Note on the shape circularity measure method based on radial moments

Joviša Žunić *  Kaoru Hirota†  Paul L. Rosin‡

Abstract

In this note we show that the, so called, circularity measures based on radial moments, as defined in [1], are a particular case of the circularity measures introduced by [2].

Keywords: Shape, Circularity measure, Hu moment invariants, Pattern recognition, Image processing.

1 Introduction

A family of circularity measures $C_\beta(S)$ was introduced recently in [2]. More precisely, if $S$ denotes a planar shape, $\mu_{0,0}(S)$ is the area of $S$, and $\beta$ is a number from the interval $(-1, \infty)$, then the quantities $C_\beta(S)$ indicate/measure how much the considered shape $S$ differs from a planar circular disc, of the same area as the given shape $S$. The formal definition, of the circularity measures $C_\beta(S)$, is as follows.

Definition 1 Let $S$ be a given shape whose centroid coincides with the origin and a real $\beta$ such that $-1 < \beta$ and $\beta \neq 0$. Then the circularity measure $C_\beta(S)$ is defined as

$$C_\beta(S) = \begin{cases} \frac{\mu_{0,0}(S)^{\beta+1}}{(\beta+1)\pi^\beta \int_S (x^2+y^2)^\beta dxdy}, & \beta > 0 \\ \frac{(\beta+1)\pi^\beta \int_S (x^2+y^2)^\beta dxdy}{\mu_{0,0}(S)^{\beta+1}}, & \beta \in (-1, 0) \end{cases}$$ (1)

*J. Žunić is with the Department of Computer Science, University of Exeter, Exeter EX4 4QF, U.K.
e-mail: J.Zunic@ex.ac.uk
†K. Hirota is with the Graduate School of Science and Engineering, Tokyo Institute of Technology, G3-49, 4259 Nagatsuta, Modori-ku, Yokohama 226-8502, Japan.e-mail: hirota@hrt.dis.titech.ac.jp
‡P.L. Rosin is with Cardiff University, School of Computer Science, Cardiff CF24 3AA, Wales, U.K.e-mail: Paul.Rosin@cs.cf.ac.uk
The formula in (1) is given in Cartesian coordinates. If the polar coordinate system is involved (instead of the Cartesian coordinate system):

\[ x = r \cos \theta, \quad y = r \sin \theta \quad \text{ (the Jacobian for this coordinate transformation is } |J| = r \) then

\[
\int \int_S (x^2 + y^2)^\beta \, dx \, dy = \int_\theta \int_r ((r \cos \theta)^2 + (r \sin \theta)^2)^\beta \cdot r \, dr \, d\theta = \int_\theta \int_r r^{2\beta + 1} \, dr \, d\theta
\]

and consequently (1) becomes

\[
C_\beta(S) = \begin{cases} 
\frac{\mu_{0,0}(S)^{\beta+1}}{(\beta + 1)\pi^\beta \int_\theta \int_r r^{2\beta + 1} \, dr \, d\theta}, & \beta > 0 \\
\frac{(\beta + 1)\pi^\beta \int_\theta \int_r r^{2\beta + 1} \, dr \, d\theta}{\mu_{0,0}(S)^{\beta+1}}, & \beta \in (-1, 0).
\end{cases}
\] (2)

Now, by setting \( \beta = \frac{p}{2} \), the first expression (for \( \beta > 0 \)) in (2) becomes

\[
C_\beta(S) = C_{p/2}(S) = \frac{2^{p/2} \cdot \pi^{-p/2} \cdot \mu_{0,0}(S)^{p+2}}{\int_\theta \int_r r^{p+1} \, dr \, d\theta}.
\] (3)

Circularity measures based on radial moments, introduced in [1], are denoted by \( \zeta_p(D) \) and formally defined, by the expression in (9) from [1], as

\[
\zeta_p(D) = \frac{2^{p/2} \cdot \pi^{-p/2} \cdot [u_0(D)]^{p+2}}{u_p(D)}.
\] (4)

Further, [1] uses the following denotation

- \( u_p(D) = \int \int_D (r - \bar{r})^p ds \), with \( ds = r \cdot dr \cdot d\theta \), \( \bar{r} = \sqrt{x_c^2 + y_c^2} \), and

\[
(x_c, y_c) = \left( \frac{\int \int_D x \, ds}{\int \int_D \, ds}, \frac{\int \int_D y \, ds}{\int \int_D \, ds} \right) \quad \text{being the centroid of the considered shape } D.
\]

(Notice: \( u_0(D) = \mu_{0,0}(D) \) and \( u_p(D) = \int_\theta \int_r r^{p+1} \, dr \, d\theta \), if \( \bar{r} = 0 \), i.e. \((x_c, y_c) = (0, 0)\).)

Finally, since \( \zeta_p(D) \) is translation invariant (see Theorem 2 from [1]) we can set \( \bar{r} = 0 \) (i.e we can assume that the shape \( D \) is translated such that its gravity center \((x_c, y_c)\) coincides with the origin \((0, 0)\)), and deduce that (for \( p > 0 \))

\[
\zeta_p(D) = C_{p/2}(D).
\] (5)
In other words, the formula in (4) is equivalent to the formula in (3), and further, shape circularity measures $\zeta_p(D)$ based on radial moments, from [1], are particular subcases of the family of circularity measures $C_\beta(S)$, introduced by [2] (measures from [2] are defined for $\beta = \frac{p}{2}$ negative, as well).

It is worth mentioning that the identity in (5) is evident in the experimental results from Table 1 in [1], which includes $C_p(D)$ denoted by $H_p(D)$. Although there is a systematic offset between $\zeta_p(D)$ and $C_{p/2}(D)$, possibly caused by digitization and numerical errors, the results for $\zeta_{p=2}(D)$ are similar to $C_{p=1}(D)$, such that their ratios are all the same to within 3 significant places. Likewise, the ratios of $\zeta_{p=4}(D)$ and $C_{p=2}(D)$ are the same to within 3 significant places.

References
