Fractal Activity Time and Integer Valued Models in Finance

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Declaration

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This thesis is dedicated in memory of my mother, Neal Kerss.

List of publications

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List of presentations

i. Fractal activity time geometric Brownian motion, *Cardiff Business School*, April 2012.

ii. Risky asset models with tempered stable distributions and long range dependence, *Gregynog*, May 2013.

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Summary

The role of the financial mathematician is to find solutions to problems in finance through the application of mathematical theory. The motivation for this work is specification of models to accurately describe the price evolution of a risky asset, a risky asset is for example a security traded on a financial market such as a stock, currency or benchmark index. This thesis makes contributions in two classes of models, namely *activity time models* and *integer valued models*, by the discovery of new real valued and integer valued stochastic processes. In both model frameworks applications to option pricing are considered.

Chapter one defines activity time models and lists well known properties of asymmetry, leptokurtic and dependent distributions. An equivalent starting point for such models in the form of a stochastic integral and differential equation is discussed, stating the conditions needed for stochastic calculus to be used. An empirical investigation also illustrates the reasons why activity time models may be a more suitable description for the price evolution of a risky asset.

The main contributions appear in chapter two when construction of the

activity time processes are specified, with three types being given. Firstly via superpositions of positive tempered stable Ornstein-Uhlenbeck type processes, producing either a short or long range dependent process. Secondly a new approach is developed in terms of a fractional tempered stable motion with exact tempered stable distributions and asymptotic long range dependence. Thirdly the inverse stable subordinator is used for a slightly modified activity time model which has links to integer valued models in the final chapter.

In the third chapter, the pricing of European call options is explored. Fitting the model is discussed and statistical parameters are computed using method of moments estimators. An option pricing formula is derived, derivatives computed and risk neutral parameters are calibrated by matching formula prices to market prices, which achieves a closer fit than the classical model. The improved performance is minor but for large institutional trades a small price discrepancy directly creates undesirable profit or loss while hedging a written contract.

The fourth chapter concerns high frequency financial data and proposes new integer valued models. Here the motivation comes from inter arrival times between trades for which the exponential distribution may not always be suitable. Instead the Mittag-Leffler law is proposed for the waiting times and the associated fractional Skellam models are constructed. The term fractional relates to the probability mass function being the solution to fractional differential equations. An empirical investigation confirms the benefits of this framework. For high frequency algorithmic traders, even a small misjudgment of a few nanoseconds may prove costly, which underlines the practical relevance of our work.

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Chapter 1

Activity time models

1.1 Introduction

It has become popular for researchers in financial mathematics to propose models that describe the price evolution through time of a risky asset. A risky asset is an exchange traded security such as a stock, currency, commodity or benchmark index. Such models include an array of exponential Lévy models, stochastic volatility models, diffusion or pure jumps models. The classical model of geometric Brownian motion introduced by Samuelson (1965a) is of diffusion type and was used by Black and Scholes (1973) in their celebrated option pricing formula. Exponential Lévy models may be diffusion plus jump or just pure jump models and were introduced as far back as Mandelbrot (1963), Merton (1976) and Clark (1973) for example. Stochastic volatility models describe a process where the volatility (variance) is random through time, these type of models were first proposed by Heston (1993).

These model classes have been introduced to better fit the empirical realities of risky assets, for which the classical diffusion model fails to capture. Indeed there are hundreds of proposed models available and it could be argued that at most only one must be correct and it may be that all are incorrect. It is unlikely that at least in the foreseeable future a full understanding of the underlying process that drives asset prices will be discovered. This is because market participants who trade such assets rarely make available their information that led them to buy or sell the asset, which leaves the true factors affecting price movement unobservable. This does not mean we should give up all hope of modeling such assets but rather take the approach to accurately describe their evolution in terms of statistical properties observed from empirical data. This approach differs from that of an economist, who looks to model the factors that effect price. Whereas a mathematician takes the view all available information is reflected in the current market value and looks to model the price.

At present there is a large literature on models that incorporate suitable distributions to match what is empirically found and stochastic volatility models have also been extensively explored. One feature observed statistically that has received considerable less attention by researchers is the dependence structure of risky assets. It has been observed, see Granger (1966), Granger (2005) and references therein, that transformations of daily price returns, such as absolute daily returns exhibit memory through time. To this end this thesis looks to investigate models that describe well the distributional and dependence properties that are clearly observable in real data. We do not suggest that our models are superior to all others but rather the models described in this thesis should be viewed as advisors and it is left to the practitioner which advisor (model) to implement and make inferences from.

To incorporate dependence (memory) whilst retaining distributions, the class of activity time models was proposed by Heyde (1999) and was subsequently developed by Heyde and Leonenko (2005), Leonenko et al. (2011) and Finlay and Seneta (2006) for example. The activity time models under review in this thesis are closely related to the stochastic volatility model proposed in Barndorff-Nielsen and Shephard (2001). However, although addressing the issue of dependence in financial time series, they where not able to give exact distributions. Here, in this work we are able to compute exact distributions for the logarithm of the price and logarithm of price returns. We feel this feature allows significant tractability in terms of applications.

The rest of the chapter is organized as follows. Section 1.2 gives notation on stochastic processes that will be used throughout before defining the well known Brownian motion in section 1.3. We give a brief historical overview of the classical model for risky assets in section 1.4. An empirical investigation for a cross section of exchange traded securities is presented in section 1.5. We describe some statistical features of risky assets, the so-called stylized facts, which are in fact true for a wide range of assets. This leads us to formally define in section 1.6, *activity time models* with their distinct property of dependence, we list some well known properties of the model and give an alternative starting point in terms of a stochastic differential equation in section 1.7.

1.2 Stochastic processes: basic definitions and notation

Before introducing even the classical model of geometric Brownian motion we will require some notation. This section provides basic terminology and definitions for stochastic modeling that will be used throughout this work, we follow commonly used notation found in the literature.

