variance its covariance structure is completely determined. It follows that,

$$\begin{aligned}
EY(t)Y(s) &= \frac{1}{2}\{E[Y(s)]^2 + E[Y(t)]^2 - E[Y(t) - Y(s)]^2\} \\
&= \frac{1}{2}\{E[Y(s)]^2 + E[Y(t)]^2 - E[Y(t-s) - Y(0)]^2\} \\
&= \frac{1}{2}\{E[Y(s)]^2 + E[Y(t)]^2 - E[Y(t-s)]^2\} \\
&= \frac{1}{2}\{|s|^{2H} + |t|^{2H} - |t-s|^{2H}\}EY(1)^2 \ .
\end{aligned}$$

Since the mean-covariance structure completely determines the finite-dimensional distributions at a Gaussian process there is only one Gaussian H-self-similar process for each $H \in (0,1)$. This process is known as fractional Brownian motion $B_H(t), t \in \mathbb{R}$. As an example, standard Brownian motion is self-similar with $H = \frac{1}{2}$,

$$\{B(at)\} \stackrel{d}{=} \{\sqrt{a}B(t)\}, \ a > 0, \ t \geq 0.$$

From the investigation of fractal activity time $\{T_t\}$ in Heyde and Liu (2001) [67], we will state that, to a good degree of first approximation, the process $\{T_t - t\}$ is asymptotically self-similar, i.e.

$$T_{ct} - ct \overset{d}{\cong} c^H(T_t - t), \;\; 0 < H < 1, \tag{2.14}$$

for positive c, meaning that, for $c = \frac{1}{t}$,

$$T_t \overset{d}{\cong} t + t^H(T_1 - 1).$$

In [67] the authors check for approximate self-similarity via a crude estimation of Hurst parameter H, over a wide range of time scales. They then claim (to a good degree of first approximation) that the process $\{T_t - t\}$ is asymptotically self-similar with index $\frac{1}{2} < H < 1$. Exact self-similarity of $\{T_t - t\}$, on the other hand, is not possible when T is said to be non-decreasing (Heyde and Leonenko (2005) [66]). It is stated that if,

$$T_{ct} - ct \overset{d}{=} c^H(T_t - t),$$

holds for all $t > 0$ and $c > 0$, then for any $0 < \Delta < 1$,

$$T_{t+\Delta} - T_t - \Delta \overset{d}{=} T_\Delta - \Delta \overset{d}{=} \Delta^H(T_1 - 1),$$

and,

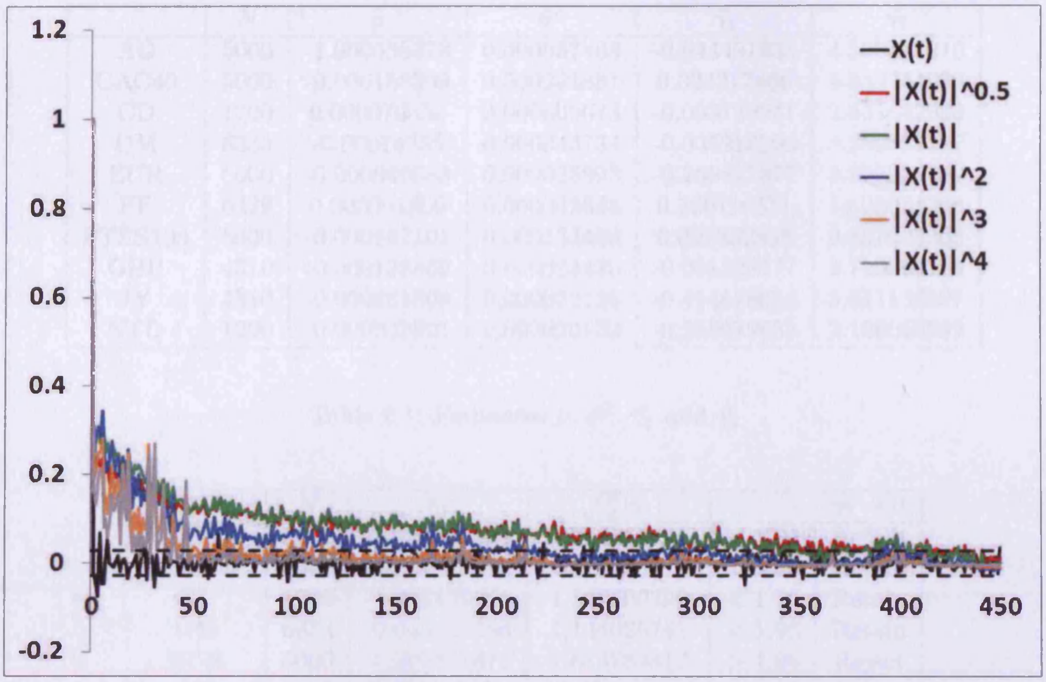
$$P(T_{t+\Delta} - T_t < 0) = P(\Delta^H T_1 < \Delta^H - \Delta) = P(T_1 < 1 - \Delta^{1-H}) > 0.$$

The concept of self-similarity paves the way to the development of fractal geometry (see Mandelbrot et al (1997) [95] and references therein). This means that it is sensible (and progressive) to consider the case where $\{T_t - t\}$ is taken to be a multifractal process. The kind of multifractal behaviour which might be expected is a relationship of the form,

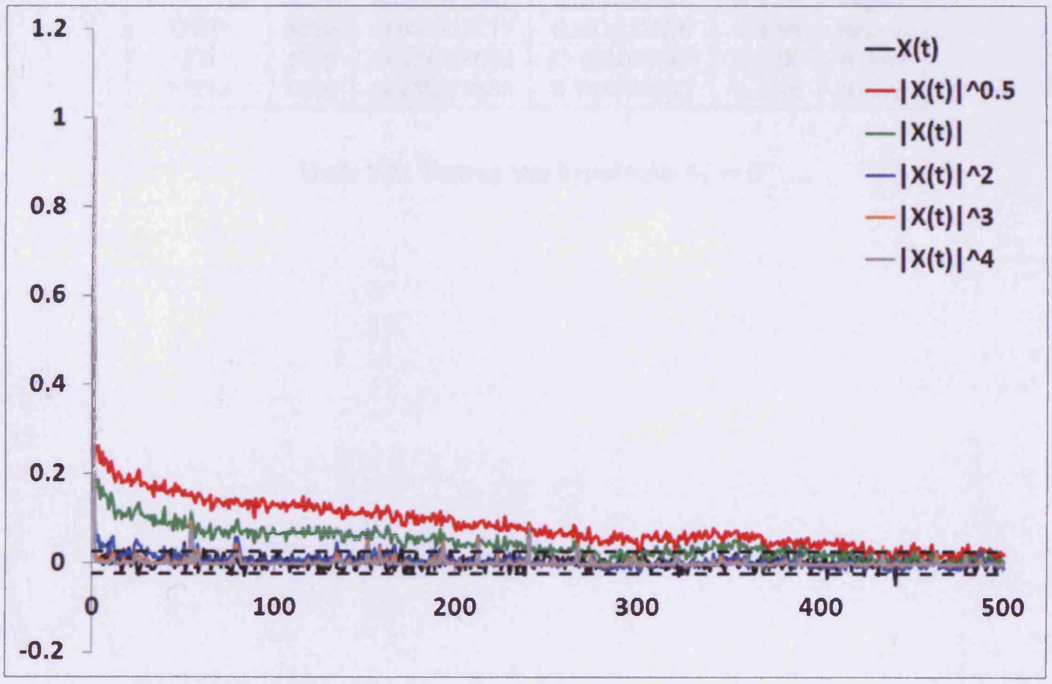
$$T_{ct} - ct \overset{d}{=} M(c)(T_t - t),$$

for positive c, where M and T are independent random functions, where we can reduce to a self-similar behaviour if $M(c)$ is of the form c^H.

We will see later in Chapter 5 how asymptotic self-similarity can be used to evaluate the call price of an European Call option expiring at time t by taking the expectation of the Black-Scholes pricing formula evaluated at T_t. In Chapter 6, we will study the more general multifractal approach to model risky assets.



(a) FTSE100



(b) GBP

Figure 2.1: Empirical autocorrelation of X and $|X|^d$ for $d = \frac{1}{2}, 1, 2, 3, 4$

	N	$\hat{\mu}$	$\hat{\sigma}^2$	$\hat{\gamma_1}$	$\hat{\gamma_2}$
AD	5000	-0.000035478	0.000037468	-0.044491933	4.508451210
CAC40	5000	-0.000158399	0.000201881	0.023217466	4.882744979
CD	1700	0.000079553	0.000009673	-0.093129341	2.651642996
DM	6333	-0.00014255	0.000043734	-0.035213296	5.289831117
EUR	5000	-0.000049583	0.000025993	-0.265827877	3.355937792
FF	6429	0.000001850	0.000042848	0.320116571	14.08034788
FTSE100	5000	-0.000167101	0.000134436	0.097352885	6.357003965
GBP	4510	-0.000138866	0.000064435	-0.001525777	2.720264185
JY	4510	-0.000281808	0.000072120	-0.414678054	5.611152007
NTD	1200	-0.000042921	0.000020132	-0.265079853	2.106045092

Table 2.1: Estimates $\hat{\mu}$, $\hat{\sigma}^2$, $\hat{\gamma_1}$ and $\hat{\gamma_2}$

	N	$\hat{\gamma_1}$	$\sqrt{\frac{n}{6}}\lvert\hat{\gamma_1}\rvert$		$\gamma_1 = 0$
AD	5000	-0.044491933	1.284371475	< 1.96	Retain
CAC40	5000	0.023217466	0.670230512	< 1.96	Retain
CD	1700	-0.093129341	1.567600399	< 1.96	Retain
DM	6333	-0.035213296	1.144025741	< 1.96	Retain
EUR	5000	-0.265827877	7.673789817	> 1.96	Reject
FF	6429	0.320116571	10.478623683	> 1.96	Reject
FTSE100	5000	0.097352885	2.810335718	> 1.96	Reject
GBP	4510	-0.001525777	0.041831526	< 1.96	Retain
JY	4510	-0.414678054	11.369037463	> 1.96	Reject
NTD	1200	-0.265079853	3.748795232	> 1.96	Reject

Table 2.2: Testing the hypothesis $\gamma_1 = 0$

	N	$\hat{\gamma_2}$	$\sqrt{\frac{n}{24}}\lvert\hat{\gamma_2}\rvert$		$\gamma_2 = 0$
AD	5000	4.508451210	65.073887993	> 1.96	Reject
CAC40	5000	4.882744979	70.476353200	> 1.96	Reject
CD	1700	2.651642996	22.316901252	> 1.96	Reject
DM	6333	5.289831117	85.929231975	> 1.96	Reject
EUR	5000	3.355937792	48.438789690	> 1.96	Reject
FF	6429	14.08034788	230.451466947	> 1.96	Reject
FTSE100	5000	6.357003965	91.755448761	> 1.96	Reject
GBP	4510	2.720264185	37.290115946	> 1.96	Reject
JY	4510	5.611152007	76.919186779	> 1.96	Reject
NTD	1200	2.106045092	14.891987660	> 1.96	Reject

Table 2.3: Testing the hypothesis $\gamma_2 = 0$

	n	χ^2 statistic		Good Fit?
AD	5000	327.718	>62.725	Reject
CAC40	5000	515.213	>55.758	Reject
CD	1700	425.847	>38.885	Reject
DM	6333	527.045	>55.758	Reject
EUR	5000	327.718	>55.758	Reject
FF	6429	373.276	>62.725	Reject
FTSE100	5000	439.991	>55.758	Reject
GBP	4510	340.177	>38.885	Reject
JY	4510	402.841	>40.113	Reject
NTD	1200	398.265	>37.652	Reject

Table 2.4: Chi-Square Goodness of Fit of the Gaussian distribution

Chapter 3

Activity Time

"In a strict sense, there wasn't any risk - if the world had behaved as it did in the past." (Merton Miller).

We have shown so far that the FATGBM model has the capability to allow for the empirical realities of risky asset returns. Such suitability to the data, however, is heavily reliant on the fractal activity time process $\{T_t\}$ being flexible enough to encompass important attributes. In particular, we require a process $\{T_t\}$ to have pre-specified unit increments with either gamma or reciprocal gamma or inverse Gaussian distribution, a dependence structure, and a self-similar limit as $t \to \infty$.

In order to incorporate these characteristics, we will present three separate constructions of activity time. They include; a process involving the use of chi-squared processes, the superposition of diffusion-type processes (each driven by standard Brownian motion), and lastly, the superposition of Ornstein-Uhlenbeck processes (each driven by Lévy noise).

3.1 Method I: Chi-Squared processes

This is a continuous approach first introduced by Heyde and Leonenko (2005) [66], which can be adapted to allow for gamma or reciprocal gamma processes resulting in VG or Student returns respectively. Indeed it was Heyde and Leonenko (2005) [66] who advocated the use of the reciprocal gamma construction to promote their Student model. They show that the resulting activity time process has a flexible correlation structure and converges to a self-similar process when appropriately normalized (standard Brownian motion for weak dependence and Rosenblatt-type process for long-range dependence). We will now provide the details of this construction. For a full discussion on the gamma construction leading to VG returns see Finlay and Seneta (2006) [50].

3.1.1 The construction of unit increments (I)

To begin we first need to define a stationary chi-squared process $\chi^2_\nu(t)$,

$$\chi^2_\nu(t) = \frac{1}{2}(\eta_1^2(t) + \cdots + \eta_\nu^2(t)), \ t \in \mathbb{R}, \tag{3.1}$$

where $\eta_1(t), \ldots, \eta_\nu(t),\ t \in \mathbb{R}$ are independent copies of a stationary Gaussian process $\eta(t),\ t \in \mathbb{R}$ with zero mean, unit variance and continuous monotonic correlation function $\rho_\eta(s) \geq 0,\ s \in \mathbb{R}$.

From Heyde and Leonenko (2005) [66] and references therein, we see that the strictly stationary chi-square process has many elegant properties:

1) From (3.1),

$$E\left\{\chi^2_\nu(t)\right\} = \frac{\nu}{2},\ Var\left\{\chi^2_\nu(t)\right\} = \frac{\nu}{2},$$

and,

$$Cov(\chi^2_\nu(t), \chi^2_\nu(t+s)) = \frac{\nu}{2}\rho^2_\eta(s),\ s \in \mathbb{R}.$$

2) The pdf of $\chi^2_\nu(t)$ is of the form,

$$p_{\frac{\nu}{2}}(x) = \frac{x^{\frac{\nu}{2}-1}e^{-x}}{\Gamma\left(\frac{\nu}{2}\right)},\ x > 0,$$

while the bivariate pdf of the random vector $(\chi^2_\nu(t), \chi^2_\nu(t+s))$ takes the form,

$$p_{\frac{\nu}{2}}(x, y; \gamma) = \left(\frac{xy}{\gamma}\right)^{\frac{\nu-2}{4}} e^{-\frac{x+y}{1-\gamma}} I_{\frac{\nu-2}{2}}\left(2\frac{\sqrt{xy\gamma}}{1-\gamma}\right)\frac{1}{\Gamma\left(\frac{\nu}{2}\right)(1-\gamma)},\ x, y > 0.$$

Here $\gamma = \rho^2_\eta(s)$, where $I_\lambda(z)$ is the modified Bessel function of the first kind,

$$\begin{aligned} I_\lambda(z) &= \sum_{k=0}^{\infty}\left(\frac{z}{2}\right)^{2k+\lambda}\frac{1}{k!\Gamma(k+\lambda+1)} \\ &= \frac{\left(\frac{z}{2}\right)^\lambda}{\sqrt{\pi}\Gamma(\lambda+\frac{1}{2})}\int_1^{-1}(1-t^2)^{\lambda-\frac{1}{2}}e^{zt}\,dt. \end{aligned}$$

3) For $k = 1, 2, \ldots,$

$$
\begin{aligned}
Cov(e_k(\chi^2_\nu(t)), e_m(\chi^2_\nu(t+s))) &= E\left\{e_k(\chi^2_\nu(t))e_m(\chi^2_\nu(t+s))\right\} - Ee_k(\chi^2_v(t))Ee_m(\chi^2_v(t+s)) \\
&= \int_0^\infty \int_0^\infty e_k(u)e_m(v)p_{\frac{\nu}{2}}(u,v,\rho^2(s))\,du\,dv \\
&- Ee_k(\chi^2_v(t))Ee_m(\chi^2_v(t+s)) \\
&= \int_0^\infty \int_0^\infty e_k(u)e_m(v)p_{\frac{\nu}{2}}(u)p_{\frac{\nu}{2}}(v)[1+\sum_{j=1}^\infty \rho^{2j}(s)e_j(u)e_j(v)]\,du\,dv \\
&- Ee_k(\chi^2_v(t))Ee_m(\chi^2_v(t+s)) \\
&= \sum_{j=1}^\infty \rho^{2j}(s)\int_0^\infty e_k(u)e_j(u)p_{\frac{\nu}{2}}(u)\,du \int_0^\infty e_m(v)e_j(v)p_{\frac{\nu}{2}}(v)\,dv \\
&= \delta_k^m \rho_\eta^{2k}(s), \qquad (3.2)
\end{aligned}
$$

where δ_k^m is the Kronecker symbol,

$$
e_k(u) = L_k^{(\nu/2)-1}(u)\left\{\frac{k!\Gamma\left(\frac{\nu}{2}\right)}{\Gamma\left(\frac{\nu}{2}+k\right)}\right\}^{1/2},
$$

and,

$$
L_k^\beta(u) = \frac{1}{k!}u^{-\beta}e^u\frac{d^k}{du^k}\left\{u^{\beta+k}e^{-u}\right\},
$$

are the generalized Laguerre polynomials of index β for $k \geq 0$. Here we note that (3.2) follows from the Hille-Hardy formula (see Bateman and Erdélyi (1953) [17], which can be written in the form,

$$
p_\beta(x,y;\gamma) = p_\beta(x)p_\beta(y)\left[1+\sum_{k=1}^\infty \gamma^k e_k(x)e_k(y)\right],\ x,y>0,
$$

with $\beta = \nu/2,\ \gamma = \rho^2(s),\ 0<\gamma<1.$

If we define $e_0(x) = 1$, then $\{e_k(x)\}_{k=0}^\infty$ is a complete orthogonal system of functions in the Hilbert space $L_2((0,\infty), p_{\frac{\nu}{2}}(x)dx)$ (see Courant and Hilbert (1953) [37] or Leonenko (1999) [84]). Thus, if a non-random function,

$$
G(x) \in L_2((0,\infty),\ p_{\frac{\nu}{2}}(x)dx),
$$

then the following expansion holds:

$$
G(x) = \sum_{k=1}^\infty C_k e_k(x), \qquad (3.3)
$$

with coefficients,

$$C_k = \int_0^\infty G(x)e_k(x)p_{\frac{\nu}{2}}(x)dx,$$

$\{e_k(x), e_0(x) = 1\}$ basis in $L_2((0,\infty), p_{\frac{\nu}{2}}(x)\,dx)$, and,

$$\int_0^\infty e_k(x)e_m(x)p_{\frac{\nu}{2}}(x)\,dx = \delta_k^m, \quad \sum_{k=1}^{\infty} C_k^2 = \int_0^\infty G^2(x)p_{\frac{\nu}{2}}(x)dx < \infty,$$

where δ_k^m is the kronecker-delta symbol, and,

$$C_0 = 1,\ C_1 = \sqrt{\frac{2}{v}},$$

$$e_0(x) = 1,\ e_1(x) = \sqrt{\frac{2}{v}}\left(\frac{v}{2} - x\right),\ e_2(x) = \frac{x^2 - 2(\frac{v}{2}+1)x + (\frac{v}{2}+1)\frac{v}{2}}{\sqrt{v(\frac{v}{2}+1)}}.$$

Lemma Note that $\tau(t) = G(\chi_\nu^2(t))$ with,

$$G(x) = \left(\frac{\nu}{2} - 1\right)\frac{1}{x} \in L_2\left((0,\infty), p_{\nu/2}(x)dx\right), \tag{3.4}$$

there exists a strictly stationary process which has marginal inverse gamma distribution $R\Gamma\left(\frac{\nu}{2}, \frac{\nu}{2} - 1\right)$ (with parameters chosen so that $E\tau_t = 1$) for every integer $\nu \geq 1$.

If we consider the reciprocal gamma case only, then it follows from (3.3) that,

$$\tau(t) = \sum_{k=0}^{\infty} C_k(\nu)e_k(\chi_\nu^2(t)),\ C_k(\nu) = (\frac{\nu}{2} - 1)\int_0^\infty \frac{p_{\frac{\nu}{2}}(x)e_k(x)dx}{x},$$

and,

$$\sum_{k=0}^{\infty} C_k^2(\nu) = \left(\frac{\nu}{2} - 1\right)^2 \int_0^\infty \frac{p_{\frac{\nu}{2}}(x)e_k(x)dx}{x^2} < \infty,\ \nu > 4.$$

Thus, we obtain the following properties of the inverse gamma process (3.4):

$$E\tau_t = 1, \qquad Var\tau_t = \frac{2}{\nu - 4},\ \nu > 4,$$

and,

$$Cov(\tau_t, \tau_{t+s}) = \sum_{k=1}^{\infty} C_k^2(\nu)\rho_\eta^{2k}(s),\ \nu > 4.$$

Note that $\rho_\eta(s),\ s \in \mathbb{R}$ is a correlation function of the Gaussian process $\eta(t),\ t \in \mathbb{R}$, and,

$$\rho_\tau(s) = \frac{cov(\tau_t, \tau_{t+s})}{var\{\tau_t\}} = \frac{\nu - 4}{2}\sum_{k=1}^{\infty} C_k^2(\nu)\rho_\eta^{2k}(s),$$

where $\nu > 4, \; s \in \mathbb{R}$.

Lemma For the gamma case in [50], we instead need to take the strictly stationary process with marginal gamma distribution $\Gamma(\frac{\nu}{2}, \frac{\nu}{2})$ (again, with parameters chosen so that $E\tau_t = 1$) for every integer $\nu \geq 1$, and consider $\tau(t) = G(\chi^2_\nu(t))$ with,

$$G(x) = \frac{2}{\nu} x \in L_2\left((0,\infty), p_{\nu/2}(x)dx\right).$$

This construction of $\{T_t\}$ turns out to be somewhat simpler than in Heyde and Leonenko (2005) [66], due to the fact that the consequences of having a gamma distribution for τ_t are easier to handle than the consequences of having an inverse gamma distribution.

3.1.2 Convergence to a self-similar limit (I)

We may choose as ρ_η any consistent Gaussian autocorrelation function and so we have a fair degree of flexibility in choosing the autocorrelation function of $G(\chi^2_\nu(t))$. We can then obtain the summable correlation function $\sum_{s=1}^{\infty} |\rho_\tau(s)| < \infty$ (short-range dependence), or correlation function that is not summable (long-range dependence), for example, when $\rho_\eta(s) = (1+s^2)^{-\frac{\alpha}{2}}, \; 0 < \alpha < \frac{1}{2}$.
Both Heyde and Leonenko (2005) [66] and Finlay and Seneta (2006) [50] show that the process $\{T_t - ET_t\} = \{T_t - t\}$ is asymptotically self-similar. When the increments $\tau(t)$ have short-range dependence, the fractal activity time, appropriately normalized, converges to the Brownian motion, and therefore asymptotic self-similarity holds with $H = \frac{1}{2}$. When the increments $\tau(t)$ have long-range dependence, the fractal activity time, appropriately normalized, converges to a Rosenblatt-type process. We now aim to show that $\{T_{[nt]} - [nt]\}$ (appropriately normed) has an asymptotic self-similar limit under both weak and long-range dependence, where $[\cdot]$ is the integer part.

Theorem 1. *Suppose $\eta_1(t), \ldots, \eta_\nu(t), \; t \in \mathbb{R}$ are independent copies of a stationary Gaussian process $\eta(t), \; t \in \mathbb{R}$ with zero mean, unit variance and continuous monotonic correlation function $\rho_\eta(s) \geq 0, \; s \in \mathbb{R}$, then the following process,*

$$T_{[nt]} - [nt] = \sum_{s=1}^{[nt]} \frac{(\nu/2 - 1)}{\frac{1}{2}\left(\eta_1^2(s) + \cdots + \eta_\nu^2(s)\right)} - [nt], \tag{3.5}$$

is asymptotically self-similar.

Proof Reconsider the strictly stationary process $\tau(t) = G(\chi^2_\nu(t))$, where $G(x) = \left(\frac{\nu}{2} - 1\right)\frac{1}{x}$, with marginal $R\Gamma\left(\frac{\nu}{2}, \frac{\nu}{2} - 1\right)$ distribution and a dependence structure. In our construction we have assumed that ν is an integer. Denote the density of χ^2_ν by $p_{\frac{\nu}{2}}$. If $\nu > 4$, the first two coefficients of the expansion of function G with respect to generalized Laguerre polynomials are,

$$C_0(\nu) = \left(\frac{\nu}{2} - 1\right) \int_0^\infty p_{\frac{\nu}{2}}(x) \frac{dx}{x} = 1,$$

$$C_1(\nu) = \left(\frac{\nu}{2} - 1\right) \int_0^\infty p_{\frac{\nu}{2}}(x) e_1(x) \frac{dx}{x} = \sqrt{\frac{2}{\nu}} \neq 0.$$

The first generalized Laguerre polynomial is,

$$e_1(x) = \sqrt{\frac{2}{\nu}} \left(\frac{\nu}{2} - x\right),$$

and hence,

$$e_1(\chi^2_\nu(t)) = -\frac{1}{\sqrt{2\nu}} \sum_{j=1}^{\nu} (\eta_j^2(t) - 1),$$

where $\eta_1(t), \ldots, \eta_\nu(t)$ are independent copies of the stationary Gaussian process $\eta(t)$ with zero mean and covariance function $\rho_\eta(s) = (1 + s^2)^{-\frac{\alpha}{2}}$. We will consider both the case of weak dependence ($\alpha > \frac{1}{2}$) and the case of long-range dependence ($0 < \alpha < \frac{1}{2}$).
Firstly, for weak dependence,

$$\sum_{t=1}^{\infty} |\rho_\eta(t)|^2 < \infty,$$

holds. Since the Laguerre rank of function G is 1, its Hermite rank of is equal to 2, and we apply Arcones (1994) [9] which gives,

$$\frac{1}{\sqrt{n}}(T_{[nt]} - [nt]) \to \sigma W(t), \; n \to \infty, \; t \geq 0,$$

where $\sigma^2 = VarG(\chi^2_\nu(t))$, and $W(t)$ is a (self-similar) Wiener process.

For long-range dependence, we begin with,

$$\begin{aligned}
T_{[nt]} - [nt] &= \sum_{s=1}^{[nt]} G(\chi_\nu^2(s)) - [nt] \\
&= \sum_{k=0}^{\infty}(\sum_{s=1}^{[nt]} C_k e_k(\chi_\nu^2(s))) - [nt] \\
&= \sum_{k=1}^{\infty}\sum_{s=1}^{[nt]} C_k e_k(\chi_\nu^2(s)) \\
&= \sum_{s=1}^{[nt]} C_1 e_1(\chi_\nu^2(s)) + \sum_{k=2}^{\infty}\sum_{s=1}^{[nt]} C_k e_k(\chi_\nu^2(s)) \\
&= \sum_{s=1}^{[nt]} C_1 e_1(\chi_\nu^2(s)) + R_{[nt]}.
\end{aligned}$$

Because of orthogonality of Laguerre polynomials, the variance of $T_t - t$ is,

$$\begin{aligned}
Var(T_t - t) &= Var\sum_{s=1}^{t} C_1 e_1(\chi_\nu^2(s)) + Var(R_t) \\
&= \sum_{s=1}^{t}\sum_{s^*=1}^{t} C_1^2\rho^2(|s-s^*|) + VarR_t \\
&\sim C_1^2 c(H) t^{2H} + VarR_t,
\end{aligned}$$

where $H = 1-\alpha$, $1/2 < H < 1$, and $c(H) = \frac{1}{(2H-1)H}$. It follows from Taqqu (1975) [119], Berman (1992) [21] and Leonenko (1999) [84] that, in the case $\frac{1}{2} < H < 1$, we have,

$$\frac{1}{t^{2H}} VarR_t \to 0, \; t \to \infty. \tag{3.6}$$

Under long range dependence, we have,

$$\begin{aligned}
Var\left(\frac{1}{n^H}(\sum_{s=1}^{[nt]} G(\chi_\nu^2(s)) - [nt]) - \frac{C_1}{n^H}\sum_{s=1}^{[nt]} e_1(\chi_\nu^2(s))\right) &= \frac{t^{2H}}{(nt)^{2H}} VarR_{[nt]} \\
&= \frac{1}{n^H} VarR_{[nt]} \\
&\to 0, \text{ as } n \to \infty,
\end{aligned}$$

for all $t > 0$, and by Slutsky lemma the weak limit of $\frac{1}{n^H}\left(\sum_{s=1}^{[nt]} G(\chi_\nu^2(s)) - [nt]\right)$ is the same as the weak limit of $\frac{C_1}{n^H}\sum_{s=1}^{[nt]} e_1(\chi_\nu^2(s))$.

Recall that $e_1(x) = \sqrt{\frac{2}{\nu}}(\frac{\nu}{2} - x)$, $C_1 = \sqrt{\frac{2}{\nu}}$. As a direct result of a proposition in Taqqu (1975) [119] with $H \in (\frac{1}{2}, 1)$, we have,

$$\begin{aligned}
\frac{C_1}{n^H}\sum_{s=1}^{[nt]} e_1(\chi_\nu^2(s)) &= -\frac{1}{\nu}\sum_{i=1}^{\nu}\left(\frac{1}{n^H}\sum_{s=1}^{[nt]}(\eta_i^2(s) - 1)\right) \\
&\to -\frac{1}{\nu}\sum_{i=1}^{\nu} R_i(t), \text{ as } n \to \infty,
\end{aligned}$$

where R_i, for $i = 1, \dots, \nu$, are independent (self-similar) Rosenblatt processes (see Appendix D) so,

$$\frac{1}{n^H}(T_{[nt]} - [nt]) \xrightarrow{d} -\frac{1}{\nu}\sum_{i=1}^{\nu} R_i(t), \text{ as } n \to \infty. \tag{3.7}$$

Note that although Heyde and Leonenko (2005) [66] constructed a long-range dependent symmetric t-model via a self-similar process $\{T_t - t\}$, in which τ_t was inverse gamma distributed, the extension to a skew t-process is trivial. □

The similar construction for the VG model is outlined in Finlay and Seneta (2006) [50]. If we again choose ρ_η appropriately to obtain weak or long-range dependence, then we arrive at,

$$\frac{1}{n^H}(T_{[nt]} - [nt]) \xrightarrow{d} \frac{1}{\nu}\sum_{i=1}^{\nu} R_i(t), \text{ as } n \to \infty,$$

Hence we can also construct a dependent symmetric VG model via a self-similar process $\{T_t - t\}$, in which τ_t is gamma distributed. The existence of a Student and VG subordinator models for risky asset returns with the desirable empirically supported properties follows. Note also that the distribution of the random variable $R(t)$ is not symmetric, as claimed in Finlay and Seneta (2006) [50]. The two limit processes for the inverse gamma and gamma cases are not in fact distributionally equivalent, (see the correction paper of Finlay and Seneta (2006) [51] for details).

3.2 Method II: Diffusion-type processes

We now look to promote the use of a fractal activity time construction based on diffusion-type processes driven by standard Brownian motion. The standard Brownian motion is such that the distribution of the increments of activity time can be inverse Gaussian. This allows us to incorporate NIG returns.

To obtain the desired properties of returns such as a flexible dependence structure, we need to consider the superposition of such diffusion-type processes. We will provide proof of convergence when appropriately normalized to a self-similar process (standard Brownian motion for finite superposition and weak dependence). Note here that we cannot consider reciprocal gamma or gamma increments, since these distributions do not behave well under superpositions of diffusion-type processes.

3.2.1 The construction of unit increments (II)

For our construction, we will first recall the background of diffusion-type processes along with some important properties (see Bibby Skovgaard and Sørensen (2005) [23]).
Consider an interval (l, r), $-\infty \leq l < r \leq \infty$, and the process y with the state space (l, r) that satisfies the following stochastic differential equation (SDE) with linear drift:

$$dy(t) = -\lambda(y(t) - \mu)dt + \sqrt{v(y(t))}dW(t), \quad t \geq 0, \; y(0) = y_0. \tag{3.8}$$

Here $\lambda > 0$, $\mu \in (l, r)$, $W = \{W(t), t \geq 0\}$ is a standard Brownian motion and y_0 is constant or a random variable independent of W. We want to choose the function v in such a way that y is ergodic with invariant density equal to a preliminary given density function f. As in [23], define the function v via f and μ as follows:

$$v(x) = 2\lambda \int_l^x (\mu - u)f(u)du/f(x), \quad l < x < r. \tag{3.9}$$

Then under some conditions on the density f, the solution of (3.8) is a mean-reverting stationary process with density function f. Namely, the following theorem was proved in [23]:

Theorem 2. *Suppose that the probability density function f is continuous, bounded, and strictly positive on (l, r), zero outside (l, r), and has finite expectation μ and finite variance. Then the following properties hold:*

1. *The SDE* (3.8) *has a unique Markovian weak solution, and the diffusion coefficient $v(\cdot)$ given by* (3.9) *is strictly positive on (l, r).*

2. *The diffusion process y that solves* (3.8) *is ergodic with invariant density f.*

3. *The function vf satisfies,*

$$\int_l^r v(x)f(x)dx < \infty, \tag{3.10}$$

and,

$$\mathrm{E}\left(y(s+t)|y(s)=x\right) = xe^{-\lambda t} + \mu\left(1-e^{-\lambda t}\right).$$

If the initial condition y_0 is a random variable with density f, then the process $y = (y(t), t \geq 0)$ is stationary, and it's autocorrelation function is given by,

$$\rho_y(h) = Corr(y(t), y(t+h)) = e^{-\lambda h},\ t, h \geq 0. \tag{3.11}$$

4. *If $-\infty < l$ or $r < \infty$, then the diffusion given by* (3.8) *is the only ergodic diffusion with drift $-\lambda(y-\mu)$ and invariant density f. If the state space is $\mathbb{R}$, it is the only ergodic diffusion with drift $-\lambda(y-\mu)$ and invariant density f for which* (3.10) *is satisfied.*

Remark When f has infinite second moment but finite first moment, the SDE (3.8) has a unique Markovian weak solution with invariant density, but (3.10) is not satisfied. The condition of the existence of the first moment can not be relaxed because function v has to be well-defined through (3.9).

Consider a stationary diffusion process with the IG invariant density, that is pdf $f = f_{IG}$ is given by (2.10). This inverse Gaussian diffusion process with ergodic density (2.10) has the correlation function (3.11) and satisfies SDE (3.8) with,

$$\mu = \frac{\delta}{\gamma} \quad \text{and} \quad v(x) = \frac{4\lambda\delta}{\gamma f(x)} e^{2\gamma\delta}\Phi\left(-\frac{\gamma x+\delta}{\sqrt{x}}\right), \tag{3.12}$$

where,

$$\Phi(x) = \int_{-\infty}^{x} \frac{e^{-t^2/2}}{\sqrt{2\pi}} dt,\ x \in \mathbb{R}.$$

is the standard normal distribution function.
The inverse Gaussian diffusion process constructed as a solution of (3.8) satisfies the conditions that ensure ρ-mixing and α-mixing with exponential rate. All definitions of mixing required in this thesis are collected in Appendix E, and will be used later to establish the asymptotic self-similarity of the activity time in the FATGBM model.

