Transient solution of the M/Ek/1 queueing system

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I would like to dedicate this thesis to my loving parents and my brother,

Leonenko Nikolaj, Olga

and Eugenij.

Thanks for everything!

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SUMMARY OF THESIS

In this thesis, the Erlang queueing model $M/E_k/1$, where customers arrive at random mean rate λ and service times have an Erlang distribution with parameter k and m service rate μ, has been considered from different perspectives. Firstly, an analytic meth of obtaining the time-dependent probabilities, $p_{n,s}(t)$ for the $M/E_k/1$ system have b proposed in terms of a new generalisation of the modified Bessel function when initia there are no customers in the system. Results have been also generalised to the case wh initially there are a customers in the system.

Secondly, a new generalisation of the modified Bessel function and its generating fu tion have been presented with its main properties and relations to other special functi (generalised Wright function and Mittag-Leffler function) have been noted.

Thirdly, the mean waiting time in the queue, $W_q(t)$, has been evaluated, using Lucha results. The double-exponential approximation of computing $W_q(t)$ has been proposed different values of ρ, which gives results within about 1% of the 'exact' values obtained fr numerical solution of the differential-difference equations. The advantage of this approxim tion is that it provides additional information, via its functional form of the characterist of the transient solution.

Fourthly, the inversion of the Laplace transform with the application to the queues h been studied and verified for $M/M/1$ and $M/E_k/1$ models of computing $W_q(t)$.

Finally, an application of the $M/E_2/1$ queue has been provided in the example of hour traffic flow for the Severn Bridge. One of the main reasons for studying queue models from a theoretical point of view is to develop ways of modelling real-life syste The analytic results have been confirmed with the simulation.

CONTENTS

CHAPTER 2. BESSEL FUNCTIONS AND THEIR GENERALISATIONS

CHAPTER 3. ERLANG QUEUEING MODEL

Abstract

This research is concerned with the study of the Erlang service queueing model in the time-dependent case. The novel aspects of the thesis are as follows:

1) The transient probabilities $p_{n,s}(t)$ are obtained in terms of the generalised modified Bessel function of the second type, where $p_{n,s}(t)$ represents the probabilities that there are n customers in the system at time t, with the customer in service being in phase s, $s = 1, 2, ..., k$.

2) Also it presents a new generalisation of the modified Bessel function, its generating function and its relation to other special functions. This function has played an important role in deriving the transient solution of the $M/E_k/1$ queue.

3) A simple method of computing $W_q(t)$ - the mean waiting time of a customer arriving in the queue at time t, based on a double-exponential approximation, is also proposed.

Finally, the research is supported with an application of the $M/E_k/1$ queueing model to a real-life problem.

PUBLICATIONS AND PRESENTATIONS

1. University of Wales Intercollegiate Colloquium in Mathematics in Gregynog, 2004. With Jeff Griffiths and Janet Williams. The Time-Dependent Solution for the $M/E_k/1$ System.

2. Young OR 14 Conference. Bath, April 4-6 April, 2005. Approximation to the Transient Solution of the Erlang Model.

3. Gregynog Mathematical Colloquium 2005, 23-25 May. The Transient behaviour of the Erlang queue.

4. J.D. Griffiths, G.M. Leonenko, J.E. Williams, Approximation to the Transient Solution of the $M/E_k/1$ Queue, Mathematical Methods of Operational Research, 2005, submitted.

5. J.D. Griffiths, G.M. Leonenko, J.E. Williams, Generalization of the Modified Bessel Function and Its Generating Function, Fractional Calculus and Applied Analysis, 2005, accepted.

6. J.D. Griffiths, G.M. Leonenko, J.E. Williams, A Transient behaviour of the Erlang Queue, Operations Research Letters, 2005, accepted.

INTRODUCTION

Queueing theory was developed to provide models to predict behavior of systems that attempt to provide service for randomly arising demands.

Any system in which arrivals place demands upon a finite capacity resource may be termed a queueing system. A queueing system can be described as customers arriving for service, waiting for service, if it is not immediate, and leaving the system after being served. A typical example is the telephone exchange which services callers requesting connection with some distant point. Supermarkets, restaurants, car parks, many aspects of hospital and airport operations, are self-evident demand/supply situations.

A feature of all situations is the increasing delay suffered by customers as the mean demand rate approaches the mean capability of the service to satisfy it. The theory is essentially stochastic; that is to say it considers a stream of demands occurring in a chance-dependent manner, serviced by a mechanism such that the duration of each service is also chance-dependent. The theory provides a description of the consequent chance fluctuations in queue length, a customer's waiting time, the busy period of the service facility, and other features of interest to both users and operators of a service-providing facility. With this knowledge it is possible to change the decision-making process, with respect to waiting lines.

The first person investigate the telephone traffic problem was Danish mathematician A.K. Erlang, who published "The Theory of Probabilities and Telephone Conversations" in 1909. One of the most important developments in queueing theory occurred with the publication of a paper by Kendall (1951) introducing the idea of regeneration points, and demonstrating how this concept could be used to embed a continuous time scale with a chain of points which possess the characteristic Markovian property.

E.C. Molina, F. Pollaczek, A.N. Kolmogorov, A. Khintchine and C.A. Crommelin have undertaken pioneering work in Poisson processes and have developed some of the basic approaches in queueing theory.

More contributions have been developed in different directions in queueing theory by D.R. Lindley on integral equations, N.T.J. Bailey on the study of bulk-service mechanisms, (that is a system where several customers may be served simultaneously), W. Lederman and G.E. Reuter on time-dependent solutions, L. Takacs on waiting time, D.R. Cox on supplementary variables, D.G. Kendall on embedded chains, D.G. Champernowne on the use of random walks, and S. Karlin and J.L. McGregor on birth-death processes.

More recent work has been presented by V.E. Beneš, U.N. Bhat, R.W. Conway, D.P. Gaver, J.D.C. Little, W.L. Maxwell, T.L. Saaty and R. Syski.

Queueing theorists tended initially to develop processes that had random arrivals, but their efforts were seriously hampered by the complex mathematics required for solution.

The transient analysis of Markov chains is much more difficult than the corresponding steady-state analysis of such chains. Due to this difficulty, there are few explicit expressions available even for simple queueing models. Considerable attention has been paid to obtain the transient solution for the $M/M/1$ queueing system. All the methods of solving this system (see Champernowne (1956), Conolly (1956, 1958), Ledermann and Reuter (1954), Parthasarathy (1987), Sharma (1990)) involve the modified Bessel function, $I_k(t)$, resulting from the structure of the appropriate generating function.

One method of extending the $M/M/1$ queueing model is to model the Erlang service distribution by means of a series of identical exponential phases. The Erlang family of probability distributions provides far greater flexibility in modelling real-life patterns than does the exponential.

This thesis is concerned with an examination of such Erlang server queues in steady-state and in time-dependent cases, and the application of the theory to real-life situations.

Chapter 1 considers the $M/M/1$ queueing system. It examines the background to the study of the simple queue and presents results for the steady-state case, and the best known methods of solving the differential-difference equations in the time-dependent case.

Chapter 2 is devoted to a study of the modified Bessel function and its different generalisations. This chapter presents a new generalisation of the modified Bessel function of the second type and its generating function, which is then used in obtaining the transient solution for the Erlang server queue.

Chapter 3 examines the $M/E_k/1$ system in the steady-state and transient cases. In this chapter we present new work and new results for the Erlang system. Also it contains a review of the literature and presents the main results, the transient probabilities for the Erlang queue determined in terms of the generalised modified Bessel function of the second type. The result is supported with various numbers of graphs, tables and comparisons.

Chapter 4 investigates the transient behaviour of one of the main queue characteristics $W_q(t)$-the mean waiting time of a customer arriving in the queue at time t. It presents a novel but simple method of computing $W_q(t)$, based on an double-exponential approximation. Numerical comparisons and graphs are also provided.

Chapter 5 is devoted to the numerical inversion of the Laplace transform, since queueing theory is closely related to Laplace transform theory. Different methods of inversion are presented and one of the main queueing characteristics is evaluated for the $M/M/1$ and $M/E_k/1$ systems.

Finally, in Chapter 6 an application of the Erlang queue is illustrated for a 24-hour traffic flow profile on the Severn Bridge. Analytic results are compared with simulations.

GLOSSARY OF TERMS

Definition of Queueing Systems

The notation employed to classify a queue is based upon that suggested by D.G. Kendall. He proposed the notation

Input Distribution/Service Distribution/ Number of Servers

where particular distributions were indicated by:

M for a negative exponential distribution,

D for a constant time distribution,

E_k for an Erlang distribution with parameter k,

GI for a general independent distribution,

G for a general distribution.

This notation does not allow bulk arrivals or services to be accommodated. Thus, the notation has been extended in the following way:

$M^{[X]}$ indicates bulk input, the arrival stream forms a Poisson process and the actual number of customers in the arriving module is a random variable, X, which may take on any positive integral value less than ∞ with probability c_X.

The notation can be extended to cover six general characteristics. These are as follows:

1) input distribution;

2) service distribution;

3) queue discipline;

4) system capacity;

5) number of service channels;

6) number of service stages.

Queue discipline refers to the manner in which customers are selected for service when a queue has formed. There are five basic queue disciplines: first come, first served (FIFO), last in, first out (LIFO), selection for service in random order independent of the time of arrival at the queue (SIRO), shortest service time (SPT) and general service discipline (GD).

System capacity refers to the physical limitation to the amount of waiting room, so that when the line reaches a certain length, no further customers are allowed to enter until space becomes available by a service completion.

The number of service channels is the number of parallel service stations which can service customers simultaneously.

General Definitions

$p_n(t)$	=	probability that there are n customers in the system at time t
$p_{n,s}(t)$	=	probability of n customers in the system at time t, with the customer in service being in phase s, $s = 1, ..., k$;
$G(y,t)$	=	generating function;
$G^*(y,t)$	=	Laplace transform of a generating function;
$J_n(t)$	=	Bessel function;
$I_n(t)$	=	modified Bessel function;
$I_n^k(t)$	=	generalised modified Bessel function of the first type;
$\tilde{I}_n^k(t)$	=	generalised modified Bessel function of the second type;
$W(t,\rho,\beta)$	=	Wright function;
$\Psi_{p,q}(t)$	=	generalised Wright function;
$E_{\beta,\mu}(t)$	=	Mittag-Leffler function;
$E_{(\beta_1,...,\beta_m)(\mu_1,...,\mu_m)}(t)$	=	multi-index Mittag-Leffler function;
$\tilde{f}$ or f^{ap}	=	approximation to a function f

Definition of Queue Measures

λ	=	mean rate per unit time at which arrival instants occur;
μ	=	mean rate of service time;
$\rho = \lambda/\mu$	=	traffic intensity or utilisation factor;
C_A	=	coefficient of variation for the interarrival times;
C_S	=	coefficient of variation for the service times;
$L(t)$	=	mean number of customers in the system at time t;
$L_q(t)$	=	mean number of customers in the queue at time t;
$W(t)$	=	mean waiting time in the system of a customer arriving in the queue at time t;
$W_q(t)$	=	mean waiting time in the queue of a customer arriving in the queue at time t.

CHAPTER 1: M/M/1 QUEUEING SYSTEM

1.1 Review of Literature for $M/M/1$ Queueing System

Historically, Poisson queues were initially observed to form in telephone systems, where calls originated by a Poisson process and the duration of calls was experimentally verified to have an exponential distribution.

Let us begin with the study of the simplest probabilistic queueing model which can be treated analytically, namely, a single-channel model with exponential interarrival times, exponential service times, and FIFO queue discipline.

A number of methods have been put forward to solve

$$\begin{cases} \frac{dp_n(t)}{dt} = \mu p_{n+1}(t) - (\lambda+\mu)p_n(t) + \lambda p_{n-1}(t), & n = 1,2,3.. \\ \frac{dp_0(t)}{dt} = \mu p_1(t) - \lambda p_0(t), & n = 0, \end{cases}$$

where $p_n(t)$ denotes the probability that there are n customers in the system at time t. These differential-difference equations of the birth-death process play an important role in queueing theory.

The solution for the steady-state probabilities ($p_n = \lim_{n\to\infty} p_n(t)$) will be illustrated in Section 1.2 and it follows from solving the steady-state differential equations using iterative, generating function or operators methods. The transient analysis of $M/M/1$ system is much more difficult than the corresponding steady-state analysis of this system. Several methods have been used to solve this problem since Clarke (1956) gave his time-dependent solution.

The known classical solution through the Laplace transform of the generating function may be found in Saaty (1961) and it takes the form

$$\begin{aligned} p_n(t) &= e^{-(\lambda+\mu)t}\left[\left(\sqrt{\frac{\mu}{\lambda}}\right)^{a-n} I_{n-a}(2\sqrt{\lambda\mu}t) + \left(\sqrt{\frac{\mu}{\lambda}}\right)^{a-n+1} I_{n+a+1}(2\sqrt{\lambda\mu}t) \right. \\ &\quad \left. + \left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n \sum_{k=n+a+2}^{\infty}\left(\sqrt{\frac{\mu}{\lambda}}\right)^k I_k(2\sqrt{\lambda\mu}t)\right], \end{aligned}$$

where a is the initial condition that there are a customers in the system at time $t = 0$.

Conolly (1958) has solved the $M/M/1$ system by applying the Laplace transform to the system of equations rather than to the equation for the generating function, with a customers in the system initially. Champernowne (1956) has solved the same problem using a random-walk method.

An interesting approach has been made by Sharma, Shobna (1984), who have written $p_n(t)$ in the following form using a two dimensional state model (n, k) representing the number of arrivals to and departures from the system at a given time.

$$p_n(t) = (1-\rho)\rho^n + e^{-(\lambda+\mu)t}\rho^n \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} \sum_{m=0}^{n+k} (k-m) \frac{(\mu t)^{m-1}}{m!}$$

with $p_0(0) = 1$ and λ, μ are the expected number of arrivals per unit time and expected rate of service, respectively. Evidently, when $\rho < 1$ and $t \to \infty$ we get the well-known one steady state solution, i.e.

$$p_n = (1-\rho)\rho^n,\ \rho = \lambda/\mu,\ n = 0, 1, 2, ...$$

The summary of this approach which can be applied to Poisson queues one can be found in Sharma (1990).

Parthasarathy (1987) proposed a simple approach to the transient solution of the $M/M/1$ system based on the introduction of the function

$$q_n(t) = \begin{cases} e^{-(\lambda+\mu)t}(\mu p_n(t) - \lambda p_{n-1}(t)), & n = 1, 2, 3.. \\ 0, & n = 0, -1, .. \end{cases}$$

He has shown that the solution can be written as

$$p_n(t) = \frac{e^{-(\lambda+\mu)t}}{\mu} \sum_{k=1}^{n} q_k(t) \left(\frac{\lambda}{\mu}\right)^{n-k} + \left(\frac{\lambda}{\mu}\right) p_0(t),$$

where

$$
\begin{aligned}
q_n(t) &= \mu\beta^{n-a}(1-\delta_{0a})[I_{n-a}(\alpha t) - I_{n+a}(\alpha t)] + \lambda\beta^{n-a-1}(I_{n+a+1}(\lambda t) - I_{n-a-1}(\alpha t)), \\
\alpha &= 2\sqrt{\lambda\mu}, \quad \beta = \sqrt{\frac{\lambda}{\mu}}
\end{aligned}
$$

and

$$p_0(t) = \int_0^t q_1(y)e^{-(\lambda+\mu)y}dy + \delta_{0a}.$$

An alternative new solution was presented by Conolly and Langaris (1993) in the form

$$p_n(t) = \left(1 - \frac{(\lambda+\mu) - |\lambda-\mu|}{2\mu}\right)\rho^n + e^{-(\lambda+\mu)t}\sum_{m\geq 0} C_m^{(n)} t^m,$$

where

$$
\begin{aligned}
C_m^{(1)} &= \frac{m+1}{\mu}C_{m+1}^{(0)} - C_m^{(0)}, \\
C_m^{(n)} &= \frac{m+1}{\mu}C_{m+1}^{(n-1)} - C_m^{(n-2)}.
\end{aligned}
$$

Sharma and Bunday (1997) have obtained the state probabilities in closed form by considering an initial empty system, $p_0(0) = 1$

$$
\begin{aligned}
p_n(t) &= e^{-(\lambda+\mu)t}\rho^n + \sum_{m=0}^{\infty}\sum_{r=0}^{[\frac{m-n}{2}]} A(m,r)\rho^r \frac{(\mu t)^m}{m!}, \\
A(m,s) &= \binom{m}{s} - \binom{m}{s-1}.
\end{aligned}
$$

Krinik (1992) gives an explicit Taylor series solution for $p_0(t)$ and describes an iterative scheme for obtaining $p_n(t)$.

Finally, Tarabia (2002) presents transient state probabilities with any arbitrary number a of customers being present in the system initially

$$p_n(t) = \begin{cases} (1-\rho)\rho^n + e^{-(\lambda+\mu)t}\rho^{n-a}\sum_{m=0}^{\infty}\{(\rho-1)(1+\rho)^m\rho^a \\ +a(m,n)\}\frac{(\mu t)^m}{m!}, \quad \rho \neq 1 \\ e^{-2\mu t}\sum_{m=0}^{\infty} a'(m,n)\frac{(\mu t)^m}{m!}, \quad \rho = 1, \end{cases}$$

where

$$a(m,n) = \begin{cases} \sum_{r=a}^{s} A(m, r-a)\rho^r, & m-n+a \text{ odd} \\ \sum_{r=a}^{s} A(m, r-a)\rho^r + AS(m,s)\rho^s, & m-n+a \text{ even} \end{cases} \quad \rho \neq 1$$

and

$$a'(m,n) = \begin{cases} \binom{m}{s-a}, & m-n+a \text{ odd} \\ \binom{m}{s}, & m-n+a \text{ even} \end{cases} \quad \text{for } \rho \to 1.$$

$$\begin{aligned} AS(m,r) &= \binom{m}{r} - \binom{m}{r-a-1} \\ s &= \left[\frac{m-n+a}{2}\right]. \end{aligned}$$

Also the $M/M/1$ queue has been studied in the case when parameters λ and μ are allowed to depend on time by Clarke (1956) and more recently by Zhang, Coyle, Edward (1991).

The problem of obtaining the transient solution for the $M/M/1/N$ queueing system is quite complicated as well. Takacs (1960) obtained the transient solution of this system using eigenvectors and eigenvalues. Morse (1958) also derived the same result, using a different approach in terms of trigonometric functions. Sharma and Gupta (1992) obtained the result of the queue length of the $M/M/1/N$ queue applying Chebychev polynomials. However, the use of Chebychev polynomials reveals some difficulties in mathematical manipulation. Recently Tarabia (2001) obtained an alternative simple approach to the same system. The results allow for an arbitrary number of initial customers in the system and it is shown that the measures of effectiveness can be easily written in an elegant closed form.

Time-dependent solutions have been presented for many-server $M/M/c$ queueing systems by Jackson, Henderson (1960), where probabilities are obtained in terms of Laplace transforms. Parthasarathy, Sharafali (1989) have obtained the inverse Laplace transform for the probabilities and have written the solution in explicit form.

1.2 Steady-State Solution of the M/M/1 Queue

To set the scene for future work, we first study the simplest probabilistic queueing model, namely, the $M/M/1$ single-channel system. We would like to find the steady-state solution $p_n = \lim_{t \to \infty} p_n(t)$, where $p_n(t)$ =Pr{ n customers in the system at time t} and various measures of effectiveness.

1. If λ is the expected number of arrivals per unit time, then the probability of n arrivals occurring in the interval $(0, t)$ is

$$e^{-\lambda t} \frac{(\lambda t)^n}{n!}, \qquad n = 0, 1, 2, ..., \infty.$$

2. If μ is the expected service rate, then the probability function for the number of departures n in time t, given that customers are available to be served throughout time t, is

$$\Pr\{\text{a service completion in } \Delta t\} = e^{-\mu t} \frac{(\mu t)^n}{n!}, \qquad n = 0, 1, 2, ..., \infty.$$

We can describe the $M/M/1$ system as Poisson input with mean λ and exponential service time at mean rate μ.

Using Markov chain property one can write the following equations (see Saaty (1961), Gross, Harris (1974))

$\Pr\{\text{an arrival occurs in } \Delta t\} = \lambda \Delta t + o(\Delta t)$

$\Pr\{\text{more that one arrival occurs in } \Delta t\} = o(\Delta t)$

$\Pr\{\text{a service completion in } \Delta t\} = \mu \Delta t + o(\Delta t)$

$\Pr\{\text{more than one service completion in } \Delta t\} = o(\Delta t)$

The relevant differential-difference equations are:

$$\begin{cases} \frac{dp_0(t)}{dt} = -\lambda p_0(t) + \mu p_1(t), & n = 0, \\ \frac{dp_n(t)}{dt} = -(\lambda+\mu)p_n(t) + \mu p_{n+1}(t) + \lambda p_{n-1}(t), & n \geq 1. \end{cases} \tag{1.2.1}$$

To obtain the steady-state solution for p_n, that is the probability of n customers in the system at an arbitrary point of time, we should equake the left hand side of the equations (1.2.1) to zero.

$$\begin{cases} p_1 = -\rho p_0, & n = 0, \\ p_{n+1} = (\rho+1)p_n - \rho p_{n-1}, & n \geq 1, \end{cases} \tag{1.2.2}$$

where $\rho = \lambda/\mu$ is the utilization factor. Let us also assume that $\lambda < \mu$, so that $\sum_{n=0} \rho^n$ converges.

The solution of (1.2.2) can be found using generating function methods. As mentioned before it can be also found using iterative or operator methods.

We define the probability generating function as

$$G(y) = \sum_{n=0}^{\infty} p_n y^n, \tag{1.2.3}$$

where y is complex with $|y| \leq 1$.

For some models, it is relatively easy to find a closed expression for $G(y)$, but quite difficult to find its series expansion to obtain the p_n.

For the model under consideration here, we can completely determine the p_n using $G(y)$. Multiplying the both sides of the (1.2.2) by y^n and summing over n it is found that

$$\frac{G(y) - p_1 y - p_0}{y} = (\rho+1)[G(y) - p_0] - \rho y G(y). \tag{1.2.4}$$

Using the initial condition we find $p_0 = 1 - \rho$, and because $\rho < 1$ we have the following expression for the generating function

$$G(y) = \frac{1-\rho}{1-y\rho}. \qquad (1.2.5)$$

We can rewrite the generating function through the geometric series and then equate the coefficients of y^n to get the solution in the form

$$p_n = \begin{cases} (1-\rho)\rho^n, & \rho < 1, \\ 0 & \rho > 1. \end{cases} \qquad (1.2.6)$$

The steady-state probability distribution for the system size allows us to calculate what are commonly called measures of effectiveness.

Let N represent the random variable "number of customers in the system at steady-state" and L represent its expected value. Then we can write

$$L = E[N] = \sum_{n=0}^{\infty} np_n = (1-\rho)\sum_{n=0}^{\infty} n\rho^n. \qquad (1.2.7)$$

Thus,

$$L = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu - \lambda}, \qquad (1.2.8)$$

and L_q - expected number of customers in the queue is

$$L_q = \frac{\rho^2}{1-\rho} = \frac{\lambda^2}{\mu(\mu-\lambda)}. \qquad (1.2.9)$$

Let T_q denote the random variable "time spent waiting in the queue" and $W_q(t)$ denote its cumulative probability distribution that is the probability of a customer waiting a time less than or equal to t in the queue, then

$$\begin{aligned} W_q(t) &= \Pr\{T_q \leq t\} \\ &= \sum_{n=1}^{\infty} (\Pr\{n \text{ completions in } \leq t | \text{arrival found } n \text{ in system}\} p_n) + W_q(0). \end{aligned}$$

We can calculate W_q as

$$W_q = E[T_q] = \int_0^\infty t dW_q(t) = \frac{\lambda}{\mu(\mu - \lambda)}. \tag{1.2.10}$$

The waiting time in the system, W, is equal to

$$W = \frac{1}{\mu - \lambda}. \tag{1.2.11}$$

Note that we can write the relationships among the measures of effectiveness as follows

$$W = W_q + \frac{1}{\mu}, \tag{1.2.12}$$

$$L_q = \lambda W_q, \tag{1.2.13}$$

$$L = L_q + \frac{\lambda}{\mu}, \tag{1.2.14}$$

$$L = \lambda W. \tag{1.2.15}$$

Equation (1.2.15) is generally known as Little's formula, because of the work of Little (1961). The measures of effectiveness as functions of ρ are given in Table 1.2.1 ($\mu = 1$).

Table 1.2.1 Measures of effectiveness for the steady-state $M/M/1$ system

ρ	0.1	0.2	0.3	0.4	0.6	0.7	0.9
W	1.111	1.250	1.428	1.666	2.500	3.333	10.00
W_q	0.111	0.250	0.428	0.666	1.500	2.333	9.000
L	0.111	0.250	0.428	0.666	1.500	2.333	9.000
L_q	0.011	0.050	0.128	0.266	0.900	1.633	8.100

1.3 Transient Solution for the M/M/1 Queue

1.3.1 Clarke's Approach

The first time-dependent solution of the $M/M/1$ system was obtained by Clarke (1956) via derivation of the generating function from the differential-difference equations.

Consider equations (1.2.1) with intially a customers are presented in the system at time $t = 0$. The generating function such is

$$G(y,t) = \sum_{n=0}^{\infty} y^n p_n(t), \tag{1.3.1}$$

which must converge within the unit circle $|y| = 1$. Multiplying each equation by y with an appropriate power and summing over n we get

$$\sum_{n=0}^{\infty} p_n'(t)y^n = -\lambda \sum_{n=0}^{\infty} p_n(t)y^n - \mu \sum_{n=1}^{\infty} p_n(t)y^n + \lambda \sum_{n=1}^{\infty} p_{n-1}(t)y^n + \mu \sum_{n=0}^{\infty} p_{n+1}(t)y^n. \tag{1.3.2}$$

After rearranging (1.3.2) we have

$$\frac{\partial G(y,t)}{\partial t} = \frac{1-y}{y}\left[(\mu - \lambda y)G(y,t) - \mu p_0(t)\right] \tag{1.3.3}$$

with the initial condition

$$G(y,0) = y^a. \tag{1.3.4}$$

We define the Laplace transforms

$$G^*(y,z) = \int_0^{\infty} e^{-zt} G(y,t)dt \tag{1.3.5}$$

and

$$p_a^*(z) = \int_0^{\infty} e^{-zt} p_a(t)dt.$$

Note that

$$\int_0^\infty e^{-zt}\frac{\partial G(y,t)}{\partial t}dt = \left[e^{-zt}G(y,t)\right]_0^\infty + z\int_0^\infty e^{-zt}G(y,t)dt$$

$$= -y^a + zG^*(y,z).$$

Using (1.3.3) and (1.3.5) we can write an expression for the Laplace transform of the generating function

$$G^*(y,z) = \frac{y^{a+1} - \mu(1-y)p_0^*(z)}{(\lambda+\mu+z)y - \mu - \lambda y^2} = \frac{y^{a+1} - [(1-y)y_1^{a+1}/(1-y_1)]}{-\lambda(y-y_1)(y-y_2)}. \quad (1.3.6)$$

Since the Laplace transform of the $G^*(y,z)$ converges in the region $|y| \le 1$ and $Re(z) > 0$, wherever the denominator of the right-hand side of (1.3.6) has zeros in that region, so must the numerator. The denominator has two zeros since it is a quadratic in y and they are

$$y_1 = \frac{\lambda+\mu+z-\sqrt{(\lambda+\mu+z)^2-4\lambda\mu}}{2\lambda},$$

$$y_2 = \frac{\lambda+\mu+z+\sqrt{(\lambda+\mu+z)^2-4\lambda\mu}}{2\lambda}. \quad (1.3.7)$$

Since $|y_1| < 1$, the numerator must vanish at $y = y_1$, and we can find the expression for the $p_0^*(z)$

$$p_0^*(z) = \frac{y_1^{a+1}}{\mu(1-y_1)}. \quad (1.3.8)$$

Substituting (1.3.8) in (1.3.6) and rearranging $G^*(y,z)$ we have

$$G^*(y,z) = \frac{1}{\lambda y_2}\left(y^a + y_1 y^{a-1} + \cdots + y_1^a\right)\sum_{k=0}^{\infty}\left(\frac{y}{y_2}\right)^k + \frac{y_1^{a+1}}{\lambda y_2(1-y_1)}\sum_{k=0}^{\infty}\left(\frac{y}{y_2}\right)^k, \quad (1.3.9)$$

where $\left|\frac{y_1}{y_2}\right| < 1$. Equating the coefficients of y^n it turns out that

$$p_n^*(z) = \frac{1}{\lambda}\left[\frac{1}{y_2^{n-a+1}} + \frac{\mu/\lambda}{y_2^{n-a+3}} + \frac{(\mu/\lambda)^2}{y_2^{n-a+5}} + \cdots \right.$$

$$\left. + \frac{(\mu/\lambda)^a}{y_2^{n+a+1}} + \left(\frac{\lambda}{\mu}\right)^{n+1} \sum_{k=n+a+2}^{\infty} \left(\frac{\mu}{\lambda y_2}\right)^k \right] \tag{1.3.10}$$

for $n \geq a$.

Inverting the Laplace transform we can find the solution in terms of the modified Bessel function (see section 2 for further properties and details)

$$p_n(t) = e^{-(\lambda+\mu)t}\left[\left(\frac{\mu}{\lambda}\right)^{(a-n)/2} I_{n-a}\left(2\sqrt{\lambda\mu t}\right) + \left(\frac{\mu}{\lambda}\right)^{(a-n+1)/2} I_{n+a+1}\left(2\sqrt{\lambda\mu t}\right)\right.$$

$$\left. + \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n \sum_{k=n+a+2}^{\infty} \left(\frac{\mu}{\lambda}\right)^{k/2} I_k\left(2\sqrt{\lambda\mu t}\right)\right], \; n \geq 1 \tag{1.3.11}$$

for $n \geq a$. It can be proved that the foregoing is also the solution for $n < a$.

1.3.2 Sharma's Studies

Sharma and Shobna (1984) studied the classic problem with the help of a two-dimensional state model. The two dimensional state model (n, l) represents respectively the number of arrivals at, and departures from, the system at a given time. Their difference is the number in the system.

Consider the $M/M/1$ system when inter-arrival and service times are negative exponentially distributed with means $1/\lambda$ and $1/\mu$, respectively. The system is taken to be empty at $t = 0$.

Define $p_{n,l}(r,t) = \Pr\{n$ customers arrive and l leave the system after getting served in the time interval $(0,t)$, and r customers remain in the system at time $t\}$ and $p_{n,l}(r,t) = 0$ if n, l or $r < 0$ or if $n - l \neq r$.