Denote by Ω the sample space of all possible events ω, denote by $\mathcal{F}$ a sigma algebra on Ω. A *random variable* Y is a measurable function from Ω onto $\mathbb{R}$, the set of real numbers. That is for each event $\omega \in \Omega$, $Y(\omega)$ is some real number. By measurable we mean, for some set of events A in $\mathcal{F}$ the inverse $Y^{-1}(A)$ is also in $\mathcal{F}$,

$$Y^{-1}(A) := \{Y \in A\} \in \mathcal{F}, \tag{1.2.1}$$

with the notation

$$\{Y \in A\} = \{\omega \in \Omega : Y(\omega) \in A\} \tag{1.2.2}$$

and we say Y is $\mathcal{F}$-measurable. Every random variable induces a *probability measure*, denoted by $\mathbb{P}$. The probability of the event or set of events A occurring is denoted by $\mathbb{P}(Y \in A)$ or in shorthand notation $\mathbb{P}(A)$ and is a number in the closed interval $[0, 1]$, for the entire sample space $\mathbb{P}(\Omega) = 1$. If an event $A \in \mathcal{F}$ satisfies $\mathbb{P}(Y \in A) = 1$, we say that A occurs almost surely. We call the triple $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space for which our random variable Y is defined upon. The probability measure is often given in terms of the *distribution function* F, namely the probability that Y is less than or equal to some real number y,

$$F(y) = \mathbb{P}(Y(\omega) \le y). \tag{1.2.3}$$

We refer to this function as the cumulative distribution function which is an increasing, right continuous function. For a continuous random variable Y, there is a function $f : \mathbb{R} \to [0, \infty]$, such that

$$\mathbb{P}(a \leq Y(\omega) \leq b) = \int_a^b f(y)dy, \tag{1.2.4}$$

which if it exists is called the *probability density function* of Y, with the relationship

$$F(y) = \int_{-\infty}^y f(u)du. \tag{1.2.5}$$

In summary, we say a random variable Y has density f and distribution F. The *expectation* of $g(Y)$, where g is a Borel measurable function from $\mathbb{R}$ onto $\mathbb{R}$, is computed as the Lebesgue integral

$$\mathbb{E}[g(Y)] := \int_\Omega g(Y(\omega))d\mathbb{P}(\omega)$$

or the Lebesgue-Stieltjes integral

$$\mathbb{E}[g(Y)] := \int_\mathbb{R} g(y)dF(y)$$

provided the integral exists. In the case when the random variable has a density function, $\mathbb{E}[g(Y)] := \int_\mathbb{R} g(y)f(y)dy$.

The *variance* is defined by

$$\mathbb{V}\text{ar}[Y] = \mathbb{E}[(Y - \mathbb{E}[Y])^2]. \tag{1.2.6}$$

A measure of statistical dependence between two random variables X and Y is the *covariance*,

$$\mathbb{C}\text{ov}[X, Y] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y],$$

which, when normalized is called the *correlation*,

$$\mathbb{C}\text{orr}[X, Y] = \frac{\mathbb{C}\text{ov}[X, Y]}{\sqrt{\mathbb{V}\text{ar}[X]\mathbb{V}\text{ar}[Y]}}.$$

A *stochastic process* is a time parametrized collection of real valued random variables,

$$\{Y(t,\omega), t \geq 0\}, \qquad Y(t,\omega) : [0,\infty] \times \Omega \to \mathbb{R}. \tag{1.2.7}$$

We will often drop the argument ω and simply write $\{Y(t), t \geq 0\}$ to indicate a stochastic process. For a fixed $t \geq 0$ we have a random variable $Y(t,\omega)$, and we write $f(y,t) = \mathbb{P}(Y(t) \leq y)$ to denote the marginal probability density of this random variable. Conversely for a fixed ω we have a function of time with map $t \mapsto Y(t,\omega)$. The stochastic process is called $\mathcal{F}_t$-*adapted* if for any fixed $t > 0$, the random variable is $\mathcal{F}_t$-measurable,

$$Y^{-1}(B) := \{(t,\omega) \in \Omega : Y(t,\omega) \in B\} \in \mathcal{F}_t. \tag{1.2.8}$$

The increasing family of σ-algebras $\{\mathcal{F}_t\}_{t\geq 0}$ is referred to as the *filtration* generated by the stochastic process $\{Y(t), t \geq 0\}$. We say the filtration is complete when $\mathcal{F}_0$ contains all the $\mathbb{P}$-null sets of $\mathcal{F}$ and $\mathcal{F}_t := \bigcap_{u>t} \mathcal{F}_u$, for all $t \in [0,\infty)$, that is, a right continuous filtration. We call the quadruplet

$$\Big(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P}\Big) \tag{1.2.9}$$

a stochastic basis and throughout this thesis we assume a complete stochastic basis to be given for which we define stochastic processes upon.

A stochastic process is said to be *cádlág* if it is *right-continuous* with left limits, that is, for each $t > 0$ the limits

$$Y(t-) = \lim_{s\to t, s<t} Y(s) \qquad Y(t+) = \lim_{s\to t, s>t} Y(s)$$

exists and $Y(t) := Y(t+)$. A *cáglád* function is defined intuitively as a *left-continuous* function with right limits.

By the *finite dimensional distributions* of the stochastic process $\{Y(t), t \geq 0\}$

we mean

$$\mathbb{P}\Big(\omega \in \Omega | (Y(t_1), \ldots, Y(t_k)) \in A\Big), \tag{1.2.10}$$

the joint distribution of $\{Y(t), t \geq 0\}$ at times $t_1, \ldots, t_k$. A stochastic process $\{Y(t), t \geq 0\}$ is said to be *stationary* if for all k, for all h and for all $t_1, \ldots, t_k$

$$\mathbb{P}\Big((Y(t_1 + h), \ldots, Y(t_k + h)) \in A\Big) = \mathbb{P}\Big((Y(t_1), \ldots, Y(t_k)) \in A\Big) \tag{1.2.11}$$

so h does not effect $F(\cdots)$, i.e. F is not a function of time.

A process is said to be stationary in the wide sense (or second-order stationary) if $\mathbb{E}[Y(t)^2] < \infty$ and if the expectation $m(t) = \mathbb{E}[Y(t)]$ and covariance $\mathbb{C}\mathrm{ov}[Y(t), Y(s)]$ are invariant with respect to group shifts in $\mathbb{R}$. In this case $\mathbb{E}[X(t)] = m =$ constant and the covariance function $\mathbb{C}\mathrm{ov}[Y(t), Y(s)] = \mathcal{R}(t - s)$ is a function of the difference $t - s$.

