Topics in the Theory of Arithmetic Functions

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Abstract

Selberg's upper bound method provides rather good results in certain circumstances. We wish to apply ideas from this upper bound method to that of the lower bound sifting problem.

The sum $G(x)$ arises in Selberg's method and in this account we study the related sum $H_z(x)$. We provide an asymptotic estimate for the sum $H_z(x)$ by investigating the residual sum $I_z(x) = H_z(\infty) - H_z(x)$ and transferring back to $H_z(x)$.

We obtain a lower bound for the sum which counts the number of $a \in \mathcal{A}$ with the logarithmic weight $\log p / \log z$ attached to the smallest prime factor of the number a subject to the condition $\nu(D, A) \leq R$ combining ideas from Selberg's $\Lambda^2\Lambda^-$ method with Richert's weights. $\nu(D, A)$ counts the number of prime factors p of a number a according to multiplicity when $p \geq D$ but counting each p at most once when $p < D$.

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Chapter 1

Introduction

Sieve theory is a set of general techniques in number theory, designed to count, or more realistically to estimate the size of, sifted sets of integers. The first known sieve technique is that of Eratosthenes. He used a brute force method which involved starting with a list of odd numbers and deleting all the multiples of 3, then deleting all the multiples of 5, and so on.

1	~~2~~	~~3~~	~~4~~	~~5~~	~~6~~
7	~~8~~	~~9~~	~~10~~	11	~~12~~
13	~~14~~	~~15~~	~~16~~	17	~~18~~
19	~~20~~	~~21~~	~~22~~	23	~~24~~
~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~

Modern sieve methods were, it appears, introduced in the hope that certain conjectures (such as Goldbach's) could be proved. While the original aims of sieve theory are largely unachieved, there have been some partial successes. For example, the Twin-Prime Conjecture stated below has yet to be proven, but partial approaches to it have been obtained.

Conjecture 1.0.1.

There are infinitely many prime twins: numbers p and $p+2$ both of which are prime.

However, Chen's Theorem states that there are infinitely many primes p such that $p+2$ is either a prime or almost prime (the product of two primes).

Modern day sieve methods are more sophisticated and include the Brun sieve, Rosser's Sieve and the Selberg sieve. In this account we concentrate on the Selberg Sieve which is simpler to understand and implement than other sieve methods. Although it is perhaps more elementary than other methods, Selberg's sieve still provides us with rather good results in certain circumstances. Selberg's upper bound method is described in Chapter 3. This method provides satisfactory results but only provides us with an upper bound. One of the objectives of this thesis is to apply Selberg's ideas to the lower bound sifting problem.

In Chapter 2 we introduce some definitions and notation which will be used throughout. We also introduce the concept of the sieve arguments which we shall use in later chapters.

In Chapter 3 we describe Selberg's Upper Bound Method, [21]. We state it here as some of the results will be used in later chapters and it also provides a level of understanding of what we are doing.

The sum $G(x)$ which arises in Selberg's upper bound method is of particular interest. In Chapter 4 we use Rankin's trick to provide an asymptotic estimate of a related sum $H_z(x)$, where we have extended the sum to include values at prime powers.

We wish to use the ideas arising in Selberg's upper bound method to produce a satisfactory lower bound method. In Chapter 5 we obtain a lower bound for the sum

$$\sum_{\substack{a\in\mathcal{A}\\ \nu(D,a)\le R}} w(q_a)$$

which counts the numbers a with the logarithmic weight $\log p/\log z$ attached to the smallest prime factor of the number a subject to the condition $\nu(D,a)\le R$. $\nu(D,a)$ counts the number of prime factors p of a number a according to multiplicity when $p\ge D$ but counting each p at most once when $p<D$,

$$\nu(D,a)=\sum_{\substack{p<D\\ p|a}} 1+\sum_{\substack{p,\alpha\\ p^\alpha|a;p\ge D}} 1. \tag{1.0.1}$$

We achieve this by combining Selberg's $\Lambda^2\Lambda^-$ method from [22], with Richert's weight [20], with $U=1$, $V=0$. The lower bound for the sum

$$\sum_{\substack{a\in\mathcal{A}\\ \nu(D,a)\le R}} w(q_a)$$

leads us to a lower bound for the number of a in $\mathcal{A}$, counted with constant weight 1, subject to the same condition on the prime factors of a. In other words, we obtain a lower bound for the number of $a\in\mathcal{A}$ having at most R prime factors.

Chapter 2

Prerequisites

2.1 Introduction

This chapter is used to introduce the background material required for the remainder of this thesis. In particular, we introduce the concepts of sifting limit β_κ, level of a sieve D, and sifting density κ. Specifically, the sifting density of the multiplicative function ρ plays an important role in sieve theory.

We will require the following hypotheses and definition throughout this thesis.

Hypothesis 2.1.1.

Assume $0 \leq \rho(p) < p$ *when* $p|P(z)$.

Hypothesis 2.1.2.

Assume $\rho(p) = 0$ *when* $p \nmid P(z)$.

Definition 2.1.3.

Define the expression $V(P(z))$ by

$$V(P(z)) = \sum_{d|P(z)} \frac{\mu(d)\rho(d)}{d} = \prod_{p|P(z)} \left(1 - \frac{\rho(p)}{p}\right). \tag{2.1.1}$$

2.2 Fundamentals

We use this section to introduce various notations which will be used without reference throughout. Theorem 2.2.7 provides an estimate for $S(\mathcal{A}, P(z))$. When suitable functions λ^+ and λ^- have been constructed, the problem of estimating $S(\mathcal{A}, P(z))$ is then reduced to that of estimating the sums $V^\pm$ and $R^\pm$.

Definition 2.2.1.

Denote by $\mathcal{A}$ a finite set of integers. Then we define by $\mathcal{A}_d$ those a in $\mathcal{A}$ which are divisible by d:

$$\mathcal{A}_d = \{a \in \mathcal{A} : a \equiv 0 (\text{mod } d)\}. \tag{2.2.1}$$

Definition 2.2.2.

We denote the product of primes less than z by

$$P(z) = \prod_{p<z} p. \tag{2.2.2}$$

Definition 2.2.3.

Write

$$S(A) = \sum_{d|A} \mu(d) \tag{2.2.3}$$

where μ is the Möbius function defined by

$$\begin{aligned} \mu(1) &= 1 \\ \mu(p_1 p_2 \dots p_r) &= (-1)^r \qquad \text{if } p_1, \dots, p_r \text{ are distinct primes} \\ \mu(d) &= 0 \qquad \text{if } p^2|d \text{ for some prime } p. \end{aligned}$$

Definition 2.2.4.

Define $S(\mathcal{A}, P(z))$ by

$$S(\mathcal{A}, P(z)) = \sum_{a \in \mathcal{A}} S((a, P(z))) \tag{2.2.4}$$

so that $S(\mathcal{A}, P(z))$ is the number of a in $\mathcal{A}$ not divisible by any prime from the product $P(z)$. Then we have

$$\begin{aligned} S(\mathcal{A}, P(z)) &= \sum_{a \in \mathcal{A}} \sum_{\substack{d|a \\ d|P(z)}} \mu(d) \\ &= \sum_{d|P(z)} \mu(d) |\mathcal{A}_d|. \end{aligned} \tag{2.2.5}$$

Definition 2.2.5.

When $\mathcal{A}_d$, X, D and $\rho(d)$ are specified, define the remainder $r_{\mathcal{A}}(d)$ by

$$|\mathcal{A}_d| = X \frac{\rho(d)}{d} + r_{\mathcal{A}}(d) \qquad \text{if } d|P(z), d \le D. \tag{2.2.6}$$

For equation (2.2.6) to be useful we require X and the multiplicative function ρ to be chosen so that the remainder term, $r_{\mathcal{A}}(d)$, is comparatively small.

Definition 2.2.6.

We consider functions λ_D^- and λ_D^+ with the properties

$$\sum_{d|A} \lambda_D^-(d) \leq \sum_{d|A} \mu(d) \leq \sum_{d|A} \lambda_D^+(d) \qquad \text{when } A|P(z). \tag{2.2.7}$$

When (2.2.7) holds, we say that λ^+, λ^- are upper and lower sifting functions of level D for the product $P(z)$, where D satisfies

$$\lambda_D^{\pm}(d) \neq 0 \Rightarrow d \leq D. \tag{2.2.8}$$

Theorem 2.2.7.

When $\mathcal{A}$ satisfies (2.2.6) *and $\lambda^{\pm}$ satisfies* (2.2.7) *we obtain*

$$XV^-(D,P(z)) + R^-(D,P(z)) \leq S(\mathcal{A},P(z)) \leq XV^+(D,P(z)) + R^+(D,P(z)) \tag{2.2.9}$$

where S is as in (2.2.4),

$$V^{\pm}(D,P(z)) = \sum_{d|P(z)} \frac{\lambda_D^{\pm}(d)\rho(d)}{d} \tag{2.2.10}$$

and

$$R^{\pm}(D,P(z)) = \sum_{d|P(z)} \lambda_D^{\pm}(d) r_{\mathcal{A}}(d). \tag{2.2.11}$$

Proof

Take $A = (a, P(z))$ in (2.2.7) and sum over $a \in \mathcal{A}$. We obtain

$$\begin{aligned} \sum_{a\in\mathcal{A}} \sum_{\substack{d|a \\ d|P(z)}} \lambda_D^-(d) &\le S(\mathcal{A}, P(z)) \le \sum_{a\in\mathcal{A}} \sum_{\substack{d|a \\ d|P(z)}} \lambda_D^+(d) \\ \sum_{d|P(z)} \lambda_D^-(d)|\mathcal{A}_d| &\le S(\mathcal{A}, P(z)) \le \sum_{d|P(z)} \lambda_D^+(d)|\mathcal{A}_d|. \end{aligned}$$

Using (2.2.6) this gives (2.2.9) as required.

We normalise our constructions so that

$$\lambda_D^{\pm}(1) = 1 \tag{2.2.12}$$

throughout. We will prove in Theorem 3.3.1 that, for the λ_D^+ appearing in Selberg's upper bound construction at (2.2.7),

$$|\lambda_D^{\pm}(d)| \le 1 \qquad \text{for all } d. \tag{2.2.13}$$

Corollary 2.2.8 then follows by estimating the terms involving $r_{\mathcal{A}}(d)$ in (2.2.11) by the following "trivial treatment" of the remainder term:

$$\left| \sum_{d|P(z)} \lambda_D^{\pm}(d) r_{\mathcal{A}}(d) \right| \le \sum_{\substack{d|P(z) \\ d\le D}} |r_{\mathcal{A}}(d)|. \tag{2.2.14}$$

Corollary 2.2.8.

Suppose (2.2.7), (2.2.13) *and* (2.2.6) *hold. Then*

$$XV^{-}(D,P(z)) - \sum_{\substack{d|P(z)\\ d\leq D}} |r_{\mathcal{A}}(d)| \leq S(\mathcal{A},P(z)) \leq XV^{+}(D,P(z)) + \sum_{\substack{d|P(z)\\ d\leq D}} |r_{\mathcal{A}}(d)|. \tag{2.2.15}$$

2.3 Level of Distribution

The level of distribution is a number D for which the r-terms in Corollary 2.2.8 are insignificantly small. Choose $\varepsilon > 0$, and say that $\mathcal{A}$ has level D if

$$\sum_{d<D} |r_{\mathcal{A}}(d)| < \varepsilon X \tag{2.3.1}$$

when (2.2.6) applies, with an appropriately chosen ρ.

2.4 Sifting Density

Here we introduce the sifting density, κ, for the multiplicative function ρ. This is a very important concept as many of the results appearing in the latter stages of this thesis require knowledge about the function ρ.

Definition 2.4.1.

Say that κ is a sifting density (or dimension of the sieve) for the function ρ if there exists a constant $L > 1$ (depending on κ) such that

$$\frac{V(P(w))}{V(P(z))} = \prod_{p|P(z)/P(w)} \left(1 - \frac{\rho(p)}{p}\right)^{-1} \le \left(\frac{\log z}{\log w}\right)^{\kappa} \left(1 + \frac{L}{\log w}\right) \quad \text{if } 2 \le w < z, \tag{2.4.1}$$

It should be noted that this does not define a sifting density κ uniquely; if (2.4.1) holds for $\kappa = \kappa_0$ then it holds when $\kappa \ge \kappa_0$. It is sometimes necessary to make the following assumption bounding $\rho(p)$ from both sides, in which case κ, if it exists, will be specified uniquely by ρ.

Definition 2.4.2.

Say that κ is a two-sided sifting density for ρ if it satisfies Definition 2.4.1 and, additionally, there exists $L' \ge L$ such that

$$\frac{V(P(w))}{V(P(z))} \ge \left(1 - \frac{L'}{\log w}\right) \left(\frac{\log z}{\log w}\right)^{\kappa} \qquad \text{if } 2 \le w < z. \tag{2.4.2}$$

In the situations which arise in this thesis it will be more convenient to replace the hypothesis in Definition 2.4.1 by a slightly stronger one. Let $g(d)$ be the multiplicative function defined for squarefree d by specifying

$$g(p) = \frac{\rho(p)}{p - \rho(p)} \tag{2.4.3}$$

for primes p and $g(d) = 0$ if $p^2|d$.

Then in place of (2.4.1) we may specify a constant $A > 1$ such that

$$\sum_{w \le p < z} g(p) \log p \le \kappa \log \frac{z}{w} + A \qquad \text{when } 2 \le w < z. \tag{2.4.4}$$

When $z \to w = p$, we obtain $g(p) \log p \le A$, so that (2.4.3) gives

$$\begin{aligned} \rho(p) &\le \frac{A}{\log p}(p - \rho(p)) \\ &\le \frac{p}{1 + A^{-1} \log p} \end{aligned}$$

and $\rho(p)$ is bounded away from p.

2.5 Sifting Limit

Suppose that the function ρ has a finite sifting density. We wish to characterise those pairs D, z for which $V^-(D, P(z)) > 0$ so that the lower bound in (2.2.9) is non-trivial. We are interested in the infimum of those s for which the desired inequality

$$V^-(z^s, P(z)) > 0 \tag{2.5.1}$$

holds. Consider the class $\mathcal{C}$ of those ρ which satisfy (2.4.4).

Definition 2.5.1.

The sifting limit $\beta(\kappa)$ is the infimum over $\mathcal{C}$ of those s for which (2.5.1) holds (for some function λ_D^-, depending on ρ, that obeys (2.2.7)).

Chapter 3

Selberg's Upper Bound Method

3.1 Introduction

Here we take the opportunity to introduce Selberg's Upper Bound Method. Although the material in this chapter previously appeared in Chapter 2 of [8], we include it here as it helps to explain the relevance of the methods appearing in later chapters. As the methods in later chapters stem from Selberg's ideas many of the results appearing here will be used throughout.

The problem of constructing a satisfactory upper bound sieve is that of satisfying the right hand inequality of (2.2.7) while keeping the level of support D on which $\lambda^+(d) \neq 0$ satisfactorily small. This ensures that the effect of the remainder term $r_{\mathcal{A}}(d)$ does not become excessively large.

So we have to satisfy

$$S(a, P(z)) \leq \sum_{d|(a,P(z))} \lambda_D^+(d). \tag{3.1.1}$$

In Selberg's upper bound method this requirement is guaranteed by arranging that the sum over d is a square:

$$\sum_{d|A} \lambda_D^+(d) = \Bigg(\sum_{\substack{d_1 < \sqrt{D} \\ d_1 | A}} \lambda(d_1) \Bigg)^2 \qquad \text{with } \lambda(1) = 1. \tag{3.1.2}$$

3.2 The Sifting Argument

Selberg's result requires Hypothesis 2.1.1, $0 \leq \rho(p) < p$, and the definition of $r_{\mathcal{A}}(d)$ in terms of $\mathcal{A}_d$ and ρ, given by (2.2.6). The question of estimating the sum $G(x)$ arises. This is done by using information about the sifting density of ρ to obtain an appropriate bound on $G(x)$.

Definition 3.2.1.

The multiplicative function ρ^* is defined by

$$\rho^*(p) = p - \rho(p). \tag{3.2.1}$$

We then define the function g by

$$g(n) = \begin{cases} \dfrac{\rho(n)}{\rho^*(n)} & \text{if } n \text{ is squarefree} \\ 0 & \text{otherwise.} \end{cases} \tag{3.2.2}$$

We note that the function g is multiplicative since both ρ and ρ^* are multiplicative. We shall denote

$$G(x) = \sum_{n<x} g(n) \tag{3.2.3}$$

The following theorem is from [21] although the proof is not the same as that appearing here.

Theorem 3.2.2.

Suppose (2.2.6) *holds:*

$$|\mathcal{A}_d| = X\frac{\rho(d)}{d} + r_{\mathcal{A}}(d) \qquad \text{if } d|P(z), d < D. \tag{3.2.4}$$

Then

$$S(\mathcal{A}, P) \leq \frac{X}{G(\sqrt{D})} + E(D, P) \tag{3.2.5}$$

where G is given by (3.2.3),

$$E(D, P) = \frac{1}{\lambda^2(1)} \sum_{\substack{d_i<\sqrt{D} \\ d_i|P}} \lambda(d_1)\lambda(d_2)r_{\mathcal{A}}([d_1, d_2]), \tag{3.2.6}$$

and the real numbers $\lambda(d)$ are given, for some $C \neq 0$, by

$$\frac{\lambda(d)\rho(d)}{d} = C\mu(d) \sum_{\substack{h\equiv 0 (\bmod d) \\ h<\sqrt{D}, h|P}} g(h) \qquad \text{if } \rho(d) \neq 0, \tag{3.2.7}$$

with $\lambda(d) = 0$ if $\rho(d) = 0$.

Taking $d = 1$ in (3.2.7) shows

$$C = \frac{\lambda(1)}{G(\sqrt{D})}. \tag{3.2.8}$$

The usual normalisation in Theorem 3.2.2 is to make $\lambda(1) = 1$.

The value of $\lambda(d)$ when $\rho(d) = 0$ is irrelevant to the coefficient of X in Theorem 3.2.2. The choice $\lambda(d) = 0$ for these d is the most efficient one when the error term $E(D, P)$ is taken into consideration.