Also

$$p(r,t) = \sum_{l=0}^{\infty} p_{r+l,l}(r,t), \ r \geq 1. \tag{1.3.12}$$

The differential-difference equations satisfied by $p_{n,l}(r,t)$ are given by

$$\begin{cases} \frac{dp_{n,l}(0,t)}{dt} = -\lambda p_{n,l}(0,t) + \mu p_{n,l-1}(1,t), \\ \frac{dp_{n,l}(r,t)}{dt} = -(\lambda+\mu) p_{n,l}(r,t) + \lambda p_{n-1,l}(r-1,t) + \mu p_{n,l-1}(r+1,t), \ r \geq 1. \end{cases} \tag{1.3.13}$$

Taking Laplace Transforms of these equations we get

$$\begin{cases} (\lambda + z)\psi_{l,l}(0,z) = \mu\psi_{l,l-1}(1,z) + p_{l,l}(0,0), \\ (\lambda + \mu + z)\psi_{r+l,l}(r,z) = \lambda\psi_{r+l-1,l}(r-1,z) + \mu\psi_{r+l,l-1}(r+1,z), \end{cases} \tag{1.3.14}$$

where

$$\psi_{n,l}(r,z) = \int_0^{\infty} e^{-zt} p_{n,l}(r,t)dt.$$

Now setting $a = \frac{\lambda}{\lambda+\mu+z}$, $b = \frac{\mu}{\lambda+\mu+z}$ and $c = \frac{\mu}{\lambda+z}$, multiplying the first $r+1$ equations by $a^r, a^{r-1}, \ldots, a, 1$ respectively we obtain

$$\psi_{r+l,l}(r,z) = b\sum_{j=0}^{r-1} a^j \psi_{r+l-j,k-1}(r+1-j,z) + ca^r\psi_{l,l-1}(1,z) + \frac{a^r}{\lambda+z}p_{l,l}(0,0), \quad (1.3.15)$$

where $r = 0,1,2\ldots,\ l = 0,1,2\ldots$

We now determine $p_{r+l,l}(r,t)$ by inverting $\psi_{r+l,l}(r,\theta)$. Rearranging the expression for $\psi_{r+l,l}(r,\theta)$ we get

$$\psi_{r+l,l}(r,\theta) = \frac{\lambda^{r+l}}{(r+l)!}\sum_{j=0}^{l}\frac{(r+l-j)(r+l+j-1)!}{j!(\lambda+\mu+\theta)^{r+l+j}(\lambda+\theta)^{l+1-j}}.$$

Using the convolution theorem on the Laplace transform of a product we get

$$\begin{aligned} p_{r+l,l}(r,t) &= \frac{\lambda^{r+l}\mu^l}{(r+l)!}\sum_{j=0}^{l}\frac{(r+l-j)(r+l+j-1)!}{j!(l-j)!(r+l+j-1)!} \\ &\quad\times \int_0^t (t-\tau)^{l-j}e^{-\lambda(t-\tau)}\tau^{r+l+j-1}e^{-(\lambda+\mu)\tau}d\tau. \end{aligned}$$

Evaluating the integral using the hypergeometric function, finally, we can write the solution in the following form

$$\begin{aligned} p(r,t) &= \sum_{l=0}^{\infty} p_{r+l,l}(r,t) \\ &= (1-\rho)\rho^r + \rho^r e^{-(\lambda+\mu)t}\sum_{l=0}^{\infty}\frac{(\lambda t)^k}{l!}\sum_{m=0}^{r+l}(l-m)\frac{(\mu t)^{m-1}}{m!} \end{aligned} \quad (1.3.16)$$

with $p(0,0) = 1$ and $\rho = \lambda/\mu$.

Evidently when $\rho < 1$ and $t \to \infty$ we get the well-known steady-state solution (1.2.6).

1.3.3 Parthasarathy's Approach

Parthasarathy (1987) has obtained a new approach for the $M/M/1$ system in terms of the function $q_n(t)$. We include this method because it takes an important role in obtaining the transient solution for the Erlang model (method 2). Parthasarathy introduced the function $q_n(t)$ as follows

$$q_n(t) = \begin{cases} e^{-(\lambda+\mu)t}((\mu p_n(t) - \lambda p_{n-1}(t)), & n = 1, 2, 3, \ldots \\ 0, & n = 0, -1, -2, -3, \ldots \end{cases} . \tag{1.3.17}$$

The generating function is

$$G(y,t) = \sum_{n=-\infty}^{\infty} q_n(t) y^n. \tag{1.3.18}$$

Assuming that initially there are a customers, the equations (1.2.1) we can rewrite as

$$\begin{cases} \frac{\partial G(y,t)}{\partial t} = \left(\lambda y + \frac{\mu}{y}\right) G(y,t) - \mu q_1(t) \\ G(y,0) = y^a[\mu(1 - \delta_{0a}) - \lambda y], \end{cases} \tag{1.3.19}$$

where δ_{0a} is the Kronecker delta.

Solving the ordinary differential equation (1.3.19) we have

$$G(y,t) = G(y,0) \exp\left\{\left(\lambda y + \frac{\mu}{y}\right) t\right\} - \mu \int_0^t q_1(t) \exp\left\{\left(\lambda y + \frac{\mu}{y}\right)(t-x)\right\} dx. \tag{1.3.20}$$

We note the property for generating functions (see Section 2 for details)

$$\exp\left\{(\lambda y + \frac{\mu}{y})t\right\} = \sum_{n=-\infty}^{\infty} (\beta y)^n I_n(\alpha t), \tag{1.3.21}$$

where $I_n(t)$ is a modified Bessel function of the first kind and $\alpha = 2\sqrt{\lambda\mu}$, $\beta = \sqrt{\lambda/\mu}$. Comparing the coefficients of y^n on both sides of (1.3.20) and using the property of modified Bessel function that $I_{-r} = I_r$, we can write the solution as follows

$$p_n(t) = \frac{\exp(\lambda+\mu)t}{\mu} \sum_{k=1}^{n} q_k(t) \left(\frac{\lambda}{\mu}\right)^{n-k} + \left(\frac{\lambda}{\mu}\right)^{n} p_0(t), \qquad (1.3.22)$$

where

$$q_n(t) = \mu\beta^{n-a}(1-\delta_{0a})[I_{n-a}(\alpha t) - I_{n+a}(\alpha t)] + \lambda\beta^{n-a-1}(I_{n+a+1}(\lambda t) - I_{n-a-1}(\alpha t)),$$

$$p_0(t) = \int_0^t q_1(y)\exp\{-(\lambda+\mu)y\}\,dy + \delta_{0a}. \qquad (1.3.23)$$

1.3.4 Euler's Method and Comparison of Results

This section compares results for the probabilities $p_n(t)$ using Clark's and Parthasarathy's methods and the so-called Euler's technique, which computes probabilities numerically from the differential-difference equations for the $M/M/1$ queueing system.

As, before, let λ be the mean rate of arrival, and let μ be the mean service rate. Starting with zero customers in the system at time $t = 0$ gives $p_0(0) = 1$ and $p_n(0) = 0$, $n \geq 1$. We can calculate $p_0(t)$ using the queueing equations for $p_0(\delta t)$, $p_0(2\delta t)$, $p_0(3\delta t)$, etc... These

$$\begin{aligned} p_0(\delta t) &= p_0(0) + \delta t(-\lambda p_0(0) + \mu p_1(0)) \\ p_0(2\delta t) &= p_0(\delta t) + \delta t(-\lambda p_0(\delta t) + \mu p_1(\delta t)) \\ &\textit{etc...} \end{aligned}$$

Then, by using increments of δt such as 0.1, 0.01, etc... and substitution, $p_0(t)$ can be found.

To compute $p_1(t)$, we can use the same technique

$$\begin{aligned} p_1(\delta t) &= p_1(0) + \delta t(-(\lambda + \mu)p_1(0) + \lambda p_0(0) + \mu p_2(0)) \\ &= \lambda \delta t \\ &\textit{etc...} \end{aligned}$$

Thus, the $p_n(t)$ can be found. The following results have been calculated with $\delta t = 0.01$. One can see comparison of the results for the $p_n(t)$ with $\rho = 0.2, 0.5$, 0.9, 1.3 in the Tables 1.3.1-1.3.12. Figures 1.3.1, 1.3.2 show the difference of the behaviour of the probabilities when $\rho < 1$ and $\rho > 1$.

Table 1.3.1 Computing probabilities for the $M/M/1$ system using Euler's method with $\rho = 0.2$

t	0.5	0.9	1.5	2.0	2.5	3.1	3.5
$p_0(t)$	0.92395	0.88734	0.85412	0.83762	0.82684	0.81842	0.81442
$p_1(t)$	0.07266	0.10430	0.13005	0.14131	0.14790	0.15255	0.15450
$p_2(t)$	0.00328	0.00789	0.01450	0.01887	0.02219	0.02525	0.02646
$p_3(t)$	0.00010	0.00042	0.00122	0.00200	0.00274	0.00351	0.00402

Table 1.3.2 Computing probabilities for the $M/M/1$ system using Clarke's method with $\rho = 0.2$

t	0.5	0.9	1.5	2.0	2.5	3.1	3.5
$p_0(t)$	0.92426	0.88764	0.85441	0.83732	0.82701	0.81854	0.81452
$p_1(t)$	0.07232	0.10396	0.12980	0.14113	0.14778	0.15244	0.15434
$p_2(t)$	0.00330	0.00787	0.01446	0.01882	0.02215	0.02503	0.02644
$p_3(t)$	0.00010	0.00043	0.00108	0.00184	0.00263	0.00340	0.00397

Table 1.3.3 Computing probabilities for the $M/M/1$ system using Parthasarathy's method with $\rho = 0.2$

t	0.5	0.9	1.5	2.0	2.5	3.1	3.5
$p_0(t)$	0.92426	0.88768	0.86343	0.84361	0.83080	0.82094	0.81642
$p_1(t)$	0.07221	0.10382	0.12301	0.13718	0.14537	0.15102	0.15337
$p_2(t)$	0.00331	0.00790	0.01242	0.01722	0.02090	0.02418	0.02578
$p_3(t)$	0.00010	0.00043	0.00098	0.00169	0.00209	0.00328	0.00376

Table 1.3.4 Computing probabilities for the $M/M/1$ system using Euler's method with $\rho = 0.5$

t	0.5	0.9	1.5	2.0	2.5	3.1	3.5
$p_0(t)$	0.82029	0.74044	0.66956	0.63336	0.60824	0.59001	0.57601
$p_1(t)$	0.16018	0.21364	0.24702	0.25751	0.26179	0.26437	0.26350
$p_2(t)$	0.01804	0.03993	0.06693	0.08258	0.09357	0.10248	0.10679
$p_3(t)$	0.00139	0.00536	0.01386	0.02116	0.02771	0.03424	0.03784

Table 1.3.5 Computing probabilities for the $M/M/1$ system using Clarke's method with $\rho = 0.5$

t	0.5	0.9	1.5	2.0	2.5	3.1	3.5
$p_0(t)$	0.82108	0.74121	0.67013	0.63379	0.60853	0.59014	0.57613
$p_1(t)$	0.15928	0.21290	0.24668	0.25726	0.26168	0.26334	0.26347
$p_2(t)$	0.01809	0.03984	0.06671	0.08224	0.09341	0.10244	0.10679
$p_3(t)$	0.00143	0.00541	0.01386	0.02113	0.02776	0.03421	0.03782

Table 1.3.6 Computing probabilities for the $M/M/1$ system using Parthasarathy's method with $\rho = 0.5$

t	0.5	0.9	1.5	2.0	2.5	3.1	3.5
$p_0(t)$	0.82108	0.74121	0.68948	0.64662	0.61766	0.59342	0.58131
$p_1(t)$	0.15903	0.21262	0.23872	0.25383	0.26013	0.26279	0.26321
$p_2(t)$	0.01813	0.03986	0.06062	0.07849	0.08349	0.09984	0.10478
$p_3(t)$	0.00144	0.00542	0.01091	0.01829	0.02517	0.03219	0.03609

Table 1.3.7 Computing probabilities for the $M/M/1$ system using Euler's method with $\rho = 0.9$

t	0.5	0.9	1.5	2.0	2.5	3.1	3.5
$p_0(t)$	0.69947	0.57993	0.47837	0.42664	0.38998	0.35753	0.33901
$p_1(t)$	0.24370	0.29426	0.30573	0.29865	0.28826	0.27665	0.26772
$p_2(t)$	0.04922	0.09739	0.14382	0.16427	0.17517	0.18119	0.18280
$p_3(t)$	0.00682	0.02333	0.05225	0.07275	0.08860	0.10234	0.10881

Table 1.3.8 Computing probabilities for the $M/M/1$ system using Clarke's method with $\rho = 0.9$

t	0.5	0.9	1.5	2.0	2.5	3.1	3.5
$p_0(t)$	0.70090	0.58115	0.47918	0.42723	0.39043	0.35794	0.34088
$p_1(t)$	0.24209	0.29326	0.30537	0.29853	0.28825	0.27560	0.26770
$p_2(t)$	0.04917	0.09697	0.14334	0.16390	0.17490	0.18108	0.18274
$p_3(t)$	0.00698	0.02340	0.05210	0.07256	0.08839	0.10211	0.10877

Table 1.3.9 Computing probabilities for the $M/M/1$ system using Parthasarathy's method with $\rho = 0.9$

t	0.5	0.9	1.5	2.0	2.5	3.1	3.5
$p_0(t)$	0.70090	0.58115	0.49740	0.43561	0.40381	0.36770	0.34904
$p_1(t)$	0.24168	0.29286	0.30499	0.30169	0.29222	0.27949	0.27134
$p_2(t)$	0.04926	0.09702	0.13112	0.15716	0.17137	0.17964	0.18211
$p_3(t)$	0.00701	0.02344	0.04465	0.06695	0.08262	0.09811	0.10567

Table 1.3.10 Computing probabilities for the $M/M/1$ system using Euler's method with $\rho = 1.3$

t	0.5	0.9	1.5	2.0	2.5	3.1	3.5
$p_0(t)$	0.59599	0.45273	0.33777	0.28114	0.24103	0.20680	0.18901
$p_1(t)$	0.29734	0.32458	0.30109	0.27302	0.24721	0.22099	0.20580
$p_2(t)$	0.08641	0.15276	0.19798	0.20773	0.20620	0.19847	0.19152
$p_3(t)$	0.01729	0.05234	0.10122	0.12803	0.14382	0.15281	0.15517

Table 1.3.11 Computing probabilities for the $M/M/1$ system using Clarke's method with $\rho = 1.3$

t	0.5	0.9	1.5	2.0	2.5	3.1	3.5
$p_0(t)$	0.59803	0.45428	0.33872	0.28180	0.24219	0.20773	0.18982
$p_1(t)$	0.29521	0.32360	0.30096	0.27313	0.24739	0.22100	0.20593
$p_2(t)$	0.08602	0.15187	0.19729	0.20730	0.20606	0.19823	0.19146
$p_3(t)$	0.01757	0.05229	0.10082	0.12759	0.14345	0.15268	0.15500

Table 1.3.12 Computing probabilities for the $M/M/1$ system using Parthasarathy's method with $\rho = 1.3$

t	0.5	0.9	1.5	2.0	2.5	3.1	3.5
$p_0(t)$	0.59803	0.45428	0.35832	0.29437	0.25653	0.21802	0.19836
$p_1(t)$	0.29467	0.32318	0.31097	0.28389	0.25702	0.22903	0.21303
$p_2(t)$	0.08616	0.15196	0.18764	0.20525	0.20741	0.20130	0.19495
$p_3(t)$	0.01763	0.05236	0.00901	0.11842	0.13829	0.15055	0.15416

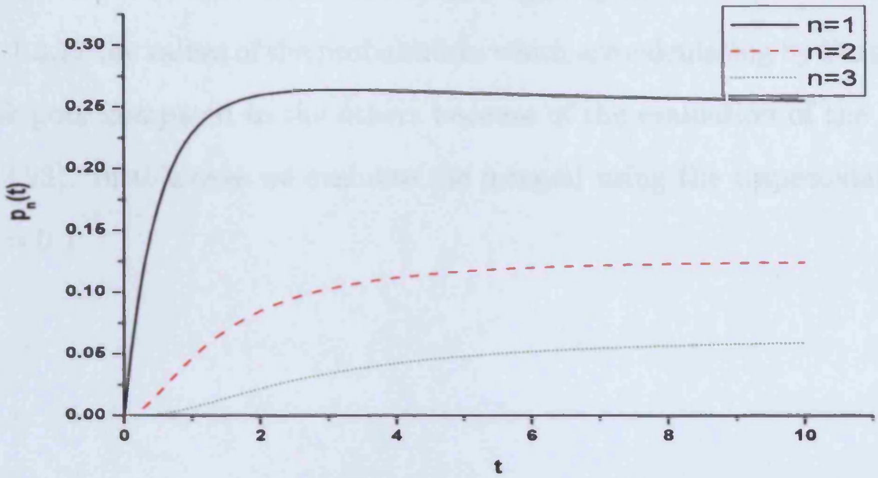

Figure 1.3.1 Graph for $p_1(t)$, $p_2(t)$, $p_3(t)$ with $\rho = 0.5$ to show convergence to the steady-state solution as t increases.

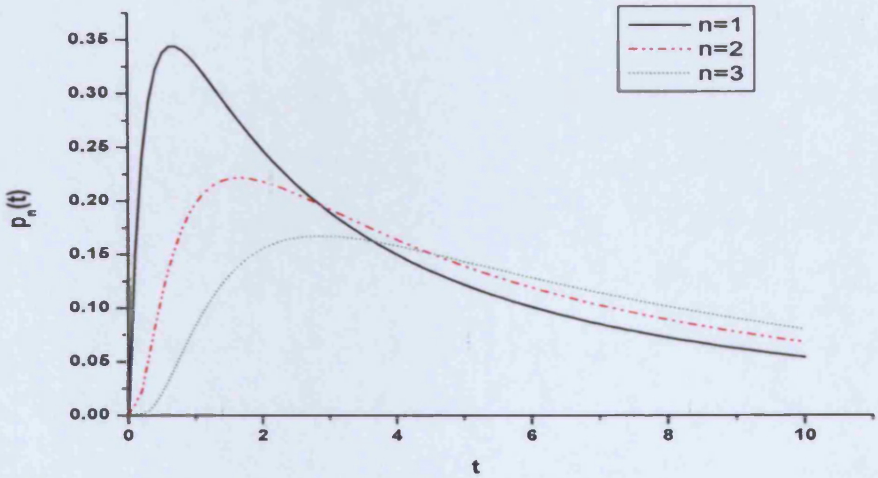

Figure 1.3.2 Graph to show the effect on $p_1(t)$, $p_2(t)$, $p_3(t)$ for $\rho > 1$ ($\rho = 1.5$) when t increases.

As you can seen from the tables the results agree quite well. But in Tables 1.3.3, 1.3.6, 1.3.9, 1.3.12 the values of the probabilities which are calculating by Parthasarathy's method look poor compared to the others because of the evaluation of the integral in formula (1.3.23). In this case we evaluate the integral using the trapezoidal rule with the step $\delta t = 0.1$.

CHAPTER 2: BESSEL FUNCTIONS AND THEIR GENERALISATIONS

2.1 Background of Bessel Functions and Their Main Properties

The theory of Bessel functions is intimately connected with the theory of a certain type of differential equation of the first order, known as Riccati's equation.

In fact a Bessel function is usually defined as a particular solution of a linear differential equation of the second order (known as Bessel's equation (2.1.1)) which is derived from Riccati's equation by an elementary transformation.

$$y^2\frac{d^2t}{d^2y} + y\frac{dt}{dy} + (y^2 - n^2)t = 0. \tag{2.1.1}$$

The earliest appearance of an equation of Riccati's type occurs in paper "Acta Eruditorum publicata Lipside" which was published by John Bernoulli in 1694. Five years later he succeeded in reducing the equation to a linear equation of the second order and then he obtained the solution in series.

The memoir in which Bessel examined in detail the function which now bears his name was written in 1824. Bessel functions play an important role in different parts of mathematics and applications. There are several ways of defining these functions. The generating function of the Bessel coefficient is

$$\exp\left(\frac{1}{2}t(y - \frac{1}{y})\right).$$

This function can be developed into a Laurent series; the coefficient of y^n in the expansion is called the Bessel coefficient of argument t, and it is denoted by the symbol $J_n(t)$, that is

$$\exp\left(\frac{1}{2}t(y - \frac{1}{y})\right) = \sum_{n=-\infty}^{\infty} y^n J_n(t). \tag{2.1.2}$$

In (2.1.2) we substitute t with $-1/t$ and get

$$\exp\left(\frac{1}{2}t(y-\frac{1}{y})\right) = \sum_{n=-\infty}^{\infty}(-y)^{-n}J_n(t) = \sum_{n=-\infty}^{\infty}(-y)^{n}J_{-n}(t).$$

Since the Laurent expansion of a function is unique, we obtain the next property of the Bessel function

$$J_{-n}(t) = (-1)^n J_n(t). \tag{2.1.3}$$

To obtain (2.1.2), observe that $\exp\left(\frac{1}{2}t(y-\frac{1}{y})\right)$ can be expanded into series of ascending powers of y

$$\exp\left(\frac{1}{2}t(y-\frac{1}{y})\right) = \sum_{r=0}^{\infty}\frac{(\frac{1}{2}t)^r y^r}{r!}\sum_{m=0}^{\infty}\frac{(-\frac{1}{2}t)^m y^{-m}}{m!}. \tag{2.1.4}$$

Let $r = n + m$, where $n \in (-\infty, +\infty)$.

Changing the variables we can rewrite (2.1.4) in the form

$$\sum_{n=-\infty}^{\infty}\sum_{m=0}^{\infty}\frac{(\frac{1}{2}t)^{n+m}}{(n+m)!}\frac{(-\frac{1}{2}t)^m}{m!}y^n = \sum_{n=-\infty}^{\infty} y^n J_n(t), \tag{2.1.5}$$

where

$$J_n(t) = \sum_{m=0}^{\infty}\frac{(-1)^m(\frac{1}{2}t)^{n+2m}}{(n+m)!m!}. \tag{2.1.6}$$

$J_n(t)$ is defined as a Bessel function.

We can write the following main properties of the function (2.1.6) (see Watson (1966)):

1) $J_{n-1}(t) + J_{n+1}(t) = \frac{2n}{t}J_n(t)$;

2) $J_{n-1}(t) - J_{n+1}(t) = 2J'_n(t)$;

3) $tJ'_n(t) + nJ_n(t) = tJ_{n-1}(t)$;

4) $tJ'_n(t) - nJ_n(t) = -tJ_{n+1}(t)$.

Also Bessel's integral for the Bessel coefficients is

$$J_n(t) = \frac{1}{2\pi}\int_0^{2\pi}\cos(n\theta - t\sin\theta)d\theta. \tag{2.1.7}$$

In general, a wide variety of special functions (such as beta, gamma, Bessel, modified Bessel, hypergeometric, Wright) and their generalisations are very useful in the derivation of solutions in queueing theory. But unfortunately, just a few generating functions are available in an explicit form. The best known one is the modified Bessel function, which plays a fundamental role in solving $M/M/1$ queueing equations. It can be expressed as

$$I_n(t) = \sum_{m=0}^{\infty} \frac{(\frac{1}{2}t)^{n+2m}}{m!\Gamma(n+m+1)}, \quad t \in C, \quad n \in Z. \tag{2.1.8}$$

The relationship with the Bessel function is

$$I_\nu(t) = e^{-\frac{1}{2}\nu\pi i} J_\nu(te^{\frac{1}{2}\pi i}), \quad -\pi < \arg t \le \tfrac{1}{2}\pi$$
$$I_\nu(t) = e^{\frac{3}{2}\nu\pi i} J_\nu(te^{-\frac{3}{2}\pi i}), \quad \tfrac{1}{2}\pi < \arg t \le \pi.$$

The following are some properties of the modified Bessel function:

$$\begin{aligned}
&1) \quad I_{n-1}(t) + I_{n+1}(t) = 2I_n'(t);\\
&2) \quad I_{n-1}(t) - I_{n+1}(t) = \tfrac{2n}{t} I_n(t);\\
&3) \quad tI_n'(t) + nI_n(t) = tI_{n-1}(t);\\
&4) \quad tI_n'(t) - nI_n(t) = tI_{n+1}(t);\\
&5) \quad \left(\tfrac{d}{tdt}\right)^m \{t^n I_n(t)\} = t^{n-m} I_{n-m}(t);\\
&6) \quad \left(\tfrac{d}{tdt}\right)^m \left\{\tfrac{I_n(t)}{t^n}\right\} = \tfrac{I_{n+m}(t)}{t^{n+m}};\\
&7) \quad I_0' = I_1(t);\\
&8) \quad I_{-n}(t) = I_n(t).
\end{aligned} \tag{2.1.9}$$

In the following text, we will use the fact that the Gamma function $\Gamma(z)$ is a meromorphic function of z. Its reciprocal is

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right) e^{-\frac{z}{k}}, \tag{2.1.10}$$

(where $\gamma = 0.577721566\ldots$ is Euler's constant) is an entire function. If $Re z > 0$, then

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$$

and $\Gamma(n+1) = n!$ for an integer $n = 1, 2, 3\ldots$

We use the factorial notation $\alpha! = \Gamma(\alpha + 1)$, where α is not a positive integer, because the function $\Gamma(z)$ has only single poles at the points $z = -n$, $n = 0, 1, 2, \ldots$

Similarly to (2.1.2) it may be proved that

$$e^{\frac{1}{2}t(y+\frac{1}{y})} = \sum_{n=-\infty}^{\infty} y^n I_n(t). \tag{2.1.11}$$

2.2 Generalised Modified Bessel Function of the First Type

Some fundamental results were obtained by Luchak (1956,1958), who solved a batch queueing system in the time-dependent case. He introduced a new generalisation of the modified Bessel function (2.1.8), and found the solution in terms of this function (see section (3.3.1) for details). In our terminology the function $I_n^k(t)$ is called the generalised modified Bessel function of the first type and it is given by

$$I_n^k(t) = \left(\frac{t}{2}\right)^n \sum_{r=0}^{\infty} \frac{(\frac{t}{2})^{r(k+1)}}{r!\Gamma(n+rk+1)}, \tag{2.2.1}$$

where $t \in C$, $n = 0, 1, 2 \ldots$, $k \in \{1, 2 \ldots\}$.

The function $I_n^k(t)$ reduces to the modified Bessel function $I_n(t)$ when $k = 1$. Also we can introduce some properties of this function (2.2.1):

1)

$$I_n^k(t) = \frac{(\frac{1}{2}t)^n}{2\pi i} \int^{(0+)} x^{-n-1} \exp\left(x + \frac{(\frac{1}{2}t)^{k+1}}{x^k}\right) dx, \tag{2.2.2}$$

where the contour of integration is any loop around the origin once clock wise;

2)

$$I_{n+k}^k(t) = \frac{1}{k}\left(I_{n-1}^k(t) - \frac{2n}{t} I_n^k(t)\right);$$

3)

$$\frac{d}{dt}\left(t^{-n} I_n^k(t)\right) = \left(\frac{1}{2}(k+1)\right) t^{-n} I_{n+k}^k(t);$$

4)

$$\frac{d}{dt} I_n^k(t) = \frac{1}{k}\left[\left(\frac{1}{2}(k+1)\right) I_{n-1}^k(t) - \frac{n}{t} I_n^k(t)\right];$$

5)

$$\frac{d}{dt}\left(t^n I_n^k(t)\right) = t^n \left[\left(\frac{k-1}{k}\right)\left(\frac{n}{t}\right) I_n^k(t) + \left(\frac{k+1}{2k}\right) I_{n-1}^k(t)\right];$$

6) $I_n^k(t)$ converges for all t.

The differential equation satisfied by $I_n^k(t)$ is

$$\left(\frac{1}{2}(k+1)\right) t^{k-n} I_{n+k}^k(t) = \frac{d}{dt}\left\{\left(\frac{2k}{k+1}\right)^k t^{2k-2n} \prod_{s=0}^{k-1}\left[\frac{1}{t}\frac{d}{dt} - \frac{k+1}{k}\frac{n-s}{t^2}\right] t^n I_n^k(t)\right\}.$$

Proof.

1) Consider the integral

$$\left(\frac{2}{2\pi i}\right)\left(\frac{1}{2}t\right)^n \int^{(0+)} x^{-n-1}\exp(x + \frac{(\frac{1}{2}t)^{k+1}}{x^k})dx$$

for which the contour of integration is any loop around the origin once clock wise. The integral can be evaluated by taking the contour of integration to be the unit circle $|x| = 1$. The integrand can now be expanded in a power series in t which is uniformly convergent on this contour. Thus the integral is

$$\frac{1}{2\pi i}\sum_{r=0}^{\infty}\frac{(\frac{1}{2}t)^{n+r(k+1)}}{r!}\int^{(0+)} x^{-rk}x^{-n-1}e^x dx.$$

The residual of the integrand at $x = 0$ is $(n + rk)!$. Then the first property is proved.

2) Consider

$$0 = \int^{(0+)} \frac{d}{dt}\left[x^{-n}\exp(x + \frac{(\frac{1}{2}t)^{k+1}}{x^k})\right] dx. \qquad (2.2.3)$$

When differentiations have been effected and (2.2.3) is written out, we get

$$0 = \int^{(0+)}\left(-nx^{-n-1}e^{x+\frac{(\frac{1}{2}t)^{k+1}}{x^k}} + x^{-n}\left[1 - k\frac{(\frac{1}{2}t)^{k+1}}{x^{k+1}}\right]\exp(x + \frac{(\frac{1}{2}t)^{k+1}}{x^k})\right) dx$$

or

$$I_{n+k}^k(t) = \frac{1}{k}\left(I_{n-1}^k(t) - \frac{2n}{t}I_n^k(t)\right).$$

Similar, you can prove the properties (3)-(5).

Note that both the modified Bessel function (2.1.8) and the generalized Bessel function (2.2.1) can be expressed in terms of the Wright function:

$$W(t,\rho,\beta) = \sum_{r=0}^{\infty} \frac{t^r}{r!\Gamma(\rho r+\beta)}, \; t \in C, \; \beta \in C, \; \rho > -1, \tag{2.2.4}$$

which was introduced by the British mathematician E.W. Wright (1933, 1935). It appeared for the first time for the case $\rho > 0$ in connection with his investigations in the asymptotic theory of partitions. Later on, many other applications have been found, first of all in the Mikusinski operational calculus analysis in the theory of integral transforms of Hankel type. Recently this function has appeared in papers relating to partial differential equations of fractional order.

Indeed, in terms of the Wright function, $I_n(t)$ and $I_n^k(t)$ can be written as

$$I_n(t) = \left(\frac{t}{2}\right)^n W\left(\frac{t^2}{4}, 1, n+1\right), \tag{2.2.5}$$

$$I_n^k(t) = \left(\frac{t}{2}\right)^n W\left(\frac{t^{k+1}}{2^{k+1}}, k, n+1\right). \tag{2.2.6}$$

In the general case of arbitrary real $\rho > -1$, the Wright function is a particular case of the Fox H-function. Unfortunately, since the Fox H-function is a very general object this representation is not especially informative. It turns out that if ρ is a positive rational number the Wright function can be represented in terms of the more familiar generalised hypergeometric functions.

Some further properties of the Wright function can be found in the paper by Gorenflo, Luchko and Mainardi (1999).

2.3 New Generalisation of the Modified Bessel Function of the Second Type and Its Generating Function

Generating functions play an important role in the investigation of various problems (including, for example, queueing theory and related stochastic processes).

It can be shown, see Sections 3.3.5 and 3.3.6, that for the more general case of the $M/E_k/1$ queue, which reduces to the $M/M/1$ model for $k = 1$, an important role is played by the extended generating function

$$\exp\left(\frac{1}{2}t(y^k + \frac{1}{y})\right) = \sum_{n=-\infty}^{\infty} \sum_{s=1}^{k} y^{k(n-1)+s} f_{n,k}(t). \tag{2.3.1}$$

The analytic form of the coefficients set $\{f_n(t)\}_{n=-\infty}^{\infty}$ has potential interest in the study of the transient solution to the $M/E_k/1$ system. That is why it is important to present the generating function (2.3.1) in the form of a double sum with corresponding power.

We now introduce a new generalisation of the modified Bessel function (2.1.8). Consider the function

$$\tilde{I}_n^{k,s}(t) = \left(\frac{t}{2}\right)^{n+k-s} \sum_{r=0}^{\infty} \frac{(\frac{t}{2})^{r(k+1)}}{(k(r+1)-s)!\Gamma(n+r+1)}, \tag{2.3.2}$$

where $t \in C,\ s \in \{1, 2, \ldots, k\},\ n = 0, \pm 1, \pm 2, \ldots,\ k = \{1, 2, \ldots\}$.

