THE EXTENDED SPECTRAL ELEMENT METHOD FOR THE APPROXIMATION OF DISCONTINUOUS FUNCTIONS

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Abstract. High-order polynomial approximations of discontinuous functions give rise to oscillations in the vicinity of the discontinuity known as Gibbs phenomenon. Enrichment of the basis using discontinuous functions is shown to remove these oscillations and to recover the convergence properties generally associated with the spectral approximation of smooth functions. The convergence properties of the enriched method, known as the extended spectral element method (XSEM) are studied and optimal error estimates are derived. The extension of these ideas to the immersed boundary method (IBM) is considered. The IBM is typically used for problems with an interface or discontinuity that is unfitted to the underlying computational mesh. An extended basis is used to approximate the pressure and the implication of this for the inf-sup for the Stokes problem is investigated.

Key words. Extended methods, XFEM, Spectral Elements, Immersed Boundary Method

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1. Introduction. Spectral methods are used extensively in many fields of scientific computing (such as computational fluid dynamics) due to their high-order accuracy and computational efficiency. If a function is sufficiently smooth, then a spectral approximation yields so-called spectral accuracy where the approximation convergences faster than algebraically. Traditional spectral methods are limited to problems defined in simple geometries. Patera [18] introduced the spectral element method (SEM) which combines the geometric flexibility of a finite element method (FEM) with the accuracy of a spectral method. In principle, SEM is similar to *hp* - FEM. SEM can obtain superior orders of convergence at a smaller computational time provided the solution is sufficiently smooth and the accepted error level is sufficiently stringent [18]. However, the convergence properties of both FEM and SEM deteriorate significantly when approximating a function which is discontinuous.

When approximating a discontinuous function using either finite elements or spectral elements, oscillations (known as Gibbs phenomenon) are present local to the discontinuity. The maximum amplitude of the oscillation (or overshoot) closest to the discontinuity tends to a finite limit and the location of the overshoot tends to the discontinuity. However, as $N \to \infty$ the amplitude of the overshoot can diverge to $+\infty$, where N is the degree of the polynomial used in the approximation. These oscillations are undesirable as they influence the approximation not only in a neighbourhood around the discontinuity but also over the entire domain. The spectral accuracy which can be obtained by spectral methods is lost when approximating a discontinuous function. The order of convergence for the h-version of FEM is also impaired when approximating a discontinuous function [21]. Hence, it is desirable to try to remove or at least control the oscillations present when approximating a discontinuous function. It is natural to attempt a smoothing (or filtering) process. In a smoothing process, the basis functions are multiplied by a so-called smoothing factor, e.g. Cesaro sums, which dampen the oscillations local to the discontinuity. However,

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too strong a smoothing could result in a smeared solution. For a more complete discussion on Gibbs phenomenon and its removal using smoothing processes, the reader is referred to the monograph of Canuto *et al.* [8, p. 56–68].

An alternative approach, common in FEM (and by natural extension to SEM), is to fit the computational mesh to the interface (and hence discontinuity) between two or more regions. This allows each region to be considered separately and thus the discontinuity can be treated in a natural way (e.g. using a discontinuous approximation). Further, any essential boundary conditions can be enforced strongly on the interface. However, due to the large computational times associated with remeshing, methods that attempt to do away with the mesh became popular (see e.g. Belytschko *et al.* [3]). These so-called meshless methods adopted the partition of unity method (PUM) [1] to enforce continuity of the approximation. An alternative to meshless methods are so-called unfitted methods. As the name suggests, in this approach any interface, discontinuity or singularity is unfitted to a background computational mesh or grid. This alleviates the need for remeshing but introduces additional complexities such as Gibbs phenomenon (briefly discussed above). The generalised finite element method (GFEM) [23] is an unfitted method which is defined as a combination of the classical FEM and the PUM. In GFEM, the classical FEM approximation is augmented (or enriched) by the addition of special functions. These special functions are defined using known information about the singularity or discontinuity.

The extended finite element method (XFEM) proposed by Moës *et al.* [17] can be considered as a specific formulation of GFEM. In XFEM, the classical FEM approximation space is augmented by the span (over a particular subset of nodes) of the standard FEM basis functions multiplied by a so-called enrichment function. The particular choice of enrichment function depends on the kind of enrichment that is required. The XFEM has been applied successfully to a wide range of problems including crack propagation problems [17] and two-phase flows [21, 13]. In 2006, Legay *et al.* [14] applied so-called spectral finite elements to the XFEM formulation. This was the first article, as far as we are aware, to consider a higher-order approximation within the XFEM framework. Chebyshev polynomials with a maximum degree of $N = 4$ were considered and convergence was studied with respect to mesh width. Optimal order of convergence was found for strong discontinuities and nearly optimal convergence was found for straight weak discontinuities. However, for curved weak discontinuities, suboptimal convergence was obtained. Legay *et al.* [14] attributed this suboptimal rate of convergence to errors in the quadrature scheme. Cheng and Fries [10] also considered higher-order XFEM applied to curved weak discontinuities in which they proposed an alternative quadrature scheme to the one considered by Legay *et al.* [14]. The modifications to the quadrature scheme are fairly complex and did not produce optimal rates of convergence for curved weak discontinuities. However, Cheng and Fries [10] introduced a modified XFEM which produced optimal rates of convergence for curved weak discontinuities. Both Legay *et al.* [14] and Cheng and Fries [10] approached the problem from the perspective of higher-order FEM and in both cases, the maximum polynomial degree considered was $N = 4$ and all convergence studies were carried out with respect to mesh width; so-called h-type convergence. This is contrary to standard spectral (and hence spectral element) studies where convergence is studied with respect to polynomial degree; so-called p-type convergence.

In this article, we consider spectral elements within the XFEM framework and hence name the method: the extended spectral element method (XSEM). This article

is not concerned with implementation issues such as blending problems. Instead, it is concerned solely with the theory of XSEM. In particular, we present convergence results for the approximation of a function in a broken Sobolev space by an extended spectral element representation. We also consider the approximation of the Stokes problem using an enriched basis for the pressure and investigate the implication of this for the inf-sup condition. The motivation for the interest in enriched methods such as XSEM stems from applications in computational fluid dynamics, such as multi-phase flow and fluid-structure interaction, in which fields are not necessarily smooth. Therefore, the use of an enriched method - which changes the approximation space - could impact any compatibility conditions present. A future article will be dedicated to the implementation and application of the method.

This article is structured as follows: in Section 2 we provide the mathematical statement of the problem under consideration. In Section 3 we introduce the spectral element discretisation followed by the extended spectral element discretisation in Section 4. Section 5 discusses the approximation results and finally Section 6 proposes a value for the discrete inf-sup parameter for the Stokes problem.

2. Mathematical Statement of the Problem. Consider the approximation of a function $u\colon \Omega \to \mathbb{R}$, where $\Omega \subset \mathbb{R}^d$ $(1 \leq d \leq 2)$, which is discontinuous across an interface Γ separating subdomains Ω_1 and Ω_2 with $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2$ and $\Omega_1 \cap \Omega_2 = \emptyset$. Let $u_i = u|_{\Omega_i}$, $i = 1, 2$, denote the restriction of u to Ω_i. Therefore, we can write:

$$u := \begin{cases} u_1 & \text{in } \Omega_1 \\ u_2 & \text{in } \Omega_2 \end{cases} \tag{2.1}$$

Furthermore, we assume that $u_i \in H^m(\Omega_i)$, where $i = 1, 2$ and $m \geq 1$. Therefore, we can define the function $u \in \mathcal{V} = H^m(\Omega_1 \cup \Omega_2)$ where $H^m(\Omega_1 \cup \Omega_2)$ is a broken Sobolev space defined by:

$$H^m(\Omega_1 \cup \Omega_2) = \left\{ u \in L^2(\Omega)\colon \ u|_{\Omega_i} \in H^m(\Omega_i), i = 1, 2 \right\} \tag{2.2}$$

and equipped with the norm

$$\|u\|^2_{H^1(\Omega_1 \cup \Omega_2)} = \sum_{i=1}^{2} \|u\|^2_{H^1(\Omega_i)} \tag{2.3}$$

In the following sections, we discuss the standard spectral element approximation, illustrate the motivation for choosing an enriched method and present the eXtended Spectral Element Method (XSEM) before finally discussing the spectral equivalent of the approximation result derived by Reusken [21] and the inf-sup condition for the velocity-pressure formulation for Stokes flow.

3. Spectral Element Method. The spectral element method (SEM) was first proposed by Patera [18] to extend the application of spectral methods to problems defined in complex geometries. It is well known that SEM *should* perform better than traditional finite elements both in terms of accuracy and efficiency provided the solution is sufficiently regular and that the accepted error level is taken to be sufficiently small. However, if the regularity of the solution is low then the spectral element method will only perform as well as finite elements [18].

