ALTERNATING SIGN MATRIX ENUMERATION INVOLVING NUMBERS OF INVERSIONS AND −1's AND POSITIONS OF BOUNDARY 1's

ROGER E. BEHREND

Abstract. This paper consists of a review of results for the exact enumeration of alternating sign matrices of fixed size with prescribed values of some or all of the following six statistics: the numbers of generalized inversions and −1's, and the positions of the 1's in the first and last rows and columns. Many of these results are expressed in terms of generating functions.

Contents

1. Preliminaries

1.1. **Introduction.** This paper provides a summary of certain results for the exact enumeration of alternating sign matrices (ASMs) of arbitrary fixed size with prescribed values of some or all of six specific statistics. These statistics are the numbers of generalized inversions and -1's in an ASM, and the positions of the 1's in the first and last rows and columns of an ASM. Many of the results are expressed in terms of polynomial generating functions, whose variables are associated with these statistics, so that the actual numbers of ASMs with prescribed values of the statistics appear as the coefficients in these polynomials.

This is entirely a review paper, with all of the results which are presented having already appeared elsewhere. Almost none of the details of the proofs of these results are given in this paper, but full references to proofs in the literature are provided. For other reviews of aspects of ASMs, see, for example, Bressoud [6, 7], Bressoud and Propp [8], Di Francesco [16, Sec. 4], [17], Hone [23], Zeilberger [40], or Zinn-Justin [41].

Much of the content of this paper is based on the author's recent paper [2], although [2] includes more material and detail, and uses a different order of presentation. The paper is also based on a talk given by the author at the workshop *Algebraic Combinatorics Related to Young Diagrams and Statistical Physics*, held at the International Institute for Advanced Study, Japan, from 6 to 12 August, 2012, and supported by the Research Institute for Mathematical Sciences (RIMS) at Kyoto University. The author is very grateful to the workshop's organizers, Masao Ishikawa and Soichi Okada, and to RIMS.

1.2. **Definitions.** An ASM, as first defined by Mills, Robbins and Rumsey [29, 30], is a square matrix in which each entry is 0, 1 or -1, and along each row and column the nonzero entries alternate in sign and have a sum of 1.

It follows that any permutation matrix is an ASM, and that, for any ASM A, each partial row sum $\sum_{j'=1}^{j} A_{ij'}$ and each partial column sum $\sum_{i'=1}^{i} A_{i'j}$ is 0 or 1. Also, in any ASM, the first and last rows and columns each contain a single 1, which will be referred to as a boundary 1, with all of their other entries being 0's.

For each positive integer n, the set of all $n \times n$ ASMs will be denoted as $\mathrm{ASM}(n)$. For example, for $n = 1, 2, 3$, these sets are

$$\mathrm{ASM}(1) = \{(1)\},$$
$$\mathrm{ASM}(2) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\},$$
$$\mathrm{ASM}(3) = \left\{ \begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix}, \begin{pmatrix} 0&1&0\\1&0&0\\0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0\\0&0&1\\0&1&0 \end{pmatrix}, \begin{pmatrix} 0&1&0\\0&0&1\\1&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1\\1&0&0\\0&1&0 \end{pmatrix}, \begin{pmatrix} 0&0&1\\0&1&0\\1&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0\\1&-1&1\\0&1&0 \end{pmatrix} \right\}. \quad (1)$$

The six statistics on $\mathrm{ASM}(n)$ which will be considered in this paper will now be introduced. For any $A \in \mathrm{ASM}(n)$, statistics which depend on the bulk structure of A are defined as

$$\nu(A)=\sum_{\substack{1\le i<i'\le n\\ 1\le j'\le j\le n}} A_{ij}\,A_{i'j'},\qquad \mu(A)=\text{number of }-1\text{'s in }A, \tag{2}$$

and statistics which describe the configuration of A at its top, right, bottom and left boundaries are defined as, respectively,

$$\begin{aligned}
\rho_{\mathrm{T}}(A)&=\text{number of 0's to the left of the 1 in the top row of }A,\\
\rho_{\mathrm{R}}(A)&=\text{number of 0's below the 1 in the right-most column of }A,\\
\rho_{\mathrm{B}}(A)&=\text{number of 0's to the right of the 1 in the bottom row of }A,\\
\rho_{\mathrm{L}}(A)&=\text{number of 0's above the 1 in the left-most column of }A.
\end{aligned} \tag{3}$$

The statistics of (3) can be depicted diagrammatically as

$$\begin{array}{c}
\overset{\longleftarrow\,\rho_{\mathrm{T}}(A)\,\longrightarrow}{}\\
\rho_{\mathrm{L}}(A)\updownarrow\begin{pmatrix}
0&0&0&0&1&0&0&0&0\\
0&&&&&&&&0\\
0&&&&&&&&0\\
1&&&&&&&&0\\
0&&&&A&&&&1\\
0&&&&&&&&0\\
0&&&&&&&&0\\
0&&&&&&&&0\\
0&0&1&0&0&0&0&0&0
\end{pmatrix}\updownarrow\rho_{\mathrm{R}}(A)\\
\longleftarrow\,\rho_{\mathrm{B}}(A)\,\longrightarrow
\end{array}. \tag{4}$$

The statistic $\nu(A)$ in (2) is a nonnegative integer for any $A\in\mathrm{ASM}(n)$, since it can be written as $\nu(A)=\sum_{i,j=1}^{n}(\sum_{i'=1}^{i-1}A_{i'j})(\sum_{j'=1}^{j}A_{ij'})$, where each factor in the summand (being a partial row or column sum of an ASM) is 0 or 1. This statistic can also be written as $\nu(A)=\sum_{1\le i\le i'\le n;\;1\le j'<j\le n}A_{ij}\,A_{i'j'}=\sum_{i,j=1}^{n}(\sum_{i'=1}^{i}A_{i'j})(\sum_{j'=1}^{j-1}A_{ij'})$.

If A is a permutation matrix, then it can be seen from (2) that $\nu(A)$ is the number of inversions in the permutation π given by $\delta_{\pi_i,j}=A_{ij}$. Accordingly, for any ASM A, $\nu(A)$ is referred to as the number of generalized inversions in A. This statistic was first defined and used by Robbins and Rumsey [34, Eq. (18)], who referred to it as the number of positive inversions in an ASM [34, p. 182]. A closely-related statistic, $\sum_{1\le i<i'\le n;\;1\le j'<j\le n}A_{ij}\,A_{i'j'}=\nu(A)+\mu(A)$ for each $A\in\mathrm{ASM}(n)$, was previously defined and used by Mills, Robbins and Rumsey [30, p. 344], and is sometimes also referred to in the literature as the number of generalized inversions in A.

It can be seen that transposition or 90° rotation of an ASM give another ASM with the same number of -1's, and with the positions of the boundary 1's simply reflected or rotated. It can also be checked straightforwardly that the number of generalized inversions is invariant under transposition of an ASM, and that if A and A' are $n\times n$ ASMs related by 90° rotation, then $\nu(A)+\nu(A')=\frac{n(n-1)}{2}-m$, where m is the number of -1's in A or A'.

It follows from these properties of transposition and rotation of ASMs that, with regards to the boundaries involved, there are essentially only six different types of ASM enumeration, as listed in Table 1, where T, R, B and L denote the top (first) row, right-most (last) column, bottom (last) row and left-most (first) column, respectively. The sections of the paper in which these types of enumeration are considered are also listed in Table 1.

Type of enumeration	Boundaries involved	Sec.
Unrefined	none	2
Singly-refined	T, R, B or L	3
Opposite-boundary doubly-refined	T & B; or L & R	4
Adjacent-boundary doubly-refined	L & T; T & R; R & B; or B & L	5
Triply-refined	B, L & T; L, T & R; T, R & B; or R, B & L	6
Quadruply-refined	T, R, B & L	7

Table 1. Categorization of ASM enumeration according to the boundaries involved.

Various ASM generating functions involving the statistics of (2) and (3) will now be introduced. Each of these generating functions will be labelled by a particular type of boundary refinement, as described in Table 1. In addition to being associated with certain boundary statistics from (3), corresponding to the boundary refinement label, each generating function will also be associated with the two bulk statistics of (2).