Remark We consider diffusion processes with inverse Gaussian distributions. These distributions have the additivity property in one of the parameters, namely if independent random variables X_1 and X_2 have $IG(\delta_1, \gamma)$ and $IG(\delta_2, \gamma)$ distributions respectively, then $X_1 + X_2$ has $IG(\delta_1 + \delta_2, \gamma)$ distribution. In addition, the variance of inverse Gaussian distribution is proportional to the parameter in which the additivity property holds. These properties are needed for the construction of superpositions of diffusion processes that are described in the next section. Note that while Gamma distribution also has these properties with respect to its shape parameter, the density is not bounded when the shape parameter is less than 1, and the solution of (3.8) is not ergodic. When the shape parameter is greater than 1, only finite superpositions that correspond to models for log returns with weak dependence and Gamma marginal distribution can be constructed using diffusion-type processes. In the case of NIG marginal distributions, processes with either weak and long range dependence can be constructed as described below.

Let $\{\tau^{(k)}(t),\ k \geq 1\}$ be a sequence of independent processes such that each $\tau^{(k)}$ is solution of the equation,

$$d\tau^{(k)}(t) = -\lambda^{(k)}(\tau^{(k)}(t) - \mu^{(k)})dt + \sqrt{v^{(k)}(\tau^{(k)}(t))}dW^{(k)}(t), \quad t \geq 0, \tag{3.13}$$

in which the Brownian motions $W^{(k)}$ are independent and the coefficients in (3.13) are such that the distribution of $\tau^{(k)}$ is $IG(\delta_k, \gamma)$. In other words, each of the stationary diffusion processes $\tau^{(k)}(t)$ has given IG marginals. Define the process τ_t using superpositions of stationary diffusion processes, either finite for an integer m,

$$\tau_t^m = \sum_{k=1}^{m} \tau^{(k)}(t),$$

or infinite,

$$\tau_t^\infty = \sum_{k=1}^{\infty} \tau^{(k)}(t). \tag{3.14}$$

The construction with infinite superposition is well-defined in the sense of mean-square or almost-sure convergence provided $\sum_{k=1}^{\infty} \delta_k < \infty$. The marginal distribution of τ_t^m is $IG(\sum_{k=1}^{m} \delta_k, \gamma)$ in the case of finite superposition, and the marginal distribution of τ_t^∞ is $IG(\sum_{k=1}^{\infty} \delta_k, \gamma)$ in the case of infinite superposition.

In the case of finite superposition, the covariance function of process τ^m is,

$$\begin{aligned} R_{\tau^m}(t) = Cov(\tau_s^m, \tau_{t+s}^m) &= \sum_{k=1}^{m} Var(\tau^{(k)}(t))e^{-\lambda^{(k)}t} \\ &= \sum_{k=1}^{m} \frac{\delta_k}{\gamma^3} e^{-\lambda^{(k)}t}. \end{aligned}$$

In the case of infinite superposition, the covariance function is,

$$R_{\tau^\infty}(t) = Cov(\tau_s^\infty, \tau_{t+s}^\infty) = \sum_{k=1}^{\infty} Var(\tau^{(k)}(t))e^{-\lambda^{(k)}t}$$
$$= \sum_{k=1}^{\infty} \frac{\delta_k}{\gamma^3} e^{-\lambda^{(k)}t}.$$

For infinite superposition, choose $0 < H < 1$, $\delta_k = k^{-(1+2(1-H))}$ and let,

$$\alpha(H) = \sum_{k=1}^{\infty} \frac{1}{k^{1+2(1-H)}},$$

be Riemann zeta-function $\zeta(z)$ with $z = 1 + 2(1 - H)$. Then, with $\lambda^{(k)} = 1/k$,

$$R_{\tau^\infty}(t) = \frac{1}{\gamma^3} \sum_{k=1}^{\infty} \frac{1}{k^{1+2(1-H)}} e^{-t/k}.$$

Lemma For infinite superposition, the covariance function of τ^∞ can be written as,

$$R_{\tau^\infty}(t) = \frac{L(t)}{t^{2(1-H)}},$$

where L is a slowly varying at infinity function, bounded on every bounded interval.

Proof Set,

$$L(t) = \frac{t^{2(1-H)}}{\gamma^3} \sum_{k=1}^{\infty} \frac{1}{k^{1+2(1-H)}} e^{-\frac{t}{k}}.$$

We estimate the sum appearing in the expression for L as follows:

$$\int_1^\infty \frac{e^{-\frac{t}{u}}}{u^{1+2(1-H)}} du \leq \sum_{k=1}^{\infty} \frac{1}{k^{1+2(1-H)}} e^{-\frac{t}{k}} \leq \int_1^\infty \frac{e^{-\frac{t}{u}}}{u^{1+2(1-H)}} du + e^{-t}.$$

Transforming the variables $t/u = s$, we get,

$$\int_0^t e^{-s} s^{2(1-H)-1} ds \leq L(t) \leq \int_0^t e^{-s} s^{2(1-H)-1} ds + e^{-t} t^{2(1-H)}.$$

Since,

$$\int_0^t e^{-s} s^{2(1-H)-1} ds \to \Gamma(2(1-H)), \text{ as } t \to \infty,$$

we have that $\lim_{t\to\infty} \frac{L(tv)}{L(t)} = 1$ for any fixed $v > 0$. □

Remark If $\frac{1}{2} < H < 1$, the process τ_t^∞ has long range dependence.

3.2.2 Convergence to a self-similar limit (II)

The activity time process T_t, constructed using the diffusion-type approach, is asymptotically self-similar. In the case of finite superposition we will use the notation,

$$T_t^m = \sum_{i=1}^{t} \tau_i^m,$$

and T_t^m has the Brownian motion as the weak limit as $t \to \infty$. In the case of infinite superpositions and $\frac{1}{2} < H < 1$, the weak limit is fractional Brownian motion. Namely, let $D[0,1]$ be Skorokhod space, and for $t \in [0,1]$ consider random functions $T_{[Nt]}^m$ and $T_{[Nt]}^\infty$.

Theorem 3. *For a fixed $m < \infty$ (finite superposition),*

$$\frac{1}{c_m\sqrt{N}}\left(T_{[Nt]}^m - ET_{[Nt]}^m\right) \to B(t), \quad t \in [0,1], \text{ as } N \to \infty, \tag{3.15}$$

in the sense of weak convergence in $D[0,1]$. The process $B = B(t), t \in [0,1]$ is Brownian motion, and the norming constant c_m is given by

$$c_m = \left(\sum_{k=1}^{m} \frac{\delta_k}{\gamma^3} \frac{1 - e^{-\lambda^{(k)}}}{1 + e^{-\lambda^{(k)}}}\right)^{\frac{1}{2}}.$$

Proof The proof of this theorem uses strong mixing properties of the components of the finite superposition. First we check the conditions of Proposition 2.8 and Corollary 2.1 in Genon-Catalot et al. (2000) [57] to show that each $\tau^{(k)}$ satisfies ρ-mixing mixing condition with exponential rate. Condition (A4) in [57] requires that,

$$\lim_{x\to 0+} \sqrt{\upsilon^{(k)}(x)} f^{(k)}(x) = 0, \quad \lim_{x\to +\infty} \sqrt{\upsilon^{(k)}(x)} f^{(k)}(x) = 0.$$

Here $f^{(k)}$ is the $IG(\delta_k, \gamma)$ density. This condition holds since from (3.12),

$$\lim \sqrt{\upsilon^{(k)}(x) f^{(k)}(x)} = \lim \sqrt{f^{(k)}(x)\Phi(-\gamma\sqrt{x} - \delta_k/\sqrt{x})}.$$

Condition (A5) in [57] requires that the function,

$$g^{(k)}(x) = \frac{2\sqrt{\upsilon^{(k)}(x)}}{\upsilon^{(k)\prime}(x) + 4\lambda^{(k)}(x - \mu^{(k)})},$$

has finite limits as $x \to 0+$ and as $x \to +\infty$.

In our case,

$$\lim_{x\to 0+} g^{(k)}(x) = 0, \quad \lim_{x\to +\infty} g^{(k)}(x) = 0,$$

for each $k = 1, \ldots, m$. Therefore if $\rho^{(k)}$ is the ρ-mixing coefficient of $\tau^{(k)}$ then,

$$\frac{\rho^{(k)}(t)}{4} = O(e^{-\epsilon_k t}),$$

for some $\epsilon_k > 0$.

Next, we will use the fact that ρ-mixing implies α-mixing (strong mixing) (see Bradley (2005) [28]). Denote by $\alpha^{(k)}(t)$ the strong mixing coefficient of the process $\tau^{(k)}$, that is (A.5) holds in Appendix E, where $\mathcal{F}_t = \sigma(\tau^{(k)}(s), s \leq t)$, $\mathcal{F}^t = \sigma(\tau^{(k)}(s), s \geq t)$, $t \geq 0$. We have that, for $k = 1, \ldots, m$,

$$\alpha^{(k)}(t) \leq \frac{\rho^{(k)}(t)}{4} = O(e^{-\epsilon_k t}),$$

The finite sum of m strong mixing processes is also strong mixing [28]. Since all moments exist for IG distribution, and the series $\sum_{i=1}^{\infty} \left(\alpha^{(k)}(i)\right)^{\frac{\delta}{2(2+\delta)}}$ converges for any $\delta > 0$ and $k = 1, \ldots, m$, Theorem 4.2 in Davydov (1970) [39] yields weak convergence in $D[0,1]$.

The norming constant is,

$$c_m^2 = Var(\tau_1^m) + 2\sum_{i=1}^{\infty} cov(\tau_1^m, \tau_{i+1}^m), \tag{3.16}$$

where,

$$\tau_1^m = \sum_{k=1}^{m} \tau^{(k)}(1).$$

In case when $\tau^{(k)}$ have IG distribution,

$$Var(\tau_1^m) = \sum_{k=1}^{m} \frac{\delta_k}{\gamma^3},$$

and,

$$Cov(\tau_1^m, \tau_{i+1}^m) = \sum_{k=1}^{m} \frac{\delta_k}{\gamma^3} e^{-\lambda^{(k)} i}.$$

The proof is completed by substituting these expressions in (3.16) and computing the norming constant. □

Remark The infinite superposition can be approximated as precisely as required by the finite superposition, and the asymptotic self-similarity can be used for option pricing as discussed in Chapter 5.

3.3 Method III: OU-type processes

Thirdly, we will describe a construction of the activity time that uses superpositions of Ornstein-Uhlenbeck (OU-type) processes. This idea originates from the work of Barndorff-Nielsen (2001) [13]. In many ways, this is a direct alternative to the diffusion-type construction. The main difference being that standard Browinan motion is replaced by Lévy noise as the driving process. The Lévy noise is such that the distribution of the activity time can be either gamma or inverse Gaussian. This allows for either VG or NIG marginal distribution of returns, along with a dependence structure that includes asymptotic self-similarity (to standard Brownian motion for finite superposition and weak dependence).

3.3.1 The construction of unit increments (III)

We first recall some definitions and known results on Lévy processes (Skorokhod (1991) [118], Bertoin (1996) [22], Sato (1999) [114], Kyprianou (2006) [81]) and Ornstein-Uhlenbeck type processes (Barndorff-Nielsen (2001) [13], Barndorff-Nielsen and Shephard (2001) [16]) which are needed for our construction.

A random variable X is said to be infinitely divisible if its cumulant function has the Lévy-Khintchine form,

$$\kappa_X(u) = iau - \frac{d}{2}u^2 + \int_{\mathbb{R}} \left(e^{iux} - 1 - iux\mathbf{1}_{[-1,1]}(x)\right) \nu(dx), \ u \in \mathbb{R}, \tag{3.17}$$

where $a \in \mathbb{R}$, $d \geq 0$ and ν is the Lévy measure, i.e. a non-negative measure on $\mathbb{R}$ with,

$$\nu(\{0\}) = 0, \qquad \int_{\mathbb{R}} \min\left(1, x^2\right) \nu(dx) < \infty.$$

The triplet (a, d, ν) uniquely determines the random variable X. For a Gaussian random variable $X \sim \mathcal{N}(a, d)$, the Lévy triplet takes the form $(a, d, 0)$.

If X is a self-decomposable random variable, then there exists a stationary stochastic process $\{X(t), t \geq 0\}$, such that $X(t) \overset{d}{=} X$ and,

$$X(t) = e^{-\lambda t}X(0) + \int_0^t e^{-\lambda(t-s)} dZ(\lambda s), \tag{3.18}$$

for all $\lambda > 0$ (see Barndorff-Nielsen (1998) [12]). Conversely, if $\{X(t), t \geq 0\}$ is a stationary process and $\{Z(t), t \geq 0\}$ is a Lévy process, independent of $X(0)$, such that $X(t)$ and $Z(t)$ satisfy the Itô stochastic differential equation,

$$dX(t) = -\lambda X(t)\, dt + dZ(\lambda t), \tag{3.19}$$

for all $\lambda > 0$, then $X(t)$ is self-decomposable. A stationary process $X(t)$ of this kind is said to be an OU-type process.

The scaling in equation (3.19) is such that the marginal distribution of the solution does not depend on λ, and the law of Lévy process is determined uniquely by the distribution of X. From Sato (1999) [114] (Theorem 17.5 and Corollary 17.9), it is known that the strong stationary solution of this equation exists if Lévy measure ρ of $Z(1)$ satisfies,

$$\int_2^\infty (\log x)\,\rho(dx) < \infty. \tag{3.20}$$

For our construction, we will take the increments of activity time $\{\tau_t\}$ to be an OU-type as defined in the above formulations. Here, the marginal distribution of the solution of (3.19) is self-decomposable, and so are the marginal distributions we are considering, $\Gamma(\alpha, \beta)$ or $IG(\delta, \gamma)$. As a special case of a more general result we have the following theorem.

Theorem 4. *There exists a stationary process $\tau(t), t \geq 0$, which has marginal $\Gamma(\alpha, \beta)$ or $IG(\delta, \gamma)$ distribution and satisfies equation* (3.19)*. The process τ has all moments, and the correlation function of τ is given as follows:*

$$\rho_\tau(s) = Corr(\tau(t), \tau(t+s)) = e^{-\lambda s}, \ s \geq 0.$$

Since $\tau > 0$ in (3.18), we can take the cumulant function of τ to be of the form,

$$\kappa_\tau(u) = iau - \int_0^\infty (e^{iux} - 1)Q(dx), \ u \in \mathbb{R},$$

where $\int_0^\infty \max(x, 1)Q(dx)$, $Q(-\infty, 0) = 0$. Since both Gamma and inverse Gaussian distributions are self-decomposable, measure Q has density with respect to Lebesgue measure, and this density has the form: $Q(dx) = \frac{q(x)}{x}dx$, where function q is decreasing on $(0, \infty)$ and is called the canonical function. The explicit forms of function q for $\Gamma(\alpha, \beta)$ or $IG(\delta, \gamma)$ distribution of y are,

$$q_\Gamma(x) = \alpha e^{-\beta x} 1_{\{x>0\}}, \ q_{IG}(x) = \frac{\delta x^{-1/2}}{\sqrt{2\pi}} e^{-\gamma^2 x/2} 1_{\{x>0\}}.$$

Further, Lévy measure of Z (i.e. Lévy measure of $Z(1)$), ρ, satisfies $\rho(x, \infty) = q(x)$. For both Gamma and inverse Gaussian cases,

$$\int_2^\infty (\log x)\,\rho(dx) = -\int_2^\infty (\log x)\,q(x)dx < \infty,$$

which verifies condition (3.20).

Since κ_τ is differentiable for $u \neq 0$ and $u\kappa_\tau(u) \to 0$ for $u \to 0$, $u \neq 0$, the cumulant transform of $Z(1)$ can be written as,

$$\kappa_{Z(1)}(u) = \log E e^{iuZ(1)} = u\frac{\partial}{\partial u}\kappa_{\tau(t)}(u).$$

In the case when τ has Gamma marginals with cumulant function,

$$\begin{aligned}
\kappa_{Z(1)}(u) &= u\frac{\partial}{\partial u}\log\left(1-\frac{iu}{\beta}\right)^{-\alpha} \\
&= \frac{\alpha ui}{\beta}\left(1-\frac{iu}{\beta}\right)^{-1} \\
&= \alpha(\phi_1(u)-1),
\end{aligned} \tag{3.21}$$

where $\phi_1(u) = (1 - iu/\beta)^{-1}$ is the characteristic function of a $\Gamma(1, \beta)$ random variable. Therefore in the case of Gamma distribution, Lévy process $Z(t)$ is a compound Poisson process $Z(t) = \sum_{n=1}^{N(t)} Z_n$, where $N(t)$ is a homogeneous Poisson process with intensity α, and Z_n are independent identically distributed $\Gamma(1, \beta)$ random variables.

In the case when X has inverse Gaussian marginals, Lévy measure of $Z(1)$ is

$$\begin{aligned}
\rho(dx) &= \frac{\delta}{2}\frac{x^{-3/2}}{\sqrt{2\pi}}e^{-\gamma^2 x/2}dx + \frac{\delta\gamma^2}{2\sqrt{2\pi}}e^{-\gamma^2 x/2}dx \\
&= \rho_1(dx) + \rho_2(dx).
\end{aligned}$$

Therefore process Z can be represented as a sum of two independent Lévy processes, Z_1 and Z_2. For the first component,

$$\kappa_{Z_1(1)}(u) = \int_0^\infty (e^{iux} - 1)\rho_1(dx) = \int_0^\infty (e^{iux} - 1)\frac{q_1(x)}{x}dx,$$

where function q_1 is a canonical function of the $IG(\frac{\delta}{2}, \gamma)$ distribution, and process Z_1 is a Lévy process with IG marginals. The second component Z_2 is a compound Poisson process (see [72]),

$$Z_2(t) = \frac{1}{\gamma^2}\sum_{k=1}^{N(t)} W_n^2,$$

where $N(t), t \geq 0$ is a homogeneous Poisson process with intensity $\frac{\delta\gamma}{2}$, and $W_1, W_2, \dots$ are independent standard normal random variables (each W_k^2 is $\Gamma(\frac{1}{2}, \frac{1}{2})$).

These specifications of the Lévy processes are important for modelling the fractal activity times that are solutions of equation (3.19). In addition, the transition function of the solution is needed for deriving pricing formulae discussed later in Chapter 5.

Remark We consider OU processes with Gamma or inverse Gaussian marginals. These distributions have the additivity property in one of the parameters, namely if independent random variables X_1 and X_2 have $\Gamma(\alpha_1, \beta)$ and $\Gamma(\alpha_2, \beta)$ distributions respectively, then

$X_1 + X_2$ has $\Gamma(\alpha_1 + \alpha_2, \beta)$ distribution. For X_1 and X_2 inverse Gaussian $IG(\delta_1, \gamma)$ and $IG(\delta_2, \gamma)$ respectively $X_1 + X_2$ has $IG(\delta_1 + \delta_2, \gamma)$ distribution. In addition, the variances of Gamma and inverse Gaussian distributions are proportional to the parameter in which the additivity property holds. These properties are needed for the construction of superpositions of OU processes that are described in the next section. Another distribution that can be considered for the increments of activity time, reciprocal Gamma distribution (leading to Student's t-distribution of the returns), does not have these properties. For reciprocal Gamma distribution of the unit increments of the activity time process, a different construction that uses chi-square processes is available (see [15]). The construction via chi-square processes also works for Gamma distribution of the increments of activity time [50] [52]. In contrast to [50] we do not need any of the parameters to be integers.

We use a discrete version of superposition introduced in [12], to define the increments of the activity time. Let $\tau^{(k)}(t)$, $k \geq 1$ be the sequence of independent processes such that each $\tau^{(k)}(t)$ is solution of the equation,

$$d\tau^{(k)}(t) = -\lambda^{(k)}\tau^{(k)}(t) + dZ^{(k)}(\lambda^{(k)}t), \quad t \geq 0,$$

in which the Lévy processes $Z^{(k)}$ are independent and are such that the distribution of $\tau^{(k)}$ is either $\Gamma(\alpha_k, \beta)$ or $IG(\delta_k, \gamma)$. In other words, each of processes $\tau^{(k)}(t)$ is of OU type with given marginals. Define the process τ_t using superpositions of OU processes, either finite for an integer m,

$$\tau_t^m = \sum_{k=1}^{m} \tau^{(k)}(t), \tag{3.22}$$

or infinite,

$$\tau_t^\infty = \sum_{k=1}^{\infty} \tau^{(k)}(t). \tag{3.23}$$

The construction with infinite superposition is well-defined in the sense of mean-square or almost-sure convergence provided that $\sum_{k=1}^{\infty} \alpha_k < \infty$ in case of the VG model, and $\sum_{k=1}^{\infty} \delta_k < \infty$ in case of NIG model. For VG model, the marginal distribution of τ_t^m is $\Gamma(\sum_{k=1}^{m} \alpha_k, \beta)$ and for NIG model, the marginal distribution of τ_t^m is $IG(\sum_{k=1}^{m} \delta_k, \gamma)$. In the case of infinite superpositions the summation in the first parameter goes to infinity.

Lemma In the case of finite superposition, the covariance function of τ^m is,

$$R_{\tau^m}(t) = Cov(\tau_s^m, \tau_{t+s}^m) = \sum_{k=1}^{m} Var(\tau^{(k)}(t))e^{-\lambda^{(k)}t}. \tag{3.24}$$

For the VG model, $Var(\tau^{(k)}) = \alpha_k/\beta^2$, and for NIG model $Var(\tau^{(k)}) = \delta_k/\gamma^3$. In the case of infinite superposition, the respective summations are to infinity instead of m, and we choose $\alpha_k = k^{-(1+2(1-H))}$ in case of VG model, and choose $\delta_k = k^{-(1+2(1-H))}$ in case of NIG model. Let $\alpha(H) = \sum_{k=1}^{\infty} \frac{1}{k^{1+2(1-H)}}$ be Riemann zeta-function. Then, with $\lambda^{(k)} = \frac{1}{k}$,

$$R_{\tau^\infty}(t) = c\sum_{k=1}^{\infty} \frac{1}{k^{1+2(1-H)}} e^{-t/k}.$$

The constant c equals $\frac{1}{\beta^2}$ in VG model, and $\frac{1}{\gamma^3}$ in NIG model.

Lemma For infinite superposition, the covariance function of τ^∞ can be written as,

$$R_{\tau^\infty}(t) = \frac{L(t)}{t^{2(1-H)}},$$

where L is a slowly varying at infinity function, bounded on every bounded interval.

Proof As in the proof for the diffusion-type construction, we set,

$$L(t) = ct^{2(1-H)} \sum_{k=1}^{\infty} \frac{1}{k^{1+2(1-H)}} e^{-\frac{t}{k}}.$$

We estimate the sum appearing in the expression for L as follows:

$$\int_1^\infty \frac{e^{-\frac{t}{u}}}{u^{1+2(1-H)}} du \leq \sum_{k=1}^{\infty} \frac{1}{k^{1+2(1-H)}} e^{-\frac{t}{k}} \leq \int_1^\infty \frac{e^{-\frac{t}{u}}}{u^{1+2(1-H)}} du + e^{-t}.$$

Transforming the variables $\frac{t}{u} = s$ we get,

$$c\int_0^t e^{-s} s^{2(1-H)-1} ds \leq L(t) \leq c\int_0^t e^{-s} s^{2(1-H)-1} ds + ce^{-t} t^{2(1-H)}.$$

Since,

$$\int_0^t e^{-s} s^{2(1-H)-1} ds \to \Gamma(2(1-H)),$$

as $t \to \infty$, we have that $\lim_{t\to\infty} \frac{L(tv)}{L(t)} = 1$ for any fixed $v > 0$. □

Remark If $\frac{1}{2} < H < 1$, the process τ_t^∞ has long range dependence.

3.3.2 Convergence to a self-similar limit (III)

We will now show that the activity time process T_t constructed using the OU-type approach is asymptotically self-similar. In the case of finite superposition we will use the notation $T_t^m = \sum_{i=1}^t \tau_i^m$, and in the case of infinite superposition we will use the notation $T_t^\infty = \sum_{i=1}^t \tau_i^\infty$. As before, let $D[0,1]$ be Skorokhod space, and for $t \in [0,1]$ consider random functions $T_{[Nt]}^m$ and $T_{[Nt]}^\infty$.

Theorem 5. *For a fixed $m < \infty$ (finite superposition),*

$$\frac{1}{c_m\sqrt{N}}\left(T^m_{[Nt]} - ET^m_{[Nt]}\right) \to B(t), \quad t \in [0,1], \text{ as } N \to \infty,$$

in the sense of weak convergence in $D[0,1]$. The process $B(t), t \in [0,1]$ is Brownian motion, and the norming constant c_m is given by,

$$c_m = \left(\sum_{k=1}^{m} Var(\tau^{(k)})\frac{1-e^{-\lambda^{(k)}}}{1+e^{-\lambda^{(k)}}}\right)^{\frac{1}{2}},$$

where $Var(\tau^{(k)}) = \alpha_k/\beta^2$ for the VG model, and $Var(\tau^{(k)}) = \delta_k/\gamma^3$ for the NIG model.

Proof The proof of this theorem has two steps. First, each OU process in the finite superposition is β-mixing (absolutely regular, see Appendix E) under condition (3.20) [72] (Theorem 3.1). Under a stronger condition of existence of the absolute moment of order $p > 0$ of the marginal distribution, there exists $a > 0$ such that the mixing coefficient $\beta_y(t) = O(e^{-at})$ ([99], Theorem 4.3). This condition is satisfied for both Gamma and inverse Gaussian distributions that have all moments.
Second, a finite sum of β-mixing processes is also β-mixing [28]. Denote by $\alpha^{(k)}(t)$ the strong mixing coefficient of the process $\tau^{(k)}$, i.e. (A.5) holds in Appendix E, where $\mathcal{F}_t = \sigma(\tau^{(k)}(s), s \le t)$, $\mathcal{F}^t = \sigma(\tau^{(k)}(s), s \ge t)$, $t \ge 0$. As discussed in [28], β - mixing implies α-mixing. Since for each k, $2\alpha^{(k)}(t) \le \beta^{(k)}(t) \le C_k e^{-a_k t}$, Theorem 4.2 in [39] yields weak convergence in $D[0,1]$.
The norming constants for Gamma and inverse Gaussian cases are computed using equation (3.24). □

Remark The infinite superposition can be approximated as precisely as required by the finite superposition, and the asymptotic self-similarity can be used for option pricing as discussed in Chapter 5.

Chapter 4

Data Fitting

"Life always has a fat tail." (Eugene F. Fama).

Here we want to validate our approach for risky asset modelling. To do this, we will estimate the parameters of suitable marginals to the best of our ability whilst promoting the use of a flexible dependence structure and non-Gaussian marginals for returns. Our chosen method will be non-linear least squares to fit; the Student distribution, the VG distribution and the NIG distribution to the data. The objective will be to minimise the sum of squared residuals (errors). All initial parameter values will first be derived from the method of moments. Thus hopefully reducing the number of iterative refinements needed to achieve convergence and the best fit possible. Note that we cannot use the popular maximum likelihood estimation method as the independence assumption for returns does not hold.
We will then focus on the symmetric Student fit only. Here we carry out hypothesis testing to determine whether the parameter estimators are good enough. This fit is tested by using a brand new expression for the characteristic function of the Student's t-distribution.

4.1 Parameter estimation using method of moments

From a Chebyshev type argument, we can prove that sample moments are consistent estimators of unknown central moments by applying the method of moments. We consider the following sample moments:

$$\hat{\mu}_1 = \frac{1}{N}\sum_{t=1}^{N} X_t = \bar{X}, \quad \hat{\mu}_2 = \frac{1}{N}\sum_{t=1}^{N}(X_t - \bar{X})^2 = s^2,$$

$$\hat{\mu}_3 = \frac{1}{N}\sum_{t=1}^{N}(X_t - \bar{X})^3, \quad \hat{\mu}_4 = \frac{1}{N}\sum_{t=1}^{N}(X_t - \bar{X})^4,$$

which gives us the sample skewness and kurtosis:

$$\hat{\gamma}_1 = \frac{\hat{\mu}_3}{(\hat{\mu}_2)^{\frac{3}{2}}}, \text{ and } \hat{\gamma}_2 = \frac{\hat{\mu}_4}{(\hat{\mu}_2)^2} - 3. \tag{4.1}$$

Ideally we would have liked to use $S(\mu, \theta, \sigma^2, \nu, \delta)$, $VG(\mu, \theta, \sigma^2, \alpha, \beta)$ and $NIG(\mu, \theta, \sigma^2, \delta, \gamma)$ for any parameter estimation, but is not possible due to the number of unknown parameters and the limited number of equations. For the method of moments we will need to make some small simplifications to obtain some meaningful results. Note that for the following cases there is no need for any assumptions of the independence of risky asset increments.

Case I: The Student model

Instead of $S(\mu, \theta, \sigma^2, \alpha, \beta)$, we will consider the (symmetric scaled) Student's t-distribution $S(\mu, \delta, \nu)$ advocated by Heyde and Leonenko (2005) [66] with $\nu > 0$ degrees of freedom. It can be defined by the probability density function (pdf)

$$f_S(x) = c(\nu, \delta)\frac{1}{\left(1 + \left(\frac{x-\mu}{\delta}\right)^2\right)^{\frac{\nu+1}{2}}}, \quad x \in \mathbb{R} \tag{4.2}$$

where $\mu \in \mathbb{R}$ is location parameter, $\delta > 0$ is scaling parameter, and

$$c(\delta, \nu) = \frac{\Gamma(\frac{\nu+1}{2})}{\delta\sqrt{\pi}\Gamma(\frac{\nu}{2})}.$$

Thus, a random variable $X \sim T(\mu, \delta, \nu)$ can be represented by $X|V \sim N(\mu, \sigma^2 V)$, where V has an inverse (reciprocal) gamma distribution. The expectation of X exists when $\nu > 1$, the variance when $\nu > 2$ and the n-th moment when $\nu > n$. By using the formula,

$$\int_0^\infty \frac{x^{\lambda-1}}{(1-x)^\mu}\,dx = \frac{\Gamma(\lambda)\Gamma(\mu-\lambda)}{\Gamma(\mu)}, \; 0 < \lambda < \mu, \tag{4.3}$$

we obtain the following expressions for the central moments of random variable $X \sim T(\mu, \delta, \nu)$:

$$E(X-\mu)^n = 0\,, \quad n = 2k-1 < \nu, \; k = 1, 2, \ldots$$

$$
\begin{aligned}
E(X-\mu)^n &= \int_{-\infty}^{\infty} (x-\mu)^n c(\delta,\nu) \frac{1}{\left[1+\left(\frac{x-\mu}{\delta}\right)^2\right]^{\frac{\nu+1}{2}}} dx \\
&= \int_{-\infty}^{\infty} \left(\frac{x}{\delta}\right)^n \delta^n c(\delta,\nu) \frac{1}{\left[1+\left(\frac{x}{\delta}\right)^2\right]^{\frac{\nu+1}{2}}} dx \\
&= \delta^{n+1} c(\delta,\nu) \int_0^{\infty} \frac{z^{\frac{n+1}{2}-1}}{[1+z]^{\frac{\nu+1}{2}}} dz \\
&= \delta^{n+1} c(\nu,\delta) \frac{\Gamma(\frac{n+1}{2})\Gamma(\frac{\nu-n}{2})}{\Gamma(\frac{\nu+1}{2})}\ , \ n=2k<\nu,\ k=1,2,\ldots \qquad (4.4)
\end{aligned}
$$

From the expression (4.4) and straightforward calculation, we conclude that the first two even central moments are:

$$
\mu_2 = E(X-\mu)^2 = \frac{\delta^2\Gamma(\frac{\nu-2}{2})}{2\Gamma(\frac{\nu}{2})} = \frac{\delta^2}{\nu-2},\ \nu>2, \qquad (4.5)
$$

$$
\mu_4 = E(X-\mu)^4 = \frac{3\delta^4}{(\nu-4)(\nu-2)},\ \nu>4, \qquad (4.6)
$$

with the coefficient of kurtosis γ_2,

$$
\gamma_2 = \frac{6}{\nu-4},\ \nu>4.
$$

Using Chebyshev's inequality we may prove that sample moments give consistent estimators for μ_2 and μ_4 and thus for γ_2. Hence, we may apply the method of moments and get consistent estimators,

$$
\hat{\mu} = \bar{X},\quad \hat{\nu} = \frac{6}{\hat{\gamma}_2}+4,\quad \hat{\delta} = s\sqrt{\hat{\nu}-2} \qquad (4.7)
$$

Case II: The VG model

If we say that the increments of activity time follow a marginal gamma distribution $\Gamma(\alpha,\alpha)$, then returns will have a marginal (skew) VG distribution (see Madan et al (1998) [88]) with pdf,

$$
f_{VG}(x) = \sqrt{\frac{2}{\pi}} \frac{\alpha^\alpha e^{\frac{(x-\mu)\theta}{\sigma^2}}}{\sigma\Gamma(\alpha)} \left(\frac{|x-\mu|}{\sqrt{\theta^2+2\alpha\sigma^2}}\right)^{\alpha-\frac{1}{2}} K_{\alpha-\frac{1}{2}}\left(\frac{|x-\mu|\sqrt{\theta^2+2\alpha\sigma^2}}{\sigma^2}\right) \qquad (4.8)
$$

This distribution was called VG because for random variable $X \sim VG(\mu,\theta,\sigma^2,\alpha)$ we can consider $X|V \sim N(\mu,\sigma^2 V)$, where V has a gamma distribution. The symmetric case in which $\theta = 0$ was first introduced in a financial context by Madan and Seneta (1990) [89] and has pdf,

$$f_{VG}(x) = \sqrt{\frac{2}{\pi}} \frac{\alpha^\alpha}{\sigma\Gamma(\alpha)} \left(\frac{|x-\mu|}{\sqrt{2\alpha\sigma^2}}\right)^{\alpha-\frac{1}{2}} K_{\alpha-\frac{1}{2}}\left(\frac{|x-\mu|\sqrt{2\alpha\sigma^2}}{\sigma^2}\right), \; x \in \mathbb{R}.$$

Here $K_\lambda(.)$ is a modified Bessel function of the third kind with index λ, given for $x > 0$ (see Appendix C).

Recall that the distribution of $\Gamma \sim \Gamma(\alpha, \alpha)$ is given by the pdf (2.8), so that expectation is 1, as required in the chi-squared construction and variance is $\frac{1}{\alpha}$. For $\theta = 0$, we have the simplified real-valued characteristic function,

$$\varphi_X(u) = \left(1 - \frac{\sigma^2 u^2}{2\alpha}\right)^{-\alpha}$$

Letting $\alpha \to \infty$ results in $\varphi_X(u) = e^{-\frac{\sigma^2 u^2}{2}}$, so X has an $N(0, \sigma^2)$ distribution.

Consequently, we obtain the following (Madan et al (1998) [88]),

$$EX = \mu + \theta, \tag{4.9}$$

$$VarX = \sigma^2 + \frac{\theta^2}{\alpha}, \tag{4.10}$$

$$E(X - EX)^3 = \frac{2\theta^3}{\alpha^2} + \frac{3\sigma^2\theta}{\alpha}, \tag{4.11}$$

$$E(X - EX)^4 = 3\sigma^4\left(1 + \frac{1}{\alpha}\right) + \frac{6\sigma^2\theta^2}{\alpha}\left(1 + \frac{2}{\alpha}\right) + \frac{3\theta^4}{\alpha^2}\left(1 + \frac{2}{\alpha}\right), \tag{4.12}$$

From this, we can obtain expressions for the coefficients of skewness γ_1 and of kurtosis γ_2. From (4.9)-(4.12), and ignoring terms in θ^2, θ^3, θ^4, we have,

$$EX = \mu + \theta, \quad \mu_2 = \sigma^2, \quad \gamma_1 = \frac{3\theta}{\alpha\sigma}, \quad \gamma_2 = 3\left(1 + \frac{1}{\alpha}\right). \tag{4.13}$$

Thus, if $\hat{\theta}$ is small, we may successfully obtain approximations to $\hat{\sigma}$, $\hat{\alpha}$, $\hat{\theta}$, and $\hat{\mu}$:

$$\hat{\sigma} = s, \quad \hat{\alpha} = \frac{3}{\hat{\gamma}_2}, \quad \hat{\theta} = \frac{\hat{\gamma}_1\hat{\alpha}\hat{\sigma}}{3}, \quad \hat{\mu} = \bar{X} - \hat{\theta}. \tag{4.14}$$

Additionally, these estimators can be simply adjusted for the symmetric case (taking $\theta = 0$ in (4.13)).