Since g is as in (3.2.2), substituting $h = dk$ in (3.2.7) gives

$$\lambda(d) = C\mu(d)\frac{d}{\rho^*(d)} \sum_{\substack{(k,d)=1 \\ k<\sqrt{D}/d;k|P}} g(k). \tag{3.2.9}$$

We will need the following identity, in which the numbers $\lambda(d)$ may be arbitrary, subject only to a restriction that they are supported on an interval $1 \leq d < \sqrt{D}$, so that all the sums appearing in Lemma 3.2.3 are finite.

Lemma 3.2.3.

Denote

$$V^+(\lambda) = \sum_{d_1|P} \sum_{d_2|P} \frac{\lambda(d_1)\lambda(d_2)\rho([d_1, d_2])}{[d_1, d_2]}, \tag{3.2.10}$$

where $[.,.]$ *in its usual notation denotes the least common multiple. Then*

$$V^+(\lambda) = \sum_{\substack{h<\sqrt{D} \\ \rho(h)\neq 0}} \frac{x^2(h)}{g(h)}, \tag{3.2.11}$$

where

$$x(h) = \sum_{d \equiv 0 (\mathrm{mod} h)} \frac{\lambda(d)\rho(d)}{d}. \tag{3.2.12}$$

Moreover

$$\frac{\lambda(d)\rho(d)}{d} = \sum_{k} \mu(k)x(kd). \tag{3.2.13}$$

Proof

Firstly, we observe that $(d_1, d_2)[d_1, d_2] = d_1 d_2$, where $(.,.)$ denotes the greatest common divisor. Using this in equation (3.2.10) gives

$$V^+(\lambda) = \sum_{d_1} \sum_{\substack{d_2 \\ \rho((d_1,d_2)) \neq 0}} \frac{\lambda(d_1)\rho(d_1)}{d_1} \frac{\lambda(d_2)\rho(d_2)}{d_2} \frac{(d_1, d_2)}{\rho((d_1, d_2))}.$$

We observe that (d_1, d_2) is squarefree so we may express the last fraction here as $\sum_{h|(d_1,d_2)} 1/g(h)$ since

$$\begin{aligned} \frac{f}{\rho(f)} &= \prod_{p|f} \frac{p}{\rho(p)} \\ &= \prod_{p|f} \left(1 + \frac{\rho^*(p)}{\rho(p)}\right) \\ &= \sum_{h|f} \frac{1}{g(h)}. \end{aligned}$$

So we obtain

$$\begin{aligned}
V^+(\lambda) &= \sum_{\substack{d_1 \quad d_2 \\ \rho((d_1,d_2))\neq 0}}\sum \frac{\lambda(d_1)\rho(d_1)}{d_1}\frac{\lambda(d_2)\rho(d_2)}{d_2}\sum_{h|(d_1,d_2)}\frac{1}{g(h)} \\
&= \sum_{\substack{h<\sqrt{D} \\ \rho(h)\neq 0}}\frac{1}{g(h)}\sum_{d_1\equiv 0(\text{mod } h)}\frac{\lambda(d_1)\rho(d_1)}{d_1}\sum_{d_2\equiv 0(\text{mod } h)}\frac{\lambda(d_2)\rho(d_2)}{d_2} \\
&= \sum_{\substack{h<\sqrt{D} \\ \rho(h)\neq 0}}\frac{x^2(h)}{g(h)}
\end{aligned}$$

which is (3.2.11), where $x(h)$ is as in (3.2.12). Lastly using (3.2.12) we obtain

$$\begin{aligned}
\sum_k \mu(k)x(kN) &= \sum_{kN|d}\mu(k)\frac{\lambda(d)\rho(d)}{d} \\
&= \sum_{kNl=d}\mu(k)\frac{\lambda(d)\rho(d)}{d} \\
&= \sum_{mN=d}\frac{\lambda(d)\rho(d)}{d}\sum_{k|m}\mu(k) \\
&= \frac{\lambda(N)\rho(N)}{N}
\end{aligned}$$

which is (3.2.13). Here we used the characteristic property

$$\sum_{d|A}\mu(d) = \begin{cases} 1 & \text{if } A = 1 \\ 0 & \text{if not} \end{cases} \tag{3.2.14}$$

of the Möbius function which appeared previously in (2.2.3).

The following proof appears in Section 2.1 of [8]. We include it here for the purpose of providing a full account.

Proof of Theorem 3.2.2

Suppose, for the moment that $\lambda(d)$ are arbitrary real numbers supported on squarefree $d < \sqrt{D}$. The starting point is the inequality

$$\lambda^2(1)S(a, P(z)) \leq \Big(\sum_{d|(a,P(z))} \lambda(d) \Big)^2. \tag{3.2.15}$$

This holds because if $(a, P(z)) = 1$ then $S(a, P(z)) = 1$ and both sides of (3.2.15) take the value $\lambda^2(1)$, but if $(a, P(z)) > 1$ then $S(a, P(z)) = 0$ and the right side is non-negative.

The right side of (3.2.15) can be written

$$\sum_{d_1|(a,P(z))} \sum_{d_2|(a,P(z))} \lambda(d_1)\lambda(d_2) = \sum_{d_1|P(z)} \sum_{d_2|P(z)} \lambda(d_1)\lambda(d_2) \sum_{[d_1,d_2]|a} 1.$$

On summing over a in $\mathcal{A}$ and expressing the result in terms of ρ this gives

$$\lambda^2(1)S(\mathcal{A}, P(z)) \leq \sum_{d_1|P(z))} \sum_{d_2|P(z))} \lambda(d_1)\lambda(d_2) \sum_{a\in\mathcal{A}} \sum_{[d_1,d_2]|a} 1.$$

Using (2.2.6) we obtain

$$\begin{aligned} \lambda^2(1)S(\mathcal{A}, P(z)) &\leq \sum_{d_1|P(z))} \sum_{d_2|P(z))} \lambda(d_1)\lambda(d_2) \Big(X \frac{\rho([d_1, d_2])}{[d_1, d_2]} + r_{\mathcal{A}}([d_1, d_2]) \Big) \\ &\leq XV^+(\lambda) + \lambda^2(1)E(D, P), \end{aligned} \tag{3.2.16}$$

with V^+ as in (3.2.10) and $E(D, P)$ as stated in (3.2.6).

In discussing V^+ we may streamline the notation by using the convention that $\rho(d) = 0$ if $d \nmid P(z)$, together with a natural one that terms with $\rho(d) = 0$ are not to be included in summations over d. This will save a certain amount of repetition of the conditions $d|P(z)$ or $\rho(d) \neq 0$, where these are implied. Similarly, the conditions $d < \sqrt{D}$, $\mu^2(d) = 1$ need not be explicit in a sum involving $\lambda(d)$.

Theorem 3.2.2 will follow by choosing $x(k)$ so as to minimise, for given $\lambda(1)$, the expression (3.2.11) for the quantity V^+. The constraint

$$\sum_{k<\sqrt{D}} \mu(k)x(k) = \lambda(1) \tag{3.2.17}$$

is now required by (3.2.13).

The following argument is attributed to P. Turan (see [7] or [18]). Recall Cauchy's inequality

$$\Big(\sum_{h<H} a_h b_h\Big)^2 \leq \sum_{h<H} a_h^2 \sum_{h<H} b_h^2, \tag{3.2.18}$$

which holds with equality when there is a constant C with $a_h = Cb_h$ whenever $h < H$. This is an immediate consequence of the identity

$$\frac{1}{2}\Big(\sum_{\substack{h<H\\k<H}} a_h b_k - a_k b_h\Big)^2 = \sum_{h<H} a_h^2 \sum_{k<H} b_k^2 - \Big(\sum_{h<H} a_h b_h\Big)^2.$$

Apply Cauchy's inequality to the condition (3.2.17) in a way that relates to (3.2.11):

$$\begin{aligned}\lambda^2(1) &= \Big(\sum_{k<\sqrt{D}} \frac{\mu(k)x(k)}{\sqrt{g(k)}}\sqrt{g(k)}\Big)^2 \\ &\le \Big(\sum_{k<\sqrt{D}} \frac{x^2(k)}{g(k)}\Big)\Big(\sum_{k<\sqrt{D}} g(k)\Big) \\ &\le V^+(\lambda)G(\sqrt{D}), \qquad (3.2.19)\end{aligned}$$

where $G(\sqrt{D})$ is as stated in (3.2.3).

Equality occurs in (3.2.19) if

$$x(k) = C\mu(k)g(k) \qquad \text{when } k < \sqrt{D}, \qquad (3.2.20)$$

in which case (3.2.16) gives the conclusion (3.2.5) required in Theorem 3.2.2. This situation is attained if we make $\lambda(d)$ satisfy (3.2.13) with these $x(k)$, so that

$$\begin{aligned}\frac{\lambda(d)\rho(d)}{d} &= C \sum_{k<\sqrt{D}/d} \mu(k)\mu(kd)g(kd) \\ &= C\mu(d) \sum_{\substack{h\equiv 0 (\text{mod } d)\\ h<\sqrt{D}}} g(h).\end{aligned}$$

This is the value stated in (3.2.7), in which the implied condition $h|P(z)$ was written explicitly. This completes the proof of Theorem 3.2.2.

3.3 The Numbers $\lambda(d)$

Theorem 3.3.1 below will give a convenient property of the numbers $\lambda(d)$ appearing in Theorem 3.2.2. Observe from (3.2.1) that for squarefree d

$$\sum_{h|d} g(h) = \prod_{p|d}\left(1 + \frac{\rho(p)}{p-\rho(p)}\right) = \frac{d}{\rho^*(d)}. \tag{3.3.1}$$

The following inequality appears in [28].

Theorem 3.3.1.

The numbers $\lambda(d)$ *described in Theorem* 3.2.2 *satisfy the inequality*

$$|\lambda(d)| \le |\lambda(1)|. \tag{3.3.2}$$

Proof

Let C be the constant (3.2.8), so that

$$\lambda(1) = C \sum_{h<\sqrt{D}} g(h).$$

We need the inequality

$$|\lambda(1)| \ge |C| \sum_{f|d} g(f) \sum_{\substack{(n,d)=1 \\ n<\sqrt{D}/d}} g(n). \tag{3.3.3}$$

With the identity (3.3.1) this gives

$$\begin{aligned} |\lambda(1)| &\geq |C|\frac{d}{\rho^*(d)} \sum_{\substack{(n,d)=1 \\ n<\sqrt{D}/d}} g(n) \\ &= |\lambda(d)|, \end{aligned}$$

by the expression (3.2.9) for $\lambda(d)$. This proves Theorem 3.3.1.

Corollary 3.3.2.

In Theorem 3.2.2, *the error term satisfies*

$$E(D,P) \leq \sum_{\substack{d_i<\sqrt{D} \\ d_i|P(z)}} |r_{\mathcal{A}}([d_1,d_2])|. \tag{3.3.4}$$

Proof

$$E(D,P) = \frac{1}{\lambda^2(1)} \sum_{\substack{d_i<\sqrt{D} \\ d_i|P}} \lambda(d_1)\lambda(d_2)r_{\mathcal{A}}([d_1,d_2]) \qquad \text{from (3.2.6).} \tag{3.3.5}$$

Since $|\lambda(d)| \leq |\lambda(1)|$ from Theorem 3.3.1 we have

$$\begin{aligned} E(D,P) &\leq \frac{|\lambda(1)|^2}{\lambda^2(1)} \sum_{\substack{d_i<\sqrt{D} \\ d_i|P}} |r_{\mathcal{A}}([d_1,d_2])| \\ &\leq \sum_{\substack{d_i<\sqrt{D} \\ d_i|P(z)}} |r_{\mathcal{A}}([d_1,d_2])| \end{aligned}$$

as required.

3.4 An Estimate for $G_z(x)$

Consider the incomplete sum

$$G_z(x) = \sum_{d \le x} g_z(d) \tag{3.4.1}$$

where g_z is multiplicative and satisfies

$$g_z(n) = \begin{cases} g(n) & \text{if } p|n \Rightarrow p < z; \\ 0 & \text{otherwise.} \end{cases}$$

Recall that $g(p) = \rho(p)/(p - \rho(p))$ and can be written as

$$\left(1 - \frac{\rho(p)}{p}\right) g(p) = \frac{\rho(p)}{p}, \qquad 1 + g(p) = \left(1 - \frac{\rho(p)}{p}\right)^{-1}.$$

We note that $G_z(x)$ increases with z, so that

$$G(x) = G_x(x) \ge G_z(x) \qquad \text{when } z \le x. \tag{3.4.2}$$

From Theorem 3.2.2 we require a lower bound for $G(\sqrt{D})$. From (3.4.2) we can see that a lower bound for $G_z(\sqrt{D})$ will provide us with a lower bound for $G(\sqrt{D})$.

We denote

$$G_z(\infty) = \sum_{d|P(z)} g(d),$$

so that $G_z(x) = G_z(\infty)$ whenever $x \ge P(z)$.

Observe that

$$G_z(x) \leq G_z(\infty) = \prod_{p|P(z)} (1 + g(p)) = \prod_{p|P(z)} \left(1 - \frac{\rho(p)}{p}\right)^{-1} = \frac{1}{V(P(z))} \tag{3.4.3}$$

where $V(P(z))$ is as in (2.1.1).

This gives a useful connection between the sum $G_z(x)$ and the product $V(P(z))$.

For Theorem 3.4.1 we require Hypothesis 2.1.1, $0 \leq \rho(p) < p$, and an upper bound

$$B(z) \leq B; \qquad \text{where} \quad B(z) = \frac{1}{\log z} \sum_{p<z} \frac{\rho(p) \log p}{p}. \tag{3.4.4}$$

Theorem 3.4.1.

Suppose that (3.4.4) *holds, where* $z \geq 2$, *and write* $z = D^{1/s}$. *Then* $G_z(\sqrt{D})$, *as defined by* (3.4.1), *satisfies*

$$\frac{1 - \exp(-\psi_B(\frac{1}{2}s))}{V(P(z))} \leq G_z(\sqrt{D}) \leq \frac{1}{V(P(z))}, \tag{3.4.5}$$

where, for each $v > 0$,

$$\psi_B(v) = \max\{0, v \log \frac{v}{B} - v + B\} = \int_{B<t<v} \log \frac{t}{B}\, dt. \tag{3.4.6}$$

Theorem 3.4.1 will follow from (3.4.3) and Lemma 3.4.2.

Lemma 3.4.2.

Suppose that $B(z)$ is as defined in (3.4.4), *and denote*

$$I_z(x) = G_z(\infty) - G_z(x) = \sum_{\substack{d \geq x \\ d|P(z)}} g(d), \tag{3.4.7}$$

and write $x = z^v$. Then for each $v > 0$

$$I_z(x) \leq \frac{\exp(-\psi_B(v))}{V(P(z))}, \tag{3.4.8}$$

where ψ_B is as in (3.4.6).

Proof

Lemma 3.4.2 follows using Rankin's trick, which appears in [11] and [23]. Take $\varepsilon \geq 0$. Then

$$I_z(x) \leq \frac{1}{x^\varepsilon} \sum_{d|P(z)} g(d) d^\varepsilon = \frac{1}{x^\varepsilon} \prod_{p|P(z)} (1 + p^\varepsilon g(p)).$$

Hence, for $V(P(z))$ as defined in (2.1.1) we obtain

$$\begin{aligned} I_z(x)V(P(z)) &\leq \frac{1}{x^\varepsilon} \prod_{p|P(z)} \left(\frac{p-\rho(p)}{p}\right)\left(1 + \frac{p^\varepsilon \rho(p)}{p-\rho(p)}\right) \\ &= \frac{1}{x^\varepsilon} \prod_{p|P(z)} \left(1 + \frac{\rho(p)}{p}(p^\varepsilon - 1)\right) \\ &\leq \frac{1}{x^\varepsilon} \exp\left(\sum_{p|P(z)} \frac{\rho(p)}{p}(p^\varepsilon - 1)\right). \end{aligned}$$

We may choose $\varepsilon = c/\log z$, provided $c \geq 0$. Observe that $(e^t - 1)/t$ increases when $t > 0$.

Hence when $p < z$

$$\frac{p^{\varepsilon}-1}{\varepsilon \log p} = \frac{e^{c\log p/\log z}-1}{c\log p/\log z} \leq \frac{e^{c}-1}{c}.$$

When $z = x^{1/v}$ use of (3.4.4) now gives

$$\begin{aligned} I_z(x)V(P(z)) &\leq \frac{1}{e^{cv}} \exp\left(\frac{e^c-1}{\log z}\sum_{p<z}\frac{\rho(p)}{p}\log p\right) \\ &\leq \exp(-cv + B(e^c-1)). \end{aligned}$$

The optimal choice of c satisfies $v = Be^c$, i.e. $c = \log v - \log B$, provided $v > B$. If $v \leq B$ then the best permissible choice is $c = 0$. This gives (3.4.8) as required by Lemma 3.4.2.

Proof of Theorem 3.4.1

The upper bound for $G_z(\sqrt{D})$ in (3.4.5) is provided immediately from (3.4.3). The lower bound for $G_z(\sqrt{D})$ is provided by noting that (3.4.7), (3.4.3) and (3.4.8) give

$$G_z(x) = G_z(\infty) - I_z(x) \geq \frac{1-\exp(-\psi_B(v))}{V(P(z))},$$

and taking $x = \sqrt{D}$. Since $x = z^v$ and $z = D^{1/s}$ we obtain

$$v = \frac{\log x}{\log z} = s\frac{\log x}{\log D} = \frac{s}{2}.$$

This completes the proof of (3.4.5).

Chapter 4

Sums of Multiplicative Functions Over Friable Integers

4.1 Introduction

We would like to use ideas from Selberg's upper bound method to seek a lower bound sieve. In this chapter we look at one method of achieving this goal which involves the use of Buchstab's Identity

$$S(\mathcal{A}, P(z)) = S(\mathcal{A}, P(w)) - \sum_{\substack{w \le p < z \\ p|P}} S(\mathcal{A}_p, P(p)) \qquad \text{if } w < z. \tag{4.1.1}$$

From this expression it is clear that an upper bound for the second sum appearing on the right would lead us to a lower bound for the sum on the left. Successive iterations of this process will provide improved upper and lower bound sieve results.