For each positive integer n, a quadruply-refined ASM generating function, which involves all six statistics of (2) and (3), and associated indeterminates x, y, z_1, z_2, z_3 and z_4, is defined as

$$Z_n^{\mathrm{quad}}(x,y;z_1,z_2,z_3,z_4)=\sum_{A\in\mathrm{ASM}(n)} x^{\nu(A)}\,y^{\mu(A)}\,z_1^{\rho_{\mathrm{T}}(A)}\,z_2^{\rho_{\mathrm{R}}(A)}\,z_3^{\rho_{\mathrm{B}}(A)}\,z_4^{\rho_{\mathrm{L}}(A)}. \tag{5}$$

Therefore, $Z_n^{\mathrm{quad}}(x,y;z_1,z_2,z_3,z_4)$ is a polynomial in x, y, z_1, z_2, z_3 and z_4, in which, for any nonnegative integers p, m, k_1, k_2, k_3 and k_4, the coefficient of $x^p y^m z_1^{k_1} z_2^{k_2} z_3^{k_3} z_4^{k_4}$ is the number of $n\times n$ ASMs A with $\nu(A)=p$, $\mu(A)=m$, $\rho_{\mathrm{T}}(A)=k_1$, $\rho_{\mathrm{R}}(A)=k_2$, $\rho_{\mathrm{B}}(A)=k_3$ and $\rho_{\mathrm{L}}(A)=k_4$. It also follows that x and y can be regarded as bulk parameters or weights, that z_1, z_2, z_3 and z_4 can be regarded as boundary parameters or weights, and (see Behrend [2, Eq. (5)]) that $Z_n^{\mathrm{quad}}(x,y;z_1,z_2,z_3,z_4)$ has degree $\frac{n(n-1)}{2}$ in x, degree $\lfloor\frac{(n-1)^2}{4}\rfloor$ in y, and degree $n-1$ in each of z_1, z_2, z_3 and z_4.

Examples of the quadruply-refined ASM generating function (5), for $n=1,2,3$, are

$$\begin{aligned}
&Z_1^{\mathrm{quad}}(x,y;z_1,z_2,z_3,z_4)=1,\\
&Z_2^{\mathrm{quad}}(x,y;z_1,z_2,z_3,z_4)=1+x\,z_1\,z_2\,z_3\,z_4,\\
&Z_3^{\mathrm{quad}}(x,y;z_1,z_2,z_3,z_4)=1+x\,z_1\,z_4+x\,z_2\,z_3+x^2\,z_1\,z_2\,z_3^2\,z_4^2+x^2\,z_1^2\,z_2^2\,z_3\,z_4+\\
&\qquad\qquad x^3\,z_1^2\,z_2^2\,z_3^2\,z_4^2+x\,y\,z_1\,z_2\,z_3\,z_4,
\end{aligned}\tag{6}$$

where the terms are written in orders which correspond to those used in (1).

Triply-refined, adjacent-boundary doubly-refined, opposite-boundary doubly-refined, singly-refined and unrefined ASM generating functions can now be defined as, respectively,

$$\begin{aligned}
Z_n^{\mathrm{tri}}(x,y;z_1,z_2,z_3) &= Z_n^{\mathrm{quad}}(x,y;z_1,1,z_2,z_3) = \textstyle\sum_{A\in\mathrm{ASM}(n)} x^{\nu(A)}\, y^{\mu(A)}\, z_1^{\rho_{\mathrm{T}}(A)}\, z_2^{\rho_{\mathrm{B}}(A)}\, z_3^{\rho_{\mathrm{L}}(A)},\\
Z_n^{\mathrm{adj}}(x,y;z_1,z_2) &= Z_n^{\mathrm{quad}}(x,y;z_1,1,1,z_2) = \textstyle\sum_{A\in\mathrm{ASM}(n)} x^{\nu(A)}\, y^{\mu(A)}\, z_1^{\rho_{\mathrm{T}}(A)}\, z_2^{\rho_{\mathrm{L}}(A)},\\
Z_n^{\mathrm{opp}}(x,y;z_1,z_2) &= Z_n^{\mathrm{quad}}(x,y;z_1,1,z_2,1) = \textstyle\sum_{A\in\mathrm{ASM}(n)} x^{\nu(A)}\, y^{\mu(A)}\, z_1^{\rho_{\mathrm{T}}(A)}\, z_2^{\rho_{\mathrm{B}}(A)},\\
Z_n(x,y;z) &= Z_n^{\mathrm{quad}}(x,y;z,1,1,1) = \textstyle\sum_{A\in\mathrm{ASM}(n)} x^{\nu(A)}\, y^{\mu(A)}\, z^{\rho_{\mathrm{T}}(A)},\\
Z_n(x,y) &= Z_n^{\mathrm{quad}}(x,y;1,1,1,1) = \textstyle\sum_{A\in\mathrm{ASM}(n)} x^{\nu(A)}\, y^{\mu(A)},
\end{aligned} \tag{7}$$

where z is a further indeterminate.

Finally, alternative quadruply-refined and alternative adjacent-boundary doubly-refined ASM generating functions are defined as, respectively,

$$\begin{aligned}
\widetilde{Z}_n^{\mathrm{quad}}(x,y;z_1,z_2,z_3,z_4) &= (z_2z_4)^{n-1}\, Z_n^{\mathrm{quad}}(x,y;z_1,\tfrac{1}{z_2},z_3,\tfrac{1}{z_4})\\
&= \textstyle\sum_{A\in\mathrm{ASM}(n)} x^{\nu(A)}\, y^{\mu(A)}\, z_1^{\rho_{\mathrm{T}}(A)}\, z_2^{n-\rho_{\mathrm{R}}(A)-1}\, z_3^{\rho_{\mathrm{B}}(A)}\, z_4^{n-\rho_{\mathrm{L}}(A)-1},\\
\widetilde{Z}_n^{\mathrm{adj}}(x,y;z_1,z_2) &= Z_n^{\mathrm{quad}}(x,y;z_1,z_2,1,1) = \textstyle\sum_{A\in\mathrm{ASM}(n)} x^{\nu(A)}\, y^{\mu(A)}\, z_1^{\rho_{\mathrm{T}}(A)}\, z_2^{\rho_{\mathrm{R}}(A)}.
\end{aligned} \tag{8}$$

Note that $\widetilde{Z}_n^{\mathrm{quad}}(x,y;z_1,z_2,z_3,z_4)$ is a generating function in which the positions of the 1's in the first and last columns of an ASM are measured relative to the opposite ends of the columns to those used in (3) and (4), i.e., in this generating function, the statistics associated with z_2 and z_4 are, respectively, the numbers of 0's above the 1 in the right-most column, and below the 1 in the left-most column of an ASM. Due to certain differences in the symmetry properties of the quadruply-refined and alternative quadruply-refined ASM generating functions $Z_n^{\mathrm{quad}}(x,y;z_1,z_2,z_3,z_4)$ and $\widetilde{Z}_n^{\mathrm{quad}}(x,y;z_1,z_2,z_3,z_4)$ (see Behrend [2, Eq. (12), first 4 lines]), it will be more convenient to use the former function for the case in which x and y are arbitrary, and the latter function for the case $x=y=1$.

It follows from the properties of 90° rotation of ASMs that the adjacent-boundary doubly-refined and alternative adjacent-boundary doubly-refined ASM generating functions are related by

$$Z_n^{\mathrm{adj}}(x,y;z_1,z_2) = x^{n(n-1)/2}\,(z_1z_2)^{n-1}\,\widetilde{Z}_n^{\mathrm{adj}}\bigl(\tfrac{1}{x},\tfrac{y}{x};\tfrac{1}{z_1},\tfrac{1}{z_2}\bigr). \tag{9}$$

It will sometimes be convenient to refer to the boundary parameter coefficients in the singly-refined ASM generating function. These will be denoted as

$$Z_n(x,y)_k = \text{coefficient of } z^k \text{ in } Z_n(x,y;z). \tag{10}$$

It follows that $Z_n(x,y)_k = \sum_{\substack{A\in\mathrm{ASM}(n)\\ A_{1,k+1}=1}} x^{\nu(A)}\, y^{\mu(A)}$ for $0\le k\le n-1$, and that

$$Z_n(x,y) = \textstyle\sum_{k=0}^{n-1} Z_n(x,y)_k. \tag{11}$$

When considering ASM enumeration with $x=y=1$, it will be useful to refer to certain numbers of ASMs, in addition to the ASM generating functions. In particular, adjacent-boundary doubly-refined, opposite-boundary doubly-refined, singly-refined and unrefined