We say that a Borel measurable function $R : [A, \infty) \to (0, \infty)$, for some $A > 0$ varies regularly with index ℓ, if

$$\lim_{x \to \infty} \frac{R(\lambda x)}{R(x)} = \lambda^{\ell}, \quad \text{for all } \lambda \neq 0. \tag{1.2.12}$$

If $\ell = 0$ we say that $R(x)$ is *slowly varying*.

A stationary stochastic process $\{Y(t), t \geq 0\}$ is *long range dependent* if its *autocorrelation function*

$$\begin{aligned} \rho(u) &:= \mathbb{C}\mathrm{orr}[Y(t), Y(t + u)] \\ &:= \frac{\mathbb{C}\mathrm{ov}[Y(t), Y(t + u)]}{\sqrt{\mathbb{V}\mathrm{ar}[(Y(t)]\mathbb{V}\mathrm{ar}[Y(t + u)]}} \end{aligned} \tag{1.2.13}$$

decays as a power of lag u

$$\rho(u) \underset{u \to \infty}{=} \frac{R(u)}{u^{1-2H}}, \qquad H \in (0, \tfrac{1}{2}) \tag{1.2.14}$$

where $R(u)$ is a slowly varying function at infinity.

We call a real valued adapted stochastic process a *martingale* with respect to its filtration, if almost surely $\mathbb{E}[|Y(t)|] < \infty$ and if for $s < t$ we have

$$\mathbb{E}[Y(t)|\mathcal{F}_s] = Y(s). \tag{1.2.15}$$

1.3 Brownian motion

Arguably the most important stochastic process is the so-called *standard Brownian motion* $\{B(t,\omega), t \geq 0\}$ (or simply $\{B(t), t \geq 0\}$ as we will commonly omit the argument ω) which by definition has the properties:

i. $B(0,\omega) = 0$ for all $\omega \in \Omega$, (with probability 1).

ii. The map $t \mapsto B(t,\omega)$ is a continuous function of t for all ω.

iii. For every t, $u \geq 0$, $B(u+t) - B(u)$ has a Gaussian distribution with mean 0 and variance t and is independent of $\{B(u), 0 \leq u \leq t\}$.

For a Brownian motion the marginal probability density function and moments of even orders for $k = 1, 2, \ldots$ can be computed by

$$f(x,t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}, \qquad \mathbb{E}[B(t)^{2k}] = \frac{(2k)!}{k!2^k} t^k$$

and we denote such a random variable $B(t) \sim N(0,t)$, to indicate that $B(t)$ has a normal distribution with zero mean and variance t.

1.4 Geometric Brownian motion as a risky asset model

This section gives a brief historical overview of the origins of financial mathematics in terms of stochastic modeling of risky assets. For a detailed report see Jarrow and Protter (2004).

The historical roots of risky asset modeling lie in the early twentieth century with the publication of *The Theory of Speculation* by the French mathematician Louis Bachelier, see Bachelier (1900). The motivation came from the Paris stock exchange, where traded assets appeared to move in a random fashion. Bachelier deployed the central limit theorem and assumed independence in returns. His work proposed what is now known as *arithmetic Brownian motion*,

$$P(t) = P(0) + \mu t + \sigma B(t) \tag{1.4.1}$$

as a reasonable model for a risky asset. Where $P(t)$ is a random variable representing the price of the risky asset at time t, the constants $\mu \in \mathbb{R}$ and $\sigma > 0$ are drift and diffusion parameters and $B(t)$ is a standard Brownian motion. However this was not the start of a flow of academic literature in financial mathematics, in fact no further work seems to have taken place in the proceeding decades. Although Bachelier was referenced by Kolmogorov and Doob, it was not untill the 1950's when Jimmie Savage suggested to Paul Samuelson to look at Bachelier's thesis, that further investigations into financial mathematics was to start.

Samuelson gave economic reasoning why stock prices move in a random fashion, see Samuelson (1965b). Working alongside Eugen Fama they formed the basis of what has come to be known as *the efficient market hypothesis*, see Fama (1965). This idea postulates that the information in the past has no influence on future price movements, discounted futures prices follow a martingale and so do arbitrary functions of the spot price.

For risky asset modeling arithmetic Brownian motion has the inherent flaw that it can take negative values, whereas stock prices must remain positive or they will cease trading on the exchange. Samuelson noticing this, went on to show that a good model for stock price movements is what we call today *geometric Brownian motion*,

$$P(t) = P(0)\exp\Big\{\mu t + \sigma B(t)\Big\}. \tag{1.4.2}$$

Samuelson also gave option valuation formulas which were nearly the same as those of Black and Scholes, but derived from the point of view that the discounted option payoff is a martingale (see Samuelson (1965a) p. 19).

The pricing formula for the valuation of options, a contingent payoff claim on an underlying security is attributed to Fischer Black and Myron Scholes in their 1973 paper titled “The Pricing of Options and Corporate Liabilities”, published in the Journal of Political Economy. At the time they worked closely with Robert C. Merton, who expanded their ideas and together with Scholes received the 1997 Nobel Prize in Economics for their work. Essentially they were the first to explicitly solve the problem of valuation of options which came at a time when the Chicago Board Options Exchange and other options markets around the world were in their infancy.

With these new pricing tools, practitioners (brokers) were able to give realistic prices to their clients, even if the option contract was not heavily traded and a market price had not been quoted. On the other hand it also allowed buyers of options the tools to check that the prices on offer were reasonable to trade at. This led to the growth of options exchange markets which today are equal in volume to stock exchanges and in some cases larger.

1.5 Empirical realities of risky assets

This section details known properties that hold true to a certain extent for all risky assets, these features have been referred to as the *stylized facts*. Before proceeding further, let us fix some notation. Related to the price process is the *log return* process $\{X(t),\, t \geq 0\}$ representing the sequence of unit increments of width $\Delta t > 0$ of the logarithm of the price, namely

$$X(t) := \log\Big(P(t)\Big) - \log\Big(P(t - \Delta t)\Big) \tag{1.5.1}$$

In many econometric studies, Δt is set implicitly equal to one in appropriate units. We will not conserve the variable Δt, instead we will use $\Delta t = 1$ day, i.e. we are measuring daily log returns. For the short investigations in this section we have collected a cross section of empirical samples of risky assets for indexes, currencies, stocks and commodities, see table 1.1. All samples were obtained from Thompson Reuters datastream terminal and reflect daily market price on close for each asset. The RI series was downloaded when applicable which readjusts for dividend days, index recomposition days, etc. We use the notation $p(1), p(2), \ldots, p(n)$ to represent the observed sequence of prices from a sample of size n, from which the sample log return sequence, denoted by $x(1), x(2), \ldots, x(n-1)$, can be obtained.