Note that the series (2.3.2) converges absolutely for all $t \in C$ (see below). The function $\tilde{I}_n^{k,s}(t)$ reduces to the modified Bessel function (2.1.8), when $k = s = 1$.

In our terminology the function (2.3.2) is called the generalised modified Bessel function of the second type. We are now able to generalise the formula (2.1.11).

Lemma 2.3.1: The generating function of the generalised modified Bessel function of the second type takes the form

$$e^{\frac{1}{2}t(y^k+\frac{1}{y})} = \sum_{n=-\infty}^{\infty} \sum_{s=1}^{k} y^{k(n-1)+s} \tilde{I}_n^{k,s}(t). \tag{2.3.3}$$

Proof. By using the Laurent expansion, we obtain

$$e^{\frac{1}{2}t(y^k+\frac{1}{y})} = \sum_{m=0}^{\infty} \sum_{r=0}^{\infty} \frac{(\frac{1}{2}ty^k)^m}{m!} \frac{(\frac{1}{2}\frac{t}{y})^r}{r!}. \tag{2.3.4}$$

We need the following identity (see Srivastava, Kashyap (1982)):

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \psi(m,n) = \sum_{j=0}^{M-1} \sum_{k=0}^{N-1} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \psi(mM+j, nN+k), \; M, N \in \{1, 2, \ldots\}, \tag{2.3.5}$$

where ψ is an arbitrary function such that the series (2.3.5) converges. Putting $M = 1, n = r, k = l$ we get

$$\sum_{m=0}^{\infty} \sum_{r=0}^{\infty} \psi(m,r) = \sum_{m=0}^{\infty} \sum_{r=0}^{\infty} \sum_{l=1}^{k} \psi(m, rk+l-1). \tag{2.3.6}$$

Thus, (2.3.4) can be rewritten as

$$\sum_{m=0}^{\infty} \sum_{r=0}^{\infty} \frac{(\frac{1}{2}ty^k)^m}{m!} \frac{(\frac{1}{2}\frac{t}{y})^r}{r!} = \sum_{m=0}^{\infty} \sum_{r=0}^{\infty} \sum_{l=1}^{k} \frac{(\frac{1}{2}ty^k)^m}{m!} \frac{(\frac{1}{2}\frac{t}{y})^{rk+l-1}}{(rk+l-1)!}$$

$$= \sum_{m=0}^{\infty} \sum_{r=0}^{\infty} \sum_{l=1}^{k} \frac{(\frac{1}{2}t)^{m+rk+l-1} y^{km-rk-l+1}}{m!(rk+l-1)!}. \tag{2.3.7}$$

Changing the variables $n = m - r, \; n \in (-\infty, \infty)$ in (2.3.7) gives

$$\sum_{m=0}^{\infty} \sum_{r=0}^{\infty} \sum_{l=1}^{k} \frac{(\frac{1}{2}t)^{n+r+rk+l-1} y^{kn-l+1}}{(r+n)!(rk+l-1)!}.$$

If we change the variables $s = k - l + 1, \; s = 1, \ldots, k$ we obtain

$$e^{\frac{1}{2}t(y^k+\frac{1}{y})} = \sum_{n=-\infty}^{\infty}\sum_{s=1}^{k} y^{k(n-1)+s} \sum_{r=0}^{\infty} \frac{(\frac{1}{2}t)^{n+r(k+1)+k-s}}{(r+n)!(k(r+1)-s)!}$$

$$= \sum_{n=-\infty}^{\infty}\sum_{s=1}^{k} y^{k(n-1)+s} \tilde{I}_n^{k,s}(t).$$

□

Note that (2.3.2) cannot be expressed in terms of the Wright function. However we are able to find an analogous property to 8) (2.1.9).

Let us consider (2.3.2) for positive integers $n = 1, 2, \ldots$. Then for $p = -n$ we obtain

$$\tilde{I}_{-n}^{k,s}(t) = \sum_{r=0}^{\infty} \frac{(\frac{1}{2}t)^{-n+r(k+1)+k-s}}{\Gamma(r-n+1)(k(r+1)-s)!} = \sum_{r=n}^{\infty} \frac{(\frac{1}{2}t)^{-n+r(k+1)+k-s}}{\Gamma(r-n+1)(k(r+1)-s)!}.$$

Since the function $\frac{1}{\Gamma(r-n+1)(k(r+1)-s)!} = 0$ (by the definition (2.1.10)), for r such that $r - n + 1 \leq 0$, we obtain for $l = r - n$

$$\tilde{I}_{-p}^{k,s}(t) = \sum_{l=0}^{\infty} \frac{(\frac{1}{2}t)^{-p+(l+p)(k+1)+k-s}}{\Gamma(l+1)(k(l+p+1)-s)!}$$

$$= \left(\frac{t}{2}\right)^{k-s+pk} \sum_{l=0}^{\infty} \frac{(\frac{t}{2})^{l(k+1)}}{l!(k(l+p+1)-s)!}. \tag{2.3.8}$$

Note that for $k = s = 1$ the relation (2.3.8) reduces to property 8) in (2.1.9). The right hand side of (2.3.8) can be considered as another generalisation of the modified Bessel function since it is easy to reduce it to (2.1.8) when $k = s = 1$.

The function (2.3.8) can be expressed in terms of the Wright function as follows for positive n

$$\left(\frac{t}{2}\right)^{nk+k-s} \sum_{r=0}^{\infty} \frac{(\frac{t}{2})^{r(k+1)}}{r!\Gamma(kr+k(n+1)+1-s)} = \left(\frac{t}{2}\right)^{nk+k-s} W\left(\frac{t^{k+1}}{2^{k+1}}, k, k(n+1)-s+1\right).$$

2.4 Relations of the Generalised Modified Bessel Function of the Second Type to other Special Functions

2.4.1 Generalised Wright Function

The function (2.3.2) is related to other special functions. The paper by Kilbas, Saigo and Trujillo (2002) deals with the generalised Wright function defined for $t \in C$, $a_j, b_j \in C$, $\alpha_j,\beta_j \in R\,(\alpha_j,\beta_j \neq 0,\ i = 1,..,p,\ j = 1,...,q)$ by the series

$$\psi_{p,q}(t) = \psi_{p,q}\left(\begin{array}{c|c} (a_i,\alpha_i)_{1,p} & \\ & t \\ (b_i,\beta_i)_{1,q} & \end{array}\right) = \sum_{r=0}^{\infty} \frac{\prod_{i=1}^{p}\Gamma(a_i+\alpha_i r)}{\prod_{j=1}^{q}\Gamma(b_i+\beta_i r)}\frac{t^r}{r!} \qquad (2.4.1)$$

which was introduced by Wright (1935) for p=1, q=2. This function generalises many special functions. It is known that if

$$\Delta = \sum_{j=1}^{q}\beta_j - \sum_{i=1}^{p}\alpha_i > -1, \qquad (2.4.2)$$

then the series (2.4.1) is absolutely convergent for all $z \in C$.

Also the paper by Kilbas, Saigo and Trujillo (2002) studies the properties of the generalised Wright function (2.4.1) and establishes the conditions for the existence of $\psi_{p,q}(t)$ and proves an integral representations in terms of the Mellin-Barnes integral. Some other special cases of the generalised Wright function involving the Mittag-Leffler function and of its generalisations are also presented, which is widely used in statistics and queueing theory (see Mathi, Saxena (1978), Srivastava, Kashyap (1982)).

In terms of the generalised Wright function (2.4.1), the function (2.3.2) can be expressed as follows:

$$\tilde{I}_n^{k,s}(t) = \left(\frac{t}{2}\right)^{n+k-s} \sum_{r=0}^{\infty} \frac{\left[\left(\frac{t}{2}\right)^{k+1}\right]^r}{r!} \frac{\Gamma(1+r)}{\Gamma(n+r+1)\Gamma(k-s+kr)}$$

$$= \left(\frac{t}{2}\right)^{n+k-s} \psi_{1,2}\left(\begin{matrix} (1,1) \\ (n+1,1)(k-s,k) \end{matrix} \left| \frac{t^{k+1}}{2^{k+1}} \right. \right), \tag{2.4.3}$$

with $p = 1$, $a_i = 1$, $\alpha_i = 1$, $q = 2$, $b_1 = n+1$, $b_2 = 1$, $\beta_1 = k - s$, $\beta_2 = k$. In our case $\Delta = 1 + k - 1 > -1$ and by (2.4.2) the series (2.3.2) is absolutely convergent for all $t \in C$.

2.4.2 Generalised Mittag-Leffler Function

Special attention and detailed studies of Mittag-Leffler functions have found place in numerous works and papers by Dzbashian (1966), Gorenflo, Mainardi (1997), Kiryakova (1997) and others.

Recently, the attention towards the Mittag-Leffler type functions and the recognition of their importance has increased from both analytical and numerical points of view.

The Mittag-Leffler functions $E_{\beta,\mu}(t)$ defined by the power series

$$E_{\beta,\mu}(t) = \sum_{r=0}^{\infty} \frac{t^r}{\Gamma(\mu + \beta r)}, \quad t \in C, \quad \mu > 0, \quad \beta > 0. \tag{2.4.4}$$

are natural extensions of the exponential function and trigonometric function.

Let us introduce so-called multi-index (multiple) Mittag-Leffler functions (see Kiryatova (2000))

$$E_{(\beta_1,...,\beta_m),(\mu_1,...,\mu_m)}(t) = \tag{2.4.5}$$

$$\sum_{r=0}^{\infty} \varphi_r t^r = \sum_{r=0}^{\infty} \frac{t^r}{\Gamma(\mu_1 + \beta_1 r)...\Gamma(\mu_m + \beta_m r)},$$

$t \in C$, $m \geq 1$ is an integer, $\beta_1, ..., \beta_m > 0$ and $\mu_1, ..., \mu_m$ are arbitrary real numbers.

The radius of convergence of series (2.4.5), by the Cauchy-Hadamard formula, is $R > 0$, where

$$\frac{1}{R} = \lim_{r\to\infty} \sup \sqrt[r]{|\varphi_r|} = \lim_{r\to\infty} \sup \left[\prod_{i=1}^{m} \Gamma(\mu_i + \beta_i r)\right]^{-\frac{1}{r}} = 0. \tag{2.4.6}$$

By Stirling's asymptotic formula for the Γ-function

$$\Gamma(r) \sim \sqrt{2\pi} r^{r-\frac{1}{2}} e^{-r}, \quad r \to \infty$$

we have

$$\Gamma(\mu_i + \beta_i r) \sim \sqrt{2\pi} (\mu_i + \beta_i r)^{\mu_i + r\beta_i - \frac{1}{2}} e^{-\mu_i} e^{\beta_i r}$$

and

$$\frac{1}{R} = \lim_{r\to\infty} \prod_{i=1}^{m} \left[(r\beta_i)^{-\beta_i} e^{\beta_i}\right] = \lim_{r\to\infty} r^{-(\beta_1+\ldots+\beta_m)} e^{(\beta_1+\ldots+\beta_m)} \beta_1^{-\beta_1} \ldots \beta_m^{-\beta_m} = 0.$$

Thus, the generalised modified Bessel function of the second type can be written in terms of (2.4.5) as follows

$$\tilde{I}_n^{k,s}(t) = \left(\frac{t}{2}\right)^{n+k-s} \sum_{r=0}^{\infty} \left[\left(\frac{t}{2}\right)^{k+1}\right]^r \frac{1}{\Gamma(n+1+r)\Gamma(k-s+kr)}$$

$$= E_{(1,k),(n+1,k-s)}\left(\frac{t^{k+1}}{2^{k+1}}\right). \tag{2.4.7}$$

In the paper by Kiryakova (2000) one can find various relationships between the multi-index Mittag-Leffler function (2.4.5) and other special functions, such as Fox's H-functions, Bessel-Maitlard functions, Struve and Lommel functions as well as generalized fractional calculus.

CHAPTER 3: ERLANG QUEUEING MODEL

3.1 Review of Literature for the Erlang Model

Up to now, we have considered the queueing models with Poisson input and exponential service times.

One method of extending the $M/M/1$ queueing system is to model identical exponential phases. The Erlang family of probability distributions provides far greater flexibility in modelling real-life service patterns than does exponential.

Only a few papers deal with the approximation to the transient solution of this queueing system.

The differential-difference equations for the $M/E_k/1$ model we can be written in the following form (see section (3.2))

$$\begin{cases} \frac{dp_0(t)}{dt} = -\lambda p_0(t) + k\mu p_{1,1}(t) & n = 0; \\ \frac{dp_{1,s}(t)}{dt} = -(\lambda + k\mu)p_{1,s}(t) + k\mu p_{1,s+1}(t) & n = 1,\ 1 \le s \le k-1; \\ \frac{dp_{1,k}(t)}{dt} = -(\lambda + k\mu)p_{1,k}(t) + \lambda p_0(t) + k\mu p_{2,1}(t) & n = 1, s = k; \\ \frac{dp_{n,s}(t)}{dt} = -(\lambda + k\mu)p_{n,s}(t) + k\mu p_{n,s+1}(t) + \lambda p_{n-1,s}(t) & n > 1,\ 1 \le s \le k-1; \\ \frac{dp_{n,k}(t)}{dt} = -(\lambda + k\mu)p_{n,k}(t) + k\mu p_{n+1,1}(t) + \lambda p_{n-1,k}(t) & n > 1,\ s = k; \end{cases} \tag{3.1.1}$$

where $p_{n,s}(t)$ represents the probability that there are n customers in the system at time t with the customer in service being in phase s $(1 \le s \le k)$.

These equations are not particularly easy to handle even for the steady-state case (available results can be seen in section 3.2 and 4.2).

Since deriving the exact solution of (3.1.1) in the time-dependent case is a difficult problem, different methods of computing probabilities as well as computing mean queueing characteristics have been proposed.

In the paper by B.W. Conolly (1960) the Busy Period of the $GI/E_k/1$ is investigated and the Laplace transform of the generating function has been obtained. Because of the difficulty of inverting the Laplace transform this result can only be checked for the $M/M/1$ case.

Diffusion approximations for the Busy Period in $M/G/1$ queues were studied in papers by Gaver (1962, 1968). The results were extended by Heyman (1974). Also the diffusion approximation for an $M/G/m$ queue in the steady-state case has been presented by T. Kimura (1983).

Some fundamental results have been obtained by G. Luchak (1956, 1958), who solved a batch queueing system in the transient case. He introduced a new generalisation of the modified Bessel function (2.2.1) and found an inverse Laplace transform for $p_0(t)$ (probability that there are no customers in the system at time t).

$$p_0(t) = \frac{1}{\mu t} \sum_{m=1}^{\infty} m \left(\frac{(\mu t)^m}{m!} + \sum_{n=1}^{\infty} \frac{(\rho \mu t)^n}{n!} \sum_{j=0}^{\infty} b_{nj} \frac{(\mu t)^{j+n+m}}{(j+n+m)!} \right) \exp(-(1+\rho)\mu t, \quad (3.1.2)$$

where $\rho = \lambda/\mu$ and b_{nj} is related to binomial coefficients. Also, $p_0(t)$ can be written down directly for the Erlang service system, as it is a particular case of the batch queueing system (see section 3.3). Note, that by means of $p_0(t)$ we can write exact formula for $W_q(t)-$ the mean waiting time of a customer arriving in the queue at time t. Even knowing $p_0(t)$ it is difficult to compute and to write down the explicit form for other probabilities $p_{n,s}(t)$, because of the complex structure of (3.1.1).

E. Roth (1983) examines the transient behaviour of finite-capacity, single-server, Markovian queueing systems. He estimated the manner in which $L_q(t)$ approaches $L_q(\infty)$, where $L_q(t)$ is the expected number of customers in a queue at time t, by using a decaying exponential function.

$$L_q(t) = L_q(\infty)[1 - \exp(-t/\tau)], \quad t \geq 0 \qquad (3.1.3)$$

where $L_q(\infty)$ and τ are constants that depend on system parameters as well as the particular interarrival and service time distributions.

W.D. Kelton (1985) proposed the computational algorithms for obtaining the required probabilities as well as illustrating the application in calculating a variety of system performance measures.

Calculating transient characteristics of queues and probability distributions by numerical Laplace transform inversion has been discussed in many papers, for example J. Abate (1993,1995), J Abate, W. Whitt (1988, 1998). The most outstanding results were published in 1993 by G.L. Choughury, D.M. Lucantoni, W. Whitt. They developed an algorithm for numerically inverting multidimenstional transforms with applications to the transient $M/G/1$ queue. They also applied this method to calculate time-dependent distributions in the transient $BMAP/G/1$ queue (with a batch Markovian arrival process) and the piecewise-stationary $M_t/G_t/1$ queue.

In 1974 J.A. Murphy and H.R. O'Donohoe approximated the numerical solution of the Kolmogorov equation for the generalised birth and death process by use of continued fractions. M. Mederer (2003) generalised this approach by suggesting an algorithm for $q-$matrices of lower band structure $(n, 1)$. Applications involving $q-$matrices of this type include, for example, many types of queueing systems with batch processing.

3.2 Erlang Model $M/E_k/1/\infty/FIFO$ in the Steady-State

3.2.1 Solution for the $M/E_k/1$ Queue in the Steady-State

We consider a single-server queueing system which has Poisson input and Erlang service times. The Erlang model is a well-known queueing system which is applicable to many real-life situations. The relation of the Erlang to the exponential distribution also allows us to describe queueing models where the service may be a series of identical phases. For example, a hospital where people need to have a medical examination, followed by identical phases such as eye examination, x-ray, blood test, etc.

By using the Erlang distribution as the service distribution, a customer entering service may be considered to generate a set of k phases of service. The phases have identical exponential distributions with parameter μk. For the total service time of a customer we have the Erlang distribution

$$\frac{(\mu k)^k}{(k-1)!} x^{k-1} e^{-k\mu x}, \quad 0 \leq x < \infty.$$

Let us assume that arrivals occur at random at mean rate λ, and that service times have an Erlang distribution with parameter k. This allows us to describe queueing models, where the service facility may be a series of identical phases and it should be noted following:

1) all steps (or phases) of the service are independent and identical;

2) only one customer at a time is allowed in the service facility as whole.

This queueing model can be illustrated as follows (see Figure 3.2.1).

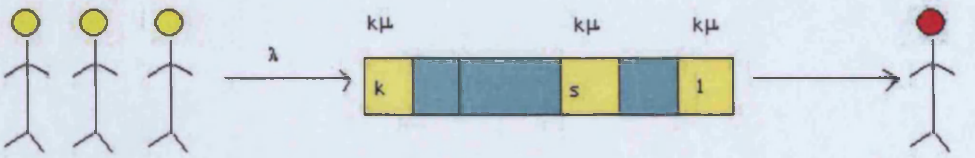

Figure 3.2.1 Erlang service queueing model

Let us denote $p_{n,s}(t)$ as the probability of n customers in the system with the customer is service being in phase s, $s = 1, 2, \ldots, k$, where we now number the phases backwards; that is, k is the first phase of service and 1 is the last (a customer leaving phase 1 actually leaves the system).

Let $p_0(t)$ denote the probability that there are no customers in the system at time t. Initially we assume that there are no customers in the system, that is $p_0(0) = 1$ for $n = 0$, and $p_{n,s}(0) = 0$ for all s and $n > 0$.

Thus we have

$\Pr\{\text{one arrival in } (t, t + \Delta t)\} = \lambda \Delta t + o(\Delta t)^2$

$\Pr\{> 1 \text{ arrival in } (t, t + \Delta t)\} = o(\Delta t)^2$

$\Pr\{\text{no arrival in } (t, t + \Delta t)\} = 1 - \lambda \Delta t + o(\Delta t)^2$

$\Pr\{\text{an existing phase service finishes in next } \Delta t\} = (k\mu)\Delta t + o(\Delta t)^2$

$\Pr\{> 1 \text{ existing phase service finishes in next } \Delta t\} = o(\Delta t)^2$

$\Pr\{\text{an existing phase service does not finishes in next } \Delta t\} = 1 - (k\mu)\Delta t + o(\Delta t)^2$.

Using the usual technique, we set up the following equations

$$
\begin{cases}
p_0(t+\Delta t) = & p_0(t)(1-\lambda\Delta t) + p_{1,1}(t)k\mu\Delta t, \\
& n = 0; \\
p_{1,s}(t+\Delta t) = & p_{1,s}(t)(1-\lambda\Delta t - k\mu\Delta t) + p_{1,s+1}(t)k\mu\Delta t, \\
& n = 1, 1 \leq s \leq k-1; \\
p_{1,k}(t+\Delta t) = & p_{1,k}(t)(1-\lambda\Delta t - k\mu\Delta t) + p_{2,1}(t)k\mu\Delta t + p_0(t)\lambda\Delta t, \\
& n = 1, s = k; \\
p_{n,s}(t+\Delta t) = & p_{n,s}(t)(1-\lambda\Delta t - k\mu\Delta t) + p_{n,s+1}(t)k\mu\Delta t + p_{n-1,s}(t)\lambda\Delta t; \\
& n \geq 2, 1 \leq s \leq k-1; \\
p_{n,k}(t+\Delta t) = & p_{n,k}(t)(1-\lambda\Delta t - k\mu\Delta t) + p_{n+1,1}(t)k\mu\Delta t + p_{n-1,k}(t)\lambda\Delta t, \\
& n \geq 2, s = k.
\end{cases}
\tag{3.2.1}
$$

where all $o(\Delta t)$ terms are ignored.

First, we will study the steady-state difference equations. Equating the left side of the equations (3.2.1) to zero, we have

$$
\begin{cases}
0 = & -\lambda p_0 + k\mu p_{1,1} & n = 0; \\
0 = & -(\lambda + k\mu)p_{1,s} + k\mu p_{1,s+1} & n = 1,\ 1 \leq s \leq k-1; \\
0 = & -(\lambda + k\mu)p_{1,k} + k\mu p_{2,1} + \lambda p_0 & n = 1,\ i = k \\
0 = & -(\lambda + k\mu)p_{n,s} + k\mu p_{n,s+1} + \lambda p_{n-1,s} & n \geq 2,\ 1 \leq s \leq k-1; \\
0 = & -(\lambda + k\mu)p_{n,k} + k\mu p_{n,1} + \lambda p_{n-1,k} & n \geq 2,\ s = k;
\end{cases}
\tag{3.2.2}
$$

We note that the above equations take on a simple form if we divide throughout by

$k\mu$. For convenience we shall make the substitution

$$\theta = \frac{\lambda}{k\mu} = \frac{\rho}{k}.$$

This parameter θ is closely related to the parameter ρ which known as the relative traffic intensity or utilization factor. Then we obtain

$$p_{1,1} = \theta p_0;$$

$$\begin{cases} p_{1,2} = & (1+\theta)p_{1,1}; \\ p_{1,3} = & (1+\theta)p_{1,2}; \\ \vdots & \vdots \\ p_{1,k} = & (1+\theta)p_{1,k-1}; \end{cases}$$

$$p_{2,1} = (1+\theta)p_{1,k} - \theta p_0;$$

$$\begin{cases} p_{2,2} = & (1+\theta)p_{2,1} - \theta p_{1,1}; \\ p_{2,3} = & (1+\theta)p_{2,2} - \theta p_{1,2}; \\ \vdots & \vdots \\ p_{2,k} = & (1+\theta)p_{2,k-1} - \theta p_{1,k-1}; \end{cases} \tag{3.2.3}$$

$$p_{3,1} = (1+\theta)p_{2,k} - \theta p_{1,k};$$

etc.

These equations are not particularly easy to solve successively. Furthermore, after obtaining the $p_{n,s}$ to get the steady-state probabilities of n customers in the system, p_n, it is necessary to calculate

$$p_n = \sum_{s=1}^{k} p_{n,s}. \tag{3.2.4}$$

Let us use the following generating function

$$G(y) = \sum_{n=1}^{\infty} \sum_{s=1}^{k} y^{k(n-1)+s} p_{n,s} + p_0. \tag{3.2.5}$$

Multiplying each queue equation (3.2.3) by the appropriate power of y, we obtain

$yp_{1,1} = \theta y p_0;$

$y^2 p_{1,2} = (1+\theta) y^2 p_{1,1};$

$y^3 p_{1,3} = (1+\theta) y^3 p_{1,2};$

. . . .

$y^k p_{1,k} = (1+\theta) y^k p_{1,k-1};$

$y^{k+1} p_{2,1} = (1+\theta) y^{k+1} p_{1,k} - \theta y^{k+1} p_0;$

$y^{k+2} p_{2,2} = (1+\theta) y^{k+2} p_{2,1} - \theta y^{k+2} p_{1,1};$

. . . .

$y^{2k} p_{2,k} = (1+\theta) y^{2k} p_{2,k-1} - \theta y^{2k} p_{1,k-1};$

$y^{2k+1} p_{3,k} = (1+\theta) y^{2k+1} p_{2,k} - \theta y^{2k+1} p_{1,k};$

etc.

Summing over n and s we can write the following equation for the generating function

$$G(y) = \frac{p_0(1-y)}{1-(1+\theta)y+\theta y^{k+1}} = \frac{p_0}{1-\theta\sum_{s=1}^{k} y^s}. \tag{3.2.6}$$

To determine the value of p_0 we can use the initial condition, that is $G(1) = 1$, thus

$$1 = \frac{p_0}{1-k\theta} \Rightarrow p_0 = 1 - k\theta.$$

Substituting p_0 into (3.2.6) we can write the following expression for the generating function

$$G(y) = \frac{1 - k\theta}{1 - \theta \sum_{s=1}^{k} y^s}. \tag{3.2.7}$$

We should remember that $G(y)$ generates the phase probabilities $p_{n,s}$. Thus, to find the state probabilities we need to expand $G(y)$ as a power series in y and then pick out all the terms involving appropriate powers to determine $p_{n,s}$. For example to define $p_{1,3}$ we need to pick out all the terms in the (3.2.6) which contain y^3. That is

$$\begin{aligned} G(y) &= (1 - k\theta)(1 - \theta(y + y^2 + ... + y^k)^{-1} = \\ &\quad (1 - \theta k)\left[1 + \theta(y + ... + y^n) + \theta^2(y + ... + y^n)^2 + ...\right]. \end{aligned}$$

And

$$p_{1,3} = (1 - k\theta)\theta(1 + \theta)^2.$$

But the phase probabilities $p_{n,s}$ are not always of prime interest. The phase p.g.f. $G(y)$, is useful in determining the usual four summary measures of a queue.

To find the overall mean waiting time, we must sum over all possible states which the arriving customer might find.

Thus,

$$W_q = \frac{1}{k\mu} \sum_{n=1}^{\infty} \sum_{s=1}^{k} [(n-1)k + s] p_{n,s}. \tag{3.2.8}$$

We can find the relationship between $G(y)$ and W_q by differentiating (3.2.5) and putting $y = 1$.

$$\left. \frac{dG(y)}{dy} \right|_{y=1} = \sum_{n=1}^{\infty} \sum_{s=1}^{k} [(n-1)k + s] p_{n,s} \tag{3.2.9}$$

Then from (3.2.8) and (3.2.9) we can write an expression for W_q

$$W_q = \frac{1}{k\mu} G'(1). \tag{3.2.10}$$

Differentiating (3.2.7) and putting $y = 1$ we can write

$$W_q = \frac{(k+1)\rho}{2k\mu(\mu-\rho)} = \frac{k+1}{2k}\frac{\lambda}{\mu(\mu-\lambda)}. \tag{3.2.11}$$

Also we can get W−the main waiting time in the system

$$W = W_q + \frac{1}{\mu} = \frac{(k+1)\rho}{2k\mu(\mu-\rho)} + \frac{1}{\mu}. \tag{3.2.12}$$

Using Little's formulas one can write an expression for the mean number of customers in a queue (L_q) and the mean number customers in the system (L) as follows

$$L_q = \lambda W_q = \frac{k+1}{2k}\frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{k+1}{2k}\frac{\rho^2}{1-\rho}, \tag{3.2.13}$$

$$L = \lambda W = \frac{k+1}{2k}\frac{\rho^2}{1-\rho} + \rho. \tag{3.2.14}$$

For illustration, numerical results for the Erlang service queueing characteristics in the steady-state case are given in Table 3.2.1

Table 3.2.1 Queueing characteristics for Erlang model with $k = 2$, $\mu = 1$.

λ	0.1	0.2	0.3	0.5	0.7	0.8	0.9
W_q	0.0833	0.1875	0.3214	0.7500	1.7500	3.0000	6.7500
W	1.0833	1.1875	1.3414	1.7500	2.7500	4.0000	7.7500
L_q	0.0083	0.0375	0.0964	0.3750	1.2250	2.4000	6.0750
L	0.1083	0.2375	0.3964	0.8750	1.9250	3.2000	6.9750

3.2.2 $M/E_k/1$ System in Bulk Arrival Terms

The Bulk queueing system is a more general case than the Erlang service model. We can rewrite the $M/E_k/1$ system in bulk terms and also solve the state-state equations to obtain the same results for the mean queueing characteristics.

We can describe the Bulk queue in the following way. The arrival stream of the queueing system $M^{[X]}/M/1$ forms a Poisson process and the actual number of customers in the arriving module is a random variable X, which may take on any positive integral value less than ∞ with probability c_X.

Let λ_X be the arrival rate of the Poisson process of batches of size X, then $c_X = \lambda_X/\lambda$, where λ is the composite arrival rate of all batches and

$$\lambda = \sum_{i=1}^{\infty} \lambda_i.$$

A set of Chapman-Kolmogorov equations can be derived for this problem in the usual manner

$$\begin{cases} \frac{dp_0(t)}{dt} = & -\lambda p_0(t) + \mu p_1(t), \\ \frac{dp_n(t)}{dt} = & -(\lambda+\mu)p_n(t) + \mu p_{n+1}(t) + \lambda \sum_{j=1}^{n} c_j p_{n-j}(t),\ n \geq 1, \end{cases} \tag{3.2.15}$$

where a negative subscript indicates the term is zero. We are able to rewrite the Erlang equations in the (3.2.15) form.