ASM numbers are defined as, respectively,

$$\begin{aligned}
\mathcal{A}^{\mathrm{adj}}_{n,k_1,k_2} &= |\{A \in \mathrm{ASM}(n) \mid A_{1,k_1+1} = A_{k_2+1,1} = 1\}|,\\
\mathcal{A}^{\mathrm{opp}}_{n,k_1,k_2} &= |\{A \in \mathrm{ASM}(n) \mid A_{1,k_1+1} = A_{n,n-k_2} = 1\}|,\\
\mathcal{A}_{n,k} &= |\{A \in \mathrm{ASM}(n) \mid A_{1,k+1} = 1\}|,\\
\mathcal{A}_n &= |\mathrm{ASM}(n)|, \qquad (12)
\end{aligned}$$

for $0 \le k, k_1, k_2 \le n-1$, with the numbers being 0 for k, k_1 or k_2 outside this range. These numbers are therefore related to functions of (7)–(10) by

$$\begin{aligned}
Z^{\mathrm{adj}}_n(1,1;z_1,z_2) &= \textstyle\sum_{k_1,k_2=0}^{n-1} \mathcal{A}^{\mathrm{adj}}_{n,k_1,k_2}\, z_1^{k_1} z_2^{k_2},\\
\widetilde{Z}^{\mathrm{adj}}_n(1,1;z_1,z_2) &= \textstyle\sum_{k_1,k_2=0}^{n-1} \mathcal{A}^{\mathrm{adj}}_{n,n-1-k_1,n-1-k_2}\, z_1^{k_1} z_2^{k_2},\\
Z^{\mathrm{opp}}_n(1,1;z_1,z_2) &= \textstyle\sum_{k_1,k_2=0}^{n-1} \mathcal{A}^{\mathrm{opp}}_{n,k_1,k_2}\, z_1^{k_1} z_2^{k_2},\\
Z_n(1,1;z) = \textstyle\sum_{k=0}^{n-1} \mathcal{A}_{n,k}\, z^k, &\quad Z_n(1,1)_k = \mathcal{A}_{n,k}, \quad Z_n(1,1) = \mathcal{A}_n. \qquad (13)
\end{aligned}$$

Various simple identities satisfied by the functions (5)–(10) and the numbers (12) can be obtained from their definitions, and by considering the properties of transposition or rotation of ASMs, the properties of ASMs with a 1 as a corner entry, or the properties of ASMs in which a boundary 1 is separated from a corner by a single zero. Summaries of such identities, and their derivations, are given by Behrend [2, Secs. 2.2 & 3.3].

1.3. **Structure of the paper.** The structure of the remaining sections of this paper will now be outlined.

The primary currently-known results for unrefined, singly-refined, opposite-boundary doubly-refined, adjacent-boundary doubly-refined, triply-refined and quadruply-refined exact enumeration of ASMs of arbitrary fixed size are reviewed in Sections 2, 3, . . . , 7, respectively, as also indicated in Table 1. Hence, these sections are structured according to which of the boundary statistics of (3) are included in the enumeration.

Each of the main sections is then divided into two subsections, with Sections 2.1, 3.1, . . . , 7.1 concerned with enumeration which involves both of the bulk statistics of (2), i.e., in which the bulk parameters x and y are both arbitrary, and Sections 2.2, 3.2, . . . , 7.2 concerned with enumeration which does not involve either of the bulk statistics of (2), i.e., in which the bulk parameters x and y are both 1.

Some currently-known results which do not fall into this scheme are mentioned briefly in the final Section 8.

It might seem that any result in Sections 2–6 could be obtained from a result in Section 7 by setting appropriate boundary parameters to 1, and that any result in Sections 2.2, 3.2,. . . , 7.2 could be obtained from a result in Sections 2.1, 3.1,. . . , 7.1, respectively, by setting the bulk parameters x and y to 1. However, only some derivations of this type are currently known. For example, derivations of (21), (22), (28), (34), (35), (42) and (43) in which boundary parameters are set to 1 in (46) are given by Behrend [2, Sec. 4.2]. On the other hand, for many of the results in Sections 2.2, 3.2, . . . , 7.2, derivations which involve setting x and y

to 1 in results of Sections 2.1, 3.1, ..., 7.1 are not currently known. For all of the results in this paper, references to the currently-published proofs are given, so by following these, the derivations in which parameters in a more general result are set to 1 could be identified. Some derivations of this type are also identified explicitly in the subsequent sections.

Finally, note that, in the subsequent sections, many of the identities will be valid only for all $n \geq 2$, or for all $n \geq 3$, where n denotes the size of the associated ASMs. This will usually be due to their containing terms (such as $Z_{n-1}(x, y)$ or $Z_{n-2}(x, y)$) which are not defined if n is taken to be 1 or 2.

2. Unrefined enumeration

2.1. Arbitrary bulk parameters.

It was shown by Behrend, Di Francesco and Zinn-Justin [3, Eq. (29) & Props. 1–3] that the unrefined ASM generating function is given by the determinant formula

$$Z_n(x, y) = \det_{0 \leq i,j \leq n-1} \bigl(K_n(x, y)_{ij}\bigr), \tag{14}$$

where

$$K_n(x, y)_{ij} = -\delta_{i,j+1} + \textstyle\sum_{k=0}^{\min(i,j+1)} \binom{i-1}{i-k}\binom{j+1}{k} x^k y^{i-k}. \tag{15}$$

For alternative versions of (14), involving transformations of the matrix $K_n(x, y)$, and related to formulae of Colomo and Pronko [10, Eqs. (23) & (24)], [11, Eqs. (4.3)–(4.7)], Lalonde [28, Thm. 3.1] and Mills, Robbins and Rumsey [30, p. 346], see Behrend, Di Francesco and Zinn-Justin [3, Eqs. (28), (65) & (66)].

An alternative method for obtaining $Z_n(x, y)$, involving a recursive approach, will be described in Section 3.1.

2.2. Bulk parameters $x = y = 1$.

An explicit formula for the number of $n \times n$ ASMs is

$$\mathcal{A}_n = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}. \tag{16}$$

These numbers, for $n = 1, \dots, 8$, are given in Table 2.

n	1	2	3	4	5	6	7	8
$\mathcal{A}_n$	1	2	7	42	429	7436	218348	10850216

Table 2. $\mathcal{A}_n$, for $n = 1, \dots, 8$.

The product formula (16) was conjectured by Mills, Robbins and Rumsey [29, 30, Conj. 1], and first proved by Zeilberger [38] and, shortly thereafter, using a different method, by Kuperberg [27].

Setting $x = y = 1$ in (14) gives the determinant formula

$$\mathcal{A}_n = \det_{0 \leq i,j \leq n-1} \Bigl(-\delta_{i,j+1} + \binom{i+j}{i}\Bigr), \tag{17}$$

as also obtained by Gessel and Xin [22, Rem. 5.2]. Alternative determinantal formulae for $\mathcal{A}_n$ can be obtained by setting $x = y = 1$ in the alternative versions of (14) mentioned in Section 2.1.

It was shown by Okada [31, Thm. 1.2 (A1)] that

$$\mathcal{A}_n = 3^{-n(n-1)/2} \times \big(\text{number of semistandard Young tableaux of shape } (n-1, n-1, \ldots, 2, 2, 1, 1) \text{ with entries from } \{1, \ldots, 2n\}\big). \quad (18)$$

The equality between the RHS of (16) and the RHS of (18) can be obtained directly using the hook-content formula for semistandard Young tableaux.

3. Singly-refined enumeration

3.1. **Arbitrary bulk parameters.** It was shown by Behrend, Di Francesco and Zinn-Justin [3, Eq. (74), Props. 4–6 & Eqs. (97) & (98)] that the singly-refined ASM generating function is given by the determinant formula

$$Z_n(x, y; z) = \det_{0 \le i,j \le n-1} \big(K_n(x, y; z)_{ij}\big), \quad (19)$$

where

$$K_n(x, y; z)_{ij} = -\delta_{i,j+1} + \begin{cases} \sum_{k=0}^{\min(i,j+1)} \binom{i-1}{i-k}\binom{j+1}{k} x^k y^{i-k}, & j \le n-2, \\ \sum_{k=0}^{i} \sum_{l=0}^{k} \binom{i-1}{i-k}\binom{n-l-1}{k-l} x^k y^{i-k} z^l, & j = n-1. \end{cases} \quad (20)$$

For alternative versions of (19), involving transformations of the matrix $K_n(x, y; z)$, and related to formulae of Lalonde [28, Thm. 3.1] and Mills, Robbins and Rumsey [30, p. 346], see Behrend, Di Francesco and Zinn-Justin [3, Eqs. (73), (87) & (88)].

It was shown by Behrend [2, Cor. 8] that the boundary parameter coefficients in the singly-refined ASM generating function, as defined in (10), satisfy

$$Z_n(x, y)_k = Z_{n-1}(x, y)\,\delta_{k,0} + Z_{n-1}(x, y) \sum_{i=0}^{k-1} \Bigg(y^{i+1} \binom{k-1}{i} \binom{n-1}{i+1} + y^i \sum_{j_1=0}^{k-i-1} \sum_{j_2=0}^{n-i-2} \frac{Z_{n-i-1}(x, y)_{j_1}\, Z_{n-i-1}(x, y)_{j_2}}{Z_{n-i-1}(x, y)\, Z_{n-i-2}(x, y)} \bigg(x \binom{k-j_1-2}{i-1} \binom{n-j_2-2}{i} - y \binom{k-j_1-1}{i} \binom{n-j_2-1}{i+1} \bigg) \Bigg), \quad (21)$$

where $Z_0(x, y)$, if it appears, is taken to be 1.