The object of this chapter is to study the sums of multiplicative functions over integers whose prime factors are relatively small. Such integers are referred to as friable integers. We aim to find a lower bound for the sum

$$H_z(x) = \sum_{1 \le n < x} h_z(n) \tag{4.1.2}$$

where $h_z(n) \ge 0$ for all n. h_z is multiplicative and satisfies

$$h_z(n) = \begin{cases} h(n) & \text{if } p|n \Rightarrow p < z \\ 0 & \text{otherwise.} \end{cases}$$

Throughout this chapter we use the notation

$$x = z^s \tag{4.1.3}$$

and we will be interested in the case $z \le x$, i.e. $s \ge 1$.

The method of proof we use in this chapter arises from a combination of ideas from Greaves [9] and Song [24]. Greaves obtained the following result

$$\left| \frac{G_z(z^s)}{G_z(\infty)} - \sigma(s) \right| \le \frac{s e^{-\psi_B(s) + O(A)}}{\log z} \qquad \text{when } s \ge 1, \tag{4.1.4}$$

where g is a multiplicative function which is non-zero only for squarefree integers and satisfies

$$\sum_{p<v} g(p) \log p \le \kappa \log v + \eta(v) \qquad \text{when } v > 1, \tag{4.1.5}$$

where $\eta(v)$ satisfies $|\eta(v)| \le A$ when $v > 1$.

G_z is the sum

$$G_z(x) = \sum_{\substack{n<x \\ p|n \Rightarrow p<z}} g(n), \tag{4.1.6}$$

and σ is the continuous function defined by requiring

$$\frac{d}{ds}\left(\frac{\sigma(s)}{s^\kappa}\right) + \frac{\kappa\sigma(s-1)}{s^{\kappa+1}} = 0 \quad \text{if} \quad s \neq 0 \tag{4.1.7}$$

where $\sigma(s) = 0$ if $s < 0$,

$$\sigma(s) = Cs^\kappa \quad \text{when} \quad 0 \leq s \leq 1; \qquad C = \frac{e^{-\gamma\kappa}}{\Gamma(\kappa+1)} \tag{4.1.8}$$

In this account, we extend the results from [9] to the context where $g(p^k)$ is not assumed to be 0 when $k \geq 2$.

Tenenbaum proved in [27] that for $0 < \delta < 1$ the result

$$\frac{H_z(z^s)}{H_z(\infty)} = \sigma(s) + O\left(\frac{1}{(\log z)^{1-\delta}}\right) \tag{4.1.9}$$

is valid for h supported on all n using a more analytic method. Prior to this Song proved that

$$\frac{H_z(z^s)}{H_z(\infty)} = \sigma(s) + O\left(\frac{\log(s+1)}{(\log z)^{1-\delta}}\right) \tag{4.1.10}$$

by making a similar extension to a result of Halberstam [24]. Song studied the sum $H_z(z^s)$ whereas here we shall investigate the residual sum $I_z(z^s)$ where

$$I_z(x) = H_z(\infty) - H_z(x) = \sum_{n \geq x} h_z(n) \tag{4.1.11}$$

and then transfer back to $H_z(z^s)$.

The first stage in the proof which appears in Section 4.4 is to obtain an integral equation for $I_z(x)$ derived by a procedure attributed to Chebychev. We then use Rankin's trick to obtain an initial upper bound for $I_z(z^s)$. Section 4.6 contains a change of variables which leads us to investigate the inner product between ξ_z and r defined in (4.6.11) and (4.2.8) respectively. In Section 4.7 we discuss the inductive argument which is taken from [9]. This inductive argument and the information gathered about the size of the inner product $\langle \xi_z, r\rangle(s)$ is then used to prove Theorem 4.3.4.

4.2 The Function $\sigma(s)$

We define $\sigma(s)$, which is used in the main result, to be the continuous solution of the differential-difference equation

$$\frac{d}{ds}\left(\frac{\sigma(s)}{s^{\kappa}}\right) + \frac{\kappa\sigma(s-1)}{s^{\kappa+1}} = 0 \qquad \text{if } s \neq 0, \tag{4.2.1}$$

and

$$\sigma(s) = \begin{cases} 0 & \text{if } s < 0; \\ C_\kappa s^\kappa & \text{when } 0 \leq s \leq 1 \end{cases} \tag{4.2.2}$$

where C_κ is the constant

$$C_\kappa = e^{-\gamma\kappa}/\Gamma(\kappa+1), \tag{4.2.3}$$

κ is as in (4.3.1), γ is Euler's constant and Γ is Euler's Gamma function.

Observe that (4.2.1) can be rewritten as

$$s\sigma'(s) = \kappa(\sigma(s) - \sigma(s-1)) \quad \text{if } s \neq 0 \tag{4.2.4}$$

which can be integrated to give

$$s\sigma(s) = \int_0^s \sigma(u)du + \kappa \int_{s-1}^s \sigma(u)du \qquad \text{for all } s. \tag{4.2.5}$$

Lemma 4.2.1.

The continuous expression $\sigma(s)$ defined by (4.2.1) *and* (4.2.2) *increases when $s \geq 0$ and satisfies*

$$\sigma(s) = 1 + O(e^{-s\log s}) \tag{4.2.6}$$

where the implied constant may depend on κ.

A proof of this lemma can be found in Lemma 7.1.1 of [8].

The continuous function $i(s) = 1 - \sigma(s)$ arises as the leading term in an approximation to $I_z(x)$. Using the properties of $\sigma(s)$ described above we then have

$$-si(s) = \kappa - \int_0^s i(u)du - \kappa \int_{s-1}^s i(u)du.$$

Letting $s \to \infty$ shows

$$\int_0^\infty i(u)du = \kappa,$$

thus

$$si(s) + \int_s^\infty i(u)du = \kappa \int_{s-1}^s i(u)du. \tag{4.2.7}$$

Equation (4.2.7) has an "adjoint" satisfying

$$\frac{d}{ds}(sr(s)) = -\kappa r(s) + \kappa r(s+1). \tag{4.2.8}$$

This equation has a solution for positive s,

$$r(s) = \int_0^\infty \exp\left(- sx + \kappa \int_0^x \frac{1-e^{-t}}{t} dt \right) dx, \tag{4.2.9}$$

for which $r(s) \sim 1/s$ as $s \to \infty$.

Define an "inner product"

$$\langle R, r \rangle (s) = sr(s)R(s) - \kappa \int_{s-1}^s r(x+1)R(x)dx. \tag{4.2.10}$$

The inner product $\langle i, r \rangle (s)$ is constant but we will instead look at the inner product between i_z and r and show that this is suitably small for our purposes.

Remark 4.2.2.

Note that

$$\begin{aligned} \langle 1, r \rangle (s) &= sr(s) - \kappa \int_{s-1}^s r(t+1)dt \\ &= 1. \end{aligned} \tag{4.2.11}$$

This follows from (4.2.8) and since $sr(s) \sim 1$ as $s \to \infty$.

4.3 Assumptions

Let h be a multiplicative function such that $h(n) \geq 0$ whenever $n \geq 1$. In what follows we assume that h satisfies the following conditions:

There exist constants δ, $0 < \delta < 1$, and $\kappa > 0$ such that

$$\sum_{p \leq z} h(p) \log p = \kappa \log z + \eta(z) \quad \text{for } z \geq 1 \tag{4.3.1}$$

where

$$\eta(v) \begin{cases} \ll (\log v)^{1-\delta} & \text{for } v \geq 1; \\ = 0 & \text{if } v < 1, \end{cases} \tag{4.3.2}$$

and *there exist constants $b > 0$, $A_\alpha > 0$ and $0 < \alpha < 1/2$ such that*

$$\sum_{\substack{p, k \geq 2 \\ p < z}} h(p^k) \log p^k \leq b \tag{4.3.3}$$

and

$$\sum_{\substack{p, k \geq 2 \\ p < z}} h(p^k) p^{k\alpha} < A_\alpha. \tag{4.3.4}$$

Remark 4.3.1.

Note that (4.3.3) is a consequence of (4.3.4) because

$$\begin{aligned}\sum_{\substack{p,k\geq 2\\ p<z}} h(p^k)\log p^k &\leq \sum_{\substack{p^k\geq 4\\ k\geq 2}} h(p^k)p^{k\alpha}\frac{\log p^k}{p^{k\alpha/2}}\frac{1}{4^{\alpha/2}}\\ &< A_\alpha \frac{1}{4^{\alpha/2}}c_\alpha \qquad (4.3.5)\end{aligned}$$

where $c_\alpha = \sup_{x\geq 1}\log x/x^{\alpha/2}$.

The definition of $I_z(x)$ raises a convergence question about $H_z(\infty)$, where H_z is as defined in (4.1.2).

From (4.3.4) the sum over prime powers is convergent and since

$$H_z(\infty) = \prod_{p<z}\left(1 + h(p) + \sum_{k\geq 2} h(p^k)\right)$$

we can see that it is indeed convergent. Thus the definition of $I_z(x)$ given in (4.1.11) is certainly valid.

Remark 4.3.2.

The multiplicative function ρ arises when $h(p)$ is expressed in the form

$$h(p) = \frac{\rho(p)}{p-\rho(p)}.$$

For the proof of Lemma 4.5.1 we need to assume Hypothesis 2.1.1, $0 \leq \rho(p) < p$ and the upper bound (3.4.4),

$$B(z) \le B, \quad (B \ge 1), \qquad \text{where} \quad B(z) = \frac{1}{\log z}\sum_{p<z}\frac{\rho(p)\log p}{p}. \tag{4.3.6}$$

We remind the reader of the function ψ defined in (3.4.6)

$$\psi_B(s) = \int_B^s \log\frac{t}{B}dt = \begin{cases} s\log\dfrac{s}{B} - s + B & \text{if } s > B; \\ 0 & \text{otherwise.} \end{cases} \tag{4.3.7}$$

The following result will be useful throughout this chapter.

Lemma 4.3.3.

For ψ_B defined in (4.3.7) *we have*

$$\exp(-\psi_B(t)) \le \exp(-\psi_B(s) + (s-t)\log s) \qquad \text{for } s \ge 1 \tag{4.3.8}$$

provided that $t < s$.

Proof

From (4.3.7) we can see that for $t < s$

$$\begin{aligned} \psi_B(s) - \psi_B(t) &= \int_t^s \log\frac{t}{B}dt \\ &\le (s-t)\log\frac{s}{B} \\ &\le (s-t)\log s. \end{aligned}$$

Accordingly we obtain

$$\exp\Big(-\psi_B(t)\Big) \le \exp\Big(-\psi_B(s) + (s-t)\log s\Big)$$

as required.

We denote the expression $V(P(z))$ appearing in Definition 2.1.3 by

$$V(P(z)) = \prod_{p<z} \left(1 - \frac{\rho(p)}{p}\right). \tag{4.3.9}$$

Observe that

$$\frac{1}{V(P(z))} = \prod_{p<z} (1 + h(p)) \leq H_z(\infty). \tag{4.3.10}$$

Using this notation our main result is as follows.

Theorem 4.3.4.

Suppose that h is multiplicative, $h(n) \geq 0$, and h satisfies (4.3.1), (4.3.2), (4.3.3) *and* (4.3.4). *Then the sum $H_z(z^s)$ defined in* (4.1.2) *satisfies*

$$\frac{H_z(z^s)}{H_z(\infty)} = \sigma(s) + O\left(\frac{s^{E-1} e^{-\psi_B(s)/3 + O(B)}}{(\log z)^\delta}\right) \qquad \text{when } 1 \leq s \leq z^\alpha \tag{4.3.11}$$

where the implied constant may depend on κ and A_α, B satisfies (4.3.6), $c_\alpha = 2/(e\alpha)$ *and where*

$$E = \max\left\{2, \frac{1}{3} + \frac{\log A_\alpha c_\alpha}{\log z^{\alpha/2}}\right\}.$$

4.4 Preliminary Lemmas

In this section we obtain the integral equation using a fairly straightforward procedure which is analogous to that used in [24] and [9].

Remark 4.4.1.

Note that by applying (4.3.1) for $z = p$ and $z = p - 1$, and then taking the difference, we obtain

$$h(p)\log p \ll (\log p)^{1-\delta} \tag{4.4.1}$$

$$h(p) \ll (\log p)^{-\delta} \tag{4.4.2}$$

and using the notation of Stieltjes integrals we get

$$\begin{aligned}\sum_{p<z} h(p)(\log p)^{1-\delta} &= \int_2^z (\log t)^{-\delta} d\Big(\sum_{2\le p<t} h(p)\log p\Big) \\ &\ll \int_2^z (\log t)^{-\delta} d(\kappa \log t + \eta(t)) \qquad \text{from (4.3.1).}\end{aligned}$$

Using (4.3.2) and integrating gives

$$\sum_{p<z} h(p)(\log p)^{1-\delta} \ll (\log z)^{1-\delta}. \tag{4.4.3}$$

Lemma 4.4.2.

The sum $I_z(x)$ *defined in* (4.1.11) *satisfies the integral equation*

$$I_z(x)\log x + \int_x^\infty I_z(t)\frac{dt}{t} = \kappa \int_{x/z}^x I_z(t)\frac{dt}{t} + \Delta_z(x) \tag{4.4.4}$$

where

$$\begin{aligned}\Delta_z(x) &= \sum_{m\ge x/z} h_z(m)\Big(\eta(z) - \eta(x/m)\Big) - \sum_{\substack{mp\ge x \\ p\mid m}} h_z(m)h_z(p)\log p \\ &\quad + \sum_{\substack{mp^k\ge x \\ p\nmid m, p, k\ge 2}} h_z(m)h_z(p^k)\log p^k. \end{aligned}\tag{4.4.5}$$

Proof

Since we have that $\log n = \sum_{p^k \| n} \log p^k$,

$$\begin{aligned}
I_z(x)\log x &= \sum_{n \geq x} h_z(n) \log \frac{x}{n} + \sum_{n \geq x} h_z(n) \log n \\
&= \sum_{n \geq x} h_z(n) \log \frac{x}{n} + \sum_{n \geq x} h_z(n) \sum_{p^k \| n} \log p^k \\
&= \sum_{n \geq x} h_z(n) \log \frac{x}{n} + \sum_{\substack{mp^k \geq x \\ p \nmid m}} h_z(m) h_z(p^k) \log p^k.
\end{aligned}$$

We observe that

$$\begin{aligned}
-\sum_{n \geq x} h_z(n) \log \frac{x}{n} &= \sum_{n \geq x} h_z(n) \log \frac{n}{x} \\
&= \sum_{x \leq t \leq n} \int h_z(n) \frac{dt}{t}
\end{aligned}$$

which gives

$$-\sum_{n \geq x} h_z(n) \log \frac{x}{n} = \int_x^\infty I_z(t) \frac{dt}{t}. \tag{4.4.6}$$

Thus we can write

$$I_z(x)\log x + \int_x^\infty I_z(t) \frac{dt}{t} = \sum_{\substack{mp^k \geq x \\ p \nmid m}} h_z(m) h_z(p^k) \log p^k. \tag{4.4.7}$$

The sum on the right of (4.4.7) can be split up as follows:

$$\begin{aligned}
\sum_{\substack{mp^k \geq x \\ p \nmid m}} h_z(m) h_z(p^k) \log p^k &= \sum_{\substack{mp \geq x \\ p \nmid m}} h_z(m) h_z(p) \log p + \sum_{\substack{mp^k \geq x \\ p \nmid m, p, k \geq 2}} h_z(m) h_z(p^k) \log p^k \\
&= \sum_{mp \geq x} h_z(m) h_z(p) \log p - \sum_{\substack{mp \geq x \\ p \mid m}} h_z(m) h_z(p) \log p \\
&\quad + \sum_{\substack{mp^k \geq x \\ p \nmid m, p, k \geq 2}} h_z(m) h_z(p^k) \log p^k.
\end{aligned} \tag{4.4.8}$$

The first sum on the right is

$$\begin{aligned}\sum_{mp\geq x} h_z(m)h_z(p)\log p &= \sum_{m\geq x/z} h_z(m) \sum_{x/m\leq p<z} h(p)\log p \\ &= \sum_{x/z\leq m<x} h_z(m)\left[\kappa \log \frac{z}{x/m} + \eta(z) - \eta(x/m)\right] \\ &\quad + \sum_{m\geq x} h_z(m)\left[\kappa \log z + \eta(z)\right]\end{aligned}$$

by (4.3.1), and since $\eta(x/m) = 0$ when $m \geq x$ we obtain

$$\begin{aligned}\sum_{mp\geq x} h_z(m)h_z(p)\log p &= \kappa \sum_{x/z\leq m<x} h_z(m)\log \frac{m}{x/z} + \kappa \sum_{m\geq x} h_z(m)\log z \\ &\quad + \sum_{m\geq x/z} h_z(m)\Big(\eta(z) - \eta(x/m)\Big).\end{aligned}$$

Momentarily ignoring the last term in this we get

$$\begin{aligned}&\kappa \sum_{x/z\leq m<x} h_z(m)\log \frac{m}{x/z} + \kappa \sum_{m\geq x} h_z(m)\log z \\ &= \kappa \sum_{m\geq x/z} h_z(m)\log \frac{m}{x/z} - \kappa \sum_{m\geq x} h_z(m)\log \frac{m}{x/z} + \kappa \sum_{m\geq x} h_z(m)\log z \\ &= \kappa \sum_{m\geq x/z} h_z(m)\log \frac{m}{x/z} - \kappa \sum_{m\geq x} h_z(m)\log \frac{m}{x} \\ &= -\kappa \sum_{m\geq x/z} h_z(m)\log \frac{x/z}{m} + \kappa \sum_{m\geq x} h_z(m)\log \frac{x}{m}\end{aligned}$$

which after comparison with (4.4.6) gives

$$\sum_{mp \geq x} h_z(m)h_z(p)\log p = \kappa \int_{x/z}^{\infty} I_z(t)\frac{dt}{t} - \kappa \int_{x}^{\infty} I_z(t)\frac{dt}{t} + \sum_{m \geq x/z} h_z(m)\Big(\eta(z) - \eta(x/m)\Big). \quad (4.4.9)$$

Combining (4.4.7), (4.4.8) and (4.4.9) we have deduced the result as stated:

$$I_z(x)\log x + \int_x^{\infty} I_z(t)\frac{dt}{t} = \kappa \int_{x/z}^{x} I_z(t)\frac{dt}{t} + \Delta_z(x). \quad (4.4.10)$$

4.5 An Upper Bound for $I_z(x)$

Here we obtain an upper bound for $I_z(x)$ by applying Rankin's trick. Although the result we obtain via this method is rather weak, it is useful in the proofs of Lemmas 4.6.1 and 4.6.2.