We have k stages. If we consider the state at a time when the system contains n customers and when the sth stage of service contains the customer in service, we then have that the number of stages contained in the total system is

$$j = (n-1)k + (k-s+1) = kn - s + 1.$$

The relationship between customers and stages allows us to write

$$p_n = \sum_{j=(n-1)k+1}^{nk} p_j,\ n = 1, 2, 3, \ldots \tag{3.2.16}$$

and where $p_j = \Pr\{j \text{ stages in system}\}$. Making the transformation $(n, s) = (n-1)k+s$ we can write the differential-difference equations in the following way

$$\begin{cases} \lambda p_0 = k\mu p_1, \\ (\lambda + k\mu)p_j = \lambda p_{j-k} + k\mu p_{j+1}, \; j = 1, 2, \ldots \end{cases} \tag{3.2.17}$$

We define the generating function of the p_j by

$$G(y) = \sum_{j=0}^{\infty} p_j y^j. \tag{3.2.18}$$

As usual, we multiply the $j-$th equation in (3.2.17) by y^j and sum over all applicable j. This yields

$$\sum_{j=1}^{\infty}(\lambda + k\mu)p_j y^j = \sum_{j=1}^{\infty} \lambda p_{j-k} y^j + \sum_{j=1}^{\infty} k\mu p_{j+1} y^j. \tag{3.2.19}$$

Recognizing $G(y)$ we obtain

$$(\lambda + k\mu)\left[G(y) - p_0\right] = \lambda y^k G(y) + \frac{k\mu}{y}\left[G(y) - p_0 - p_1 y\right] \Longrightarrow$$

$$G(y) = \frac{p_0\left[\lambda + k\mu - \left(\frac{k\mu}{y}\right)\right] - k\mu p_1}{\lambda + k\mu - \lambda y^k - \left(\frac{k\mu}{y}\right)} = \frac{k\mu p_0[1-y]}{k\mu + \lambda y^{k+1} - (\lambda + k\mu)y}. \tag{3.2.20}$$

We can calculate p_0 from the initial condition that is $G(1) = 1$ and using L'Hospital's rule

$$G(1) = \frac{k\mu p_0}{k\mu - \lambda k} \Longrightarrow p_0 = 1 - \frac{\lambda}{\mu}. \tag{3.2.21}$$

Substituting p_0 into (3.2.20) we have

$$\begin{aligned} G(y) &= \frac{k\mu(1-\rho)(1-y)}{k\mu + \lambda y^{k+1} - (\lambda + k\mu)y} \\ &= \frac{1-\rho}{(1 - y/y_1)(1 - y/y_2)\cdots(1 - y/y_k)} \end{aligned}$$

$$= (1-\rho)\sum_{s=1}^{k} \frac{B_s}{(1 - y/y_s)}, \tag{3.2.22}$$

where

$$B_s = \prod_{m=1, m\neq s} \frac{1}{(1 - y_s/y_m)} \tag{3.2.23}$$

and $y_1, ..., y_k$ are zeros of the polynomial $k\mu + \lambda y^{k+1} - (\lambda + k\mu)y$ which lie outside of the Unit Circle. Finally we have the solution for the distribution of the number of stages in the form

$$p_j = (1-\rho)\sum_{s=1}^{k} \frac{B_s}{(y_s)^j}, \; j = 1, 2, \ldots, k. \tag{3.2.24}$$

Also, we can obtain the same measures of effectiveness (3.2.11)-(3.2.14).

3.3 Transient Solutions for the $M/E_k/1$ Model in Terms of Generalised Modified Bessel Function of the First Type

As can be seen from the previous section, the steady-state probabilities can be computed for the Erlang service system. The outstanding results for the transient solution of the Bulk queue have been obtained by Luchak (1956, 1958). It can be shown that the Erlang service queue can be rewritten in the bulk queue terms and consequently one can obtain the solution of $M/E_k/1$ system in terms of generalised modified Bessel function of the first type.

Let us return to the differential difference equations, which we have derived in Section 3.2.

$$\begin{cases} \frac{dp_0(t)}{dt} = -\lambda p_0(t) + k\mu\, p_{1,1}(t) & n = 0; \\ \frac{dp_{1,s}(t)}{dt} = -(\lambda + k\mu)p_{1,s}(t) + k\mu\, p_{1,s+1}(t) & n = 1,\ 1 \le s \le k-1; \\ \frac{dp_{1,k}(t)}{dt} = -(\lambda + k\mu)p_{1,k}(t) + \lambda p_0(t) + k\mu\, p_{2,1}(t) & n = 1, s = k; \\ \frac{dp_{n,s}(t)}{dt} = -(\lambda + k\mu)p_{n,s}(t) + k\mu\, p_{n,s+1}(t) + \lambda p_{n-1,s}(t) & n > 1,\ 1 \le s \le k-1; \\ \frac{dp_{n,k}(t)}{dt} = -(\lambda + k\mu)p_{n,k}(t) + k\mu\, p_{n+1,1}(t) + \lambda p_{n-1,k}(t) & n > 1,\ s = k; \end{cases} \tag{3.3.1}$$

and we can define the generating function

$$G(y,t) = \sum_{n=0}^{\infty}\sum_{s=1}^{k} y^{(n-1)k+s} p_{n,s}(t), \tag{3.3.2}$$

which must be analytic within the unit circle $|y| = 1$.

To solve the equations (3.3.1) we multiply the first equation by y, the second by y^2, third by y^3 and etc.

$$\begin{cases} \frac{dp_0(t)}{dt}y = -\lambda p_0(t)y + k\mu\, p_{1,1}(t)y \\ \frac{dp_{1,1}(t)}{dt}y^2 = -(\lambda+k\mu)p_{1,1}(t)y^2 + k\mu\, p_{1,2}(t)y^2 \\ \frac{dp_{1,2}(t)}{dt}y^3 = -(\lambda+k\mu)p_{1,2}(t)y^3 + k\mu\, p_{1,3}(t)y^3 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \frac{dp_{1,k}(t)}{dt}y^{k+1} = -(\lambda+k\mu)p_{1,k}(t)y^{k+1} + \lambda p_0(t)y^{k+1} + k\mu\, p_{2,1}(t)y^{k+1} \\ \frac{dp_{2,1}(t)}{dt}y^{k+2} = -(\lambda+k\mu)p_{2,1}(t)y^{k+2} + k\mu\, p_{2,2}(t)y^{k+2} \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \frac{dp_{n,1}(t)}{dt}y^{(n-1)k+2} = -(\lambda+k\mu)p_{n,1}(t)y^{(n-1)k+2} + k\mu\, p_{n,2}(t)y^{(n-1)k+2} + +\lambda p_{n-1,1}(t)y^{(n-1)k+2} \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \frac{dp_{n,k}(t)}{dt}y^{nk+1} = -(\lambda+k\mu)p_{n,k}(t)y^{nk+2} + k\mu\, p_{n+1,1}(t)y^{nk+2} + +\lambda p_{n-1,k}(t)y^{nk+2} \end{cases}$$

Dividing each equation by $k\mu$ and summing over n and s we have the partial differential equation

$$y\frac{\partial G(y,t)}{\partial t} = (1-y)\{G(y,t)[k\mu - \lambda(y+\cdots+y^k)] - k\mu\, p_0(t)\}, \tag{3.3.3}$$

where $\theta = \lambda/k\mu$.

Note that we specify no customers present at $t=0$, so the initial conditions are

$$\begin{cases} p_0(0) = 1, \quad n = 0, \\ p_{n,s}(0) = 0, \ \forall\ s, n > 0. \end{cases} \tag{3.3.4}$$

To solve these equations we take the Laplace transforms.

Let us define the Laplace transform with respect to time of $G(y,t)$ and $p_{n,s}(t)$ as

$$G^*(y,z) = \int_0^\infty e^{-zt}G(y,t)dt \tag{3.3.5}$$

and

$$p^*_{n,s}(z) = \int_0^\infty e^{-zt} p_{n,s}(t)dt, \tag{3.3.6}$$

where $Re(z) > 0$. Noting that

$$\int_0^\infty e^{-zt}\frac{\partial G}{\partial t}dt = e^{-zt}G|_0^\infty + z\int_0^\infty e^{-zt}Gdt = -1 + zG^*(y,t) \tag{3.3.7}$$

and applying the Laplace transform to (3.3.3) we obtain

$$y(-1 + zG^*(y,z)) = (1-y)\{G^*(y,z)[k\mu - \lambda(y + \cdots + y^k)] - k\mu p_0^*(z)\}. \tag{3.3.8}$$

Hence,

$$\begin{aligned} G^*(y,z) &= \frac{y - (1-y)k\mu p_0^*(z)}{yz - (1-y)(k\mu - \lambda(y + \cdots + y^k))} \\ &= \frac{y - (1-y)k\mu p_0^*(z)}{-\lambda y^{k+1} + (z + k\mu + \lambda)y - k\mu}. \end{aligned} \tag{3.3.9}$$

The main problem which arises now is how to find the inverse Laplace transform and this is a difficult problem.

If we inspect the denominator of (3.3.9) we can see that it is a polynomial in y of degree $(k+1)$ and hence it will have $(k+1)$ zeros. It is well known that there is no analytic expression for the roots of a polynomial degree k, for $k > 4$.

Here we can apply Rouche's Theorem to the denominator of (3.3.9). That is, if $f(y)$ and $g(y)$ are functions analytic inside and on a closed contour C and if $|g(y)| < |f(y)|$ on C, then $f(y)$ and $f(y) + g(y)$ have the same number of zeros inside C.

Let us investigate conditions of this theorem.

Assume that

$$|f(y)| = \left|\left(\frac{z}{\lambda} + \frac{1}{\theta} + 1\right)y\right| \quad \text{and} \quad |g(y)| = \left|y^{k+1} + \frac{1}{\theta}\right|,$$

where $Re(z) > 0$. Define the contour C here: $|y| = 1$.

Hence on C

$$|f(y)| = \left|\frac{z}{\lambda} + \frac{1}{\theta} + 1\right| > \frac{1}{\theta} + 1 \geq \left|y^{k+1} + \frac{1}{\theta}\right| \equiv |g(y)|.$$

Since $f(y)$ clearly has 1 zero inside C, then so has $f(y) + g(y)$.

Rouche's Theorem shows that the denominator of (3.3.9) has exactly 1 zero within the Unit Circle, and hence there are k zeros outside the Unit Circle. Since $G^*(y,z)$ must be analytic within and on the Unit Circle, the numerator must vanish at the zero of the denominator which lies within the Unit Circle, i.e. we may write the numerator in the form $A(y-y_0)$, where y_0 is the zero of denominator which lies within Unit Circle, and A is a constant.

Thus, from (3.3.9) we can find the expression for the Laplace transform of the probability that there are no customers in the system

$$p_0^*(z) = \frac{y_0(z)}{k\mu(1 - y_0(z))}. \tag{3.3.10}$$

The method proposed by Luchak (1956, 1958) can be used to calculate the inverse Laplace transform for $p_0^*(z)$, but not for other probabilities.

Let put

$$\zeta = \frac{k\mu}{\lambda + k\mu + z}, \tag{3.3.11}$$

$$\eta = \frac{\lambda}{\lambda + k\mu + z}, \tag{3.3.12}$$

$$\Phi(y) = y^{k+1}. \tag{3.3.13}$$

The Lagrange expansion of an arbitrary holomorphic function of y_0 can be written

$$F(y_0) = F(\zeta) + \sum_{n=1}^{\infty} \frac{\eta^n}{n!} \left(\frac{d}{d\zeta}\right)^{n-1} [F'(\zeta)(\Phi(\varsigma))^n]. \tag{3.3.14}$$

For all further work it is convenient to choose

$$F(y_0) = y_0^\nu,$$

where ν is an arbitrary integral constant.

Thus, we get

$$\left(\frac{d}{d\zeta}\right)^{n-1}[F'(\zeta)(\Phi(\varsigma))^n] = \frac{\nu(nk+n+\nu-1)!}{(nk+\nu)!}\zeta^{nk+\nu}. \tag{3.3.15}$$

Putting $\theta = \frac{\lambda}{k\mu} = \frac{\eta}{\varsigma}$ and apply (3.3.14) we have

$$y_0^\nu = \zeta^\nu + \nu\sum_{n=1}^{\infty}\frac{\theta^n}{n!}\frac{(n(k+1)+\nu-1)!}{(nk+\nu)!}\zeta^{n(k+1)+\nu}. \tag{3.3.16}$$

Inverting the Laplace transform of (3.3.10), we derive (3.3.17) using the fact that

$$p_0^*(z) = \frac{y_0(z)}{k\mu(1-y_0(z))} = \frac{1}{k\mu}\sum_{\nu=1}^{\infty} y_0^\nu(z).$$

$$L^{-1}(y_0^\nu) = \frac{k\mu\nu}{\tau}\left\{\frac{\tau^\nu}{\nu!} + \sum_{n=1}^{\infty}\frac{\theta^n}{n!}\frac{\tau^{n(k+1)\nu}}{(nk+\nu)!}\right\}e^{-(1+\theta)\tau}, \tag{3.3.17}$$

where $\tau = k\mu t$. To find the inversion of the Laplace transform of (3.3.17) we use that

$$L^{-1}\left(\frac{1}{(z+a)^n}\right) = \frac{t^{n-1}}{(n-1)!}e^{-at}, \quad a = const.$$

The formulae for $p_0(t)$ can be written as follows

$$p_0(t) = \frac{1}{\tau}\sum_{\nu=1}^{\infty}\nu\left\{\frac{\tau^\nu}{\nu!} + \sum_{n=1}^{\infty}\frac{(\theta\tau)^n}{n!}\frac{\tau^{nk+\nu}}{(nk+\nu)!}\right\}e^{-(1+\theta)\tau}$$

$$= 2\sum_{\nu=1}^{\infty}\nu\theta^{-\frac{\nu-1}{k+1}}\frac{I_\nu^k(r)}{r}e^{-(1+\theta)\tau}, \tag{3.3.18}$$

where

$$r = 2\theta^{\frac{1}{1+k}}\tau$$

and $I_\nu^k(r)$ is the generalised modified Bessel function of the first type (2.2.1) with properties(1)-(6), see also Luchak (1956, 1958).

Knowing $p_0(t)$ gives us the possibility of computing all the other probabilities from the differential-difference equations. Because of the complex structure of the Erlang service equations (3.3.1) the computation of these probabilities becomes extremely difficult.

3.4 Transient Solution to the Erlang Model Through the Generalised Modified Bessel Function of the Second Type. First Method

This section contains our main results in which we obtain time-dependent probabilities for the Erlang queueing system.

We assume Poisson input (exponential interarrival times) with parameter λ and service times having an Erlang type k distribution with parameter μ.

Let $p_{n,s}(t)$ represent the probability that there are n customers in the system at time t with the customer in service being in phase s $(1 \leq s \leq k)$ and let $p_0(t)$ represent the probability that there are no customers in the system at time t.

The differential-difference equations for the $M/E_k/1$ system are given by (3.3.1). Let us define the following generating function (note that summation over n is from $-\infty$ to ∞)

$$G(y,t) = \sum_{n=-\infty}^{\infty} \sum_{s=1}^{k} y^{(n-1)k+s} p_{n,s}(t), \tag{3.4.1}$$

where we define $p_{n,s}(t) = 0$ for $n = 0, -1, ...;\ s = 1, ...k$. Note that $G(y,t)$ is analytic within and on the unit circle $|y| = 1$.

Using the same methodology as in Section 3.3 we obtain the partial differential equation (3.3.3).

We specify that there are no customers present at $t = 0$, although this condition can be generalised (see Section 3.8). The same initial condition is given in (3.3.4). We may write (3.3.3) and (3.3.4) in the form

$$\begin{cases} \frac{\partial G(y,t)}{\partial t} = G(y,t)\varphi(y) - \frac{1-y}{y} k\mu p_0(t), \\ G(y,0) = 1, \end{cases} \tag{3.4.2}$$

where

$$\varphi(y) = \frac{1-y}{y}\left[k\mu - \lambda(y + \cdots + y^k)\right] = \lambda y^k + \frac{k\mu}{y} - (k\mu + \lambda). \tag{3.4.3}$$

Solving (3.4.2) we obtain

$$G(y,t) = A(t)\exp(t\varphi(y)). \tag{3.4.4}$$

To find $A(t)$ we need to differentiate (3.4.4) w.r.t t and substitute it into (3.4.2)

$$A'(t)\exp(t\varphi(y)) = -\frac{1-y}{y}k\mu p_0(t).$$

Therefore

$$A(t) - A(0) = -\frac{1-y}{y}k\mu \int_0^t p_0(z)\exp(-z\varphi(y))dz.$$

Thus, the solution is

$$G(y,t) = e^{-t(\lambda+k\mu)}e^{t(\lambda y^k + \frac{k\mu}{y})}$$

$$-\frac{1-y}{y}k\mu \int_0^t p_0(z)e^{-t(\lambda+k\mu)}e^{z(\lambda+k\mu)}e^{(t-z)(\lambda y^k+\frac{k\mu}{y})}dz. \tag{3.4.5}$$

Here we use the generalisation of the modified Bessel function of the second kind, which has been introduced in Chapter 2.

$$\tilde{I}_n^{k,s}(z) = \left(\frac{z}{2}\right)^{n+k-s}\sum_{r=0}^{\infty}\frac{(\frac{z}{2})^{r(k+1)}}{(k(r+1)-s)!\Gamma(n+r+1)}, \tag{3.4.6}$$

where $z \in C,\ s \in \{1, 2, \ldots, k\},\ n = 0, \pm 1, \pm 2, \ldots,\ k = \{1, 2, \ldots\}$.

Note that the generating function of the generalised modified Bessel function of the second type takes the form

$$e^{\frac{1}{2}z(\lambda y^k+\frac{k\mu}{y})} = \sum_{n=-\infty}^{\infty}\sum_{s=1}^{k}(\beta y)^{k(n-1)+s}\tilde{I}_n^{k,s}(\alpha z), \tag{3.4.7}$$

where

$$\alpha = 2\left(\lambda(k\mu)^k\right)^{\frac{1}{k+1}},\ \beta = \left(\frac{\lambda}{k\mu}\right)^{\frac{1}{k+1}}. \tag{3.4.8}$$

For the proof of this fact and for further properties of the function (3.4.6) and relations to other special functions, see Chapter 2.

Applying property (3.4.7) to (3.4.5) gives us

$$G(y,t) = \sum_{n=-\infty}^{\infty} \sum_{s=1}^{k} y^{k(n-1)+s} p_{n,s}(t) \tag{3.4.9}$$

$$= e^{-t(\lambda+k\mu)} \left(\sum_{n=-\infty}^{\infty} \sum_{s=1}^{k} \beta^{k(n-1)+s} y^{k(n-1)+s} \tilde{I}_n^{k,s}(\alpha t) \right.$$
$$+ k\mu \sum_{n=-\infty}^{\infty} \sum_{s=1}^{k} \beta^{k(n-1)+s} y^{k(n-1)+s} \int_0^t p_0(z) e^{z(\lambda+k\mu)} \tilde{I}_n^{k,s}(\alpha(t-z)) dz$$
$$\left. -k\mu \sum_{n=-\infty}^{\infty} \sum_{s=1}^{k} \beta^{k(n-1)+s} y^{k(n-1)+s-1} \int_0^t p_0(z) e^{z(\lambda+k\mu)} \tilde{I}_n^{k,s}(\alpha(t-z)) dz \right).$$

We can calculate the integrals in (3.4.9) by using an expression for $p_0(t)$, which has been derived by Luchak (1956, 1958). For the Erlang service system we have seen that it takes the form given in (3.3.18).

Now we are able to write down the probabilities $p_{n,s}(t)$ for the Erlang service system. Equating two parts of equation (3.4.9) we have

$$p_0(t) = 2\sum_{\nu=1}^{\infty} \nu \theta^{-\frac{\nu-1}{k+1}} \frac{I_\nu^k(r)}{r} e^{-(1+\theta)\tau}, \tag{3.4.10}$$

when $n = 0$, and

$$p_{n,s}(t) = e^{-t(\lambda+k\mu)} \left(\beta^{k(n-1)+s} \tilde{I}_n^{k,s}(\alpha t) + k\mu\beta^{k(n-1)+s} \int_0^t p_0(z) e^{z(\lambda+k\mu)} \tilde{I}_n^{k,s}(\alpha(t-z)) dz \right.$$

$$\left. -k\mu\beta^{k(n-1)+s+1} \int_0^t p_0(z) e^{z(\lambda+k\mu)} \tilde{I}_n^{k,s+1}(\alpha(t-z)) dz \right), \tag{3.4.11}$$

when $1 \le s \le k-1$, $1 \le n < \infty$, and

$$p_{n,k}(t) = e^{-t(\lambda+k\mu)} \left(\beta^{kn} \tilde{I}_n^{k,k}(\alpha t) + k\mu\beta^{kn} \int_0^t p_0(z) e^{z(\lambda+k\mu)} \tilde{I}_n^{k,k}(\alpha(t-z)) dz \right.$$

$$\left. - k\mu\beta^{kn+1} \int_0^t p_0(z) e^{z(\lambda+k\mu)} \tilde{I}_{n+1}^{k,1}(\alpha(t-z)) dz \right), \tag{3.4.12}$$

when $n \geq 1, \ s = k$.

3.5 Transient Solution to the Erlang Model Through the Generalised Modified Bessel Function of the Second Type. Second method

In this section we propose another method of obtaining the solution to the Erlang queueing model. It is based on the method which was introduced by Parthasarathy (1987), see Section 1.3.3.

Assuming the Erlang differential-difference equations (3.3.1) we define new functions $q_{n,s}(t)$ and $\nu_{n,s}(t)$ as follows

$$q_{n,s}(t) = \begin{cases} e^{t(\lambda+k\mu)}(k\mu p_{1,1}(t) - \lambda p_0(t)), & n=1,\ s=1, \\ e^{t(\lambda+k\mu)}(k\mu p_{n,1}(t) - \lambda p_{n-1,k}(t)), & n>1,\ s=1, \\ e^{t(\lambda+k\mu)}(k\mu p_{n,s}(t) - \lambda p_{n,s-1}(t)), & n\geq 1,\ 2\leq s\leq k, \\ 0, & n=0,-1,-2,\dots; \end{cases} \tag{3.5.1}$$

$$\nu_{n,s}(t) = \begin{cases} 0, & n=1,\ s=1, \\ e^{t(\lambda+k\mu)}(k\mu p_{1,s-1}(t)), & n=1,\ 2\leq s\leq k, \\ e^{t(\lambda+k\mu)}(k\mu p_{n-1,k}(t) - \lambda p_{n-2,k}(t)), & s=1,\ n\geq 2, \\ e^{t(\lambda+k\mu)}(k\mu p_{n,s-1}(t) - \lambda p_{n-1,s-1}(t)), & n\geq 2,\ 2\leq s\leq k, \\ 0, & n=0,-1,-2,\dots \end{cases} \tag{3.5.2}$$

Let us also define the following generating function

$$H(y,t) = \sum_{n=-\infty}^{\infty}\sum_{s=1}^{k} y^{(n-1)k+s}\left(q_{n,s}(t) - \nu_{n,s}(t)\right). \tag{3.5.3}$$

Note that this representation gives us that $p_{n,s}(t)$ can be expressed in terms of the differences $q_{n,s}(t) - \nu_{n,s}(t)$ for each n and s.

Indeed, differentiating the difference $q_{n,s}(t) - \nu_{n,s}(t)$ we obtain for $n \geq 1,\ 1 \leq s \leq k$

$$\begin{aligned}
&\frac{d}{dt}\left(q_{n,s}(t) - \nu_{n,s}(t)\right) \\
= \ & e^{t(\lambda+k\mu)}[(\lambda + k\mu)(k\mu p_{n,s}(t) - \lambda p_{n,s-1}(t)) + k\mu\frac{d}{dt}p_{n,s}(t) - \lambda\frac{d}{dt}p_{n,s-1}(t)] \\
& -e^{t(\lambda+k\mu)}[(\lambda + k\mu)(k\mu p_{n,s-1}(t) - \lambda p_{n-1,s-1}(t)) + k\mu\frac{d}{dt}p_{n,s-1}(t) - \lambda\frac{d}{dt}p_{n-1,s-1}(t)] \\
= \ & (\lambda + k\mu)\left(q_{n,s}(t) - \nu_{n,s}(t)\right) + k\mu\left(q_{n,s+1}(t) - \nu_{n,s+1}(t)\right) \\
& -\lambda\left(q_{n,s}(t) - \nu_{n,s}(t)\right) - k\mu\left(q_{n,s}(t) - \nu_{n,s}(t)\right) + \lambda\left(q_{n-1,s}(t) - \nu_{n-1,s}(t)\right) \\
= \ & k\mu\left(q_{n,s+1}(t) - \nu_{n,s+1}(t)\right) + \lambda\left(q_{n-1,s}(t) - \nu_{n-1,s}(t)\right).
\end{aligned}$$

Multiplying each appropriate equation by $y,\ y^2, \ldots$ and then summing over n and s, we derive the following differential equation

$$\frac{dH(y,t)}{dt} = \left(\lambda y^k + \frac{\mu k}{y}\right) H(y,t) - k\mu q_{1,1}(t). \tag{3.5.4}$$

It is easy to show that for $s = k = 1$ we obtain the appropriate result for the $M/M/1$ system, see (1.3.19).

The initial condition gives us

$$H(y,0) = \sum_{n=-\infty}^{\infty}\sum_{s=1}^{k} y^{(n-1)k+s}\left(q_{n,s}(0) - \nu_{n,s}(0)\right) = -\lambda y. \tag{3.5.5}$$

Solving the equation (3.5.4) with initial condition (3.5.5) we have

$$H(y,t) = -k\mu\int_0^t q_{1,1}(x)e^{-y\varphi(y)}dx - \lambda z e^{t\varphi(y)}$$

$$= H(y,0)e^{t\left(\lambda y^k + \frac{k\mu}{y}\right)} - k\mu\int_0^t q_{1,1}(x)e^{(t-x)\left(\lambda y^k + \frac{k\mu}{y}\right)}dx, \tag{3.5.6}$$

where

$$\varphi(y) = \lambda y^k + \frac{\mu k}{y}.$$

Using Lemma 2.3.1 and equating second parts of the generating function we obtain

$$\sum_{n=-\infty}^{\infty}\sum_{s=1}^{k} y^{k(n-1)+s}(q_{n,s}(t)-\nu_{n,s}(t)) \tag{3.5.7}$$

$$= -\lambda y \sum_{n=-\infty}^{\infty} (\beta y)^{k(n-1)+s}\tilde{I}_n^{k,s}(t) - k\mu \int_0^t q_{1,1}(x) \sum_{n=-\infty}^{\infty}\sum_{s=1}^{k}(\beta y)^{k(n-1)+s}\tilde{I}_n^{k,s}(\alpha(t-x))dx.$$

where

$$\alpha = 2[\lambda(k\mu)^k]^{\frac{1}{k+1}}, \quad \beta = \left(\frac{\lambda}{k\mu}\right)^{\frac{1}{k+1}},$$

and $\tilde{I}_n^{k,s}(\alpha t)$ is the generalised modified Bessel function of the second type, see (2.3.2).

Comparing the coefficients of $y^{k(n-1)+s}$ on both sides for $n = 1, 2, 3, \dots$ we obtain

$$q_{n,s}(t) - \nu_{n,s}(t) = \begin{cases} -\beta^{kn-1}\lambda \tilde{I}_{n-1}^{k,k}(\alpha t) - k\mu\beta^{kn}\int_0^t q_{1,1}(x)\tilde{I}_n^{k,k}(\alpha(t-x))dx, \\ s=1, n\geq 2; \\ -\beta^{k(n-1)+s-1}\lambda \tilde{I}_n^{k,s-1}(\alpha t) - k\mu\beta^{k(n-1)+s}\int_0^t q_{1,1}(x)\tilde{I}_n^{k,s}(\alpha(t-x))dx, \\ 2\leq s\leq k,\ n\geq 1. \end{cases} \tag{3.5.8}$$

To find $q_{1,1}(x)$ in (3.5.8) we use again the results from Luchak (1956, 1958). Applying the formulae for $p_0(t)$, see (3.4.13), and using the first differential-difference in equation (3.3.1) we can find the expression for $q_{1,1}(x)$.

$$p_{1,1}(t) = \frac{1}{k\mu}\left(\frac{dp_0(t)}{dt} + \lambda p_0(t)\right). \tag{3.5.9}$$

Let us rewrite $p_0(t)$ as

$$p_0(t) = \sum_{\nu=1}^{\infty} \nu\theta^{-\frac{\nu-1}{k+1}} \sum_{n=0}^{\infty} \frac{1}{n!(\nu+nk)!} e^{-(1+\theta)k\mu t}\left(\theta^{\frac{1}{k+1}}k\mu t\right)^{\nu+n(k+1)-1}$$

$$= \sum_{\nu=1}^{\infty}\sum_{n=0}^{\infty} \nu\theta^n e^{-(k\mu+\lambda)t}\frac{(k\mu)^{\nu+n(k+1)-1}}{n!(\nu+nk)!}t^{\nu+n(k+1)-1}, \tag{3.5.10}$$

and differentiating (3.5.10)

$$\frac{dp_0(t)}{dt} = -(k\mu+\lambda)e^{-(k\mu+\lambda)t}\sum_{\nu=1}^{\infty}\sum_{n=0}^{\infty}\nu\theta^n\frac{(k\mu)^{\nu+n(k+1)-1}}{n!(\nu+nk)!}t^{\nu+n(k+1)-1}$$

$$+\sum_{\nu=1}^{\infty}\sum_{n=0}^{\infty}\nu\theta^n e^{-(k\mu+\lambda)t}\frac{(k\mu)^{\nu+n(k+1)-1}}{n!(\nu+nk)!}(\nu+n(k+1)-1)t^{\nu+n(k+1)-2} \tag{3.5.11}$$

we can substitute (3.5.10) and (3.5.11) into (3.5.9) and write $p_{1,1}(t)$ as follows

$$p_{1,1}(t) = \frac{1}{k\mu}e^{-(k\mu+\lambda)t}\sum_{\nu=1}^{\infty}\sum_{n=0}^{\infty}\frac{\nu\theta^n(k\mu)^{\nu+n(k+1)-1}}{n!(\nu+nk)!}t^{\nu+n(k+1)-1}\left(-k\mu+\frac{1}{t}(\nu+n(k+1))\right). \tag{3.5.12}$$

Finally, using the first expression in (3.5.1) we can obtain $q_{1,1}(t)$

$$q_{1,1}(t) = -(k\mu+\lambda)\sum_{\nu=1}^{\infty}\sum_{n=0}^{\infty}\frac{\theta^n(k\mu)^{\nu+n(k+1)-1}}{n!(\nu+nk)!}t^{\nu+n(k+1)-1}$$

$$+\sum_{\nu=1}^{\infty}\sum_{n=0}^{\infty}\frac{\theta^n(k\mu)^{\nu+n(k+1)-1}}{n!(\nu+nk)!}(\nu+n(k+1)-1)t^{\nu+n(k+1)-2}. \tag{3.5.13}$$

The solution for $p_{n,s}(t)$ in terms of $q_{n,s}(t)-\nu_{n,s}(t)$ can be written from the equations (3.5.1) and (3.5.2). For example,

$$q_{1,1}(t)-\nu_{1,1}(t) = e^{(k\mu+\lambda)t}[k\mu p_{1,1}(t)-\lambda p_0(t)]$$

and

$$p_{1,1}(t) = \frac{e^{-(k\mu+\lambda)t}}{k\mu}q_{1,1}(t)+\frac{\lambda}{k\mu}p_0(t).$$

Similarly,

$$q_{1,2}(t)-\nu_{1,2}(t) = e^{(k\mu+\lambda)t}[k\mu p_{1,2}(t)-(\lambda+k\mu)p_{1,1}(t)]$$

$$p_{1,2}(t) = \frac{e^{-(k\mu+\lambda)t}}{k\mu}(q_{1,2}(t)-\nu_{1,2}(t))+\frac{(\lambda+k\mu)}{k\mu}p_{1,1}(t).$$

Then the transient probabilities $p_{n,s}(t)$ satisfy the following recursive relation:

for $n = 1,\ 2 \leq s \leq k$ we have

$$p_{1,s}(t) = \frac{e^{-(k\mu+\lambda)t}}{k\mu} \sum_{l=1}^{s} (q_{1,s}(t) - \nu_{1,s}(t)) \left(\frac{\lambda + k\mu}{k\mu}\right)^{s-l} + \frac{\lambda(\lambda + k\mu)^{s-l}}{(k\mu)^s} p_0(t); \quad (3.5.14)$$

for $n \geq 2,\ s = 1$

$$p_{n,1}(t) = \frac{e^{-(k\mu+\lambda)t}}{k\mu} (q_{n,1}(t) - \nu_{n,1}(t)) + \frac{(\lambda + k\mu)}{k\mu} p_{n-1,k}(t) - \frac{\lambda}{k\mu} p_{n-2,k}(t); \quad (3.5.15)$$

for $n \geq 2,\ 1, s \leq k$

$$p_{n,s}(t) = \frac{e^{-(k\mu+\lambda)t}}{k\mu} (q_{n,s}(t) - \nu_{n,s}(t)) + \frac{(\lambda + k\mu)}{k\mu} p_{n,s-1}(t) - \frac{\lambda}{k\mu} p_{n-1,s-1}(t), \quad (3.5.16)$$

where (3.5.8) is the formula for the differences $q_{n,s}(t) - \nu_{n,s}(t)$, (3.3.18) is a formula for the $p_0(t)$ and (3.5.13) is a formula for the $q_{1,1}(t)$.