Summing (21) over k, using (11), and again taking $Z_0(x,y)$ to be 1, gives

$$Z_n(x,y)=Z_{n-1}(x,y)\Biggl(1+\sum_{i=0}^{n-2}\Biggl(y^{i+1}\binom{n-1}{i+1}^2+\\ \frac{xy^i\bigl(\sum_{j=0}^{n-i-2}\binom{n-j-2}{i}Z_{n-i-1}(x,y)_j\bigr)^2-y^{i+1}\bigl(\sum_{j=0}^{n-i-2}\binom{n-j-1}{i+1}Z_{n-i-1}(x,y)_j\bigr)^2}{Z_{n-i-1}(x,y)\,Z_{n-i-2}(x,y)}\Biggr)\Biggr), \quad (22)$$

as obtained by Behrend [2, Cor. 9].

It can be seen that (21) and (22) give $Z_n(x,y)_k$ and $Z_n(x,y)$ in terms of $Z_i(x,y)_j$ and $Z_i(x,y)$ for $i=1,\dots,n-1$, thereby enabling the singly-refined and unrefined ASM generating functions to be computed recursively, and providing an alternative method to that of using the determinantal formulae (14) and (19).

3.2. **Bulk parameters $x=y=1$.** An explicit formula for the singly-refined ASM numbers is

$$\mathcal{A}_{n,k}=\begin{cases}\frac{(n+k-1)!\,(2n-k-2)!}{k!\,(n-k-1)!\,(2n-2)!}\prod_{i=0}^{n-2}\frac{(3i+1)!}{(n+i-1)!}, & 0\le k\le n-1,\\ 0, & \text{otherwise}.\end{cases} \quad (23)$$

Examples of these numbers, for $n=1,\dots,5$, are given in Table 3.

$\mathcal{A}_{n,k}$	$k=0$	1	2	3	4
$n=1$	1				
2	1	1			
3	2	3	2		
4	7	14	14	7	
5	42	105	135	105	42

TABLE 3. $\mathcal{A}_{n,k}$, for $n=1,\dots,5$ and $k=0,\dots,n-1$

The formula (23) was first proved by Zeilberger [39], and confirms the validity of conjectures of Mills, Robbins and Rumsey [29, 30, Conj. 2]. Alternative proofs of (23) have been given by Colomo and Pronko [14, Sec. 5.3], [15, Sec. 4.2], Fischer [19], and Stroganov [36, Sec. 4]. See also Razumov and Stroganov [32, Sec. 2], [33, Sec. 2] for additional details related to the third of these proofs.

The singly-refined ASM generating function at $x=y=1$ can be written explicitly using (23) and the fourth equation of (13). Alternatively, it was observed by Colomo and Pronko [12, Eq. (2.16)], [14, Eq. (5.43)], [15, Eq. (4.19)] that it can be expressed in terms of the Gaussian hypergeometric function as

$$Z_n(1,1;z)=\mathcal{A}_{n-1}\,{}_2F_1\left[{1-n,\;n \atop 2-2n};z\right]. \quad (24)$$

Various further expressions for $Z_n(1,1;z)$ can be obtained by setting all but one boundary parameter to 1 in certain subsequent formulae, such as (33), for ASM generating functions at $x=y=1$.

4. Opposite boundary doubly-refined enumeration

4.1. **Arbitrary bulk parameters.** It was shown by Behrend, Di Francesco and Zinn-Justin [4, Eqs. (21) & (22)] that the opposite-boundary doubly-refined ASM generating function is given by the determinant formula

$$Z_n^{\mathrm{opp}}(x,y;z_1,z_2)=\det_{0\le i,j\le n-1}\bigl(K_n(x,y;z_1,z_2)_{ij}\bigr), \tag{25}$$

where

$$K_n(x,y;z_1,z_2)_{ij}= \\ -\delta_{i,j+1}+\begin{cases}\sum_{k=0}^{\min(i,j+1)}\binom{i-1}{i-k}\binom{j+1}{k}x^ky^{i-k}, & j\le n-3,\\ \sum_{k=0}^{i}\sum_{l=0}^{k}\binom{i-1}{i-k}\binom{n-l-2}{k-l}x^ky^{i-k}z_2^{l+1}, & j=n-2,\\ \sum_{k=0}^{i}\sum_{l=0}^{k}\sum_{m=0}^{l}\binom{i-1}{i-k}\binom{n-l-2}{k-l}x^ky^{i-k}z_1^m z_2^{l-m}, & j=n-1.\end{cases} \tag{26}$$

Note that $K_n(x,y;z,1)=K_n(x,y;z)$ and $K_n(x,y;1,1)=K_n(x,y)$ (with $K_n(x,y;z)$ and $K_n(x,y)$ defined in (20) and (15), respectively), and that setting $z_2=1$ or $z_1=z_2=1$ in (25) gives (19) or (14), respectively.

For an alternative version of (25), involving a transformation of the matrix $K_n(x,y;z_1,z_2)$, see Behrend, Di Francesco and Zinn-Justin [4, Eqs. (65) & (66)].

The opposite-boundary doubly-refined ASM generating function satisfies

$$(z_1-z_2)\,(z_3-z_4)\,Z_n^{\mathrm{opp}}(x,y;z_1,z_2)\,Z_n^{\mathrm{opp}}(x,y;z_3,z_4)- \\ (z_1-z_3)\,(z_2-z_4)\,Z_n^{\mathrm{opp}}(x,y;z_1,z_3)\,Z_n^{\mathrm{opp}}(x,y;z_2,z_4)+ \\ (z_1-z_4)\,(z_2-z_3)\,Z_n^{\mathrm{opp}}(x,y;z_1,z_4)\,Z_n^{\mathrm{opp}}(x,y;z_2,z_3)=0, \tag{27}$$

and it can be expressed in terms of singly-refined and unrefined ASM generating functions as

$$(z_1-z_2)\,Z_n^{\mathrm{opp}}(x,y;z_1,z_2)\,Z_{n-1}(x,y)=(z_1-1)\,z_2\,Z_n(x,y;z_1)\,Z_{n-1}(x,y;z_2)- \\ z_1\,(z_2-1)\,Z_{n-1}(x,y;z_1)\,Z_n(x,y;z_2). \tag{28}$$

The identities (27) and (28) are essentially equivalent, as discussed by Behrend, Di Francesco and Zinn-Justin [4, p. 415] or Behrend [2, pp. 459–460].

A result which is equivalent to (28) with $x=y=1$ was obtained by Stroganov [36, Eq. (34)], and a result which is equivalent to (28) with arbitrary x and y was obtained by Colomo and Pronko [13, Eq. (5.32)], [15, Eq. (3.32)]. Alternative proofs of (27) and (28) have been given by Behrend, Di Francesco and Zinn-Justin [4, Sec. 5], and Behrend [2, Cor. 5, or Eqs. (72) & (73) with $m=2$, $k_1=1$, $k_2=n$].

4.2. **Bulk parameters $x = y = 1$.** It was shown by Stroganov [36, Eq. (34)] that the opposite-boundary doubly-refined, singly-refined and unrefined ASM numbers satisfy

$$(\mathcal{A}^{\text{opp}}_{n,k_1-1,k_2} - \mathcal{A}^{\text{opp}}_{n,k_1,k_2-1})\,\mathcal{A}_{n-1} = \mathcal{A}_{n,k_1-1}\,\mathcal{A}_{n-1,k_2-1} - \mathcal{A}_{n,k_1}\,\mathcal{A}_{n-1,k_2-1} - \mathcal{A}_{n-1,k_1-1}\,\mathcal{A}_{n,k_2-1} + \mathcal{A}_{n-1,k_1-1}\,\mathcal{A}_{n,k_2}. \quad (29)$$

Examples of opposite-boundary doubly-refined ASM numbers, for $n = 3, 4, 5$, are given in Table 4.

$\mathcal{A}^{\text{opp}}_{3,k_1,k_2}$	$k_1=0$	1	2
$k_2=0$	1	1	0
1	1	1	1
2	0	1	1

$\mathcal{A}^{\text{opp}}_{4,k_1,k_2}$	$k_1=0$	1	2	3
$k_2=0$	2	3	2	0
1	3	5	4	2
2	2	4	5	3
3	0	2	3	2

$\mathcal{A}^{\text{opp}}_{5,k_1,k_2}$	$k_1=0$	1	2	3	4
$k_2=0$	7	14	14	7	0
1	14	30	33	21	7
2	14	33	41	33	14
3	7	21	33	30	14
4	0	7	14	14	7

TABLE 4. $\mathcal{A}^{\text{opp}}_{n,k_1,k_2}$, for $n = 3, 4, 5$ and $k_1, k_2 = 0, \ldots, n-1$.