The spectral element approximation is chosen to be in a conforming discrete subspace, $\mathcal{V}_N \subset \mathcal{V}$. The domain Ω is divided into K uniform, non-overlapping, quadrilateral spectral elements Ω_e, $e = 1, \ldots, K$, such that

$$\bar{\Omega} = \bigcup_{e=1}^{K} \bar{\Omega}_e \tag{3.1}$$

Let $P_N(\Omega_e)$ denote the space of all polynomials on Ω_e of degree less than or equal to N and define:

$$\mathrm{P}_N(\Omega) := \left\{\phi\colon\ \phi|_{\Omega_e} \in P_N(\Omega_e)\right\} \tag{3.2}$$

Thus the approximation space may then be defined as:

$$\mathcal{V}_N := \mathcal{V} \cap [\mathrm{P}_N(\Omega)]^d \tag{3.3}$$

where we assume $1 \leq d \leq 2$. Note that we have assumed a uniform discretisation of our domain Ω and therefore, the interface Γ is completely unfitted.

To illustrate the spectral element approximation consider the case $d = 1$. Each spectral element Ω_e, $e = 1, \ldots, K$, is mapped to the parent or reference interval $D = [-1, 1]$ using the transfinite mapping, F, of Schneidesch and Deville [22], where for each point $\xi \in D$ there exists a point $x = F(\xi) \in \Omega_e$. The approximation of a function u on the element Ω_e is then given by:

$$u_N^e(\xi) = \sum_{i=0}^{N} \hat{u}_i^e h_i(\xi) \tag{3.4}$$

where $\hat{u}_i = u(F(\xi_i))$ denotes the value of the function u at the node $x = F(\xi_i)$ $(i = 0, \ldots, N)$, $h_i(\xi)$ $(i = 0, \ldots, N)$ are the Lagrange interpolants defined on the parent interval $\xi \in [-1, 1]$ by:

$$h_i(\xi) = -\frac{(1 - \xi^2)L'_N(\xi)}{N(N+1)L_N(\xi_i)(\xi - \xi_i)} \tag{3.5}$$

and the points ξ_i, $i = 0, \ldots, N$, are the collocation points on the Gauss-Lobatto Legendre grid.

The spectral element method described above is a high-order method based on polynomial interpolation using Legendre polynomials. Therefore, within an element Ω_e, the approximation of a function u is continuous (in fact, the approximation is continuous everywhere). As we have assumed u is discontinuous across an interface Γ (which is unfitted to the underlying mesh), we expect that spurious oscillations will be produced local to the discontinuity. This phenomenon is well known and is called the Gibbs phenomenon. Gibbs phenomenon can be classified, formally, as the inability to approximate a discontinuity using continuous functions. To illustrate this phenomenon we consider the spectral interpolation of a function f on a grid of uniformly spaced points. For simplicity, we assume that $\Omega \subset \mathbb{R}$ (i.e. $d = 1$) and that we have a *single* spectral element. Define $\Omega = [-1, 1]$ and consider the interpolation of the piecewise constant function:

$$f(x) = \begin{cases} 0 & \forall x \in [-1, 10^{-4}) \\ 1 & \forall x \in [10^{-4}, 1] \end{cases} \tag{3.6}$$

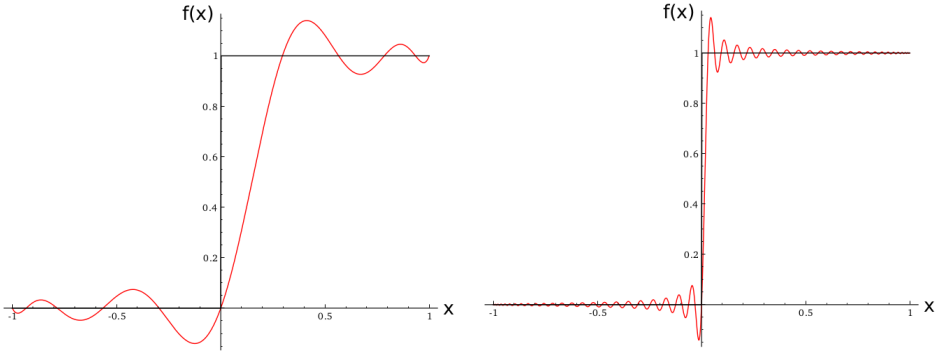
Note that the point of discontinuity is chosen to be $x = 10^{-4}$ since $x = 0$ is a member of the Gauss-Lobatto Legendre grid. We evaluate the spectral element interpolant of this function on a uniform grid

$$D_u = \{x_j \in \Omega \colon x_j = -1 + j\Delta x, j = 0, \ldots, M\}$$

where $\Delta x = 2/M$ is the constant mesh width, $x_0 = -1$ and $x_M = 1$. The interpolant is then:

$$f_N(x_k) = \sum_{i=0}^{N} f_i h_i(x_k) \qquad \forall x_k \in D_u \tag{3.7}$$

Fig. 1 shows the spurious oscillations present around the discontinuity when the number of uniformly spaced points $M = 1000$ and the polynomial degree is $N = 10$ and $N = 100$. It is evident that the Gibbs phenomenon becomes more local as N increases.



(a) $N = 10$ (b) $N = 100$

Fig. 1: Spectral interpolation of a discontinuous function with (a) $N = 10$, (b) $N = 100$.

Although this is a very simple problem, it illustrates the problems which occur when spectral methods are used to approximate a discontinuous function. The same phenomenon is seen in finite elements although it is not as severe due to the lower-order polynomial interpolation. One could argue that one should fit the discontinuity to the mesh. However, this would greatly increase the computational time, particularly if the discontinuity was allowed to move freely within the computational domain. This motivates the use of an enriched or extended method which we discuss in the next section.

4. eXtended Spectral Element Method. As for the standard spectral element approximation defined in §3, we choose a conforming discrete subspace for our approximation, $\mathcal{V}_N^\Gamma \subset \mathcal{V}$. Note that the superscript Γ denotes the enriched approximation space. As before, the domain is decomposed into K uniform, non-overlapping, quadrilateral spectral elements Ω_e, $e = 1, \ldots, K$, which are *not* fitted to Γ, such that (3.1) is satisfied. Additionally, following Groß and Reusken [13], we introduce the set

of elements containing the discontinuity Γ:

$$\Omega^\Gamma := \{\Omega_e : \Omega_e \cap \Gamma \neq \emptyset,\, e = 1, \ldots, K\} \tag{4.1}$$

In §3, the approximation space $\mathcal{V}_N$ was defined by the intersection of $\mathcal{V}$ with the polynomial space (3.2). Unfortunately, the approximation space $\mathcal{V}_N$ is not suitable for the enriched approximation since the enrichment is discontinuous inside an element $\Omega_e \in \Omega^\Gamma$. If the approximation space $\mathcal{V}_N$ is used, in the case of finite elements, we cannot expect a better bound than [21]:

$$\inf_{u_h \in \mathcal{V}_h^\Gamma} \|u - u_h\|_{L^2(\Omega)} \leq C\sqrt{h}\, \|u\|_{H^1(\Omega_1 \cup \Omega_2)} \tag{4.2}$$

We expect a similar bound for spectral elements since they will not necessarily perform any better than finite elements when the regularity of the solution decreases. However, we are unaware of any such bound for spectral elements. Therefore, we need to define a suitable approximation space, $\mathcal{V}_N^\Gamma$.

Let $\{\Psi_i,\, i \in \mathcal{I}\}$ denote the global basis functions of $\mathcal{V}_N$, with the nodal index set $\mathcal{I} = \{1, \ldots, N_V\}$ where N_V denotes the dimension of $\mathcal{V}_N$. Define the space

$$\mathcal{P}(\Omega^\Gamma) := \text{ span } \left\{\Psi_i \Phi_i,\, i \in \mathcal{I}^\Gamma\right\} \tag{4.3}$$

where $\mathcal{I}^\Gamma \subset \mathcal{I}$ is a subset of nodal points which require *enrichment* and Φ_i are global enrichment functions, which for the moment, are undefined. The enriched approximation space is then defined as:

$$\mathcal{V}_N^\Gamma := \mathcal{V}_N \oplus \mathcal{P}(\Omega^\Gamma) \tag{4.4}$$

Therefore, the global XSEM approximation $u_N^\Gamma \in \mathcal{V}_N^\Gamma$ of a function $u \in \mathcal{V}$ is

$$\begin{aligned} u_N^\Gamma(\boldsymbol{x}) &= u_N(\boldsymbol{x}) + u_N^X(\boldsymbol{x}) \\ &= \sum_{i \in \mathcal{I}} \hat{u}_i \Psi_i(\boldsymbol{x}) + \sum_{j \in \mathcal{I}^\Gamma} \alpha_j \Psi_j(\boldsymbol{x}) \Phi_j(\boldsymbol{x}) \end{aligned} \tag{4.5}$$

where u_N is the continuous (or standard) part, u_N^X is termed the discontinuous (or extended) part of the enriched approximation u_N^Γ, $\hat{u}_i$ $(i \in \mathcal{I})$ is the value of the function u at node i and α_j $(j \in \mathcal{I}^\Gamma)$ are additional degrees of freedom at the nodal points which have been enriched. The global XSEM approximation is very similar to its XFEM counterpart [13].