Case III: The NIG model

Suppose that the conditional distribution of X given Z is $N(\mu + \kappa Z, Z)$. If Z itself follows an $IG(\delta, \gamma)$ distribution with parameters δ and γ, then the resulting mixed distribution is $NIG(\mu, \kappa, \alpha, \delta)$, where $\alpha = \sqrt{\gamma^2 + \kappa^2}$. The pdf of X is then given by,

$$f_{NIG}(x) = \frac{\alpha}{\pi} e^{\delta\sqrt{\alpha^2-\kappa^2}-\kappa\mu} \phi(x)^{-\frac{1}{2}} K_1(\delta\alpha\phi(x)^{\frac{1}{2}}) e^{\kappa x}, \ x \in \mathbb{R}, \tag{4.15}$$

where $\phi(x) = 1 + (\frac{x-\mu}{\delta})^2$ and $K_r(x)$ denotes the modified Bessel function of the third kind of order r evaluated at x.

This distribution is a flexible, four parameter distribution that can describe a wide range of shapes. Some properties are summarized using the moments. For further information see Barndorff-Nielsen (1997) [11] and Lillestol (2000) [87].

After some work, it can be shown that,

$$EX = \mu + \frac{\delta\kappa}{\gamma}, \quad \mu_2 = \frac{\delta\alpha^2}{\gamma^3}, \quad \gamma_1 = \frac{3\kappa}{\alpha(\gamma\delta)^{\frac{1}{2}}}, \quad \gamma_2 = 3(\frac{1 + \frac{4\kappa^2}{\alpha^2}}{\delta\gamma}). \tag{4.16}$$

The moments given in (4.16) have a rather simple form allowing for simple moment estimation. Equating the theoretical moments with their sample counterparts, and solving for the parameters one obtains that,

$$\hat{\gamma} = \frac{3}{s\sqrt{3\hat{\gamma}_2 - 5\hat{\gamma}_1^2}}, \quad \hat{\kappa} = \frac{\hat{\gamma}_1 s \hat{\gamma}^2}{3}, \quad \hat{\delta} = \frac{s^2\hat{\gamma}^3}{\hat{\kappa}^2 + \hat{\gamma}^2}, \quad \hat{\mu} = \bar{X} - \hat{\kappa}\frac{\hat{\delta}}{\hat{\gamma}}.$$

Note that, the moment estimates do not exist if $3\hat{\gamma}_2 < 5\hat{\gamma}_1^2$. Also, these parameter estimates can be easily adjusted for the symmetric case by taking $\kappa = 0$ in (4.16), (??), and (??).

To illustrate the fit to the data, the parameter estimates in *Tables 4.2, 4.3 and 4.4* are used to plot the Student, VG and NIG probability density curves alongside the inferior corresponding Gaussian. In addition to higher peaks than Gaussian distribution (see *Figure 4.1*), we observe that typical returns data has heavier tails also (see *Figure 4.2*). The parabolic decay of the tails for the logarithm of Gaussian density is too fast, while the hyperbolic decay of the log-Student, log-VG and log-NIG provides a much better fit to the data. Whether the semi-heavy tails of the Student distribution are substantial enough to model typical returns data is unclear. The heavy tails of VG and NIG may be more suitable. As we discussed previously, further study into the actual tailweight of risky asset returns is required.

4.2 Testing the parameter estimates of the Student model

To directly compare the fits of the symmetric Student distribution with the Gaussian, one may consider applying the chi-square goodness of fit test. When testing the goodness-of-fit for Gaussian distribution, under the null hypothesis that the returns are Gaussian and therefore independent since they are uncorrelated in our model. The test provides a valid procedure, and the hypothesis of the Gaussian distribution is rejected for all 10 risky assets (see *Table 2.4*). However, the goodness of fit test that states that the marginal distribution of the returns is Student has not been developed. This test will be treated formally under the null hypothesis the returns are dependent, and may be even long-range dependent. An associated test has been applied formally, under the hypothesis that returns are dependent, by Finlay and Seneta (2006) [50] for the VG model. However, the development of the goodness-of-fit test for dependent data is ongoing in the literature.

Instead we present a testing procedure based on the characteristic function for the following null hypothesis:

$$H_0\colon \nu = \hat{\nu} \ \text{ and } \ \delta = \hat{\delta}. \tag{4.17}$$

4.2.1 The characteristic function of the Student's t-distribution

The characteristic function of the Student t-distribution has been a topic of some controversy and difficulties in statistical literature for the last 30 years (see Ifram (1970) [71], Pastena (1991) [105], Hurst (1995) [70] and Dreier and Kotz (2002) [43] for the survey of explicit expressions). In this section we focus on the case when $\mu = 0$, and use notation $T_{\nu,\delta}$ for symmetric Student.

We note that when the degrees of freedom ν is an integer n, and $\delta = \nu = n$, the density of t-variable T_n is,

$$f_n(x) = \frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi n}\Gamma(\frac{n}{2})}\left(1 + \frac{x^2}{n}\right)^{-\frac{n+1}{2}}, \ x \in \mathbb{R}\,. \tag{4.18}$$

For the integer degrees of freedom n and $\delta = n$, Dreier and Kotz (2002) [43] developed the following expression for the characteristic function:

$$\phi_{T_n}(t) = Ee^{itT_n} = \frac{2^n\sqrt{n^n}}{\Gamma(n)}\int_0^\infty e^{-\sqrt{n}(2x+|t|)}(x(x+|t|))^{\frac{n-1}{2}}\,dx,\ t \in \mathbb{R}\,. \tag{4.19}$$

On the other hand, in the more general case of Student's t-variable $T_{\nu,\delta}$ with the density,

$$f_{\nu,\delta}(x) = \frac{\Gamma(\frac{\nu+1}{2})}{\delta\sqrt{\pi}\Gamma(\frac{\nu}{2})}(1 + (\frac{x}{\delta})^2)^{-\frac{\nu+1}{2}}, \ x \in \mathbb{R}\,, \tag{4.20}$$

the following elegant expression for the characteristic function is known,

$$\phi_{T_{\nu,\delta}}(t) = Ee^{itT_{\nu,\delta}} = \frac{K_{\frac{\nu}{2}}(\delta|t|)(\delta|t|)^{\frac{\nu}{2}}2^{1-\frac{\nu}{2}}}{\Gamma(\frac{\nu}{2})}, \; t \in \mathbb{R} \; , \tag{4.21}$$

where $K_\lambda(\cdot)$ is the Bessel function of the third kind of order λ (see Appendix C).

Note that the parameter $\nu > 0$ in (4.20) and (4.21) is not necessarily integer while the scaling parameter $\delta > 0$. The formula (4.21) was derived by Hurst (1995) [70]. Next, we will prove that the formula (4.19) can be generalized to the Student t-distribution $T_{\nu,\delta}$ with the density (4.20).

Theorem 6. *The characteristic function of the t-variable $T_{\nu,\delta}$ with the density* (4.20) *can be written as follows:*

$$\varphi_{T_{\nu,\delta}}(t) = \frac{(2\delta)^\nu}{\Gamma(\nu)} \int_0^\infty e^{-\delta(2x+|t|)}(x(x+|t|))^{\frac{\nu-1}{2}}\, dx \; , t \in \mathbb{R} \; . \tag{4.22}$$

Proof Let X be a Gamma distributed random variable $\Gamma(\alpha, \beta)$ with the parameters $\alpha = \frac{\nu+1}{2}$ and $\beta = \frac{1}{\delta}$, whose density is given by,

$$f_X(x) = \left(\frac{1}{\beta}\right)^\alpha \frac{1}{\Gamma(\alpha)} e^{-\frac{x}{\beta}} x^{\alpha-1} = \frac{\delta^{\frac{\nu+1}{2}}}{\Gamma(\frac{\nu+1}{2})} e^{-\delta x} x^{\frac{\nu-1}{2}} \; , \quad x \geq 0,$$

and characteristic function,

$$\varphi_X(t) = \left(1 - \frac{it}{\delta}\right)^{-\frac{(\nu+1)}{2}} .$$

Consider now the random variable Y independent of X such that $(-Y)$ has the same distribution as the random variable X.

Finally, we consider the sum $Z = X + Y$. The characteristic function of Z is,

$$\varphi_Z(t) = \left(1 - \frac{it}{\delta}\right)^{-\frac{(\nu+1)}{2}} \left(1 + \frac{it}{\delta}\right)^{-\frac{(\nu+1)}{2}} = \left(1 + \frac{t^2}{\delta^2}\right)^{-\frac{(\nu+1)}{2}} .$$

Thus, the characteristic function of Z is the density of the t-variable $T_{\nu,\delta}$ normalized in such a manner that it is equal to 1 at 0.

As in Dreier and Kotz (2002) [43] we will now use the duality theorem for characteristic functions of the symmetric distributions and probability density functions (see Appendix F). This theorem implies that the density of Z normed such that it is equal to 1 at 0 ($f_T(t) = \frac{f_Z(t)}{f_Z(0)}$, $t \in \mathbb{R}$) is the characteristic function of $T_{\nu,\delta}$.

The density of Z is now found as convolution of the densities f_X and f_Y,

$$\begin{aligned} f_Z(z) = f_{X+Y}(z) &= \int_{\mathbb{R}} p_X(x) p_Y(z-x)\, dx \ , \\ &= \left(\frac{\delta^{\frac{\nu+1}{2}}}{\Gamma(\frac{\nu+1}{2})} \right)^2 \int_z^\infty e^{-\delta x} x^{\frac{\nu-1}{2}} e^{\delta(z-x)} (x-z)^{\frac{\nu-1}{2}}\, dx \ , \\ &= \frac{\delta^{\nu+1}}{(\Gamma(\frac{\nu+1}{2}))^2} \int_0^\infty e^{-\delta(2x+z)} (x(x+z))^{\frac{\nu-1}{2}}\, dx \ , \quad z > 0 \ . \end{aligned}$$

Since,

$$f_Z(0) = \frac{\delta^{\nu+1}}{(\Gamma(\frac{\nu+1}{2}))^2} \int_0^\infty e^{-2\delta x} x^{\nu-1}\, dx = \frac{\delta^{\nu+1}}{(\Gamma(\frac{\nu+1}{2}))^2} \frac{\Gamma(\nu)}{(2\delta)^\nu} \ ,$$

and due to the symmetry of the characteristic function of the t-variable $T_{\nu,\delta}$, we arrive at the following representation of the characteristic function of the t-variable:

$$\varphi_T(t) = \frac{f_Z(t)}{f_Z(0)} = \frac{(2\delta)^\nu}{\Gamma(\nu)} \int_0^\infty e^{-\delta(2x+|t|)} (x(x+|t|))^{\frac{\nu-1}{2}}\, dx \ , \quad t \in \mathbb{R} \ .$$

□

Note that from (4.22), to the best of our knowledge, we have obtained a new expression for the modified Bessel function of the third kind (see Appendix G).

Remark The obtained expression is computationally friendly. For $\nu = 1$,

$$\phi_T(t) = e^{-|t|}, \ t \in \mathbb{R}, \tag{4.23}$$

which is the characteristic function of the Cauchy distribution. For $\delta = \sqrt{\nu}$ and $\nu \to \infty$, we obtain,

$$\phi_T(t) = e^{-\frac{t^2}{2}}, \ t \in \mathbb{R}, \tag{4.24}$$

the well-known expression for the characteristic function of the normal distribution.

4.2.2 Hypothesis testing using the characteristic function

We will use the result from Theorem 1 as our characteristic function of the Student's t-distribution,

$$\varphi(u;\nu,\delta) = \varphi_{T_{\nu,\delta}}(u) = \frac{(2\delta)^\nu}{\Gamma(\nu)} \int_0^\infty e^{-\delta(2x+|u|)} (x(x+|u|))^{\frac{\nu-1}{2}}\, dx \ , u \in \mathbb{R} \ . \tag{4.25}$$

For a real-valued (and therefore symmetric) characteristic function,

$$\varphi(u;\nu,\delta) = E e^{iuT_{\nu,\delta}} = E\cos(uT_{\nu,\delta}).$$

So let $\hat{\varphi_N}(u)$ be the empirical characteristic function based on the observations of the returns $X_1, X_2, \ldots, X_N$,

$$\hat{\varphi_N}(u) = \frac{1}{N}\sum_{j=1}^{N} e^{iuX_j} = \frac{1}{N}\sum_{j=1}^{N}\cos(uX_j). \tag{4.26}$$

Theorem 7. *Let $X_1, X_2, \ldots, X_N$ be observations of the returns that follow model (2.5) with the activity time increments constructed via (3.5) so that the characteristic function of the marginal distribution of the returns is (4.25), and the empirical characteristic function is (4.26). Assume that the process η has a monotonic correlation function $\rho_\eta(t) = \frac{L(t)}{t^\alpha}$, where L is a function slowly varying at infinity and bounded on every bounded interval. Then we have the following:*

1. *$E\hat{\varphi_N}(u) = \varphi(u;\theta)$, where $\theta = (\nu, \delta)$ and $\hat{\varphi_N}(u) \to \varphi(u;\theta)$ as $N \to \infty$ almost surely and in the mean square for each $u \in \mathbb{R}$. In addition, for a fixed $0 \leq T < \infty$,*

$$P\left[\limsup_{N\to\infty |u|\leq T} |\hat{\varphi_N}(u) - \varphi(u)| = 0\right] = 1.$$

2. *If $\alpha > 1/2$, the empirical characteristic function, appropriately normalized, is asymptotically normal. Namely, for any $u_1, \ldots, u_m$, $m \geq 1$, let $\Sigma^{(N)} = (\sigma_{ij}^{(N)})$ be the covariance matrix with $\sigma_{ij}^{(N)} = Cov(\hat{\varphi_N}(u_i), \hat{\varphi_N}(u_j))$, $1 \leq i, j \leq m$, and $(\hat{\varphi_N}(u_1), \ldots \hat{\varphi_N}(u_m))'$ be a vector-column. Then,*

$$\left(\Sigma^{(N)}\right)^{-\frac{1}{2}} [(\hat{\varphi_N}(u_1), \ldots \hat{\varphi_N}(u_m))' - (\varphi(u_1;\theta), \ldots, \varphi(u_m;\theta))'] \xrightarrow{d} N(0, I),$$

where, $\Sigma^{(N)} = \left(\Sigma^{(N)}\right)^{\frac{1}{2}} \left(\Sigma^{(N)}\right)^{\frac{1}{2}}$.

3. *If $0 < \alpha < \frac{1}{2}$, the stochastic process,*

$$\frac{N^\alpha}{\sigma L(N)}(\hat{\varphi}_{[Nt]}(u) - \varphi(u;\theta)),$$

converges in distribution as $N \to \infty$ to,

$$R(t) = \sum_{i=1}^{\nu} R_i(t),$$

where R_i, $i = 1, \ldots, \nu$ are independent copies of Rosenblatt process (see Appendix D), and σ is a constant so that,

$$\sigma^2 = \frac{1}{(1-\alpha)(1-2\alpha)} \sum_{k_1+\cdots+k_\nu=2} \frac{C^2_{k_1,\ldots,k_\nu,0}}{k_1!\cdots k_\nu!}.$$

Proof The first part of Theorem 9 follows from ergodic theorem for stationary processes. The proof of parts 2 and 3 of the theorem is based on the results of Arcones (1994) [9]. Let $\{\xi_j\}_{j=1}^{\infty}$ be a $\mathbb{R}^d$-valued stationary mean-zero Gaussian sequence, i.e. $\xi_j = (\xi_j^{(1)}, \ldots, \xi_j^{(d)})$. Given a measurable function $G : \mathbb{R}^d \to \mathbb{R}^1$, Arcones (1994) [9] considers conditions on the convergence to the normal vector,

$$\frac{1}{\sqrt{n}} \sum_{j=1}^{n} (G(\xi_j) - EG(\xi_j)) \xrightarrow{d} \mathrm{N}(0, \Sigma), \text{ as } n \to \infty, \tag{4.27}$$

where $\Sigma = (\sigma_{i,j})_{1 \le i,j \le d}$.

Let us assume that $G(\cdot) \in L_2(\mathbb{R}^d, \prod_{j=1}^{d} \phi(x_j)\, dx_j)$, where,

$$\phi(x_j) = \frac{e^{-x_j^2/2}}{\sqrt{2\pi}}.$$

The orthonormal basis in $L_2(\mathbb{R}^d, \prod_{j=1}^{d} \phi(x_j)\, dx_j)$ consists of multidimensional Hermite polynomials,

$$e_{k_1,\ldots,k_d}(x_1, \ldots, x_d) = \prod_{j=1}^{d} H_{k_j}(x_j), \tag{4.28}$$

where $k_1, \ldots k_d$ are non-negative integers, and H_{k_j} are one-dimensional Hermite polynomials. A function $G(\cdot) \in L_2(\mathbb{R}^d, \prod_{j=1}^{d} \phi(x_j)\, dx_j)$ can be expanded,

$$G(x_1, \ldots, x_d) = \sum_{m=0}^{\infty} \sum_{k_1+\ldots+k_d=m} \frac{C_{k_1,\ldots,k_d}}{k_1! \cdots k_d!} e_{k_1,\ldots,k_d}(x_1, \ldots, x_d),$$

where,

$$C_{k_1,\ldots,k_d} = \int_{\mathbb{R}^d} G(x_1, \ldots, x_d) e_{k_1,\ldots,k_d}(x_1, \ldots, x_d) \prod_{j=1}^{d} \phi(x_j)\, dx_1 \cdots dx_d.$$

The function G has a Hermite rank equal to m, if there is an integer $m \ge 1$, such that at least one coefficient $C_{k_1,\ldots,k_d} : k_1 + \ldots + k_d = m$, is not zero, and $C_{k_1,\ldots,k_d} = 0, 1 \le k_1 + \ldots + k_d \le m - 1$.

Consider a Gaussian vector $\xi(t) = (\xi_1(t), \ldots \xi_\nu(t), \xi_{\nu+1}(t))$ in $\mathbb{R}^{\nu+1}$, where the first ν components are independent copies of the Gaussian process η with monotonic correlation function ρ_η, and the last component $\xi_{\nu+1}$ is a sequence of independent $N(0,1)$ random variables that are also independent of the process η. Then with our construction of the Student process,

$$X_t = G(\xi(t)) = G(\xi_1(t), \ldots \xi_\nu(t), \xi_{\nu+1}(t)),$$

where the non-random function G is given by the following expression:

$$G(x_1, x_2, \ldots, x_\nu, x_{\nu+1}) = \mu + \theta \frac{1}{\frac{2}{\nu-2}[\frac{1}{2}(x_1^2 + \ldots + x_\nu^2)]} + \sigma \sqrt{\frac{1}{\frac{2}{\nu-2}[\frac{1}{2}(x_1^2 + \ldots + x_\nu^2)]}} x_{\nu+1}.$$

Then for fixed u we consider the function,

$$\tilde{G}_u(x_1, \ldots, x_\nu, x_{\nu+1}) = \cos(uG(x_1, x_2, \ldots, x_\nu, x_{\nu+1})) \in L_2(\mathbb{R}^{\nu+1}, \prod_{j=1}^{\nu+1} \phi(x_j)).$$

We have,

$$\begin{aligned} \hat{\varphi}_N(u) - \varphi(u;\theta) &= \frac{1}{N}\sum_{j=1}^{N}(\cos(uX_j) - E\cos(uX_j)) \\ &= \frac{1}{N}\sum_{j=1}^{N}\Big(\tilde{G}_u(\xi_1(j), \ldots \xi_\nu(j), \xi_{\nu+1}(j)) - E\tilde{G}_u(\xi_1(j), \ldots \xi_\nu(j), \xi_{\nu+1}(j))\Big). \end{aligned}$$

For fixed u the function $\tilde{G}_u$ can be expanded:

$$\tilde{G}_u(x_1, \ldots, x_\nu, x_{\nu+1}) = \sum_{m=0}^{\infty} \sum_{k_1+\ldots k_{\nu+1}=m} \frac{C_{k_1,\ldots,k_{\nu+1}}}{k_1! \cdots k_{\nu+1}!} e_{k_1,\ldots,k_{\nu+1}}(x_1, \ldots, x_\nu, x_{\nu+1}),$$

and,

$$\tilde{G}_u(\xi_1(t), \ldots \xi_\nu(t), \xi_{\nu+1}(t)) - E\tilde{G}_u(\xi_1(t), \ldots, \xi_\nu(t), \xi_{\nu+1}(t)) =$$
$$\sum_{m=1}^{\infty} \sum_{k_1+\ldots,k_{\nu+1}=m} \frac{C_{k_1,\ldots,k_{\nu+1}}}{k_1! \cdots k_{\nu+1}!} e_{k_1,\ldots,k_\nu}(\xi_1(t), \ldots, \xi_\nu(t)) H_{k_{\nu+1}}(\xi_{\nu+1}(t)).$$

We separate the term where $k_{\nu+1} = 0$ and write,

$$\frac{1}{N}\sum_{t=1}^{N}\Big(\tilde{G}_u(\xi_1(t), \ldots, \xi_\nu(t), \xi_{\nu+1}(t)) - E\tilde{G}_u(\xi_1(t), \ldots \xi_\nu(t), \xi_{\nu+1}(t))\Big) =$$
$$\frac{1}{N}\sum_{t=1}^{N}\sum_{m=1}^{\infty} \sum_{k_1+\cdots+k_\nu=m} \frac{C_{k_1,\ldots,k_\nu,0}}{k_1! \cdots k_\nu!} e_{k_1,\ldots,k_\nu}(\xi_1(t), \ldots, \xi_\nu(t)) +$$
$$\frac{1}{N}\sum_{t=1}^{N}\sum_{k_{\nu+1}=1}^{\infty} \sum_{k_1+\cdots+k_\nu \geq 0} \frac{C_{k_1,\ldots,k_{\nu+1}}}{k_1! \cdots k_{\nu+1}!} e_{k_1,\ldots,k_\nu}(\xi_1(t), \ldots, \xi_\nu(t)) H_{k_{\nu+1}}(\xi_{\nu+1}(t)) =$$
$$A_N + B_N.$$

Due to orthogonality of Hermite polynomials,

$$Var\bigg(\frac{1}{N}\sum_{t=1}^{N}\Big(\tilde{G}_u(\xi_1(t), \ldots, \xi_\nu(t), \xi_{\nu+1}(t)) - E\tilde{G}_u(\xi_1(t), \ldots \xi_\nu(t), \xi_{\nu+1}(t))\Big)\bigg) = Var(A_N) + Var(B_N),$$

and we evaluate these variances below. The variance of the second term is,

$$Var(B_N) = \frac{1}{N^2}\sum_{t,s=1}^{N}\sum_{k_{\nu+1}=1}^{\infty} \sum_{k_1+\cdots+k_\nu \geq 0} \frac{C^2_{k_1,\ldots,k_{\nu+1}}}{(k_1!)^2 \cdots (k_{\nu+1}!)^2} \times$$

$$Ee_{k_1,\dots,k_\nu}(\xi_1(t),\dots,\xi_\nu(t))e_{k_1,\dots,k_\nu}(\xi_1(s),\dots,\xi_\nu(s))\times$$
$$EH_{k_{\nu+1}}(\xi_{\nu+1}(t))H_{k_{\nu+1}}(\xi_{\nu+1}(s)).$$

Since $\xi_{\nu+1}$ are i.i.d. N(0,1), the last expectation of the product of Hermite polynomials is not zero only when $t = s$, and so,

$$Var(B_N) = \frac{1}{N}\sum_{k_{\nu+1}=1}^{\infty}\sum_{k_1+\dots+k_\nu\ge 0}\frac{C^2_{k_1,\dots,k_{\nu+1}}}{k_1!\cdots k_{\nu+1}!}.$$

Since,

$$c_B = \sum_{k_{\nu+1}=1}^{\infty}\sum_{k_1+\dots+k_\nu\ge 0}\frac{C^2_{k_1,\dots,k_{\nu+1}}}{k_1!\cdots k_{\nu+1}!}\le\sum_{k_1,\dots k_{\nu+1}}\frac{C^2_{k_1,\dots,k_{\nu+1}}}{k_1!\cdots k_{\nu+1}!}<\infty,$$

$$Var(B_N) = \frac{c_B}{N}.$$

In the first term, A_N, the summation with respect to m can begin at $m = 2$, because when considered as a function of the first ν coordinates, $\tilde{G}_u$ has Hermite rank 2 (it is an even function of each of the first ν coordinates). We have,

$$\begin{aligned}
Var(A_N) &= \frac{1}{N^2}\sum_{t,s=1}^{N}\sum_{m=2}^{\infty}\sum_{k_1+\dots+k_\nu=m}\frac{C^2_{k_1,\dots,k_\nu,0}}{(k_1!)^2\cdots(k_\nu!)^2}Ee_{k_1,\dots,k_\nu}(\xi_1(t),\dots,\xi_\nu(t))e_{k_1,\dots,k_\nu}(\xi_1(s),\dots,\xi_\nu(s))\\
&= \frac{1}{N^2}\sum_{t,s=1}^{N}\sum_{m=2}^{\infty}\sum_{k_1+\dots+k_\nu=m}\frac{C^2_{k_1,\dots,k_\nu,0}}{k_1!\cdots k_\nu!}\rho_\eta^m(|t-s|)\\
&= \sum_{k_1+\dots+k_\nu=2}\frac{C^2_{k_1,\dots,k_\nu,0}}{k_1!\cdots k_\nu!}\frac{1}{N^2}\sum_{t,s=1}^{N}\rho^2(|t-s|)+\frac{1}{N^2}\sum_{t,s=1}^{N}\sum_{m=3}^{\infty}\sum_{k_1+\dots+k_\nu=m}\frac{C^2_{k_1,\dots,k_\nu,0}}{k_1!\cdots k_\nu!}\rho^m(|t-s|).
\end{aligned}$$

When $\alpha < \frac{1}{2}$, from Lemma 3.1 in Taqqu (1975) [119],

$$\frac{1}{N^2}\sum_{t,s=1}^{N}\rho_\eta^2(|t-s|)\sim\frac{1}{(1-\alpha)(1-2\alpha)}\frac{L^2(N)}{N^{2\alpha}},$$

and,

$$\frac{1}{N^2}\sum_{t,s=1}^{N}\sum_{m=3}^{\infty}\sum_{k_1+\dots+k_\nu=m}\frac{C^2_{k_1,\dots,k_\nu,0}}{k_1!\cdots k_\nu!}\rho_\eta^m(|t-s|) = o\left(\frac{L^2(N)}{N^{2\alpha}}\right).$$

Therefore with,

$$\sigma^2 = \frac{1}{(1-2\alpha)(1-\alpha)}\sum_{k_1+\dots+k_\nu=2}\frac{C^2_{k_1,\dots,k_\nu,0}}{k_1!\cdots k_\nu!},$$

we have,

$$Var(A_N) = \sigma^2\frac{L^2(N)}{N^{2\alpha}}(1+o(1)),\ \text{as } N\to\infty.$$

Since the variance of B_N is of order $\frac{1}{N}$, $\frac{N^\alpha}{L(N)} B_N \to 0$ in the mean square if $0 < \alpha < \frac{1}{2}$. Therefore the limit distribution of $\frac{N^\alpha}{\sigma L(N)}(\hat{\varphi}(u) - \varphi(u;\theta))$ is the same as the limit distribution of,

$$\frac{N^\alpha}{\sigma L(N)} \frac{1}{N} \sum_{t=1}^{N} \Big(F_u(\xi_1(t), \ldots \xi_\nu(t)) - EF_u(\xi_1(t), \ldots \xi_\nu(t)) \Big),$$

where $F_u(\cdot) = E\tilde{G}_u(\cdot, \xi_{\nu+1}(t))$ so that,

$$A_N = \frac{1}{N} \sum_{t=1}^{N} \Big(F_u(\xi_1(t), \ldots \xi_\nu(t)) - EF_u(\xi_1(t), \ldots \xi_\nu(t)) \Big).$$

When $0 < \alpha < \frac{1}{2}$, the limit distribution of A_N follows from a multivariate generalization of Theorem 6.1 in Taqqu (1975) [119], and from Theorem 6 in Arcones (1994) [9]:

$$\frac{N^\alpha}{\sigma L(N)} A_{[Nt]} \xrightarrow{d} \sum_{i=1}^{\nu} R_i(t), \text{ as } N \to \infty.$$

When $\alpha > \frac{1}{2}$, the asymptotic normality of $\sqrt{N} A_N$ follows from the result of Arcones (1994) [9]. The Hermite rank of function F_u is 2 and,

$$\sum_{k=-\infty}^{\infty} |\rho_\eta(k)|^2 < \infty.$$

The asymptotic normality of $\sqrt{N} B_N$ follows from the fact that $\xi_{\nu+1}(t)$ for $t = 1, 2, \ldots$ are i.i.d. N(0,1). Therefore when $\alpha > \frac{1}{2}$, $\left(\sigma_u^{(N)}\right)^{-1/2} (\hat{\varphi_N}(u) - \varphi(u;\theta)) \xrightarrow{d} \mathrm{N}(0,1)$, $N \to \infty$. $\square$

To carry out the test of hypotheses about the parameters of the t-distribution when $\alpha > \frac{1}{2}$, we need to compute $\sigma_{ij}^{(N)}$:

$$\begin{aligned}
\sigma_{ij}^{(N)} &= \frac{1}{N^2} Cov\Big(\sum_{r=1}^{N} e^{iu_i X_r}, \sum_{s=1}^{N} e^{iu_j X_s}\Big) \\
&= \frac{1}{N^2} \sum_{r=1}^{N} \sum_{s=1}^{N} Cov(e^{iu_i X_r}, e^{iu_j X_s}) \\
&= \frac{1}{N} Cov(e^{iu_i X_1}, e^{iu_j X_1}) + \frac{1}{N^2} \sum_{r=1}^{N} \sum_{s=1, s\neq r}^{N} Cov(e^{iu_i X_r}, e^{iu_j X_s}) \\
&= \frac{1}{N} \{\varphi(u_i + u_j;\theta) - \varphi(u_i;\theta)\varphi(u_j;\theta)\} + \frac{1}{N^2} \sum_{r=1}^{N} \sum_{s=1, s\neq r}^{N} Cov(e^{iu_i X_r}, e^{iu_j X_s}) \quad (4.29)
\end{aligned}$$

Remark Without the second term in equation (4.29) that corresponds to summation where $r \neq s$, we find that,

$$\frac{1}{N} \{\varphi(u_i + u_j;\theta) - \varphi(u_i;\theta)\varphi(u_j;\theta)\},$$

may not be positive definite.

Consider the following stationary process,

$$Y_j(u) = \cos(uX_j),\ j = 1, \dots N,\ u \in \mathbb{R}, \tag{4.30}$$

so that,

$$\begin{aligned}
\frac{1}{N^2}\sum_{r=1}^{N}\sum_{s=1,s\neq r}^{N} Cov(e^{iu_iX_r}, e^{iu_jX_s}) &= \frac{1}{N^2}\sum_{r=1}^{N}\sum_{s=1,s\neq r}^{N} Cov(\cos(u_iX_r), \cos(u_jX_s)) \\
&= \frac{1}{N^2}\sum_{r=1}^{N}\sum_{s=1}^{N} Cov(Y_r(u_i), Y_s(u_j)) \\
&= \frac{1}{N^2}\sum_{r=1}^{N}\sum_{s=1,s\neq r}^{N} R_{i,j}(s-r) \\
&= \frac{2}{N}\sum_{k=1}^{N-1}(1-\frac{k}{N})R_{i,j}(k),
\end{aligned} \tag{4.31}$$

where $R_{i,j}(k) = Cov(Y_t(u_i), Y_{t+k}(u_j)) = Cov(\cos(u_iX_t), \cos(u_jX_{t+k}))$. We consistently estimate this covariance with,

$$\hat{R}_{ij}^{(N)}(k) = \frac{1}{N}\sum_{t=1}^{N-k}(Y_t(u_i) - \bar{Y}_N(u_i))(Y_{t+k}(u_j) - \bar{Y}_N(u_j)).$$

Hence,

$$\begin{aligned}
\sigma_{ij}^{(N)} &= \frac{1}{N}\{\varphi(u_i + u_j;\theta) - \varphi(u_i;\theta)\varphi(u_j;\theta)\} + \frac{2}{N}\sum_{k=1}^{N-1}(1-\frac{k}{N})\hat{R}_{ij}^{(N)}(k) \\
&+ \frac{2}{N}\sum_{k=1}^{N-1}(1-\frac{k}{N})(R_{i,j}(k) - \hat{R}_{ij}^{(N)}(k)).
\end{aligned}$$

The last term converges to zero almost surely due to ergodic theorem, and by Slutsky's lemma we have that the results on asymtotic distribution for the normalized empirical characteristic function stated in Theorem 7 hold when $\sigma_{ij}^{(N)}$ is replaced with $\tilde{\sigma}_{ij}^{(N)}$ where,

$$\tilde{\sigma}_{ij}^{(N)} = \frac{1}{N}\{\varphi(u_i + u_j;\theta) - \varphi(u_i;\theta)\varphi(u_j;\theta)\} + \frac{2}{N}\sum_{k=1}^{N-1}(1-\frac{k}{N})\hat{R}_{ij}^{(N)}(k).$$

Under the assumption of short-range dependence ($\alpha > \frac{1}{2}$), we use the last expression to carry out the test of the null hypothesis, $H_0:\ \nu = \hat{\nu}$ and $\delta = \hat{\delta}$. For a fixed u, $\hat{\theta} = (\hat{\nu}, \hat{\delta})$ and large N, we can compute that,

$$\tilde{\sigma}_u^{(N)}(\hat{\theta}) = \frac{1}{N}\{\varphi(2u;\hat{\theta}) - \varphi^2(u;\hat{\theta})\} + \frac{2}{N}\sum_{k=1}^{N-1}(1-\frac{k}{N})\hat{R}_u^{(N)}(k),$$

where,

$$\hat{R}_u^{(N)}(k) = \frac{1}{N}\sum_{t=1}^{N-k}(Y_t(u) - \bar{Y}_N(u))(Y_{t+k}(u) - \bar{Y}_N(u)).$$

We have,

$$P\{|(\tilde{\sigma}_u^{(N)}(\hat{\theta}))^{-\frac{1}{2}}(\hat{\varphi_N}(u) - \varphi(u;\hat{\theta}))| \leq u_\epsilon\} \approx P\{|\mathrm{N}(0,1)| < u_\epsilon\} = 1 - \epsilon.$$

Set $\epsilon = 0.05$ so $u_\epsilon = 1.96$, and test whether the test statistic,

$$\tilde{\sigma}_u^{(N)}(\hat{\theta}))^{-\frac{1}{2}}|\hat{\varphi_N}(u) - \varphi(u;\hat{\delta},\hat{\nu})|,$$

is below the critical value of the normal distribution, namely whether,

$$\tilde{\sigma}_u^{(N)}(\hat{\theta}))^{-\frac{1}{2}}|\hat{\varphi_N}(u) - \varphi(u;\hat{\delta},\hat{\nu})| < 1.96. \tag{4.32}$$

	u=1000	u=600	u=200	u=-200	u=-600	u=-1000
GBP	0.475	0.299	0.184	0.184	0.299	0.475

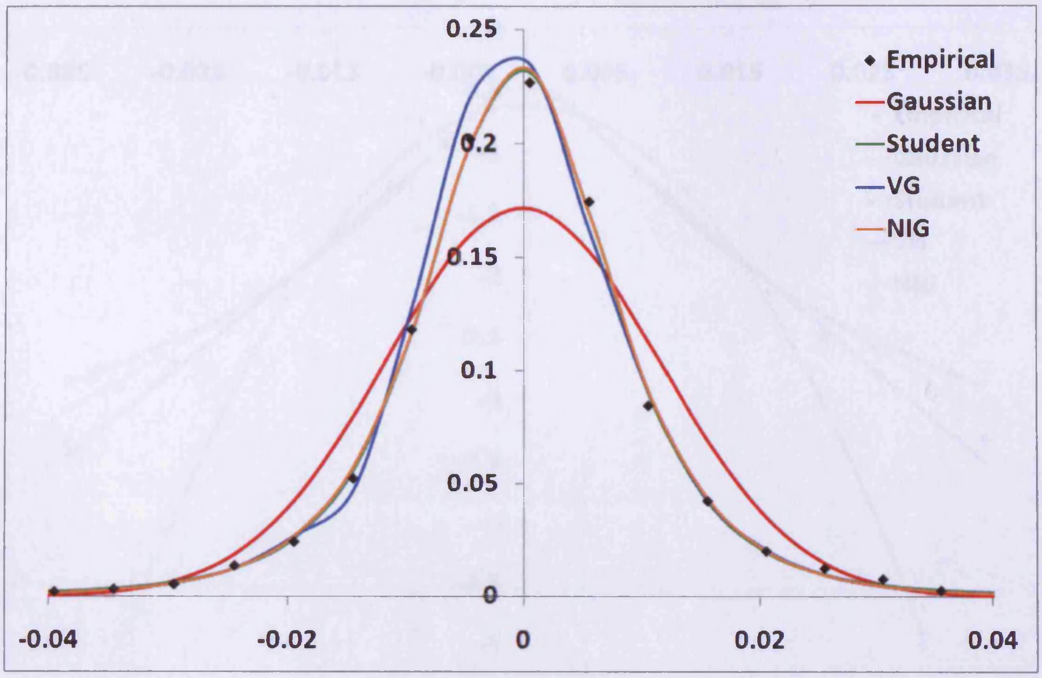
Table 4.1: Values of the test statistic for the tests of Student distribution parameters for GBP

We use the numerical values of the parameter estimates (see *Table 4.2*) obtained from method of moments as $\hat{\nu}$ and $\hat{\delta}$, and present the values of the test statistics for the hypotheses tests in *Table 4.1*. We see that the inequality (4.32) is satisfied for,

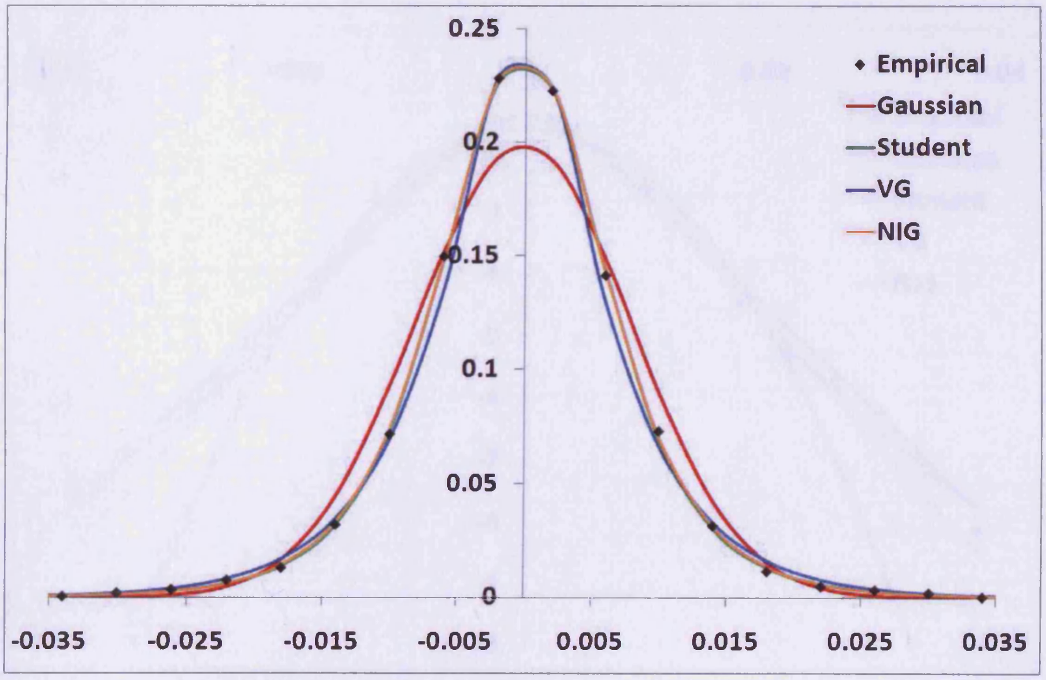
$$u = \{1000, 600, 200, -200, -600, -1000\},$$

and therefore we retain H_0 at a 5% significance level. *Figure 4.3* is the plot of both theoretical characteristic function (4.25) and empirical characteristic function (4.26).