It can be seen, using (13), that (29) is equivalent to (28) at $x = y = 1$, i.e., to

$$(z_1-z_2)\,Z^{\text{opp}}_n(1,1;z_1,z_2)\,\mathcal{A}_{n-1} = (z_1-1)\,z_2\,Z_n(1,1;z_1)\,Z_{n-1}(1,1;z_2) - z_1\,(z_2-1)\,Z_{n-1}(1,1;z_1)\,Z_n(1,1;z_2). \quad (30)$$

The relation (29) can easily be solved for the opposite-boundary doubly-refined ASM numbers, giving

$$\mathcal{A}^{\text{opp}}_{n,k_1,k_2} = \tfrac{1}{\mathcal{A}_{n-1}}\textstyle\sum_{i=0}^{\min(k_1,n-k_2-1)}\big(\mathcal{A}_{n,k_1-i}\,\mathcal{A}_{n-1,k_2+i} + \mathcal{A}_{n-1,k_1-i-1}\,\mathcal{A}_{n,k_2+i} - \mathcal{A}_{n,k_1-i-1}\,\mathcal{A}_{n-1,k_2+i} - \mathcal{A}_{n-1,k_1-i-1}\,\mathcal{A}_{n,k_2+i+1}\big). \quad (31)$$

See also Ayyer and Romik [1, Eq. 1.3], and Karklinsky and Romik [25, p. 32].

It was shown by Biane, Cantini and Sportiello [5, Thm. 1] that the opposite-boundary doubly-refined and unrefined ASM numbers also satisfy

$$\det_{0\le k_1,k_2\le n-1}(\mathcal{A}^{\text{opp}}_{n,k_1,k_2}) = (-1)^{n(n+1)/2+1}\,(\mathcal{A}_{n-1})^{n-3}. \quad (32)$$

The opposite-boundary doubly-refined ASM generating function $Z^{\text{opp}}_n(1,1;z_1,z_2)$ can be computed using (30) or (31), together with (13), (16) and (23).

This function can also be expressed as

$$Z^{\text{opp}}_n(1,1;z_1,z_2) = 3^{-n(n-1)/2}\,\big(q^2(z_1+q)(z_2+q)\big)^{n-1}\times s_{(n-1,n-1,\ldots,2,2,1,1)}\big(\tfrac{qz_1+1}{z_1+q},\tfrac{qz_2+1}{z_2+q},\underbrace{1,\ldots,1}_{2n-2}\big)\big|_{q=e^{\pm 2\pi i/3}}, \quad (33)$$

where $s_{(n-1,n-1,\dots,2,2,1,1)}\bigl(\frac{qz_1+1}{z_1+q},\frac{qz_2+1}{z_2+q},1,\dots,1\bigr)$ is the Schur function indexed by the double-staircase partition $(n-1,n-1,\dots,2,2,1,1)$, evaluated at the $2n$ parameters $\frac{qz_1+1}{z_1+q}$, $\frac{qz_2+1}{z_2+q}$, $1,\dots,1$. The expression (33) was obtained by Di Francesco and Zinn-Justin [18, Eqs. (2.2) & (2.4)], using a result of Okada [31, Thm. 2.4(1), second equation].

Finally, it should be noted that certain further expressions for the ASM numbers or ASM generating functions of Sections 2.2, 3.2 and 4.2, i.e., $\mathcal{A}_n$, $\mathcal{A}_{n,k}$, $Z_n(1,1;z)$, $\mathcal{A}^{\mathrm{opp}}_{n,k_1,k_2}$ or $Z^{\mathrm{opp}}_n(1,1;z_1,z_2)$, follow from results obtained in the context of totally symmetric self-complementary plane partitions, together with a result of Fonseca and Zinn-Justin [21, Thm.] that $Z^{\mathrm{opp}}_n(1,1;z_1,z_2)$ is equal to a certain doubly-refined generating function for such plane partitions. For example, for $\mathcal{A}_n$, $Z_n(1,1;z)$ or $Z^{\mathrm{opp}}_n(1,1;z_1,z_2)$, Pfaffian expressions follow from results of Ishikawa [24, Thms. 1.2 & 1.4, & Sec. 7] and Stembridge [35, Thm. 8.3], constant-term expressions follow from results of Ishikawa [24, Sec. 8], Krattenthaler [26, Thm.] and Zeilberger [37], [38, Sublems. 1.1 & 1.2], and integral expressions (which can easily be converted to constant-term expressions) follow from results of Fonseca and Zinn-Justin [21, Eqs. (4.9) & (4.14)] and Zinn-Justin and Di Francesco [42, Eqs. (37) & (39)]. Note that some of these results are expressed in terms of certain triangles of positive integers (specifically, monotone or Gog triangles for ASMs, and Magog triangles for totally symmetric self-complementary plane partitions), or closely related integer arrays. Also, many such results are stated in more general forms, which involve certain entries of such arrays being prescribed to take certain values, or being bounded by certain values.

5. Adjacent boundary doubly-refined enumeration

5.1. **Arbitrary bulk parameters.** It was shown by Behrend [2, Cor. 3] that the adjacent-boundary doubly-refined and alternative adjacent-boundary doubly-refined ASM generating functions satisfy the recursion relations

$$\begin{aligned}(z_1-1)(z_2-1)\,Z^{\mathrm{adj}}_n(x,y;z_1,z_2)\,Z_{n-2}(x,y) &= yz_1z_2\,Z^{\mathrm{adj}}_{n-1}(x,y;z_1,z_2)\,Z_{n-1}(x,y)\,+\\ &\quad\bigl(x(z_1-1)(z_2-1)-y\bigr)\,z_1z_2\,Z_{n-1}(x,y;z_1)\,Z_{n-1}(x,y;z_2)\,+\\ &\quad(z_1-1)(z_2-1)\,Z_{n-1}(x,y)\,Z_{n-2}(x,y),\end{aligned}\tag{34}$$

$$\begin{aligned}(z_1-1)(z_2-1)\,\widetilde{Z}^{\mathrm{adj}}_n(x,y;z_1,z_2)\,Z_{n-2}(x,y) &= yz_1z_2\,\widetilde{Z}^{\mathrm{adj}}_{n-1}(x,y;z_1,z_2)\,Z_{n-1}(x,y)\,+\\ &\quad\bigl((z_1-1)(z_2-1)-yz_1z_2\bigr)\,Z_{n-1}(x,y;z_1)\,Z_{n-1}(x,y;z_2)\,+\\ &\quad(z_1-1)(z_2-1)\,(xz_1z_2)^{n-1}\,Z_{n-1}(x,y)\,Z_{n-2}(x,y).\end{aligned}\tag{35}$$

If $Z_0(x,y)$ is taken to be 1, then (34) and (35) hold for all $n\ge 2$.

It was also shown by Behrend [2, Cor. 6] that (34) and (35) can be solved for the adjacent-boundary doubly-refined ASM generating functions, giving

$$Z_n^{\mathrm{adj}}(x,y;z_1,z_2) = Z_{n-1}(x,y)\left(1+\sum_{i=1}^{n-1}\left(\frac{y\,z_1\,z_2}{(z_1-1)(z_2-1)}\right)^{n-i}\times\right.$$
$$\left.\left(1+\frac{(x(z_1-1)(z_2-1)-y)\,Z_i(x,y;z_1)\,Z_i(x,y;z_2)}{y\,Z_{i-1}(x,y)\,Z_i(x,y)}\right)\right), \quad (36)$$

$$\widetilde{Z}_n^{\mathrm{adj}}(x,y;z_1,z_2) = Z_{n-1}(x,y)\left((xz_1z_2)^{n-1}+\sum_{i=1}^{n-1}\left(\frac{y}{(z_1-1)(z_2-1)}\right)^{n-i}\times\right.$$
$$\left.\left(x^{i-1}(z_1z_2)^{n-1}+\frac{(z_1z_2)^{n-i-1}((z_1-1)(z_2-1)-yz_1z_2)\,Z_i(x,y;z_1)\,Z_i(x,y;z_2)}{y\,Z_{i-1}(x,y)\,Z_i(x,y)}\right)\right), \quad (37)$$

where, in the sums over i, $Z_0(x,y)$ is taken to be 1.