Again, to illustrate the approximation, we let $d = 1$. Just as was done with the standard spectral element approximation, each spectral element Ω_e is mapped on to the parent domain $D = [-1, 1]$ via the transfinite map. Note that over an element $\Omega_e \notin \Omega^\Gamma$, the local enriched approximation must satisfy $u_N^\Gamma(\xi) = u_N(\xi)$, where $u_N(\xi)$ is defined as in (3.4), so that in elements without a discontinuity the method reduces to standard SEM. If we restrict (4.5) to an element $\Omega_e \in \Omega^\Gamma$, then the local enriched approximation is:

$$\begin{aligned} u_N^\Gamma(\xi) &= u_N(\xi) + u_N^X(\xi) \\ &= \sum_{i=0}^{N} \hat{u}_i h_i(\xi) + \sum_{k=0}^{N} \alpha_k h_k(\xi) \phi_k(\xi) \end{aligned} \tag{4.6}$$

The function ϕ_k $(k = 0, \ldots, N)$ is a local version of the enrichment function Φ_j $(j \in \mathcal{I}^\Gamma)$ present in the discontinuous part of the global enriched approximation (4.5).

The choice of enrichment function depends on the type of discontinuity, or singularity, being enriched. For example, if the function under consideration has a strong discontinuity (e.g. the function f defined by (3.6)) then a common choice for the enrichment function is based on the Heaviside function. In this article, we choose the same global enrichment function as defined by Groß and Reusken [13], i.e. $\Phi_i(x) = H(x) - H(x_i)$ $(i \in \mathcal{I}^\Gamma)$, where $H(x)$ is the Heaviside function defined by:

$$H(x) = \begin{cases} 0 & x \in \Omega_1 \\ 1 & x \in \Omega_2 \end{cases} \tag{4.7}$$

Therefore, through the transfinite map, it makes sense for us to define our the version ϕ_k by $\phi_k(\xi) = H(\xi) - H(\xi_k)$ where the Heaviside function is now defined as:

$$H(\xi) = \begin{cases} 0 & x = F(\xi) \in \Omega_1 \\ 1 & x = F(\xi) \in \Omega_2 \end{cases} \tag{4.8}$$

Alternatively, if the function under consideration has a weak (or gradient) discontinuity then a common choice is the so-called *abs-enrichment*. An example of a global abs-enrichment function is given by Legay *et al.* [14]: $\Phi_i(x) = |g(x) - g(x_i)|$ $(i \in \mathcal{I}^\Gamma)$ where g is a function which gives the location of the interface Γ - such as a level set function. If polynomials of degree N are used for the continuous part of the enriched approximation, then Legay *et al.* [14] found that polynomials of degree $N - 1$ are required for the discontinuous part u_N^X in order to achieve optimal convergence. Additional considerations are required when enriching a gradient discontinuity. The elements adjacent to the enriched elements Ω^Γ have some nodes which have been enriched and others which have not; these elements are known as blending elements. Legay *et al.* [14] stated that higher-order terms appear in the blending elements which must be cancelled by the continuous field. They found that for linear spectral elements ($N = 1$) the assumed strain method [11] is required to obtain good convergence. However, for higher-order spectral elements ($N \geq 2$) Legay *et al.* [14] found good convergence, without the assumed strain method when polynomials of one degree less are used for the discontinuous part of the enriched approximation. In this article, we do not consider weak discontinuities or blending problems. Instead we focus on some theoretical questions when strong discontinuities are considered.

5. An Approximation Result for XSEM. The analysis of enriched methods of the type discussed in this article can be difficult. In fact, the only error estimates for the extended finite element method (XFEM) that we are aware of can be found in Reusken [21]. Some of the difficulty in analysing the method is due to the dependence of the approximation on the enrichment function Φ. Depending on the *kind* of enrichment which is required, this function can vary quite dramatically. Even in the case of strong and weak discontinuities at an interface, the enrichment function is completely different. Therefore, it is difficult to analyse the method in a unified manner independent of the enrichment function considered. The framework introduced by Reusken [21] provides a unified treatment of the XFEM approximation, for functions with strong discontinuities, by removing reference to the enrichment function. Before we state the results, we briefly summarize the Reusken framework [21].

Let the space of functions $\mathcal{V}$ be a broken Sobolev space of order $m \geq 1$ defined in (2.2). When $m = 0$, we define:

$$H^0(\Omega_1 \cup \Omega_2) = L^2(\Omega_1 \cup \Omega_2) = \left\{u \in L^2(\Omega) \colon u \in L^2(\Omega_i),\ i = 1, 2\right\} = L^2(\Omega) \tag{5.1}$$

The construction of the approximation space in the Reusken framework [21] is slightly different to that considered earlier in this article §4. Once again, let N_V denote the dimension of $\mathcal{V}_N$ and let $\{\Psi_i, i \in \mathcal{I}\}$, where $\mathcal{I} = \{1, \ldots, N_V\}$, denote the global basis functions spanning $\mathcal{V}_N$, where $\mathcal{V}_N$ is defined in (3.3). Additionally, let $\mathcal{X} = \{\boldsymbol{x}_k, k \in \mathcal{I}\}$ be the set of all nodal points. The enriched approximation space is defined as the restriction of the original approximation space to each side of the interface Γ. The restriction operator, $R_i \colon L^2(\Omega) \to L^2(\Omega)$, $i = 1, 2$, is defined as:

$$R_i u = \begin{cases} u|_{\Omega_i} & \text{in } \Omega_i \\ 0 & \text{otherwise} \end{cases} \tag{5.2}$$

Hence the enriched approximation space is defined as: $\mathcal{V}_N^\Gamma = R_1\mathcal{V}_N \oplus R_2\mathcal{V}_N$. In Theorem 2 of [21], Reusken showed that

$$\mathcal{V}_N^\Gamma = R_1\mathcal{V}_N \oplus R_2\mathcal{V}_N = \mathcal{V}_N \oplus \mathcal{V}_N^{\Gamma,\,1} \oplus \mathcal{V}_N^{\Gamma,\,2}$$

where

$$\mathcal{V}_N^{\Gamma,\,i} = \text{ span } \left\{R_i\Psi_j \colon j \in \mathcal{I}_i^\Gamma\right\} \qquad i = 1, 2$$

and

$$\mathcal{I}_1^\Gamma = \{j \in \mathcal{I} \colon \boldsymbol{x}_j \in \Omega_2 \quad \underline{\text{and}} \quad \text{supp}(\Psi_j) \cap \Gamma \neq \emptyset\} \tag{5.3}$$

$$\mathcal{I}_2^\Gamma = \{j \in \mathcal{I} \colon \boldsymbol{x}_j \in \Omega_1 \quad \underline{\text{and}} \quad \text{supp}(\Psi_j) \cap \Gamma \neq \emptyset\} \tag{5.4}$$

Additionally, it was shown that one may write the approximation $u_N^\Gamma \in \mathcal{V}_N^\Gamma$ in the form

$$u_N^\Gamma = u_N + \sum_{k \in \mathcal{I}_1^\Gamma} \beta_k^{(1)} R_1\Psi_k + \sum_{k \in \mathcal{I}_2^\Gamma} \beta_k^{(2)} R_2\Psi_k \tag{5.5}$$

where $u_N \in \mathcal{V}_N$ is the standard continuous approximation and $\beta_k^{(i)}$, $i = 1, 2$, are additional degrees of freedom. We are now in a position to present the XSEM equivalent of the approximation results derived by Reusken [21] for the case of a single spectral element.