For the Dow Jones Industrial average the empirical trajectory (sample path) is displayed in figure 1.1 and the corresponding log return sequence in figure 1.2. Notice that although the sample paths of geometric Brownian motion look similar to the empirical path observed for the Dow Jones in figure 1.1, on closer inspection the corresponding log return process shows that they are in fact quite different. The empirical log returns show higher variability

Risky asset	Mean	Variance	Skewness	Kurtosis
FTSE 100	0.00035	0.00013	-0.47	12.1
FTSE ALL SHARE	0.00036	0.00011	-0.59	13.0
ASX 200	0.00034	0.00009	-0.45	9.1
DOW JONES	0.00035	0.00012	-1.78	46.3
SP 500 COMPOSITE	0.00035	0.00013	-0.29	12.0
NASDAQ 100	0.00032	0.00020	-0.11	10.4
HANG SENG	0.00049	0.00027	0.00	12.7
USD:EUR	-0.00002	0.00004	0.07	6.8
GBP:EUR	0.00002	0.00002	0.37	8.6
YEN:EUR	-0.00004	0.00006	-0.29	7.5
USD:GBP	-0.00004	0.00003	-0.06	7.3
GOLDBLN	0.00018	0.00011	-0.22	11.7
CRUDOIL	0.00018	0.00064	-0.77	18.3
SLVCASH	0.00026	0.00055	0.14	37.1
GLAXOSMITHKLINE	0.00027	0.00005	0.34	11.8
HSBC	0.00023	0.00006	-0.19	11.7
WAL MART STORES	0.00030	0.00007	0.03	8.3
GENERAL ELECTRIC	0.00016	0.00005	-0.10	11.5
PFIZER	0.00019	0.00006	-0.19	7.3

Table 1.1: Log returns statistics

than would be expected for geometric Brownian motion, see figure 1.2. When returns with similar magnitude cluster through time, we say the asset exhibits *volatility clustering*. In other words, the assets time series displays a sequence of consecutive trading days or even weeks with high variance, followed by periods of low variance. It is evident from figure 1.2 that the Dow Jones index displays empirical volatility clustering.

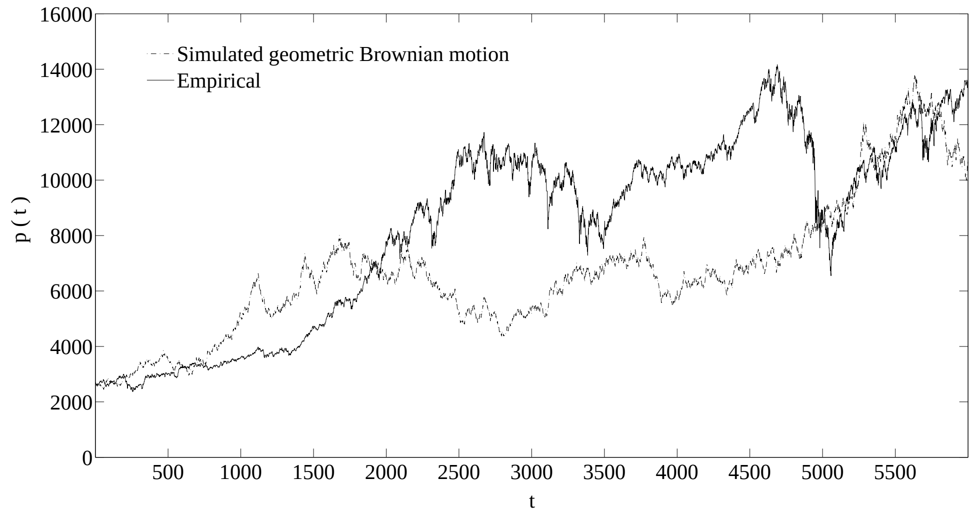

Figure 1.1: Empirical price path for Dow Jones Industrial index

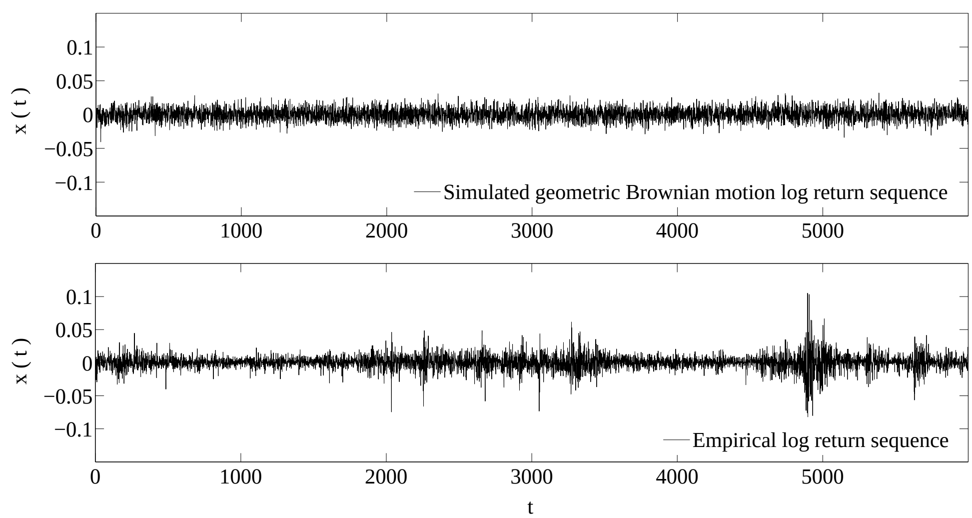

Figure 1.2: Dow Jones empirical log return sequence

It is widely accepted that risky asset log returns do not follow the normal distribution when measured at a daily frequency, this has been noted as far back as Mandelbrot (1963). A simple way to quantify the distribution of log returns is by computation of the kurtosis κ,