Remark We carried out the hypothesis test for each of the values of u listed above separately. Using multivariate normal distribution, a single test could be carried out, and for our data it results in retaining the null hypothesis.

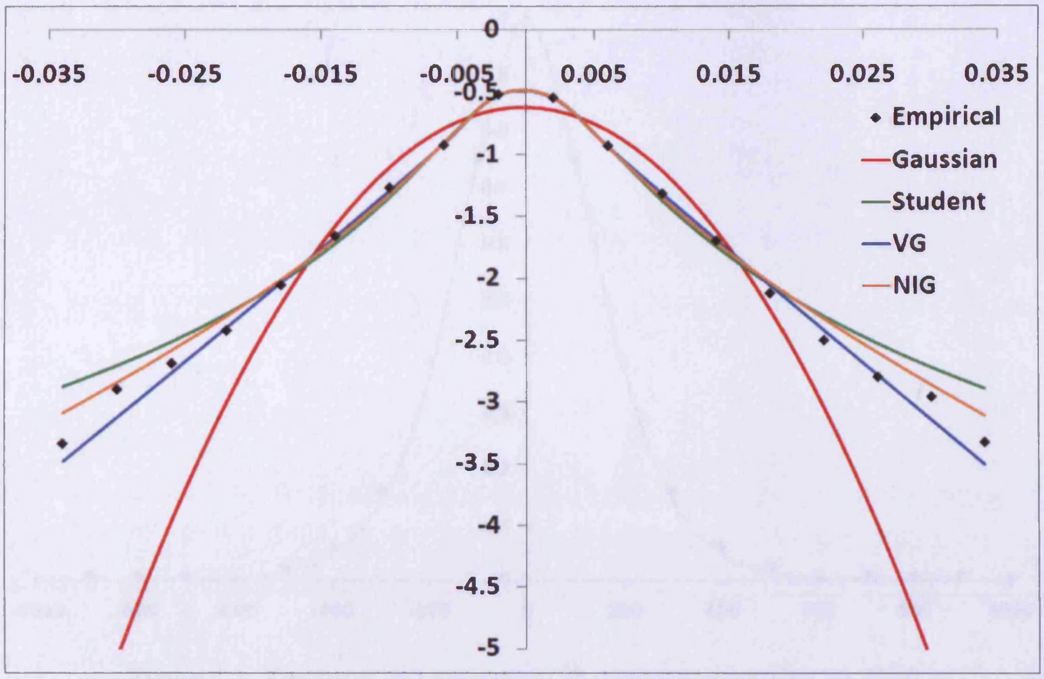


(a) FTSE100

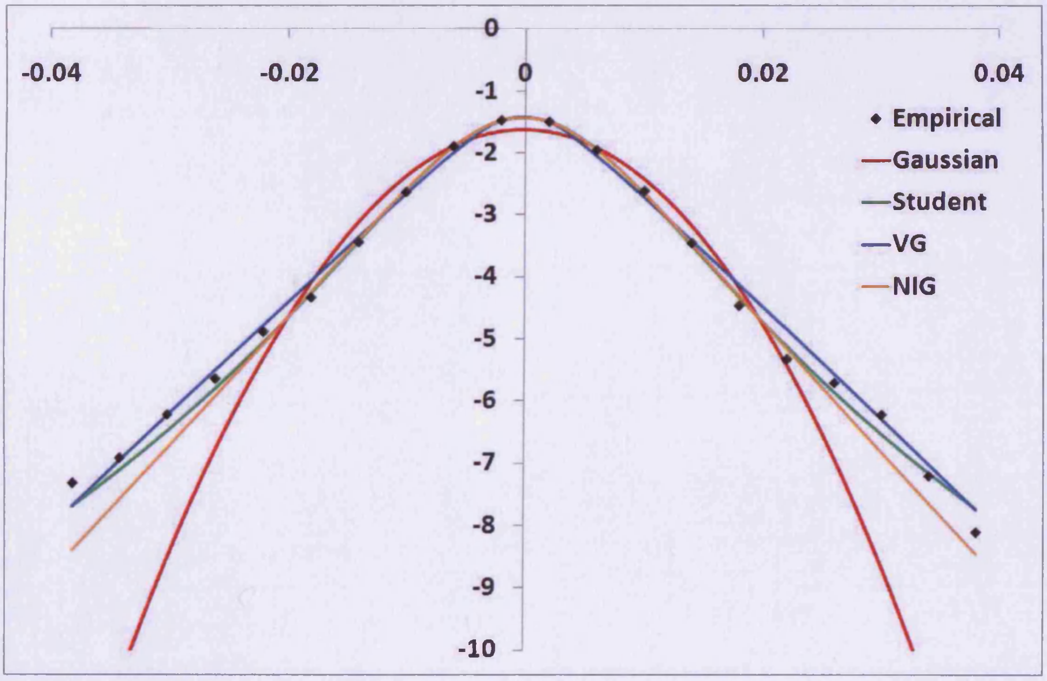


(b) GBP

Figure 4.1: Empirical density of X, Gaussian, Student, VG and NIG densities



(a) FTSE100



(b) GBP

Figure 4.2: Logarithm of empirical density of X, Gaussian, Student, VG and NIG densities

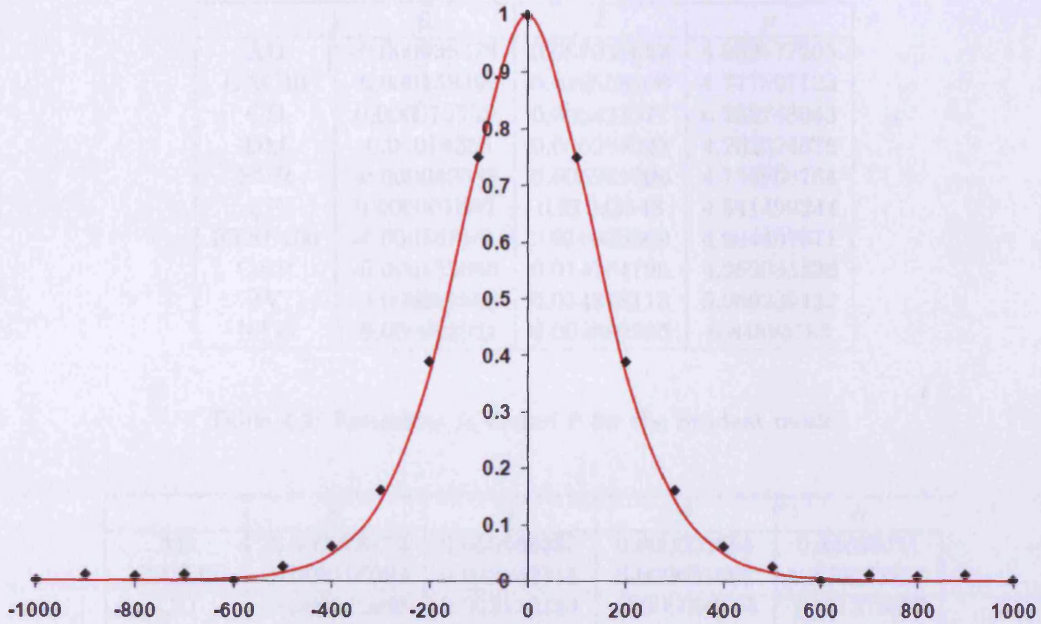


Figure 4.3: Empirical and theoretical characteristic function for GBP

	$\hat{\mu}$	$\hat{\delta}$	$\hat{\nu}$
AD	-0.000035478	0.007630623	4.060577205
CAC40	-0.000158399	0.019858306	4.777597123
CD	0.000079553	0.006421377	6.262748043
DM	-0.00014255	0.006288023	4.262224676
EUR	-0.000049583	0.005931796	4.736966764
FF	0.000001850	0.01043548	4.541499244
FTSE100	-0.000167101	0.016036669	4.994407871
GBP	-0.000138866	0.014164796	4.969083328
JY	-0.000281808	0.014878115	5.069299137
NTD	-0.000042921	0.009880295	6.84894185

Table 4.2: Estimates $\hat{\mu}$, $\hat{\delta}$, and $\hat{\nu}$ for the Student model

	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\theta}$	$\hat{\alpha}$
AD	-0.000038133	0.005988855	0.000002655	0.84605741
CAC40	-0.000196061	0.012933311	0.000037661	0.872473786
CD	-0.000076908	0.003110164	-0.000002645	1.131374022
DM	-0.000141552	0.006358803	-0.000000998	0.831701019
EUR	-0.000013025	0.005117664	-0.000036558	0.947375313
FF	0.000200842	0.006545871	-0.000198992	0.270749622
FTSE100	-0.00019587	0.042539066	0.000028769	1.499993761
GBP	-0.000100097	0.007973439	-0.000038769	1.094880615
JY	-0.000206056	0.008492358	-0.000075752	1.148918175
NTD	-0.000075047	0.0044869	0.000032126	1.424470925

Table 4.3: Estimates $\hat{\mu}$, $\hat{\sigma}$, $\hat{\theta}$ and $\hat{\alpha}$ for the VG model

	$\hat{\mu}$	$\hat{\gamma}$	$\hat{\kappa}$	$\hat{\delta}$
AD	0.000129091	133.3145319	0.000015789	0.005603404
CAC40	-0.000684482	81.01104609	0.000024876	0.015046163
CD	0.000056844	116.9964824	0.000002544	0.00330816
DM	-0.000144536	95.31123022	0.000029843	0.004895101
EUR	0.000207405	128.7901444	0.0000995796	0.005005469
FF	-0.000189629	80.23584843	0.000008776	0.003421896
FTSE100	-0.000439758	112.3870192	0.000264433	0.012295845
GBP	-0.000137952	169.8717431	0.000055798	0.010523985
JY	0.001196689	135.4103476	0.000054329	0.009415426
NTD	-0.000075047	0.0044869	0.000008264	0.004470925

Table 4.4: Estimates $\hat{\mu}$, $\hat{\gamma}$, $\hat{\kappa}$ and $\hat{\delta}$ for the NIG model

Chapter 5

Option Pricing

"[The GBM model] was of the most elegant and precise models that any of us has ever seen ... What Mr. Merton and Mr. Scholes did, back in 1973, was to put a price on risk." (Gregg Jarrell).

We now venture into the heartland of modern finance, where the spotlight here is on the pricing of derivatives. Our objective is to derive a valid alternative to the European Call option pricing formula of Black and Scholes (1973) [26] and Merton (1973) [100] based on the FATGBM model. This will be done by following the proposed ideas of Heyde and Gay (2002) [58] to describe an approach for an option pricing formula using probability density functions where no specific assumptions are made on the independence of returns or which distribution they follow. An exact approach based on the densities of the finite-dimensional distributions, and an approximate approach based on the asymptotic self-similarity of activity time, are both considered. We also sketch an alternative approach of Carr and Madan (1999) [32] which uses the characteristic function.

Before we start to implement the approach of Heyde and Gay (2002) [58] for option pricing, we first need to consider the risk-neutral measures. The overall idea is to move from a real-world model of a stock price process $\{P_t\}$ and the associated historical returns, to a risk-free model of a discount stock price process $\{e^{-rt}P_t\}$ where r is the interest rate. Our way is to impose parameter restrictions to ensure that $\{e^{-rt}P_t\}$ is a martingale. This will allow us to reduce the pricing of options on the risky asset to calculating the expected values of the discounted payoffs. Note that the existence of this martingale is related to the absence of arbitrage.

If B_t is the price of a non-risky asset and r is the interest rate: $B_t = B_0 e^{rt}$, then we need to show that $\{e^{-rt}P_t\}$ is a martingale where P_t is the price of a risky asset and e^{-rt} is the discounting factor. Consider the σ-algebra $\mathcal{F}_s$ of the information available up until time s:

$$\mathcal{F}_s = \sigma\{\{B(u), u \leq T_s\}, \{T_u, u \leq s\}\},$$

and the σ-algebra $\mathcal{F}_*$ defined as follows:

$$\mathcal{F}_* = \sigma\{\{B(u), u \leq T_s\}, \{T_u, u \leq s\}, T_t\}, \ s < t.$$

We have (Finlay and Seneta (2006) [50]), that a.s.,

$$\begin{aligned} E(e^{-rt}P_t|\mathcal{F}_s) &= P_0 E(e^{(\mu-r)t+\theta(T_t-T_s+T_s)+\sigma(B(T_t)-B(T_s)+B(T_s))}|\mathcal{F}_s) \\ &= P_s e^{(\mu-r)t-\mu s} E E((e^{\theta(T_t-T_s)+\sigma(B(T_t)-B(T_s))}|\mathcal{F}_*)|\mathcal{F}_s) \\ &= e^{-rs}P_s e^{(\mu-r)(t-s)} E(e^{(\theta+\frac{1}{2}\sigma^2)(T_t-T_s)}|\mathcal{F}_s) \quad \text{since } \mathcal{F}_s \subset \mathcal{F}_*, \end{aligned}$$

where we use a moment generating function of normal variable.
If we introduce a sigma-algebra $\mathcal{F}^*_{s,t} = \sigma\{\{B(u), u \leq T_s\}, \{T_u, u \leq s\}, T_t\}, \ s \leq t$, then since $\mathcal{F}_s \subset \mathcal{F}^*_{s,t}$,

$$E(e^{-rt}P_t|\mathcal{F}_s) = e^{-rs}P_s e^{(\mu-r)(t-s)} E(e^{(\theta+\frac{1}{2}\sigma^2)(T_t-T_s)}|\mathcal{F}_s), \text{ a.s.},$$

where again we used a moment generating function of normal variable.

5.1 Mean-correcting martingales

In this subsection we review the mean-correcting martingale approach of Madan, Carr, and Chang (1998) [88] and Finlay and Seneta (2008) [53], where the activity time was assumed to have independent and identically-distributed (i.i.d.) increments.
If we use the sigma-algebra introduced above then since $\mathcal{F}_s \subset \mathcal{F}^*_{s,t}$, we have,

$$E(e^{-rt}P_t|\mathcal{F}_s) = e^{-rs}P_s e^{(\mu-r)(t-s)} E(e^{(\theta+\frac{1}{2}\sigma^2)(T_t-T_s)}|\mathcal{F}_s), \text{ a.s.},$$

where again we use a moment generating function of normal variable. The mean-correcting martingale approach to pricing is restricting the mean parameter μ so that,

$$E(e^{(\theta+\frac{1}{2}\sigma^2)(T_t-T_s)}|\mathcal{F}_s) = e^{(r-\mu)(t-s)} \text{ a.s.} \tag{5.1}$$

We will now show that this approach does not always work when the i.i.d. assumption is dropped.

With our construction of activity time using superpositions of OU-type processes, the moment generating function of $T_t - T_s$ can be obtained using the moment generating functions of the corresponding Lévy processes Z. Recall that in the case of finite superposition,

$$T_t^m - T_s^m = \sum_{k=1}^{m} \sum_{i=s}^{t} \tau^{(k)}(i).$$

Using Lemma 2.1 in [102] (proved in [44]), we obtain that in the case of finite superposition, the moment generating function of $T_t - T_s$ is as follows:

$$Ee^{b(T_t - T_s)} = \prod_{k=1}^{m} Ee^{b\sum_{i=s}^{t} \tau^{(k)}(i)} = \prod_{k=1}^{m} E\left[\exp\left(\int_0^{t-s} f_k(u) dZ^{(k)}(\lambda^{(k)} u)\right)\right],$$

where $f_k(u) = be^{-\lambda^{(k)}(t-s-u)}$. Applying Lemma 2.1 in [102], we have,

$$E\left[\exp\left(\int_0^{t-s} f_k(u) dZ^{(k)}(\lambda^{(k)} u)\right)\right] = \exp\left(\lambda^{(k)} \int_0^{t-s} C_k(f(u)) du\right),$$

where C_k is the moment generating function of $Z^{(k)}$, the Lévy process from equation (5.6). When increments of the activity time have Gamma distribution (VG model), then $C_k(u) = \dfrac{\alpha_k u}{\beta - u}$, and the moment generating function of $T_t - T_s$ is,

$$Ee^{b(T_t - T_s)} = \prod_{k=1}^{m} \exp\left(\alpha_k \log \frac{\beta - be^{-\lambda^{(k)}(t-s)}}{\beta - b}\right).$$

When increments of the activity time have IG distribution (NIG model), then $C_k(u) = \dfrac{\delta_k u}{\sqrt{\gamma^2 - 2u}}$, and the moment generating function of $T_t - T_s$ is,

$$Ee^{b(T_t - T_s)} = \prod_{k=1}^{m} \exp\left(\frac{\delta_k}{2}\left[\sqrt{\gamma^2 + 2be^{-\lambda^{(k)}(t-s)}} - \sqrt{\gamma^2 - 2b}\right]\right).$$

Thus, the parameter restriction to satisfy (5.1) for all $t > s$ is not possible, and the "mean-correcting martingale" approach, where a risk-neutral model is obtained from a real-world model by restricting the mean parameter, does not work when dependence in the activity time is introduced through superpositions of OU-type processes.

5.2 Skew-correcting martingales

An alternative approach to obtaining a martingale was proposed by Heyde and Leonenko (2005) [66] and used in [50]. With this approach, parameter restrictions $\mu = r$ and $\theta = -\frac{1}{2}\sigma^2$ are imposed in the identity,

$$E(e^{-rt} P_t | \mathcal{F}_s) = e^{-rs} P_s e^{(\mu - r)(t-s)} E(e^{(\theta + \frac{1}{2}\sigma^2)(T_t - T_s)} | \mathcal{F}_s),$$

so that a.s. $E(e^{-rt}P_t|\mathcal{F}_s) = e^{-rs}P_s$ as desired. This approach is simple and quite general, as it does not in fact depend on the distribution of T_t (distribution of T_t is of course needed when one comes to actually compute the price of the option.) It is somewhat restrictive however, in that two parameters μ and θ are constrained. As parameter θ determines skewness, we call the approach "skew-correcting martingale".

5.3 Pricing formula using a probability density function

We now look to obtain a pricing formula for a European Call option, denoted by $C(Y, K)$, which gives the holder the right but not an obligation to buy stock for a fixed (strike) price K at expiry (maturity) time Y. Our approach is based on a Black-Scholes type method using probability density functions. As we have seen, the original Black-Scholes formula is valid for the model that suggests that the log returns behave according to a Normal distribution, so instead will consider a pricing formula using the FATGBM model.

Let $A^+ = \max(A, 0)$, $Z \sim N(0,1)$, and for $a, b, c > 0$ the inequality $ae^{-\frac{1}{2}c^2+cZ} > b$ holds if and only if $Z > \frac{1}{2}c - \frac{1}{c}\log\frac{a}{b}$. Then with $\mu = r$ and $\theta = -\frac{1}{2}\sigma^2$, the price of an European call option is,

$$\begin{aligned}
C(Y,K) &= e^{-rY}E(P_Y - K)^+ \\
&= E((P_0e^{-\frac{1}{2}\sigma^2T_Y+\sigma B(T_Y)} - Ke^{-rY})^+) \\
&= E(E((P_0e^{-\frac{1}{2}\sigma^2T_Y+\sigma\sqrt{T_Y}Z} - Ke^{-rY})\mathbf{1}_{(Z>-d_2)})|T_Y) \\
&= E(P_0E(\mathbf{1}_{(Z>-d_1)}|T_Y) - Ke^{-rY}E(\mathbf{1}_{(Z>-d_2)}|T_Y)) \\
&= E(P_0E(\mathbf{1}_{(Z<d_1)}|T_Y) - Ke^{-rY}E(\mathbf{1}_{(Z<d_2)}|T_Y)) \\
&= E(P_0\Phi(d_1) - Ke^{-rY}\Phi(d_2)),
\end{aligned}$$

where $\Phi(\cdot)$ is a cumulative distribution function of $N(0,1)$ and,

$$d_1 = \frac{\log\frac{P_0}{K} + rY + \frac{1}{2}\sigma^2T_Y}{\sigma\sqrt{T_Y}} \quad d_2 = \frac{\log\frac{P_0}{K} + rY - \frac{1}{2}\sigma^2T_Y}{\sigma\sqrt{T_Y}}$$

are both functions of T_Y.

In the above chain of calculation we have used the fact that for a function F such that

$EF(Z) < \infty$, we have,

$$\begin{aligned} EF(Z+c) &= \int_{-\infty}^{\infty} F(Z+c)\Phi(Z)\,dZ \\ &= \int_{-\infty}^{\infty} F(y)\Phi(y-c)\,dy \\ &= \int_{-\infty}^{\infty} e^{-\frac{1}{2}c^2+cy} F(y)\Phi(y)\,dy \\ &= E(e^{-\frac{1}{2}c^2+cZ} F(Z)), \end{aligned}$$

where in our case $F(Z) = \mathbf{1}_{Z>-d_2}$ and,

$$EF(-Z) = \int_{-\infty}^{\infty} F(-Z)\Phi(Z)\,dZ = EF(Z).$$

Also note that we obtain the same expression as in Black-Scholes formula,

$$\tilde{d}_1 = \frac{\log\frac{P_0}{K} + rY + \frac{1}{2}\sigma^2 t}{\sigma\sqrt{t}},\ \tilde{d}_2 = \frac{\log\frac{P_0}{K} + rY - \frac{1}{2}\sigma^2 t}{\sigma\sqrt{t}}. \tag{5.2}$$

If we assign a distribution to T_Y with pdf $f_{T_Y}(t)$, then the call price becomes,

$$C(Y,K) = \int_0^{\infty} (P_0\Phi(d_1) - Ke^{-rY}\Phi(d_2)) f_{T_Y}(t)\,dt. \tag{5.3}$$

Given P_0, K, Y, r, σ, and f_{T_Y}, this expectation (5.3) can be numerically evaluated. Note that we made no assumptions about the distribution of T_Y, so as long as our model has the subordinator structure and all expectations are finite, then this pricing formula is valid.

5.3.1 Via an exact approach

To compute the prices using formula (5.3), the explicit expression for the density of T_Y is needed. As mentioned in Chapter 3, the transition function of the solution of (3.19) can be used.

From Lemma 17.1 of [114] (see also [123, 124]) one can obtain that the temporally homogeneous transition function $P_t(x, B)$ for the solution of OU-type process satisfies,

$$\int_R e^{izy} P_t(x,dy) = \exp\left\{ize^{-\lambda t}x + \lambda\int_0^t \kappa_{Z(1)}(e^{-\lambda s}z)ds\right\},$$

where $\kappa_{Z(1)}$ is the cumulant function of $Z(1)$.

For a Gamma OU-type process, $\kappa_{Z(1)}(z)$ is given by equation (3.21). It was shown in [123] that temporally homogeneous transition function $P_t(x, B) = P(t, y; x, \lambda, \alpha, \beta)$ from x to

$y(t) \leq y$ after time interval t is 0 if $y < e^{-\lambda t}x$, $P(t, y, x, \lambda, \alpha, \beta) = e^{-\lambda \alpha t}$ if $y = e^{-\lambda t}x$, and if $y > e^{-\lambda t}x$ then,

$$P(t, y; x, \lambda, \alpha, \beta) = e^{-\lambda \alpha t} + \sum_{n=1}^{\infty} \frac{(\lambda \alpha t)^n e^{-\lambda \alpha t}}{n!} \int_0^{y - e^{-\lambda t}x} f_n(u) du, \tag{5.4}$$

where

$$f_1(x) = f(x),\ f_n(x) = \int_0^{\infty} f(y) f_{n-1}(x - y) dy,\ f(w) = \frac{e^{-\beta w} - e^{-\beta w e^{\lambda t}}}{\lambda t w}, w > 0,$$

and $f(w) = 0, w \leq 0$. Thus the transition density function (of $y(t)$ from x to y after a time interval t) of Gamma OU-type process can be expressed as follows:

$$p(t, y; x, \lambda, \alpha, \beta) = e^{-\lambda \alpha t} \left[\delta(y - e^{-\lambda t}x) \mathbf{1}_{y = e^{-\lambda t}x}\right] + \sum_{n=1}^{\infty} \frac{(\lambda \alpha t)^n}{n!} f_n(y - e^{-\lambda t}x) \mathbf{1}_{y > e^{-\lambda t}x},$$

where $\delta(\cdot)$ represents the probability density function concentrated at 0.

For an $IG(\delta, \gamma)$ case, reference [124] provides the representation of $\tilde{Y}_t$, namely,

$$\tilde{Y}_t = \int_0^t e^{-\lambda(t-s)} dZ(\lambda s)$$

as the sum of independent IG random variable and a compound Poisson process. Using this representation and the fact that,

$$\int_R e^{izy} P_t(x, dy) = E e^{iz(e^{-\lambda t}x + \tilde{Y}_t)},$$

the transition probability of the IG OU-type process can be expressed as follows:

$$P(t, y; x, \lambda, \gamma, \delta) = \sum_{n=1}^{\infty} \frac{\exp\{-\delta\gamma t(1 - e^{-1/2\lambda t})\}(\delta\gamma t(1 - e^{-1/2\lambda t}))^n}{n!} \int_0^{y - e^{-\lambda t}x} f_n(u) du, \tag{5.5}$$

for $y > e^{-\lambda t}x$, and $P(t, x; y, \lambda, \gamma, \delta) = 0$ if $y \leq e^{-\lambda t}x$. Function f_1 is the IG density with parameters $(\delta(1 - e^{-1/2\lambda t}), \gamma)$ and,

$$f_n(u) = \int_0^{\infty} f_{n-1}(u - x) f(x) dx,\ n \geq 2,$$

where,

$$f(u) = \frac{e^{-1/2\gamma^2 u} - e^{-1/2\gamma^2 u e^{\lambda t}}}{\sqrt{2\pi u^3}\gamma(e^{1/2\lambda t} - 1)}, \quad u > 0.$$

In the case of finite superposition, $T_Y^m = \sum_{k=1}^{m} \sum_{t=1}^{Y} \tau^{(k)}(i)$, where $\tau^{(k)}$ are independent OU-type processes that solve equations (3.19). The density of T_Y^m can be computed as a convolution of densities of $\sum_{i=1}^{Y} \tau^{(k)}(i)$. Each $\tau^{(k)}$ is a Markov process with transition probability given by (5.4) in the VG case, and (5.5) in the NIG case. Therefore,

$$P\left(\sum_{i=1}^{Y} \tau^{(k)}(i) \leq x\right) = \int_{x_1+x_2+\ldots+x_Y \leq x} f(x_1;k)\, dx_1 P(1,\, dx_2; x_1) P(1,\, dx_3; x_2) \ldots P(1,\, dx_Y; x_{Y-1}),$$

where $f(\cdot;k)$is either $\Gamma(\alpha_k, \beta)$ or $IG(\delta_k, \gamma)$ density for VG or NIG models, respectively.

5.3.2 Via an approximation using asymptotic self-similarity

As suggested in [66], the density f_{T_Y} can be approximated using asymptotic self-similarity, by taking the density of $YE\tau_1 + Y^H(T_1 - E\tau_1)$ with Hurst parameter H.
Under the chi-squared construction, the distribution of $T_1 \stackrel{d}{=} \tau_1$ can be either $R\Gamma(\frac{\nu}{2}, \frac{\nu}{2} - 1)$ or $\Gamma(\alpha, \alpha)$, so that, $E\tau_1 = 1$. Therefore, one can use either,

$$f_{T_Y}(u) = Y^{-H} f_{R\Gamma}\left(\frac{u + Y^H - Y}{Y^H}\right), \tag{5.6}$$

for the Student model or,

$$f_{T_Y}(u) = Y^{-H} f_{\Gamma}\left(\frac{u + Y^H - Y}{Y^H}\right),$$

for the VG model, with appropriate parameters. The expression for the $R\Gamma$ density is given by (2.7), and the gamma density by (2.8).
For the diffusion-type and the OU-type construction, the density f_{T_Y} can be taken as approximately the density of $YE\tau_1 + Y^H(T_1 - E\tau_1)$ with $H = \frac{1}{2}$. In the case of the VG model under the OU-type construction, we take $E\tau_1 = \sum_{k=1}^{m} \frac{\alpha_k}{\beta}$. In the case of the NIG model under either OU-type or diffusion-type construction, we have instead $E\tau_1 = \sum_{k=1}^{m} \frac{\delta_k}{\gamma}$.
In the VG model, the distribution of T_1^m is $\Gamma(\sum_{k=1}^{m} \alpha_k, \beta)$. The corresponding distributions for the NIG model are $IG(\sum_{k=1}^{m} \delta_k, \gamma)$. Therefore an approximation to $f_{T_Y}(u)$, one can use,

$$f_{T_Y}(u) = Y^{-\frac{1}{2}} f_{\Gamma}\left(\frac{u + E\tau_1(Y^{\frac{1}{2}} - Y)}{Y^{\frac{1}{2}}}\right),$$

for the VG model or,

$$f_{T_Y}(u) = Y^{-\frac{1}{2}} f_{IG}\left(\frac{u + E\tau_1(Y^{\frac{1}{2}} - Y)}{Y^{\frac{1}{2}}}\right),$$

for the NIG model, with the appropriate parameters. The expressions for the Gamma density f_{Γ} and IG density f_{IG} are given in sections (2.8) and (2.10).

5.4 The comparison with the Black-Scholes formula

Modern day investors regularly check for inconsistencies between the financial market prices and those calculated using a pricing formula. If prices seem to be in conflict with each other, then there might be an opportunity for arbitrage. We will follow the arbitrage-free methodology we have outlined and first engendered by Black, Scholes and Merton. Therefore, our goal is to calibrate a risky asset model such that any estimated prices match closely with those of the current market. We will do this whilst comparing our pricing formula (5.3) with the classical Black-Scholes pricing formula.

We will analyse the mid-prices of European Call options on the S&P500 Index. The S&P500 Index (ticker SPX) is one of many options offered on the Chicago Board Options Exchange. The prices were taken at the close of market on 18 April 2002, where the index closed at 1124.47 with an interest rate of 1.9%.

Note that a European Put option can be calculated simply from the price of a Call option with the same strike and maturity, by using the Put-Call parity relation:

$$\text{Put price} = \text{Call price} - \text{Stock (risky asset) price} + \text{Strike price}$$

Here though we analyse the Call option prices only. The dataset can be taken from Schoutens (2003) [116], and we will consider the case where maturity is one year (T=1), corresponding to an expiry of 18 April 2003.

To begin, the underlying risky asset (in our case the S&P500 Index) is investigated. We take a large enough set of historical stock prices (N=2000) at the close of market up until 18 April 2002. The corresponding returns turn out to follow symmetric Student closer than VG and NIG, and all necessary parameters were estimated using method of moments. The activity time process T_Y is given by,

$$T_Y = G(\chi^2_\nu(1)) = \frac{\frac{\nu}{2} - 1}{\frac{1}{2}(\eta_1^2(1) + \ldots + \eta_\nu^2(1))}$$

from chapter 3, and the density f_{T_Y} can then be approximated using asymptotic self-similarity (5.6). We also find that $H \approx \frac{1}{2}$ using the methods in Appendix B.

This leaves volatilty σ as the only parameter left to estimate before we can implement our pricing formula (5.3). According to the Black-Scholes formula, this expected volatility of the underlying asset is the key element in pricing an option. The more an asset fluctuates, the more likely it is to rise above the strike price, and so, the higher the price of the option becomes.