Corollary 5.1. *Let $\Omega = [-1, 1]^d$ with $1 \leq d \leq 2$. Define $\mathcal{E}_i^m \colon H^m(\Omega_i) \to H^m(\Omega)$ to be an extension operator such that*

$$(\mathcal{E}_i^m v)|_{\Omega_i} = v \qquad \textit{and} \qquad \|\mathcal{E}_i^m v\|_{H^m(\Omega)} \leq c\,\|v\|_{H^m(\Omega_i)} \qquad \forall v \in H^m(\Omega_i)$$

Let $\pi_N^m \colon H^m(\Omega) \to \mathcal{V}_N := H^m(\Omega) \cap P_N(\Omega)$ be a projection operator satisfying

$$\|w - \pi_N^m w\|_{L^2(\Omega)} \leq C_1 N^{-m}\,\|w\|_{H^m(\Omega)} \qquad (m \geq 0) \tag{5.6}$$

$$|w - \pi_N^m w|_{H^1(\Omega)} \leq C_2 N^{1-m}\,\|w\|_{H^m(\Omega)} \qquad (m \geq 2) \tag{5.7}$$

$\forall w \in H^m(\Omega)$, *[8, p. 314]. Then*

$$\inf_{u_N^\Gamma \in \mathcal{V}_N^\Gamma} \left\| u - u_N^\Gamma \right\|_{L^2(\Omega_1 \cup \Omega_2)} \leq C N^{-m} \left\| u \right\|_{H^m(\Omega_1 \cup \Omega_2)} \qquad (m \geq 0) \tag{5.8}$$

$$\inf_{u_N^\Gamma \in \mathcal{V}_N^\Gamma} \left\| u - u_N^\Gamma \right\|_{H^1(\Omega_1 \cup \Omega_2)} \leq C N^{1-m} \left\| u \right\|_{H^m(\Omega_1 \cup \Omega_2)} \qquad (m \geq 2) \tag{5.9}$$

$\forall u \in H^m(\Omega_1 \cup \Omega_2)$.

Proof. The steps required in proving the above approximation result are identical to the steps used by Reusken [21] and therefore the details are omitted. The only difference between the proofs is the use of the projection operator π_N^m and the use of the H^1-semi-norm. Note that either a projection operator or an interpolation operator can be used in the analysis since the asymptotic behaviour of the interpolation and projection errors is the same [8] on a Gauss-Lobatto-Legendre grid. □

5.1. Approximation of a discontinuous function. We compare the accuracy of the standard spectral element method against the extended spectral element method when approximating a discontinuous, one dimensional function. We choose a non-polynomial, smooth function $f\colon \Omega \to \mathbb{R}$ and add a discontinuity at a particular point. Let $\Omega = [-1, 1]$ contain two subdomains $\Omega_1 = [-1, -1/3)$ and $\Omega_2 = [-1/3, 1]$ with interface

$$\Gamma = \partial\Omega_1 \cap \partial\Omega_2 = -\frac{1}{3}$$

Consider the function f defined by

$$f(x) = \begin{cases} \sin x & \text{in } \Omega_1 \\ \sin x + 3.3 & \text{in } \Omega_2 \end{cases} \tag{5.10}$$

The spectral element and extended spectral element approximations, denoted f_N and f_N^Γ respectively, on the domain Ω are given by:

$$f_N(x) = \sum_{i=0}^{N} f_i h_i(x) \tag{5.11}$$

$$f_N^\Gamma(x) = \sum_{i=0}^{N} f_i h_i(x) + \sum_{j=0}^{N} \alpha_j h_j(x) \phi_j(x) \tag{5.12}$$

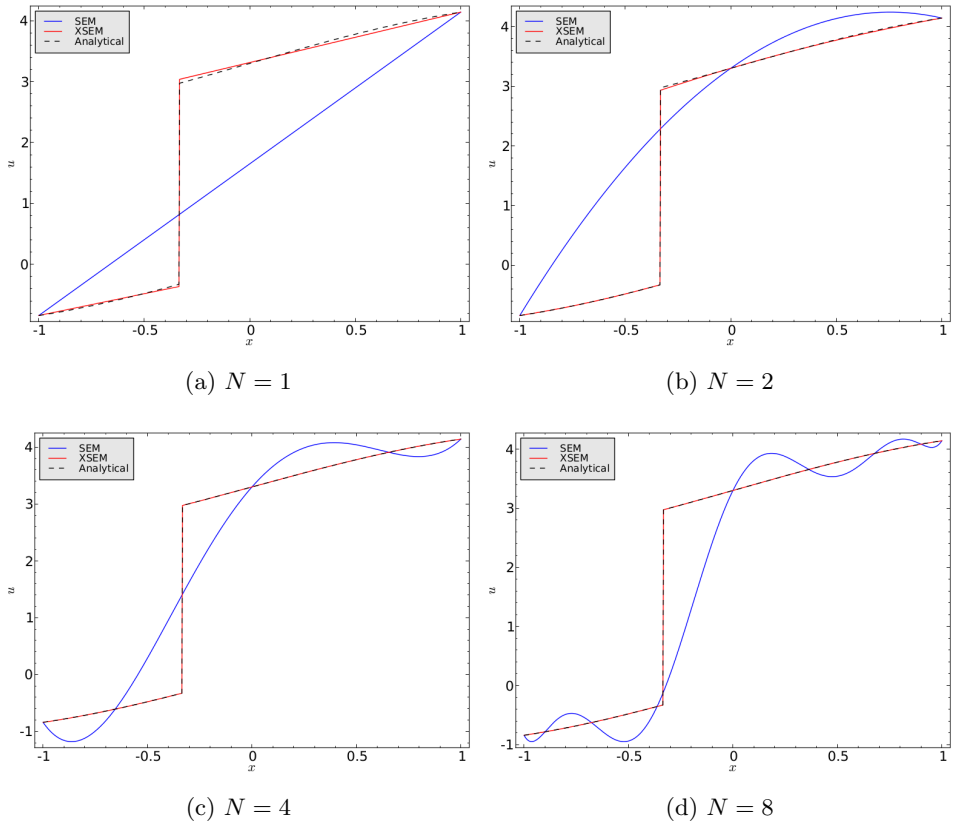
where $f_i = f(x_i)$, $i = 0, \ldots, N$, α_j are the additional degrees of freedom as a result of the enrichment and ϕ_j is the enrichment function defined by: $\phi_j(x) = H(x) - H(x_j)$, where $H(x)$ is the Heaviside step-function defined in (4.7). Note that for simplicity, we consider a single spectral element.

The coefficients α_j are additional degrees of freedom that need to be determined. In order to calculate them, we assume that $f_N^\Gamma(x_k) \equiv f(x_k)$, $\forall x_k \in D_{GLL}$, where D_{GLL} is the Gauss-Lobatto Legendre grid of order N_Q, where $N_Q \gg N$. The coefficients α_j are then found from the residual of the standard SEM approximation:

$$\sum_{j=0}^{N} M_{k,j} \alpha_j = f(x_k) - \sum_{i=0}^{N} f_i h_i(x_k) \qquad \forall x_k \in D_{GLL} \tag{5.13}$$

where $M_{k,j} = h_j(x_k)\phi_j(x_k)$. Note that as $N_Q \gg N$, the matrix M is not square. Therefore it is inverted by multiplying by its transpose to produce a square matrix

$M^T M$, which is then inverted by means of a Choleski factorisation. Fig. 2 compares the SEM and XSEM approximations of the discontinuous function. We can see clearly the standard SEM approximation is characterised by oscillations around the discontinuity (when $N = 8$). However, there are no oscillations present in the XSEM approximation and the discontinuity is captured exactly.



(a) $N = 1$ (b) $N = 2$

(c) $N = 4$ (d) $N = 8$

Fig. 2: Comparison of the SEM and XSEM approximations of the discontinuous function f with the analytical solution for (a) $N = 1$, (B) $N = 2$, (c) $N = 4$, (d) $N = 8$.

For smooth functions, spectral methods can obtain exponential order of convergence, see e.g. [8, 9]. Table 1 presents the order of convergence for both SEM and XSEM approximation of the discontinuous function f defined in (5.10) with respect to the L^2 norm. The L^2 norm is approximated using Gauss-Lobatto Legendre quadrature. Firstly we note that the SEM approximation does not converge monotonically to the asymptotic value, instead it seems to oscillate towards a value of approximately 0.5. These oscillations are most likely due to Gibbs phenomenon. It is clear from Table 1, that the order of convergence for the XSEM approximation of this discontinuous function is exponential. This illustrates the power of an enriched method and validates the approximation estimate given in §5.

N	$\|f - f_N\|_{L^2(\Omega)}$	Order	$\|f - f_N^\Gamma\|_{L^2(\Omega)}$	Order
1	1.57870	-	0.04709	-
2	1.37584	0.19843	0.01407	1.74328
4	0.80355	0.77586	6.0328×10^{-5}	7.86509
8	1.06387	-0.40487	2.6811×10^{-10}	17.77967
16	0.62365	0.77051	-	-
32	0.32542	0.93845	-	-
64	0.43223	-0.40950	-	-
128	0.28190	0.61661	-	-
256	0.16528	0.77031	-	-

Table 1: Order of convergence in the L^2 norm with respect to N for both SEM and XSEM.