Using (36) and (37), or (34) and (35), it follows that, in addition to being related by (9), the adjacent-boundary doubly-refined and alternative adjacent-boundary doubly-refined ASM generating functions are also related by

$$((z_1-1)(z_2-1)-yz_1z_2)\,Z_n^{\mathrm{adj}}(x,y;z_1,z_2) - (x(z_1-1)(z_2-1)-y)\,z_1z_2\,\widetilde{Z}_n^{\mathrm{adj}}(x,y;z_1,z_2)$$
$$= (z_1-1)(z_2-1)\big(1-(xz_1z_2)^n\big)\,Z_{n-1}(x,y), \quad (38)$$

as shown by Behrend [2, Cor. 7].

It follows that the adjacent-boundary doubly-refined ASM generating functions can be computed using relations from this section, together with the methods given in Sections 2.1 and 3.1 for obtaining the unrefined and singly-refined ASM generating functions.

5.2. **Bulk parameters $x=y=1$.** It was shown by Stroganov [36, p. 61] that the adjacent-boundary doubly-refined, opposite-boundary doubly-refined and unrefined ASM numbers satisfy

$$\mathcal{A}^{\mathrm{adj}}_{n,k_1-1,k_2}+\mathcal{A}^{\mathrm{adj}}_{n,k_1,k_2-1}-\mathcal{A}^{\mathrm{adj}}_{n,k_1,k_2} = \mathcal{A}^{\mathrm{opp}}_{n,k_1-1,n-k_2}-(\delta_{k_1,1}-\delta_{k_1,0})(\delta_{k_2,1}-\delta_{k_2,0})\,\mathcal{A}_{n-1}. \quad (39)$$

This relation is also a special case of a formula obtained by Fischer [20, Thm. 1].

Examples of adjacent-boundary doubly-refined ASM numbers, for $n=3,4,5$, are given in Table 5.

It can be seen, using (13), that (39) is equivalent to a relation satisfied by the adjacent-boundary and opposite-boundary doubly-refined ASM generating functions at $x=y=1$ and the unrefined ASM numbers, specifically

$$(z_1+z_2-1)\,Z_n^{\mathrm{adj}}(1,1;z_1,z_2) = z_1\,z_2^n\,Z_n^{\mathrm{opp}}(1,1;z_1,\tfrac{1}{z_2})-(z_1-1)(z_2-1)\,\mathcal{A}_{n-1}. \quad (40)$$

$\mathcal{A}^{\text{adj}}_{3,k_1,k_2}$	$k_1=0$	1	2
$k_2=0$	2	0	0
1	0	2	1
2	0	1	1

$\mathcal{A}^{\text{adj}}_{4,k_1,k_2}$	$k_1=0$	1	2	3
$k_2=0$	7	0	0	0
1	0	7	5	2
2	0	5	6	3
3	0	2	3	2

$\mathcal{A}^{\text{adj}}_{5,k_1,k_2}$	$k_1=0$	1	2	3	4
$k_2=0$	42	0	0	0	0
1	0	42	35	21	7
2	0	35	49	37	14
3	0	21	37	33	14
4	0	7	14	14	7

TABLE 5. $\mathcal{A}^{\text{adj}}_{n,k_1,k_2}$, for $n = 3, 4, 5$ and $k_1, k_2 = 0, \ldots, n-1$.

The relations (39) or (40) can be solved for the adjacent-boundary doubly-refined ASM numbers, giving

$$\mathcal{A}^{\text{adj}}_{n,k_1,k_2} = \begin{cases} \mathcal{A}_{n-1}, & k_1 = k_2 = 0, \\ \binom{k_1+k_2-2}{k_1-1}\mathcal{A}_{n-1} - \sum_{i=1}^{k_1}\sum_{j=1}^{k_2}\binom{k_1+k_2-i-j}{k_1-i}\mathcal{A}^{\text{opp}}_{n,i-1,n-j}, & 1 \le k_1, k_2 \le n-1, \\ 0, & \text{otherwise}. \end{cases} \tag{41}$$

This formula was obtained by Fischer [20, p. 570]. See also Ayyer and Romik [1, p. 164].

6. Triply-refined enumeration

6.1. Arbitrary bulk parameters. It was shown by Behrend [2, Cors. 2 & 4] that the triply-refined ASM generating function satisfies

$$\begin{aligned}(z_2-z_1)(z_3-1)\, Z^{\text{tri}}_n(x,y;z_1,z_2,z_3)\, Z_{n-2}(x,y) = \\ \big((z_2-1)(z_3-1)-yz_2z_3\big)\, z_1\, Z^{\text{adj}}_{n-1}(x,y;z_1,z_3)\, Z_{n-1}(x,y;z_2) - \\ \big(x(z_1-1)(z_3-1)-y\big)\, z_1z_2z_3\, \widetilde{Z}^{\text{adj}}_{n-1}(x,y;z_2,z_3)\, Z_{n-1}(x,y;z_1) - \\ (z_1-1)(z_3-1)\, z_2\, Z_{n-1}(x,y;z_2)\, Z_{n-2}(x,y) + \\ (z_2-1)(z_3-1)\, z_1\, (xz_2z_3)^{n-1}\, Z_{n-1}(x,y;z_1)\, Z_{n-2}(x,y),\end{aligned} \tag{42}$$

and

$$\begin{aligned}y(z_2-z_1)z_3\, Z^{\text{tri}}_n(x,y;z_1,z_2,z_3)\, Z_{n-1}(x,y) = \\ (z_1-1)\big((z_2-1)(z_3-1)-yz_2z_3\big)\, Z^{\text{adj}}_n(x,y;z_1,z_3)\, Z_{n-1}(x,y;z_2) - \\ (z_2-1)\big(x(z_1-1)(z_3-1)-y\big)\, z_1z_3\, \widetilde{Z}^{\text{adj}}_n(x,y;z_2,z_3)\, Z_{n-1}(x,y;z_1) - \\ (z_1-1)(z_2-1)(z_3-1)\, Z_{n-1}(x,y;z_2)\, Z_{n-1}(x,y) + \\ (z_1-1)(z_2-1)(z_3-1)\, z_1z_2^{n-1}(xz_3)^n\, Z_{n-1}(x,y;z_1)\, Z_{n-1}(x,y).\end{aligned} \tag{43}$$

The triply-refined ASM generating function can be computed using either (42) or (43), together with the methods given in Sections 2.1, 3.1 and 5.1 for obtaining the unrefined, singly-refined and adjacent-boundary doubly-refined ASM generating functions.

6.2. **Bulk parameters $x = y = 1$.** The triply-refined ASM generating function at $x = y = 1$ satisfies

$$
\begin{aligned}
&(z_1z_3-z_3+1)(z_2z_3-z_2+1)z_3^{n-1}Z_n^{\mathrm{tri}}(1,1;z_1,z_2,\tfrac{1}{z_3}) = \\
&\qquad \frac{z_1\,z_3\,\det_{1\le i,j\le 3}\bigl(z_i^{j-1}\,(z_i-1)^{3-j}\,Z_{n-j+1}(1,1;z_i)\bigr)}{\mathcal{A}_{n-1}\,\mathcal{A}_{n-2}\,\prod_{1\le i<j\le 3}(z_i-z_j)} + \\
&\qquad\qquad (z_2-1)(z_3-1)(z_1z_3-z_3+1)z_1z_2^{n-1}Z_{n-1}(1,1;z_1) + \\
&\qquad\qquad\qquad (z_1-1)(z_3-1)(z_2z_3-z_2+1)z_3^{n-1}Z_{n-1}(1,1;z_2), \qquad (44)
\end{aligned}
$$

and

$$
\begin{aligned}
&(z_1z_3-z_3+1)(z_2z_3-z_2+1)z_3^{n-1}Z_n^{\mathrm{tri}}(1,1;z_1,z_2,\tfrac{1}{z_3}) = \\
&\quad \frac{z_1\,z_3}{\mathcal{A}_{n-2}\,(z_1-z_2)(z_3-1)}\bigl((z_1z_3-z_1+1)(z_1z_3-z_3+1)z_2\,Z_{n-1}(1,1;z_1)\,Z_{n-1}^{\mathrm{opp}}(1,1;z_2,z_3) - \\
&\qquad\qquad (z_2z_3-z_2+1)(z_2z_3-z_3+1)z_1\,Z_{n-1}(1,1;z_2)\,Z_{n-1}^{\mathrm{opp}}(1,1;z_1,z_3)\bigr) + \\
&\qquad\qquad (z_2-1)(z_3-1)(z_1z_3-z_3+1)z_1z_2^{n-1}Z_{n-1}(1,1;z_1) + \\
&\qquad\qquad\qquad (z_1-1)(z_3-1)(z_2z_3-z_2+1)z_3^{n-1}Z_{n-1}(1,1;z_2). \qquad (45)
\end{aligned}
$$

Note that (44) and (45) differ only in the first terms on each RHS.