3.6 The Mean Waiting Time of a Customer Arriving in the Queue at time t

In many situations $p_{n,s}(t)$, the individual probabilities of there being n customers in the system and one being in phase s, $s = 1, ..., k$, at time t, are not of prime interest to a practitioner; rather it is the summary measures, such as the mean waiting time in a queue or mean number of customers in the system for a customer arriving at time time t which are of great importance.

Unfortunately, Little's formulae work just for the steady-state case and thus other time-dependent queueing characteristics have to be written in termsof the probabilities $p_{n,s}(t)$.

In this section we will write an expression just for the one summary measure $W_q(t)$. As we did in Section 3.2 we can find the mean waiting time in the queue through the generating function as

$$W_q(t) = \frac{1}{k\mu}\frac{\partial G(y,t)}{\partial y}\bigg|_{y=1}. \tag{3.6.1}$$

Also note that $W_q(t)$ can be represented as

$$W_q(t) = \frac{1}{k\mu}\sum_{n=1}^{\infty}\sum_{s=1}^{k}[(n-1)k+s]p_{n,s}(t). \tag{3.6.2}$$

Let us return to the partial differential equation (3.4.2) where $\varphi(y)$ is defined in (3.4.3). The solution gives the explicit expression for $G(y,t)$

$$G(y,t) = \exp\{t\varphi(y)\} -$$

$$\frac{1-y}{y}k\mu\int_0^t p_0(z)\exp\left\{\frac{1-y}{y}\varphi(y)(t-z)\right\}dz. \tag{3.6.3}$$

Applying (3.6.1) to (3.6.3) we have

$$\frac{\partial G(y,t)}{\partial y} = \exp\left\{t\left(\lambda y^k + \frac{k\mu}{y} - (\lambda + k\mu)\right)\right\} t\left(\lambda k y^{k-1} - \frac{k\mu}{y^2}\right)$$

$$+\frac{k\mu}{y^2}\int_0^t p_0(z)\exp\left\{\left(\lambda y^k + \frac{k\mu}{y} - (\lambda + k\mu)\right)(t-z)\right\}dz$$

$$-\frac{1-y}{y}k\mu\int_0^t p_0(z)\exp\left\{\left(\lambda y^k + \frac{k\mu}{y} - (\lambda + k\mu)\right)(t-z)\right\}(t-z)\left(\lambda k y^{k-1} - \frac{k\mu}{y^2}\right)dz.$$

Putting $y = 1$ we obtain

$$W_q(t) = \frac{t\lambda}{\mu} - t + \int_0^t p_0(z)dz. \qquad (3.6.4)$$

We can also obtain this result directly.

Substituting the expression for $p_0(t)$ from (3.3.18), finally we derive

$$W_q(t) = \frac{t\lambda}{\mu} - t + \int_0^t \frac{1}{k\mu z}\sum_{\nu=1}^{\infty}\nu\left\{\frac{(k\mu z)^{\nu}}{\nu!} + \sum_{n=1}^{\infty}\frac{(\lambda z)^n}{n!}\frac{(k\mu z)^{nk+\nu}}{(nk+\nu)!}\right\}e^{-(1+\theta)k\mu z}dz. \qquad (3.6.5)$$

3.7 Queueing Characteristics and Their Relationships

In this section we provide formulae for calculating other transient queueing characteristics and their relationships between each other. As was mentioned before, all these queueing measures contain the probabilities $p_{n,s}(t)$.

Rewriting $L(t)$, the mean number of customers in the system at time t, and $L_q(t)$, the mean number of customers in the queue at time t, we have

$$L(t) = 1\left[p_{1,1}(t) + p_{1,2}(t) + \cdots + p_{1,k}(t)\right] + \tag{3.7.1}$$

$$+2\left[p_{2,1}(t) + p_{2,2}(t) + \cdots + p_{2,k}(t)\right] +$$
$$+3\left[p_{3,1}(t) + p_{3,2}(t) + \cdots + p_{3,k}(t)\right] + \ldots$$

$$L_q(t) = 1\left[p_{2,1}(t) + p_{2,2}(t) + \cdots + p_{2,k}(t)\right] + \tag{3.7.2}$$

$$+2\left[p_{3,1}(t) + p_{3,2}(t) + \cdots + p_{3,k}(t)\right] +$$
$$+3\left[p_{4,1}(t) + p_{4,2}(t) + \cdots + p_{4,k}(t)\right] + \ldots$$

Combining (3.7.1) and (3.7.2) we obtain

$$\begin{aligned} L(t) &= L_q(t) + \left[p_{1,1}(t) + p_{1,2}(t) + \cdots + p_{1,k}(t)\right] \\ &\quad +2\left[p_{2,1}(t) + p_{2,2}(t) + \cdots + p_{2,k}(t)\right] + \ldots \end{aligned}$$

$$L(t) = L_q(t) + \sum_{n=1}^{\infty}\sum_{s=1}^{k} p_{n,s}(t) = L_q(t) + (1 - p_0(t)). \tag{3.7.3}$$

Similar we can write an expressions for the $W(t)$ and $W_q(t)$.

$$W(t) = \frac{1}{k\mu}\sum_{n=1}^{\infty}\sum_{s=1}^{k}[nk+s]p_{n,s}(t) \tag{3.7.4}$$

$$\begin{aligned} = & \frac{1}{k\mu}[\{(k+1)p_{1,1}(t) + (k+2)p_{1,2}(t) + (k+3)p_{1,3}(t) + \cdots\} + \\ & +\{(2k+1)p_{2,1}(t) + (2k+2)p_{2,2}(t) + (2k+3)p_{2,3}(t) + \cdots\} + \cdots] \end{aligned}$$

$$W_q(t) = \frac{1}{k\mu}[\{1 \cdot p_{1,1}(t) + 2 \cdot p_{1,2}(t) + \cdots + kp_{1,k}(t)\} + \tag{3.7.5}$$

$$\begin{aligned} & +\{(k+1)p_{2,1}(t) + (k+2)p_{2,2}(t) + \cdots + 2kp_{2,k}(t)\} + \\ & +\{(2k+1)p_{3,1}(t) + (2k+2)p_{3,2}(t) + \cdots + 3kp_{3,k}(t)\} + \cdots] \end{aligned}$$

Combining (3.7.4) and (3.7.5) we can write

$$W(t) = W_q(t) + \frac{1}{\mu}\sum_{n=1}^{\infty}\sum_{s=1}^{k}p_{n,s}(t). \tag{3.7.6}$$

Also $W_q(t)$ is related to the $L_q(t)$ in the following way

$$W_q(t) = \frac{1}{\mu}L_q(t) + \frac{1}{k\mu}\sum_{n=1}^{\infty}\sum_{s=1}^{k}sp_{n,s}(t). \tag{3.7.7}$$

Proof.

$$W_q(t) = \frac{1}{k\mu}\sum_{n=1}^{\infty}\sum_{s=1}^{k}[(n-1)k+s]\,p_{n,s}(t),$$

$$\begin{aligned} L_q(t) \quad = \quad & [p_{2,1}(t) + p_{2,2}(t) + \cdots + p_{2,k}(t)] + \\ & +2\,[p_{3,1}(t) + p_{3,2}(t) + \cdots + p_{3,k}(t)] + \\ & +3\,[p_{4,1}(t) + p_{4,2}(t) + \cdots + p_{4,k}(t)] + \ldots \end{aligned}$$

We can rearrange $W_q(t)$ in the following way

$$
\begin{aligned}
W_q(t) &= \frac{1}{k\mu}[\{1p_{1,1}(t) + 2p_{1,2}(t) + \cdots + kp_{1,k}(t)\} + \\
&\quad +\{(k+1)p_{2,1}(t) + (k+2)p_{2,2}(t) + \cdots + 2kp_{2,k}(t)\} + \\
&\quad +\{(2k+1)p_{3,1}(t) + (2k+2)p_{3,2}(t) + \cdots + 3kp_{3,k}(t)\} + \cdots] \\
&= \frac{k}{k\mu}[\{p_{2,1}(t) + p_{2,2}(t) + \cdots + p_{2,k}(t)\} + \\
&\quad +2\{p_{3,1}(t) + p_{3,2}(t) + \cdots + p_{3,k}(t)\} + \\
&\quad +\frac{1}{k\mu}[\{p_{1,1}(t) + 2p_{2,1}(t) + p_{3,1}(t) + \cdots\} + \\
&\quad +2\{p_{1,2}(t) + p_{2,2}(t) + p_{3,2}(t) + \cdots\} + \\
&\quad +3\{p_{1,3}(t) + p_{2,3}(t) + p_{3,3}(t) + \cdots\} + \cdots].
\end{aligned}
$$

Then,

$$W_q(t) = \frac{1}{\mu}L_q(t) + \frac{1}{k\mu}\sum_{n=1}^{\infty}\sum_{s=1}^{k} sp_{n,s}(t).$$

□

3.8 Extended Transient Solution in Terms of the Generalised Modified Bessel Function of the Second Type

In this section we will rewrite the initial condition for the Erlang service model for a customers present in the system present at time $t = 0$.

Assuming differential-difference equations (3.3.1) for the $M/E_k/1$ system with the generating function (3.4.1), we can rewrite (3.3.3) in the form

$$y\frac{\partial G(y,t)}{\partial t} = (1-y)\left(G(y,t)\left[k\mu - \lambda(y+\cdots+y^k)\right] - k\mu p_0(t)\right). \tag{3.8.1}$$

The initial condition in this case can be presented as

$$\begin{cases} p_n(0) = 1, \ n = a, \\ p_{n,s}(0) = 0, \ \text{for all other } n, s. \end{cases} \tag{3.8.2}$$

The differential equation for the generating function is

$$\begin{cases} \frac{\partial G(y,t)}{\partial t} = G(y,t)\varphi(y) - \frac{1-y}{y}k\mu p_0(t), \\ G(y,0) = y^{ak}, \end{cases} \tag{3.8.3}$$

where

$$\varphi(y) = \lambda y^k + \frac{k\mu}{y} - (k\mu + \lambda).$$

Using the same methodology as in Section 3.4 the solution of (3.8.3) can be written as follows

$$\begin{aligned} G(y,t) &= y^{ak}e^{-t(\lambda+k\mu)}e^{t(\lambda y^k+\frac{k\mu}{y})} \\ &- \frac{1-y}{y}k\mu\int_0^t p_0(z)e^{-t(\lambda+k\mu)}e^{z(\lambda+k\mu)}e^{(t-z)(\lambda y^k+\frac{k\mu}{y})}dz. \end{aligned} \tag{3.8.4}$$

Applying Lemma 2.3.1 with appropriate coefficients (3.4.8) and generalised modified Bessel function of the second type (2.3.2) we obtain

$$G(y,t) = \sum_{n=-\infty}^{\infty} \sum_{s=1}^{k} y^{k(n-1)+s} p_{n,s}(t) \tag{3.8.5}$$

$$= e^{-t(\lambda+k\mu)} \left(\sum_{n=-\infty}^{\infty} \sum_{s=1}^{k} \beta^{k(n-1)+s} y^{k(n-1)+s} \tilde{I}_n^{k,s}(\alpha t) \right.$$

$$+ k\mu \sum_{n=-\infty}^{\infty} \sum_{s=1}^{k} \beta^{k(n-1)+s} y^{k(n-1)+s+ak} \int_0^t p_0(z) e^{z(\lambda+k\mu)} \tilde{I}_n^{k,s}(\alpha(t-z))dz$$

$$\left. -k\mu \sum_{n=-\infty}^{\infty} \sum_{s=1}^{k} \beta^{k(n-1)+s} y^{k(n-1)+s-1} \int_0^t p_0(z) e^{z(\lambda+k\mu)} \tilde{I}_n^{k,s}(\alpha(t-z))dz \right).$$

We calculate the integrals in (3.8.5) by using an expression for $p_0(t)$, which has been obtained by Luchak (1956, 1958). When initially there are a customers in the system it takes the form

$$p_0(t) = \frac{1}{\tau} \sum_{\nu=ak+1}^{\infty} \nu \left\{ \frac{\tau^\nu}{\nu!} + \sum_{n=1}^{\infty} \frac{(\theta\tau)^n}{n!} \frac{\tau^{nk+\nu}}{(nk+\nu)!} \right\} e^{-(1+\theta)\tau} \tag{3.8.6}$$

where

$$r = 2\theta^{\frac{1}{1+k}}\tau, \quad \tau = k\mu t \text{ and } \theta = \frac{\lambda}{k\mu},$$

and $I_\nu^k(r)$ is the generalisation of the modified Bessel function of the first kind given by (2.2.1).

Equating the two parts of equation (3.8.5) we have

$$p_0(t) = 2 \sum_{\nu=ak+1}^{\infty} \nu\theta^{-\frac{\nu-1}{k+1}} \frac{I_\nu^k(r)}{r} e^{-(1+\theta)\tau}, \tag{3.8.7}$$

when $n = 0$, and

$$p_{n,s}(t) = e^{-t(\lambda+k\mu)}\left(\beta^{k(n-1)+s}\tilde{I}^{k,s}_{n-a}(\alpha t) + k\mu\beta^{k(n-1)+s}\int_0^t p_0(z)e^{z(\lambda+k\mu)}\tilde{I}^{k,s}_{n}(\alpha(t-z))dz\right.$$

$$\left.-k\mu\beta^{k(n-1)+s+1}\int_0^t p_0(z)e^{z(\lambda+k\mu)}\tilde{I}^{k,s+1}_{n}(\alpha(t-z))dz\right), \tag{3.8.8}$$

when $1 \leq s \leq k-1,\ 1 \leq n < \infty$, and

$$p_{n,k}(t) = e^{-t(\lambda+k\mu)}\left(\beta^{kn}\tilde{I}^{k,k}_{n-a}(\alpha t) + k\mu\beta^{kn}\int_0^t p_0(z)e^{z(\lambda+k\mu)}\tilde{I}^{k,k}_{n}(\alpha(t-z))dz\right.$$

$$\left.-k\mu\beta^{kn+1}\int_0^t p_0(z)e^{z(\lambda+k\mu)}\tilde{I}^{k,1}_{n+1}(\alpha(t-z))dz\right). \tag{3.8.9}$$

when $n \geq 1,\ s = k$.

Note that in this situation we calculate the generalised modified Bessel function of the second type by the formulae

$$\tilde{I}^{s,k}_{n-a}(t) = \left(\frac{t}{2}\right)^{(n-a)+k-s}\sum_{r=0}^{\infty}\frac{\left(\frac{t}{2}\right)^{r(k+1)}}{(k(r+1)-s)!(n-a+r)!},\ t \in C, \text{ if } n-a > 0, \tag{3.8.10}$$

and

$$\tilde{I}^{s,k}_{n-a}(t) = \left(\frac{t}{2}\right)^{k-s-(n-a)k}\sum_{r=0}^{\infty}\frac{\left(\frac{t}{2}\right)^{r(k+1)}}{r!(k(r-(n-a)+1)-s)!},\ t \in C, \text{ if } n-a \leq 0. \tag{3.8.11}$$

To obtain (3.8.11) we have used the property (2.3.8) in the Section 2.3.

In this case $W_q(t)$ takes the form

$$W_q(t) = \frac{a}{\mu} + \frac{t\lambda}{\mu} - t + \int_0^t \frac{1}{k\mu z}\sum_{\nu=ak+1}^{\infty}\nu\left\{\frac{(k\mu z)^{\nu}}{\nu!} + \sum_{n=1}^{\infty}\frac{(\lambda z)^n}{n!}\frac{(k\mu z)^{nk+\nu}}{(nk+\nu)!}\right\}e^{-(1+\theta)k\mu z}dz. \tag{3.8.12}$$

3.9 Numerical Calculations and Comparisons

This section contains numerical calculations and comparisons for the probabilities $p_{n,s}(t)$ and queueing characteristics.

1) Euler's technique

Euler's method computes probabilities $p_{n,s}(t)$ numerically from the differential-difference equations (3.3.1). Assuming $p_0(0) = 1$ and $p_{n,s}(0) = 0$ for all $n \geq 1,\ s = 1, ..k$ we can calculate $p_0(t)$ as

$$\begin{aligned} p_0(\delta t) &= p_0(0) + \delta t(-\lambda p_0(0) + k\mu p_{1,1}(0)) \\ p_0(2\delta t) &= p_0(\delta t) + \delta t(-\lambda p_0(\delta t) + k\mu p_{1,1}(\delta t)) \\ & \quad etc... \end{aligned}$$

and other probabilities can be obtained from the appropriate queueing equations. For example, to find $p_{1,1}(t)$ we apply the second equation from (3.3.1)

$$\begin{aligned} p_{1,1}(\delta t) &= p_{1,1}(0) + \delta t(-(\lambda + k\mu)p_{1,1}(0) + k\mu p_{1,2}(0)) \\ p_{1,1}(2\delta t) &= p_{1,1}(\delta t) + \delta t(-(\lambda + k\mu)p_{1,1}(\delta t) + k\mu p_{1,2}(\delta t) \\ & \quad etc... \end{aligned}$$

where δt is a small increment such as $0.1,\ 0.01$.

2) Calculating probabilities through the generalised modified Bessel function of the second type

This method calculates probabilities $p_{n,s}(t)$ using formulas (3.4.10)-(3.4.12).

Tables 3.9.1-3.9.8 show comparisons of calculated probabilities for the $M/E_k/1$ queueing system for different ρ and k, using these two methods. As one can see from the tables, these results agree quite well.

Table 3.9.1 Computed probabilities for the $M/E_k/1$ queue using Euler's method with $\rho = 0.5,\ k = 3,\ \delta t = 0.01$

t	0.2	0.5	0.9	1.2	1.5	1.9	2.2
$p_0(t)$	0.90506	0.78941	0.69127	0.64779	0.61886	0.59274	0.57864
$p_{1,1}(t)$	0.00320	0.02485	0.05589	0.06949	0.07662	0.08107	0.08270
$p_{1,2}(t)$	0.01822	0.05837	0.08491	0.09183	0.09488	0.09694	0.09783
$p_{1,3}(t)$	0.06906	0.10323	0.11060	0.11188	0.11306	0.11432	0.11491
$p_{2,1}(t)$	0.00020	0.00413	0.01561	0.02388	0.03001	0.03540	0.03811
$p_{2,2}(t)$	0.00107	0.00830	0.01897	0.02428	0.02787	0.03129	0.03325
$p_{2,3}(t)$	0.00303	0.00981	0.01513	0.01775	0.02006	0.02276	0.02443
$p_{3,1}(t)$	$6.6-06$	0.00037	0.00245	0.00475	0.00701	0.00952	0.01104
$p_{3,2}(t)$	$3.4-05$	0.00069	0.00026	0.00418	0.00552	0.00705	0.00807
$p_{3,3}(t)$	$8.9-05$	0.00069	0.00170	0.00242	0.00317	0.00420	0.00493

Table 3.9.2 Computed probabilities for the $M/E_k/1$ queue using the generalised modified Bessel function of the second type method with $\rho = 0.5,\ k = 3$

t	0.2	0.5	0.9	1.2	1.5	1.9	2.2
$p_0(t)$	0.90541	0.79047	0.69248	0.64874	0.61955	0.59320	0.57901
$p_{1,1}(t)$	0.00324	0.02509	0.05558	0.06910	0.07635	0.08094	0.08263
$p_{1,2}(t)$	0.01977	0.05902	0.08526	0.09236	0.09552	0.09762	0.09851
$p_{1,3}(t)$	0.06718	0.10145	0.10956	0.11106	0.11228	0.11356	0.11416
$p_{2,1}(t)$	0.00022	0.00428	0.01560	0.02375	0.02983	0.03529	0.03803
$p_{2,2}(t)$	0.00116	0.00836	0.01886	0.02417	0.02782	0.03124	0.03321
$p_{2,3}(t)$	0.00310	0.00974	0.01508	0.01773	0.02005	0.02273	0.02440
$p_{3,1}(t)$	$8.6-06$	0.00040	0.00248	0.00475	0.00699	0.00949	0.01101
$p_{3,2}(t)$	$4.1-05$	0.00070	0.00026	0.00417	0.00551	0.00704	0.00806
$p_{3,3}(t)$	$9.9-05$	0.00070	0.00170	0.00243	0.00318	0.00420	0.00493

Table 3.9.3 Computed probabilities for the $M/E_k/1$ queue using Euler's method with $\rho = 0.95,\ k = 3,\ \delta t = 0.01$

t	0.2	0.5	0.9	1.2	1.5	1.9	2.2
$p_0(t)$	0.82700	0.63756	0.49382	0.43263	0.39083	0.35116	0.32312
$p_{1,1}(t)$	0.00563	0.03808	0.07285	0.08236	0.08456	0.08345	0.08291
$p_{1,2}(t)$	0.03185	0.08937	0.11285	0.11366	0.11163	0.10816	0.10725
$p_{1,3}(t)$	0.12029	0.15891	0.15326	0.14813	0.14428	0.13940	0.13818
$p_{2,1}(t)$	0.00068	0.01209	0.03860	0.05336	0.06213	0.06826	0.06921
$p_{2,2}(t)$	0.00357	0.02420	0.04774	0.05678	0.06220	0.06689	0.06775
$p_{2,3}(t)$	0.01009	0.02876	0.04034	0.04619	0.05141	0.05686	0.05794
$p_{3,1}(t)$	$4.2-05$	0.00206	0.01151	0.02009	0.02731	0.03447	0.03592
$p_{3,2}(t)$	0.00021	0.00384	0.01264	0.01849	0.02332	0.02877	0.02998
$p_{3,3}(t)$	0.00056	0.00389	0.00866	0.01220	0.01595	0.02087	0.02197

Table 3.9.4 Computed probabilities for the $M/E_k/1$ queue using the generalised modified Bessel function of the second type method with $\rho = 0.95,\ k = 3$

t	0.2	0.5	0.9	1.2	1.5	1.9	2.2
$p_0(t)$	0.82795	0.63962	0.49562	0.43388	0.39172	0.35178	0.32917
$p_{1,1}(t)$	0.00563	0.03817	0.07228	0.08192	0.08435	0.08340	0.08168
$p_{1,2}(t)$	0.03134	0.09008	0.11342	0.11449	0.11251	0.10898	0.10620
$p_{1,3}(t)$	0.11671	0.15623	0.15211	0.14721	0.14339	0.13858	0.13501
$p_{2,1}(t)$	0.00074	0.01237	0.03839	0.05299	0.06184	0.06809	0.07056
$p_{2,2}(t)$	0.00384	0.02423	0.04741	0.05656	0.06208	0.06680	0.06911
$p_{2,3}(t)$	0.01025	0.02849	0.04026	0.04617	0.05135	0.05677	0.05845
$p_{3,1}(t)$	$5.4-05$	0.00221	0.01158	0.02002	0.02729	0.03437	0.03598
$p_{3,2}(t)$	0.00026	0.00395	0.01261	0.01843	0.02328	0.02872	0.03008
$p_{3,3}(t)$	0.00062	0.00390	0.00867	0.01222	0.01596	0.02080	0.02209

Table 3.9.5 Computed probabilities for the $M/E_k/1$ queue using Euler's method with $\rho = 0.2,\ k = 5,\ \delta t = 0.01$

t	0.2	0.5	0.9	1.2	1.5	1.9	2.2
$p_0(t)$	0.96076	0.90670	0.85527	0.83406	0.82226	0.81361	0.80969
$p_{1,1}(t)$	$9.9-05$	0.00377	0.01581	0.02308	0.02706	0.02940	0.03025
$p_{1,2}(t)$	0.00061	0.00871	0.02236	0.02763	0.02993	0.03130	0.03190
$p_{1,3}(t)$	0.00291	0.01668	0.02820	0.03097	0.03211	0.03303	0.03351
$p_{1,4}(t)$	0.02018	0.02614	0.03226	0.03327	0.03396	0.03473	0.03511
$p_{1,5}(t)$	0.02468	0.03352	0.03459	0.03509	0.03576	0.03643	0.03673
$p_{2,1}(t)$	$2.5-06$	0.00027	0.00198	0.00356	0.00472	0.00559	0.00596
$p_{2,2}(t)$	$1.6-05$	0.00060	0.00253	0.00372	0.00442	0.00494	0.00519
$p_{2,3}(t)$	$7.3-05$	0.00104	0.00269	0.00337	0.00375	0.00408	0.00427
$p_{2,4}(t)$	0.00023	0.00133	0.00228	0.00259	0.00281	0.00306	0.00321
$p_{2,5}(t)$	0.00040	0.00104	0.00135	0.00152	0.00170	0.00191	0.00202

Table 3.9.6 Computed probabilities for the $M/E_k/1$ queue using the generalised modified Bessel function of the second type method with $\rho = 0.2,\ k = 5$

t	0.2	0.5	0.9	1.2	1.5	1.9	2.2
$p_0(t)$	0.96081	0.90708	0.85602	0.83468	0.82257	0.81246	0.80381
$p_{1,1}(t)$	$4.1-05$	0.00394	0.01580	0.02288	0.02687	0.02931	0.03022
$p_{1,2}(t)$	0.00057	0.00891	0.02216	0.02740	0.02982	0.03125	0.03188
$p_{1,3}(t)$	0.00311	0.01672	0.02796	0.03085	0.03210	0.03304	0.03353
$p_{1,4}(t)$	0.01122	0.02657	0.03278	0.03390	0.03462	0.03536	0.03574
$p_{1,5}(t)$	0.02356	0.03250	0.03388	0.03442	0.03508	0.03573	0.03599
$p_{2,1}(t)$	$1.8-06$	0.00030	0.00200	0.00354	0.00468	0.00556	0.00595
$p_{2,2}(t)$	$1.7-05$	0.00063	0.00252	0.00369	0.00440	0.00493	0.00518
$p_{2,3}(t)$	$8.4-05$	0.00106	0.00266	0.00335	0.00374	0.00407	0.00426
$p_{2,4}(t)$	0.00025	0.00132	0.00226	0.00258	0.00281	0.00306	0.00320
$p_{2,5}(t)$	0.00041	0.00103	0.00135	0.00152	0.00171	0.00191	0.00201

Table 3.9.7 Computed probabilities for the $M/E_k/1$ queue using Euler's method with $\rho = 2,\ k = 5,\ \delta t = 0.01$

t	0.2	0.5	0.9	1.2	1.5	1.9	2.2
$p_0(t)$	0.66771	0.37274	0.20446	0.14788	0.11216	0.08097	0.06497
$p_{1,1}(t)$	0.00075	0.01610	0.03272	0.03034	0.02545	0.02001	0.01685
$p_{1,2}(t)$	0.00454	0.03659	0.04694	0.03930	0.03257	0.02576	0.02165
$p_{1,3}(t)$	0.02114	0.06919	0.06174	0.04990	0.04173	0.03307	0.02783
$p_{1,4}(t)$	0.07260	0.10781	0.07718	0.06353	0.05361	0.04247	0.03583
$p_{1,5}(t)$	0.17337	0.13988	0.09615	0.08175	0.06889	0.05457	0.04619
$p_{2,1}(t)$	0.00019	0.01196	0.04139	0.04647	0.04347	0.03784	0.03364
$p_{2,2}(t)$	0.00120	0.02577	0.05332	0.05230	0.04794	0.04213	0.03761
$p_{2,3}(t)$	0.00545	0.04397	0.05891	0.05476	0.05122	0.04567	0.04102
$p_{2,4}(t)$	0.01691	0.05566	0.05571	0.05412	0.05263	0.04763	0.04324
$p_{2,5}(t)$	0.02904	0.04438	0.04488	0.04991	0.05045	0.04679	0.04329

Table 3.9.8 Computed probabilities for the $M/E_k/1$ queue using the generalised modified Bessel function of the second type method with $\rho = 2,\ k = 5$

t	0.2	0.5	0.9	1.2	1.5	1.9	2.2
$p_0(t)$	0.67050	0.37700	0.20691	0.14919	0.11306	0.08155	0.06533
$p_{1,1}(t)$	0.00060	0.01605	0.03220	0.03016	0.02550	0.02013	0.01691
$p_{1,2}(t)$	0.00398	0.03627	0.04642	0.03931	0.03267	0.02583	0.02172
$p_{1,3}(t)$	0.02174	0.06824	0.06169	0.05017	0.04188	0.03320	0.02796
$p_{1,4}(t)$	0.07830	0.10793	0.07854	0.06496	0.05432	0.04325	0.03648
$p_{1,5}(t)$	0.17423	0.13668	0.09511	0.08063	0.06813	0.05413	0.04581
$p_{2,1}(t)$	0.00014	0.01224	0.04064	0.04596	0.04337	0.03784	0.03366
$p_{2,2}(t)$	0.00122	0.02588	0.05246	0.05209	0.04795	0.04211	0.03763
$p_{2,3}(t)$	0.00587	0.04339	0.05842	0.05480	0.05118	0.04563	0.04104
$p_{2,4}(t)$	0.01769	0.05461	0.05585	0.05415	0.05248	0.04759	0.04325
$p_{2,5}(t)$	0.02986	0.04388	0.04498	0.04962	0.05029	0.04673	0.04326

3) Calculated probabilities using Euler's and generalised modified Bessel function of the second type methods for the case when initially there are a customers in the system

The Tables 3.9.9-3.9.12 show numerical results for the Erlang service system with initial condition $p_a(0) = 1$, where a is a number of customers at time $t = 0$ in the queue. The calculations have been undertaken using Euler's method and generalised modified Bessel function of the second type method, applying formulas (3.8.7)-(3.8.9).