6. Discrete Inf-Sup Condition for the Stokes Problem. Our motivation for considering an enriched method stems from fluid-structure interaction problems using the immersed boundary method (IBM) [19]. Any weak or strong discontinuities that occur are unfitted in the IBM and hence, introduce oscillations in the approximation of the fluid variables. One of the common examples used throughout the immersed boundary literature is that of an incompressible elastic boundary or fibre immersed in a Newtonian fluid. Such an example is known as a co-dimension one example because the immersed boundary is one dimension less than the surrounding fluid. In such an example, the pressure has a strong discontinuity across the fluid-structure interface and hence its approximation needs to be enriched. However, using an enriched approximation space for the pressure introduces the question of whether the approximation spaces are still compatible. Reusken [21] and Groß and Reusken [13] considered the inf-sup condition for XFEM numerically and found that so-called *regions of small support* needed to be removed in order to improve inf-sup stability.

We begin by recalling the continuous inf-sup condition for Stokes flow and stating some results of Maday *et al.* [15, 16] for the discrete inf-sup condition for the standard SEM approximation, before finally discussing the discrete inf-sup condition for the XSEM approximation. Throughout this section, we assume that $\Omega = [-1, 1]^2$ and consider only a single spectral element. The velocity-pressure formulation of Stokes flow is given by:

$$-\nabla^2 \boldsymbol{u} + \nabla p = \boldsymbol{f} \qquad \text{in } \Omega \tag{6.1a}$$

$$\nabla \cdot \boldsymbol{u} = 0 \qquad \text{in } \Omega \tag{6.1b}$$

$$\boldsymbol{u} = \boldsymbol{u}_D \qquad \text{on } \partial\Omega \tag{6.1c}$$

where $\boldsymbol{u}$ is the velocity field, p is the pressure, $\boldsymbol{f}$ is a source term composed of body forces and $\boldsymbol{u}_D$ is the Dirichlet boundary condition.

The spectral element method is based on solving the equations of motion in their

equivalent weak form and therefore, we define velocity and pressure spaces:

$$\mathcal{V} := [H_0^1(\Omega)]^2 = \left\{\boldsymbol{v} \in [H^1(\Omega)]^2 \colon \boldsymbol{v} = 0 \text{ on } \partial\Omega\right\} \qquad \subset [H^1(\Omega)]^2 \tag{6.2a}$$

$$\mathcal{Q} := L_0^2(\Omega) = \left\{q \in L^2(\Omega) \colon \int_\Omega q \, \mathrm{d}\Omega = 0\right\} \qquad \subset L^2(\Omega) \tag{6.2b}$$

respectively. The zero mean pressure condition present in the definition of $\mathcal{Q}$ is required in order to remove any indeterminacy in the pressure. The corresponding weak formulation of Stokes problem is: find $(\boldsymbol{u}, p) \in \mathcal{V} \times \mathcal{Q}$ such that

$$a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle \qquad \forall \boldsymbol{v} \in \mathcal{V} \tag{6.3a}$$

$$b(\boldsymbol{u}, q) = 0 \qquad \forall q \in \mathcal{Q} \tag{6.3b}$$

where $\langle \cdot, \cdot \rangle$ denotes a duality pairing between $\mathcal{V}'$ and $\mathcal{V}$ (where $\mathcal{V}'$ is the dual of $\mathcal{V}$). The bilinear forms $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ induce continuous linear operators $A \colon \mathcal{V} \to \mathcal{V}'$ and $B \colon \mathcal{V} \to \mathcal{Q}'$, respectively, such that

$$\langle A\boldsymbol{u}, \boldsymbol{v} \rangle_{\mathcal{V}' \times \mathcal{V}} = a(\boldsymbol{u}, \boldsymbol{v}) \qquad \forall \boldsymbol{u} \in \mathcal{V},\ \forall \boldsymbol{v} \in \mathcal{V} \tag{6.4}$$

$$\langle B\boldsymbol{v}, q \rangle_{\mathcal{Q}' \times \mathcal{Q}} = \langle \boldsymbol{v}, B^* q \rangle_{\mathcal{V} \times \mathcal{V}'} = b(\boldsymbol{u}, q) \qquad \forall \boldsymbol{v} \in \mathcal{V},\ \forall q \in \mathcal{Q} \tag{6.5}$$

where $B^* \colon \mathcal{Q} \to \mathcal{V}'$ is the transpose, or adjoint, of the operator B. These operators allow us to write (6.3) in an equivalent dual formulation:

$$A\boldsymbol{u} + B^* p = \boldsymbol{f} \qquad \text{in } \mathcal{V}' \tag{6.6a}$$

$$B\boldsymbol{u} = 0 \qquad \text{in } \mathcal{Q}' \tag{6.6b}$$

The inf-sup condition arises from consideration of the existence and uniqueness of the solution to (6.3) and is dependent on the properties of the operator B - specifically its range and kernel, $\mathrm{Rg}(B)$ and $\ker(B)$, respectively. Thus, we summarize Theorem 1.1 of Brezzi and Fortin [7]:

THEOREM 6.1. *Let the bilinear forms $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ be continuous on $\mathcal{V} \times \mathcal{V}$ and $\mathcal{V} \times \mathcal{Q}$, respectively. Let $\mathrm{Rg}(B)$ be closed in $\mathcal{Q}'$; that is, there exists $k_0 > 0$ such that*

$$\sup_{\boldsymbol{v} \in \mathcal{V}} \frac{b(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_{\mathcal{V}}} \geq k_0 \|q\|_{\mathcal{Q}/\ker(B^*)} \tag{6.7}$$

Moreover, let the bilinear form $a(\cdot, \cdot)$ be coercive on $\ker(B)$; that is, there exists α_0 such that

$$a(\boldsymbol{v}_0, \boldsymbol{v}_0) \geq \alpha_0 \|\boldsymbol{v}_0\|_{\mathcal{V}}^2 \qquad \forall \boldsymbol{v}_0 \in \ker(B) \tag{6.8}$$

Then there exists a unique solution $(\boldsymbol{u}, p) \in \mathcal{V} \times \mathcal{Q}/\ker(B^*)$ to (6.3) for any $\boldsymbol{f} \in \mathcal{V}'$ provided $0 \in \mathrm{Rg}(B)$.

The statement (6.7) is known as the continuous inf-sup condition and mathematically it describes compatibility between the velocity and pressure spaces.

We now turn to the finite-dimensional setting of the discrete problem. We require suitable approximation spaces for the velocity and pressure which we denote $\mathcal{V}_N \subset \mathcal{V}$ and $\mathcal{Q}_N \subset \mathcal{Q}$, respectively, and for the moment, we leave them undefined. The discrete problem is then: find $(\boldsymbol{u}_N, p_N) \in \mathcal{V}_N \times \mathcal{Q}_N$ such that

$$a(\boldsymbol{u}_N, \boldsymbol{v}_N) + b(\boldsymbol{v}_N, p_N) = \langle \boldsymbol{f}_N, \boldsymbol{v}_N \rangle \qquad \forall \boldsymbol{v}_N \in \mathcal{V}_N \tag{6.9a}$$

$$b(\boldsymbol{u}_N, q_N) = 0 \qquad \forall q_N \in \mathcal{Q}_N \tag{6.9b}$$

Once again the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ induce linear operators A_N and B_N, respectively. The corresponding discrete dual formulation can now be expressed as:

$$A_N \boldsymbol{u}_N + B_N^* p_N = \boldsymbol{f}_N \qquad \text{in } \mathcal{V}_N' \tag{6.10a}$$

$$B_N \boldsymbol{u}_N = 0 \qquad \text{in } \mathcal{Q}_N' \tag{6.10b}$$

In Theorem 6.1, the pressure solution is uniquely determined up to an element of $\ker(B^*)$ (a constant), whereas for the discrete problem, the pressure solution is uniquely determined up to an element of $\ker(B_N^*)$. Therefore we require that $\ker(B_N^*) \subseteq \ker(B^*)$, otherwise spurious pressure modes of the discrete problem may be generated. We begin by considering the standard spectral method based on Legendre polynomials, with a single spectral element, over the domain $\Omega = [-1,1]^2$ before considering an extended spectral method.