Note that we are unable to fix $\theta = -\frac{1}{2}\sigma^2$ because the symmetric Student case demands $\theta = 0$. Instead, we estimated σ using least squares to give the prices in *Figure 5.1.* The market prices are denoted by a circle, with the model prices by a square for the GBM model and a triangle for the FATGBM model. From this comparison, we can see that the FATGBM gives us a superior fit to the market price of a European Call option.

5.5 Pricing formula using a characteristic function

There are two stock price process representations that result in VG distributional returns; the subordinator model representation (2.5), and the difference of two gamma processes representation.

Consider what we shall call the "Difference of Gammas" process:

$$\log P_t = \log P_0 + \mu t + \Gamma^{(1)}_{a,b}(t) - \Gamma^{(2)}_{c,d}(t), \tag{5.7}$$

where $\Gamma^{(1)}_{a,b}(t) \sim \Gamma(at, b)$ and $\Gamma^{(2)}_{c,d}(t) \sim \Gamma(ct, d)$ denote independent-increment Lévy Gamma processes for any given t. For each t the returns or log price increments from P_t have characteristic function,

$$\phi_{DG}(u; \mu, a, b, c, d) = e^{i\mu u}(1 - \frac{iu}{b})^{-a}(1 + \frac{iu}{d})^{-c}. \tag{5.8}$$

Combining (5.8) and (2.9) it is clear that choosing $c = a$ results with VG distribution with parameters $\alpha = a$, $\theta = \beta(\frac{1}{b} - \frac{1}{a})$, $\sigma^2 = \frac{2\beta}{bd}$. Madan, Carr, Chang (1998) [88] considered this restricted version as well as Finlay and Seneta (2008) [53].

We drop the $c = a$ restriction from the Difference of Gammas and works with the process described by Carr and Madan (1999) [32]. This introduces one extra degree of freedom into the model while still retaining most of the properties of the VG model, such as its simple characteristic function. This means that we can no longer write down a closed-form expression for the pdf of returns. It is not a problem for us, however, since for option pricing in this method we need the characteristic function only.

For construction we also drop the assumptions that T_t has independent increments, and instead look at models to price options with strictly stationary return process. For $C(Y, K)$, the price of an European call option with expiry Y (time to mature) and strike price K, where $\kappa = \log K$ as in [32], we define the modified call price as follows:

$$c(Y, K) = e^{-\gamma\kappa}C(Y, \kappa),$$

for some $\gamma: \; EP_Y^{\gamma+1} < \infty$.

The Fourier transform of $c(Y, \kappa)$ is then given by,

$$\begin{aligned}
\psi_Y(x) &= \int_{-\infty}^{\infty} e^{ix\kappa} C(Y,\kappa)\, d\kappa \\
&= \int_{-\infty}^{\infty} e^{ix\kappa} \int_{\kappa}^{\infty} e^{\gamma\kappa} e^{-rY}(e^p - e^\kappa) q_Y(p)\, dp\, d\kappa \\
&= \int_{-\infty}^{\infty} e^{-rY} q_Y(p) \int_{-\infty}^{p} (e^{p+\gamma\kappa} - e^{(1+\gamma)\kappa}) e^{ix\kappa}\, d\kappa\, dp \\
&= \int_{-\infty}^{\infty} e^{-rY} q_Y(p) (\frac{e^{(\gamma+1+ix)p}}{\gamma+ix} - \frac{e^{(\gamma+1+ix)p}}{\gamma+1+ix})\, dp \\
&= \frac{e^{-rY}\phi_Y(x-(\gamma+1)i)}{\gamma^2+\gamma-x^2+ix(2\gamma+1)},
\end{aligned}$$

where for the specific model under consideration, $q_Y(p)$ is the risk-neutral density of $\log P_Y$, the log stock price at time Y,

$$\int_{\kappa}^{\infty} e^{-rY}(e^p - e^\kappa) q_Y(p)\, dp = e^{-rY} E(P_Y - \kappa)^+ = C(Y,\kappa),$$

and $\phi_Y(x)$ is the characteristic function of the log stock price at time Y. Since $C(Y,\kappa)$ is real, the real part of $\psi_Y(x)$ is even while the imaginary part is odd, so taking the inverse transform of $\psi_Y(x)$ gives,

$$\begin{aligned}
C(Y,K) &= \frac{e^{-\gamma\kappa}}{2\pi} \int_{-\infty}^{\infty} e^{-ix\kappa} \psi_Y(x)\, dx \\
&= \frac{e^{-\gamma\kappa}}{\pi} \int_{0}^{\infty} \Re\{e^{-ix\kappa}\psi_Y(x)\}\, dx, \qquad (5.9)
\end{aligned}$$

which can be computed by numerical integration.

In fact we use a modified version of the above suggested in Lee (2004) [83] and given by,

$$C(Y,K) = R_\gamma + \frac{e^{-\gamma\kappa}}{\pi} \int_{0}^{\infty} \Re\{e^{-ix\kappa}\psi_Y(x)\}\, dx. \qquad (5.10)$$

Here the R_γ term results from shifting an integral through or across a pole in the complex plane, and is given by,

$$R_\gamma = \begin{cases} 0 \quad \text{for } \gamma > 0 \\ \frac{\phi_Y(-i)}{2} \quad \text{for } \gamma = 0 \\ \phi_Y(-i) \quad \text{for } -1 < \gamma < 0 \\ \phi_Y(-i) - \frac{e^\kappa \phi_Y(0)}{2} \quad \text{for } \gamma = -1 \\ \phi_Y(-i) - e^\kappa \phi_Y(0) \quad \text{for } \gamma < -1 \end{cases}$$

The choice of γ impacts on the error generated by the numerical approximation of (5.10).

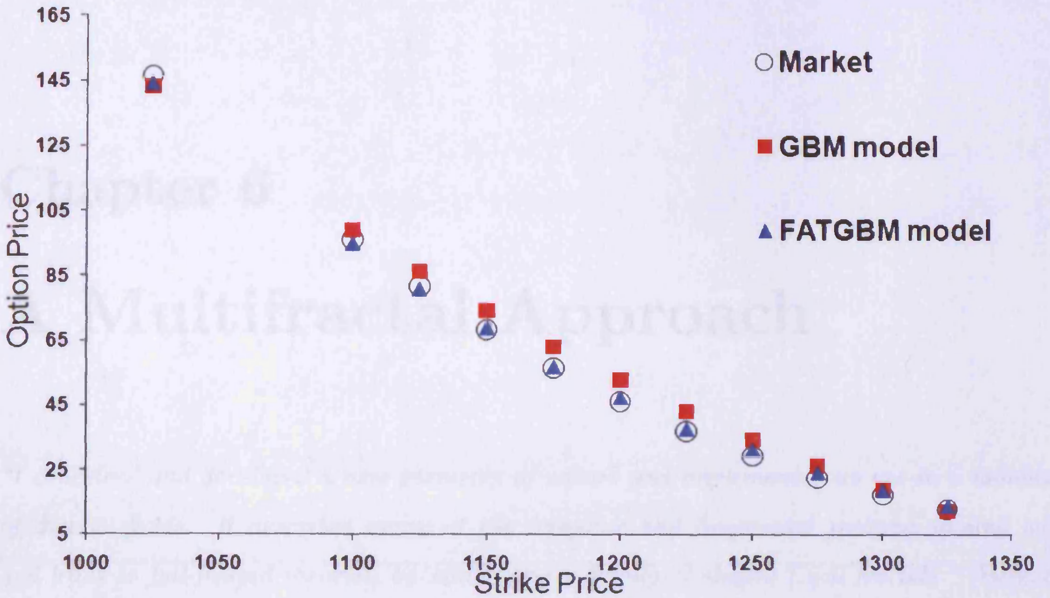


Figure 5.1: Estimates of the fair (S&P500) market price for both GBM and FATGBM models

Chapter 6

A Multifractal Approach

"I conceived and developed a new geometry of nature and implemented its use in a number of diverse fields. It describes many of the irregular and fragmental patterns around us, and leads to full-fledged theories, by identifying a family of shapes I call fractals." (Benoit Mandlebrot).

Since Mandelbrot (in particular 1977, 1982, and 1997) [93] [94] [95] developed and popularized the concept of fractals and multifractals, and advocated their use in the explanation of observed features of time series arising in natural sciences, there has been ongoing interest by researchers in a variety of disciplines in widening their application.

In a finance setting, there is already evidence (see for example Schmitt et al (1999) [115] or Calvet and Fisher (2002) [29]), that through the use of fractals (in particular, multifractals), one can remedy some of the empirically established and occasionally puzzling shortcomings of the paradigm Black-Scholes option pricing model. We have already seen, from the empirical investigation in chapter 2, that the GBM model departs from the realities of risky asset returns, but here we will focus on the scaling nature of returns data. The catalyst for a multifractal approach arises from the fact that non-decreasing $\{T_t\}$ in the FATGBM model can only be asymptotically self-similar, and not exactly self-similar (see Heyde and Leonenko (2005) [66]).

We will begin this chapter by giving a short description of the main features of fractals and multifractals. The next section will introduce a financial model which can incorporate multiscaling. We will equip such a model with a multifractal process construction based on the products of geometric OU-type processes. In particular, we are going to discuss a class of multifractal models originally introduced by Anh et al (2008) [4], and show they provide a useful and flexible family of models for applications.

We then consider some cases of infinitely divisible distributions for the background driving process of OU-type processes, along with their Rényi function and dependence structure. By including some empirical evidence that multifractality exists for real financial data, we can end by validating this approach by testing the fit of the model (Rényi function) to the data.

6.1 Background theory and motivation

There are two main models for fractals that occur in nature. Generally speaking, fractals are either statistically self-similar or they are multifractals.

Multifractals were introduced in Mandelbrot (1972) [91] as measures to model turbulence. The concept was extended in Mandelbrot et al. (1997) [95] to stochastic processes as a generalisation of self-similar stochastic processes. The definition of a multifractal is motivated by that of a stochastic process X_t which satisfies a relationship of the form,

$$\{X(ct)\} \stackrel{d}{=} \{M(c)X(t)\}, \ t \geq 0, \tag{6.1}$$

for $0 < c < 1$ where M is a random variable independent of X and equality is in finite-dimensional distributions.

In the special case $M(c) = c^H$, the multifractal reduces to a self-similar fractal where the parameter $0 < H < 1$ is known as the Hurst parameter named after the British engineer Harold Hurst (whose work on Nile river data played an important role in the development of self-similar processes). For a more detailed review of self-similar processes see Embrechts and Maejima (2002) [46].

It is assumed further that,

$$M(ab) = M_1(a)M_2(b), \ a, b > 0,$$

where M_1 and M_2 are independent random variables with the common distribution of M.

The definition of a multifractal process, as in Mandelbrot et al. (1997) [95], is given in terms of the moments of the process and includes processes satisfying the following statement.

A stochastic process $X = (X(t), t \geq 0)$ is multifractal if it has stationary increments and there exist functions $c(q)$ and $\tau(q)$ and positive constants $q_- < q_+$ and T such that $\forall q \in$

$[q_-, q_+], \forall t \in [0, T]$,

$$E(|X(t)|^q) = c(q)t^{\tau(q)+1}, \; q > 0. \tag{6.2}$$

Here $\tau(q)$ and $c(q)$ are both deterministic functions of q. $\tau(q)$ is called the scaling function and takes into account the influence of the time t on the moments of order q, and $c(q)$ is called the prefactor.

While this definition is the standard for a multifractal process, most processes studied as multifractals only obey it for particular values of t or sometimes for asymptotically small t. The condition of stationary increments is also quite often relaxed.

Conversely, Taqqu et al (1997) [122] tests the scaling properties of the increments of $X(t)$ instead of the process itself. If this method is used then the subtraction of the mean $E(X(t+1)-X(t))$ from $X(t+1)-X(t)$ may be required to ensure a fair investigation, because such a stationary process cannot be self-similar or even asymptotically self-similar if it has non-zero mean. For our investigation, we have $E(X(t+1) - X(t)) = 0$ for each of our data sets.

It follows from (6.2) that,

$$\log E(|X(t)|^q) = \log c(q) + (\tau(q) + 1)\log t,$$

and so $X(t)$ is multifractal if for each q, $\log E|X(t)|^q$ scales linearly with $\log t$ and the slope is $\tau(q) + 1$.

To explain the notion of the scaling function $\tau(q)$, consider the particular case of the fractional Brownian motion - a self-similar process. A fractional Brownian motion, with a Hurst exponent H, satisfies,

$$X(t) \stackrel{d}{=} t^H X(1),$$

which implies that,

$$E(|X(t)|^q) = t^{Hq} E(|X(1)|^q).$$

Here we obtain the prefactor,

$$c(q) = E(|X(1)|^q),$$

and the scaling function,

$$\tau(q) = Hq - 1.$$

So the scaling function is linear if the process is self-similar. Alternatively, the process is multifractal if it has the multiscaling properties that imply nonlinearity of the scaling function.

If we define a log-price process $\{Y(t) = \log P(t) - \log P(0), 0 \le t \le T\}$, where $P(t)$ is the price of a risky asset at time t, we can study the scaling behaviour of our FATGBM model. Firstly, under the GBM model (Black and Scholes (1973) [26]) with zero drift term,

$$Y(t) = \sigma B(t),\ t \ge 0,$$

where $\sigma > 0$ is a fixed constant and $\{B(t), t \ge 0\}$ is a standard Brownian motion (a self-similar process with $H = \frac{1}{2}$). By the scaling property of standard Brownian motion,

$$\begin{aligned} E(|Y(t)|^q) &= t^{\frac{q}{2}} E(|\sigma B(1)|^q) \\ &= t^{\frac{q}{2}} (\sqrt{2\sigma^2})^q \frac{\Gamma(\frac{1+q}{2})}{\sqrt{\pi}}. \end{aligned}$$

Here the scaling function $\tau_Y(q) = \frac{q}{2} - 1$ is linear.

Under the FATGBM model (Heyde (1999) [63]), again with zero drift,

$$Y(t) = \sigma B(T(t)),\ t \ge 0, \tag{6.3}$$

where $\sigma > 0$ is a fixed constant, and $\{B(t), t \ge 0\}$ is a standard Brownian motion (or could be extended to fractional Brownian motion, see various authors including Mandelbrot et al (1997) [95] and Elliott and Van Der Hoek (2003) [45]), and $\{T(t)\}$ is our random activity time process independent of $\{B(t)\}$. The linearity of the corresponding scaling function will be investigated later to determine whether $\{T(t)\} = \{A(t)\}$, where $\{A(t)\}$ is a multifractal process.

Mandelbrot et al. (1997) [95] showed that the scaling function is concave for all multifractals with the following argument. Let ω_1, ω_2 be positive weights with $\omega_1 + \omega_2 = 1$ and let $q_1, q_2 \ge 0$ and $q = q_1\omega_1 + q_2\omega_2$. Then by Hölder inequality,

$$E|X(t)|^q \le (E|X(t)|^{q_1})^{\omega_1} (E|X(t)|^{q_2})^{\omega_2},$$

and hence,

$$\log c(q) + \tau(q) \log t \le (\omega_1 \tau(q_1) + \omega_2 \tau(q_2)) \log t + (\omega_1 \log c(q_1) + \omega_2 \log c(q_2)).$$

Letting t go to zero we have $\tau(q) \ge \omega_1\tau(q_1) + \omega_2\tau(q_2)$, so τ is concave. If $T = \infty$ we can let t go to ∞ and we get the reverse inequality $\tau(q) \le \omega_1\tau(q_1) + \omega_2\tau(q_2)$. It follows that $T = \infty$ implies that τ is linear and so $X(t)$ is self-similar.

An important associated concept is the multifractal spectrum. It is the Legendre transform of the scaling function $\tau(q)$ and is given by,

$$f(\alpha) = \inf_q [q\alpha - \tau(q)],$$

where it is defined. For self-similar processes it is only defined at H with $f(H) = 1$. The multifractal spectrum plays an important role in multifractal measures where it represents the fractal dimensions of sets where the measure has certain limiting intensities. The analogous definition for multifractal processes is the dimension of sets with local Hölder exponent α (see Calvet et al. (1997) [30] for details). However, for multifractal processes the multifractal spectrum is only used as a tool for fitting the model to data.

The motivating example of a multifractal process is the cascade. They were first introduced as measures in Mandelbrot (1974) [92] and can be defined on the interval $[0, 1]$ as follows. Define a sequence of random measures μ_n by,

$$\mu_n(dt) = \prod_{i=1}^{n} M_{\eta_1, \eta_2, \ldots, \eta_i}(dt),$$

where t has expansion $t = 0.\eta_1\eta_2 \ldots$ in base b and the $M_{\eta_1, \eta_2, \ldots, \eta_i}$ are a collection of positive i.i.d random variables with distribution M where $EM = 1$. Kahane and Peyrière (1976) [75] showed that the almost sure vague limit of μ_n exists, denoted as μ. The stochastic process $X(t)$ is defined as $X(t) = \mu([0, t])$. It is easy to check that (6.2) holds when $t = b^{-n}$. Of course $X(t)$ does not fully satisfy the definition of a multifractal as equation (6.2) does not hold except when t is of the form b^{-n} and $X(t)$ does not even have stationary increments. Even though cascades do not satisfy the formal definition they remain the prototype model for multifractal processes.

Multifractals overcome an important limitation of self-similar stochastic processes which is they can be positive and still have finite mean as in the case of cascades. When $X(t)$ is positive and $EX(1) < \infty$ equation (6.2) implies that $\tau(1) = 0$.

6.2 The construction of the multifractal model

Models with multifractal scaling have been used in many applications in hydrodynamic turbulence, genomics, computer network traffic, etc. (see Kolmogorov (1941, 1962) [78] [79], Gupta and Waymire (1993) [62], Novikov (1994) [103], Frisch (1995) [55], Anh et al (2001) [2]). The application to finance was first investigated by Mandelbrot et al (1997) [95], where it is established that most multifractal models are not designed to cover important features of financial data, such as a tractable dependence structure.

To surmount these problems, Anh et al (2008) [4] considered multifractal products of stochastic processes as defined in Kahane (1985, 1987) [73] [74] and Mannersalo et al (2002) [98]. These multifractals are based on products of geometric Ornstein-Uhlenbeck processes driven by Lévy motion were constructed, and several cases of infinitely divisible distributions for the background driving Lévy process are studied. The behaviour of the q-th order moments and Rényi functions were found to be nonlinear, hence displaying the multifractality as required. We will replicate this methodology and look to integrate this construction into the model (6.3).

6.2.1 Multifractal products of stochastic processes

We begin by recapturing some basic results on multifractal products of stochastic processes as developed in Kahane (1985) [73] and Mannersalo et al (2002) [98]. The following conditions hold:

C1 Let $\Lambda(t), \ t \in \mathbb{R}_+ = [0, \infty)$, be a measurable, separable, strictly stationary, positive stochastic process with $\mathrm{E}\Lambda(t) = 1$.

We call this process the mother process and consider the following setting:

C2 Let $\Lambda^{(i)}$, $i = 0, 1, ...$ be independent copies of the mother process Λ, and $\Lambda_b^{(i)}$ be the rescaled version of $\Lambda^{(i)}$,

$$\Lambda_b^{(i)}(t) \stackrel{d}{=} \Lambda^{(i)}(tb^i), \quad t \in \mathbb{R}_+, \quad i = 0, 1, 2, \ldots,$$

where the scaling parameter $b > 1$.

C3 For $t \in \mathbb{R}_+$, let $\Lambda(t) = \exp\{X(t)\}$, where $X(t)$ is a stationary process with $\mathrm{E}X^2(t) < \infty$. We denote by $\theta \in \Theta \subseteq \mathbb{R}^p, p \geqslant 1$ the parameter vector of the distribution of the process $X(t)$ and assume that there exist a marginal probability density function $p_\theta(x)$ and a bivariate probability density function $p_\theta(x_1, x_2; t_1 - t_2)$ such that the moment generating function,

$$M(\zeta) = \mathrm{E}e^{\zeta X(t)},$$

and the bivariate moment generating function,

$$M(\zeta_1, \zeta_2; t_1 - t_2) = \mathrm{E}e^{\zeta_1 X(t_1) + \zeta_2 X(t_2)},$$

exist.

The conditions **C1-C3** yield,

$$\mathrm{E}\Lambda_b^{(i)}(t) = M(1) = 1,$$

$$\mathrm{Var}\Lambda_b^{(i)}(t) = M(2) - 1 = \sigma_\Lambda^2 < \infty,$$

$$\mathrm{Cov}(\Lambda_b^{(i)}(t_1), \Lambda_b^{(i)}(t_2)) = M(1, 1; (t_1 - t_2)b^i) - 1, \; b > 1. \tag{6.4}$$

We define the finite product processes,

$$\Lambda_n(t) = \prod_{i=0}^{n} \Lambda_b^{(i)}(t) = e^{\sum_{i=0}^{n} X(tb^i)}, \tag{6.5}$$

and the cumulative processes,

$$A_n(t) = \int_0^t \Lambda_n(s)ds, \quad n = 0, 1, 2, \ldots, \tag{6.6}$$

We also consider the corresponding positive random measures defined on Borel sets B of $\mathbb{R}_+$,

$$\mu_n(B) = \int_B \Lambda_n(s)ds, \quad n = 0, 1, 2, \ldots \tag{6.7}$$

Kahane (1987) [74] proved that the sequence of random measures μ_n converges weakly a.s. to a random measure μ. Moreover, given a finite or countable family of Borel sets B_j on $\mathbb{R}_+$, it holds that $\lim_{n\to\infty} \mu_n(B_j) = \mu(B_j)$ for all j with probability one. The a.s. convergence of $A_n(t)$ in countably many points of $\mathbb{R}_+$ can be extended to all points in $\mathbb{R}_+$ if the limit process $A(t)$ is a.s. continuous. In this case, $\lim_{n\to\infty} A_n(t) = A(t)$ with probability one for all $t \in \mathbb{R}_+$. As noted in Kahane (1987) [74], there are two extreme cases: (i) $A_n(t) \to A(t)$ in L_1 for each given t, in which case $A(t)$ is not a.s. zero and is said to be fully active (non-degenerate) on $\mathbb{R}_+$; (ii) $A_n(1)$ converges to 0 a.s., in which case $A(t)$ is said to be degenerate on $\mathbb{R}_+$. Sufficient conditions for non-degeneracy and degeneracy in a general situation and relevant examples are provided in Kahane (1987) [74].

The Rényi function, also known as the deterministic partition function, is defined for $t \in [0, 1]$ as follows:

$$\begin{aligned} R(q) &= \liminf_{n\to\infty} \frac{\log \mathrm{E} \sum_{k=0}^{2^n-1} \mu^q \left(I_k^{(n)}\right)}{\log \left|I_k^{(n)}\right|} \\ &= \liminf_{n\to\infty} \left(-\frac{1}{n}\right) \log_2 \mathrm{E} \sum_{k=0}^{2^n-1} \mu^q \left(I_k^{(n)}\right), \end{aligned}$$

where $I_k^{(n)} = [k2^{-n}, (k+1)2^{-n}]$, $k = 0, 1, \ldots, 2^n - 1$, $\left|I_k^{(n)}\right|$ is its length, and $\log_b$ is log to the base b.

Mannersalo et al. (2002) [98] presented the conditions for L_2-convergence and scaling of moments:

Theorem 8. *Suppose that the conditions **C1-C3** hold.*

If, for some positive numbers δ and γ,

$$\exp\{-\delta|\tau|\} \leqslant \rho(\tau) = \frac{M(1,1;\tau) - 1}{M(2) - 1} \leqslant |C\tau|^{-\gamma}, \tag{6.8}$$

then $A_n(t)$ converges in L_2 if and only if

$$b > 1 + \sigma_\Lambda^2 = M(2).$$

If $A_n(t)$ converges in L_2, then the limit process $A(t)$ satisfies the recursion

$$A(t) = \frac{1}{b}\int_0^t \Lambda(s) d\tilde{A}(bs), \tag{6.9}$$

where the processes $\Lambda(t)$ and $\tilde{A}(t)$ are independent, and the processes $A(t)$ and $\tilde{A}(t)$ have identical finite-dimensional distributions.

If $A(t)$ is non-degenerate, the recursion (6.9) holds, $A(1) \in L_q$ for some $q > 0$, and $\sum_{n=0}^{\infty} c(q, b^{-n}) < \infty$, where $c(q,t) = \mathrm{E}\sup_{s\in[0,t]} |\Lambda^q(0) - \Lambda^q(s)|$, then there exist constants $\overline{C}$ and $\underline{C}$ such that

$$\underline{C}t^{q-\log_b \mathrm{E}\Lambda^q(t)} \leqslant \mathrm{E}A^q(t) \leqslant \overline{C}t^{q-\log_b \mathrm{E}\Lambda^q(t)}, \tag{6.10}$$

which will be written as

$$\mathrm{E}A^q(t) \sim t^{q-\log_b \mathrm{E}\Lambda^q(t)}, \ t \in [0,1].$$

If, on the other hand, $A(1) \in L_q$, $q > 1$, then the Rényi function is given by

$$R(q) = q - 1 - \log_b \mathrm{E}\Lambda^q(t) = q - 1 - \log_b M(q).$$

If $A(t)$ is non-degenerate, $A(1) \in L_2$, and $\Lambda(t)$ is positively correlated, then

$$\mathrm{Var}A(t) \geqslant \mathrm{Var}\int_0^t \Lambda(s)ds.$$

Hence, if $\int_0^t \Lambda(s)ds$ is strongly dependent, then $A(t)$ is also strongly dependent.

6.2.2 Multifractal products of OU-type processes

We recall the definitions and known results on Lévy processes and Ornstein-Uhlenbeck type processes from Chapter 3. These are again needed to construct a class of multifractal processes.

A random variable X is said to be infinitely divisible if its cumulant function has the Lévy-Khintchine form (3.17). The resulting triplet (a, d, ν) where $a \in \mathbb{R}$, $d \geq 0$ and ν is the Lévy measure, uniquely determines the random variable X.

If X is self-decomposable, then there exists a stationary stochastic process $\{X(t), t \geq 0\}$, such that $X(t) \stackrel{d}{=} X$ and (3.18) holds for all $\lambda > 0$ (see Barndorff-Nielsen 1998). Conversely, if $\{X(t), t \geq 0\}$ is a stationary process and $\{Z(t), t \geq 0\}$ is a Lévy process, independent of $X(0)$, such that (3.19) holds for all $\lambda > 0$, then $X(t)$ is self-decomposable.

A stationary process $X(t)$ of this kind is said to be an OU-type process. The process $Z(t)$ is termed the background driving Lévy process corresponding to the process $X(t)$. In fact (3.18) is the unique (up to indistinguishability) strong solution to (3.19) [114]. Moreover, if $X(t)$ is a square integrable OU process, then it has the correlation function in Theorem 5.

The following result is needed in the construction of mutifractal processes from OU-type processes:

Theorem 9. *Let $X(t), t \in [0,1]$ be an OU type stationary process* (3.18) *such that the Lévy measure ν in* (3.17) *of the random variable $X(t)$ satisfies the condition that for some range of $q \in \mathbb{R}$,*

$$\int_{|x|\geq 1} g_q(x)\nu(dx) < \infty,$$

where $g_q(x)$ denotes any of the functions $e^{2qx}, e^{qx}, e^{qx}|x|$. Then, for the geometric OU type process $\Lambda_q(t) := e^{qX(t)}$,

$$\sum_{n=0}^{\infty} c(q, b^{-n}) < \infty,$$

where $c(q,t) = \mathrm{E}\sup_{s\in[0,t]} |\Lambda_q(0)^q - \Lambda_q(s)^q|$.

The proof this theorem is given in Anh, Leonenko and Shieh (2008) [4]. To prove that a geometric OU-type process satisfies the covariance decay condition (6.8) in Theorem 8, the following proposition gives a general decay estimate which the driving Lévy processes Z in the next subsection indeed satisfy:

Consider the stationary OU-type process X defined by (3.19) which has a stationary distribution $\pi(x)$ such that, for some $a > 0$,

$$\int |x|^a \, \pi(dx) < \infty. \tag{6.11}$$

Then there exist positive constants c and C such that,

$$\text{Cov}\left(e^{X(t)}, e^{X(0)}\right) \leq Ce^{-ct},$$

for all $t > 0$.

Masuda (2004) [99] showed that, under the assumption (6.11), the stationary process $X(t)$ satisfies the β-mixing condition with coefficient $\beta_X(t) = O(e^{-ct}), t > 0$. Note that this is also true for the stationary process $e^{X(t)}$, since the $\sigma-$algebras generated by these two processes are equivalent. Hence,

$$\beta_{e^X}(t) = O(e^{-ct}), t > 0.$$

It then follows that,

$$\text{Cov}\left(e^{X(t)}, e^{X(0)}\right) \leq const \times \beta_{e^X}(t) \leq Ce^{-ct},$$

(see Billingsley 1968 [25]).

In this section the results discussed in the previous sections are used to construct multifractal processes. The mother process of assumption **C1** will take the form,

$$\Lambda(t) = \exp\left\{X(t) - c_X\right\}, \tag{6.12}$$

where $X(t)$ is a stationary OU type process and c_X is a constant depending on the parameters of its marginal distribution such that $E\Lambda(t) = 1$.

All the definitions given in (6.5) - (6.7) and correspondingly all the statements of Theorem 8 are now understood to be in terms of the mother process (6.12). At this point however it is convenient to introduce separate notations for the moment generating function of Λ, which we denote by $M_\Lambda(\cdot)$, and the moment generating function of X, which we denote by $M(\cdot)$. Thus,

$$M_\Lambda(z) = \text{E}\exp\left(z\left(X(t) - c_X\right)\right) = \exp\{-zc_X\}M(z),$$

and,

$$\begin{aligned} M_\Lambda(z_1, z_2; (t_1 - t_2)) &= \text{E}\exp\{z_1(X(t_1) - c_X) + z_2(X(t_2) - c_X)\} \\ &= \exp\{-c_X(z_1 + z_2)\}M(z_1, z_2; (t_1 - t_2)), \end{aligned}$$

The correlation function of the mother process Λ then takes the form,

$$\rho(\tau) = \frac{M_\Lambda(1,1;\tau) - 1}{M_\Lambda(2) - 1}.$$

The constant c_X (when it exists) can be obtained as,

$$c_X = \log \mathrm{E}e^{X(t)} = \log M(1).$$

Accordingly, the Rényi function is obtained as,

$$R(q) = q\left(1 + \frac{\log M(1)}{\log b}\right) - \frac{\log M(q)}{\log b} - 1.$$

Example I: The log-gamma scenario

We will use a stationary OU-type process with marginal gamma distribution $\Gamma(\beta, \alpha)$, which is self-decomposable, and, hence, infinitely divisible. The probability density function (pdf) of $X(t), t \in \mathbb{R}_+$, is given by,

$$f(X) = \frac{\alpha^\beta}{\Gamma(\beta)} x^{\beta-1} e^{-\alpha x} \mathbf{1}_{[0,\infty)}(x), \quad \alpha > 0, \beta > 0, \tag{6.13}$$

with the Lévy triplet of the form $(0, 0, \nu)$, where,

$$\nu(du) = \frac{\beta e^{-\alpha u}}{u} \mathbf{1}_{[0,\infty)}(u) du,$$

while the Lévy process $\grave{Z}(t)$ in (3.19) is a compound Poisson subordinator,

$$\grave{Z}(t) = \sum_{n=1}^{P(t)} Z_n,$$

with the $Z_n, n = 1, 2, \ldots,$ being independent copies of the random variable $\Gamma(1, \alpha)$ and $P(t), t \geq 0$, being a homogeneous Poisson process with intensity β. The logarithm of the characteristic function of $\grave{Z}(1)$ is,

$$\kappa(z) = \log Ee^{iz\grave{Z}(1)} = \frac{i\beta z}{\alpha - iz}, \quad z \in \mathbb{R},$$

and the (finite) Lévy measure $\tilde{\nu}$ of $\grave{Z}(1)$ is,

$$\tilde{\nu}(du) = \alpha\beta e^{-\alpha u} \mathbf{1}_{[0,\infty)}(u) du.$$

C4 Consider a mother process of the form,

$$\Lambda(t) = e^{X(t) - c_X} \text{ with } c_X = \log \frac{1}{(1 - \frac{1}{\alpha})^\beta} \text{ and } \alpha > 1,$$

where $X(t), t \in \mathbb{R}_+$, is a stationary gamma OU-type stochastic process with marginal density (6.13) and covariance function,

$$r_X(t) = \frac{\beta}{\alpha^2} e^{-\lambda|t|}, \ t \in \mathbb{R}.$$

From the discussion above it follows that Theorem 8 and Theorem 9 can be applied to this setting to yield the following result:

Theorem 10. *Suppose that condition C4 holds, and let* $Q = \{q : 0 < q < \alpha, \alpha > 2\}$. *Then, for any* $b > e^{-2c\chi}(1-2\alpha)^{-\beta}$, $\beta > 0$, *the stochastic processes* $A_n(t)$ *defined by* (6.6) *converge in* L_2 *to the stochastic process* $A(t)$ *as* $n \to \infty$ *such that, if* $A(1) \in L_q$ *for* $q \in Q$,

$$EA(t)^q \sim t^{R(q)+1},$$

where the Rényi function is given by,

$$R(q) = q\left(1 + \frac{1}{\log b} \log \frac{1}{(1-\frac{1}{\alpha})^\beta}\right) + \frac{\beta}{\log b} \log\left(1 - \frac{q}{\alpha}\right) - 1, \ q \in Q,$$

(see Anh et al (2008) [4] for proof).

Example II: The log-inverse Gaussian scenario

We will use a stationary OU-type process with marginal inverse Gaussian distribution $IG(\delta, \gamma)$, which is self-decomposable and, hence, infinitely divisible. The pdf of $X(t), t \in \mathbb{R}_+$, is given by,

$$f(x) = \frac{1}{\sqrt{2\pi}} \frac{\delta e^{\delta\gamma}}{x^{\frac{3}{2}}} e^{-(\frac{\delta^2}{x} + \gamma^2 x)\frac{1}{2}} \mathbf{1}_{[0,\infty)}(x), \ \delta > 0, \gamma \geq 0, \tag{6.14}$$

with the Lévy triplet of the form $(0, 0, \nu)$, where,

$$\nu(du) = \frac{1}{\sqrt{2\pi}} \frac{\delta}{u^{\frac{3}{2}}} e^{-\frac{\gamma^2 u}{2}} \mathbf{1}_{[0,\infty)}(u) du,$$

while the Lévy process $\grave{Z}(t)$ in (3.19) has the cumulant function,

$$\kappa(z) = \log E e^{iz\grave{Z}(1)} = \frac{iz\delta}{\gamma\sqrt{1 - \frac{2iz}{\gamma^2}}}, \ z \in \mathbb{R},$$

that is, the Lévy triplet of $\grave{Z}(1)$ is of the form $(0, 0, \tilde{\nu})$, and $\grave{Z}(t)$ is the sum of two independent Lévy processes: $\grave{Z}(t) = \grave{Z}_1(t) + \grave{Z}_2(t)$. Here $\grave{Z}_1(t), t \in \mathbb{R}_+$, is an $IG(\frac{\delta}{2}, \gamma)$ subodinator with Lévy density,

$$\tilde{\nu}_1(du) = \frac{1}{2\sqrt{2\pi}} \frac{\delta}{u\sqrt{u}} e^{-\frac{\gamma^2 u}{2}} \mathbf{1}_{[0,\infty)}(u) du,$$

which has infinitely many jumps in bounded time intervals, and $\grave{Z}_2(t), t \in \mathbb{R}_+$, is a compound Poisson subordinator:

$$\grave{Z}_2(t) = \frac{1}{\gamma^2} \sum_{n=1}^{P(t)} Z_n^2,$$

where the $Z_n, n = 1, 2, \ldots$, are independent copies of the standard normal variable and $P(t), t \in \mathbb{R}_+$, is a homogeneous Poisson process with intensity $\frac{\delta\gamma}{2}$. The (finite) Lévy measure $\tilde{\nu}$ of $\grave{Z}_2(1)$ can be computed as,

$$\tilde{\nu}_2(du) = \frac{1}{2\sqrt{2\pi}} \frac{\delta\gamma^2}{\sqrt{u}} e^{-\frac{\gamma^2 u}{2}} \mathbf{1}_{[0,\infty)}(u) du.$$

C5 Consider a mother process of the form,

$$\Lambda(t) = e^{X(t) - c_X} \text{ with } c_X = \delta(\gamma - \sqrt{\gamma^2 - 2}) \text{ and } \gamma \geq \sqrt{2},$$

where $X(t), t \in \mathbb{R}_+$, is a stationary inverse Gaussian OU-type with marginal density (6.14) and covariance function,

$$r_X(t) = \frac{\delta}{\gamma^3} e^{-\lambda|t|}, \ t \in \mathbb{R}.$$

From the discussion above it follows that Theorem 8 and Theorem 9 can be applied to this setting to yield the following result:

Theorem 11. *Suppose that condition **C5** holds, and let $Q = \{q : 0 < q < \frac{\gamma^2}{2}, \alpha > 2\}$. Then, for any $b > e^{-2c_X + \delta(\gamma - \sqrt{\gamma^2 - 4})}$, the stochastic processes $A_n(t)$ defined by (6.6) converge in L_2 to the stochastic process $A(t)$ as $n \to \infty$ such that, if $A(1) \in L_q$ for $q \in Q$,*

$$EA(t)^q \sim t^{R(q)+1},$$

where the Rényi function is given by,

$$R(q) = q\left(1 + \frac{\delta(\gamma - \sqrt{\gamma^2 - 2})}{\log b}\right) + \frac{\delta}{\log b}\sqrt{\gamma^2 - 2q} - \frac{\gamma\delta}{\log b} - 1, \ q \in Q,$$

(see Anh et al (2008) [4] for proof).