Lemma 4.5.1.

The expression defined in (4.1.11) *satisfies*

$$I_z(x) \leq L_\alpha \frac{\exp(-\psi_B(s))}{V(P(z))} \qquad \text{for } s < Bz^\alpha$$

where B satisfies (4.3.6), *ψ_B is as in* (4.3.7) *and $L_\alpha = \exp(A_\alpha)$ with A_α as in* (4.3.4).

Proof

Firstly, denote

$$T_p(\lambda) = \sum_{k\geq 2} h(p^k)p^{k\lambda} \qquad \text{for some } \lambda \geq 0.$$

We have

$$\begin{aligned}
I_z(x) &= \sum_{n\geq x} h_z(n) \\
&\leq \frac{1}{x^\lambda}\prod_{p<z}\Big(1 + h(p)p^\lambda + \sum_{k\geq 2} h(p^k)p^{k\lambda}\Big) \qquad \text{by Rankin's Trick} \\
&\leq \frac{1}{x^\lambda}\prod_{p<z}\Big(1 + h(p)p^\lambda + T_p(\lambda)\Big)
\end{aligned}$$

so that for $V(P(z))$ as in (4.3.9),

$$\begin{aligned}
I_z(x)V(P(z)) &\leq \frac{1}{x^\lambda}\prod_{p<z}\Big(1 + h(p)p^\lambda + T_p(\lambda)\Big)\Big(1 - \frac{\rho(p)}{p}\Big) \\
&= \frac{1}{x^\lambda}\prod_{p<z}\Big[\Big(1 + h(p)p^\lambda\Big)\Big(1 - \frac{\rho(p)}{p}\Big) + T_p(\lambda)\Big(1 - \frac{\rho(p)}{p}\Big)\Big].
\end{aligned}$$

Now from Remark 4.3.2, we have $h(p) = \rho(p)/(p - \rho(p))$ so that

$$\begin{aligned}
I_z(x)V(P(z)) &\leq \frac{1}{x^\lambda}\prod_{p<z}\Big[\Big(1 + \frac{\rho(p)}{p}\Big(p^\lambda - 1\Big)\Big) + T_p(\lambda)\Big(1 - \frac{\rho(p)}{p}\Big)\Big] \\
&\leq \frac{1}{x^\lambda}\exp\Big(\sum_{p<z}\log\Big[\Big(1 + \frac{\rho(p)}{p}\Big(p^\lambda - 1\Big)\Big) + T_p(\lambda)\Big(1 - \frac{\rho(p)}{p}\Big)\Big]\Big).
\end{aligned}$$

In view of the fact that $\log(1+x) \le x$ we get

$$\begin{aligned} I_z(x)V(P(z)) &\le \frac{1}{x^\lambda}\exp\left(\sum_{p<z}\left[\frac{\rho(p)}{p}\left(p^\lambda - 1\right) + T_p(\lambda)\left(1-\frac{\rho(p)}{p}\right)\right]\right) \\ &\le \frac{1}{x^\lambda}\exp\left(\sum_{p<z}\frac{\rho(p)}{p}(p^\lambda - 1)\right)\exp\left(\sum_{p<z}T_p(\lambda)\left(1-\frac{\rho(p)}{p}\right)\right). \end{aligned}$$

We may choose $\lambda = d/\log z$, provided $d \ge 0$. Observing that $(e^t-1)/t$ increases when $t>0$, then for $p<z$ we have,

$$\frac{p^\lambda - 1}{\lambda \log p} = \frac{e^{d\log p/\log z}-1}{d\log p/\log z} \le \frac{e^d-1}{d}.$$

When $x = z^s$ we have

$$\begin{aligned} \frac{1}{x^\lambda}\exp\left(\sum_{p<z}\frac{\rho(p)}{p}(p^\lambda - 1)\right) &= \frac{1}{e^{ds}}\exp\left(\sum_{p<z}\frac{\rho(p)\log p}{p}\frac{(p^\lambda-1)}{\log p}\right) \\ &\le \frac{1}{e^{ds}}\exp\left(\frac{e^d-1}{\log z}\sum_{p<z}\frac{\rho(p)}{p}\log p\right) \\ &\le \exp(-ds + B(e^d-1)). \end{aligned}$$

By differentiating we can see that the optimal choice of d satisfies $s = Be^d$, provided $s > B$. If $s \le B$ then the best permissible choice is $d = 0$ since $\lambda \ge 0$. Therefore,

$$\frac{1}{x^\lambda}\exp\left(\sum_{p<z}\frac{\rho(p)}{p}(p^\lambda - 1)\right) \le \exp(-\psi_B(s)).$$

Consequently,

$$\begin{aligned} I_z(x)V(P(z)) &\leq \exp(-\psi_B(s))\exp\left(\sum_{p<z} T_p(\lambda)\left(1-\frac{\rho(p)}{p}\right)\right). \\ &\leq \exp(-\psi_B(s))\exp\left(\sum_{p<z} T_p(\lambda)\right) \quad \text{since } (1-\rho(p)/p) \leq 1. \\ &\leq \exp(-\psi_B(s))\exp\left(\sum_{\substack{p<z \\ k\geq 2}} h(p^k)p^{k\lambda}\right). \end{aligned}$$

If we assume that $s < Bz^{\alpha}$, then as a consequence of (4.3.4) we have that

$$\begin{aligned} I_z(x)V(P(z)) &\leq \exp(-\psi_B(s))\exp(A_\alpha) \\ &\leq L_\alpha \exp(-\psi_B(s)). \end{aligned}$$

Consequently

$$I_z(x) \leq L_\alpha \frac{\exp(-\psi_B(s))}{V(P(z))} .$$

4.6 Change of Variables

In this section we look at the integral equation obtained in Lemma 4.4.2. The first thing we need to do is to obtain an asymptotic estimate for the third sum appearing in the definition of $\Delta_z(x)$, (4.4.5). The result of this appears in Lemma 4.6.1. We then define a change of variables which will enable us to use the integral equation from Lemma 4.4.2 to provide information about the size of an inner product involving ξ, which arises in the change of variables. The size of this inner product is the foundation of the proof of Theorem 4.3.4.

Lemma 4.6.1.

The last sum appearing in the term $\Delta_z(x)$ *of Lemma* 4.4.2 *satisfies*

$$\sum_{\substack{mp^k \geq x \\ p \nmid m, p, k \geq 2}} h_z(m) h_z(p^k) \log p^k \ll H_z(\infty) s^{D_\alpha} e^{-\psi_B(s)/3} \qquad \text{when } s < z^\alpha \tag{4.6.1}$$

where $D_\alpha = \log A_\alpha c_\alpha / \log z^{\alpha/2}$, A_α *is as in* (4.3.4) *and* $c_\alpha = 2/(e\alpha)$.

Proof

By following the same argument as Remark 4.3.1 we observe

$$\begin{aligned} \sum_{\substack{p^k \geq t \\ k \geq 2}} h_z(p^k) \log p^k &< \sum_{\substack{p^k \geq t \\ k \geq 2}} h_z(p^k) p^{k\alpha} \frac{\log p^k}{p^{k\alpha/2}} \frac{1}{t^{\alpha/2}} && (4.6.2) \\ &< A_\alpha \frac{1}{t^{\alpha/2}} c_\alpha && (4.6.3) \end{aligned}$$

where $c_\alpha = \sup_{x \geq 1} \log x / x^{\alpha/2} = 2/(e\alpha)$.

To evaluate the sum we consider two cases. First we look at those m such that $m < T$, where a suitable $T > 0$ will be chosen later. This gives

$$\begin{aligned} \sum_{\substack{mp^k \geq x \\ k \geq 2, m < T}} h_z(m) h_z(p^k) \log p^k &\leq \sum_{m<T} h_z(m) \sum_{\substack{p^k \geq x/m \\ k \geq 2}} h_z(p^k) \log p^k \\ &\leq \frac{A_\alpha c_\alpha}{x^{\alpha/2}} \sum_{m<T} h_z(m) m^{\alpha/2} \qquad \text{by (4.6.2)} \\ &\leq \frac{A_\alpha c_\alpha}{x^{\alpha/2}} T^{\alpha/2} \sum_m h_z(m). \end{aligned}$$

So we have

$$\sum_{\substack{mp^k \geq x \\ k \geq 2, m < T}} h_z(m) h_z(p^k) \log p^k \leq \frac{A_\alpha c_\alpha}{x^{\alpha/2}} T^{\alpha/2} H_z(\infty). \tag{4.6.4}$$

We now consider the contribution from those $m \geq T$.

$$\begin{aligned} \sum_{\substack{mp^k \geq x \\ k \geq 2, m \geq T}} h_z(m) h_z(p^k) \log p^k &\leq \sum_{p, k \geq 2} h_z(p^k) \log p^k \sum_{m \geq T} h_z(m) \\ &\leq \sum_{p, k \geq 2} h_z(p^k) \log p^k I_z(T) \end{aligned}$$

which after appealing to (4.3.3) gives

$$\sum_{\substack{mp^k \geq x \\ k \geq 2, m \geq T}} h_z(m) h_z(p^k) \log p^k \leq b I_z(T). \tag{4.6.5}$$

Using (4.3.10) and the result from Lemma 4.5.1 we obtain

$$\sum_{\substack{mp^k \geq x \\ k \geq 2, m \geq T}} h_z(m) h_z(p^k) \log p^k \leq bL_\alpha H_z(\infty) \exp\left(- \psi_B\left(\frac{\log T}{\log z}\right)\right)$$

provided $(\log T)/(\log z) \leq Bz^\alpha$. If we choose $T < z^s$ then this condition is satisfied and Lemma 4.3.3 shows that

$$\exp\left(- \psi_B\left(\frac{\log T}{\log z}\right)\right) \leq \exp\left(- \psi_B(s) + \left(s - \frac{\log T}{\log z}\right) \log s\right).$$

This gives

$$\sum_{\substack{mp^k \geq x \\ k\geq 2, m\geq T}} h_z(m)h_z(p^k)\log p^k \leq bL_\alpha H_z(\infty)\exp\left(-\psi_B(s)+\left(s-\frac{\log T}{\log z}\right)\log s\right). \quad (4.6.6)$$

Combining (4.6.4) and (4.6.6) we obtain for all m,

$$\frac{1}{H_z(\infty)}\sum_{\substack{mp^k\geq x \\ k\geq 2}} h_z(m)h_z(p^k)\log p^k \ll \frac{A_\alpha c_\alpha}{x^{\alpha/2}}T^{\alpha/2} \quad (4.6.7)$$

$$+ \quad bL_\alpha \exp\left(-\psi_B(s)+\left(s-\frac{\log T}{\log z}\right)\log s\right).$$

Now we need to choose a suitable $0 < T < z^s$ so that both terms on the right hand side of this equation have the same order of magnitude with respect to s. Choose T so that it satisfies

$$\frac{A_\alpha c_\alpha}{x^{\alpha/2}}T^{\alpha/2} = \exp\left\{\frac{\log z^{\alpha/2}}{\log z^{\alpha/2}+\log s}\left(-\psi_B(s)+\frac{\log A_\alpha c_\alpha}{\log z^{\alpha/2}}\log s\right)\right\}. \quad (4.6.8)$$

Since $x = z^s$ as in (4.1.3), this says

$$\frac{\log T}{\log z} = s - \frac{\log A_\alpha c_\alpha}{\log z^{\alpha/2}} + \frac{1}{\log z^{\alpha/2}+\log s}\left(-\psi_B(s)+\frac{\log A_\alpha c_\alpha}{\log z^{\alpha/2}}\log s\right).$$

Now the exponent in (4.6.7) takes the form

$$-\psi_B(s)+\left(s-\frac{\log T}{\log z}\right)\log s = \left(1-\frac{\log s}{\log z^{\alpha/2}+\log s}\right)\left(-\psi_B(s)+\frac{\log A_\alpha c_\alpha}{\log z^{\alpha/2}}\log s\right).$$

Consequently (4.6.7) gives

$$\frac{1}{H_z(\infty)} \sum_{\substack{mp^k \geq x \\ k \geq 2}} h_z(m) h_z(p^k) \log p^k$$
$$\ll (1 + bL_\alpha) \exp\left(\frac{\log z^{\alpha/2}}{\log z^{\alpha/2} + \log s} \left(-\psi_B(s) + \frac{\log A_\alpha c_\alpha}{\log z^{\alpha/2}} \log s \right)\right).$$

Since $s < z^\alpha$ we have

$$-\frac{\log z^{\alpha/2}}{\log z^{\alpha/2} + \log s} \psi_B(s) < -\frac{1}{3}\psi_B(s)$$

which gives

$$\frac{1}{H_z(\infty)} \sum_{\substack{mp^k \geq x \\ k \geq 2}} h_z(m) h_z(p^k) \log p^k \ll \exp\left(-\frac{1}{3}\psi_B(s) + \frac{\log A_\alpha c_\alpha}{\log z^{\alpha/2}} \log s \right)$$
$$\ll e^{-\psi_B(s)/3} s^{D_\alpha},$$

and Lemma 4.6.1 follows provided we can show that $T < z^s$. From (4.6.8) it is clear that to do this we need to prove that

$$\exp\left\{ \frac{\log z^{\alpha/2}}{\log z^{\alpha/2} + \log s} \left(-\psi_B(s) + \frac{\log A_\alpha c_\alpha}{\log z^{\alpha/2}} \log s \right)\right\} < A_\alpha c_\alpha.$$

If we take logarithms and rearrange we obtain

$$\begin{aligned}\frac{\log A_\alpha c_\alpha}{\log z^{\alpha/2}+\log s}\log s-\log A_\alpha c_\alpha &< \frac{\log z^{\alpha/2}}{\log z^{\alpha/2}+\log s}\psi_B(s)\\ \log A_\alpha c_\alpha\left(\frac{\log s}{\log z^{\alpha/2}+\log s}-1\right) &< \frac{\log z^{\alpha/2}}{\log z^{\alpha/2}+\log s}\psi_B(s).\end{aligned}$$

So $T < z^s$ provided

$$\begin{aligned}-\log A_\alpha c_\alpha &< \psi_B(s)\\ \frac{1}{A_\alpha c_\alpha} &< e^{\psi_B(s)}\end{aligned}$$

which is clearly true since $\psi_B(s) \geq 0$. This can be seen from the definition of ψ_B in (4.3.7).

Using the following change of variables

$$x = z^s, \qquad t = z^u, \qquad I_z(x) = i_z(s), \qquad \Delta_z(x) = \vartheta_z(s) \tag{4.6.9}$$

we rewrite the integral equation (4.4.4) as

$$si_z(s) + \int_s^\infty i_z(u)du = \kappa\int_{s-1}^s i_z(u)du + \frac{\vartheta_z(s)}{\log z}. \tag{4.6.10}$$

Note that by ignoring the entry $\vartheta_z(s)$ in (4.6.10) we obtain the equation appearing in (4.2.7).

If we now write

$$i_z(s) = H_z(\infty)\left(i(s) + \frac{\xi_z(s)}{(\log z)^\delta}\right) \tag{4.6.11}$$

then proving Theorem 4.3.4 is equivalent to showing that

$$\xi_z(s) \ll s^{E-1} \exp\Big(-\frac{1}{3}\psi_B(s) + O(B)\Big).$$

We will use information about the inner product $\langle \xi_z, r\rangle(s)$ to gain information about $\xi_z(s)$ and subsequently prove Theorem 4.3.4.

Lemma 4.6.2.

The inner product $\langle \xi_z, r\rangle$ *satisfies*

$$\langle \xi_z, r\rangle(s) \ll s^E r(s) e^{-\psi_B(s)/3} \qquad \text{for } s < Bz^\alpha$$

where

$$E = \max\left\{2, \frac{1}{3} + \frac{\log A_\alpha c_\alpha}{\log z^{\alpha/2}}\right\},$$

ξ_z *satisfies* (4.6.11), r *satisfies* (4.2.9), B *satisfies* (4.3.6), ψ_B *is as in* (4.3.7) *and the inner product is defined by* (4.2.10).