The first step in analysing an inf-sup condition, is to guarantee that there does not exist any spurious modes. Let $\mathcal{V}_N$ and $\mathcal{Q}_N$ be defined as:

$$\mathcal{V}_N := \mathcal{V} \cap [\mathrm{P}_N(\Omega)]^2 \qquad\qquad \mathcal{Q}_N := \mathcal{Q} \cap \mathrm{P}_{N-2}(\Omega) \ , \tag{6.11}$$

respectively, where P_N is as defined in (3.2) and the subscript N denotes polynomials of degree N. Also let $\mathcal{M}_N = \mathcal{Q} \cap \mathrm{P}_N(\Omega)$ and $\mathcal{D}_N$ be the range of $\mathcal{V}_N$ by the divergence operator. Denote by $\mathcal{Z}_N$ the set of all polynomials of $\mathrm{P}_N(\Omega)$ that are orthogonal to $\mathcal{D}_N$ with respect to the $L^2(\Omega)$ inner product. Then according to Lemma 3.3 of Maday *et al.* [16], and Proposition 4.1 of Bernardi and Maday [5, p. 126], the set $\mathcal{Z}_N$ is given by:

$$\begin{aligned} \mathcal{Z}_N = \ \text{span} \ \big\{ & 1, L_N(x), L_N(y), L_N(x)L_N(y), L'_{N+1}(x)L'_{N+1}(y), \\ & L'_N(x)L'_{N+1}(y), L'_{N+1}(x)L'_N(y), L'_N(x)L'_N(y) \big\} \end{aligned} \tag{6.12}$$

If we let $p_N \in \mathcal{M}_N$, then $\mathcal{Z}_N$ is the set of spurious modes because the addition of any element of $\mathcal{Z}_N$, or linear combination thereof, to the pressure approximation will not affect the velocity solution. The approximation space $\mathcal{Q}_N$ is then chosen such that $\mathcal{M}_N = \mathcal{Q}_N \oplus \mathcal{Z}_N$, in other words the pressure is approximated using polynomials of degree $N-2$.

Remark 1. *Actually, it is not necessary for the pressure to be approximated using polynomials of degree $N-2$. It was shown by Bernardi and Maday [6] that the pressure can be a polynomial of degree m where*

- *$m = N - \lambda$ for a fixed $\lambda \geq 2$,*
- *$m = N - \lambda N^\alpha$ for two real numbers $\lambda > 0$ and $0 < \alpha < 1$,*
- *$m = \lambda\sqrt{N}$ for a positive and small enough real number λ,*
- *$m = \lambda N$ for $0 < \lambda < 1$.*

It was shown in [6] that when $m = \lambda N$ the discrete inf-sup parameter is independent of N.

This is the so-called $P_N \times P_{N-2}$ method of Maday *et al.* [16] and guarantees that $\ker(B_N^*) \subseteq \ker(B^*)$. From this we can deduce that the velocity and pressure approximation spaces are indeed compatible. According to Remark 2.10 of Brezzi and Fortin [7], if $\ker(B_N^*) \subseteq \ker(B^*)$ and the continuous inf-sup condition (6.7) holds then we can guarantee that the discrete inf-sup condition holds; that is there exists a $k_N > 0$, in general dependent on N, such that:

$$\sup_{\boldsymbol{v}_N \in \mathcal{V}_N} \frac{b(\boldsymbol{v}_N, q_N)}{\|\boldsymbol{v}_N\|_{\mathcal{V}}} \geq k_N \|q_N\|_{\mathcal{Q}_N / \ker(B_N^*)} \tag{6.13}$$

The above inequality is the mathematical statement that the velocity and pressure approximation spaces are compatible. This, together with the assumption that the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are continuous on $\mathcal{V}\times\mathcal{V}$ and $\mathcal{V}\times\mathcal{Q}$, respectively, and that $a(\cdot,\cdot)$ is coercive on $\ker(B_N)$ is sufficient to prove that the problem (6.9) is well-posed and has a unique solution $(\boldsymbol{u}_N, p_N)\in\mathcal{V}_N\times\mathcal{Q}_N/\ker(B_N^*)$ (Proposition 2.1 of Brezzi and Fortin [7]). The next step is to attempt to obtain an expression for the parameter k_N as it is crucial in determining *a-priori* error estimates. The method used in determining k_N is *very* technical and can be found in Bernardi and Maday [5] for 2D and Maday *et al.* [16] for 3D and is omitted here. The discrete inf-sup condition takes the form

$$\sup_{\boldsymbol{v}_N\in\mathcal{V}_N}\frac{b(\boldsymbol{v}_N,q_N)}{\|\boldsymbol{v}_N\|_{\mathcal{V}}}\geq cN^{\frac{1-d}{2}}\,\|q_N\|_{\mathcal{Q}_N/\ker(B_N^*)} \tag{6.14}$$

where $d=2,3$. The fact that k_N is dependent on N is a consequence of the pressure being approximated by polynomials of degree $N-2$. The factor $N^{\frac{1-d}{2}}$ impairs the order of convergence for the pressure approximation.

We now move to discuss the inf-sup analysis for XSEM, where the aim is to determine the parameter k_N. Let Ω contain two subdomains Ω_1 and Ω_2, respectively, such that $\bar{\Omega}=\bar{\Omega}_1\cup\bar{\Omega}_2$ with the interface $\Gamma=\bar{\Omega}_1\cap\bar{\Omega}_2$. Note that again we assume that we have a single spectral element and hence, Γ is unfitted the computational mesh. Let $\mathcal{Q}_N^\Gamma$ be the approximation space for the enriched pressure. In XSEM, additional basis functions are added to the original approximation; typically we would have for $p_N^\Gamma\in\mathcal{Q}_N^\Gamma$:

$$p_N^\Gamma(x,y)=\sum_{i=0}^{N}\sum_{j=0}^{N}\hat{p}_{i,j}L_i(x)L_j(y)+\sum_{i=0}^{N}\sum_{j=0}^{N}\hat{\alpha}_{i,j}L_i(x)L_j(y)\Phi(x,y) \tag{6.15}$$

where L_i, $i=0,\ldots,N$, are the Legendre polynomials up to degree N, the tensor product of which form a basis for $\mathrm{P}_N(\Omega)$, and $\Phi(x,y)$ is an enrichment function which we leave undefined for the moment. Therefore we wish to consider:

$$b(\boldsymbol{u}_N,q_N^\Gamma)=\int_\Omega(\nabla\cdot\boldsymbol{u}_N)\,L_k(x)L_l(y)\,\mathrm{d}\Omega+\int_\Omega(\nabla\cdot\boldsymbol{u}_N)\,L_k(x)L_l(y)\Phi(x,y)\,\mathrm{d}\Omega \tag{6.16}$$

where $k,l=0,\ldots,N$. Proceeding as before, we let $\mathcal{D}_N$ be the range of $\mathcal{V}_N$ by the divergence operator and denote by $\mathcal{Z}_N^\Gamma$ the set of all polynomials of $\mathcal{Q}_N^\Gamma=\mathrm{P}_N(\Omega)\oplus\mathcal{P}(\Omega^\Gamma)$ that are orthogonal to $\mathcal{D}_N$ with respect to the L^2 inner product, where

$$\mathcal{P}(\Omega^\Gamma)=\ \text{span}\ \{L_i(x)L_j(y)\Phi(x,y)\colon i,j=0,\ldots,N\}$$

It is clear that the set $\mathcal{Z}_N^\Gamma$ is dependent on the enrichment function. As we are interested in strong discontinuities here, we define $\Phi(x,y)=H(x,y)$. Substituting this into the form (6.16) gives:

$$\begin{aligned}b(\boldsymbol{u}_N,q_N^\Gamma)&=\int_\Omega(\nabla\cdot\boldsymbol{u}_N)\,L_k(x)L_l(y)\,\mathrm{d}\Omega+\int_\Omega(\nabla\cdot\boldsymbol{u}_N)\,L_k(x)L_l(y)H(x,y)\,\mathrm{d}\Omega\\&=\int_\Omega(\nabla\cdot\boldsymbol{u}_N)\,L_k(x)L_l(y)\,\mathrm{d}\Omega+\int_{\Omega_2}(\nabla\cdot\boldsymbol{u}_N)\,L_k(x)L_l(y)\,\mathrm{d}\Omega\end{aligned} \tag{6.17}$$

where $k,l=0,\ldots,N$. We can see that if k,l are chosen such that $L_k(x)L_l(y)\in\mathcal{Z}_N$, (where $\mathcal{Z}_N$ is as defined in (6.12)), then the first integral is zero. Thus the original set

$\mathcal{Z}_N$ is still a set of spurious modes for the XSEM approximation. However, we cannot guarantee that it is the only set and, in general, we can only guarantee $\mathcal{Z}_N \subset \mathcal{Z}_N^\Gamma$.