<u>Example III: The log-spectrally negative α-stable scenario</u>

We propose a stationary OU-type process satisfying the Itô stochastic differential equation (3.19), where $\{Z_t, t \geq 0\}$ is a càdlàg spectrally negative α-stable process with $1 < \alpha < 2$ and stationary and independent increments. Due to the absence of positive jumps, Patie (2007)

[106] states that it is possible to extend the characteristic exponent of $\{Z_t\}$ on the negative imaginary line to derive its Laplace exponent, $\psi(z) = Ee^{-zZ(t)} = u^\alpha,\ u \geq 0$. However, as we are interested in the case where there is an absence of negative jumps, the logarithm of the characteristic function of $Z(1)$ is,

$$\kappa_{Z(1)}(z) = \log Ee^{iZ(1)} = (iz)^\alpha,$$

and the (finite) Lévy measure $\tilde{\nu}$ of $Z(1)$ is,

$$\tilde{\nu}(du) = cu^{-\alpha-1}\mathbf{1}_{(0,\infty)}(u)du,\ c > 0.$$

The related logarithm of the characteristic function of X is,

$$\kappa_X(z) = \frac{1}{\lambda}\int_0^z \frac{\kappa_{Z(1)}(\xi)}{\xi}d\xi = \frac{(iz)^\alpha}{\alpha\lambda}.$$

C1 Consider a mother process of the form,

$$\Lambda(t) = e^{X(t)-c_X} \text{ with } c_X = \frac{1}{\alpha\lambda},$$

where $X(t), t \in \mathbb{R}$, is a stationary spectrally negative α-stable OU-type stochastic process. All conditions hold for Theorems 8 and 9, so we can now formulate the following result:

Theorem 12. *Suppose that condition **C1** holds. Then, for any $b > e^{-\frac{2^\alpha}{\alpha\lambda}-\frac{2}{\alpha\lambda}}$, $\lambda > 0$, the stochastic processes $A_n(t)$ converge in L_2 to the stochastic process $A(t)$ as $n \to \infty$ such that, if $A(1) \in L_q$ for $q \in Q$,*

$$EA(t)^q \sim t^{R(q)+1},$$

where the Rényi function is given by,

$$R(q) = q\left(1 + \frac{1}{\log b}\frac{1}{\alpha\lambda}\right) - \frac{1}{\log b}\frac{q^\alpha}{\alpha\lambda} - 1,\ q \in Q.$$

In *Table 6.1*, we have collected all corresponding Rènyi functions, and ranges of q for the L_2- convergence of A_n to A for the models discussed in this chapter. For further scenarios and a table for ready reference see the papers by Anh et al (2008, 2009a, 2009b, 2010) [4] [5] [6] [7].

For $q \in Q \cap [1,2]$, the condition $A(1) \in L_q, q > 1$ follows from the L_2 convergence; thus the above results hold at least for this range. For q outside this range, the condition is still to be verified for the validity of multifractal moment scaling. However, Anh et al (2010) [8] illustrates that through simulation experiments that convergence to multifractality should hold for values of q larger than 2. Hence there is scope for relaxing the condition $A(1) \in L_q$ for $q = 1, 2$.

6.3 Fitting multifractal scenarios to data

We will now compare the estimate of the scaling function with the Rényi function obtained for the scenarios corresponding to the gamma, inverse Gaussian and spectrally negative stable distributions in *Table 6.1.*

If we first take the activity time process $\{T(t)\}$ in (6.3) to be a multifractal process $\{A(t), t \geq 0\}$ with scaling function $\tau_A(q)$, then by (6.2),

$$\begin{aligned} E(|Y(t)|^q) &= EA(t)^{\frac{q}{2}} E(|\sigma B(1)|^q) \\ &= c_A(\frac{q}{2}) t^{\tau_A(\frac{q}{2})+1} E(|\sigma B(1)|^q). \\ &= c_A(\frac{q}{2}) t^{\tau_A(\frac{q}{2})+1} (\sqrt{2\sigma^2})^q \frac{\Gamma(\frac{1+q}{2})}{\sqrt{\pi}}. \end{aligned} \quad (6.15)$$

The scaling function is thus given by,

$$\tau_Y(q) = \tau_A(\frac{q}{2})$$

and this leads us directly to the following empirical test:

- if the scaling function $\tau(q)$ is linear then the process is self-similar.
- if the scaling function $\tau(q)$ is non-linear then the process is multifractal (always concave).

To estimate the scaling function, Calvet and Fisher (2002) [29] proposed a method based on a partition function. It allowed them to successfully detect the multifractal properties of real financial data (in their case, the CAC40 stock Index). This partition function will be denoted $\pi_\delta(Y, q)$, and defined by partitioning the series $\{Y(t)\}$ into n subintervals of length δ for each moment q,

$$\pi_\delta(Y, q) = \sum_{i=1}^{n} |Y_{\lceil i\delta \rceil} - Y_{\lceil (i-1)\delta \rceil}|^q, \quad (6.16)$$

where $\lceil \cdot \rceil$ is the integer part (ceiling) operator. By allowing this partition function to be the empirical counterpart of $E(|Y(t)|^q)$ in (6.15), we have,

$$\log \pi_\delta(Y, q) = \tau_A(\frac{q}{2}) \log \delta + \log T + const,$$

where $T = n\delta$ and $const = \log c_A(\frac{q}{2})(\sqrt{2\sigma^2})^q \Gamma(\frac{1+q}{2})/\sqrt{\pi}$. Thus by plotting $\log \pi_\delta(Y, q)$ against $\log \delta$ for various moments q (*Figure 6.1*), we can obtain $\hat{\tau}_A(\frac{q}{2})$.

In addition, the multifractal spectrum (see *Figure 6.2*) is estimated by,

$$\hat{f}(\alpha) = \inf_q [q\alpha - \hat{\tau}(q)].$$

For our empirical work, the values of the increment δ in the calculation of the partition function have been set at 1,2,3,4,5,6,7,15 and 30 (i.e. first week, two weeks and one month, respectively), together with the values of moment q ranging from 0 to 8 by 0.5 increments. This enables us to compute $\hat{\tau}(q)$ for every q and for each data set.

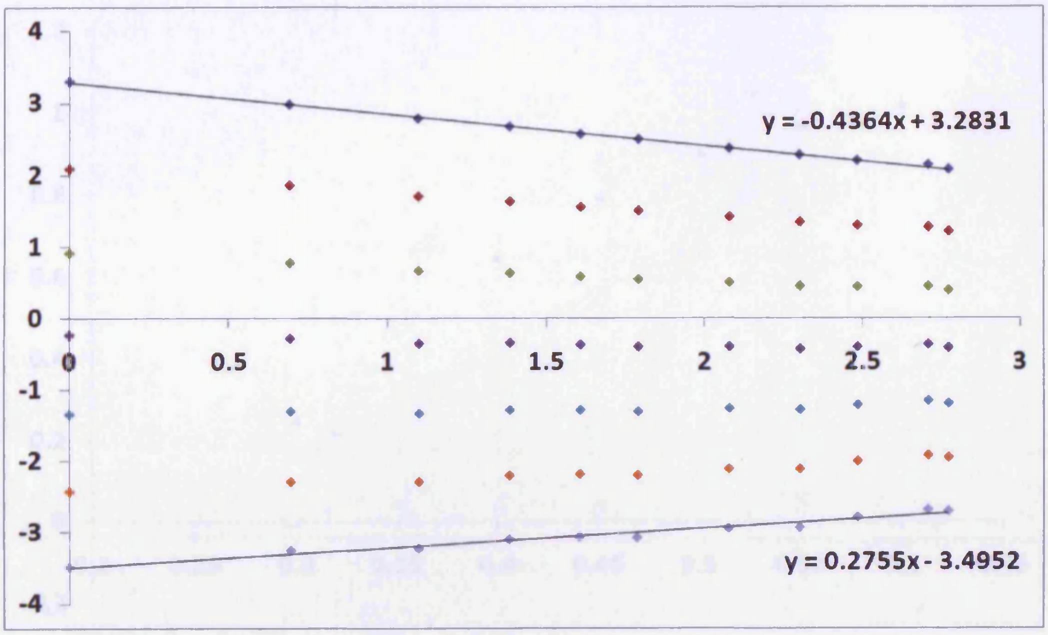
In *Figures 6.3, 6.4, and 6.5* we can show clear evidence of multifractality in real financial data based on non-parametric estimates. This supports the theory in this chapter and enables us to look at various constructions of multifractal processes which can be implemented into the risky asset model (6.3).

All parameters of the parametric Rényi functions including the scaling parameter b and parameters of the marginal distribution of $X(t)$ have been estimated using non-linear least squares. To judge about the applicability of the models discussed, we have compared the non-parametric estimate of the scaling function $\hat{\tau}(q)$ with the Rényi function obtained for the scenarios of gamma, inverse Gaussian and spectrally negative stable distributions (see *Figures 6.3, 6.4, and 6.5*).

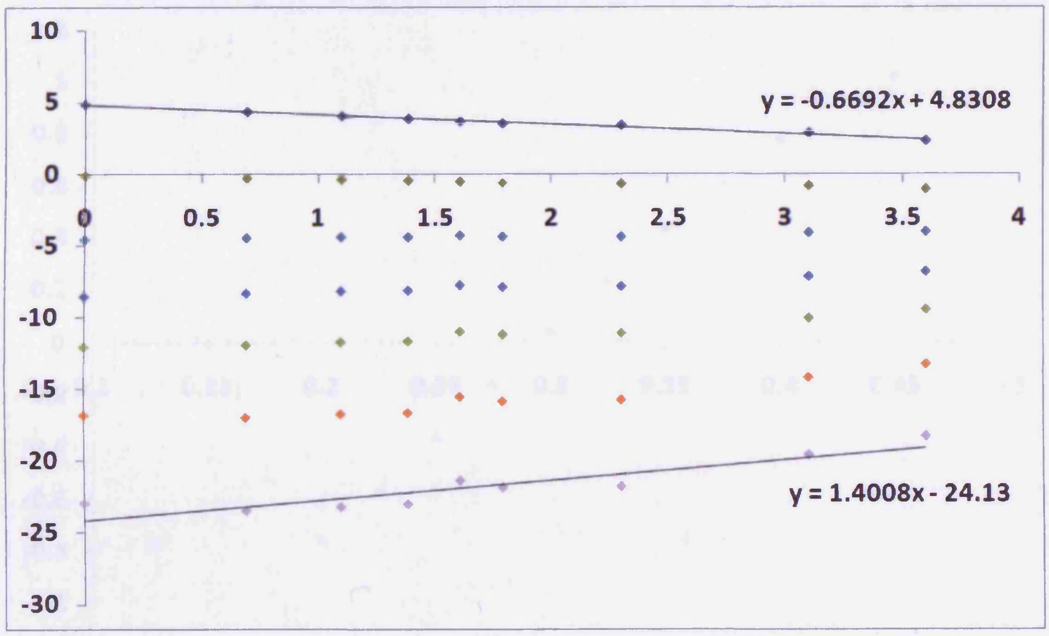
Our aim is to minimise the mean square error between the scaling function estimated from the data and the corresponding analytical forms; the data-fitted Rényi function is denoted by $\tau_{\hat{\theta}}(q)$. All fitted scenarios seem to be able to capture quite well the behaviour of the non-parametric estimate $\hat{\tau}(q)$, with some "distinguo". The Log-SNS scenario, in particular, looks quite apt for the risky asset modelling problem as it obtains good results for our data (see the residual sum of squares in *Table 6.2*).

For a comparison between the Log-Gamma, Log-Inverse Gaussian and Log-Spectrally negative α-stable multifractal scenarios, we could formally use the AIC, BIC and SBC criterions outlined in Appendix H. These criterions are worth considering as they take into account the number of parameters required to fit the data. The only issue is that the underlying models are based on the maximum likelihood function. Since we have dependence in all our data sets, the values in Appendix H should only be taken as a guide.

Remark We have reviewed a class of models based on multifractal activity time and have tested their flexibility in applications through the use of risky asset data. Multifractal processes based on products of geometric OU-type processes appear well apt for varied applications as several different scenarios are easily derived by the characteristic function of the underlying mother process.

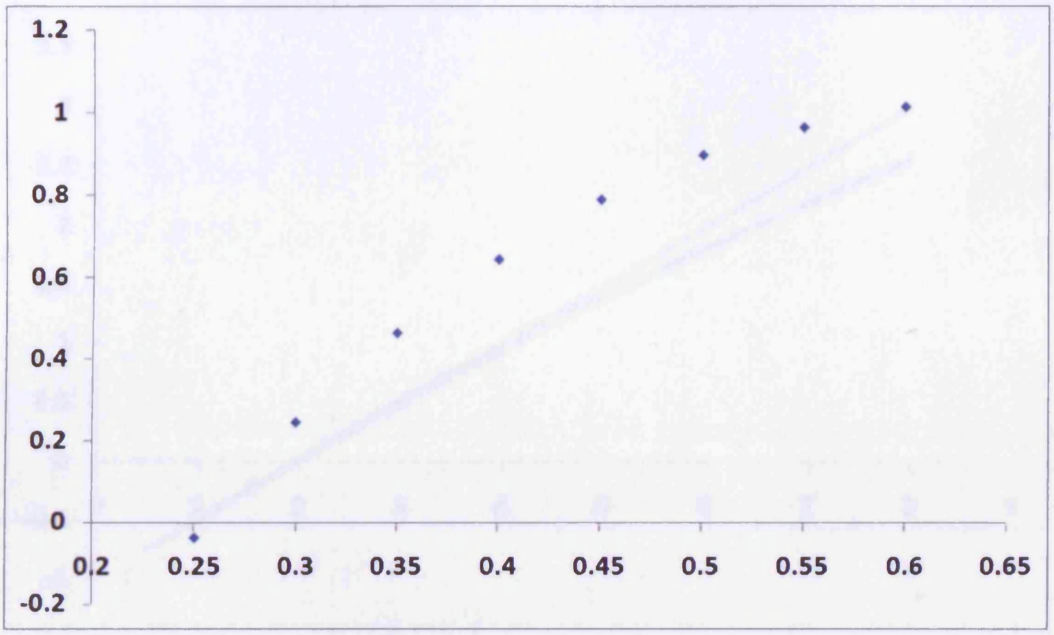


(a) FTSE100

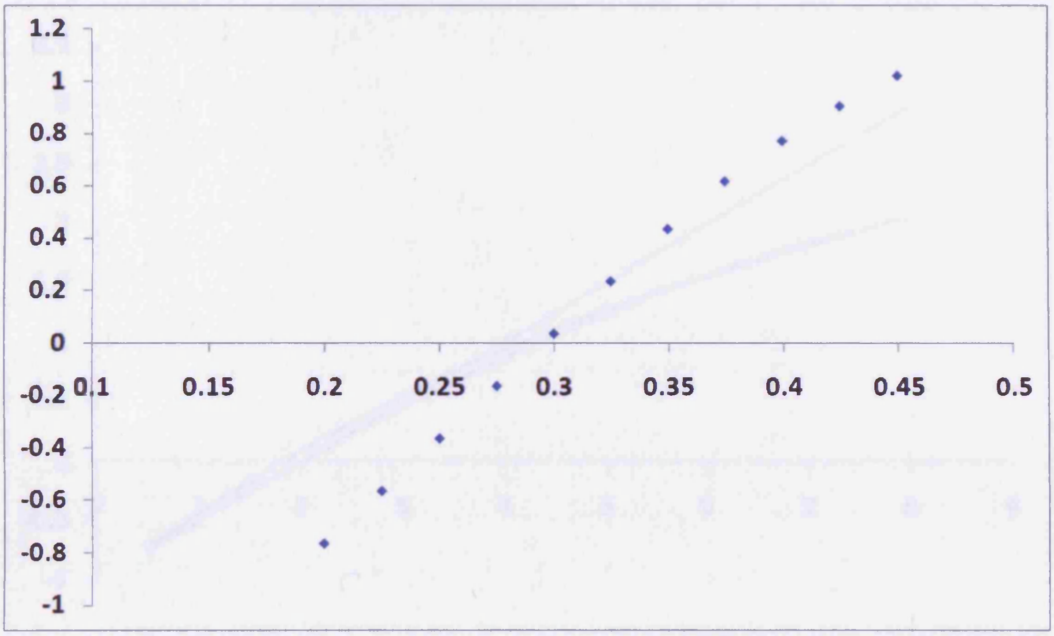


(b) GBP

Figure 6.1: The partition function

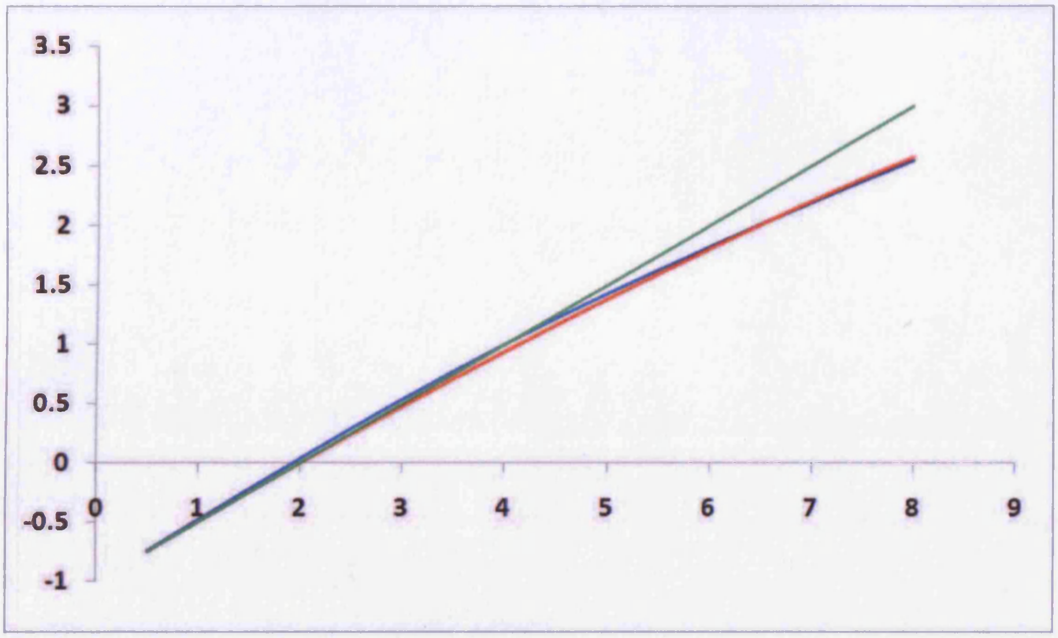


(a) FTSE100

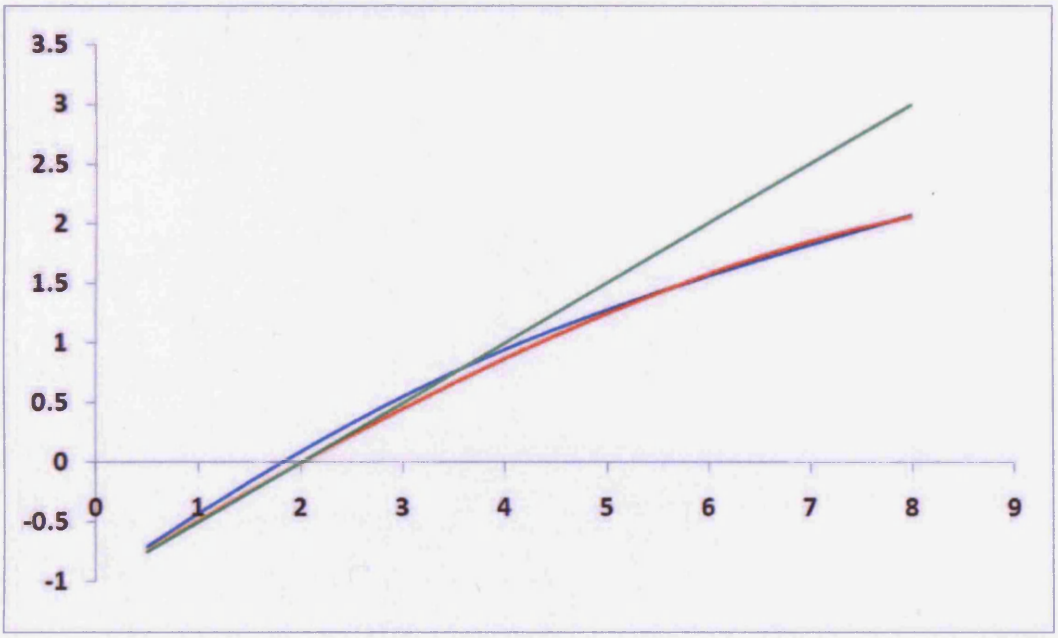


(b) GBP

Figure 6.2: Estimation of the multifractal spectrum

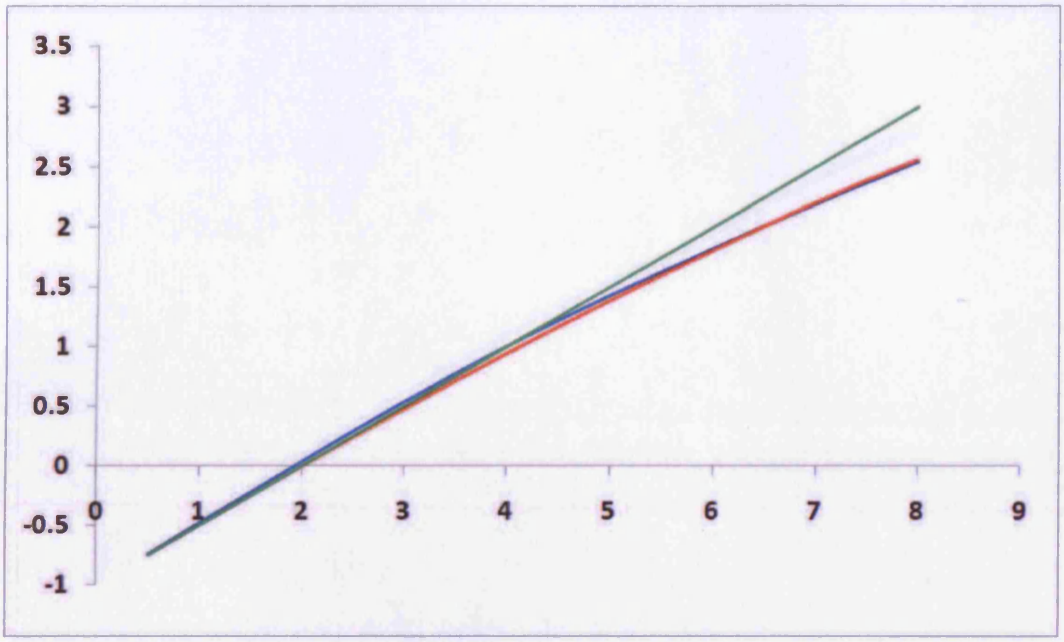


(a) FTSE100 ($\hat{\alpha} = 8.01, \hat{\beta} = 1.42, \hat{b} = 2.46$)

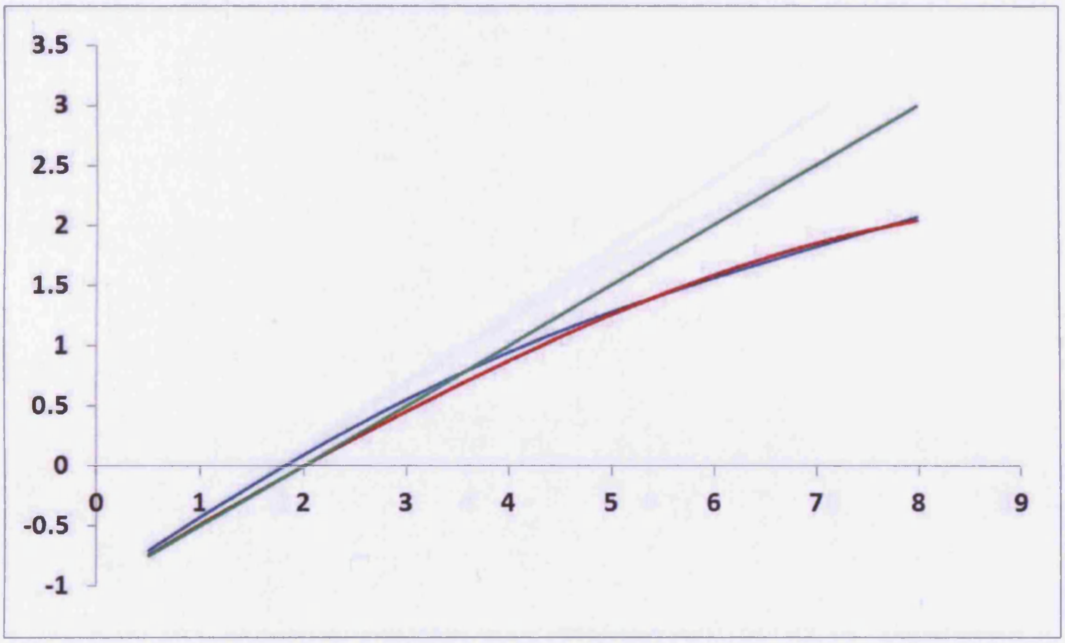


(b) GBP ($\hat{\alpha} = 8.01, \hat{\beta} = 0.40, \hat{b} = 3.30$)

Figure 6.3: Log-gamma scenario of multifractal products of geometric OU-type processes: Blue (line)- non-parametric estimate of $\tau(q)$, Red- fitted parametric estimate of $\tau(q)$, Green-Brownian motion case

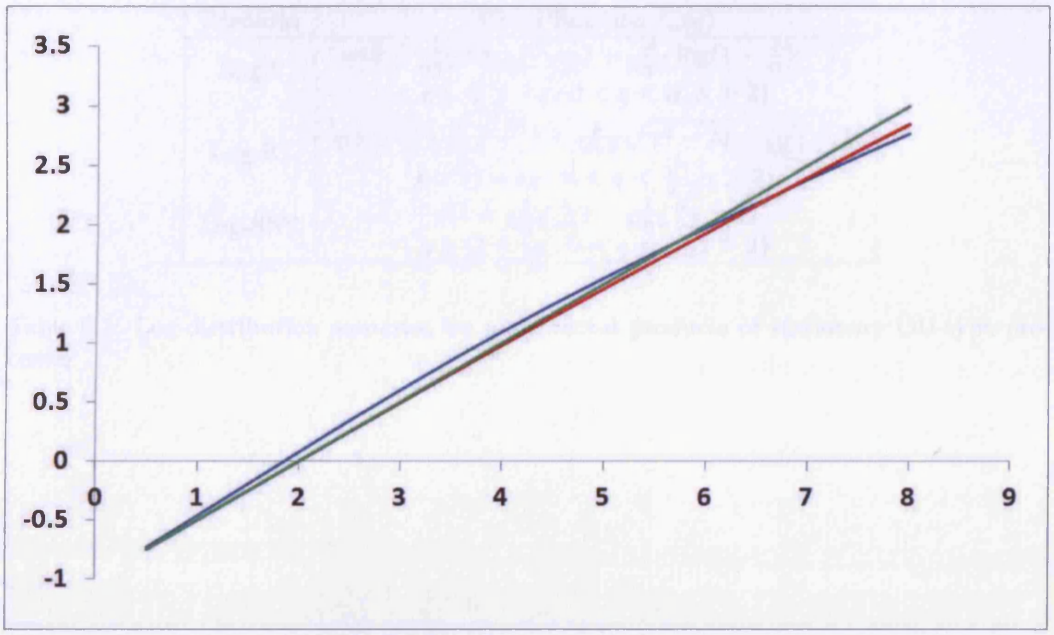


(a) FTSE100 ($\hat{\delta} = 0.89, \hat{\gamma} = 4.52, \hat{b} = 1.86$)

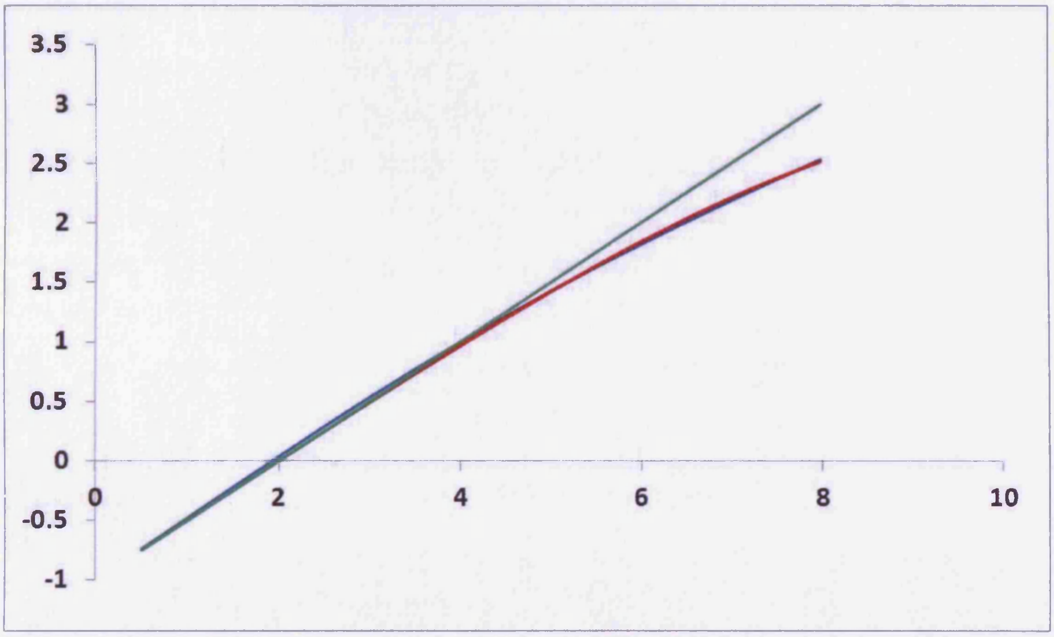


(b) GBP ($\hat{\delta} = 0.32, \hat{\gamma} = 4.01, \hat{b} = 2.31$)

Figure 6.4: Log-inverse Gaussian scenario of multifractal products of geometric OU-type processes: Blue (line)- non-parametric estimate of $\tau(q)$, Red- fitted parametric estimate of $\tau(q)$, Green- Brownian motion case



(a) FTSE100 ($\hat{\alpha} = 3.24, \hat{\lambda} = 36.11, \hat{b} = 10.73$)



(b) GBP ($\hat{\alpha} = 3.64, \hat{\lambda} = 42.11, \hat{b} = 8.10$)

Figure 6.5: Log-spectrally negative α-stable scenario of multifractal products of geometric OU-type processes: Blue (line)- non-parametric estimate of $\tau(q)$, Red- fitted parametric estimate of $\tau(q)$, Green- Brownian motion case

Scenario	Rényi function $\tau_\alpha(q)$
Log-Γ	$q(1 + \frac{1}{\log b} \log \frac{1}{(1-\frac{1}{\alpha})^\beta}) + \frac{\beta}{\log b} \log(1 - \frac{q}{\alpha}) - 1,$ $q \in Q = \{q : 0 < q < \alpha, \alpha > 2\}$
Log-IG	$q(1 + \frac{\delta(\gamma - \sqrt{\gamma^2 - 2})}{\log b}) + \frac{\delta}{\log b} \sqrt{\gamma^2 - 2q} - \frac{\gamma\delta}{\log b} - 1,$ $q \in Q = \{q : 0 < q < \frac{\gamma^2}{2}, \alpha > 2\}$
Log-SNS	$q(1 + \frac{1}{\log b} \frac{1}{\alpha\lambda}) - \frac{1}{\log b} \frac{q^\alpha}{\alpha\lambda} - 1,$ $q \in Q = \{q : 0 < q < \alpha, \alpha > 2\}$

Table 6.1: Log-distribution scenarios for multifractal products of stationary OU-type processes

Scenario	Residual Sum of Squares	
	FTSE100	GBP
Log-Γ	0.0607	0.0256
Log-IG	0.0582	0.0232
Log-SNS	0.0533	0.0136

Table 6.2: The residual sum of squares after regression

Chapter 7

Conclusion

Here we will describe the main results from each chapter in the thesis. We also discuss any opportunities for future work.

The introduction sets the scene for the rest of the thesis. We define what we mean by a risky asset, but stress that the corresponding log-price increments or risky asset returns are our primary focus. We then describe briefly the history of Mathematical Finance, by listing the key contributors and some of their work. The first chapter can thus be used as a handy resource, as it provides many important references for anyone who wishes to study this topic. Inevitably, our approach to modelling risky assets is motivated, in no short measure, by the pioneering work of Black, Scholes and Merton. Their ideas surrounding "continuous-time finance" and "risk-neutral option pricing" provide us with elegant theory which is still popular with both practitioners and academics. Their paradigm GBM (or Black-Scholes) model gave people an early quantitative insight into financial markets and how they moved over time.

To Black, Scholes and Merton, price changes in financial markets were always considered to be random. In fact [26] explicitly states, "We will assume ideal conditions in the market for the stock and for the option ... The stock price follows a random walk in continuous time." Also the word "speculative" in [100] is put into quotes. We now know this randomness to be a shortfall of the GBM model, as it is universally accepted that risky assets behave quite differently in real-life.

In Chapter 2, we outline some of the empirical features of risky asset returns which we need to incorporate into our model. Those characteristics include; no correlation but some dependence, and a leptokurtic distribution (higher peak and heavier tails than the Gaussian distribution). In addition to this, we have noticed an ocassional skewness of the distribution of real data, and evidence of aggregational Gaussianity when looking at varying sampling frequencies.

For risky asset returns, a decreasing sampling frequency results in the distribution tending towards a Gaussian. This would rule out, for example, the stable distributions as models for returns, as suitable distributions ought to be closed under convolution and contain Gaussian as a limit. We do not, however, include a study other sampling frequencies to daily in this thesis. Thus, there is an opportunity here for an extension as higher and higher frequency data is becoming widely available. Some time could be spent to study these more closely to check their characteristics.