Proof

From (4.6.10), (4.2.7) and (4.6.11) we obtain an integral equation for $\xi_z(s)$,

$$s\xi_z(s) + \int_s^\infty \xi_z(u)du = \kappa \int_{s-1}^s \xi_z(u)du + \frac{\vartheta_z(s)}{H_z(\infty)(\log z)^{1-\delta}}. \tag{4.6.12}$$

Using the notation of Stieltjes integrals, (4.2.8) and since

$$\begin{aligned} \lim_{s\to\infty} sr(s)\xi_z(s) &= \lim_{s\to\infty} \xi_z(s) \qquad \text{since } sr(s) \to 1 \text{ as } s \to \infty \\ &= \lim_{s\to\infty} \left(\frac{i_z(s)}{H_z(\infty)} - i(s)\right) \qquad \text{by (4.6.11)} \\ &= 0 \end{aligned}$$

we observe that

$$\int_s^\infty ur(u)d\xi_z(u) = -sr(s)\xi_z(s) - \int_s^\infty \xi_z(u)(-\kappa r(u) + \kappa r(u+1))du \tag{4.6.13}$$

From (4.6.12), (4.6.13) and (4.2.10) we have,

$$\begin{aligned}
\int_s^\infty r(u)\frac{d\vartheta_z(u)}{H_z(\infty)(\log z)^{1-\delta}} &= \int_s^\infty r(u)\Big(ud\xi_z(u) - \kappa\xi_z(u)du + \kappa\xi_z(u-1)du\Big) \\
&= -sr(s)\xi_z(s) + \kappa\int_{s-1}^{s} r(u+1)\xi_z(u)du \\
&= -\langle \xi_z, r\rangle(s). \qquad (4.6.14)
\end{aligned}$$

Reverting back to the original variables (4.6.9) we have

$$\int_s^\infty r(u)d\vartheta_z(u) = \int_x^\infty r\left(\frac{\log t}{\log z}\right)d\Delta_z(t) = I_1 - I_2 + I_3 \tag{4.6.15}$$

where

$$I_1 = \int_x^\infty r\left(\frac{\log t}{\log z}\right)d\Big(\sum_{m\geq t/z} h_z(m)(\eta(z) - \eta(t/m))\Big) \tag{4.6.16}$$

$$I_2 = \int_x^\infty r\left(\frac{\log t}{\log z}\right)d\Big(\sum_{\substack{mp\geq t\\ p|m}} h_z(m)h_z(p)\log p\Big) \tag{4.6.17}$$

$$I_3 = \int_x^\infty r\left(\frac{\log t}{\log z}\right)d\Big(\sum_{\substack{mp^k\geq t\\ p\nmid m, p, k\geq 2}} h_z(m)h_z(p^k)\log p^k\Big). \tag{4.6.18}$$

Firstly, we look at I_1. From (4.6.16) we get

$$\begin{aligned} I_1 &= \int_x^\infty r\left(\frac{\log t}{\log z}\right) d\Big(\sum_{m\geq t/z} h_z(m)(\eta(z)-\eta(t/m))\Big) \\ &= -r\left(\frac{\log x}{\log z}\right)\sum_{m\geq x/z} h_z(m)(\eta(z)-\eta(x/m)) \\ &\quad + \int_x^\infty \sum_{m\geq t/z} h_z(m)(\eta(z)-\eta(t/m))\left|dr\left(\frac{\log t}{\log z}\right)\right|, \end{aligned}$$

from which we can determine

$$|I_1| \ll r\left(\frac{\log x}{\log z}\right) I_z\left(\frac{x}{z}\right)(\log z)^{1-\delta}. \qquad (4.6.19)$$

The sum in the d expression of I_2 can be expressed in such a way that I_2 becomes

$$\begin{aligned} I_2 &= \int_x^\infty r\left(\frac{\log t}{\log z}\right) d\Big(\sum_{\substack{lp^{k+1}\geq t, p\nmid l \\ k\geq 1, p<z}} h_z(l)h(p^k)h(p)\log p\Big) \\ &= -r\left(\frac{\log x}{\log z}\right)\sum_{\substack{lp^{k+1}\geq x, p\nmid l \\ k\geq 1, p<z}} h_z(l)h(p^k)h(p)\log p \\ &\quad + \int_x^\infty \sum_{\substack{lp^{k+1}\geq t, p\nmid l \\ k\geq 1, p<z}} h_z(l)h(p^k)h(p)\log p\left|dr\left(\frac{\log t}{\log z}\right)\right|. \end{aligned}$$

From above and using the fact that the d expression in (4.6.17) is ≤ 0 we obtain

$$-r\left(\frac{\log x}{\log z}\right)\sum_{\substack{lp^{k+1}\geq x, p\nmid l \\ k\geq 1, p<z}} h_z(l)h(p^k)h(p)\log p \leq I_2 \leq 0. \qquad (4.6.20)$$

We now have

$$|I_2| \le r\left(\frac{\log x}{\log z}\right)\left\{\sum_{\substack{lp^2\ge x\\ p\nmid l, p<z}} h_z(l)h^2(p)\log p + \sum_{\substack{lp^{k+1}\ge x, p\nmid l\\ k\ge 2, p<z}} h_z(l)h(p^k)h(p)\log p\right\}.$$

Applying (4.4.1) to this we obtain

$$\begin{aligned}
|I_2| &\le r\left(\frac{\log x}{\log z}\right)\left\{\sum_{p<z} h(p)(\log p)^{1-\delta}\sum_{l\ge x/p^2} h_z(l) + \sum_{\substack{lp^k\ge x/z\\ k\ge 2, p<z}} h_z(l)h(p^k)(\log p)^{1-\delta}\right\}\\
&\le r\left(\frac{\log x}{\log z}\right)\left\{I_z\left(\frac{x}{z^2}\right)(\log z)^{1-\delta} + \sum_{\substack{lp^k\ge x/z\\ k\ge 2, p<z}} h_z(l)h(p^k)\log p^k\right\} \qquad \text{using (4.4.3)}\\
&\ll r\left(\frac{\log x}{\log z}\right)\left\{(\log z)^{1-\delta}\frac{s^2e^{-\psi_B(s)}}{V(P(z))} + H_z(\infty)(s-1)^{D_\alpha}e^{-\psi_B(s-1)/3}\right\},
\end{aligned}$$

for $s < Bz^\alpha$ after appealing to Lemmas 4.3.3, 4.5.1 and 4.6.1.

Thus we have achieved

$$\begin{aligned}
|I_2| &\le r(s)H_z(\infty)\left\{s^2e^{-\psi_B(s)}(\log z)^{1-\delta} + s^{1/3}s^{D_\alpha}e^{-\psi_B(s)/3}\right\}\\
&\ll r(s)H_z(\infty)s^Ee^{-\psi_B(s)/3}(\log z)^{1-\delta}, \qquad (4.6.21)
\end{aligned}$$

where $E = \max\{2, D_\alpha + \frac{1}{3}\}$.

We now turn our attention to the d expression in I_3. From (4.6.18) we have

$$I_3 = -\sum_{\substack{mp^k \geq t \\ p\nmid m, p, k\geq 2}} r\left(\frac{\log(mp^k)}{\log z}\right) h_z(m) h_z(p^k) \log p^k.$$

Using the fact that r is a decreasing function and the result from Lemma 4.6.1 we have

$$\begin{aligned} |I_3| &\ll r(s) \sum_{\substack{mp^k \geq t \\ p\nmid m, p, k\geq 2}} h_z(m) h_z(p^k) \log p^k \\ &\ll r(s) s^{D_\alpha} e^{-\psi_B(s)/3} H_z(\infty). \end{aligned} \tag{4.6.22}$$

Substituting (4.6.19), (4.6.21) and (4.6.22) into (4.6.15) and referring to (4.6.14) we conclude

$$\begin{aligned} H_z(\infty)(\log z)^{1-\delta} \langle \xi_z, r\rangle(s) &\ll |I_1| + |I_2| + |I_3| \\ &\ll r(s)\bigg(I_z(z^{s-1})(\log z)^{1-\delta} + s^{D_\alpha} H_z(\infty) e^{-\psi_B(s)/3} \\ &\qquad + s^E H_z(\infty) e^{-\psi_B(s)/3} (\log z)^{1-\delta}\bigg). \end{aligned}$$

Using Lemma 4.5.1, (4.3.10) and the fact that $E > D_\alpha$ we obtain

$$\begin{aligned} \langle \xi_z, r\rangle(s) &\ll \frac{r(s)}{H_z(\infty)} \frac{\exp(-\psi_B(s-1))}{V(P(z))} + s^E r(s) e^{-\psi_B(s)/3} \\ &\ll s^E r(s) e^{-\psi_B(s)/3}\left(\frac{1}{s^{E-1}} + 1\right), \end{aligned}$$

again using (4.3.7) to express $e^{-\psi_B(s-1)}$ in terms of $e^{-\psi_B(s)}$. Therefore we have

$$\langle \xi_z, r\rangle(s) \ll s^E r(s) e^{-\psi_B(s)/3} \tag{4.6.23}$$

as required.

4.7 The Inductive Argument

The proof of Theorem 4.3.4 requires us to take information known about the size of $\langle \xi_z, r\rangle(s)$ and transfer it to $\xi_z(s)$. This can be done using the result from Lemma 4.7.4. The inductive argument used here is almost identical to that used in [9].

Lemma 4.7.1.

Assume that $Q(s)$ is continuous in $s > 0$ apart from simple jump discontinuities. Suppose that $c \geq 0$, $Q(s) < cs^{\kappa-\delta}$ when $0 < s \leq 1$, where $\kappa > 0$, $0 < \delta < 1$ and $\langle Q, r\rangle(s) < 0$ when $1 \leq s \leq S$. Then $Q(s) < cs^{\kappa-\delta}$ for all $0 < s \leq S$.

Proof

We first prove the case $c = 0$ by contradiction. Suppose that $Q(s) \geq 0$ for some $1 \leq s \leq S$. Denote by s_1 the infimum of all such s. Then we have $1 \leq s_1 \leq S$ and $Q(s_1+) \geq 0$.

Since we have assumed that $\langle Q, r\rangle(s) < 0$ we obtain

$$sr(s)Q(s) < \kappa \int_{s-1}^{s} r(x+1)Q(x)dx \qquad \text{when } 1 \leq s \leq S. \tag{4.7.1}$$

We now consider two cases. If Q is continuous at s_1 then taking $s = s_1$ gives $Q(s_1) < 0$, a contradiction.

On the other hand, if Q is discontinuous at s_1 then

$$s_1 r(s_1)Q(s_1+) \leq \kappa \int_{s_1-1}^{s_1} r(x+1)Q(x)dx \leq 0. \tag{4.7.2}$$

Hence $Q(s_1+) \le 0$, so that in fact $Q(s_1+) = 0$, since $Q(s_1+) \ge 0$ as already noted. Now (4.7.2) implies $Q(x) = 0$ whenever $s_1 - 1 < x < s_1$. This contradicts the definition of s_1, so that the suggested discontinuity at s_1 cannot arise. This establishes the case $c = 0$ of Lemma 4.7.1.

Note that when c is positive and $s > 1$,

$$c\int_{s-1}^{s} r(x+1)\kappa x^{\kappa-\delta}dx < cr(s)\int_{s-1}^{s} \kappa x^{\kappa-\delta}dx < cr(s)s^{\kappa-\delta+1},$$

since r decreases.

With (4.7.1) this shows

$$sr(s)(Q(s) - cs^{\kappa-\delta}) < \kappa\int_{s-1}^{s} r(x+1)(Q(x) - cx^{\kappa-\delta})dx.$$

Denote $Q^*(t) = Q(t) - ct^{\kappa-\delta}$. Then $\langle Q^*, r\rangle(s) < 0$ and Q^* satisfies the hypotheses previously expressed for Q in the case $c = 0$. The corresponding conclusion now gives $Q(s) - cs^{\kappa-\delta} < 0$ whenever $0 < s \le S$. This establishes Lemma 4.7.1.

Lemma 4.7.2.

Suppose $U^+(s) \ge 0$, and that U^+ is bounded and integrable on the finite interval $[0, s]$. Define $u(x) = U^+(x)e^{\phi(x)/3}$, where $\phi(x) = x\log x - Dx$ for some constant $D > 0$. Then

$$\frac{\kappa}{s}\int_{s-1}^{s} U^+(x)dx < \frac{1}{2}e^{-\phi(s)/3}\sup_{s-1\le x\le s} u(x),$$

if se^{-D} exceeds a suitable constant depending on κ.

Proof

Use the fact that $\phi'(x) = \log x + 1 - D$ increases with x. Consequently $\phi(s) - \phi(x) \le (s - x)\phi'(s)$ if $s - 1 \le x \le s$. Here $u(x) \ge 0$, and the expression to be estimated in Lemma 4.7.2 does not exceed

$$\begin{aligned} \frac{\kappa}{s} e^{-\phi(s)/3} \int_{s-1}^{s} e^{(\phi(s)-\phi(x))/3} u(x)dx &\le \frac{\kappa}{s} e^{-\phi(s)/3} \int_{s-1}^{s} e^{\phi(s)-\phi(x)} u(x)dx \\ &\le \frac{\kappa}{s} e^{-\phi(s)/3} \int_{s-1}^{s} e^{(s-x)\phi'(s)} dx \sup_{s-1 \le x \le s} u(x) \\ &\le \frac{\kappa e^{\phi'(s)}}{s\phi'(s)} e^{-\phi(s)/3} \sup_{s-1 \le x \le s} u(x), \end{aligned}$$

provided $s > e^{D-1}$, so that $\phi'(s) > 0$. However

$$\frac{\kappa e^{\phi'(s)}}{s\phi'(s)} = \frac{\kappa s e^{1-D}}{s(\log s + 1 - D)} = \frac{\kappa e^{1-D}}{\log s + 1 - D} < \frac{1}{2}$$

if $\log s - D$ is large enough, so Lemma 4.7.2 follows.

Lemma 4.7.3.

Suppose that $h(s)$ is bounded above on each interval $(0, S_1)$. If

$$h(s) < 1 + \frac{1}{2} \sup_{s-1 < x \le s} h(x) \qquad \text{when } 0 < s \le S,$$

then $h(s)$ is bounded above on $0 < s \le S$.

Proof

Let $h^*(s) = \sup_{0 < x < s} h(x)$. Then $h^*(s) \le 1 + \frac{1}{2} h^*(s)$. This says that $h^*(s) \le 2$, so Lemma 4.7.3 follows.

Lemma 4.7.4.

Suppose that $U_z(s)$ is continuous in $s > 0$ apart from single jump discontinuities, and that there is a constant $c > 0$ such that $U_z(s) \leq cs^{\kappa-\delta}$ when $0 < s < 1$. Assume that $B \geq 1$ is as in (4.3.6) and

$$\langle U_z, r\rangle(s) < sr(s)f(s)e^{-\psi_B(s)/3} \qquad \text{when } 1 \leq s \leq S, \tag{4.7.3}$$

where $\psi_B(s)$ is as in (4.3.7), r satisfies (4.2.8), $f(s) \geq 1$, and $f(s)$ increases as s increases. Then

$$U_z(s) \leq f(s) \exp\left(-\frac{1}{3}\psi_B(s) + O(B)\right) \qquad \text{when } 1 \leq s \leq S.$$

Proof

First consider s in the range $1 \leq s \leq c_0 B$ where c_0 is a suitable constant. For s in this range we can see from (4.3.7) that $\psi_B(s) = O(B)$ so we need to prove that $U_z(s_0) \leq f(s_0)e^{O(B)}$ for $s_0 \leq c_0 B$. Define $Q(s) = U_z(s) - c_1 f(s_0)$ when $0 < s \leq s_0$, and take $Q(s) = 0$ if $s > s_0$. We can see that by choosing $c_1 = e^{O(B)}$, for $1 < s < s_0$ we have

$$\begin{aligned} \langle Q, r\rangle(s) &= \langle U_z, r\rangle(s) - c_1 f(s_0)\langle 1, r\rangle(s) \\ &< sr(s)f(s_0)e^{O(B)} - c_1 f(s_0) \qquad \text{using (4.2.11)} \\ &< 0. \end{aligned}$$

From Lemma 4.7.1 we get $Q(s_0) < cs_0^{\kappa-\delta}$, so that

$$U_z(s_0) < cs_0^{\kappa-\delta} + c_1 f(s_0) = f(s_0)e^{O(B)}$$

since $1 \leq s_0 \leq c_0 B$, as required.

Define $U^+(x) = \max\{U_z(x), 0\}$, so that $U^+(x) \geq 0$. When $s > c_0 B$, the hypothesis of Lemma 4.7.4 gives

$$sr(s)U_z(s) < \kappa \int_{s-1}^{s} U^+(x)r(x+1)dx + sr(s)f(s)e^{-\psi_B(s)/3}.$$

Since r decreases,

$$sU_z(s) \leq \kappa \int_{s-1}^{s} U^+(x)dx + sf(s)e^{-\psi_B(s)/3}. \tag{4.7.4}$$

In Lemma 4.7.2 take $D = 1 + \log B$ so that $\phi(s) + B = \psi_B(s)$ when $s > B$ and ψ_B is as in (4.3.7). Now (4.7.4) gives

$$\frac{U_z(s)}{e^{-B/3}} < \frac{\kappa}{s} \int_{s-1}^{s} \frac{U^+(x)}{e^{-B/3}} dx + f(s)e^{-\phi(s)/3} \qquad \text{when } s > B. \tag{4.7.5}$$

Define h by $U^+(s)/e^{-B/3} = h(s)e^{-\phi(s)/3}f(s)$. Thus $h(s) \geq 0$. Then $u(x)/e^{-B/3} = h(x)f(x)$ in Lemma 4.7.2, with which (4.7.5) gives

$$h(s)f(s)e^{-\phi(s)/3} < e^{-\phi(s)/3}\left(f(s) + \frac{1}{2} \sup_{s-1 \leq x \leq s} h(x)f(x)\right),$$

provided $s > c_2 e^D$, for a suitable constant $c_2 = c_2(\kappa)$. But $f(x)$ increases with x, so this gives

$$h(s) < 1 + \frac{1}{2} \sup_{s-1 \leq x \leq s} h(x),$$

when $s > c_2 eB$.

With Lemma 4.7.3, this gives $U^+(s) \ll e^{-B/3-\phi(s)/3} f(s)$ when $s > c_0 B$, where $c_0 = c_2 e$. Since $U_z(s) \le U^+(s)$, this completes the proof of Lemma 4.7.4.