The next simplest step, would be to consider $L_k(x)L_l(y)\Phi(x,y)$ as a possible spurious mode. However, the second integral is not necessarily zero because the integration is over a subset of $\Omega = [-1,1]^2$. Having said that, we note that if the second integral above is small in comparison to the first integral (or very close to zero) then one could infer a *pseudo-spurious* mode. Intuitively, this suggests that the required *amount* of enrichment is small. This is quite surprising as it implies that the enrichment could potentially result in a spurious pressure mode. However, this idea agrees with the work of Groß and Reusken [13], who (in the case of finite elements) found improved inf-sup stability if they removed some of the enrichment corresponding to regions of small support, i.e. when Ω_2 is *small* in the integral above. Hence one may, possibly, deduce $\mathcal{Z}_N^\Gamma = \mathcal{Z}_N \oplus \mathcal{Z}_N^X$ where $\mathcal{Z}_N^X$ contains any terms involving the enrichment function. As we have assumed a single spectral element, the region Ω_2 is unlikely to be *small* and therefore, can be considered to be *meaningful*. In other words, it becomes reasonable for us to assume that using polynomials of degree $N-2$ for the pressure will remove the majority of the spurious modes. However, removing $\mathcal{Z}_N^X$ is not as simple. This is a subject of future research and requires a much more in-depth analysis than presented here.

REMARK 2. *In practice, a large number of elements are used in the decomposition of* Ω. *As the number of elements increases, the mesh width of each element correspondingly decreases and therefore, it becomes increasingly likely that* pseudo-spurious *modes will be present due to regions of small support. Groß and Reusken [13] suggested skipping those extended basis functions with small contributions (or small support). However, as far as we are aware, it is currently not clear how to choose which extended basis functions to ignore.*

We wish to determine the value of the parameter k_N. In general, its existence is not guaranteed currently, as we cannot say with absolute certainty that $\ker(B_N^*) \subseteq \ker(B^*)$ due to the potential existence of spurious modes caused by the enrichment term. However, as we have assumed a single spectral element, it is a reasonable assumption and therefore, we will proceed on this basis. The XSEM approximation of the pressure is defined as in (4.5):

$$\begin{aligned} p_N^\Gamma(x,y) &= p_N(x,y) + p_N^X(x,y) \\ &= \sum_{i\in\mathcal{I}} p_i \Psi_i(x,y) + \sum_{j\in\mathcal{I}^\Gamma} \alpha_j \Psi_j(x,y)\Phi_j(x,y) \end{aligned} \tag{6.18}$$

where $\mathcal{I} = \{1,\dots,N_Q\}$, with N_Q being the dimension of $\mathcal{Q}_N$, $\{\Psi_k, k\in\mathcal{I}\}$ are the global basis functions spanning $\mathcal{Q}_N$ and $\mathcal{I}^\Gamma \subset \mathcal{I}$. The enrichment function Φ_j, $j \in \mathcal{I}^\Gamma$, once again presents some added difficulty as far as the analysis is concerned. Therefore, we look to the framework of Reusken [21] as a means of removing the dependence on the enrichment function. We postulate the following proposition.

PROPOSITION 1. *Let* $\Omega = [-1,1]^2$ *contain two sub-domains* Ω_1 *and* Ω_2 *with an interface* $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$. *Let* $(\boldsymbol{u}_N, p_N) \in \mathcal{V}_N \times \mathcal{Q}_N^\Gamma$ *denote the discrete velocity and pressure solutions to Stokes problem* (6.1). *Assume that a single spectral element is used so that* Γ *is unfitted and assume the regions* Ω_1 *and* Ω_2 *are meaningful (i.e. the extended basis functions have sufficiently large contributions in those regions). Then, the discrete inf-sup parameter is given by:* $k_N = \frac{1}{2}CN^{-1/2}$.

Proof. Using the framework of Reusken [21], the enriched pressure approximation

becomes:

$$p_N^\Gamma = \sum_{i\in\mathcal{I}} p_i\Psi_i(x,y) + \sum_{k\in\mathcal{I}_1^\Gamma} \beta_k^{(1)} E_1^0[R_1\Psi_k(x,y)] + \sum_{k\in\mathcal{I}_2^\Gamma} \beta_k^{(2)} E_2^0[R_2\Psi_k(x,y)] \tag{6.19}$$

where the sets $\mathcal{I}_i^\Gamma$, $i=1,2$, are given by:

$$\mathcal{I}_1^\Gamma = \{i\in\mathcal{I}\colon \boldsymbol{x}_i\in\Omega_2 \quad \underline{\text{and}} \quad \text{supp}(\Psi_i(x,y))\cap\Gamma\neq\emptyset\} \tag{6.20}$$

$$\mathcal{I}_2^\Gamma = \{i\in\mathcal{I}\colon \boldsymbol{x}_i\in\Omega_1 \quad \underline{\text{and}} \quad \text{supp}(\Psi_i(x,y))\cap\Gamma\neq\emptyset\} \tag{6.21}$$

with $\mathcal{I}=\{1,\ldots,N_Q\}$. The expression (6.19) can be written as: $q_N^\Gamma = q_N + q_{N,1}^X + q_{N,2}^X$ where $q_N\in\mathcal{Q}_N$ is the standard SEM approximation and $q_{N,i}^X\in\mathcal{Q}_{N,i}^X$, $i=1,2$, are the discontinuous parts, where

$$\mathcal{Q}_{N,i}^X = \text{ span } \left\{E_i^0[R_i\Psi_k(x,y)], k\in\mathcal{I}_i^\Gamma\right\} \qquad i=1,2 \tag{6.22}$$

are defined as the *extended* spaces. Note that the expression (6.19) above is slightly different to (5.5) due to the restriction operator, R_i, $i=1,2$, above being defined slightly differently here compared to (5.2). Let $R_i\colon L^2(\Omega)\to L^2(\Omega_i)$, $i=1,2$, be the restriction operator and let $E_i\colon L^2(\Omega_i)\to L^2(\Omega)$ denote an extension operator, which acts as an inverse so that

$$E_i[R_iu] = u \qquad i=1,2,\ \forall u\in L^2(\Omega) \tag{6.23}$$

Finally, denote by $E_i^0\colon L^2(\Omega_i)\to L^2(\Omega)$, $i=1,2$, the zero extension operator such that

$$E_i^0[w] = \begin{cases} w(x,y) & \text{in } \Omega_i \\ 0 & \text{in } \Omega\setminus\Omega_i \end{cases} \qquad i=1,2,\ \forall w\in L^2(\Omega_i) \tag{6.24}$$

Thus the restriction operator defined in (5.2) is equivalent to $E_i^0[R_iu]$ defined above. Clearly, we have

$$\|R_iu\|_{L^2(\Omega_i)} \leq \|u\|_{L^2(\Omega)} = \|E_i[R_iu]\|_{L^2(\Omega)} \tag{6.25}$$

Additionally, we have $\left\|E_i^0[w]\right\|_{L^2(\Omega)} = \|w\|_{L^2(\Omega_i)}$ since $\forall w\in L^2(\Omega_i)$

$$\begin{aligned}\left\|E_i^0[w]\right\|_{L^2(\Omega)}^2 &= \int_\Omega \left|E_i^0[w]\right|^2 \,\mathrm{d}\Omega = \int_{\Omega_i} \left|E_i^0[w]\right|^2 \,\mathrm{d}\Omega + \int_{\Omega\setminus\Omega_i} \left|E_i^0[w]\right|^2 \,\mathrm{d}\Omega \\ &= \int_{\Omega_i} |w|^2 \,\mathrm{d}\Omega = \|w\|_{L^2(\Omega_i)}^2 \end{aligned} \tag{6.26}$$

In particular, we have $\left\|E_i^0[R_iu]\right\|_{L^2(\Omega)} = \|R_iu\|_{L^2(\Omega_i)}$, $i=1,2$, $\forall u\in L^2(\Omega)$. We wish to show that there exists a positive constant k_N such that

$$\sup_{\boldsymbol{v}_N\in\mathcal{V}_N} \frac{b(\boldsymbol{v}_N, q_N^\Gamma)}{\|\boldsymbol{v}_N\|_{\mathcal{V}}} \geq k_N \left\|q_N^\Gamma\right\|_{\mathcal{Q}_N/\ker(B_N^*)} \tag{6.27}$$

or equivalently,

$$b(\boldsymbol{v}_N, q_N^\Gamma) \geq k_N \|\boldsymbol{v}_N\|_{\mathcal{V}} \left\|q_N^\Gamma\right\|_{\mathcal{Q}_N/\ker(B_N^*)} \tag{6.28}$$