Table 3.9.9 Computing probabilities for the $M/E_k/1$ queue using Euler's method with $\rho = 0.8,\ k = 3,\ a = 3,\ \delta t = 0.01$

t	0.7	0.9	1.3	1.5	1.9	2.2	2.5
$p_0(t)$	0.00014	0.00080	0.00639	0.01240	0.03052	0.04729	0.06455
$p_{1,1}(t)$	0.00056	0.00214	0.00988	0.01532	0.02583	0.03183	0.03609
$p_{1,2}(t)$	0.00229	0.00665	0.02113	0.02873	0.04041	0.04591	0.05144
$p_{1,3}(t)$	0.00807	0.01794	0.03992	0.04827	0.05863	0.06306	0.06802
$p_{2,1}(t)$	0.02399	0.04112	0.06467	0.06966	0.07212	0.07132	0.07241
$p_{2,2}(t)$	0.05879	0.07851	0.09068	0.08948	0.08424	0.08046	0.07794
$p_{2,3}(t)$	0.11453	0.12141	0.10929	0.10208	0.09148	0.08576	0.08195
$p_{3,1}(t)$	0.16886	0.14772	0.11491	0.10558	0.09311	0.08614	0.08107
$p_{3,2}(t)$	0.17689	0.14033	0.10994	0.10219	0.09018	0.08311	0.07954
$p_{3,3}(t)$	0.12896	0.11376	0.10041	0.09407	0.08308	0.07680	0.07512

Table 3.9.10 Computing probabilities for the $M/E_k/1$ queue using generalised modified Bessel function of the second type method with $\rho = 0.8,\ k = 3,\ a = 3$

t	0.7	0.9	1.3	1.5	1.9	2.2	2.5
$p_0(t)$	0.00019	0.00093	0.00676	0.01281	0.03078	0.04734	0.06443
$p_{1,1}(t)$	0.00065	0.00232	0.01002	0.01533	0.02565	0.03162	0.03592
$p_{1,2}(t)$	0.00251	0.00691	0.02111	0.02855	0.04018	0.04579	0.05060
$p_{1,3}(t)$	0.00842	0.01811	0.03955	0.04782	0.05830	0.06282	0.06706
$p_{2,1}(t)$	0.02417	0.04077	0.06397	0.06912	0.07192	0.07124	0.07102
$p_{2,2}(t)$	0.05808	0.07737	0.09010	0.08925	0.08428	0.08052	0.07824
$p_{2,3}(t)$	0.11265	0.12016	0.10932	0.10234	0.09166	0.08589	0.08122
$p_{3,1}(t)$	0.16767	0.14799	0.11559	0.10604	0.09330	0.08629	0.08053
$p_{3,2}(t)$	0.17902	0.14224	0.11046	0.10244	0.09035	0.08327	0.07909
$p_{3,3}(t)$	0.13172	0.11458	0.10040	0.09417	0.08326	0.07695	0.07431

Table 3.9.11 Computing probabilities for the $M/E_k/1$ queue using Euler's method with $\rho = 0.8,\ k = 5,\ a = 3,\ \delta t = 0.01$

t	0.7	0.9	1.3	1.5	1.9	2.2	2.5
$p_0(t)$	$1.0-06$	$2.1-05$	0.00084	0.00274	0.01306	0.02695	0.04378
$p_{1,1}(t)$	$4.2-06$	$5.9-05$	0.00131	0.00327	0.00982	0.01490	0.01857
$p_{1,2}(t)$	$1.9-05$	0.00020	0.00295	0.00633	0.01503	0.02025	0.02346
$p_{1,3}(t)$	$8.1-05$	0.00063	0.00616	0.01135	0.02156	0.02622	0.02921
$p_{1,4}(t)$	0.00031	0.00181	0.01178	0.01874	0.02905	0.03265	0.03402
$p_{1,5}(t)$	0.00107	0.00470	0.02055	0.02846	0.03703	0.03955	0.04124
$p_{2,1}(t)$	0.00332	0.01094	0.03245	0.03929	0.04312	0.04293	0.04294
$p_{2,2}(t)$	0.00908	0.02269	0.04624	0.04974	0.04859	0.04719	0.04585
$p_{2,3}(t)$	0.00217	0.04138	0.05921	0.05785	0.05283	0.05074	0.04822
$p_{2,4}(t)$	0.04490	0.06553	0.06825	0.06262	0.05607	0.05346	0.05066
$p_{2,5}(t)$	0.07836	0.08872	0.07169	0.06458	0.05846	0.05508	0.05174

Table 3.9.12 Computing probabilities for the $M/E_k/1$ queue using generalised modified Bessel function of the second type method with $\rho = 0.8,\ k = 5,\ a = 3$

t	0.7	0.9	1.3	1.5	1.9	2.2	2.5
$p_0(t)$	$1.5-06$	$2.9-05$	0.00099	0.00312	0.01362	0.02731	0.04382
$p_{1,1}(t)$	$4.9-06$	$6.7-05$	0.00147	0.00346	0.00981	0.01473	0.01838
$p_{1,2}(t)$	$2.5-05$	0.00027	0.00318	0.00651	0.01488	0.02000	0.02328
$p_{1,3}(t)$	$9.8-05$	0.00078	0.00641	0.01141	0.02126	0.02598	0.02871
$p_{1,4}(t)$	0.00041	0.00208	0.01191	0.01858	0.02872	0.03256	0.03485
$p_{1,5}(t)$	0.00131	0.00509	0.02039	0.02799	0.03668	0.03994	0.04141
$p_{2,1}(t)$	0.00375	0.01133	0.03186	0.03859	0.04292	0.04291	0.04252
$p_{2,2}(t)$	0.00965	0.02274	0.04529	0.04909	0.04861	0.04722	0.04605
$p_{2,3}(t)$	0.00220	0.04065	0.05830	0.05759	0.05299	0.05076	0.04890
$p_{2,4}(t)$	0.04427	0.06385	0.06791	0.06291	0.05623	0.05346	0.05090
$p_{2,5}(t)$	0.07823	0.08681	0.07223	0.06520	0.05850	0.05509	0.05184

The behaviour of the probabilities $p_{n,s}(t)$ when t tends to infinity are shown in Figures 3.9.1, 3.9.2, where $p_n(t) = \sum_{s=1}^{k} p_{n,s}(t)$. We note that the transient probabilities tend to their steady-state solution when t is large enough and $\rho < 1$.

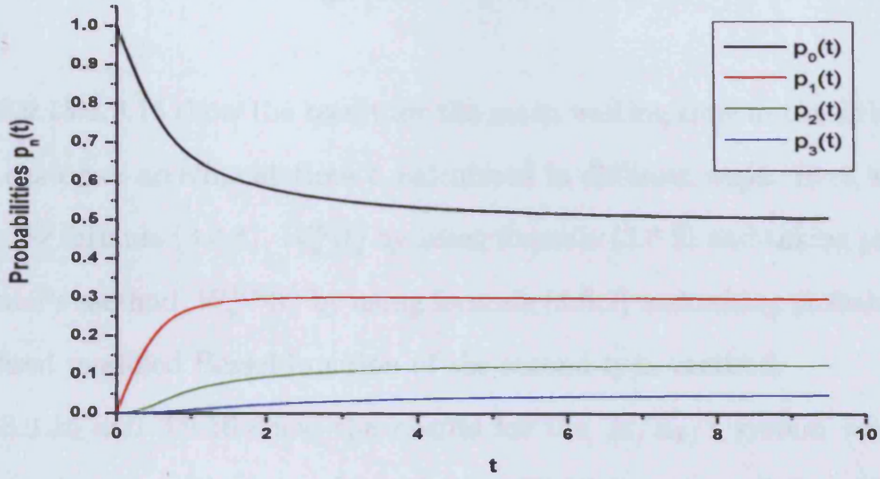

Figure 3.9.1 Transient probabilities $p_n(t)$ with $k=3,\ \rho=0.5$

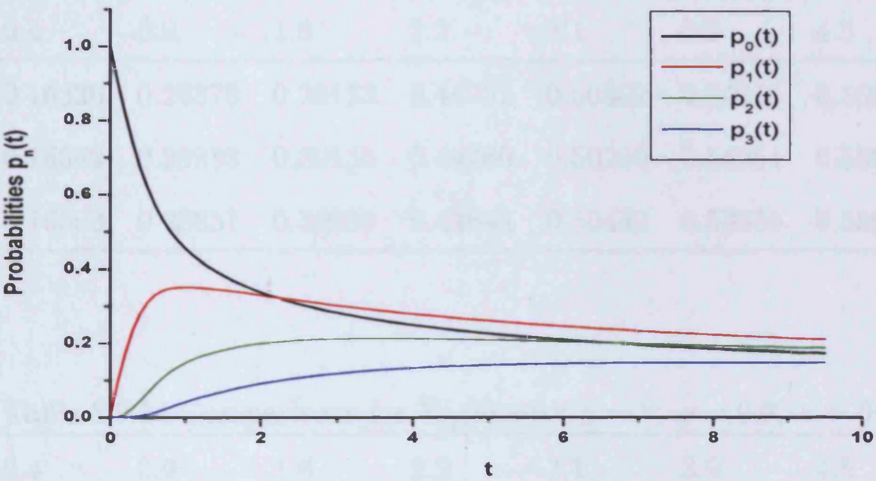

Figure 3.9.2 Transient probabilities $p_n(t)$ with $k=3,\ \rho=0.95$

4) Calculated mean waiting time in a queue for a customer arriving at time t

Tables 3.9.13-3.9.16 show the result for the mean waiting time in the Erlang service queue of a customer arriving at time t, calculated in different ways. Here we compute $W_q(t)$ by using formula (3.6.5), $W_q^n(t)$ by using formula (3.6.2) and taking probabilities from the Euler's method, $W_q^{GF}(t)$ by using formula (3.6.2) and taking probabilities from the generalised modified Bessel function of the second type method.

Tables 3.9.15 and 3.9.16 show the results for the $M/E_k/1$ system when initially there are a customers in the queue. In this case $W_q(t)$ is computed from (3.8.12).

Table 3.9.13 Comparisons for $W_q(t)$ with $k = 3,\ \rho = 0.5,\ a = 0$

t	0.4	0.9	1.6	2.2	3.1	3.9	4.5	5
$W_q(t)$	0.16320	0.28878	0.39113	0.44752	0.50502	0.54015	0.56023	0.57399
$W_q^N(t)$	0.16382	0.28933	0.39135	0.44760	0.50290	0.54251	0.56220	0.57569
$W_q^{GF}(t)$	0.16303	0.28831	0.39056	0.44694	0.50442	0.53955	0.55963	0.57338

Table 3.9.14 Comparisons for $W_q(t)$ with $k = 5,\ \rho = 0.9,\ a = 0$

t	0.4	0.9	1.6	2.2	3.1	3.9	4.5	5
$W_q(t)$	0.29637	0.53823	0.77035	0.92582	1.11758	1.26119	1.35721	1.43120
$W_q^N(t)$	0.29732	0.53884	0.77050	0.92082	1.11684	1.26098	1.35714	1.43112
$W_q^{GF}(t)$	0.29622	0.53761	0.76969	0.92515	1.11659	1.25961	1.35388	1.42522

Table 3.9.15 Comparisons for $W_q(t)$ with $k = 3,\ \rho = 0.5,\ a = 3$

t	0.4	0.9	1.6	2.2	3.1	3.9	4.5	5
$W_q(t)$	2.80000	2.55015	2.20893	1.93940	1.61449	1.40035	1.27674	1.19227
$W_q^N(t)$	2.80000	2.55011	2.20657	1.93765	1.61384	1.39979	1.27527	1.19141
$W_q^{GF}(t)$	2.80000	2.55010	2.20711	1.93889	1.61363	1.40000	1.27633	1.19179

Table 3.9.16 Comparisons for $W_q(t)$ with $k = 5,\ \rho = 0.9,\ a = 3$

t	0.4	0.9	1.6	2.2	3.1	3.9	4.5	5
$W_q(t)$	2.96000	2.91000	2.84081	2.78799	2.73487	2.71254	2.70692	2.70259
$W_q^N(t)$	2.96000	2.91000	2.84067	2.78682	2.73424	2.71218	2.70653	2.70230
$W_q^{GF}(t)$	2.96000	2.91099	2.84052	2.78676	2.73998	2.71098	2.70635	1.70226

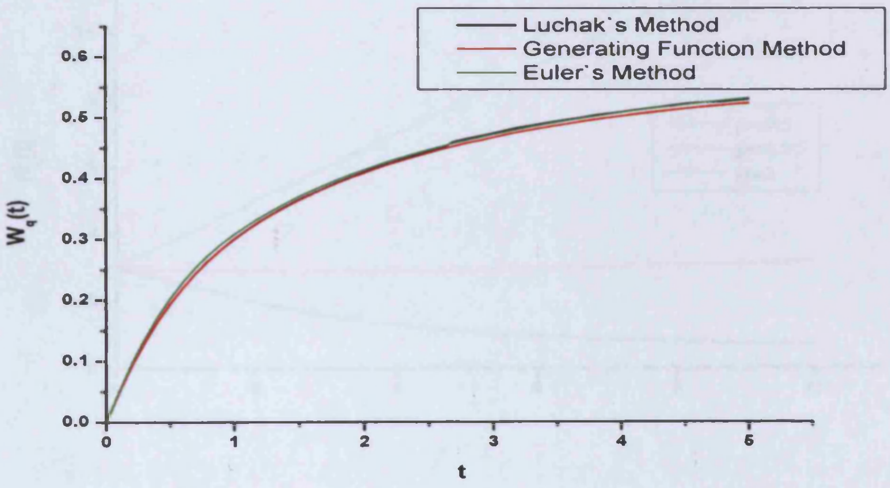

Figure 3.9.3 Graphs for $W_q(t),\ W_q^N(t),\ W_q^{GF}(t)$ with $k = 5,\ \rho = 0.5,\ a = 0$

Figures 3.9.4 and 3.9.5 illustrate the behaviour of $W_q(t)$ for different values of ρ and a.

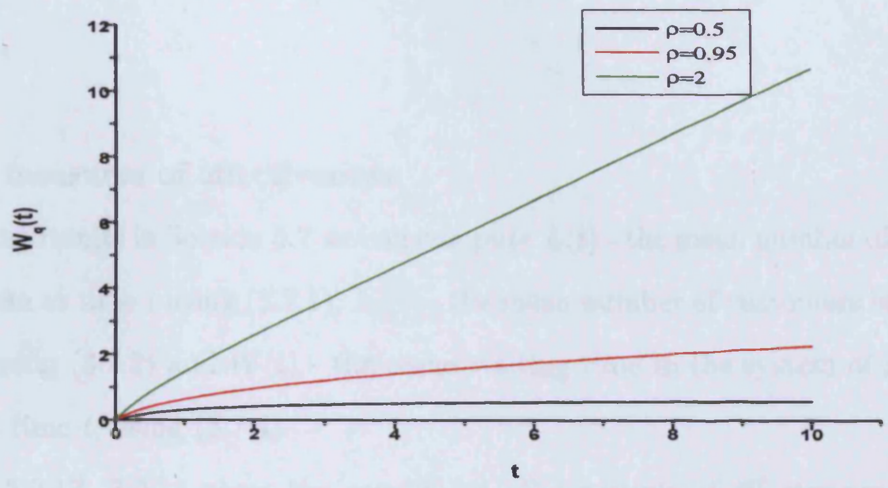

Figure 3.9.4 Graphs for $W_q(t)$ for different ρ with $k = 5,\ a = 0$

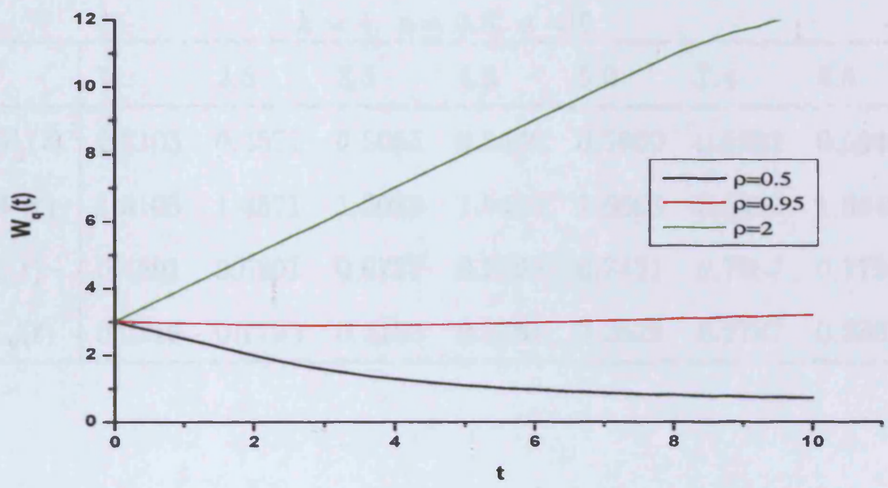

Figure 3.9.5 Graphs for $W_q(t)$ for different ρ with $k = 5,\ a = 3$

5) Other measures of effectiveness

Using the results in Section 3.7 we can compute $L(t)$ - the mean number of customers in the system at time t using (3.7.1), $L_q(t)$ - the mean number of customers in the queue at time t using (3.7.2) and $W(t)$ - the mean waiting time in the system of a customer arriving at time t, using (3.7.4).

Tables 3.9.17, 3.9.18 show the results for all measures of effectiveness applying Euler's method with $\delta = 0.1$.

Table 3.9.17 Numerical results for the measures of effectiveness with $k = 4,\ \rho = 0.5,\ a = 0$

t	1	2.5	3.5	4.8	5.9	7.4	8.5
$W_q(t)$	0.3105	0.4571	0.5055	0.5446	0.5660	0.5852	0.5948
$W(t)$	1.3105	1.4571	1.5055	1.5446	1.5660	0.5852	1.5948
$L(t)$	0.4291	0.6201	0.6777	0.7229	0.7471	0.7687	0.7794
$L_q(t)$	0.0819	0.1793	0.2153	0.2484	0.2629	0.2787	0.2867

Table 3.9.18 Numerical results for the measures of effectiveness with $k = 4,\ \rho = 0.5,\ a = 3$

t	1	2.5	3.5	4.8	5.9	7.4	8.5
$W_q(t)$	2.5000	1.7823	1.4438	1.1619	1.0097	0.8763	0.8120
$W(t)$	3.5000	2.7823	2.4438	2.1619	2.0097	1.8763	1.8120
$L(t)$	2.8699	2.1325	1.7415	1.4160	1.2420	1.0910	1.0188
$L_q(t)$	1.8699	1.2315	0.9706	0.7514	0.6308	0.5230	0.4703

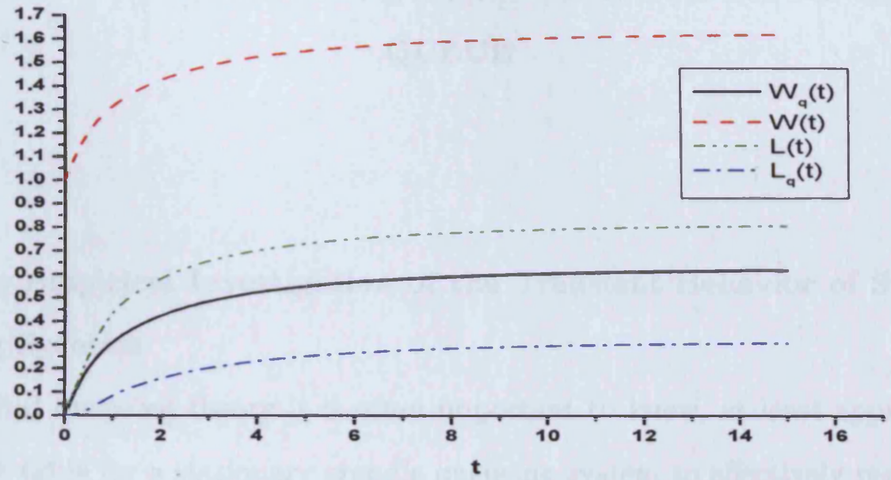

Figure 3.9.6 Measures of effectiveness for the $M/E_k/1$ system with

$k = 4,\ \rho = 0.5,\ a = 0$

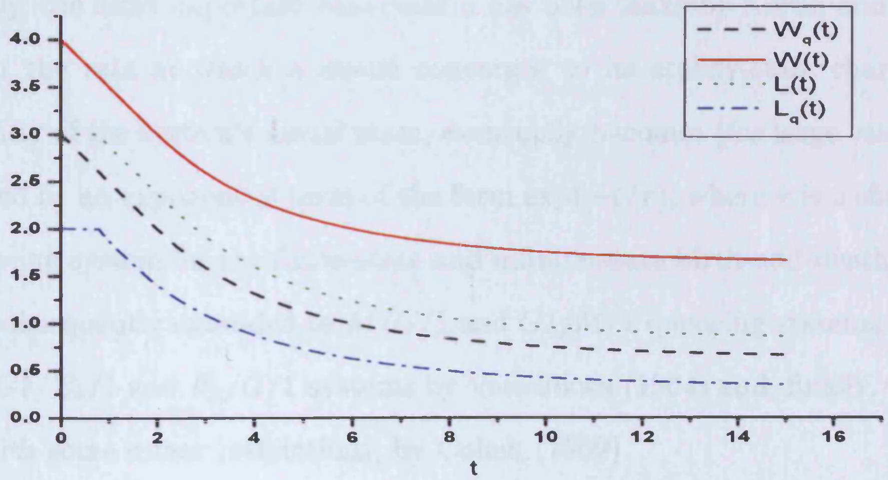

Figure 3.9.7 Measures of effectiveness for the $M/E_k/1$ system with

$k = 4,\ \rho = 0.5,\ a = 3$

CHAPTER 4: APPROXIMATIONS TO THE TRANSIENT ERLANG QUEUE

4.1 An Empirical Investigation of the Transient Behavior of Stationary Queueing Systems

In applied queueing theory it is often important to know, at least approximately, how long it takes for a stationary ergodic queueing system to effectively reach steady-state. Also it is useful to know the manner of approaching a steady-state solution.

This section is based on the work of Odoni and Roth (1983) in which they examine the transient behavior of infinite-capacity, single-server, Markovian queueing systems. They have approximated $L_q(t)$, the expected number of customers in the queue at time t, through a decaying exponential function. We will rewrite approximations for $W_q(t)$, the mean waiting time in a queue of a customer arriving at time t.

Probably, the most important observation has been make by Karlin and McGregor (1957) that the rate at which a queue converges to its steady-state characteristics, independently of the system's initial state, eventually becomes (for large values of time t) dominated by an exponential term of the form $\exp(-t/\tau)$, where τ is a characteristic of the queueing system for the finite-state and infinite-state birth-and-death processes.

It was subsequently extended to $M/G/1$ and $GI/M/1$ queueing systems by Kendall (1960), to $GI/E_k/1$ and $E_k/G/1$ systems by Vere-Jones (1964) and, finally, to $GI/G/1$ systems, with some minor restrictions, by Cohen (1969).

Further work on the transient behavior of queueing systems has concentrated increasingly on approximations and on numerical techniques. This includes Gross and Harris (1974), Newell (1971) in his work on the diffusion approximation of $GI/G/1$ queueing systems under heavy traffic, Mori (1976), developed a numerical technique for estimating the transient behavior of the expected waiting time for $M/M/1$ and $M/D/1$ queueing systems.

We need to begin our study of the transient characteristics of Markovian systems by solving the Kolmogorov forward equations numerically, using for example Euler's technique. Choosing a small step size $\delta t = 0.1$ and initial condition $p_0(0) = 1$, we can obtain an approximate time-dependent solution for all the probabilities $p_{n,s}(t)$ and the queueing characteristics. Also note, that sometimes the accuracy with step $\delta t = 0.1$ is not good enough and in this case it is better to choose $\delta t = 0.01$.

There are two basic numerical techniques, "randomization" (see Grassmann (1977)) and a technique first used successfully by Koopman, which numerically solves the Kolmogorov forward equations of a system.

Roth (1981) has programmed sets of equations for an extensive variety of Markovian queueing systems, e.g., $M/M/1$, $M/E_k/1$, $E_k/M/1$, $E_m/E_k/1$ and $M/H_2/1$.

Our primary interest is to investigate the transient behavior of $W_q(t)$, the mean waiting time of a customer arriving in the queue at time t.

In the paper by Odoni and Roth (1983), they observe systems beginning at rest, where λ and μ are constants over time and they consider the function $\log|W_q(\infty) - W_q(t)|$. As indicated by the approximately linear shape of this function, the decay of the transient part of $W_q(t)$ appears to be approximately exponential for large values of t. Moreover, the convexity of the $\log|W_q(\infty) - W_q(t)|$ for large t provides an upper bound for the rate of decay of $W_q(t)$, for all $t > 0$.

These observations suggest that for Markovian queueing systems that begin from rest, $W_q(t)$ can be bounded from below by an expression of the form

$$W_q(t) = W_\infty(t)\left[1 - \exp(-t/\tau)\right], \quad t \geq 0, \tag{4.1.1}$$

where $W_q(\infty)$ and τ are constants that depend on system parameters as well as the particular interarrival and service time distributions. The steady-state expected waiting time in the Erlang model can be calculated exactly from (3.2.11), but the empirical measurement of τ requires some insight and care. Using the diffusion approximation for queues under heavy traffic, Newell gave the following expression

$$\tau_N = \frac{\rho C_A^2 + C_S^2}{\mu(1-\rho)^2}, \tag{4.1.2}$$

where C_A and C_S, are the coefficients of variation for the interarrival and service times, respectively.

Using numerical results and a trial-and-error approach, Odoni and Roth suggested a slightly different expression

$$\tau_R = \frac{C_A^2 + C_S^2}{\mu(1-\sqrt{\rho})^2}. \tag{4.1.3}$$

For the Erlang queueing system the appropriate coefficients are $C_S^2 = 1/k$, $C_A^2 = 1$.

The numerical results comparing these two methods can be seen in Table 4.1.1 and Figure 4.1.1 for various values of ρ and k.

Table 4.1.1 Comparing results for $W_q(t)$ using Euler's method and Roth's approximation

t	0.9	1.5	2.4	5.7	6.9	7.3	15
Euler $\rho = 0.1, k = 3$	0.0565	0.0671	0.0719	0.0740	0.0741	0.0741	0.0741
Roth $\rho = 0.1, k = 3$	0.0602	0.0695	0.0732	0.0741	0.0741	0.0741	0.0741
Euler $\rho = 0.5, k = 4$	0.2937	0.3759	0.4509	0.5627	0.5798	0.5842	0.6181
Roth $\rho = 0.5, k = 4$	0.1605	0.2436	0.3410	0.5266	0.5572	0.5649	0.6151

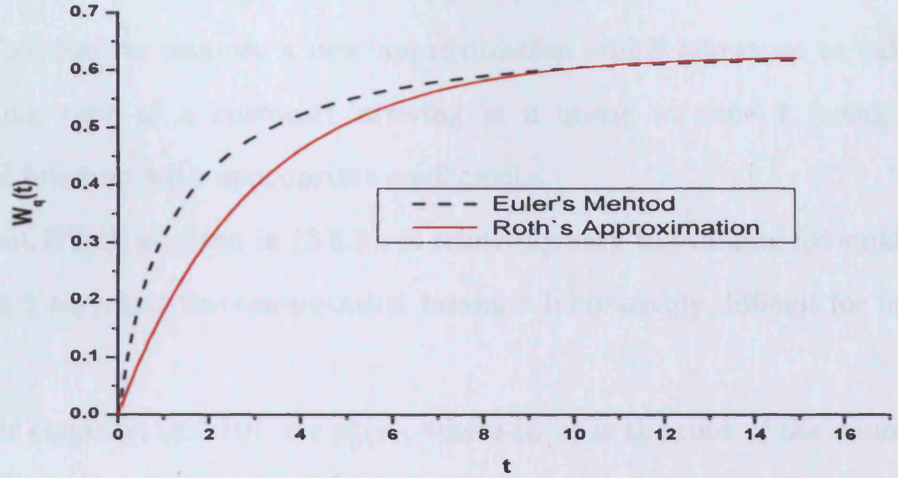

Figure 4.1.1 Graphs for $W_q(t)$, using Euler's method and Roth's approximation with $k = 4,\ \rho = 0.5$

Further investigation of this approximation and comparison results show that it works quite well when ρ is small $(0.1 \leq \rho \leq 0.4)$, but gives poor results, when ρ tends to one.

4.2 A new Double-Exponential Approximation

In this section we propose a new approximation which allows us to calculate the mean waiting time of a customer arriving in a queue at time t, using a double-exponential function with appropriate coefficients.

Note that $W_q(t)$, as given in (3.6.5), is relatively easy to evaluate for small values of k (less than 5 say), but the computation becomes increasingly difficult for large values of k.

Consider equation (3.3.10), for $p_0^*(z)$, where $y_0(z)$ is the root of the denominator of (3.3.9) which lies inside the unit circle. We can find a numerical approximation to this root using, for example, Newton's method. Thus, applying

$$y_{n+1} = y_n - \frac{f(y_n)}{f'(y_n)}, \tag{4.2.1}$$

where $f(y)$ is a polynomial and taking $y_0 = 0$, we can construct an approximation to $y_0(z)$ for each point z, see Figure 4.2.1.

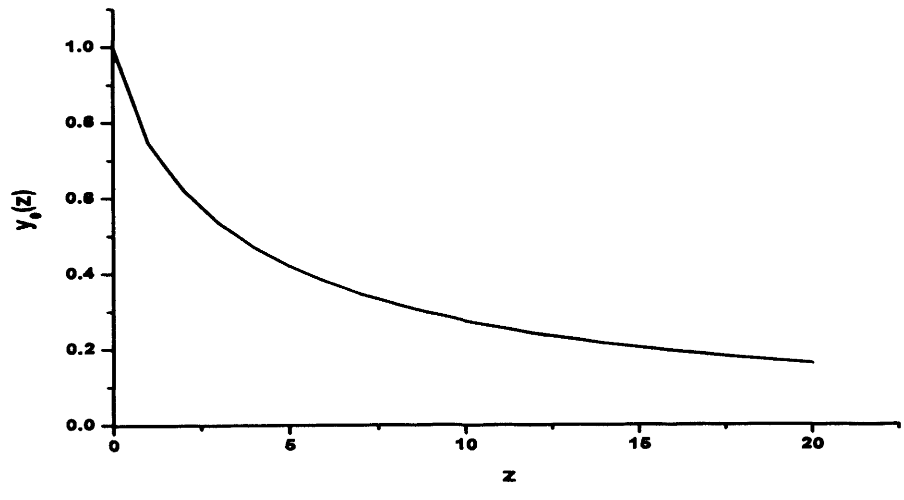

Figure 4.2.1 Graph of $y_0(z)$ with $k = 4$, $\lambda = 0.5$, $\mu = 1$

Also using (3.3.10) we can obtain an approximation to $p_0^*(z)$ for every point z, see Figure 4.2.2.

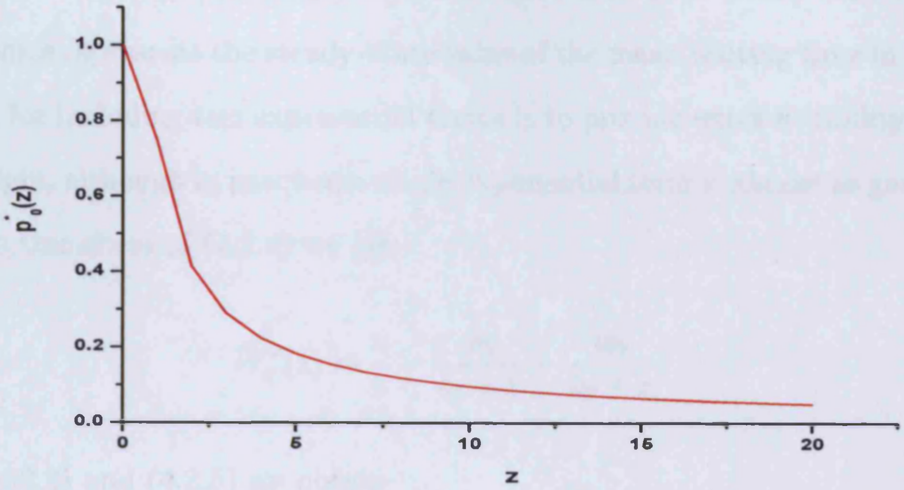

Figure 4.2.2 Graph for the approximation of $p_0^*(z)$ with $k = 4,\ \lambda = 0.5,\ \mu = 1$

Note that knowing $p_0^*(z)$ gives us the possibility of calculating an approximate value for $W_q^*(z)$, since $G^*(y, z)$ and $W_q^*(z)$ are related in a similar way to (3.6.1)

$$W_q^*(z) = \frac{1}{k\mu}\frac{\partial G^*(y,z)}{\partial y}\bigg|_{y=1} = \sum_{n=1}^{\infty}\sum_{i=1}^{k}[(n-1)k+i]p_{n,i}^*(z). \tag{4.2.2}$$

Differentiating (3.3.9) gives us

$$W_q^*(z) = \frac{p_0^*(z)}{z} - \left(\frac{1-\rho}{z^2}\right). \tag{4.2.3}$$

Also note that we cannot define the expression (3.3.10) at $z = 0$, thus in our numerical calculations we will start at point $z = 0.1$.