4.8 Proof of Theorem 4.3.4

The proof is of an inductive type. The use of Lemma 4.7.4 will require that our result should be already known when $0 < s \le 1$. Here we begin by quoting the fact that the case $s = 1$ of Theorem 4.3.4 was established by Song [24], namely,

$$\frac{H_z(z)}{H_z(\infty)} = C_\kappa\left(1 + O\left(\frac{1}{(\log z)^\delta}\right)\right).$$

When $0 < s < 1$ write $w = z^s$, and note that $H_z(w) = H_w(w)$. Then from the case $s = 1$ of Theorem 4.3.4 we get

$$\frac{H_z(z^s)}{H_w(\infty)} = \frac{H_w(w)}{H_w(\infty)} = C_\kappa\left(1 + O\left(\frac{1}{(\log w)^\delta}\right)\right). \tag{4.8.1}$$

Now we need to find $H_z(\infty)/H_w(\infty)$.

$$\begin{aligned}\frac{H_z(\infty)}{H_w(\infty)} &= \prod_{w \le p < z} (1 + h(p) + \sum_{k \ge 2} h(p^k)) \\ &= \exp \sum_{w \le p < z} \log(1 + h(p) + \sum_{k \ge 2} h(p^k)) \\ &= \exp \sum_{w \le p < z} \{h(p) + \sum_{k \ge 2} h(p^k) + O(h^2(p))\}\end{aligned}$$

since

$$x - \frac{1}{2}x^2 \le \log(1 + x) \le x.$$

By partial summation

$$\begin{aligned}
\sum_{w\le p<z} h(p) &= \sum_{w\le p<z} h(p)\log p \frac{1}{\log p} \\
&= \int_{w\le t<z} \frac{1}{\log t} d(\kappa \log t + \eta(t)) \\
&= \int_{w\le t<z} \frac{1}{\log t} d(\kappa \log t + O((\log t)^{1-\delta})) \\
&= \kappa \log\left(\frac{\log z}{\log w}\right) + O\left(\left|\int_{w\le t<z} \frac{1}{t(\log t)^{1+\delta}} dt\right|\right).
\end{aligned}$$

The integral in the O-term is

$$\begin{aligned}
\int_{w\le t<z} \frac{1}{t(\log t)^{1+\delta}} dt &= \left[\frac{\log t}{(\log t)^{1+\delta}}\right]_w^z + \int_w^z \frac{1+\delta}{t(\log t)^{1+\delta}} dt \\
&\ll \frac{1}{(\log w)^{\delta}}
\end{aligned}$$

so that

$$\sum_{w\le p<z} h(p) = \kappa \log\left(\frac{\log z}{\log w}\right) + O\left(\frac{1}{(\log w)^{\delta}}\right).$$

From (4.3.3) we get

$$\begin{aligned}
\sum_{w\le p<z} \sum_{k\ge 2} h(p^k) &= \sum_{\substack{w\le p<z \\ k\ge 2}} h(p^k)\log p^k \frac{1}{\log p^k} \\
&\le \frac{1}{\log w^2} \sum_{\substack{p,k\ge 2 \\ p<z}} h(p^k)\log p^k \\
&\le \frac{b}{2\log w}.
\end{aligned}$$

Resulting from equation (4.4.2) we obtain

$$\begin{aligned}\sum_{w\le p<z} h^2(p) &= \sum_{w\le p<z} h(p)\log p\frac{h(p)}{\log p}\\ &\ll \sum_{w\le p<z} h(p)\log p\frac{1}{(\log p)^{1+\delta}}. \qquad (4.8.2)\end{aligned}$$

Using (4.3.1), we have by partial summation

$$\begin{aligned}\sum_{w\le p<z} h(p)\log p\frac{1}{(\log p)^{1+\delta}} &= \frac{1}{(\log z)^{1+\delta}}\Big(\kappa\log z+\eta(z)\Big)-\frac{1}{(\log w)^{1+\delta}}\Big(\kappa\log w+\eta(w)\Big)\\ &\quad+\kappa(1+\delta)\int_w^z\frac{dt}{t(\log t)^{1+\delta}}+O\left(\frac{dt}{t(\log t)^{1+2\delta}}\right)\\ &= \frac{\kappa}{(\log z)^{\delta}}-\frac{\kappa}{(\log w)^{\delta}}-\frac{\kappa(1+\delta)}{\delta}\left[\frac{1}{(\log t)^{\delta}}\right]_w^z\\ &\quad+O\left(\frac{1}{(\log w)^{2\delta}}\right).\end{aligned}$$

Thus we have

$$\begin{aligned}\sum_{w\le p<z} h(p)\log p\frac{1}{(\log p)^{1+\delta}} &= \frac{\kappa}{(\log z)^{\delta}}-\frac{\kappa}{(\log w)^{\delta}}-\frac{\kappa(1+\delta)}{\delta}\left(\frac{1}{(\log z)^{\delta}}-\frac{1}{(\log w)^{\delta}}\right)\\ &\quad+O\left(\frac{1}{(\log w)^{2\delta}}\right)\\ &= \frac{\kappa}{\delta}\frac{1}{(\log w)^{\delta}}-\frac{\kappa}{\delta}\frac{1}{(\log z)^{\delta}}+O\left(\frac{1}{(\log w)^{2\delta}}\right)\\ &\ll \frac{1}{(\log w)^{\delta}}.\end{aligned}$$

Substituting this into (4.8.2) we obtain

$$\begin{aligned}\sum_{w\le p<z} h^2(p) &\ll \frac{1}{(\log w)^\delta}\\ &\ll \frac{1}{(s\log z)^\delta} \qquad \text{since } w=z^s. \end{aligned} \tag{4.8.3}$$

Therefore we now have

$$\begin{aligned}\frac{H_z(\infty)}{H_w(\infty)} &= \exp\left\{\kappa\log\frac{1}{s}+O\bigg(\frac{1}{(s\log z)^\delta}\bigg)\right\}\\ &= \frac{1}{s^\kappa}\exp\left(O\bigg(\frac{1}{(s\log z)^\delta}\bigg)\right)\end{aligned}$$

so that

$$\frac{H_{z^s}(\infty)}{H_z(\infty)} = s^\kappa\exp\left(\frac{-c_3}{(s\log z)^\delta}\right). \tag{4.8.4}$$

From (4.8.1) and (4.8.4) it follows that

$$\begin{aligned}\frac{H_z(z^s)}{H_z(\infty)} &= C_\kappa s^\kappa\left(1+O\bigg(\frac{1}{s^\delta(\log z)^\delta}\bigg)\right)\left(1+O\bigg(\frac{1}{s^\delta(\log z)^\delta}\bigg)\right)\\ &= \sigma(s)+O\bigg(\frac{s^{\kappa-\delta}}{(\log z)^\delta}\bigg) \qquad \text{when } 0<s<1.\end{aligned}$$

Thus we have found $\xi_z(s)\ll s^{\kappa-\delta}$ when $0<s\le 1$. We now see that applying Lemma 4.7.4 with $U_z(s)=\pm\xi_z(s)$, $f(s)=s^{E-1}$ and referring to Lemma 4.6.2 gives

$$\xi_z(s)\ll s^{E-1}\exp\bigg(-\frac{1}{3}\psi_B(s)+O(B)\bigg) \qquad \text{for } s<Bz^\alpha.$$

Consequently we obtain

$$\frac{H_z(z^s)}{H_z(\infty)} = \sigma(s) + O\left(\frac{s^{E-1}e^{-\psi_B(s)/3+O(B)}}{(\log z)^\delta}\right) \tag{4.8.5}$$

which is the statement claimed in Theorem 4.3.4.

Chapter 5

Weighted Sieves

5.1 Introduction

Another way of utilising Selberg's ideas involves attaching a logarithmic weight to the prime factors of the number a in a suitable sequence $\mathcal{A}$.

$$m(a) = \Big(\sum_{\substack{d|a \\ d|P(z)}} \lambda_1(d) \Big) \Big(\sum_{\substack{d|a \\ d|P(z)}} \lambda(d) \Big)^2 \tag{5.1.1}$$

where λ_1 is any suitable sifting function and λ is subject to the constraint $\lambda(1) = 1$. By attaching the weight we are allowing numbers with a small number of prime factors to survive the sifting process as opposed to numbers without small prime factors.

Selberg [22] used the sifting function defined by

$$\lambda_1(1) = 1, \qquad \lambda_1(p) = -1 \qquad \text{when } p|P(z) \tag{5.1.2}$$

and $\lambda_1(d) = 0$ otherwise to obtain a positive lower bound for the sum $W^-(D, P(z))$, where $D = z^s$, valid for

$$s > 2\kappa + \frac{19}{36}. \tag{5.1.3}$$

The sum $W^-(D, P(z))$ is described in (5.1.9). In [8], Greaves used a less precise asymptotic analysis to provide a slightly weaker inequality than (5.1.3), namely

$$s > 2\kappa + c\sqrt{\kappa \log \kappa} \tag{5.1.4}$$

where $c > 2\sqrt{2}$ and $\kappa > \kappa_0(c)$, for some $\kappa_0(c)$ depending only upon c.

Here we use a similar method to that of Greaves to do a corresponding analysis when λ_1 is the sifting function defined by

$$\lambda_1(1) = 1 \qquad \text{and} \qquad \lambda_1(p) = -\left(1 - \frac{\log p}{\log z}\right) \qquad \text{if } 2 \le p < z, \tag{5.1.5}$$

with $\lambda_1(d) = 0$ otherwise.

This choice of λ_1 appears to have been first used by Ankeny and Onishi [1], and it has also been used by Richert [20]. We, however, use Richert's weight in its simplest form, with $U = 1$ and $V = 0$.

This gives a new sifting function λ^- for which

$$m(a) = \sum_{d|a} \lambda^-(d), \qquad \lambda^-(d) = \sum_{[d_1,d_2,d_3]=d} \lambda_1(d_1)\lambda(d_2)\lambda(d_3)$$

subject to the constraint

$$\lambda^-(1) = 1. \tag{5.1.6}$$

$\lambda(d)$ is given by

$$\mu(h)g(h)y(h) = \sum_{d \equiv 0 (\bmod h)} \frac{\lambda(d)\rho(d)}{d} \tag{5.1.7}$$

for

$$y(h) = \begin{cases} 1 & \text{if } h < \xi/z \\ \dfrac{\log \xi/h}{\log z} & \text{if } \xi/z \leq h < \xi \\ 0 & \text{if } h \geq \xi. \end{cases} \tag{5.1.8}$$

Remark 5.1.1.

Note that for the constraint (5.1.6) to be satisfied we require $\lambda(1) = 1$. If the definition of $\lambda(d)$ given in (5.1.7) does not satisfy this requirement then we would need to scale the function $y(h)$ accordingly.

We need to estimate

$$W^{-}(D, P(z)) = \sum_{d|P(z)} \frac{\lambda^{-}(d)\rho(d)}{d} \tag{5.1.9}$$

where λ^{-} is a sifting function of level D.

Section 5.2 restates some notation and results from Chapters 2 and 3 which will be useful in later sections. In Section 5.3 we describe a lower bound for the number of $a \in \mathcal{A}$ with the weight $w(q_a)$ attached to the smallest prime factor, q_a, of the number a. The first stage of the proof of Theorem 5.3.3 appears in Section 5.4

where we obtain a positive lower bound for the sum $W^{-}(D, P(z))$. This sum occurs as the main term in our estimation of the sum

$$\sum_{\substack{a \in \mathcal{A} \\ \nu(D,a) \leq R}} w(q_a).$$

The results claimed in Section 5.3 are then proved in Section 5.5 using the lower bound for $W^{-}(D, P(z))$ from Section 5.4.

5.2 Notation and Preliminary Lemmas

In this section we introduce some notation and results which will be useful in subsequent sections. First of all, we reintroduce the following notation which has already appeared in Chapters 2 and 3.

Define $B(z)$ and $\psi_B(v)$ to satisfy, respectively

$$B(z) \leq B; \qquad \text{where} \quad B(z) = \frac{1}{\log z} \sum_{p<z} \frac{\rho(p) \log p}{p} \tag{5.2.1}$$

and

$$\begin{aligned} \psi_B(v) &= \max\{0, v \log \frac{v}{B} - v + B\} \\ &= \int_{B<t<v} \log \frac{t}{B}\, dt. \end{aligned} \tag{5.2.2}$$

Hypothesis 5.2.1.

The function ρ satisfies

$$\sum_{p<z} \frac{\rho(p)}{p} \log p < \kappa \log z + O(1). \tag{5.2.3}$$

We remind the reader that the function ρ^* defined in Definition 3.2.1 is given by

$$\rho^*(p) = p - \rho(p). \tag{5.2.4}$$

Because ρ^* is multiplicative this defines $\rho^*(d)$ for squarefree d. Then the function g is given by

$$g(n) = \begin{cases} \rho(n)/\rho^*(n) & \text{if } n \text{ is squarefree;} \\ 0 & \text{otherwise.} \end{cases} \tag{5.2.5}$$

The following result was proven in Chapter 3 and appears as Lemma 3.4.2.

Lemma 5.2.2.

Suppose that $B(z)$ as defined in (5.2.1) holds, and denote

$$I_z(x) = G_z(\infty) - G_z(x) = \sum_{\substack{d \geq x \\ d|P(z)}} g(d), \tag{5.2.6}$$

and write $x = z^v$. Then for each $v > 0$

$$I_z(x) \leq \frac{\exp(-\psi_B(v))}{V(P(z))}, \tag{5.2.7}$$

where ψ_B is as in (5.2.2).

5.3 A Lower Bound for $\sum w(q_a)$

In this section we describe the results obtained by combining Richert's weight with Selberg's $\Lambda^2\Lambda^-$ sieve. This result is described in Theorem 5.3.3.

Lemma 5.3.1.

For λ defined by (5.1.7) *we have*

$$|\lambda(d)| \leq |\lambda(1)|. \tag{5.3.1}$$

Proof

From (5.1.7) we have

$$\mu(h)g(h)y(h) = \sum_{d\equiv 0(\text{mod } h)} \frac{\lambda(d)\rho(d)}{d}$$

which gives

$$\lambda(d) = \frac{d\mu(d)}{\rho(d)} \sum_{\substack{h\equiv 0(\text{mod d})\\ h<\xi}} g(h)y(h). \tag{5.3.2}$$

We can see from (5.3.2)

$$\lambda(1) = \sum_{h<\xi} g(h)y(h). \tag{5.3.3}$$

For given d, a squarefree number h may be written uniquely as $h = fn$, where $f|d$ and $(n,d) = 1$. Hence

$$\begin{aligned}
\sum_{h<\xi} g(h)y(h) &= \sum_{\substack{(n,d)=1\\ f|d;fn<\xi}} g(f)g(n)y(fn) \\
&= \sum_{f|d} g(f) \sum_{\substack{(n,d)=1\\ n<\xi/f}} g(n)y(fn) \\
&\geq \sum_{f|d} g(f) \sum_{\substack{(n,d)=1\\ n<\xi/f}} g(n)y(fn) \qquad \text{since } f < d.
\end{aligned}$$

Thus we have

$$|\lambda(1)| = \sum_{h<\xi} g(h)y(h) \geq \sum_{f|d} g(f) \sum_{\substack{(n,d)=1 \\ n<\xi/f}} g(n)y(fn). \tag{5.3.4}$$

Now since y is a decreasing function, and $f < d$, we have $y(fn) < y(dn)$ so that

$$|\lambda(1)| \geq \sum_{f|d} g(f) \sum_{\substack{(n,d)=1 \\ n<\xi/f}} g(n)y(nd).$$

Now

$$\sum_{f|d} g(f) = \prod_{p|d} \left(1 + \frac{\rho(p)}{p-\rho(p)}\right) = \frac{d}{\rho^*(d)}$$

so that

$$\begin{aligned}
|\lambda(1)| &\geq \frac{d}{\rho^*(d)} \sum_{\substack{(n,d)=1 \\ n<\xi/d}} g(n)y(nd) \\
&= \frac{d}{\rho^*(d)} \sum_{\substack{h\equiv 0 (\text{mod } d) \\ h<\xi}} g(h)y(h)\frac{\rho^*(d)}{\rho(d)} \\
&= \frac{d}{\rho(d)} \sum_{\substack{h\equiv 0 (\text{mod } d) \\ h<\xi}} g(h)y(h) \\
&= |\lambda(d)|
\end{aligned}$$

as required.

We introduce the convenient notation

$$w(p) = 1 + \lambda_1(p) = \frac{\log p}{\log z}. \tag{5.3.5}$$

Lemma 5.3.2.

Suppose $m(a)$ is as in (5.1.1), with λ_1 as in (5.1.5). Define

$$\nu(D, a) = \sum_{\substack{p<D \\ p|a}} 1 + \sum_{\substack{p,\alpha \\ p^\alpha|a; p\geq D}} 1. \tag{5.3.6}$$

Suppose g is chosen so that $a \leq D^g$, and that $g \leq R/s$. Let q_a denote the smallest prime factor of a. Suppose also $m(a) > 0$. Then $\nu(D, a) \leq R$ and $m(a) \leq a^{2\epsilon} w(q_a)$, for some $\epsilon \to 0$.

Proof

When $m(a) > 0$ we find

$$0 < \left(1 - \sum_{\substack{p<z \\ p|a}} \left(1 - \frac{\log p}{\log z}\right)\right) \left(\sum_{\substack{d|a \\ d\leq\xi}} \lambda(d)\right)^2 \tag{5.3.7}$$

Now since

$$\left(\sum_{\substack{d|a \\ d\leq\xi}} \lambda(d)\right)^2 > 0 \tag{5.3.8}$$

this leads to

$$0 < \left(1 - \sum_{\substack{p<z \\ p|a}} \left(1 - \frac{\log p}{\log z}\right)\right). \tag{5.3.9}$$

Since $1 - \log p / \log z \leq 0$ for $p \geq z$, we have

$$\begin{aligned} 0 &< 1 - \sum_{\substack{p<z \\ p|a}} \left(1 - \frac{\log p}{\log z}\right) - \sum_{\substack{z \leq p < D \\ p|a}} \left(1 - \frac{\log p}{\log z}\right) \\ 0 &< 1 - \sum_{\substack{p<D \\ p|a}} \left(1 - \frac{\log p}{\log z}\right). \end{aligned}$$

If we extend this sum over all prime factors of a we obtain

$$\begin{aligned} 0 &< 1 - \sum_{\substack{p<D \\ p|a}} \left(1 - \frac{\log p}{\log z}\right) - \sum_{\substack{p,\alpha \\ p^\alpha | a; p \geq D}} \left(1 - \frac{\log p}{\log z}\right) \\ 0 &< 1 - \left(\sum_{\substack{p<D \\ p|a}} 1 + \sum_{\substack{p,\alpha \\ p^\alpha | a; p \geq D}} 1\right) + \left(\sum_{\substack{p<D \\ p|a}} \frac{\log p}{\log z} + \sum_{\substack{p,\alpha \\ p^\alpha | a; p \geq D}} \frac{\log p}{\log z}\right). \end{aligned}$$

From (5.3.6) and since $a \leq D^g$ and $D = z^s$,

$$\begin{aligned} 0 &< 1 - \nu(D, a) + \frac{\log a}{\log z} \\ \nu(D, a) &< sg + 1 \\ \nu(D, a) &< R + 1 \qquad \text{since } g = R/s \\ \nu(D, a) &\leq R. \end{aligned}$$

Now we require to show that $m(a) \leq a^{2\epsilon} w(q_a)$.

$$\begin{aligned} m(a) &= \left(1-\sum_{\substack{p<z\\ p|a}}\left(1-\frac{\log p}{\log z}\right)\right)\left(\sum_{\substack{d|a\\ d\leq\xi}}\lambda(d)\right)^2 \\ &= \left(1-1+\frac{\log q_a}{\log z}+\sum_{\substack{q_a<p<z\\ p|a}}\left(\frac{\log p}{\log z}-1\right)\right)\left(\sum_{\substack{d|a\\ d\leq\xi}}\lambda(d)\right)^2 \\ &\leq \left(\frac{\log q_a}{\log z}\right)\left(\sum_{\substack{d|a\\ d\leq\xi}}\lambda(d)\right)^2 \end{aligned}$$

since $\log p/\log z - 1 < 0$. Now, we have from Lemma 5.3.1 that $|\lambda(d)| \leq |\lambda(1)| \leq 1$ which gives

$$\begin{aligned} m(a) &\leq \left(\frac{\log q_a}{\log z}\right)\left(\sum_{d|a}1\right)^2 \\ &\leq \left(\frac{\log q_a}{\log z}\right)a^{2\epsilon} \end{aligned}$$

for some $\epsilon \to 0$.