Consider the bilinear form $b(\cdot,\cdot)$:

$$\begin{aligned} b(\boldsymbol{v}_N, q_N^\Gamma) &= b(\boldsymbol{v}_N, q_N) + b(\boldsymbol{v}_N, q_{N,1}^X) + b(\boldsymbol{v}_N, q_{N,2}^X) \\ &\geq CN^{-\frac{1}{2}} \left\|\boldsymbol{v}_N\right\|_{H^1(\Omega)^2} \left\|q_N\right\|_{L^2(\Omega)} + b(\boldsymbol{v}_N, q_{N,1}^X) + b(\boldsymbol{v}_N, q_{N,2}^X) \end{aligned} \tag{6.29}$$

where we have applied the discrete inf-sup condition (6.14) derived by Maday *et al.* [16, 5] with $d = 2$ to the first term on the right hand side. Using the restriction and extension operators (6.23) and (6.24) defined above and letting $q_N \in \mathrm{P}_{N-2}(\Omega)$ denote any polynomial of degree less than or equal to $N-2$, then we know, for $i = 1, 2$, that $E_i[R_i q_N] = q_N \in \mathcal{Q}_N$ and therefore, in particular, $E_i[R_i q_N]$, $i = 1, 2$, satisfies the discrete inf-sup condition, i.e.:

$$\begin{aligned} b(\boldsymbol{v}_N, E_i[R_i q_N]) &\geq CN^{-\frac{1}{2}} \left\|\boldsymbol{v}_N\right\|_{H^1(\Omega)^2} \left\|E_i[R_i q_N]\right\|_{L^2(\Omega)} \\ &\geq CN^{-\frac{1}{2}} \left\|\boldsymbol{v}_N\right\|_{H^1(\Omega)^2} \left\|R_i q_N\right\|_{L^2(\Omega_i)} \qquad\qquad i = 1, 2 \end{aligned} \tag{6.30}$$

where we have used (6.25). To finish, we make use of the fact that:

$$b(\boldsymbol{v}_N, E_1^0[R_1 q_N]) = b(\boldsymbol{v}_N, E_2[R_2 q_N]) - b(\boldsymbol{v}_N, E_2^0[R_2 q_N]) \tag{6.31}$$

$$b(\boldsymbol{v}_N, E_2^0[R_2 q_N]) = b(\boldsymbol{v}_N, E_1[R_1 q_N]) - b(\boldsymbol{v}_N, E_1^0[R_1 q_N]) \tag{6.32}$$

and that $b(\boldsymbol{v}_N, q_{N,i}^X) = b(\boldsymbol{v}_N, E_i^0[R_i q_N])$. We write:

$$b(\boldsymbol{v}_N, E_1^0[R_1 q_N]) + b(\boldsymbol{v}_N, E_2^0[R_2 q_N]) = \frac{1}{2}\left(b(\boldsymbol{v}_N, E_1[R_1 q_N]) + b(\boldsymbol{v}_N, E_2[R_2 q_N])\right) \tag{6.33}$$

$$\geq \frac{1}{2} CN^{-\frac{1}{2}} \left\|\boldsymbol{v}_N\right\|_{H^1(\Omega)^2} \left(\left\|R_1 q_N\right\|_{L^2(\Omega_1)} + \left\|R_2 q_N\right\|_{L^2(\Omega_2)}\right) \tag{6.34}$$

Thus we have

$$\begin{aligned} b(\boldsymbol{v}_N, q_N^\Gamma) &= b(\boldsymbol{v}_N, q_N) + b(\boldsymbol{v}_N, q_{N,1}^X) + b(\boldsymbol{v}_N, q_{N,2}^X) \\ &\geq CN^{-\frac{1}{2}} \left\|\boldsymbol{v}_N\right\|_{H^1(\Omega)^2} \left\|q_N\right\|_{L^2(\Omega)} \\ &\qquad + \frac{1}{2} CN^{-\frac{1}{2}} \left\|\boldsymbol{v}_N\right\|_{H^1(\Omega)^2} \left(\left\|R_1 q_N\right\|_{L^2(\Omega_1)} + \left\|R_2 q_N\right\|_{L^2(\Omega_2)}\right) \end{aligned} \tag{6.35, 6.36}$$

$$\geq \frac{1}{2} CN^{-\frac{1}{2}} \left\|\boldsymbol{v}_N\right\|_{H^1(\Omega)^2} \left(\left\|q_N\right\|_{L^2(\Omega)} + \left\|R_1 q_N\right\|_{L^2(\Omega_1)} + \left\|R_2 q_N\right\|_{L^2(\Omega_2)}\right) \tag{6.37}$$

Since $\left\|E_i^0[R_i q_N]\right\|_{L^2(\Omega)} = \left\|R_i q_N\right\|_{L^2(\Omega_i)}$, $i = 1, 2$, due to (6.26) we have

$$b(\boldsymbol{v}_N, q_N^\Gamma) \geq \frac{1}{2} CN^{-\frac{1}{2}} \left\|\boldsymbol{v}_N\right\|_{H^1(\Omega)^2} \left(\left\|q_N\right\|_{L^2(\Omega)} + \left\|E_1^0[R_1 q_N]\right\|_{L^2(\Omega)} + \left\|E_2^0[R_2 q_N]\right\|_{L^2(\Omega)}\right) \tag{6.38}$$

Then using Minkowski's inequality, we obtain:

$$b(\boldsymbol{v}_N, q_N^\Gamma) \geq \frac{1}{2} CN^{-\frac{1}{2}} \left\|\boldsymbol{v}_N\right\|_{H^1(\Omega)^2} \left(\left\|q_N + E_1^0[R_1 q_N] + E_2^0[R_2 q_N]\right\|_{L^2(\Omega)}\right) \tag{6.39}$$

$$= \frac{1}{2} CN^{-\frac{1}{2}} \left\|\boldsymbol{v}_N\right\|_{H^1(\Omega)^2} \left(\left\|q_N + q_{N,1}^X + q_{N,2}^X\right\|_{L^2(\Omega)}\right) \tag{6.40}$$

$$= \frac{1}{2} CN^{-\frac{1}{2}} \left\|\boldsymbol{v}_N\right\|_{H^1(\Omega)^2} \left\|q_N^\Gamma\right\|_{L^2(\Omega)} \tag{6.41}$$

□

This result is slightly disappointing (although not surprising since we assumed $q_N \in \mathrm{P}_{N-2}(\Omega)$), because the inf-sup parameter, k_N, can be seen to be dependent on N. In fact, its dependence on N is identical to the inf-sup condition for the standard SEM approximation (6.14). However, we do not believe that the bound given above is the sharpest bound. As far as we are aware, for standard XFEM, the only authors to have considered the inf-sup condition are Groß and Reusken [13] and Reusken [21]. Both articles were concerned with the numerical inf-sup condition and no analysis was given. Therefore the inf-sup condition for XFEM, and hence XSEM, remains an open problem. For Nitsche's XFEM Burman *et al.* [2] found that a stabilisation term, known as the *ghost penalty* term, was required for the inf-sup condition to be satisfied.

Computationally, the XSEM is implemented by separating the extended part of the approximation in the incompressibility constraint, i.e., numerically we enforce:

$$b(\boldsymbol{u}_N, q_N) = 0 \qquad \text{and} \qquad b(\boldsymbol{u}_N, q_N^X) = 0 \tag{6.42}$$

One could argue that this redefines the problem to be a doubly-constrained minimisation problem. A doubly-constrained minimisation was considered by Gerritsma and Phillips [12] for the inf-sup condition of the velocity-pressure-stress formulation of Stokes flow. It is possible that a similar approach can be considered for XSEM.

7. Conclusions and Future Work. The extended spectral element method (XSEM) considered in this article is a high-order enriched method based on the extended finite element method (XFEM) [17, 4]. As far as we are aware, the only *a-priori* error estimates available for XFEM were proposed by Reusken [21]. In this article, the equivalent error estimates for the XSEM have been presented. Additionally, it was shown that when XSEM is used to approximate a piecewise smooth function, exponential order of convergence is obtained.

One motivation for considering an enriched approximation stems from an interest in solving fluid-structure interaction problems using the immersed boundary method [20]. Any weak or strong discontinuities that occur are unfitted in the IBM and hence, oscillations in the approximation of the fluid variables are introduced. However, changing the approximation space by enriching the basis for either velocity or pressure prompts the question concerning the compatibility of the approximation spaces. In this article, we assumed that XSEM was applied to the pressure and discussed the inf-sup condition. We indicated that in certain cases, the enrichment could result in so-called *pseudo-spurious* pressure modes. This is in agreement with the work of Reusken [21] and Groß and Reusken [13] who found numerically that so-called *regions of small support* needed to be removed in order to improve the inf-sup stability. However, removing all the spurious pressure modes analytically, does not seem to be straightforward and is a subject for future research. If one assumes that these so-called *regions of small support* are not present (e.g. when only a single element is considered) then polynomials of degree $N-2$ may be used to approximate the pressure. In such a scenario, we showed that for 2D problems the inf-sup parameter (k_N) of an enriched pressure approximation is the same as for the standard approximation, i.e. $k_N = N^{-1/2}$.

In future articles, we will apply XSEM to some benchmark problems in computational fluid dynamics and fluid-structure interaction problems using the immersed boundary method.

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