The relation (44) was obtained by Ayyer and Romik [1, Thms. 1 & 3], with its form incorporating a suggestion of Colomo [9]. An alternative proof of (44) has been given by Behrend [2, Eqs. (49)–(50) & Sec. 5.10]. The relation (45) was obtained by Behrend [2, Cor. 11].

It was shown by Behrend [2, Eqs. (70) & (75)] that the first term on the RHS of either (44) or (45) can also be expressed as

$$
3^{-n(n-1)/2}\,z_1z_3\,\bigl(-(z_1+q)(z_2+q)(z_3+q)\bigr)^{n-1}\times \\
s_{(n-1,n-1,\dots,2,2,1,1)}\Bigl(\tfrac{qz_1+1}{z_1+q},\tfrac{qz_2+1}{z_2+q},\tfrac{qz_3+1}{z_3+q},\underbrace{1,\dots,1}_{2n-3}\Bigr)\Big|_{q=e^{\pm 2\pi i/3}},
$$

where this uses the same notation as (33).

The triply-refined ASM generating function at $x = y = 1$ can be computed using either (44) or (45), together with (16) and the methods given in Sections 3.2 and 4.2 for obtaining the singly-refined and opposite-boundary doubly-refined ASM generating functions at $x = y = 1$.

It can be seen that the identities (30) and (40), satisfied by the doubly-refined ASM generating functions at $x = y = 1$, are special cases of (44). More specifically, setting $z_3 = 1$ in (44) gives (30), while setting $z_2 = 1$ in (44), and using (30), gives (40).

7. Quadruply-refined enumeration

7.1. Arbitrary bulk parameters. It was shown by Behrend [2, Thm. 1] that the quadruply-refined ASM generating function satisfies

$$\begin{aligned}
&y(z_4-z_2)(z_1-z_3)\,Z_n^{\mathrm{quad}}(x,y;z_1,z_2,z_3,z_4)\,Z_{n-2}(x,y)=\\
&\quad\big((z_1-1)(z_2-1)-yz_1z_2\big)\big((z_3-1)(z_4-1)-yz_3z_4\big)\,Z_{n-1}^{\mathrm{adj}}(x,y;z_4,z_1)\,Z_{n-1}^{\mathrm{adj}}(x,y;z_2,z_3)-\\
&\quad\big(x(z_4-1)(z_1-1)-y\big)\big(x(z_2-1)(z_3-1)-y\big)\,z_1z_2z_3z_4\,\widetilde{Z}_{n-1}^{\mathrm{adj}}(x,y;z_1,z_2)\,\widetilde{Z}_{n-1}^{\mathrm{adj}}(x,y;z_3,z_4)-\\
&\quad(z_2-1)(z_3-1)\big((z_4-1)(z_1-1)-yz_4z_1\big)\,Z_{n-1}^{\mathrm{adj}}(x,y;z_4,z_1)\,Z_{n-2}(x,y)+\\
&\quad(z_3-1)(z_4-1)\big(x(z_1-1)(z_2-1)-y\big)\,z_1z_2\,(xz_3z_4)^{n-1}\,\widetilde{Z}_{n-1}^{\mathrm{adj}}(x,y;z_1,z_2)\,Z_{n-2}(x,y)-\\
&\quad(z_4-1)(z_1-1)\big((z_2-1)(z_3-1)-yz_2z_3\big)\,Z_{n-1}^{\mathrm{adj}}(x,y;z_2,z_3)\,Z_{n-2}(x,y)+\\
&\quad(z_1-1)(z_2-1)\big(x(z_3-1)(z_4-1)-y\big)\,z_3z_4\,(xz_1z_2)^{n-1}\,\widetilde{Z}_{n-1}^{\mathrm{adj}}(x,y;z_3,z_4)\,Z_{n-2}(x,y)+\\
&\quad(z_1-1)(z_2-1)(z_3-1)(z_4-1)\big(1-(x^2z_1z_2z_3z_4)^{n-1}\big)\,Z_{n-2}(x,y)^2. \qquad (46)
\end{aligned}$$

If $Z_0(x,y)$ is taken to be 1, then (46) holds for all $n\geq 2$.

It can be seen that (46) enables the quadruply-refined ASM generating function to be obtained recursively. More specifically, $Z_n^{\mathrm{quad}}(x,y;z_1,z_2,z_3,z_4)$ can be computed using the initial conditions (from (6)) $Z_1^{\mathrm{quad}}(x,y;z_1,z_2,z_3,z_4)=1$ and $Z_2^{\mathrm{quad}}(x,y;z_1,z_2,z_3,z_4)=1+xz_1z_2z_3z_4$, together with the definitions (from (7)–(8)) $Z_n^{\mathrm{adj}}(x,y;z_1,z_2)=Z_n^{\mathrm{quad}}(x,y;z_1,1,1,z_2)$, $\widetilde{Z}_n^{\mathrm{adj}}(x,y;z_1,z_2)=Z_n^{\mathrm{quad}}(x,y;z_1,z_2,1,1)$ and $Z_n(x,y)=Z_n^{\mathrm{quad}}(x,y;1,1,1,1)$.

Accordingly, for each $n\geq 3$, $Z_n^{\mathrm{quad}}(x,y;z_1,z_2,z_3,z_4)$ and all of the ASM generating functions of (7) and (8) which are defined in terms of $Z_n^{\mathrm{quad}}(x,y;z_1,z_2,z_3,z_4)$ are determined by (46).

Note, however, that if the generating functions are obtained recursively in this way, then, for each successive n, $Z_n^{\mathrm{quad}}(x,y;z_1,z_2,z_3,z_4)$ should first be computed for arbitrary z_1, z_2, z_3, and z_4, with the factor $(z_1-z_3)(z_4-z_2)$ being explicitly cancelled from both sides of (46), so that division by zero is avoided when boundary parameters need to be set to 1 in subsequent computations.

Alternatively, the quadruply-refined ASM generating function can be computed using (46), together with the methods given in Sections 2.1 and 5.1 for obtaining the unrefined and adjacent-boundary doubly-refined ASM generating functions.

7.2. Bulk parameters $x=y=1$. The alternative quadruply-refined ASM generating function at $x=y=1$ satisfies

$$\begin{aligned}
&(z_4z_1-z_4+1)(z_1z_2-z_1+1)(z_2z_3-z_2+1)(z_3z_4-z_3+1)\,\widetilde{Z}_n^{\mathrm{quad}}(1,1;z_1,z_2,z_3,z_4)=\\
&\quad\frac{z_1\,z_2\,z_3\,z_4\,\det_{1\leq i,j\leq 4}\big(z_i^{j-1}\,(z_i-1)^{4-j}\,Z_{n-j+1}(1,1;z_i)\big)}{\mathcal{A}_{n-1}\,\mathcal{A}_{n-2}\,\mathcal{A}_{n-3}\,\prod_{1\leq i<j\leq 4}(z_i-z_j)}+\\
&\quad(z_2-1)(z_3-1)(z_4z_1-z_4+1)(z_1z_2-z_1+1)(z_3z_4-z_3+1)(z_2z_4)^{n-1}Z_{n-1}^{\mathrm{adj}}(1,1;\tfrac{1}{z_4},z_1)+
\end{aligned}$$

$$(z_3-1)(z_4-1)(z_1z_2-z_1+1)(z_2z_3-z_2+1)(z_4z_1-z_4+1)(z_1z_3)^{n-1}Z^{\mathrm{adj}}_{n-1}(1,1;\tfrac{1}{z_1},z_2)\;+$$
$$(z_4-1)(z_1-1)(z_2z_3-z_2+1)(z_3z_4-z_3+1)(z_1z_2-z_1+1)(z_2z_4)^{n-1}Z^{\mathrm{adj}}_{n-1}(1,1;\tfrac{1}{z_2},z_3)\;+$$
$$(z_1-1)(z_2-1)(z_3z_4-z_3+1)(z_4z_1-z_4+1)(z_2z_3-z_2+1)(z_1z_3)^{n-1}Z^{\mathrm{adj}}_{n-1}(1,1;\tfrac{1}{z_3},z_4)\;-$$
$$(z_1-1)(z_2-1)(z_3-1)(z_4-1)\big((z_1z_2-z_1+1)(z_3z_4-z_3+1)(z_2z_4)^{n-1}\;+$$
$$(z_2z_3-z_2+1)(z_4z_1-z_4+1)(z_1z_3)^{n-1}\big)\mathcal{A}_{n-2}, \quad (47)$$