$$\kappa = \frac{\mathbb{E}[(X - \mathbb{E}[X])^4]}{\mathbb{E}[(X - \mathbb{E}[X])^2]^2}, \tag{1.5.2}$$

using its empirical counterpart

$$\hat{\kappa} = \frac{\frac{1}{n}\sum_{t=1}^{n}(x(t) - \bar{x})^4}{\left(\frac{1}{n}\sum_{t=1}^{n}(x(t) - \bar{x})^2\right)^2}, \tag{1.5.3}$$

where $\bar{x}$ is the empirical sample mean. The kurtosis is defined as $\kappa = 3$ for a Gaussian distribution, a positive value of $\kappa > 3$ indicating a *heavy or semi-heavy tail*, that is the log density forms a hyperbola whereas the log density of the normal distribution is a parabola. For our data sets the kurtosis is far from its Gaussian value: typical values for daily log returns are (see table 1.1): $\kappa = 12.1$ for the FTSE 100, $\kappa = 6.8$ for the USD:EUR exchange rate and $\kappa = 11.7$ for the GOLDBLN. When a distribution has excess kurtosis we say it has heavy tails and high peaks.

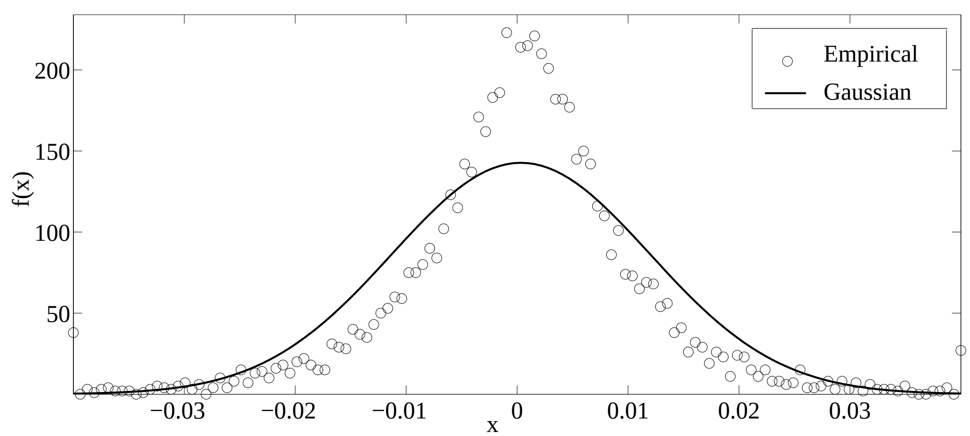

Figure 1.3: Dow Jones empirical probability density

A high peaked distribution describes log returns more likely to have a relatively small change than the normal distribution would specify, see

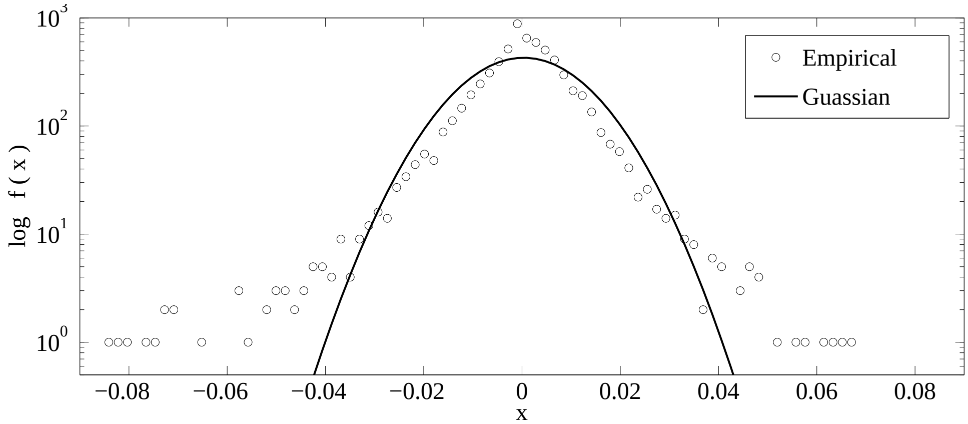

Figure 1.4: Dow Jones empirical log probability density

figures 1.3 and 1.4. Moreover the heavy tails indicate a greater chance of a large market swing in either direction, again this would be underestimated using a normal distribution. These features are not sufficient to identify the distribution of returns and leave a considerable margin for the choice of distribution.

It has been noted (see for example Cont (2001)) that location, scale, skewness and tail parameters are needed to fit risky asset log returns. Such four parameter distributions include but are not limited to: normal inverse Gaussian, generalized hyperbolic and exponentially truncated stable distributions. The correct choice is an open question and may simply be decided upon as a matter of analytical and numerical tractability as well as the quest for mathematical exploration.

A further property inherent in traded securities is independence of returns but dependence in transformations of returns. Recall that a stochastic process is long range dependent if its autocorrelation function $\rho(u)$ decays as a power of lag u, see equation (1.2.14) for details. To investigate the dependence structure we compute and plot the empirical autocovariance function $\hat{\rho}$, given a set of observations $x(1), \ldots, x(n)$,

$$\hat{\rho}(u) = \frac{1}{n} \sum_{t=1}^{n-u} (x(t) - \bar{x})(x(t+u) - \bar{x}).$$

where n is the number of observations in the sample, an $\bar{x}$ is the empirical expectation. It can be seen empirically that the log returns themselves, namely the sequence $\{x(t), t = 1, 2, \ldots\}$, do not exhibit any significant autocorrelation, see figure 1.5 where the dashed lines are Gaussian white noise bands. This is a well known fact for risky asset returns and has been used in support of the efficient market hypothesis.