Now we seek a function which is a good approximation to the function $W_q(t)$, and which has a Laplace transform which is readily invertible. Using previous experience and trial-and-error, we have chosen a double-exponential function

$$\tilde{W}_q(t) = c - w_1\exp(-k_1 t) - w_2\exp(-k_2 t), \tag{4.2.4}$$

where c, w_1, k_1, w_2, k_2 are some parameters, which we will find later. We shall see that this function works well provided $\rho < 1$. We note that as $t \to \infty$, then $\tilde{W}_q(t) \to c$, which of course represents the steady-state value of the mean waiting time in the queue. The reason for including two exponential terms is to provide extra flexibility in the fits near the origin, although in practice a single exponential term is almost as good. Taking the Laplace transform of (4.2.4) we get

$$\tilde{W}_q^*(z) = \frac{c}{z} - \frac{w_1}{k_1 + z} - \frac{w_2}{k_2 + z}. \tag{4.2.5}$$

Equating (4.2.3) and (4.2.5) we obtain

$$\frac{p_0^*(z)}{z} - \left(\frac{1-\rho}{z^2}\right) = \frac{c}{z} - \frac{w_1}{k_1 + z} - \frac{w_2}{k_2 + z}. \tag{4.2.6}$$

We can estimate the parameters c, w_1, w_2, k_1, k_2 by the least squares estimates method and then put them into (4.2.4). As we see from the numerical results in Section 4.3, this approximation works well for values $\rho < 1$.

We have an important consideration when $\rho > 1$. Although the steady-state solution of $M/E_k/1$ leads to infinite waiting times in this case, the situation where ρ is greater than 1 for short periods of time (rush hour) is of ever-increasing importance. In practice of course, infinite queue lengths and waiting times do not occur, and the relevance of the transient solution is paramount. When ρ is substantially greater than 1 (say, 2), the queue will never empty as t becomes large. Thus the accumulation of customers during time t will be $(\lambda - \mu)t$. Hence we need to include a term which is linear in t in our potential expression for $\tilde{W}_q(t)$ to cover the case when t is large. In this case we will use

$$\tilde{W}_q(t) = c - bt - w_1 \exp(-k_1 t) - w_2 \exp(-k_2 t), \tag{4.2.7}$$

when $\rho > 1$. The correspondent Laplace transform is

$$\tilde{W}_q^*(z) = \frac{c}{z} - \frac{b}{z^2} - \frac{w_1}{k_1 + z} - \frac{w_2}{k_2 + z}. \tag{4.2.8}$$

The question now arises as to how we proceed from (4.2.4) to (4.2.7), i.e. we must move from the situation where we have a dominant constant term as in (4.2.4) to the case where we have a dominant linear term $c - bt$ as in (4.2.7). Following consideration of the limiting form of the Laplace transform of $\tilde{W}_q(t)$ as $\rho \to 1$, and confirmation from experimentation, it was found that $\tilde{W}_q(t)$ could be adequately represented by a term of the form $\approx \sqrt{t}$ when t was large. Thus we use

$$\tilde{W}_q(t) = c - b\sqrt{t} - w_1 \exp(-k_1 t) - w_2 \exp(-k_2 t), \tag{4.2.9}$$

when $\rho \approx 1$. This experimentation then provides a smoother transition between (4.2.4) and (4.2.7) for ρ in the approximate range (0.9-1.1).

The Laplace transform of (4.2.9) is given by

$$\tilde{W}_q^*(z) = \frac{c}{z} - \frac{b\sqrt{\pi}}{2z^{3/2}} - \frac{w_1}{k_1 + z} - \frac{w_2}{k_2 + z}. \tag{4.2.10}$$

As a consequence we can obtain explicit expressions for $p_0^*(z)$ from (4.2.6)

$$p_0^*(z) = \begin{cases} c - \frac{w_1 z}{k_1+z} - \frac{w_2 z}{k_2+z} + \frac{(1-\rho)}{z}, & 0 \le \rho < 1, \\ c - \frac{w_1 z}{k_1+z} - \frac{w_2 z}{k_2+z} + \frac{(1-\rho)}{z} - b\frac{\sqrt{\pi}}{2} z^{-1/2}, & \rho \approx 1, \\ c - \frac{w_1 z}{k_1+z} - \frac{w_2 z}{k_2+z} + \frac{(1-\rho)}{z} - bz^{-1}, & \rho > 1. \end{cases} \tag{4.2.11}$$

Inverting the Laplace transform of (4.2.11) we obtain

$$p_0(t) = \begin{cases} (c - w_1 - w_2)\delta(t) + w_1 k_1 e^{-k_1 t} + w_2 k_2 e^{-k_2 t} + (1-\rho) & 0 \le \rho < 1, \\ (c - w_1 - w_2)\delta(t) + w_1 k_1 e^{-k_1 t} + w_2 k_2 e^{-k_2 t} + (1-\rho) - \frac{b}{2} t^{-1/2}, & \rho \approx 1, \\ (c - w_1 - w_2)\delta(t) + w_1 k_1 e^{-k_1 t} + w_2 k_2 e^{-k_2 t} + (1-\rho) - b, & \rho > 1, \end{cases} \tag{4.2.12}$$

where $\delta(t)$ is the Dirac Delta Function.

Thus, we construct $p_0^*(z)$ for each point z, using (3.3.10) and Newton's method and then approximate it with the function $p_0^*(z)$ from (4.2.11). Parameters c, b, w_1, w_2, k_1, k_2

we estimate by using the least squares method. Putting these parameters into (4.2.12) and (4.2.4), (4.2.7) and (4.2.9) give a good fit to the functions $W_q(t)$ and $p_0(t)$, see Section 4.3.

4.3 Numerical Results and Comparisons

This section contains numerical comparisons for $W_q(t)$, $p_0(t)$, $p_0^*(z)$, computed by Euler's method with $\delta t = 0.01$ and double-exponential approximation with appropriate estimated parameters for different values ρ and k.

Table 4.3.1 Comparison of $p_0^*(z)$ and its approximation with $k = 3$, $\rho = 0.3$

z	1	3	5	12	15	19
$p_0^*(z)$	0.84042	0.30608	0.1831	0.081315	0.06536	0.05181
$p_0^*(z)$ *ap*	0.84080	0.30643	0.18943	0.081200	0.06513	0.05146

Table 4.3.2 Comparison of $W_q(t)$ and its approximation with $k = 3$, $\rho = 0.3$

t	1	3	5	12	15	19
$W_q(t)$	0.17038	0.25858	0.27671	0.28536	0.28561	0.28569
$W_q(t)$ *ap*	0.17818	0.25857	0.27661	0.28539	0.28558	0.28562

Table 4.3.3 Comparison of $p_0(t)$ and its approximation with $k = 3$, $\rho = 0.3$

t	1	3	5	12	15	19
$p_0(t)$	0.79038	0.71608	0.70465	0.70014	0.70004	0.70000
$p_0(t)$ *ap*	0.79180	0.71058	0.70481	0.70011	0.70002	0.70000

Estimated parameters are

$$c = 0.2856, \quad w_1 = 0.1213, \quad k_1 = 0.5217,$$
$$w_2 = 0.1628, \quad k_2 = 1.5671.$$

As may be seen from Figures 4.3.1-4.3.3, for $\rho < 1$, and when t tends to infinity, we have a steady-state solution.

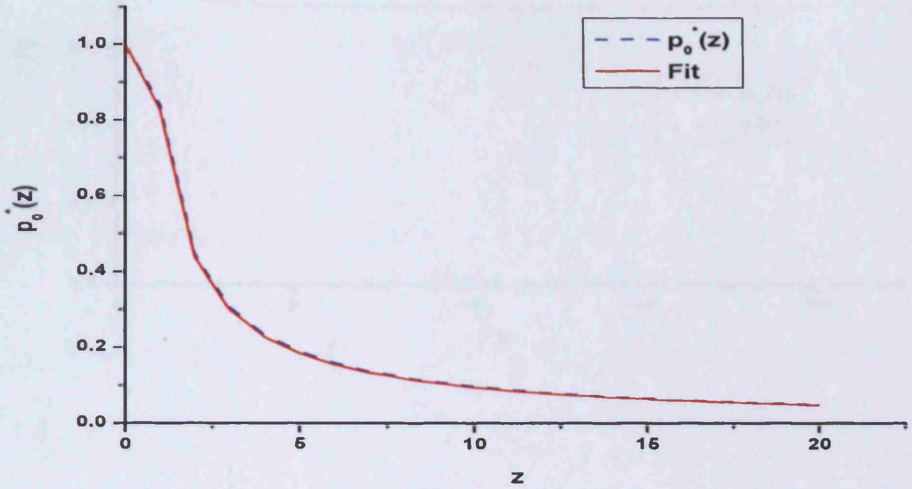

Figure 4.3.1 Graph of $p_0^*(z)$ and its approximation with $k = 3,\ \rho = 0.3$

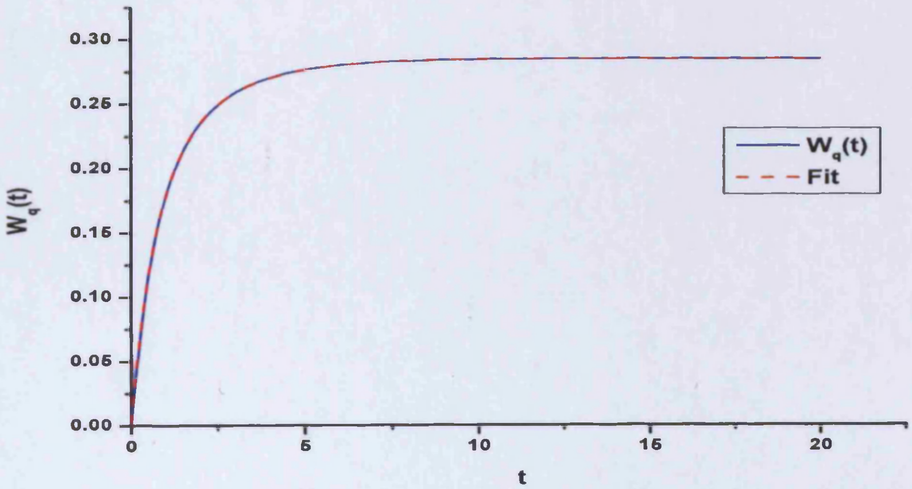

Figure 4.3.2 Graph of $W_q(t)$ and its approximation with $k = 3,\ \rho = 0.3$

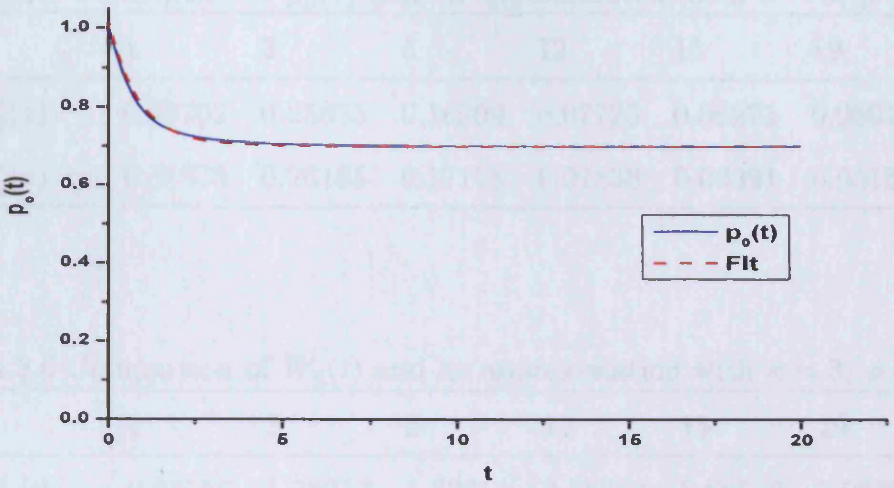

Figure 4.3.3 Graph of $p_0(t)$ and its approximation with $k = 3,\ \rho = 0.3$

Table 4.3.4 Comparison of $p_0^*(z)$ and its approximation with $k = 3,\ \rho = 0.95$

z	1	3	5	12	15	19
$p_0^*(z)$	0.58702	0.25833	0.16909	0.07725	0.06271	0.05013
$p_0^*(z)$ *ap*	0.58873	0.26165	0.17178	0.07838	0.06391	0.05155

Table 4.3.5 Comparison of $W_q(t)$ and its approximation with $k = 3,\ \rho = 0.95$

t	1	3	5	12	15	19
$W_q(t)$	0.62356	1.23213	1.63914	2.58498	2.88542	3.23321
$W_q(t)$ *ap*	0.62183	1.23011	1.64106	2.58325	2.88332	3.23595

Table 4.3.6 Comparison of $p_0(t)$ and its approximation with $k = 3,\ \rho = 0.95$

t	1	3	5	12	15	19
$p_0(t)$	0.47042	0.28592	0.22808	0.15729	0.14371	0.13091
$p_0(t)$ *ap*	0.46910	0.28875	0.22925	0.15671	0.14414	0.13294

Estimated parameters are

$$c = 0.1047, \quad w_1 = 0.2619, \quad k_1 = 0.2613,$$
$$b = -0.7187, \quad w_2 = -0.1414, \quad k_2 = 28.452.$$

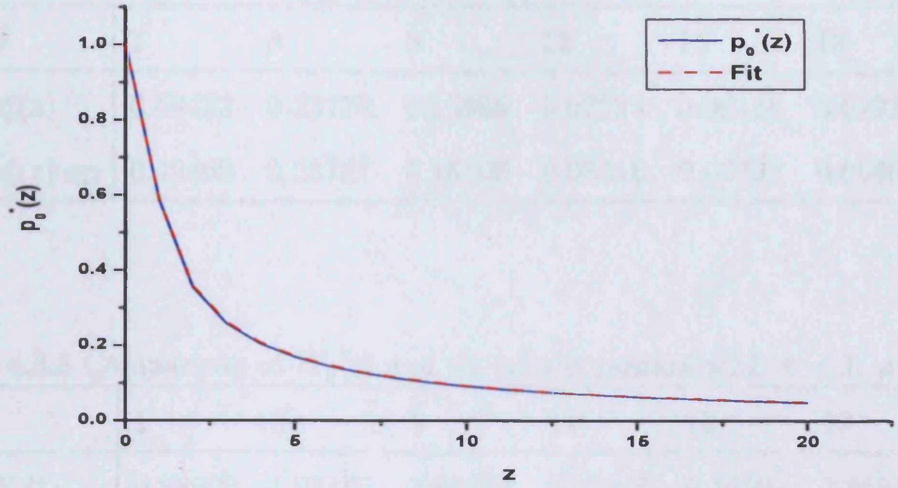

Figure 4.3.4 Graph of $p_0^*(z)$ and its approximation with $k = 3,\ \rho = 0.95$

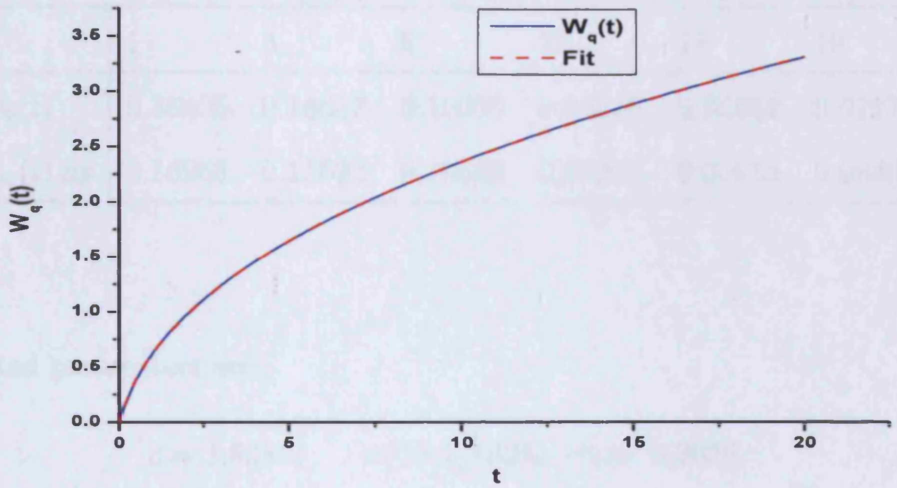

Figure 4.3.5 Graph of $W_q(t)$ and its approximation with $k = 3,\ \rho = 0.95$

Table 4.3.7 Comparison of $p_0^*(z)$ and its approximation with $k = 3,\ \rho = 1.3$

z	1	3	5	12	15	19
$p_0^*(z)$	0.49433	0.23774	0.15985	0.07523	0.06136	0.04926
$p_0^*(z)$ *ap*	0.49460	0.23767	0.16106	0.08016	0.06722	0.05400

Table 4.3.8 Comparison of $W_q(t)$ and its approximation with $k = 3,\ \rho = 1.3$

t	1	3	5	12	15	19
$W_q(t)$	0.88909	1.95197	2.80862	5.35996	6.36591	7.66914
$W_q(t)$ *ap*	0.89031	1.95062	2.80620	5.36607	6.36379	7.67093

Table 4.3.9 Comparison of $p_0(t)$ and its approximation with $k = 3,\ \rho = 1.3$

t	1	3	5	12	15	19
$p_0(t)$	0.35405	0.16657	0.10350	0.04078	0.03052	0.02174
$p_0(t)$ *ap*	0.36968	0.15693	0.10523	0.03976	0.03033	0.02407

Estimated parameters are

$$c = 1.6331, \quad w_1 = 1,1634, \quad k_1 = 0.2028,$$
$$b = -0.3190, \quad w_2 = 0.4586, \quad k_2 = 1.4091.$$

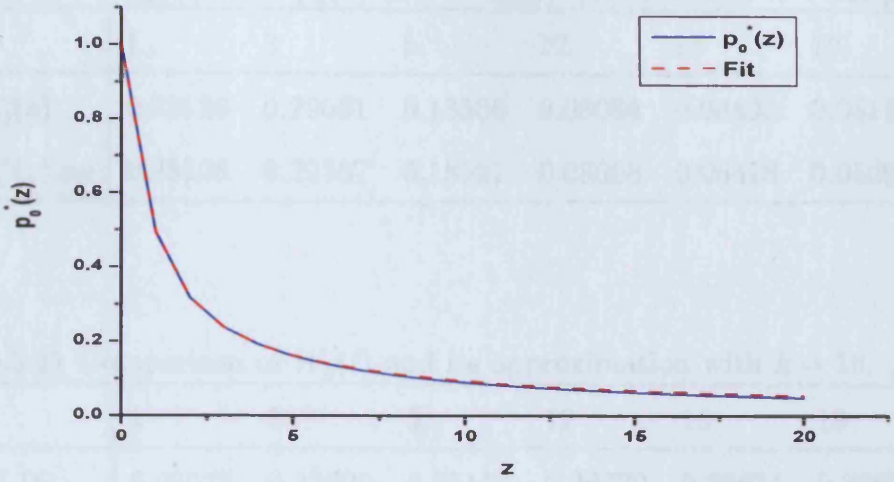

Figure 4.3.6 Graph of $p_0^*(z)$ and its approximation with $k=3,\ \rho=1.3$

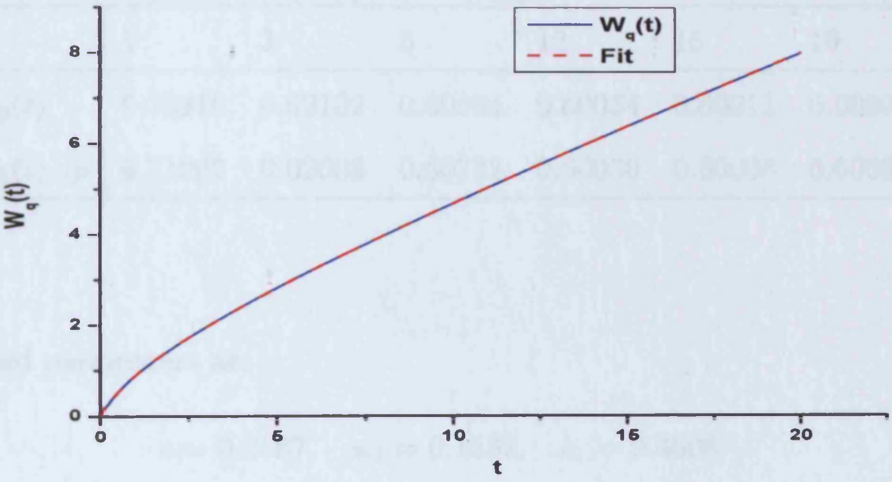

Figure 4.3.7 Graph of $W_q(t)$ and its approximation with $k=3,\ \rho=1.3$

Table 4.3.10 Comparison of $p_0^*(z)$ and its approximation with $k = 10,\ \rho = 0.4$

z	1	3	5	12	15	19
$p_0^*(z)$	0.78120	0.29601	0.18536	0.08064	0.06493	0.05154
$p_0^*(z)$ *ap*	0.78128	0.29587	0.18521	0.08008	0.06418	0.05095

Table 4.3.11 Comparison of $W_q(t)$ and its approximation with $k = 10,\ \rho = 0.4$

t	1	3	5	12	15	19
$W_q(t)$	0.23078	0.32620	0.35119	0.36770	0.36634	0.36658
$W_q(t)$ *ap*	0.22980	0.32578	0.35058	0.36603	0.36653	0.36668

Table 4.3.12 Comparison of $p_0(t)$ and its approximation with $k = 10,\ \rho = 0.4$

t	1	3	5	12	15	19
$p_0(t)$	0.70216	0.62102	0.60694	0.60034	0.60011	0.60002
$p_0(t)$ *ap*	0.71092	0.62008	0.60732	0.60030	0.60008	0.60001

Estimated parameters are

$$c = 0.3667, \quad w_1 = 0.1532, \quad k_1 = 0.4508,$$
$$w_2 = 0.2157, \quad k_2 = 1.7034.$$

Table 4.3.13 Comparison of $p_0^*(z)$ and its approximation with $k = 10$, $\rho = 0.99$

z	1	3	5	12	15	19
$p_0^*(z)$	0.55524	0.25286	0.16719	0.07698	0.06253	0.05000
$p_0^*(z)$ *ap*	0.55651	0.25517	0.16918	0.07816	0.06395	0.05103

Table 4.3.14 Comparison of $W_q(t)$ and its approximation with $k = 10$, $\rho = 0.99$

t	1	3	5	12	15	19
$W_q(t)$	0.63464	1.23291	1.64286	2.63181	3.15666	3.34345
$W_q(t)$ *ap*	0.63276	1.23197	1.64314	2.63171	3.15639	3.34361

Table 4.3.15 Comparison of $p_0(t)$ and its approximation with $k = 10$, $\rho = 0.99$

t	1	3	5	12	15	19
$p_0(t)$	0.41519	0.24516	0.19126	0.12538	0.11270	0.10072
$p_0(t)$ *ap*	0.41828	0.24618	0.19176	0.12522	0.11271	0.10105

Estimated parameters are

$$c = -0.1101, \quad w_1 = 0.0632, \quad k_1 = 0.2425,$$
$$b = -0.7924, \quad w_2 = -0.1558, \quad k_2 = 28.010.$$

Table 4.3.16 Comparison of $p_0^*(z)$ and its approximation with $k = 10,\ \rho = 1.5$

z	1	3	5	12	15	19
$p_0^*(z)$	0.43239	0.22408	0.15408	0.07407	0.06060	0.04878
$p_0^*(z)$ *ap*	0.43311	0.22427	0.15480	0.07579	0.06174	0.04973

Table 4.3.17 Comparison of $W_q(t)$ and its approximation with $k = 10,\ \rho = 1.5$

t	1	3	5	12	15	19
$W_q(t)$	1.02763	2.33899	3.47752	7.15247	8.67937	10.6999
$W_q(t)$ *ap*	1.03097	2.33615	3.47685	7.15169	8.67765	10.7017

Table 4.3.18 Comparison of $p_0(t)$ and its approximation with $k = 10,\ \rho = 1.5$

t	1	3	5	12	15	19
$p_0(t)$	0.26266	0.09465	0.05048	0.01145	0.00691	0.00374
$p_0(t)$ *ap*	0.27743	0.09332	0.05253	0.01074	0.00708	0.00528

Estimated parameters are

$$c = 1.1197, \quad w_1 = 0.7061, \quad k_1 = 0.2905,$$
$$b = -0.5044, \quad w_2 = 0.4079, \quad k_2 = 1.8344.$$

CHAPTER 5: NUMERICAL INVERSION OF THE LAPLACE TRANSFORM

5.1 The Laplace Transform And Its Main Properties

The description of the queueing systems leads to solving differential-difference equations with some initial conditions. Unhappily, in many cases of importance, the complexity of the equations far exceeds the power of our mathematical capabilities. In this impasse, our only resource is a numerical solution of the equations. Over the last decade, numerical methods have become very effective in scientific research.

The fundamental importance of the Laplace transform in solving differential equations resides in its ability to lower the transcendence level of an equation.

We will present some methods of inverting Laplace transforms and compare the results for different functions. We will then apply these methods to queueing theory, since probability distributions can often be characterized in terms of transforms.

Let $f(t)$ be a function defined for $t \geq 0$. The function $\hat{f}(s)$, introduced by means of the expression

$$\hat{f}(s) = \int_0^{\infty} e^{-st} f(t)dt \tag{5.1.1}$$

is termed the Laplace transform of f, where s is a complex variable with nonnegative real part. The transform possesses a number of simple, but extremely important properties which make it one of the most useful operations in analysis.

We suppose always that $f(t)$ satisfies a bound of the form

$$|f(t)| \leq ae^{bt},$$

for some constants a and b, as $t \to \infty$, and that

$$\int_0^T |f(t)|\, dt < \infty,$$

for any finite T. The combination of these two assumptions permits us to conclude that the integral convergence absolutely.

First, note that

$$L(e^{-at}f(t)) = \hat{f}(s+a), \tag{5.1.2}$$

because

$$\int_0^\infty e^{-st}e^{-at}f(t)dt = \int_0^\infty e^{-(s+a)t}f(t)dt.$$

The second fundamental property is associated with the expression

$$h(t) = \int_0^\infty f(x)g(t-x)dx. \tag{5.1.3}$$

This operation, creating the function $h(t)$ as a composite of the two functions $f(t)$ and $g(t)$, has Laplace transform

$$L(f*g) = L(f)L(g), \tag{5.1.4}$$

because

$$\begin{aligned} \int_0^\infty e^{-st}h(t)dt &= \int_0^\infty e^{-st}\left[\int_0^\infty f(x)g(t-x)dx\right]dt = \\ \int_0^\infty f(x)\left[\int_x^\infty e^{-st}g(t-x)dt\right]dx &= \left[\int_0^\infty e^{-sx}f(x)dx\right]\left[\int_0^\infty e^{-st}g(t)dt\right]. \end{aligned}$$

For further main properties see Widder (1946), Feller (1971).

In the next section we analyze four numerical Laplace transform inversion algorithms developed by Widder (1971), Abate, Whitt, (1988, 1992, 1995, 1998) Gaver, Stehfest (1966), Choudhury and Lucantoni (1994).

5.2 Inversion of the Laplace Transform

5.2.1 Post-Widder Method

Applying Laplace transforms is a usual method of solving differential-difference equations in queueing theory. But the main difficulty which arises in the equations is to find the inverse Laplace transform. For transforms with suitable structure the inverse function can be recovered by using tables of function-transform pairs. For more complicated transforms, numerical transform inversion may be a viable option.

Nevertheless, it is difficult to provide effective methods with simple general error bounds which can be used for all functions.

These methods work only when the Laplace transform $\hat{f}(s)$ of a function $f(t)$ is known on a complex half-plane.

Our object is to calculate values of a real-valued function $f(t)$ of a positive real variable t for various t from the Laplace transform

$$\hat{f}(s) = \int_0^\infty e^{-st} f(t)dt,$$

where s is a complex variable with nonnegative real part. In probability applications we typically know that we have a complementary odd f, so that there is nothing extra to verify.

Many methods have been developed for performing numerical transform inversion, see Papoulis' (1957), Euler method (Abate (1995)), Post-Widder method (Bellman (1966)), Gaver-Stehfest method (Gaver (1966)), Crump and Piessens-Huysmans method (Crump (1976), Piessens and Huysmans (1984)), the quotient difference method with accelerated convergence for the continued fraction expansion (d'Hoog, Knight and Stokes (1982), and d'Amore, Laccetti, Murli (1999)), Weeks method (Weeks (1966)), the Kwok-Barthez algorithm (see Kwok, Barthez (1989)), multidimentional inversion of the Laplace transform by Choudhury, Lucantoni, Whitt (1994).

The following result is based on the Post-Widder theorem (see Feller (1971)).

Theorem 5.2.1 Let $f(t)$ be bounded and continuous and let $\hat{f}(s)$ denote its Laplace transform. Then $f(t)$ equals the the point-wise limit as $n \to \infty$ of

$$f_n(t) = \frac{(-1)^n}{n!} \left(\frac{n+1}{t}\right)^{n+1} \hat{f}^{(n)} \left(\frac{n+1}{t}\right). \tag{5.2.1}$$

Moreover the convergence is uniform on bounded intervals. Here $\hat{f}^{(n)}(s)$ is the nth derivative of the Laplace transform.

It is clear that relatively small errors in the evaluation of the derivatives could seriously impair the accuracy of (5.2.1). Numerical differentiation is always a risky procedure. As stated above, it is impossible to find a completely satisfactory numerical inversion formula, so for the different classes of function the following numerical methods work differently. A crucial role in the numerical inversion of Laplace transforms is played by Euler summation.

5.2.2 Euler Method

This method was developed by Dubner and Abate (1968) and Simon, Stroot and Weiss (1972) who were the first to use Euler summation in the Fourier-series inversion algorithm, then Hosono (1984), Abate and Whitt (1992) have made refinements to the basic algorithm and pioneered new areas of application. This method is based on the Bromwich contour inversion integral, which can be expressed as the integral of a real-valued function of a real variable by choosing a specific contour.

Letting the contour be any vertical line $s = x$ such that $\hat{f}(s)$ has no singularities on or to the right of it, we obtain

$$f(t) = \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} e^{st}\hat{f}(s)ds = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(x+iy)t}\hat{f}(x+iy)dy \tag{5.2.2}$$

$$\begin{aligned}
&= \frac{e^{xt}}{2\pi}\int_{-\infty}^{\infty}(\cos yt + i\sin yt)\hat{f}(x+iy)dy \\
&= \frac{e^{xt}}{2\pi}\int_{-\infty}^{\infty}\left[Re(\hat{f}(x+iy))\cos yt - Im(\hat{f}(x+iy))\sin yt\right]dy \\
&= \frac{2e^{xt}}{\pi}\int_{0}^{\infty} Re(\hat{f}(x+iy))\cos yt dy,
\end{aligned}$$

where $i = \sqrt{-1}$ and $Re(s)$ and $Im(s)$ are the real and imaginary parts of s.

We calculate the integral (5.2.2) approximately by using Fourir-series (the Poisson summation formula) to replace the integral by a series (which corresponds to the trapezoidal rule) with a specified discretization error.

Since the series is nearly alternating, we apply Euler summation to accelerate convergence (approximately calculate the infinite sum).

The Fourer-series method for numerically inverting Laplace transforms (and identifying the discretization error) was first proposed by Dubner and Abate (see Dubner and Abate (1968)).