and

$$(z_4z_1-z_4+1)(z_1z_2-z_1+1)(z_2z_3-z_2+1)(z_3z_4-z_3+1)\,\widetilde{Z}^{\mathrm{quad}}_n(1,1;z_1,z_2,z_3,z_4)=$$
$$\frac{z_1\,z_2\,z_3\,z_4}{\mathcal{A}_{n-2}\,(z_1-z_3)(z_2-z_4)}\times$$
$$\big((z_1z_2-z_1+1)(z_1z_2-z_2+1)(z_3z_4-z_3+1)(z_3z_4-z_4+1)\,Z^{\mathrm{opp}}_{n-1}(1,1;z_4,z_1)\,Z^{\mathrm{opp}}_{n-1}(1,1;z_2,z_3)\;-$$
$$(z_4z_1-z_4+1)(z_4z_1-z_1+1)(z_2z_3-z_2+1)(z_2z_3-z_3+1)\,Z^{\mathrm{opp}}_{n-1}(1,1;z_1,z_2)\,Z^{\mathrm{opp}}_{n-1}(1,1;z_3,z_4)\big)\;+$$
$$(z_2-1)(z_3-1)(z_4z_1-z_4+1)(z_1z_2-z_1+1)(z_3z_4-z_3+1)(z_2z_4)^{n-1}Z^{\mathrm{adj}}_{n-1}(1,1;\tfrac{1}{z_4},z_1)\;+$$
$$(z_3-1)(z_4-1)(z_1z_2-z_1+1)(z_2z_3-z_2+1)(z_4z_1-z_4+1)(z_1z_3)^{n-1}Z^{\mathrm{adj}}_{n-1}(1,1;\tfrac{1}{z_1},z_2)\;+$$
$$(z_4-1)(z_1-1)(z_2z_3-z_2+1)(z_3z_4-z_3+1)(z_1z_2-z_1+1)(z_2z_4)^{n-1}Z^{\mathrm{adj}}_{n-1}(1,1;\tfrac{1}{z_2},z_3)\;+$$
$$(z_1-1)(z_2-1)(z_3z_4-z_3+1)(z_4z_1-z_4+1)(z_2z_3-z_2+1)(z_1z_3)^{n-1}Z^{\mathrm{adj}}_{n-1}(1,1;\tfrac{1}{z_3},z_4)\;-$$
$$(z_1-1)(z_2-1)(z_3-1)(z_4-1)\big((z_1z_2-z_1+1)(z_3z_4-z_3+1)(z_2z_4)^{n-1}\;+$$
$$(z_2z_3-z_2+1)(z_4z_1-z_4+1)(z_1z_3)^{n-1}\big)\mathcal{A}_{n-2}. \quad (48)$$

Note that (47) and (48) differ only in the first terms on each RHS.

Note also that (40) can be used to replace adjacent-boundary doubly-refined ASM generating functions in (47) or (48) by opposite-boundary doubly-refined ASM generating functions. For example, applying (40) to the last five terms on the RHS of (47) or (48) gives

$$(z_2-1)(z_3-1)(z_4z_1-z_4+1)(z_1z_2-z_1+1)(z_3z_4-z_3+1)(z_2z_4)^{n-1}Z^{\mathrm{adj}}_{n-1}(1,1;\tfrac{1}{z_4},z_1)\;+$$
$$(z_3-1)(z_4-1)(z_1z_2-z_1+1)(z_2z_3-z_2+1)(z_4z_1-z_4+1)(z_1z_3)^{n-1}Z^{\mathrm{adj}}_{n-1}(1,1;\tfrac{1}{z_1},z_2)\;+$$
$$(z_4-1)(z_1-1)(z_2z_3-z_2+1)(z_3z_4-z_3+1)(z_1z_2-z_1+1)(z_2z_4)^{n-1}Z^{\mathrm{adj}}_{n-1}(1,1;\tfrac{1}{z_2},z_3)\;+$$
$$(z_1-1)(z_2-1)(z_3z_4-z_3+1)(z_4z_1-z_4+1)(z_2z_3-z_2+1)(z_1z_3)^{n-1}Z^{\mathrm{adj}}_{n-1}(1,1;\tfrac{1}{z_3},z_4)\;-$$
$$(z_1-1)(z_2-1)(z_3-1)(z_4-1)\big((z_1z_2-z_1+1)(z_3z_4-z_3+1)(z_2z_4)^{n-1}\;+$$
$$(z_2z_3-z_2+1)(z_4z_1-z_4+1)(z_1z_3)^{n-1}\big)\mathcal{A}_{n-2}$$
$$=\;(z_2-1)(z_3-1)(z_1z_2-z_1+1)(z_3z_4-z_3+1)z_4z_1z_2^{n-1}Z^{\mathrm{opp}}_{n-1}(1,1;z_4,z_1)\;+$$
$$(z_3-1)(z_4-1)(z_2z_3-z_2+1)(z_4z_1-z_4+1)z_1z_2z_3^{n-1}Z^{\mathrm{opp}}_{n-1}(1,1;z_1,z_2)\;+$$
$$(z_4-1)(z_1-1)(z_3z_4-z_3+1)(z_1z_2-z_1+1)z_2z_3z_4^{n-1}Z^{\mathrm{opp}}_{n-1}(1,1;z_2,z_3)\;+$$
$$(z_1-1)(z_2-1)(z_4z_1-z_4+1)(z_2z_3-z_2+1)z_3z_4z_1^{n-1}Z^{\mathrm{opp}}_{n-1}(1,1;z_3,z_4)\;+$$

$$(z_1-1)(z_2-1)(z_3-1)(z_4-1)\big((z_1z_2-z_1+1)(z_3z_4-z_3+1)(z_2z_4)^{n-1}+ \\ (z_2z_3-z_2+1)(z_4z_1-z_4+1)(z_1z_3)^{n-1}\big)\mathcal{A}_{n-2}. \quad (49)$$

The relation (47) was obtained by Ayyer and Romik [1, Thms. 2 & 3], with its form incorporating a suggestion of Colomo [9]. An alternative proof of (47) has been given by Behrend [2, Eq. (50) & Sec. 5.10]. The relation (48) was obtained by Behrend [2, Cor. 10].

It was shown by Behrend [2, Eqs. (70) & (75)] that the first term on the RHS of either (47) or (48) can also be expressed as

$$3^{-n(n-1)/2}\, q^{4(n-1)}\, z_1z_2z_3z_4\, \big((z_1+q)\dots(z_4+q)\big)^{n-1}\times \\ s_{(n-1,n-1,\dots,2,2,1,1)}\Big(\tfrac{qz_1+1}{z_1+q},\dots,\tfrac{qz_4+1}{z_4+q},\underbrace{1,\dots,1}_{2n-4}\Big)\Big|_{q=e^{\pm 2\pi i/3}},$$

where this uses the same notation as (33).

The quadruply-refined ASM generating function at $x=y=1$ can be computed using either (47) or (48), together with (16) and the methods given in Sections 3.2, 4.2 and 5.2 for obtaining the singly-refined and doubly-refined ASM generating functions at $x=y=1$.

8. Further results

Various further results for the exact enumeration of ASMs, and involving some of the statistics of (2) and (3), are reviewed or obtained by Behrend [2, Sec. 3]. These cases include the following, where full references to the literature can be obtained using the references to [2] given here.

- Results which provide explicit expressions for all of the ASM generating functions of (5) and (7) for the case $y=0$. See [2, Sec. 3.1].
- Results which provide explicit expressions for all of the ASM generating functions of (5) and (7) for the case $y=x+1$. See [2, Sec. 3.2].
- Results for a certain ASM generating function associated with several rows (or several columns) of an ASM. (This generating function provides a certain generalization of the unrefined, singly-refined and opposite-boundary doubly-refined ASM generating functions.) See [2, Sec. 3.5].
- Results for the enumeration of ASMs with several rows or columns closest to two opposite boundaries prescribed. See [2, Sec. 3.6].
- Results for $Z_n(1,3)$ and $Z_n(1,3;z)$. See [2, Sec. 3.7].
- Results for $Z_n(1,1;-1)$. See [2, Sec. 3.8].
- Results for $Z_n(x,1)$, i.e., for the enumeration of ASMs with a prescribed number of generalized inversions. See [2, Sec. 3.9].
- Results for $Z_n(1,y)$, i.e., for the enumeration of ASMs with a prescribed number of -1's. See [2, Sec. 3.10].
- Results associated with descending plane partitions. See [2, Sec. 3.12].

- Results associated with totally symmetric self-complementary plane partitions. See [2, Sec. 3.13].
- Results associated with fully packed loop configurations and loop models. See [2, Sec. 3.14].
- Results for the enumeration of ASMs invariant under symmetry operations. See [2, Sec. 3.15].

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R. E. Behrend, School of Mathematics, Cardiff University, Cardiff, CF24 4AG, UK
E-mail address: behrendr@cardiff.ac.uk