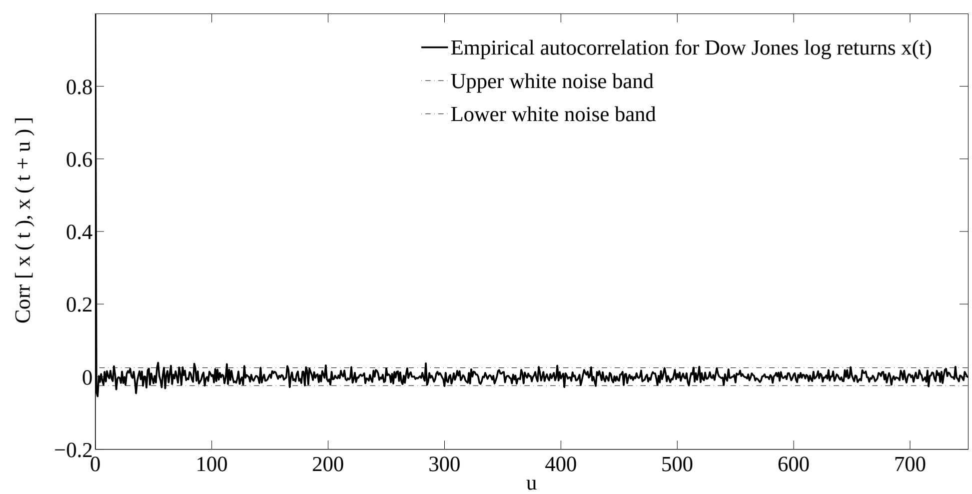

Figure 1.5: Empirical autocorrelation function using log return sequence for Dow Jones Industrial index

If the log returns are independent then the absence of any significant autocorrelations should also hold true for transformations of log returns. It has been reported in the literature that this is in fact not the case. For evidence of dependence in risky assets see Granger (2005). Figure 1.6

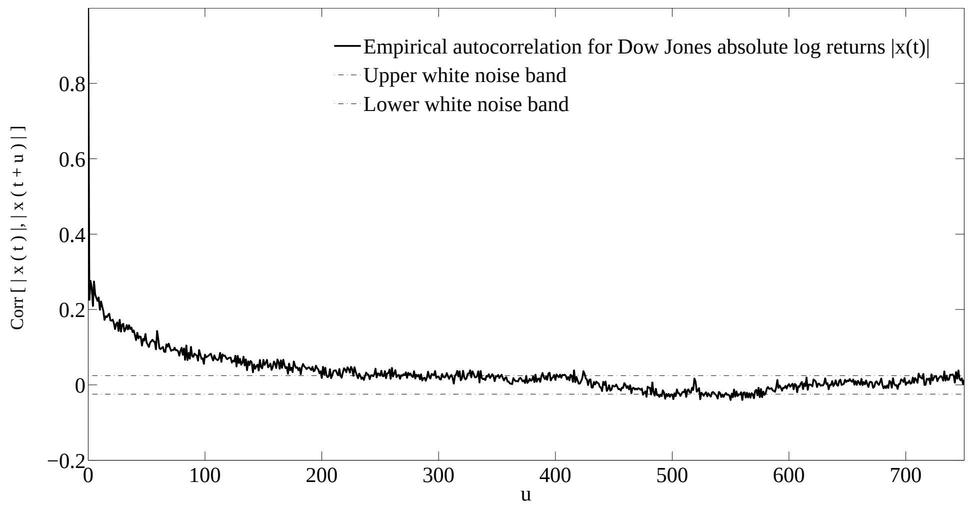

Figure 1.6: Empirical autocorrelation function using absolute log return sequence for Dow Jones Industrial index

confirms for the Dow Jones Industrial index, autocorrelation empirically for absolute returns decreases slower than independent returns which would be expected to lie inside the two dashed lines.

It is an ongoing debate whether long range dependence is present in log returns and it is not a trivial question how statistical methods could answer such a question. The question this thesis aims to answer is not the problem of proving LRD in the empirical sense but instead tackling the challenging problem of providing a mathematically rigorous model for the presence of the long-range dependence phenomenon in stock returns.

A suitable modeling framework able to capture dependence is the class of *activity time models* for which the next section will introduce and for which chapter 2 of this thesis will make contributions too.

1.6 Fractal activity time models

This section introduces *fractal activity time geometric Brownian motion* with its distinct property of *dependence*. Fractal activity time geometric Brownian motion (FATGBM) models for risky assets are due to Heyde (1999), see also Heyde and Liu (2001). The model describes the price $P(t)$ of a risky asset at time t.

Definition 1. *A fractal activity time geometric Brownian motion process* $\{P(t), t \geq 0\}$ *is defined by*

$$P(t) = P(0) \exp\left\{\mu t + \theta T(t) + \sigma B(T(t))\right\}. \tag{1.6.1}$$

where $B(t)$ *is a* $\mathcal{F}_t$*-adapted standard Brownian motion and* $T(t)$ *is the fractal activity time, a* $\mathcal{F}_t$*-adapted, right continuous, positive, increasing random process with long range dependence and* $T(0) = 0$*. With constants* $\mu \in \mathbb{R}$, $\theta \in \mathbb{R}$ *and* $\sigma > 0$ *referred to as the location, skew and scale parameters.*

The model allows considerable flexibility since the activity time process is not defined in complete detail, the following question then naturally arises.

Q. Can we construct an activity process $\{T(t), t \geq 0\}$ with long range dependence and suitable distributions?

The answer is yes and has for example been constructed under FATGBM processes for reciprocal gamma, gamma, inverse Gaussian and generalized inverse Gaussian laws in the works of Heyde and Leonenko (2005), Leonenko et al. (2011), Leonenko et al. (2012), Finlay and Seneta (2006), and Finlay and Seneta (2012). Our work differs in one respect that we will be constructing fractal activity times with tempered stable laws. Tempered stable law have been considered in risky asset models by Cont and Tankov (2003), see also references therein. Furthermore we will be adding a new construction of the time change in the form of an integral representation in chapter 2.

Some known properties of the FATGBM process given by equation (1.6.1) are now listed.

Properties 1. *A fractal activity time model described in definition 1 above, has the following properties:*

1. **Conditional distribution.** *The log returns $X(t)$ have equality in law*

$$X(t) \stackrel{d}{=} \mu + \theta\tau(t) + \sigma\sqrt{\tau(t)}\xi(t), \quad t = 1, 2, \dots, \tag{1.6.2}$$

where $\stackrel{d}{=}$ denotes equality in distribution, and $\xi(t)$ is a sequence of independent standard normal random variables independent of $\tau(t) := T(t) - T(t-1)$.