The details of the last step here can be found in Appendix B. This gives

$$m(a) \leq a^{2\epsilon} w(q_a) \tag{5.3.10}$$

as required.

Theorem 5.3.3.

Suppose that the function ρ *satisfies* (5.2.3). *Write* $z = D^{1/s}$, *and assume*

$$s > 2\kappa - 2\kappa A + c\sqrt{\kappa \log \kappa}, \tag{5.3.11}$$

where c *is a constant satisfying* $c > 2\sqrt{2}$ *and*

$$A \to \frac{\sqrt{2}\log 2}{\sqrt{\kappa \log \kappa}} \qquad \text{as } \kappa \to \infty. \tag{5.3.12}$$

Then for some $\epsilon > 0$ *we have*

$$\sum_{\substack{a \in \mathcal{A} \\ \nu(D,a) \le R}} w(q_a) \ge \frac{X z^{-2sg\epsilon}}{V(P(z))}\left(1 - \left(1 + \frac{B(z)}{4}\right) e^{-\psi_B(v)}\right) - z^{-2sg\epsilon} \sum_{d \le D} |r_{\mathcal{A}}(d)|. \tag{5.3.13}$$

Remark 5.3.4.

Note that $\epsilon \to 0$ as $D^g \to \infty$.

Remark 5.3.5.

Note that Theorem 5.3.3 counts the numbers a with the weight $w(q_a)$ attached to the smallest prime factor, q_a.

5.4 An Identity for the Main Term

To prove the result claimed in Theorem 5.3.3 we need to find a positive lower bound for the sum $W^-(D, P(z))$ given by (5.1.9), which appears as the main term in our estimate of

$$\sum_{\substack{a\in\mathcal{A}\\ \nu(D,a)\leq R}} w(q_a).$$

We obtain the following lower bound by using an analogous method to that appearing in Section 7.3 of [8].

Theorem 5.4.1.

Suppose that the function ρ satisfies (5.2.3). *Write $z = D^{1/s}$, and assume*

$$s >_{\backslash} (1-A)2\kappa + c\sqrt{\kappa\log\kappa}, \tag{5.4.1}$$

where c is a constant satisfying $c > 2\sqrt{2}$ and A satisfies

$$A = \sqrt{2}\left(\sqrt{\frac{\log\kappa}{\kappa}} - \sqrt{\frac{\log(1+\kappa/4)}{\kappa}}\right). \tag{5.4.2}$$

Then there exists a sieve of level D for which

$$\begin{aligned} W^-(D,P(z)) &> \frac{1}{V(P(z))}\left(1-\left(1+\frac{B(z)}{4}\right)e^{-\psi_B(v)}\right)\\ &> 0, \end{aligned}$$

provided $\kappa > \kappa_0(c)$ for a certain $\kappa_0(c)$ depending only upon c, where $W^-(D, P(z))$ is given by (5.1.9.)

The following lemma taken from [8] is the starting point for our proof.

Lemma 5.4.2.

Denote

$$W^{-}(D,P(z)) = \sum_{d_1}\sum_{d_2}\sum_{d_3} \lambda_1(d_1)\lambda(d_2)\lambda(d_3)\frac{\rho([d_1,d_2,d_3])}{[d_1,d_2,d_3]}$$

where $[d_1,d_2,d_3]$ *denotes the least common multiple of* d_1, d_2 *and* d_3. *Let* ρ^* *and* $g = \rho/\rho^*$ *be as in* (5.2.4) and (5.2.5) *and define* $x(h)$, $y(h)$ *so that*

$$\mu(h)g(h)y(h) = x(h) = \sum_{d\equiv 0(\mathrm{mod} h)} \frac{\lambda(d)\rho(d)}{d}. \tag{5.4.3}$$

Then

$$W^{-}(D,P(z)) = \sum_{d_1} \frac{\lambda_1(d_1)\rho(d_1)}{d_1} \sum_{(h,d_1)=1} g(h)\Big(\sum_{k|d_1}\mu(k)y(hk)\Big)^2.$$

Proof

Begin by writing

$$W^{-}(D,P(z)) = \sum_{d_1} \frac{\lambda_1(d_1)\rho(d_1)}{d_1}\sum_{d_2}\sum_{d_3}\frac{\lambda(d_2)\lambda(d_3)\rho_{d_1}([d_2,d_3])}{[d_2,d_3]} \tag{5.4.4}$$

where $\rho_{d_1}(d)$ is defined for squarefree $d = [d_2,d_3]$ so that

$$\frac{\rho(d_1)}{d_1}\frac{\rho_{d_1}(d)}{d} = \frac{\rho([d_1,d])}{[d_1,d]} = \frac{\rho([d_1,d])(d_1,d)}{d_1 d}.$$

When $\rho(d_1) \neq 0$ this implies

$$\rho_{d_1}(d) = \frac{\rho([d_1, d])(d_1, d)}{\rho(d_1)} = \prod_{\substack{p|d_1 \\ p|d}} p \prod_{\substack{p\nmid d_1 \\ p|d}} \rho(p),$$

which is satisfied if ρ_{d_1} is the multiplicative function for which

$$\rho_{d_1}(p) = \begin{cases} p & \text{if } p|d_1; \\ \rho(p) & \text{if } p \nmid d_1. \end{cases} \tag{5.4.5}$$

Thus the "conjugate" function $\rho^*_{d_1}$ satisfies $\rho^*_{d_1}(p) = p - \rho_{d_1}(p) = 0$ if $p|d_1$, so that $\rho^*_{d_1}(h) \neq 0$ only when $(h, d_1) = 1$. Furthermore

$$\frac{\rho^*_{d_1}(h)}{\rho_{d_1}(h)} = \begin{cases} 0 & \text{if } (h, d_1) > 1; \\ \rho^*(h)/\rho(h) & \text{if } (h, d_1) = 1 \text{ and } \rho(h) \neq 0. \end{cases} \tag{5.4.6}$$

Note also from (5.4.5) that when $\rho(d) \neq 0$

$$\frac{\rho_{d_1}(d)}{\rho(d)} = \frac{(d_1, d)}{\rho((d_1, d))} = \sum_{k|(d_1,d)} \frac{\rho^*(k)}{\rho(k)}. \tag{5.4.7}$$

Lemma 3.2.3 and (5.4.6) now show that the inner sum in (5.4.4) is

$$\sum_{d_2} \sum_{d_3} \frac{\lambda(d_2)\lambda(d_3)\rho_{d_1}([d_2, d_3])}{[d_2, d_3]} = \sum_{(h,d_1)=1} \frac{x^2_{d_1}(h)}{g(h)}, \tag{5.4.8}$$

where

$$x_{d_1}(h) = \sum_{d\equiv 0(\bmod h)} \frac{\lambda(d)\rho_{d_1}(d)}{d} = \sum_{d\equiv 0(\bmod h)} \frac{\lambda(d)\rho(d)}{d} \sum_{k|(d_1,d)} \frac{\rho^*(k)}{\rho(k)},$$

the last step following from (5.4.7). Thus, when $(h, d_1) = 1$ as in (5.4.8),

$$\begin{aligned} x_{d_1}(h) &= \sum_{k|d_1} \frac{\rho^*(k)}{\rho(k)} \sum_{d\equiv 0(\mathrm{mod}\, hk)} \frac{\lambda(d)\rho(d)}{d} \\ &= \sum_{k|d_1} \frac{\rho^*(k)}{\rho(k)} x(hk) \\ &= \frac{\mu(h)\rho(h)}{\rho^*(h)} \sum_{k|d_1} \mu(k) y(hk) \\ &= \mu(h) g(h) \sum_{k|d_1} \mu(k) y(hk), \end{aligned}$$

where y is as in (5.4.3). With (5.4.8) this gives

$$\sum_{d_2}\sum_{d_3} \frac{\lambda(d_2)\lambda(d_3)\rho_{d_1}([d_2,d_3])}{[d_2,d_3]} = \sum_{\substack{(h,d_1)=1 \\ \rho(h)\neq 0}} g(h) \Big(\sum_{k|d_1} \mu(k) y(hk) \Big)^2, \tag{5.4.9}$$

so that (5.4.4) gives the identity for $W^-(\lambda)$ enunciated in Lemma 5.4.2.

Lemma 5.4.3. /

Let λ_1 *be defined as in* (5.1.5). *Then using the result from Lemma* 5.4.2 *we have*

$$W^-(D, P(z)) = \Sigma_1 + \Sigma_2, \tag{5.4.10}$$

where

$$\Sigma_1 = \sum_h g(h) y^2(h) \tag{5.4.11}$$

$$\Sigma_2 = -\sum_{p|P(z)} \Big(1 - \frac{\log p}{\log z}\Big) \frac{\rho(p)}{p} \sum_{(h,p)=1} g(h) \Big(y(h) - y(ph) \Big)^2. \tag{5.4.12}$$

Proof

$$
\begin{aligned}
W^{-}(D, P(z)) &= \sum_{d_1} \frac{\lambda_1(d_1)\rho(d_1)}{d_1} \sum_{(h,d)=1} g(h) \Big(\sum_{k|d_1} \mu(k) y(hk) \Big)^2 \\
&= \sum_{h} g(h) y^2(h) \lambda_1(1) \\
&\quad + \sum_{p|P(z)} \frac{\rho(p)}{p} \lambda_1(p) \sum_{(h,p)=1} g(h) \Big(y(h) - y(ph) \Big)^2 .
\end{aligned}
$$

From (5.1.5) we have that $\lambda_1(1) = 1$ and $\lambda_1(p) = -(1 - \log p / \log z)$. This gives

$$
\begin{aligned}
W^{-}(D, P(z)) &= \sum_{h} g(h) y^2(h) \\
&\quad - \sum_{p|P(z)} \left(1 - \frac{\log p}{\log z}\right) \frac{\rho(p)}{p} \sum_{(h,p)=1} g(h) \Big(y(h) - y(ph) \Big)^2 .
\end{aligned}
$$

We define the function $y(h)$ introduced in Lemma 5.4.2 to be

$$
y(h) = \begin{cases} 1 & \text{if } h < \xi/z \\ \dfrac{\log \xi/h}{\log z} & \text{if } \xi/z \le h < \xi \\ 0 & \text{if } h \ge \xi. \end{cases} \tag{5.4.13}
$$

The proof of the following lemma is taken from [8] and will be used in the proof of Theorem 5.4.1.

Lemma 5.4.4.

The function $y(h)$ described above satisfies

$$y(h) - y(ph) = 0 \qquad \text{if } p < z \text{ and } h < \xi/z^2$$

and

$$0 \le y(h) - y(ph) \le \frac{\log p}{\log z} \qquad \text{if } p < z \text{ and } \xi/z^2 \le h < \xi.$$

Proof

Observe from (5.4.13) that

$$y(h) - y(ph) = 0 \qquad \text{if } p < z \text{ and } h < \xi/z^2. \tag{5.4.14}$$

If $p < z$ and $\xi/z^2 \le h < \xi$ then

$$0 \le y(h) - y(ph) \le \frac{\log p}{\log z}, \tag{5.4.15}$$

for $y(h) - y(ph)$, if not zero, satisfies

$$y(h) - y(ph) = \begin{cases} 1 - \dfrac{\log(\xi/ph)}{\log z} & \text{if } h < \xi/z < ph \\ \dfrac{\log(\xi/h) - \log(\xi/ph)}{\log z} & \text{if } \xi/z \le h < \xi. \end{cases} \tag{5.4.16}$$

Proof of Theorem 5.4.1

The construction used for Theorem 5.4.1 employs the very simple $\lambda_1(d_1)$ given in (5.1.5), so that $\lambda_1(1) = 1$ and $\lambda_1(p) = -(1 - \log p/\log z)$ for primes $p < z$. For the major input $\lambda(d)$ we will require

$$\lambda(d) \neq 0 \quad \text{only if} \quad d < \xi = \sqrt{D/z}, d|P(z). \tag{5.4.17}$$

Then the sifting function

$$\lambda^-(d) = \sum_{[d_1,d_2,d_3]=d} \lambda_1(d_1)\lambda(d_2)\lambda(d_3) \tag{5.4.18}$$

is of level D, as required by Theorem 5.4.1.

Now, using only that $y(h) = 1$ when $h \leq \xi/z^2$

$$\begin{aligned}
\Sigma_1 &:= \sum_h g(h)y^2(h) \\
&\geq \sum_{\substack{h<\xi/z^2 \\ h|P(z)}} g(h)y^2(h) \\
&\geq \sum_{\substack{h<\xi/z^2 \\ h|P(z)}} g(h) \\
&\geq \left(\frac{1}{V(P(z))} - I_z\left(\frac{\xi}{z^2}\right)\right).
\end{aligned}$$

We now look at

$$\Sigma_2 = -\sum_{p|P(z)} \frac{\rho(p)}{p}\left(1 - \frac{\log p}{\log z}\right) \sum_{(h,p)=1} g(h)\Big(y(h) - y(ph)\Big)^2.$$

We want to find a lower bound for Σ_2 and we shall do this by obtaining an upper bound for $-\Sigma_2$.

$$\begin{aligned}
-\Sigma_2 &= \sum_{p|P(z)} \frac{\rho(p)}{p}\left(1-\frac{\log p}{\log z}\right) \sum_{\substack{(h,p)=1\\ h<\xi, h|P(z)}} g(h)\Big(y(h)-y(ph)\Big)^2 \\
&\le \sum_{p|P(z)} \frac{\rho(p)}{p}\left(1-\frac{\log p}{\log z}\right)\left(\frac{\log p}{\log z}\right)^2 \sum_{\substack{\xi/z^2\le h<\xi\\ h|P(z)}} g(h) \qquad \text{using Lemma 5.4.4} \\
&< \sum_{p|P(z)} \frac{\rho(p)\log p}{p\log z}\left(1-\frac{\log p}{\log z}\right)\left(\frac{\log p}{\log z}\right) I_z\left(\frac{\xi}{z^2}\right) \qquad (5.4.19)
\end{aligned}$$

where I_z is as in (5.2.6). Since $\log p/\log z < 1$ we have that

$$\left(1-\frac{\log p}{\log z}\right)\left(\frac{\log p}{\log z}\right) < \frac{1}{4} \qquad (5.4.20)$$

which gives

$$\begin{aligned}
-\Sigma_2 &< \frac{1}{4}\sum_{p|P(z)} \frac{\rho(p)\log p}{p\log z} I_z\left(\frac{\xi}{z^2}\right) \\
&< \frac{B(z)}{4} I_z\left(\frac{\xi}{z^2}\right) \qquad (5.4.21)
\end{aligned}$$

$$(5.4.22)$$

where $B(z)$ is as in (5.2.1).

We note that this bound is rather approximate and can be improved (see Appendix A). We use the weaker bound here in an effort to keep the method as simple as possible. Thus we have

$$\Sigma_2 > -\frac{B(z)}{4} I_z\left(\frac{\xi}{z^2}\right).$$

Thus we obtain

$$\begin{aligned} W^-(D, P(z)) &> \left(\frac{1}{V(P(z))} - I_z\left(\frac{\xi}{z^2}\right)\right) - \frac{B(z)}{4} I_z\left(\frac{\xi}{z^2}\right) \\ &> \frac{1}{V(P(z))} - I_z\left(\frac{\xi}{z^2}\right)\left(1 + \frac{B(z)}{4}\right). \end{aligned}$$

From Lemma 5.2.2 we obtain

$$W^-(D, P(z)) > \frac{1}{V(P(z))}\left(1 - \left(1 + \frac{B(z)}{4}\right)e^{-\psi_B(v)}\right) \tag{5.4.23}$$

where

$$v = \frac{\log \xi / z^2}{\log z} = \frac{s-5}{2} \qquad \text{since } \xi \text{ is as in (5.4.17).} \tag{5.4.24}$$

Since $1/V(P(z))$ is always positive, $W^-(D, P(z)) > 0$ whenever

$$\left(1 + \frac{B(z)}{4}\right)e^{-\psi_B(v)} < 1. \tag{5.4.25}$$

We will prove (5.4.25) by induction on v. Firstly we assume that $v < 2\kappa$. By (5.2.3) we have

$$B(z) \le \kappa + O\left(\frac{1}{\log z}\right) \tag{5.4.26}$$

and since $s > (1 - A)2\kappa + c\sqrt{\kappa \log \kappa}$ then $v > B(z)$ as soon as κ is large enough. Note from (5.2.2) when $v > B$ that

$$\frac{\partial \psi_B(v)}{\partial B} = 1 - \frac{v}{B}.$$

If $B > \kappa$, then

$$\frac{\psi_B(v) - \psi_\kappa(v)}{B - \kappa} \geq 1 - \frac{v}{\kappa} \geq -1.$$

Thus from (5.4.26) we have

$$\psi_B(v) \geq \psi_\kappa(v) + O\left(\frac{1}{\log z}\right).$$

If $B \leq \kappa$ then $\psi_B(v) \geq \psi_\kappa(v)$.