The further results of Simon, Stroof (1972), Weiss and Hosono (1984) show that (5.2.2) can be expressed as following

$$f(t) = -\frac{2e^{xt}}{\pi} \int_0^\infty Im(\hat{f}(x+iy)) \sin ytdy. \tag{5.2.3}$$

We can evaluate (5.2.2) by means of the trapezoidal rule with step size h

$$f(t) \simeq f_h(t) = \frac{he^{xt}}{\pi} Re(\hat{f}(x)) + \frac{2he^{xt}}{\pi} \sum_{k=1}^{\infty} Re(\hat{f}(x+ikh)) \cos(kht). \tag{5.2.4}$$

Letting $h = \pi/2t$ and $x = A/2t$ we obtain

$$f_h(t) = \frac{e^{A/2}}{2t} Re\left(\hat{f}\left(\frac{A}{2t}\right)\right) + \frac{e^{A/2}}{t} \sum_{k=1}^{\infty} (-1)^k Re\left(\hat{f}\left(\frac{A+2k\pi i}{2t}\right)\right). \tag{5.2.5}$$

To identify the discretization error associated with (5.2.5) we use the Poisson summation formula, developed by Dubner, Abate (1968). The idea consists of replacing the damped function $g(t) = \exp(-bt)$, $b > 0$ by the periodic function

$$g_p(t) = \sum_{k=-\infty}^{\infty} g(t + \frac{2\pi k}{h}). \tag{5.2.6}$$

It can be shown that error is bounded by

$$|e_\alpha| \leq \frac{e^{-A}}{1-e^{-A}}. \tag{5.2.7}$$

Hence, to have a discretization error of most $10^{-\gamma}$, we let $A = \gamma \log 10$. To achieve a discretization error of 10^{-8} we put $A = 18.4$.

For computation we use a finite series approximation and using Euler summation we can write the following expression

$$f(t) \approx E(m,n,t) = \sum_{k=1}^{m} \binom{m}{n} 2^{-m} S_{n+k}(t), \tag{5.2.8}$$

where

$$S_n(t) = \frac{e^{A/2}}{2t} Re\left(\hat{f}\left(\frac{A}{2t}\right)\right) + \frac{e^{A/2}}{t}\sum_{k=1}^{n}(-1)^k Re\left(\hat{f}\left(\frac{A+2k\pi i}{2t}\right)\right). \quad (5.2.9)$$

Typically we use $m = 11,\ n = 15$.

5.2.3 Gaver-Stehfest Method

Gaver's method is based on a discrete analog of (5.2.1) involving finite differences, see Gaver(1966). However, the Gaver-Stehfest method is easy to program but it is much less robust than the Fourier-series method. For many problems, it works very well, but for others it does not.

Let

$$\Delta \hat{f}(n\alpha) = \hat{f}((n+1)\alpha) - \hat{f}(n\alpha), \tag{5.2.10}$$

and let $\Delta^k = \Delta(\Delta^{k-1})$, so that

$$(-1)^n \Delta^n \hat{f}(n\alpha) = \sum_{k=1}^{n} (-1)^k \binom{n}{k} 2^{-n} \hat{f}((n+k)\alpha). \tag{5.2.11}$$

Theorem 5.2.2 If f is a bounded real-valued function that is continuous at t, then

$$f(t) = \lim_{n\to\infty} \hat{f}_n(t),$$

where

$$\hat{f}_n(t) = (-1)^n \frac{\ln 2}{t} \frac{(2n)!}{n!(n-1)!} \Delta^n \hat{f}\left(n\frac{\ln 2}{t}\right), \tag{5.2.12}$$

Here is Stehfest's result.

Theorem 5.2.3 Let $\hat{f}_n(t)$ given by (5.2.12) and let

$$f_n^*(t) = \sum_{k=1}^{n} w(k,n) \hat{f}_k(t),$$

for

$$w(k,n) = (-1)^{n-k} \frac{k^n}{k!(n-k)!},$$

then $f_n^*(t) - f(t) = o(n^{-k})$, as $n \to \infty$ for all k.

5.2.4 Multidimensional Inversion

G.L. Choudhury, D.M. Lucantoni and W. Whitt (1994) developed an algorithm for numerically inverting multidimensional transforms with applications to the transient $M/G/1$ queue. Their method is a multivariate generalisation of the Euler and lattice-Poisson algorithms in Abate and Whitt (1992). They apply a multivariate version of the Poisson summation formula and damp the given function by multiplying by a two-dimensional decaying exponential function and then approximate the damped function by a periodic function constructed by aliasing.

They consider three types of two-dimensional transforms: (i) continuous-continuous, (ii) continuous-discrete and (iii) discrete-discrete. Also the formulae can be generalised to more than two dimensions with any number of continuous and discrete variables.

Here, we present just a continuous-discrete type, because this inversion allow us to calculate the time-dependent probability distributions in queueing models.

Let the function of interest be $f(n,t)$, where t is a nonnegative continuous variable and n is a nonnegative integer. We wish to calculate $f(n,t)$ by numerically inverting the two-dimensional transform

$$\hat{f}(z,s) = \int_0^\infty \sum_{n=0}^{\infty} f(n,t)\exp(-st)z^n dt. \tag{5.2.13}$$

Let $F(n,t)$ be defined for real t and integer n and let $\phi(u_1,u_2)$ be its Fourier transform, that is,

$$\phi(u_1,u_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(n,t)\exp(i(u_1 t + u_2 n))dt.$$

The bivariate mixed Poisson summation is

$$\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} F(n+km, t+\frac{2\pi j}{h}) = \frac{h}{2\pi m}\sum_{j=-\infty}^{\infty}\sum_{k=-m/2}^{m/2-1} \phi(jh, \frac{2\pi k}{m})\exp(-i(jht + \frac{2\pi kn}{m})).$$

In order to control the aliasing error, we undertake exponential damping as follows:

$$F(n,t) = \begin{cases} f(n,t)\exp(-at)r^n, & t \geq 0,\ n \geq 0 \\ 0, & \text{otherwise,} \end{cases}$$

where $a > 0,\ 0 < r < 1$. Letting $h = \pi/tl_1$, $m = 2l_2n$, $a = A/(2tl_1)$ and after some manipulation we get

$$f(t,n) = \frac{\exp(A/2l_1)}{2l_1t} \sum_{j_1=1}^{l_1} \sum_{j=-\infty}^{\infty} (-1)^j \exp(-\frac{ij_1\pi}{l_1}) \tag{5.2.14}$$

$$\times \left\{ \frac{1}{2l_2hr^n} \sum_{k_1=0}^{l_2-1} \sum_{k=-n}^{n-1} (-1)^k \exp(-\frac{ik_1\pi}{l_2}) \right.$$
$$\left. \times \hat{f}(r\exp\left(\frac{\pi i(k_1 + l_2k)}{l_2n}\right), \frac{A}{2l_1t} - \frac{ij_1\pi}{l_1t} - \frac{ji\pi}{t}) \right\}.$$

The aliasing error can be bounded by

$$|\bar{e}| = \frac{C(e^{-A} + r^{2l_2n} - e^{-A}r^{2l_2n})}{(1-e^{-A})(1-r^{2l_2n})} \approx C(e^{-A} + r^{2l_2n}), \tag{5.2.15}$$

assuming that $|f| \leq C$.

Typically they use $A = 19.114$ (or $A = 28.324$), $l_1 = l_2 = 1, 2, 3, ..., 7$.

Next we consider the transient queue-length distribution for the $M/E_k/1$ queue.

Let $L_q(t)$ represent the queue length at time t. Let there be a departure at time $t = 0$ and initially there are no customers in the system. Let $Y(n,t) = P(L_q(t) = n \mid L_q(0) = 0)$. Consider the two-dimensional transform

$$\hat{Y}(z,s) = \sum_{n=0}^{\infty} \int_{t=0}^{\infty} \exp(-st)z^nY(n,t)dt. \tag{5.2.16}$$

It can be shown that

$$\hat{Y}(z,s) = \frac{z\left(1-\hat{h}(s+\lambda-\lambda z)\right)}{(s+\lambda-\lambda z)(z-\hat{h}(s+\lambda-\lambda z)} + \frac{(z-1)\hat{p}_0(s)\hat{h}(s+\lambda-\lambda z)}{z-\hat{h}(s+\lambda-\lambda z)}, \qquad (5.2.17)$$

where

$$\hat{h}(s+\lambda-\lambda z) = \left(1+\frac{s+\lambda-\lambda z}{k\mu}\right)^{-k}, \; \hat{p}_0(s) = \frac{1}{s+\lambda-\lambda\hat{G}(s)},$$

and

$$\hat{G}(s) = \int_0^\infty \exp(-sx)dG_2(x) = \left(1+\frac{s+\lambda-s\hat{G}(s)}{k\mu}\right)^{-k},$$

where $G_2(x) = P(X \leq x)$, X is a busy period in the $M/E_k/1$ queue. In the last formula $\hat{G}(s)$ can be found iteratively using, for example, Newton's method.

5.2.5 Test Functions and Numerical Comparisons

To compare the methods described above we can use tables of known function-transform pairs, see Table 5.2.1.

Table 5.2.1 Test functions for Laplace transform

$\hat{f}(s)$	$\frac{s-1}{(s-1)^2+1} - \frac{1}{s}$	$(s^2+1)^{-1/2}$	$\frac{1}{(s^2+1)}$	s^{-2}	s^{-3}	$s/(s^2+1)$
$f(t)$	$e^t \cos(t) - 1$	$J_0(t)$	$\sin(t)$	t	$1/(2t^2)$	$\cos(t)$
$\hat{f}(s)$	$\frac{1}{s-1}$	$\ln\left(\frac{s-1}{s-2}\right)$	$\log(s+1)/s$	$1/s$	$\left(1+\frac{s}{k\lambda}\right)^{-k}$	1
$f(t)$	$\exp(t)$	$\frac{\exp(t)-\exp(2t)}{t}$	$(1-e^{-t})/t$	1	$\frac{(k\lambda)^k}{(k-1)!} t^{k-1} \exp(-k\lambda t)$	$\delta(t)$

Let us compare the results for some of these functions.

Table 5.2.2 Numerical Inversion of the Laplace transform for $\hat{f}(s) = \frac{1}{s^2}$

t	1	2	3	4	6	8	10
$f(t)$	1.0000	2.0000	3.0000	4.0000	6.0000	8.0000	10.0000
$Post - Widder,\ n = 15$	1.0000	2.0000	3.0000	4.0000	6.0000	8.0000	10.0000
$Euler,\ m = 11,\ n = 15$	1.0003	2.0010	3.0017	4.0020	5.9993	9.0042	10.0040
$Gaver,\ n = 5$	1.0001	2.0003	3.0004	4.0006	6.0009	8.0012	10.0015

Table 5.2.3 Numerical Inversion of the Laplace transform for $\hat{f}(s) = \frac{1}{s^2+1}$

t	1	2	3	6	8	10
$f(t)$	0.8414	0.9092	0.1411	−0.2794	0.9893	−0.5440
$Post - Widder,\ n = 15$	0.8278	0.8536	0.1301	−0.1973	0.3680	−0.0530
$Euler,\ m = 11,\ n = 15$	0.8411	0.9091	0.1402	−0.2769	0.9906	−0.5411
$Gaver,\ n = 3$	0.8379	0.9469	−0.0951	−0.0677	0.1152	0.0619

Table 5.2.4 Numerical Inversion of the Laplace transform for $\hat{f}(s) = \frac{1}{s-1}$

t	1	2	3	6	8	10
$f(t)$	2.7182	7.3890	20.085	403.42	2980.9	22026.5
$Post-Widder,\ n = 15$	2.7196	7.4038	20.176	410.77	3078.3	23162.4
$Euler,\ m = 11,\ n = 15$	2.7110	7.3921	20.092	404.12	3278.3	−5572.02
$Gaver,\ n = 3$	16.775	−6161.2	−29486	−1478.2	−18.543	−2.39413

Table 5.2.5 Numerical Inversion of the Laplace transform for $\hat{f}(s) = \frac{s}{s^2+1}$

t	1	2	3	6	8	10
$f(t)$	0.5403	−0.4161	−0.9899	0.9601	−0.1455	−0.8390
$Post-Widder,\ n = 15$	0.5389	−0.4119	−0.9680	0.8775	−0.1205	−0.6379
$Euler,\ m = 11,\ n = 15$	0.5400	−0.4118	−0.9891	0.9666	−0.1423	−0.8452
$Gaver,\ n = 3$	0.5643	−0.5964	−0.7334	0.1523	0.0759	0.0214

Table 5.2.6 Numerical Inversion of the Laplace transform for $\hat{f}(s) = \ln\left(\frac{s-1}{s-2}\right)$

t	1	2	3	6	8	10
$f(t)$	4.6707	23.604	127.78	27058	$0.111 \cdot 10^7$	$0.48 \cdot 10^8$
$Post-Widder,\ n = 15$	4.7112	24.516	140.12	41528	$0.248 \cdot 10^7$	$0.18 \cdot 10^9$
$Euler,\ m = 11,\ n = 15$	4.7956	23.545	127.83	33.184	−383149	−478.838
$Gaver,\ n = 3$	–	–	–	–	–	–

From the results we can conclude that the Gaver method gives poor results, but the computation is very easy to perform and we can apply it to functions with complex structure. The Post-Widder method works very well, but because we need to take n derivatives it becomes extremely difficult in computations for large n.

5.2.6 Applying Inversion of the Laplace Transform to the M/M/1 and M/E$_k$/1 Systems

Numerical Inversion of Laplace Transform for the M/M/1 System

We can apply the theory of numerical inversion of Laplace transforms to the queueing model considered previously. We first employ the $M/M/1$ queueing system, which has been described in Chapter 1. In this case we can rewrite the Laplace transform of the generating function as follows

$$G^*(y,z) = \frac{y - \left(\frac{(1-y)\alpha_2(z)}{1-\alpha_2(z)}\right)}{-\lambda(y-\alpha_1(z))(y-\alpha_2(z))}, \tag{5.2.18}$$

where $\alpha_{1,2}(z)$ are the roots

$$\alpha_{1,2}(z) = \frac{(\lambda+\mu+z) \pm \sqrt{(\lambda+\mu+z)^2 - 4\lambda\mu}}{2\lambda}. \tag{5.2.19}$$

We can find $W_q^*(z)$ the Laplace transform of the mean waiting time which customer spend in a queue by using the following result

$$W_q^*(z) = \frac{1}{\mu}\frac{\partial G^*(y,z)}{\partial y}\Big|_{y=1}. \tag{5.2.20}$$

Differentiating (5.2.18) and putting $y=1$ gives us

$$W_q^*(z) = \frac{\frac{\alpha_2(z)}{1-\alpha_2(z)}z + \lambda - \mu}{\mu z^2}. \tag{5.2.21}$$

Applying the three methods of inverting Laplace transform, we have the following results, see Tables (5.2.7)-(5.2.10), $W_q(t)$ we compute by formula (5.2.22) using Euler's method of finding probabilities $p_n(t)$ with step $\delta t = 0.1$.

$$W_q(t) = \sum_{n=1}^{\infty} \frac{n-1}{\mu} p_n(t). \tag{5.2.22}$$

Table 5.2.7 Numerical inversion of the Laplace transform for $M/M/1$ queue with $\lambda = 0.2,\ \mu = 1$

t	1	2	3	4	6	8	10
$W_q(t)$	0.1329	0.1882	0.2130	0.2292	0.2420	0.2467	0.2485
$Post - Widder$	0.1267	0.1808	0.2082	0.2234	0.2381	0.2441	0.2469
$Euler$	0.1012	0.2038	0.2150	0.2297	0.2388	0.2420	0.2387
$Gaver$	0.1308	0.1843	0.2112	0.2294	0.2351	0.2518	0.2432

Table 5.2.8 Numerical inversion of the Laplace transform for $M/M/1$ queue with $\lambda = 0.5,\ \mu = 1$

t	1	2	3	4	6	8	10
$W_q(t)$	0.3420	0.5141	0.6226	0.6983	0.7972	0.8577	0.8958
$Post - Widder$	0.329	0.4996	0.6075	0.6832	0.7826	0.8440	0.8847
$Euler$	0.3332	0.5154	0.6209	0.6954	0.7952	0.8555	0.8963
$Gaver$	0.3363	0.5089	0.6186	0.6959	0.7939	0.8560	0.8942

Table 5.2.9 Numerical inversion of the Laplace transform for $M/M/1$ queue with $\lambda = 0.9,\ \mu = 1$

t	1	2	3	4	6	8	10
$W_q(t)$	0.6366	1.0186	1.3092	1.5499	1.9427	2.2623	2.5248
$Post - Widder$	0.6193	0.9979	1.2863	1.5251	1.9146	2.2313	2.5012
$Euler$	0.6270	1.0173	1.3015	1.5460	1.9375	2.2594	2.5323
$Gaver$	0.6274	1.0107	1.3036	1.5450	1.9394	2.2611	2.5337

Table 5.2.10 Numerical inversion of the Laplace transform for $M/M/1$ queue with $\lambda = 2,\ \mu = 1$

t	1	2	3	4	6	8	10
$W_q(t)$	1.5202	2.7023	3.7999	4.8597	6.9258	8.9584	10.9757
$Post - Widder$	1.5007	2.6847	3.7840	4.8454	6.9143	8.9493	10.9686
$Euler$	1.5078	2.6942	3.7951	4.8565	6.9224	8.9560	10.9744
$Gaver$	1.5071	2.6909	3.7880	4.8467	6.9088	8.9238	10.9514

As can be seen from the tables, there are differences between the results from the various methods of inversion of Laplace transforms. In particular, it is noted that the results of Post Widder method look poor in comparison with the others. This illustrates the difficulties associated with the numerical inversion of Laplace transforms.

Numerical Inversion of Laplace Transform for the M/E_k/1 System

To apply numerical inversion methods of Laplace transforms to the Erlang queueing model is a more difficult problem than for the $M/M/1$ queue. The Laplace transform of the generating function was found in (3.3.9), where $p_0^*(z)$ is the Laplace transform of the probability that there are no customers at time t. Using identity (3.6.1) we can obtain the following formula for the Laplace transform of the mean waiting time in the queue

$$W_q^*(z) = \frac{p_0^*(z)}{z} + \frac{(\lambda/\mu - 1)}{z^2}. \qquad (5.2.22)$$

The approximation to $p_0^*(z)$ can be found by applying Newton's method. Therefore the numerical expression for $p_0^*(z)$ becomes complicated and we can only get numerical results using Gaver's method. (I have used Maple 9 for the calculations). In Tables 5.2.11-5.2.13 we calculate $W_q(t)$ by formula (3.6.2) using Euler's method of finding probabilities $p_{n,s}(t)$ with step $\delta t = 0.1$.

Table 5.2.11 Numerical inversion of the Laplace transform for $M/E_k/1$ queue with $\lambda = 0.5,\ \mu = 1, k = 2$

t	0.5	1	1.5	2	2.5	3	5
$W_q(t)$	0.20331	0.32284	0.40155	0.45870	0.50257	0.53741	0.62554
Gaver	0.19503	0.31521	0.39424	0.45321	0.49721	0.53354	0.62303

Table 5.2.12 Numerical inversion of the Laplace transform for $M/E_k/1$ queue with

$\lambda = 0.8, \ \mu = 1, k = 3$

t	0.5	1	1.5	2	2.5	3	5
$W_q(t)$	0.32636	0.52411	0.66759	0.78336	0.88119	0.96625	1.22758
Gaver	0.31383	0.51384	0.65890	0.77546	0.87549	0.96024	1.22376

Table 5.2.13 Numerical inversion of the Laplace transform for $M/E_k/1$ queue with

$\lambda = 1.1, \ \mu = 1, k = 5$

t	0.5	1	1.5	2	2.5	3	5
$W_q(t)$	0.45144	0.73471	0.95952	1.15412	1.32931	1.49054	2.04874
Gaver	0.44503	0.72359	0.95046	1.14648	1.32235	1.48557	2.04376

As can be seen from the tables, there are some differences between numerical results, in particular, it is noted that for small values t, the numerical results are not in good agreement.

CHAPTER 6: APPLICATION OF THE ERLANG MODEL

Severn Bridge 24-Hour Traffic Flow

The previous chapters have provided useful analytic results and different methods for solving the differential-difference $M/E_k/1$ equations. In this chapter consideration is given to an application of the $M/E_k/1$ system, namely 24-hour traffic flow.

A description of the system is given, as is an account of the data collection process undertaken at the "Severn Bridge" in the early 1980s. Using this data we can compare both analytic and simulation results for the $M/E_2/1$ model.

During the project of building the Second Severn Crossing, commissioned by the Government in 1984, an investigation of the traffic flow on Severn Bridge was made. The Severn Bridge was opened in 1966 to replace the ferry service from Aust to Beachle. It was plagued with structural and maintenance problems which caused lane closures as a regular feature. As the result there were delays of several hours at peak periods. On Friday evenings in the summer, delays of 2-3 hours were common, especially if accompanied by accidents or breakdowns.

A substantial amount of data was collected using direct observation and camcorder techniques, see paper by Griffiths and Williams (1984).

The data was collected for a 24-hour period of time when vehicles arrived at the 4-channel service. The times taken by vehicles for service at a toll-booth were measured under one of the following conditions:

(a) there was no queue at the toll-booth when the vehicle arrived;

(b) the arriving vehicle had to queue before it went through the toll-booth;

(c) service rate per channel is $\mu = 9$ per minute (assumed constant).

Road traffic flows tend to show two distinct peaks over a 24-hour cycle (see Figure 6.1), one corresponding to the morning rush hour and the other to the evening. This leads naturally to the idea of a demand profile; that is, the mean arrival rate of customers varies over a period of time, but is reasonably constant over short periods. The objective in modelling such cases is to estimate queue length in comparison with the underlying demand profile. To demonstrate the flexibility of the model, each case represents a differently shaped demand distribution and length of the queue changes with the different arrival rate λ. Let $\rho(t) = \lambda(t)/\mu$.

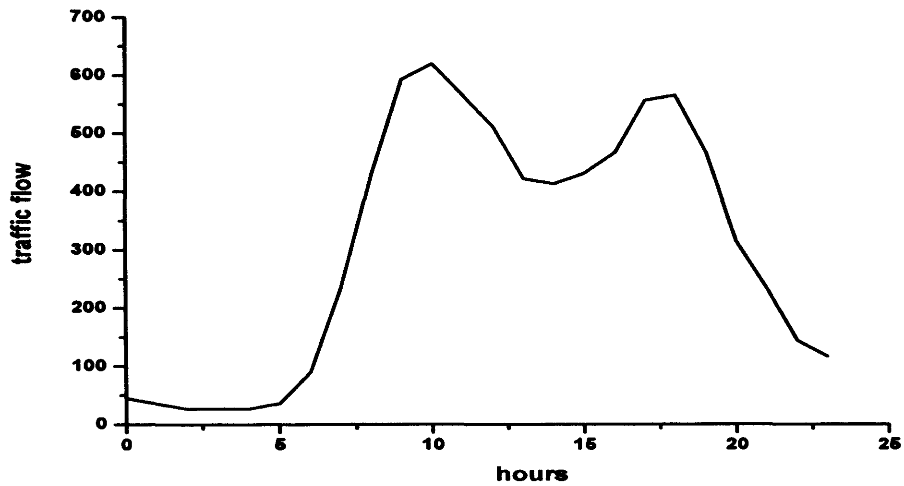

Figure 6.1 24-hour traffic flow profile on Severn Bridge

Table 6.1 24-hour traffic flow profile on Severn Bridge

Hour of day	Traffic flow	Traffic flow per channel	$\lambda(t)$ per minute	$\rho(t)$
0	180	45	0.75	0.08333
1	144	36	0.6	0.06666
2	108	27	0.45	0.05000
3	108	27	0.45	0.05000
4	108	27	0.45	0.05000
5	144	36	0.6	0.06666
6	360	90	1.5	0.16666
7	936	234	3.9	0.43333
8	1728	432	7.2	0.80000
9	2376	594	9.9	1.10000
10	2484	621	10.35	1.15000
11	2268	567	9.45	1.05000
12	2052	513	8.55	0.95000
13	1692	423	7.05	0.78333
14	1656	414	6.9	0.76666
15	1728	432	7.2	0.80000
16	1872	468	7.8	0.86666
17	2232	558	9.3	1.03333
18	2268	567	9.45	1.05000
19	1872	468	7.8	0.86666
20	1260	315	5.25	0.58333
21	936	234	3.9	0.43333
22	576	144	2.4	0.26666
23	468	117	1.95	0.21666

We propose the $M/E_2/1$ model with $\mu = 9$ per minute (assumed constant), and $\lambda(t)$ as found in Table 6.1. This is a typical example of an average weekday flow with morning and evening rush hours, where $\rho > 1$ occasionally.

To compare the results for $L_q(t)$, the mean number at customers in the queue of time t, we use

(a) simulation results, which have been computed using SIMUL8;

(b) analytic results, using the calculation of time-dependent probabilities $p_{n,s}(t)$ for the $M/E_2/1$ system and using formulae (3.8.7)-(3.8.9).

To produce one iteration in SIMUL8 we select the appropriate input parameter λ for each hour. Queue length at the start of each hour, L_q^0 is added to the program automatically. An example of one iteration is shown in Figure 6.2.

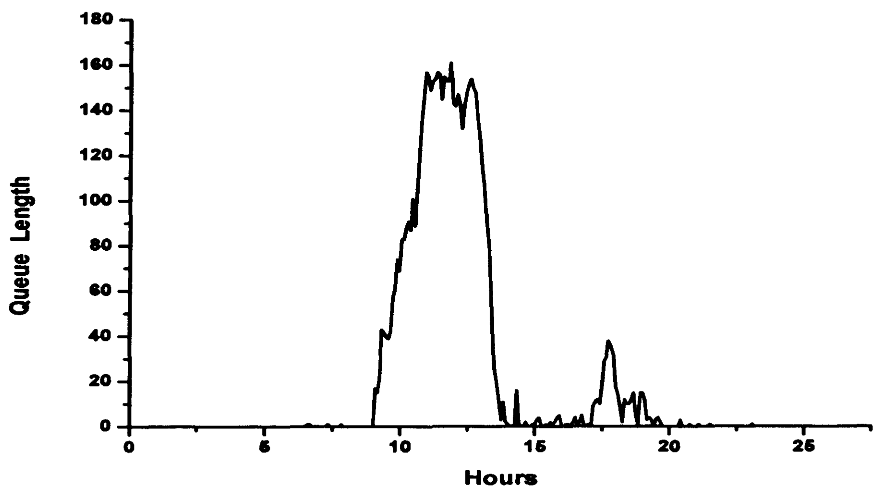

Figure 6.2 Showing how queue length vary over 24 hours

Analytic results are calculated for each hour using the results obtained in Section 3.8, see Figure 6.3. The queue length at the start of each hour is evaluated from the calculations of the previous hour.

The simulation results in Figure 6.2 are the average over 10 iterations.

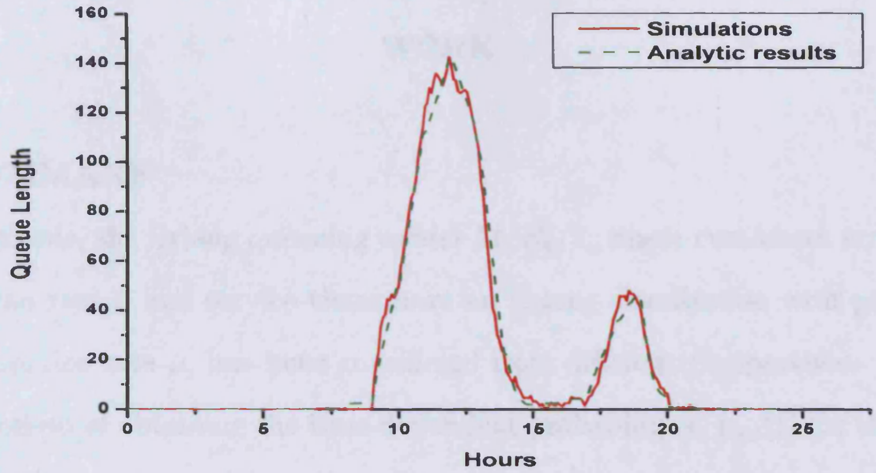

Figure 6.3 Comparisons of analytic and simulation results

As can be seen from the example, agreement between the simulation and formulaic results is remarkably good.

CHAPTER 7: SUMMARY AND SUGGESTIONS FOR FURTHER WORK

7.1 SUMMARY

In this thesis, the Erlang queueing model $M/E_k/1$, where customers arrive at random at mean rate λ and service times have an Erlang distribution with parameter k and mean service rate μ, has been considered from different perspectives. Firstly, an analytic method of obtaining the time-dependent probabilities, $p_{n,s}(t)$ for the $M/E_k/1$ system have been proposed in terms of a new generalisation of the modified Bessel function when initially there are no customers in the system. Results have been also generalised to the case when initially there are a customers in the system.

Secondly, a new generalisation of the modified Bessel function and its generating function have been presented with its main properties and relations to other special functions (generalised Wright function and Mittag-Leffler function) have been noted.

Thirdly, the mean waiting time in the queue, $W_q(t)$, has been evaluated, using Luchak's results. The double-exponential approximation of computing $W_q(t)$ has been proposed for different values of ρ, which gives results within about 1% of the 'exact' values obtained from numerical solution of the differential-difference equations. The advantage of this approximation is that it provides additional information, via its functional form of the characteristics of the transient solution.

Fourthly, the inversion of the Laplace transform with the application to the queues has been studied and verified for $M/M/1$ and $M/E_k/1$ models of computing $W_q(t)$.

Finally, an application of the $M/E_2/1$ queue has been provided in the example of 24-hour traffic flow for the Severn Bridge. One of the main reasons for studying queueing models from a theoretical point of view is to develop ways of modelling real-life systems. The analytic results have been confirmed with the simulation.

7.2 SUGGESTIONS FOR FURTHER WORK

The work presented in this thesis could be extended in a number of directions.

Firstly, the analytic results for obtaining time-dependent probabilities could be extended to a bulk queueing system, see (3.2.15)

$$\begin{cases} \frac{dp_0(t)}{dt} = -\lambda p_0(t) + \mu p_1(t), & n = 0; \\ \frac{dp_n(t)}{dt} = -(\lambda+\mu)p_n(t) + \mu p_{n+1}(t) + \lambda \sum_{j=1}^{n} c_j p_{n-j}(t), & n \geq 1. \end{cases} \tag{7.2.1}$$

The generating function for this system satisfies

$$\begin{cases} \frac{dG(y,t)}{dt} = G(y,t)\varphi(y) + \mu\frac{y-1}{y}p_0\,(t) \\ G(y,0) = 1, \end{cases} \tag{7.2.2}$$

where

$$\varphi(y) = \frac{\mu}{y} - (\lambda+\mu) + \lambda \sum_{j=1}^{n} c_j y^i.$$

Solving the differential equation (7.2.2) we obtain

$$G(y,t) = \exp(t\varphi(y)) - \mu\frac{1-y}{y}\int_0^t p_0(z)\exp(\varphi(z)(t-z)dz, \tag{7.2.3}$$

where $p_0(t)$ can be found by using the Luchak's results (1956, 1958).

The main problem here is how to find a new generating function in terms of some special functions for the function

$$\exp\left(t(\lambda\sum_{j=1}^{\infty} c_j y^j + \frac{\mu}{y})\right).$$

However this problem is unlikely to be easy to solve.

Another extension could be obtaining transient probabilities for the $M/E_k/1$ model with time-dependent arrival and service rates. Indeed, we can use the results in the paper by Ragab Omarah Al-Seedy, Fawziah M. Al-Ibraheem (2003), where they have investigated the transient solution for the $M/M/\infty$ queue with Poisson arrivals and exponential service times where the parameters of both distributions are allowed to vary with time.They have only obtained the results for the case when $\lambda(t) = \lambda t$, $\mu(t) = \mu t$ and produced an expression for the generating function which is similar to (3.4.8). For the more general case, when $\lambda(t)$ and $\mu(t)$ are unknown functions, the problem becomes more complicated.

A further interesting investigation would be the Erlang service multiserver queueing model $M/E_k/m$. The analytic results are very difficult to obtain because of the complex structure of the differential-difference equations. Even the $M/M/m$ system is difficult to analyse and an exact solution could be found only in the paper by Parthasarathy (1989).

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