2. **Moments.** *The log returns have mean*

$$\mathbb{E}[X(t)] = \mu + \theta\mathbb{E}[\tau(t)], \tag{1.6.3}$$

and variance

$$\mathbb{V}\mathrm{ar}[X(t)] = \sigma^2\mathbb{E}[\tau(t)] + \theta^2\mathbb{E}[(\tau_t - \mathbb{E}\tau(t))^2]. \tag{1.6.4}$$

Even when $\theta = 0$ the variance is time dependent, that is the model is hetroskedastic.

3. **Asymmetry.** *The distribution of $X(t)$ has skewness*

$$\vartheta_3 = \frac{3\theta\sigma^2\kappa_2 + \theta^3\kappa_3}{(\sigma^2\mathbb{E}[\tau(t)] + \theta^2\kappa_2)^{3/2}}. \tag{1.6.5}$$

where $\kappa_i := \mathbb{E}[(\tau(t) - \mathbb{E}\tau_t)^i]$. *Symmetric log returns corresponds to* $\theta = 0$.

4. **Leptokurtic.** *The kurtosis is*

$$\vartheta_4 = \frac{3\sigma^4(\kappa_2 + \mathbb{E}[\tau(t)]^2) + 6\theta^2\sigma^2(\mathbb{E}[\tau(t)]\kappa_2 + \kappa_3) + \theta^4\kappa_4}{(\sigma^2\mathbb{E}[\tau(t)] + \theta^2\kappa_2)^2}. \quad (1.6.6)$$

In the symmetric case $\theta = 0$, *the kurtosis*

$$\vartheta_4 = \frac{3\mathbb{V}\mathrm{ar}[\tau(t)]}{\mathbb{E}[\tau(t)]^2} \geq 3, \quad (1.6.7)$$

still allows for heavy tail returns over that of the Gaussian law.

5. **Dependence.** *The covariance of returns is*

$$\mathbb{C}ov[X(t), X(t+k)] = \theta^2\mathbb{C}ov[\tau(t), \tau(t+k)]. \quad (1.6.8)$$

For $\mu = \theta = 0$ *we also have*

$$\mathbb{C}ov[|X(t)|, |X(t+k)|] = \frac{2}{\pi}\sigma^2\mathbb{C}ov[\sqrt{\tau(t)}, \sqrt{\tau(t+k)}]. \quad (1.6.9)$$

The squared returns have covariance

$$\mathbb{C}ov[X^2(t), X^2(t+k)] = \sigma^4\mathbb{C}ov[\tau(t), \tau(t+k)]. \quad (1.6.10)$$

6. **Skew correcting martingale.** *Under the parameter restrictions* $\mu = r$ *and* $\theta = -\frac{1}{2}\sigma^2$ *where* $r \geq 0$, *the process* $\{e^{-rt}P(t), t \geq 0\}$ *is a martingale with respect to the filtration* $\mathcal{F}_{T(t)}$.

The result of Heyde (1999) allowed independence (when $\theta = 0$) in log returns but dependence in absolute or squared returns (see equations (1.6.8), (1.6.9) and (1.6.10)) as empirically observed for risky assets, see figures 1.5 and 1.6. The dependence property sets the model apart from other exponential Lévy models in the literature, indeed with dependence the model will no long be a Lévy process.

1.7 Governing stochastic differential and integral equations

Let us now introduce one further property in the form of Lemma 1, which is a particular case of Proposition 4.4 in Kobayashi (2011). Firstly we require the following assumption.

Assumption 1. *The activity time process $\{T(t), t \geq 0\}$ has continuous sample paths, i.e. the map $t \mapsto T(t, \omega)$ is a continuous function of t for all paths ω.*

Lemma 1. *Let $B(t)$ be a $\mathcal{F}_t$-adapted standard Brownian motion and $T(t)$ a $\mathcal{F}_t$-adapted, right continuous, positive, increasing random process with $T(0) = 0$. Let $\mu \in \mathbb{R}$, $\theta \in \mathbb{R}$ and $\sigma > 0$ be constants. Assume that assumption 1 holds, then the unique strong solution to the stochastic differential equation*

$$dP(t) = \mu P(t)dt + \Big(\theta + \frac{1}{2}\sigma^2\Big)P(t)dT(t) + \sigma P(t)dB(T(t)) \tag{1.7.1}$$

with initial condition $P(0) = p(0)$ is given by

$$P(t) = P(0)\exp\Big\{\mu t + \theta T(t) + \sigma B(T(t))\Big\}. \tag{1.7.2}$$

Proof: The homogeneous linear stochastic differential equation (1.7.1) can be represented by the stochastic integral equation

$$\begin{aligned} P(t) = P(0) + \int_0^t \mu P(s)ds + \int_0^t (\theta + \frac{1}{2}\sigma^2)P(s)dT(s) \\ + \int_0^t \sigma P(s)dB(T(s)). \end{aligned} \tag{1.7.3}$$

Since $T(t)$ is continuous, then B stays constant for all $s \in [T(t-), T(t)]$ and B is said to be in *synchronization* with T. From Kobayashi (2011), $B(T(t))$ is a semimartingale adapted to the filtration $\mathcal{F}_{T(t)}$. Moreover

$T(t)$ is a semimartingale since $T(t)$ is a $\mathcal{F}_t$-adapted, cádlág, increasing process with paths of finite variation on compacts sets. Then $P(t)$ is also a semimartingale. From Bender and Marquardt (2009), $\{P(t), t \geq 0\}$ has continuous trajectories with no jump discontinuities in its sample paths. The Ito formula tells us that for the twice differentiable continuous function $f(x) = \log(x)$, $x > 0$ we have

$$f(P(t)) - f(P(0)) = \int_0^t \frac{1}{P(s)} dP(s) + \frac{1}{2}\int_0^t \frac{1}{P^2(s)} d[P,P](s). \tag{1.7.4}$$

By the associativity of stochastic integrals, plugging the stochastic differential equation (1.7.1) into equation (1.7.4) yields

$$\begin{aligned}\log\frac{P(t)}{P(0)} &= \int_0^t \mu ds + \int_0^t (\theta + \frac{1}{2}\sigma^2) dT(s) + \int_0^t \sigma dB(T(s)) \\ &\quad + \frac{1}{2}\int_0^t \frac{1}{P^2(s)} d[P,P](s).\end{aligned} \tag{1.7.5}$$