Thus we obtain

$$\begin{aligned} \left(1 + \frac{B(z)}{4}\right)e^{-\psi_B(v)} &\leq \left(1 + \frac{\kappa}{4} + O\left(\frac{1}{\log z}\right)\right)e^{-\psi_\kappa(v)} \\ &\leq \left(1 + \frac{\kappa}{4}\right)e^{-\psi_\kappa(v)}\left(1 + O\left(\frac{1}{\log z}\right)\right). \end{aligned}$$

Since $\psi_B(v)$ is as defined in (5.2.2) this gives

$$\left(1 + \frac{B(z)}{4}\right)e^{-\psi_B(v)} \leq \left(1 + \frac{\kappa}{4}\right)\exp\left(-\kappa\int_1^{v/\kappa}\log x\,dx\right)\left(1 + O\left(\frac{1}{\log z}\right)\right). \quad (5.4.27)$$

From (5.4.24) we have

$$\frac{v}{\kappa} = \frac{s-5}{2\kappa} = 1 + h. \quad (5.4.28)$$

Then

$$\begin{aligned} \int_1^{1+h}\log x\,dx &= (1+h)\log(1+h) - h \\ &= (1+h)\left(h - \frac{h^2}{2} + \frac{h^3}{3} - \cdots\right) - h \\ &= h - \frac{h^2}{2} + \frac{h^3}{3} - \cdots + h^2 - \frac{h^3}{2} + \cdots - h \\ &= \frac{1}{2}h^2 + O(h^3). \end{aligned}$$

The last step follows since

$$1 + h = \frac{v}{\kappa} < \frac{2\kappa}{\kappa} = 2.$$

In Theorem 5.4.1 we have

$$\begin{aligned} s &> 2\kappa - 2\sqrt{2}\kappa\left(\sqrt{\frac{\log\kappa}{\kappa}} - \sqrt{\frac{\log(1+\kappa/4)}{\kappa}}\right) + c\sqrt{\kappa\log\kappa} \\ 1+h &> 1 - \sqrt{2}\left(\sqrt{\frac{\log\kappa}{\kappa}} - \sqrt{\frac{\log(1+\kappa/4)}{\kappa}}\right) + \frac{c}{2}\sqrt{\frac{\log\kappa}{\kappa}} + O\left(\frac{1}{\kappa}\right) \\ h &> \frac{c}{2}\sqrt{\frac{\log\kappa}{\kappa}} - \sqrt{2}\left(\sqrt{\frac{\log\kappa}{\kappa}} - \sqrt{\frac{\log(1+\kappa/4)}{\kappa}}\right) + O\left(\frac{1}{\kappa}\right). \end{aligned}$$

We observe that h is positive provided κ is large enough and

$$\sqrt{2}\left(\sqrt{\frac{\log\kappa}{\kappa}} - \sqrt{\frac{\log(1+\kappa/4)}{\kappa}}\right) < \frac{c}{2}\sqrt{\frac{\log\kappa}{\kappa}} \tag{5.4.29}$$

which is clearly satisfied since $c > 2\sqrt{2}$.

Since

$$\int_1^{v/\kappa} \log x\, dx > \frac{1}{2}\left(\frac{c}{2}\sqrt{\frac{\log\kappa}{\kappa}} - \sqrt{2}\left(\sqrt{\frac{\log\kappa}{\kappa}} - \sqrt{\frac{\log(1+\kappa/4)}{\kappa}}\right)\right)^2 + O\left(\left(\frac{\log\kappa}{\kappa}\right)^{3/2}\right),$$

and referring to equation (5.4.27) it remains to show that

$$\begin{aligned} \exp\Bigg(-\frac{\kappa}{2}\Bigg(\Bigg(\frac{c}{2}\sqrt{\frac{\log\kappa}{\kappa}} &- \sqrt{2}\left(\sqrt{\frac{\log\kappa}{\kappa}} - \sqrt{\frac{\log(1+\kappa/4)}{\kappa}}\right)\Bigg)^2 + O\left(\left(\frac{\log\kappa}{\kappa}\right)^{3/2}\right)\Bigg)\Bigg) \\ &\times \left(1 + \frac{\kappa}{4}\right) < 1. \end{aligned} \tag{5.4.30}$$

Now since $c > 2\sqrt{2}$, we have

$$\begin{aligned} \frac{c}{2}\sqrt{\frac{\log\kappa}{\kappa}} - \sqrt{2}\left(\sqrt{\frac{\log\kappa}{\kappa}} - \sqrt{\frac{\log(1+\kappa/4)}{\kappa}}\right) &> \sqrt{2}\sqrt{\frac{\log\kappa}{\kappa}} - \sqrt{2}\sqrt{\frac{\log\kappa}{\kappa}} \\ &\quad + \sqrt{2}\sqrt{\frac{\log(1+\kappa/4)}{\kappa}} \\ &= \sqrt{2}\sqrt{\frac{\log(1+\kappa/4)}{\kappa}}. \end{aligned}$$

Taking squares gives

$$\begin{aligned} 2\,\frac{\log(1+\kappa/4)}{\kappa} &< \left(\frac{c}{2}\sqrt{\frac{\log\kappa}{\kappa}} - \sqrt{2}\left(\sqrt{\frac{\log\kappa}{\kappa}} - \sqrt{\frac{\log(1+\kappa/4)}{\kappa}}\right)\right)^2 \\ &< \left(\frac{c}{2}\sqrt{\frac{\log\kappa}{\kappa}} - \sqrt{2}\left(\sqrt{\frac{\log\kappa}{\kappa}} - \sqrt{\frac{\log(1+\kappa/4)}{\kappa}}\right)\right)^2 \\ &\quad + O\left(\left(\frac{\log\kappa}{\kappa}\right)^{3/2}\right). \end{aligned} \tag{5.4.31}$$

Rearranging and exponentiating we obtain

$$\begin{aligned} \exp\left(\frac{\kappa}{2}\left(\left(\frac{c}{2}\sqrt{\frac{\log\kappa}{\kappa}}\right.\right.\right. &- \left.\left.\left.\sqrt{2}\left(\sqrt{\frac{\log\kappa}{\kappa}} - \sqrt{\frac{\log(1+\kappa/4)}{\kappa}}\right)\right)^2 + O\left(\left(\frac{\log\kappa}{\kappa}\right)^{3/2}\right)\right)\right) \\ &> \left(1+\frac{\kappa}{4}\right) \end{aligned} \tag{5.4.32}$$

which provides

$$\begin{aligned} \exp\left(-\frac{\kappa}{2}\left(\left(\frac{c}{2}\sqrt{\frac{\log\kappa}{\kappa}}\right.\right.\right. &- \left.\left.\left.\sqrt{2}\left(\sqrt{\frac{\log\kappa}{\kappa}} - \sqrt{\frac{\log(1+\kappa/4)}{\kappa}}\right)\right)^2 + O\left(\left(\frac{\log\kappa}{\kappa}\right)^{3/2}\right)\right)\right) \\ &\times \left(1+\frac{\kappa}{4}\right) < 1 \end{aligned} \tag{5.4.33}$$

as required.

This conclusion persists when $v > 2\kappa$ since $\psi_B(v)$ increases with v.

5.5 Proof of Theorem 5.3.3

From Lemma 5.3.2, we have that $m(a) \le a^{2\epsilon} w(q_a)$. Noting that $a \le D^g$ and $D = z^s$ we obtain

$$m(a) \le z^{2sg\epsilon} w(q_a) \qquad \text{where } \epsilon \to 0 \text{ as } a \to \infty \tag{5.5.1}$$

which leads us to

$$\sum_{a\in\mathcal{A}} w(q_a) \ge z^{-2sg\epsilon} \sum_{a\in\mathcal{A}} m(a). \tag{5.5.2}$$

Now

$$\begin{aligned}
\sum_{a\in\mathcal{A}} m(a) &= \sum_{a\in\mathcal{A}} \sum_{d|a} \lambda^-(d) \\
&= \sum_{d|P(z)} \lambda^-(d) \sum_{a\in\mathcal{A}_d} 1 \\
&= \sum_{d|P(z)} \lambda^-(d) |\mathcal{A}_d|.
\end{aligned}$$

Since

$$|\mathcal{A}_d| = X\frac{\rho(d)}{d} + r_{\mathcal{A}}(d) \tag{5.5.3}$$

we obtain

$$\begin{aligned}
\sum_{a\in\mathcal{A}} m(a) &\ge X \sum_{d|P(z)} \frac{\lambda^-(d)\rho(d)}{d} - \sum_{d|P(z)} \left|\lambda^-(d) r_{\mathcal{A}}(d)\right| \\
&= XW^-(D, P(z)) - \sum_{d|P(z)} \left|\lambda^-(d) r_{\mathcal{A}}(d)\right|.
\end{aligned} \tag{5.5.4}$$

Using the results from Theorem 5.4.1 and Lemma 5.3.2 we obtain

$$\sum_{\substack{a\in\mathcal{A}\\ \nu(D,a)\le R}} w(q_a) \ge \frac{Xz^{-2sg\epsilon}}{V(P(z))}\left(1-\left(1+\frac{B(z)}{4}\right)e^{-\psi_B(v)}\right)-z^{-2sg\epsilon}\sum_{d\le D}\left|\lambda^-(d)r_{\mathcal{A}}(d)\right|. \quad (5.5.5)$$

Since $|\lambda^-(d)| \le 1$ we can estimate the terms involving $|r_{\mathcal{A}}(d)|$ by the following "trivial treatment" of the remainder term:

$$\left|\sum_{d|P(z)} \lambda^-(d)r_{\mathcal{A}}(d)\right| \le \sum_{\substack{d|P(z)\\ d\le D}} |r_{\mathcal{A}}(d)| \quad (5.5.6)$$

from which it follows that

$$\sum_{\substack{a\in\mathcal{A}\\ \nu(D,a)\le R}} w(q_a) \ge \frac{Xz^{-2sg\epsilon}}{V(P(z))}\left(1-\left(1+\frac{B(z)}{4}\right)e^{-\psi_B(v)}\right)-z^{-2sg\epsilon}\sum_{d|P(z)}|r_{\mathcal{A}}(d)| \quad (5.5.7)$$

valid for

$$s > (1-A)2\kappa + c\sqrt{\kappa\log\kappa},$$

where c is a constant satisfying $c > 2\sqrt{2}$ and A satisfies

$$A = \sqrt{2}\left(\sqrt{\frac{\log\kappa}{\kappa}} - \sqrt{\frac{\log(1+\kappa/4)}{\kappa}}\right).$$

Note that

$$\log\left(1+\frac{\kappa}{4}\right) = \log\left\{\frac{\kappa}{4}\left(1+\frac{4}{\kappa}\right)\right\} = \log\kappa + \log\frac{1}{4} + \log\left(1+\frac{4}{\kappa}\right). \quad (5.5.8)$$

So

$$\begin{aligned}\frac{A}{\sqrt{2}} &= \sqrt{\frac{\log\kappa}{\kappa}} - \sqrt{\frac{\log(1+\kappa/4)}{\kappa}} \\ &= \sqrt{\frac{\log\kappa}{\kappa}} - \sqrt{\frac{\log\kappa - 2\log 2 + O(1/\kappa)}{\kappa}} \\ &= \sqrt{\frac{\log\kappa}{\kappa}}\left\{1 - \sqrt{1 - \frac{2\log 2}{\log\kappa} + O\left(\frac{1}{\kappa\log\kappa}\right)}\right\}.\end{aligned}$$

Now, as $\kappa \to \infty$ we have

$$1 - \sqrt{1 - \frac{2\log 2}{\log\kappa} + O\left(\frac{1}{\kappa\log\kappa}\right)} \sim \frac{\log 2}{\log\kappa} + O\left(\frac{1}{\kappa\log\kappa}\right) \tag{5.5.9}$$

which gives

$$A \sim \sqrt{2}\sqrt{\frac{\log\kappa}{\kappa}}\left\{\frac{\log 2}{\log\kappa} + O\left(\frac{1}{\kappa\log\kappa}\right)\right\} \tag{5.5.10}$$

$$\sim \frac{\sqrt{2}\log 2}{\sqrt{\kappa\log\kappa}} + O\left(\frac{1}{\kappa^{3/2}\log\kappa}\right) \tag{5.5.11}$$

which completes the proof of Theorem 5.3.3.

Appendix A

In (5.4.19) we have treated the factor

$$\sum_{p|P(z)} \frac{\rho(p)\log p}{p\log z}\left(1-\frac{\log p}{\log z}\right)\left(\frac{\log p}{\log z}\right) \tag{A.0.1}$$

to a rather rough approximation. This can be improved by partial summation as opposed to taking $(1-x)x$ at its worst value, where $x = \log p/\log z$.

Let

$$F(z) = \sum_{p<z} \frac{\rho(p)}{p} \tag{A.0.2}$$

then by partial summation we have

$$\begin{aligned}
\sum_{p|P(z)} \frac{\rho(p)}{p}\left(\frac{\log p}{\log z}\right)^2\left(1-\frac{\log p}{\log z}\right) &= F(z)\left(\frac{\log z}{\log z}\right)^2\left(1-\frac{\log z}{\log z}\right) \\
&\quad - \int_{t<z} F(t)\left(\frac{2\log t}{t\log^2 z} - \frac{3\log^2 t}{t\log^3 z}\right)dt \\
&= \frac{1}{\log^2 z}\int_{t<z} F(t)\left(\frac{3\log^2 t}{t\log z} - \frac{2\log t}{t}\right). \qquad \text{(A.0.3)}
\end{aligned}$$

Now using (5.2.3) we have

$$\begin{aligned} F(z) &= \sum_{p<z} \frac{\rho(p)}{p} \\ &= \sum_{p<z} \frac{\rho(p)\log p}{p}\frac{1}{\log p} \\ &< \kappa + \kappa \int_{t<z} \frac{\log t}{t \log^2 t} dt + O\left(\frac{1}{\log z}\right) \\ &< \kappa(1 + \log\log z). \end{aligned} \tag{A.0.4}$$

Substituting this into (A.0.3) we obtain

$$\begin{aligned} \sum_{p|P(z)} \frac{\rho(p)}{p}\left(\frac{\log p}{\log z}\right)^2\left(1 - \frac{\log p}{\log z}\right) &< \kappa \int_{t<z} (1+\log\log t)\left(\frac{3\log^2 t}{t\log^3 z} - \frac{2\log t}{t\log^2 z}\right) dt \\ &< \frac{3\kappa}{\log^3 z}\int_{t<z} \frac{\log^2 t}{t} dt - \frac{2\kappa}{\log^2 z}\int_{t<z}\frac{\log t}{t} dt \\ &\quad + \frac{3\kappa}{\log^3 z}\int_{t<z}\frac{\log^2 t \log\log t}{t} dt \\ &\quad - \frac{2\kappa}{\log^2 z}\int_{t<z}\frac{\log t\log\log t}{t} dt. \end{aligned}$$

Integration gives

$$\begin{aligned} \sum_{p|P(z)} \frac{\rho(p)}{p}\left(\frac{\log p}{\log z}\right)^2\left(1 - \frac{\log p}{\log z}\right) &< \frac{3\kappa}{\log^3 z}\frac{\log^3 z}{3} - \frac{2\kappa}{\log^2 z}\frac{\log^2 z}{2} \\ &\quad + \frac{3\kappa}{\log^3 z}\frac{\log^3 z\log\log z}{3} - \frac{3\kappa}{\log^3 z}\frac{\log^3 z}{9} \\ &\quad - \frac{2\kappa}{\log^2 z}\frac{\log^2 z\log\log z}{2} + \frac{2\kappa}{\log^2 z}\frac{\log^2 z}{4} \end{aligned}$$

so we have

$$\sum_{p|P(z)} \frac{\rho(p)}{p}\left(\frac{\log p}{\log z}\right)^2\left(1-\frac{\log p}{\log z}\right) \begin{aligned}[t] &< \kappa - \kappa + \kappa \log\log z - \frac{\kappa}{3} \\ &\quad -\kappa \log\log z + \frac{\kappa}{2} \\ &< \frac{\kappa}{6}. \end{aligned} \tag{A.0.5}$$

So we can replace the factor $B(z)/4$ in (5.4.21) by $\kappa/6$. This means that we can choose A in Theorem 5.3.3 almost as large as

$$A \sim \frac{\sqrt{2}\log\sqrt{6}}{\sqrt{\kappa\log\kappa}}. \tag{A.0.6}$$

Appendix B

First we define the notation

$$d(n) = \sum_{d|n} 1. \tag{B.0.1}$$

The following result appears in [12].

Theorem B.0.1.

$$\lim_{n\to\infty} \frac{\log d(n) \log\log n}{\log n} = \log 2; \tag{B.0.2}$$

that is, if $\alpha > 0$, *then*

$$d(n) < 2^{(1+\alpha)\log n/\log\log n} \tag{B.0.3}$$

for all $n > n_0(\alpha)$ *and*

$$d(n) > 2^{(1-\alpha)\log n/\log\log n} \tag{B.0.4}$$

for an infinity of values of n.

From this we obtain

$$\log d(n) < \frac{(1+\alpha)\log 2}{\log\log n}\log n \tag{B.0.5}$$

so that for all $n > n_0(\alpha)$ we have

$$d(n) < n^{\epsilon_n} \tag{B.0.6}$$

where

$$\epsilon_n = \frac{(1+\alpha)\log 2}{\log\log n}. \tag{B.0.7}$$

We note that since $\log\log n \to \infty$ as $n \to \infty$, we have that $\epsilon_n \to 0$ as $n \to \infty$.

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