The distributions we decided to investigate for risky asset returns were the heavy-tailed Student's t-distribution, and the semi-heavy tailed Variance Gamma and Normal Inverse Gaussian distribtuions. The reason being is that there is large amount of support in the Mathematical Finance literature over the suitability of these distributions. In depth studies into the controversy of the return's tailweight can be found in [65] [56], where a wide range of competing distributions are investigated.

Another area for debate is whether we have weak (short-range) or strong (long-range) dependence. Many papers (see [64] or [41]) have claimed that long-range dependence exists for risky assets. In this thesis, we follow the lead of [19] and [121] by using Hurst parameter as an estimator of the intensity of dependence for returns data. The difficulty lies, however, when measuring Hurst parameter from the output. From the definitions we must focus on the high lags where fewer readings are available, and as a result, where the majority of noise in the data occurs. To avoid any unnecessary scrutiny, we will only conclude that typical risky asset returns are not independent and could exhibit long-range dependence.

We do not plan to answer the question of tailweight for risky asset returns and whether long-range dependence actually exists. Instead we just conclude that such a risky asset model must allow for returns with both semi-heavy and heavy tails, along with a dependence structure. Flexibility is key here and we look to incorporate these empirical findings into the FATGBM model introduced in [63].

We feel that this model we state in chapter 2 captures enough of the reality of the actual processes to warrant the more detailed examination of this thesis, without obscuring the picture with additional sources of possible heavy tails or long-range dependence. In Kerss, Leonenko and Sikorskii (in progress) [77], they look to write a linear stochastic different equation for which the risky asset price (2.5) is the strong solution.

Before any work started on this thesis, there was already strong theoretical and empirical support of this FATGBM model. The main idea is to introduce a random activity time process to evaluate the standard Brownian motion, as opposed to just calender time in the GBM model. Some early attempts of "changed time" were noted in the introduction, but we claim that Heyde's stochastic model for risky assets is able to incorporate the properties that reflect all empirical findings. Other increasingly elaborate ways to generalise GBM can also be found in the Mathematical Finance literature, but many of these are restrained by particular assumptions which do not hold in practice. We must emphasise that models which venture away from the elegant theory of Black, Scholes and Merton are at risk of becoming statistically invalid.

We also study the scaling behaviour of this activity time process. From [66], we note that our activity time cannot be exactly self-similar. However, in [67] we see, at least to first approximation, that we have asymptotic self-similarity. We use this in our constructions of activity time in chapter 3. In the final chapter, we instead consider a more general multifractal activity time, as opposed to the monofractal (asymptotically self-similar) one.

The three constructions of fractal activity time in Chapter 3, are the primary focus of three out of the four submitted papers using material from this thesis. We look to successfully incorporate the empirical features from Chapter 2. For the FATGBM model, we require a fractal activity time process to have certain characteristics. In our constructions we need; pre-specified unit increments with either gamma or reciprocal gamma or inverse Gaussian distribution (for Student, VG or NIG returns respectively), a flexible dependence structure, and a self-similar limit.

The first construction we considered was via chi-squared processes. In this thesis, it is the only method which constructs the unit increment of a activity time process to give us Student distributed returns. The paper shows that the resulting activity time process converges to a self-similar process when appropriately normalized (standard Brownian motion for weak dependence and Rosenblatt-type process for long-range dependence).

The remaining two constructions stem from the need to construct inverse Gaussian unit increment of a activity time process. Here the parameters may also be non-integer. Notably, [52] has since presented non-integer parameters for the chi-squared process also. These second and third construction which we will consider, both have a self-similar limit when appropriately normalised (standard Brownian motion for finite superposition and weak dependence).

For both the diffusion-type and the OU-type approach, we take the superpositions. The reason being that the correlation structure found in risky asset data decays at a slower rate than the exponentially decreasing autocorrelation of the processes. We have seen that superpositions lead to a class of autocovariance functions which are flexible and can be fitted to many autocovariance functions arising in applications including finance.

In Chapter 4, we turn our attention again to fitting real financial data. This time we are interested in checking the fit of Student, VG and NIG, and comparing to that of Gaussian. To do this, any approach based on the maximum likelihood function method would be flawed as the necessary independence assumption does not hold for any of our data sets. Instead, we consider the method of moments.

We then move to consider the symmetric Student case only. This enables us to outline a hypothesis test based on the characteristic function to check the fit of the parameter estimates. A brand new derived expression for the characteristic function of the Student's t-distribution is derived and used for model validation.

The topic of option pricing was considered in the next chapter. We attempt to derive a pricing formula for a European Call option to compare with the paradigm Black-Scholes pricing formula. To achieve this we outlined our plan to follow the approach in [26] and [100], so we can consider a risk-neutral model with the notion of perfect markets and investors who always act rationally.

Similarly for standard Brownian motion in the GBM model, standard Brownian motion evaluated at our random activity time $\{T_t\}$ is a martingale. This is an important property for option pricing, but does not hold for most other risky asset models. Alternative generalisations of Brownian motion in GBM such as fractional Brownian motion, are not martingales.

One issue we do have when deriving a formula for the price of a European option is the uniqueness of the martingale measure. This uniqueness is required to move to the risk-free state of Black, Scholes and Merton, however for the FATGBM, we have two random processes. Thus we choose a particular parameter restrictions to enable us to state such a elegant formula which we can directly compare with the classical Black-Scholes pricing formula. Note here that no assumptions of independence is made for our formula, and if we take $T_Y = Y$ in (5.3) then we reduce to the original Black-Scholes formula.

In Chapter 6, we move away from any self-similar (monofractal), or any asymptotically self-similar assumption. Here, we discuss the possibility of multifractal scaling in risky asset data, and hence, we look to construct an alternative multifractal activity time process.

We started by providing some background theory of multifractal processes. This work in a finance application is still being developed, but the early results of Mandelbrot are important. Using this theory, papers such as [29] showed clear evidence of multifractal scaling in risky asset data, through the fitting of the scaling function. We use this method to investigate the scaling nature of our risky asset data sets.

For the multifractal model, we investigated the properties of products of geometric OU-type processes. We present the general conditions for the L_2 convergence of cumulative processes to limiting processes and investigate their q-th order moments and Rényi functions. We will show that these Rényi functions are non-linear, and hence display multifractality as required. We establish the corresponding scenarios for the limiting processes, such as Log-Gamma, Log-Inverse Gaussian and Log-Spectrally negative α-stable.

In the papers [4] [5] [6] [7], the scenarios were obtained for $q \in Q \cap [1, 2]$, where Q is a set of parameters of marginal distribution of an OU-type processes driven by Lévy motion. The simulation shows for q outside this range, the scenarios are still holds (see [8], and Denisov and Leonenko (in progress) [40] for a rigorous proof).

Appendix A

Appendix

For completeness, and to uphold a logical ordering, we will now present the supplement material for the thesis.

A.1 Appendix A: More pictures

Here are the empirical findings of the other 8 data sets - exchange rates from United Sates Dollar to currencies; Australian Dollar (AD), Canadian Dollar (CD), Deutsche Mark (DM), Euro (EUR), French Franc (FF), Japanese Yen (JY) and New Taiwan Dollar (NTD), and stock index CAC40.

A.2 Appendix B: Estimating the Hurst parameter

We first consider the process $\{Y(t), t \geq 0\}$ which is self-similar i.e. $\{Y(at), t \geq 0\}$ has the same finite-dimensional distributions as $\{a^H Y(t), t \geq 0\}$ for all $a > 0$ and $H \in (0, 1)$, with stationary increments $X_t = Y_t - Y_{t-1}$, $t = 1, 2, \ldots$, then it follows that $Y(t)$ is a finite variance process. The covariance between X_t and X_{t+k} $(k > 0)$ is equal to,

$$\begin{aligned}
\rho(k) &= Cov(X_t, X_{t+k}) \\
&= Cov(X_1, X_{k+1}) \\
&= \frac{1}{2} E \left[\left(\sum_{j=1}^{k+1} X_j \right)^2 + \left(\sum_{j=2}^{k} X_j \right)^2 - \left(\sum_{j=1}^{k} X_j \right)^2 - \left(\sum_{j=2}^{k+1} X_j \right)^2 \right] \\
&= \frac{1}{2} \{ E[(Y_{k+1} - Y_0)^2] + E[(Y_{k-1} - Y_0)^2] - E[(Y_k - Y_0)^2] - E[(Y_k - Y_0)^2] \} \\
&= \frac{\sigma^2}{2} \left[(k+1)^{2H} + (k-1)^{2H} - 2k^{2H} \right].
\end{aligned}$$

And the correlation between X_t and X_{t+k} $(k > 0)$ is equal to,

$$\rho(k) = \frac{1}{2}\left[(k+1)^{2H} + (k-1)^{2H} - 2k^{2H}\right].$$

We will take,

$$\rho(k) = \frac{1}{2}k^{2H} g\left(\frac{1}{k}\right),$$

where,

$$g(x) = (1+x)^{2H} + (1-x)^{2H} - 2.$$

The asymptotic behaviour of $\rho(k)$ then follows by Taylor expansion. Beran (1994) [19] noted if $0 < H < 1$ $(H \neq \frac{1}{2})$, the first non-zero term in the Taylor expansion of $g(x)$, expanded at the origin, is equal to $2H(2H-1)x^2$. Therefore,

$$\rho(k) \sim H(2H-1)k^{2H-2}, \; k \to \infty,$$

which gives us the following three cases:

1. For $0 < H < \frac{1}{2}$, the correlations are summable. In fact,

$$\sum_{k=-\infty}^{\infty} \rho(k) = 0.$$

2. For $H = \frac{1}{2}$, all correlations at non-zero lags are zero. So $X_t, t = 1, 2, \ldots$ are uncorrelated.

3. For $\frac{1}{2} < H < 1$, this means that the correlations decay to zero so slowly that,

$$\sum_{k=-\infty}^{\infty} \rho(k) = \infty.$$

   Then $X_t, t = 1, 2, \ldots$ has long-range dependence (long memory, strong memory or strong dependence).

A variety methods to empirically estimate Hurst parameter H are discussed in Beran (1994) [19], with a simulation study to statistically compare the different methods in Taqqu et al (1995) [121]. Here we will describe just a couple of these approaches.

Method I: The variance plot

The motivation behind this method is found by looking at the returns $X_1, X_2, \ldots, X_n$ more closely. In a typical sample path of the returns; there are relatively long periods where the observations tend to stay at a high level, (on the other hand, there are long periods with low levels,) there seem to be no apparent persisting trend or cycle, and the overall series looks stationary. In addition, we can also observe that the variance of the sample mean decays to zero at a slower rate than n^{-1}. In good approximation, the rate is proportional to $n^{-\alpha}$ for some $0 < \alpha < 1$.

If we let X_t be a stationary process with long-range dependence, then from Beran (1994) [19] we have,

$$\lim_{n\to\infty} \frac{Var\sum_{i=1}^{n} X_i}{cn^{2H}} = \frac{1}{H(2H-1)}$$

So,

$$Var(\bar{X}_n) \approx cn^{2H-2},$$

where $c > 0$. This suggests the following method for estimating H:

1. Let k be an integer. For different integers k in the range $2 \le k \le \frac{n}{2}$, and a sufficient number (say m_k) of subseries of length k, calculate the sample means $\bar{X}_1(k), \bar{X}_2(k), \ldots, \bar{X}_{m_k}(k)$ and the overall mean
$$\bar{X}(k) = \frac{1}{m_k}\sum_{j=1}^{m_k} \bar{X}_j(k).$$
2. For each k, calculate the sample variance of the sample means $\bar{X}_j(k)$ $(j = 1, \ldots, m_k)$
$$s^2(k) = \frac{1}{(m_k - 1)}\sum_{k=1}^{m_k}(\bar{X}_j(k) - \bar{X}(k))^2.$$
3. Plot $\log s^2(k)$ against $\log k$.

For large values of k, the points in this plot are expected to be scattered around a straight line with negative slope $2H - 2$. In the case of short-range dependence or independence, the ultimate slope is -1.

Method II: Periodogram

In order to investigate the asymptotic behaviour of the periodogram for long-memory time series, we require the assumptions that; the second moments are finite and $\lim_{k\to\infty} \rho(k) = 0$,

	Hurst parameter $\hat{H}$	
	Variance plot	Periodogram
FTSE100	0.5484	0.5541
GBP	0.5087	0.5137

Table A.1: Estimates of Hurst paramter

and hence $0 < H < 1$. Under these criteria, the spectral density of the increment process X_i can be derived.

The spectral density of X_i is given by,

$$f(\lambda) = 2c_f(1 - \cos\lambda) \sum_{j=-\infty}^{\infty} |2\pi j + \lambda|^{-2H-1} \quad \lambda \in [-\pi, \pi]$$

with $c_f = c_f(H, \sigma^2) = \sigma^2(2\pi)^{-1}\sin(\pi H)\Gamma(2H+1)$ and $\sigma^2 = Var(X_i)$.

The behaviour of f near the origin follows by Taylor expansion at zero. Under the assumptions we outlined above,

$$f(\lambda) = c_f|\lambda|^{1-2H} + o(|\lambda|^{min(3-2H,2)}).$$

The approximation of f by $c_f|\lambda|^{1-2H}$ is in fact very good in practice, even for relatively large frequencies.

The periodogram $I(\lambda)$ is the empirical counterpart of the spectral density $f(\lambda)$, which gives,

$$I(\lambda_j) = \frac{1}{2\pi n}|\sum_{t=1}^{n}(X_t - \bar{X}_n)e^{it\lambda_j}|^2 = \frac{1}{2\pi}\sum_{k=-(n-1)}^{n-1}\hat{R}(k)e^{ik\lambda_j},$$

with Fourier frequencies $\lambda_j = \frac{2\pi j}{n}$ $(j = 1, \ldots, \frac{n}{2} - \frac{1}{2})$.

If the correlations were summable, then near the origin the periodogram should be scattered randomly around a constant. Instead, for dependence, the points are scattered around a negative slope.

These methods will give a decent estimate for Hurst parameter H, and thus, a rough idea about whether there is long-range dependence in the data (see table A.1). We find $\frac{1}{2} < H < 1$ for FTSE100 and GBP data, but only slightly. With such small departures from $H = \frac{1}{2}$, we find it rather difficult to distinguish between weak and strong memory, even for rather large sample sizes.

A.3 Appendix C: Modified Bessel function of the third kind

In this appendix, a number of results concerning the modified Bessel function of the third kind or McDonalds function are collected (see Kotz et al (2001) [80]).

The modified Bessel function of the third kind, with index $\lambda \in \mathbb{R}$, can be defined by the integral representation,

$$K_\lambda(x) = \frac{1}{2}\int_0^\infty u^{\lambda-1}e^{\frac{1}{2}x(u-\frac{1}{u})}du, \ x > 0.$$

The function $K_\lambda(x)$ is a continuous, positive function of $\lambda \geq 0$ and $x > 0$. If $\lambda \geq 0$ is fixed, then for x in the interval $(0, \infty)$, the function $K_\lambda(x)$ is positive and decreasing.

If λ is fixed then, as $x \to 0+$,

$$K_\lambda(x) \sim \Gamma(\lambda)2x, \ \lambda > 0, \qquad K_0(x) \sim \log(\frac{1}{x}).$$

For $\lambda = r + \frac{1}{2}$, where r is a nonnegative integer, the function $K_\lambda(x)$ has the closed form,

$$K_{r+\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2x}}e^{-x}\sum_{k=0}^{r}\frac{(r+k)!}{(r-k)!k!}(2x)^k.$$

A.4 Appendix D: Rosenblatt Processes

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $\xi(t) = \xi(\omega, t) : \Omega \mathrm{x} \mathbb{R} \to \mathbb{R}$ be a random process in continuous time.

We first list the following relevant assumptions stated in Anh and Leonenko (2001) [3]:

A1 The process $\xi(t)$, $t \in \mathbb{R}$, is a real measurable mean-square continuous stationary Gaussian process with mean $E\xi(t) = 0$ and covariance function $r(t) = r(|t|) = Cov(\xi(0), \xi(t)),\ t \in \mathbb{R}$, such that $r(0) = 1$.

A2 The covariance function $r(t),\ t \in \mathbb{R}$, is of the form

$$r(t) = \frac{L(|t|)}{|t|^\alpha}, 0 < \alpha < 1, \tag{A.1}$$

where $L(t) : (0, \infty) \to (0, \infty)$ is bounded on each finite interval and slowly varying for large values of t; i.e. for each $\lambda > 0$, $\lim_{t \in \infty}[L(\lambda t)/L(t)] = 1$.

A3 A non-random Borel function $G : \mathbb{R} \to \mathbb{R}$ is defined such that

$$\int_{-\infty}^{\infty} G^2(u)\varphi(u)\, du < \infty$$

with

$$\varphi(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}, \; u \in \mathbb{R}.$$

The nonlinear function $G(u)$, $u \in \mathbb{R}$ can then be expanded in the series

$$G(u) = \sum_{k=0}^{\infty} C_k H_k(u)/k!, \; C_k = \int_{-\infty}^{\infty} G(u) H_k(u) \varphi(u)\, du, \; k = 0, 1, 2, \ldots \tag{A.2}$$

of orthogonal Chebyshev-Hermite polynomials

$$H_k(u) = (-1)^k e^{\frac{u^2}{2}} \frac{d^k}{du^k} e^{-\frac{u^2}{2}}, \; k = 0, 1, 2, \ldots$$

which form a complete orthogonal system in Hilbert space $L^2(\mathbb{R}, \varphi(u)\, du)$.

A4 There exists an integer $m \geq 1$ such that $C_1 = \ldots = C_{m-1} = 0, C_m \neq 0$. The integer $m \geq 1$ will be called the Hermitian rank of G.

We state the following non-central limit theorem due to Taqqu (1975) [119](see also Rosenblatt (1987) [111]):

Theorem 13. *Under assumptions **A1-A4** with $\alpha \in (0, 1/m)$, where $m \geq 1$ is the Hermitian rank of the function G, the finite-dimensional distributions of the random processes*

$$Y_n(t) = \frac{1}{d(n)} \int_0^{nt} [G(\xi(s)) - C_0]\, ds, \; t \geq 0, \tag{A.3}$$

with

$$d(n) = n^{1-\alpha m/2} L^{m/2}(n),$$

converge weakly, as $n \to \infty$, to the finite dimensional distributions of the random process

$$Y_m(t) = \frac{C_m}{m!} [c_1(\alpha)]^{m/2} \int_{\mathbb{R}^m}' \frac{e^{i(\lambda_1 + \ldots + \lambda_m)t} - 1}{i(\lambda_1 + \ldots + \lambda_m)} \frac{W(d\lambda_1) \ldots W(d\lambda_m)}{|\lambda_1 \ldots \lambda_m|^{(1-\alpha)/2}}, \; t \geq 0, \tag{A.4}$$

where C_0 and C_m are defined by (A.2) *and $\int_{\mathbb{R}^m}' \ldots$ is a multiple stochastic integral with respect to complex Gaussian white noise $W(\cdot)$ with integration on the hyperplanes $\lambda_i = \lambda_j$, $i, j = 1, \ldots, m$, $i \neq j$, being excluded (see Taqqu (1979) [120], Dobrushin and Major (1979) [42] and Major (1981) [90] for the definition and properties of the multiple stochastic integral* (A.4)*).*

For a random process in continuous time, the proof of Theorem 4 may be constructed from Taqqu (1979) [120] and Dobrushin and Major (1979) [42] by using the argument of Berman (1979) [20].

Remark The normalising factor $d(n)$ in (A.3) is chosen such that, as $n \to \infty$,

$$Var[\int_0^n H_m(\xi(s))\, ds] = d^2(n) m! c_2(m,\alpha)(1+o(1)),$$

where

$$c_2(m,\alpha) = \int_0^1 \int_0^1 \frac{dsdt}{|s-t|^{m\alpha}} = \frac{2}{(1-m\alpha)(2-m\alpha)}, \; 0 < \alpha < 1/m.$$

Note that $E|Y_m(t)|^2 < \infty$, but for $m \geq 2$ the process $Y_m(t)$ have non-Gaussian structure. The process $Y_2(t)$, $t \geq 0$, defined in (A.4) with $m = 2$, is called the Rosenblatt process (see Taqqu (1975) [119]) because it first appeared in Rosenblatt (1961) [110]. Some moment properties of these distributions can be found in Taqqu (1975) [119] and Taqqu and Goldberg (1982) [59]. In particular, the marginal distribution of the random process,

$$R(t) = Y_2(t) = \frac{C_2}{2} c_1(\alpha) \int_{\mathbb{R}^2}' \frac{e^{i(\lambda_1+\lambda_2)t} - 1}{i(\lambda_1+\lambda_2)} \frac{W(d\lambda_1)W(d\lambda_2)}{|\lambda_1\lambda_2|^{(1-\alpha)/2}}, \; t \geq 0, \; 0 < \alpha < 1/2$$

is called the Rosenblatt distribution. Note that,

$$ER^2(1) = [\frac{C_2}{2} c_1(\alpha)]^2 \int_{\mathbb{R}^2} |\frac{e^{i(\lambda_1+\lambda_2)} - 1}{i(\lambda_1+\lambda_2)}|^2 \frac{d\lambda_1 d\lambda_2}{|\lambda_1\lambda_2|^{1-\alpha}} < \infty, \; 0 < \alpha < 1/2.$$

From Rosenblatt (1961) [110], Taqqu (1975) [119] and Berman (1979) [20], we obtain the characteristic function of the random variable,

$$\bar{R} = R(1)/[C_2 c_1(\alpha)/2].$$

It has the form,

$$E\exp\{iu\bar{R}\} = \exp\{\frac{1}{2}\sum_{j=2}^{\infty} \frac{(2iu)^j}{j} \aleph_j\}, \; u \in \mathbb{R},$$

where,

$$\aleph_j = \int_{[0,1]^j} \frac{dx_1 \ldots dx_j}{\prod_{k=2}^j |x_{k-1}-x_k|^\alpha |x_j - x_1|^\alpha}, \; 0 < \alpha < 1/2.$$

From Leonenko and Taufer (2005) [85], we obtain the extension to the joint characteristic function of $Y_n(u_1), \ldots, Y_n(u_q)$, q is an integer, as $n \to \infty$ and for $0 < \alpha < \frac{1}{2}$ is,

$$E\exp\{i(u_1\bar{R}(t_1) + \ldots + u_q\bar{R}(t_q))\} = \exp\{\frac{1}{2}\sum_{j=2}^{\infty} \frac{(2i)^j}{j} \sum_{s_1,\ldots,s_j \in \{1,\ldots,q\}} u_{s_1} \ldots u_{s_q} S_\alpha(j)\},$$

where,

$$S_\alpha(j) = \int_0^{u_{s_1}} \cdots \int_0^{u_{s_j}} \frac{dx_1 \ldots dx_j}{\prod_{k=2}^j |x_{k-1}-x_k|^\alpha |x_j - x_1|^\alpha}.$$

This is a generalization of a result of Taqqu (1975) [119] (see Fox and Taqqu (1985) [54] for a corrected version).

A.5 Appendix E: Mixing definitions

Denote $\alpha^{(k)}(t)$ (strong mixing), $\beta^{(k)}(t)$ (sometimes written as $\phi^{(k)}(t)$) and $\rho^{(k)}(t)$ (see Bradley (2005) [28]) as the following:

$$\alpha^{(k)}(t) = \sup |P(A \cap B) - P(A)P(B)|, \ A \in \mathcal{F}_s, B \in \mathcal{F}^{s+t}, \tag{A.5}$$

$$\beta^{(k)}(t) = \sup |P(B/A) - P(B)|, \ A \in \mathcal{F}_s, B \in \mathcal{F}^{s+t},$$

$$\rho^{(k)}(t) = \sup |Corr(\xi, \eta)|, \ \xi \in L_2(\mathcal{F}_s), \eta \in L_2(\mathcal{F}^{s+t}),$$

where $\mathcal{F}_s = \sigma(\xi^{(k)}(s), s \leq t), \ \mathcal{F}^t = \sigma(\xi^{(k)}(s), s \geq t), \ t \geq 0.$

A.6 Appendix F: Duality Theorem

Theorem 14. *If the real-valued characteristic function φ of random variable satisfies,*

$$\varphi(t) \geq 0, \ \int_{\mathbb{R}} \varphi(t)dt < \infty,$$

(and so the random variable has bounded continuous density f that is symmetrical about 0), then $f(0) > 0$ and,

$$\frac{\varphi(x)}{2\pi f(0)}, \ x \in \mathbb{R},$$

is a probability density function of the random variable whose characteristic function is,

$$\frac{f(t)}{f(0)}, \ t \in \mathbb{R}.$$

For the proof and historical remarks on its origin, see Harrar, Seneta and Gupta (2006) [61].

A.7 Appendix G: New Expression for Bessel Function

We want to equate (4.21) and (4.22), namely,

$$\frac{K_{\frac{\nu}{2}}(\delta|t|)(\delta|t|)^{\frac{\nu}{2}} 2^{1-\frac{\nu}{2}}}{\Gamma(\frac{\nu}{2})} = \frac{(2\delta)^{\nu}}{\Gamma(\nu)} \int_0^{\infty} e^{-\delta(2x+|t|)}(x(x+|t|))^{\frac{\nu-1}{2}}\, dx$$

$$K_{\frac{\nu}{2}}(\delta|t|) = \frac{\Gamma(\frac{\nu}{2})(2\delta)^{\nu}}{\Gamma(\nu)2^{1-\frac{\nu}{2}}(\delta|t|)^{\frac{\nu}{2}}} \int_0^{\infty} e^{-\delta(2x+|t|)}(x(x+|t|))^{\frac{\nu-1}{2}}\, dx$$

Letting $\mu = \frac{\nu}{2}$, and $z = \delta|t|$

$$K_\mu(z) = \frac{\Gamma(\mu)(2\delta)^{2\mu}}{\Gamma(2\mu)2^{1-\mu}z^\mu}\int_0^\infty e^{-2\delta x - z}(\frac{\delta x(\delta x + z)}{\delta^2})^{\mu-\frac{1}{2}}\frac{d(\delta x)}{\delta}$$

Now set $u = \delta x$

$$\begin{aligned} K_\mu(z) &= \frac{\Gamma(\mu)(2\delta)^{2\mu}}{\delta\Gamma(2\mu)2^{1-\mu}z^\mu}\int_0^\infty e^{-2u-|z|}\frac{(u(u+z))^{\mu-\frac{1}{2}}}{\delta^{2(\mu-\frac{1}{2})}}du \\ &= \frac{\Gamma(\mu)2^{2\mu}}{\Gamma(2\mu)2^{1-\mu}}\frac{1}{z^\mu}\int_0^\infty e^{-2u-|z|}(u(u+z))^{\mu-\frac{1}{2}}du \\ &= \frac{2^{3\mu-1}\Gamma(\mu)}{\Gamma(2\mu)}\frac{e^{-|z|}}{z^\mu}\int_0^\infty e^{-2u}(u(u+z))^{\mu-\frac{1}{2}}du \text{ }, z \neq 0 \end{aligned}$$

A better expression would be

$$K_\mu(z)|z|^\mu = \frac{2^{3\mu-1}\Gamma(\mu)e^{-|z|}}{\Gamma(2\mu)}\int_0^\infty e^{-2u}(u(u+z))^{\mu-\frac{1}{2}}du \text{ }, \mu > 0, z \in \mathbb{R}.$$

A.8 Appendix H: AIC/BIC/SBC criterion

The Akaike information criterion (AIC) introduced in Akaike (1974) [1], is a measure of the relative goodness of fit of a statistical model. It is grounded in the concept of information entropy, in effect offering a relative measure of the information lost when a given model is used to describe reality. It can be said to describe the trade-off between bias and variance in model construction, or loosely speaking between accuracy and complexity of the model. Given the parametric fitting of the Rényi functions in *Figures 6.3, 6.4, and 6.5*, we will now look to provide a comparison of the Log-Gamma, Log-Inverse Gaussian and Log-Spectrally negative α-stable multifractal scenarios.

The values for the AIC will be calculated by,

$$AIC = N\log\left(\frac{RSS}{N}\right) + 2k,$$

where k is the number of parameters needed to fit the data and RSS is the residual sum of squares.

One benefit of this citerion is that AIC not only rewards goodness of fit, but also includes a penalty that is an increasing function of the number of estimated parameters. This penalty discourages overfitting (increasing the number of free parameters in the model improves the goodness of fit). Any results, however, should only be treated formally since the AIC is based on the maximum likelihood function, and we have clear evidence of dependence in our data.

For further testing, we will use both the Bayesian information criterion (BIC) and Schwarz's Bayesian criterion (SBC). These are also based, in part, on the likelihood function, and it is closely related to AIC. Note that the penalty terms for overfitting are larger for BIC and SBC than in AIC.

		Scenario		
		Log-Γ	Log-IG	Log-SNS
FTSE100	AIC	-83.4962	-83.7514	-90.0932
	BIC	-81.6610	-81.9191	-88.1833
	SBC	-81.1784	-81.4336	-87.0029
GBP	AIC	-98.6057	-96.9892	-99.8854
	BIC	-99.6611	-97.9342	-101.8866
	SBC	-96.2879	-94.6715	-96.7650

Table A.2: Results after applying the AIC, BIC and SBC criterions (the lower the value, the better the fit)

When comparing two estimated models, the model with the lower value of AIC/BIC/SBC is the one to be preferred. The criterions are increasing functions of the error variance and increasing functions of the number of paramters k. That is, unexplained variation in the dependent variable and the number of explanatory variables increase the value of AIC/BIC/SBC. Hence, lower values implies either fewer explanatory variables, better fit, or both. In table A.2 we see that Log-SNS gives us the lowest value for each of the criterions.

(a) AD

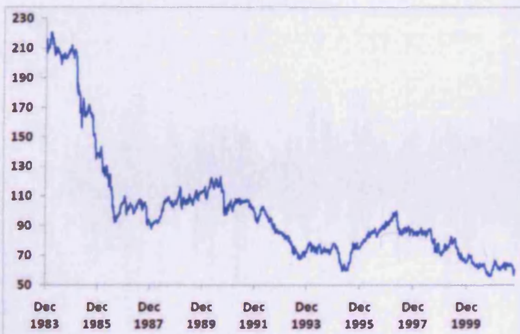
(b) CAC40

(c) CD

(d) DM

(e) EUR

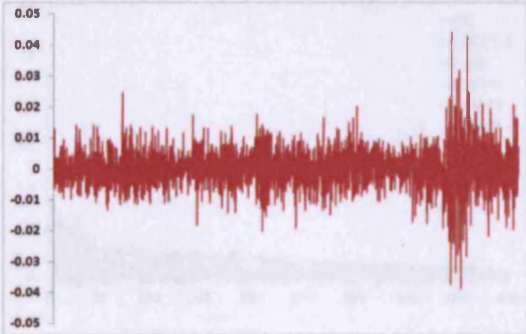
(f) FF



(g) JY

(h) NTD

Figure A.1: Risky asset price $P(t)$



(a) AD

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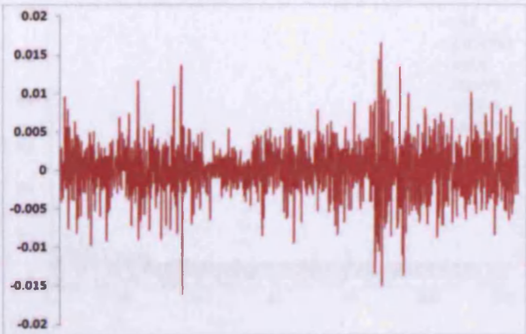

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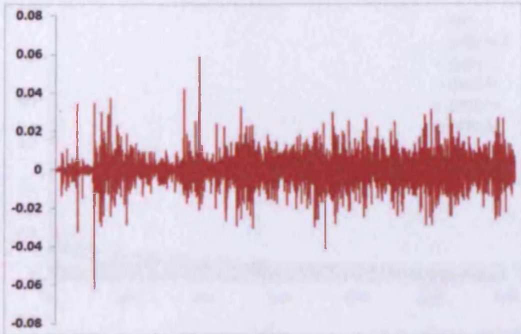

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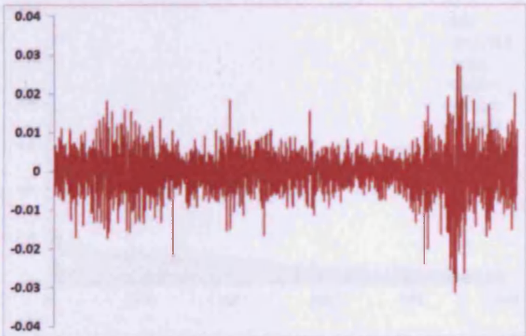

(e) EUR

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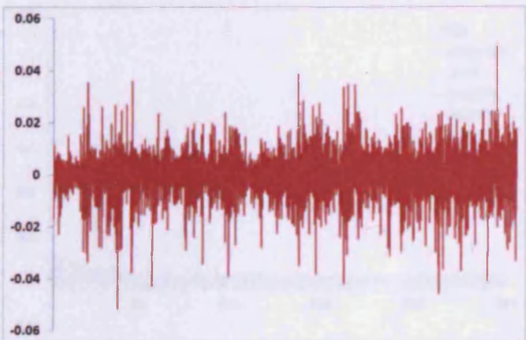

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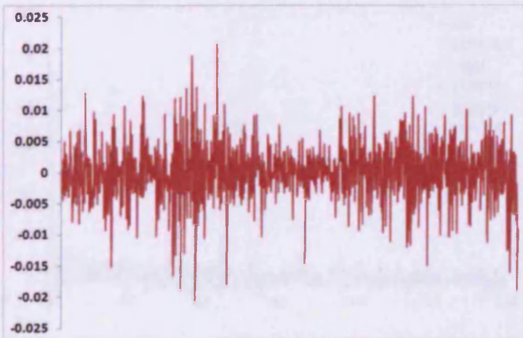
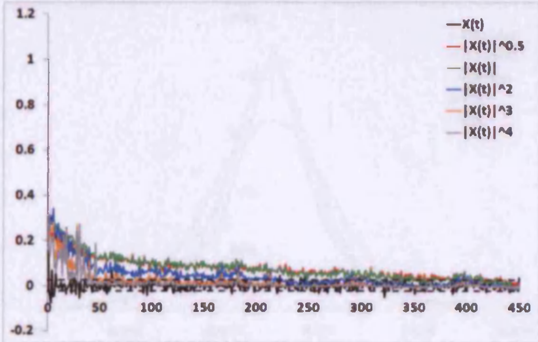
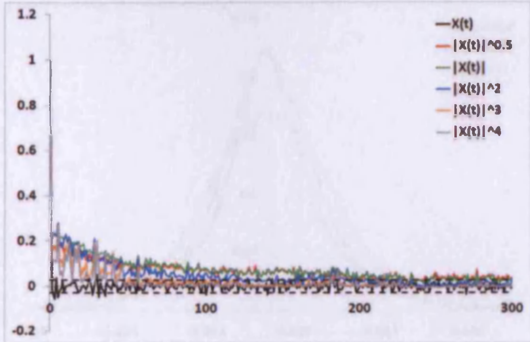


(h) NTD

Figure A.2: Risky asset returns $X(t)$



(a) AD



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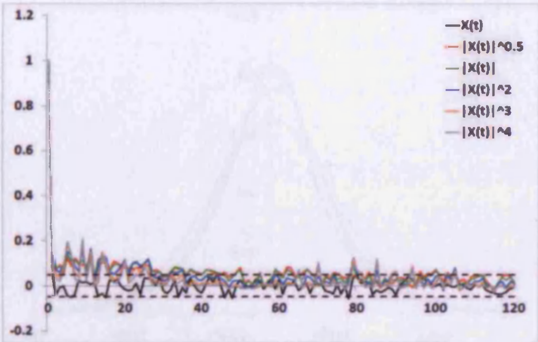

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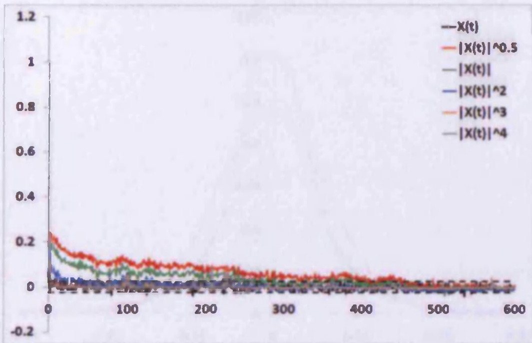

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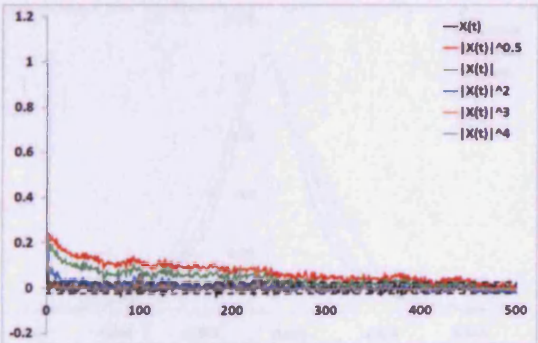

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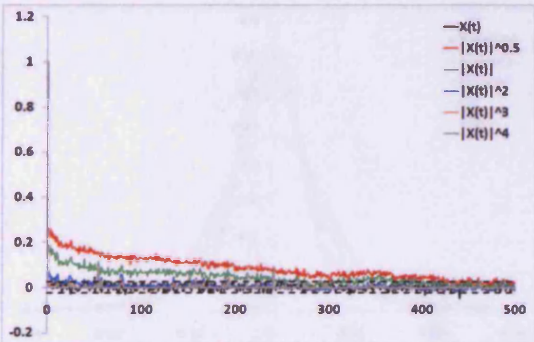

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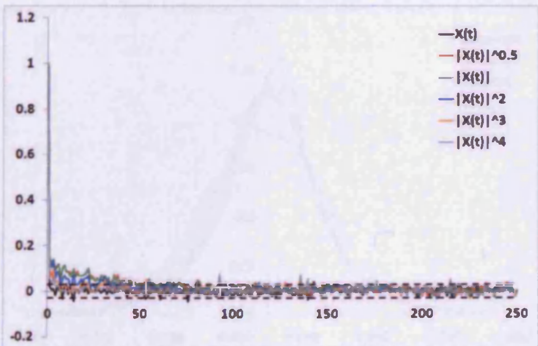

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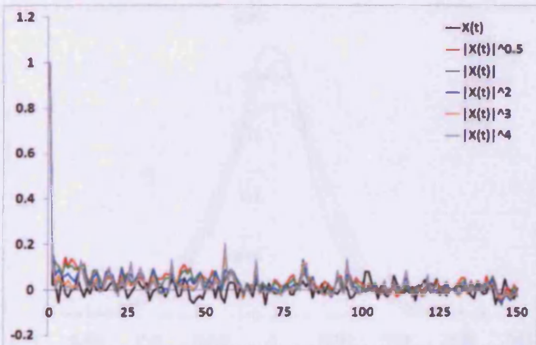
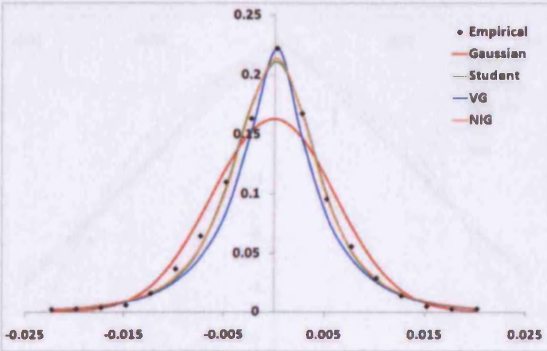
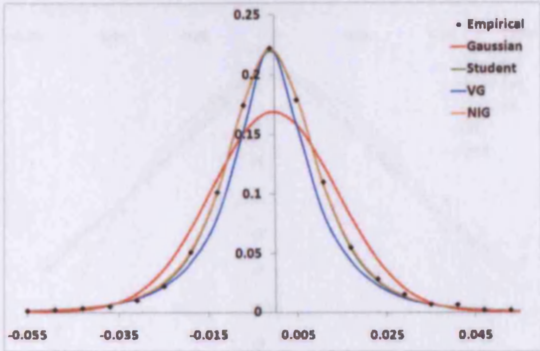


(h) NTD

Figure A.3: Empirical autocorrelation of X and $|X|^d$ for $d = \frac{1}{2}, 1, 2, 3, 4$



(a) AD



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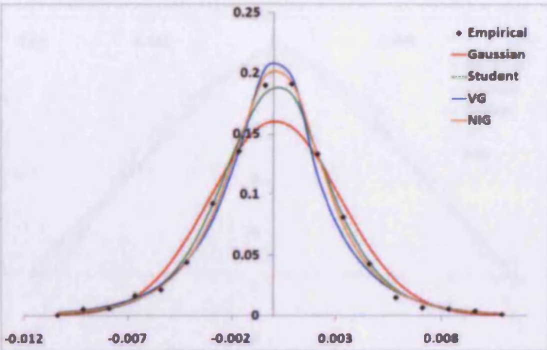

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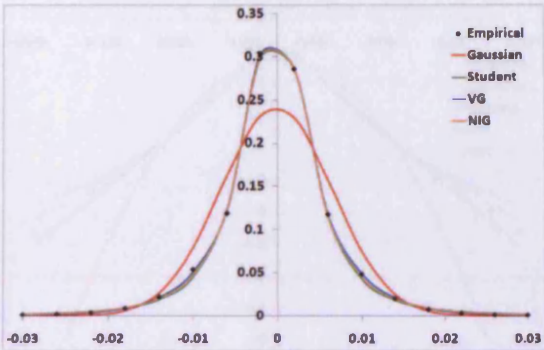

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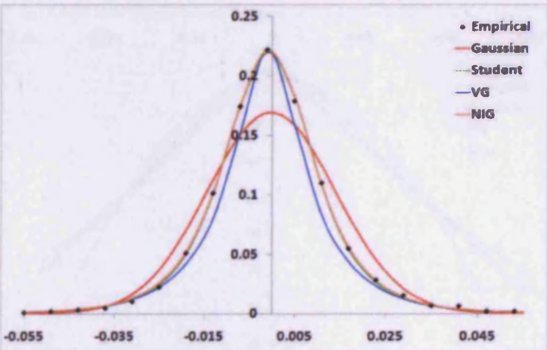

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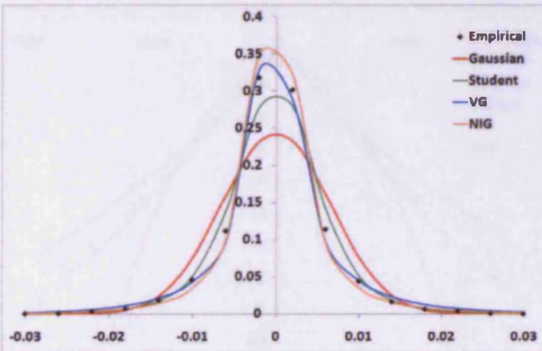

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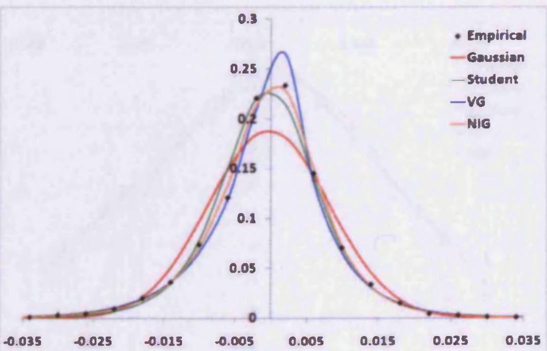

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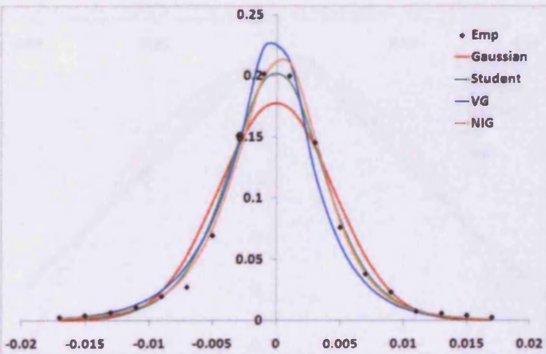
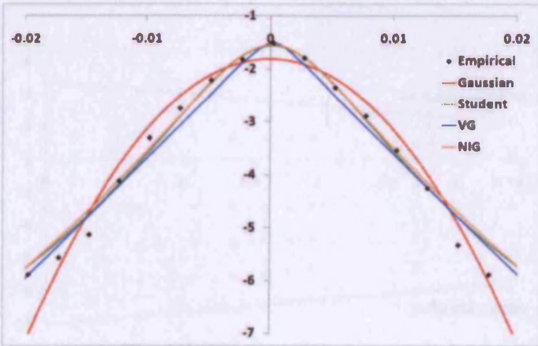
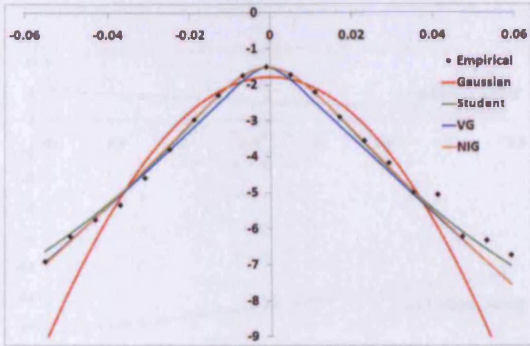


(h) NTD

Figure A.4: Empirical density of X, Gaussian, Student, VG and NIG densities



(a) AD



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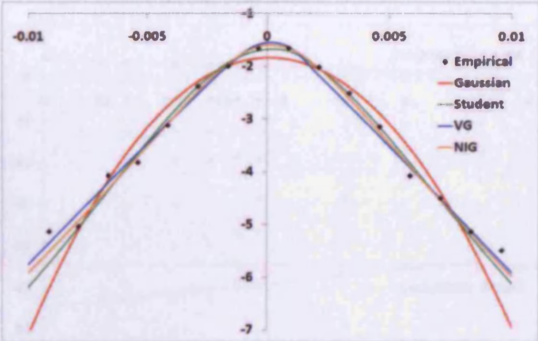

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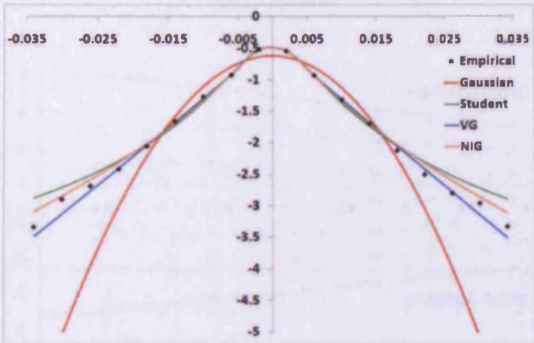

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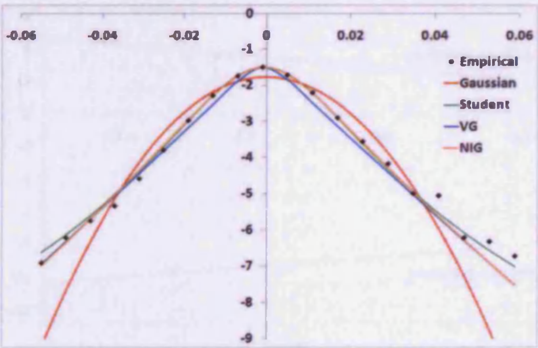

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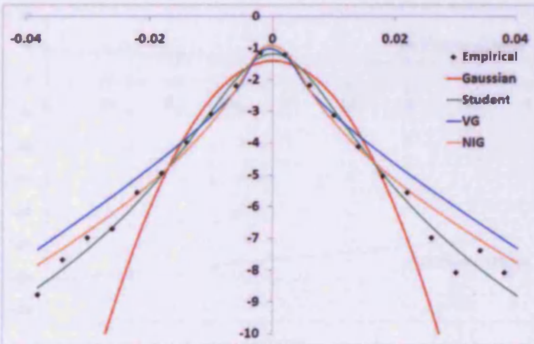
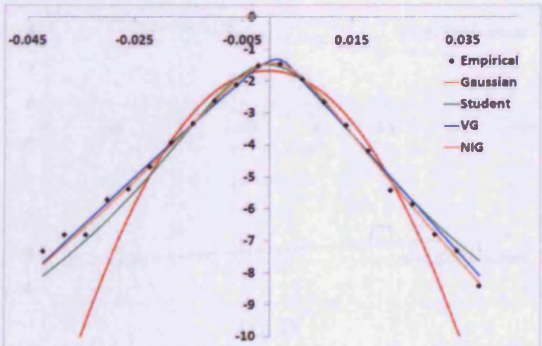
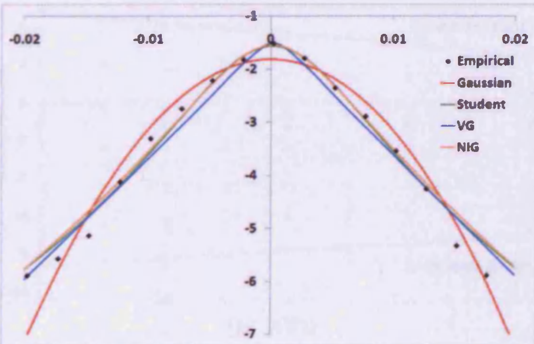


(f) FF

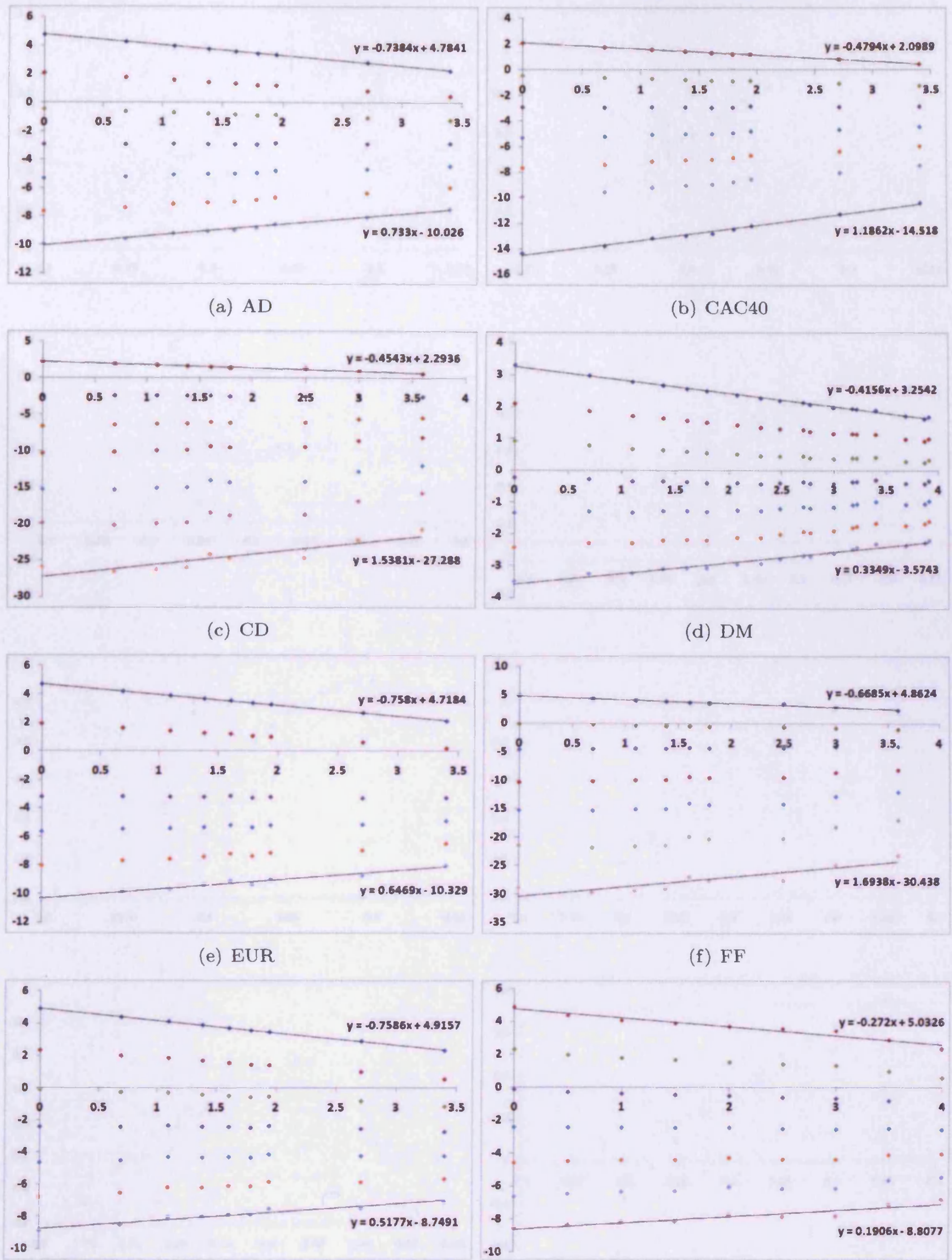


(g) JY



(h) NTD

Figure A.5: Logarithm of empirical density of X, Gaussian, Student, VG and NIG densities



(a) AD

(b) CAC40

(c) CD

(d) DM

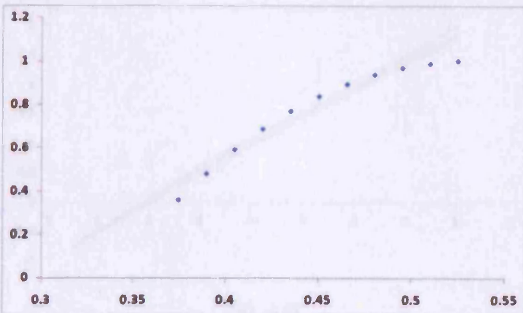
(e) EUR

(f) FF

(g) JY

(h) NTD

Figure A.6: The partition function



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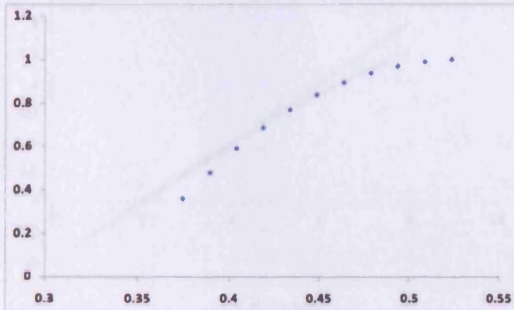

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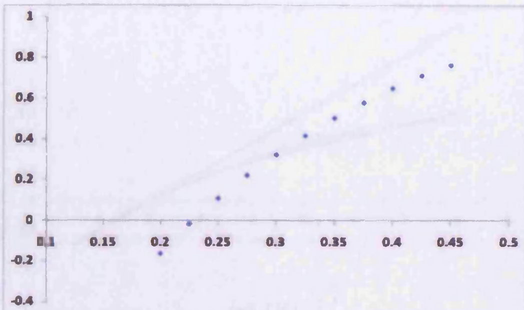

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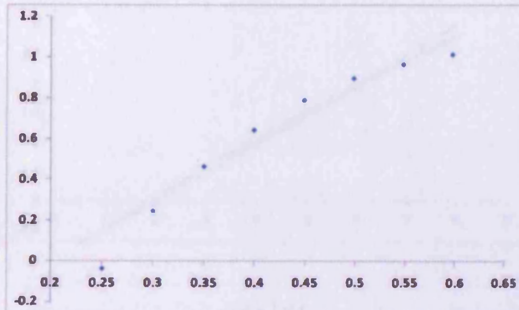

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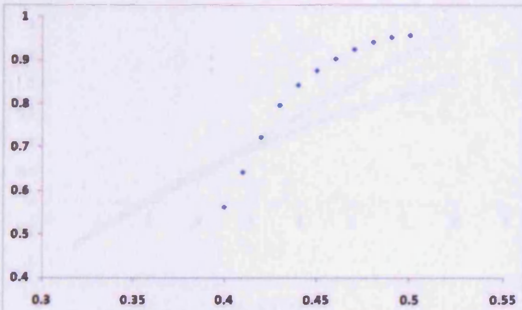

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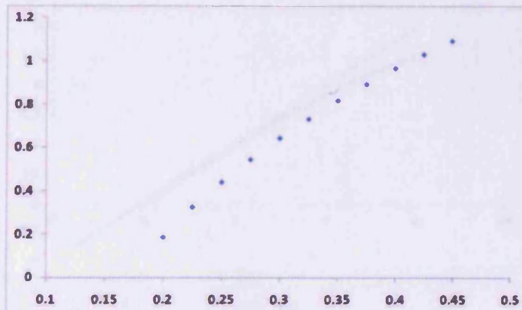

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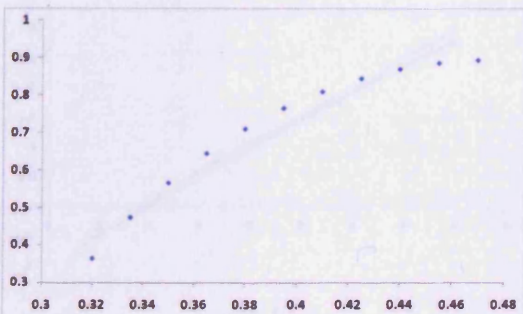

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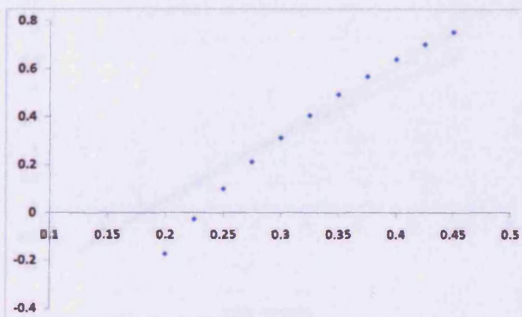
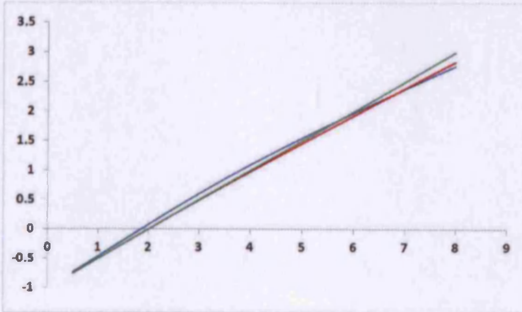


(h) NTD

Figure A.7: Estimation of the multifractal spectrum



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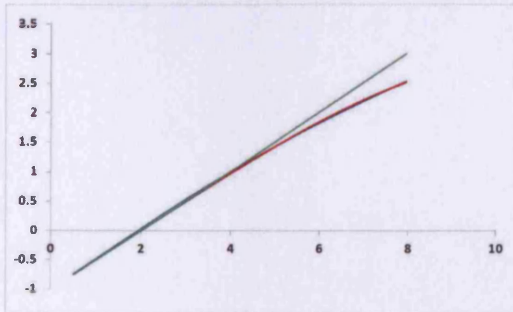

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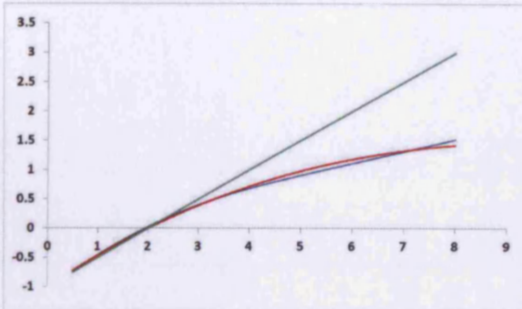

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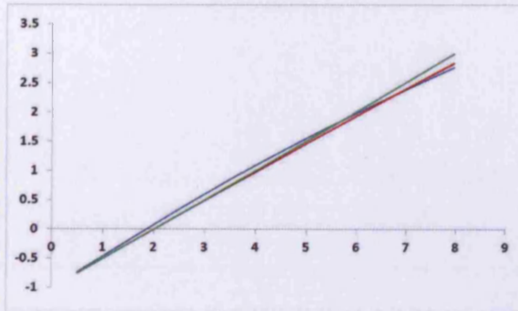

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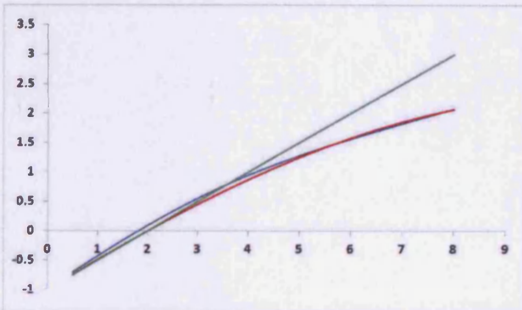

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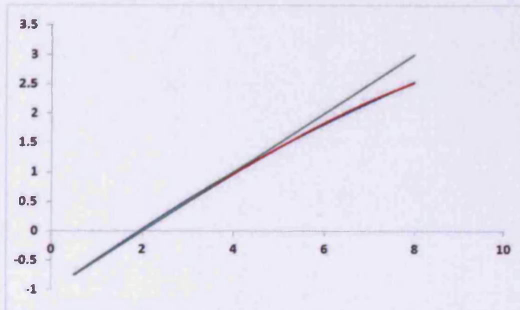

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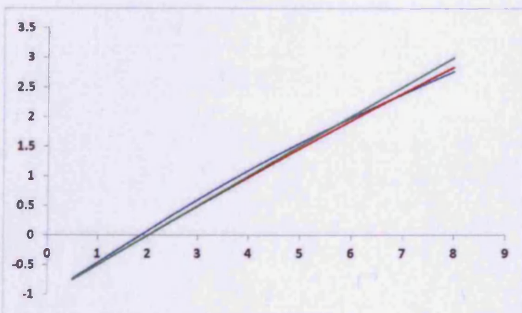

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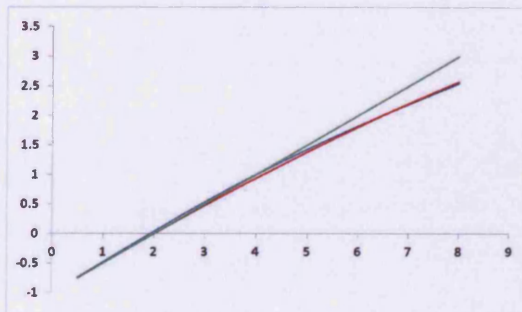
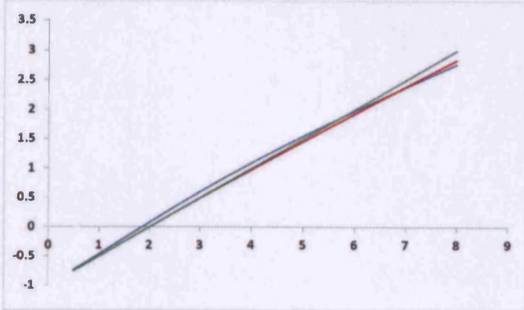
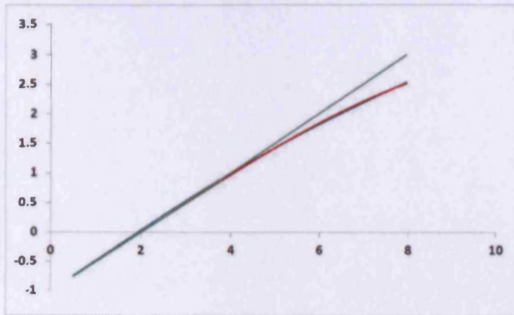


(h) NTD

Figure A.8: Log-gamma scenario of multifractal products of geometric OU-type processes: Blue (line)- non-parametric estimate of $\tau(q)$, Red- fitted parametric estimate of $\tau(q)$, Green-Brownian motion case



(a) AD



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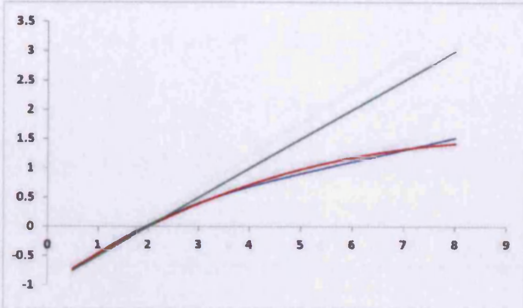

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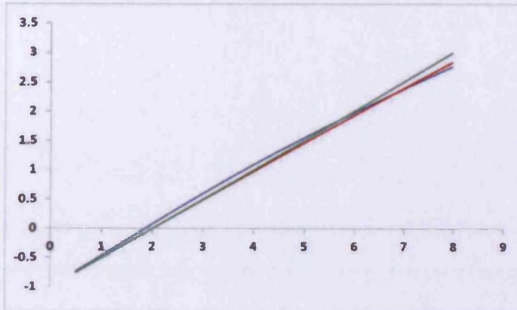

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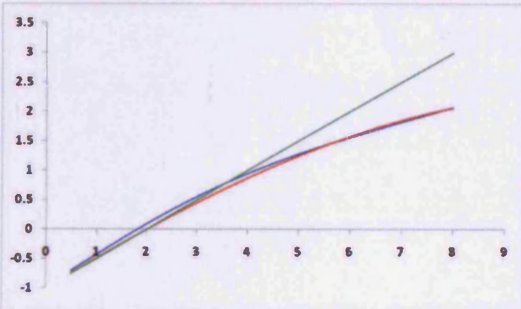

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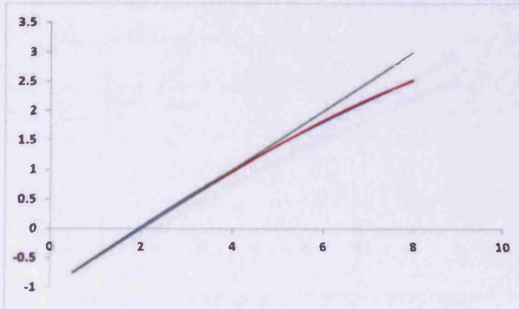

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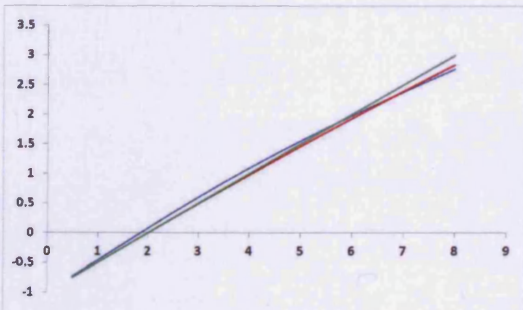

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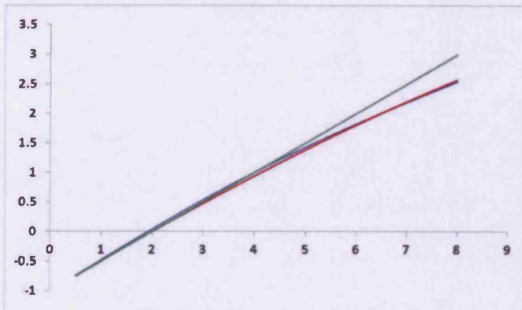
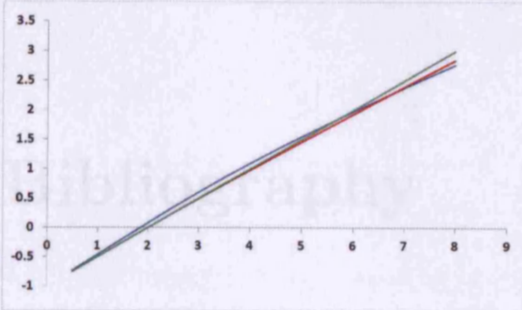
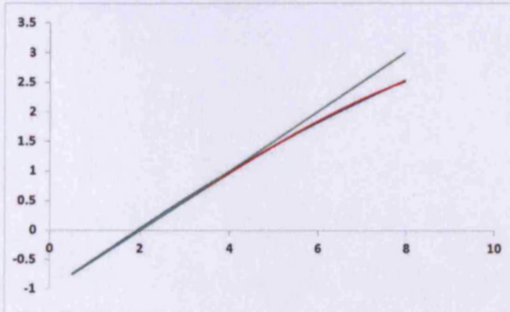


(h) NTD

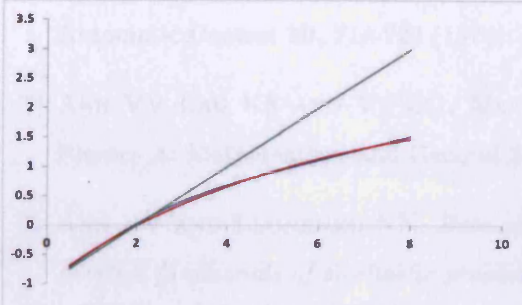
Figure A.9: Log-inverse Gaussian scenario of multifractal products of geometric OU-type processes: Blue (line)- non-parametric estimate of $\tau(q)$, Red- fitted parametric estimate of $\tau(q)$, Green- Brownian motion case



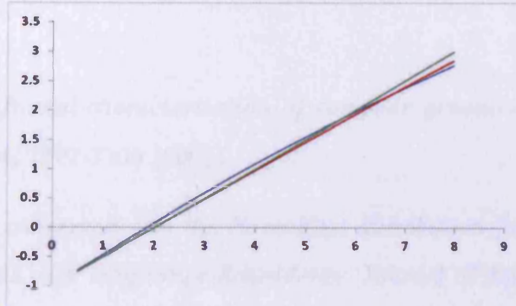
(a) AD



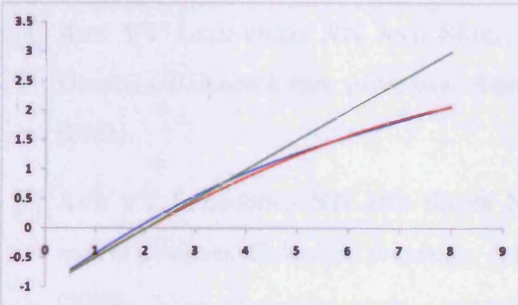
(b) CAC40



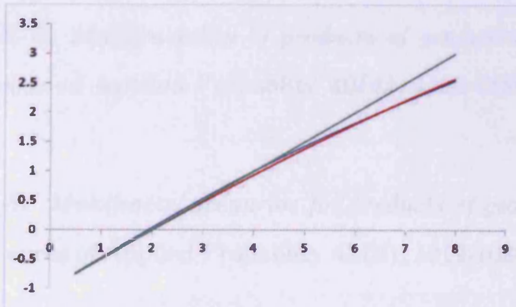
(c) CD



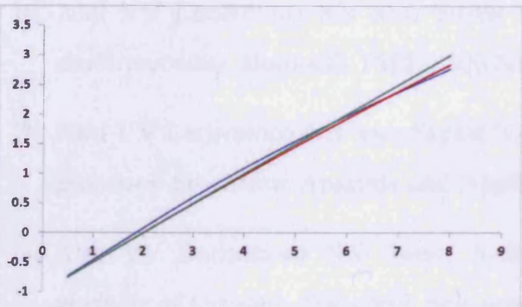
(d) DM



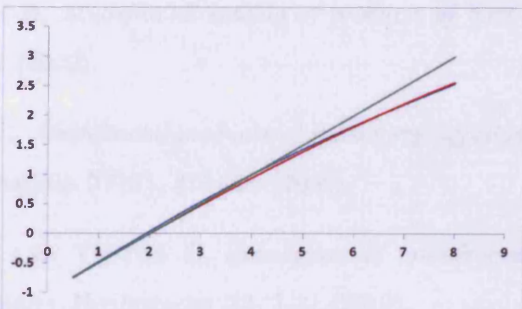
(e) EUR



(f) FF



(g) JY



(h) NTD

Figure A.10: Log-spectrally negative α-stable scenario of multifractal products of geometric OU-type processes: Blue (line)- non-parametric estimate of $\tau(q)$, Red- fitted parametric estimate of $\tau(q)$, Green- Brownian motion case

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