Where the quadratic variation $[P,P](t)$ process is defined by definition as

$$[P,P](t) := P^2(t) - 2\int_0^t P(s)dP(s), \tag{1.7.6}$$

can be computed with the calculus rules (see equation 4.11, p.13 in Kobayashi (2011))

$$[B\circ T, B\circ T](t) = [B,B](T(t)) = T(t) \tag{1.7.7}$$

$$[m, B\circ T] = [m,m] = [m,T] = [T, B\circ T] = [T,T] = 0 \tag{1.7.8}$$

where m is the identity map and $B\circ T := B(T(\cdot))$. It is easy to show the quadratic variation differential is

$$d[P,P](t) = \sigma^2 P^2(t) dT(t). \tag{1.7.9}$$

Then the stochastic integral equation (1.7.5) can be written as

$$\log\frac{P(t)}{P(0)} = \int_0^t \mu ds + \int_0^t \theta dT(s) + \int_0^t \sigma dB(T(s)). \tag{1.7.10}$$

Using the following change of variables formula

$$\int_0^t dB(T(s)) = \int_0^{T(t)} dB(s), \tag{1.7.11}$$

due to Kobayashi (2011) theorem 3.1, (which can also be found in Barndorff-Nielsen and Shiryaev (2010) corollary 1.1) the equation (1.7.10) can finally be rewritten as

$$\begin{aligned} P(t) &= P(0)\exp\left\{\int_0^t \mu ds + \int_0^{T(t)} \theta ds + \int_0^{T(t)} \sigma dB(s)\right\} \\ &= P(0)\exp\{\mu t + \theta T(t) + \sigma B(T(t))\} \end{aligned} \tag{1.7.12}$$

since $B(0) = T(0) = 0$. Uniqueness and existence follows from Lemma 4.1 Kobayashi (2011). This completes the proof. □

1.8 Concluding remarks

We have seen empirically the realities of risky asset log returns in terms of non Gaussian distributions and dependence in transformations. The classical model of geometric Brownian motion is clearly unsuitable, so the refined fractal activity time model was defined with properties which could incorporate the empirical observations. The key point for activity time models is that the activity time process should exhibit dependence, which is then inherited by the log returns. Our work now continues in chapter two in terms of a rigorous construction of the activity time process that will exhibit dependence and retain approximate or exact distributions.

Chapter 2

Activity time construction

2.1 Introduction

The goal of this chapter is construction of fractal activity time processes $\{T(t), t \geq 0\}$ with dependence and tempered stable distributions and their corresponding activity time models. We provide details for the fractal activity time in three different ways, referred to as types I, II and III. The first construction, type I is via superpositions of positive non-Gaussian Ornstein-Uhlenbeck processes, type II uses a convoluted subordinator while the third construction, type III, is by definition the inverse stable subordinator.

Both the first and second constructions involve choosing the law for a driving Lévy process in order to obtain desirable laws for the fractal activity time and both are more delicate than the third construction, although appearing more simple, is however the limit of related models in chapter 4.

The rest of this chapter is organized as follows. In section 2.2 Lévy processes are introduced and further notation is set. The stable and tempered stable Lévy processes are detailed in section 2.3 and 2.4. respectively. In section 2.5 Ornstein-Uhlenbeck processes are introduced and a discussion of existence and stationarity of the solution is presented. The specific case of tempered stable Ornstein-Uhlenbeck processes are detailed in section 2.6, a key building block for the type I fractal activity time. Superpositions of Ornstein-Uhlenbeck processes are introduced in section 2.7, a technique to build processes with a rich dependence structure. Processes with long range dependence are constructed in section 2.8, via superpositions of Ornstein-Uhlenbeck processes. The type I fractal activity time is then defined in section 2.9 and the corresponding specification of the activity time model in section 2.10. Convoluted subordinators and quantile clocks are introduced in sections 2.11 and 2.12 respectively, which will be used to define our second construction. The type II fractal activity time process is then detailed in section 2.13 along with the corresponding activity time model. Inverse stable subordinators are introduced in section 2.14 and type III fractal activity time is defined in section 2.15, again along with the resulting activity time model.

2.2 Lévy processes

This section introduces some known results for Lévy processes, see for example Sato (1999).

A cádlág stochastic process $\{L(t), t \geq 0\}$ on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ with values in $\mathbb{R}$ such that $L(0) = 0$ is called a Lévy process if

i. Independent increments; for every increasing sequence, the random variables $L(t_0), L(t_1) - L(t_2), \ldots, L(t_n) - L(t_{n-1})$ are independent.

ii. Stationary increments; the law of $L(t+h) - L(t)$ depends only on h.

iii. Stochastic continuity; for all $\epsilon > 0$, $\lim_{h \to 0} \mathbb{P}(|L(t+h) - L(t)| > \epsilon) = 0$.

The last condition does not mean sample paths are continuous, it mearly serves to exclude processes with jumps at fixed times. That is, the probability of seeing a jump at time t is zero, discontinuities occur at random times.

A key concept related to Lévy processes is the idea of infinite divisibility. A probability distribution F on $\mathbb{R}$ is infinitely divisible if for any positive n, there is a another probability distribution, say F_n on $\mathbb{R}$ such that $F = (F_n)^n$. Or in other words, the nth root of an infinity divisible distribution F exists.

For every infinitely divisible distribution F on $\mathbb{R}$ there exists a Lévy process $\{L(t), t \geq 0\}$ such that $L(1) \stackrel{d}{=} F$. The characteristic function ψ for the distribution of $L(1)$ is defined by

$$\psi_{L(1)}(\zeta) := \int_{\mathbb{R}} e^{i\zeta x} dF(x), \quad \zeta \in \mathbb{R} \tag{2.2.1}$$

where $dF(x) = f(x)dx$ and $f(x)$, if it exists, is the probability density or mass function of the distribution F. The characteristic function of the corresponding Lévy process $\{L(t), t \geq 0\}$ has the Lévy-Khinchin representation:

$$\psi_{L(t)}(\zeta) := \mathbb{E}[e^{i\zeta L(t)}] = \exp\{t\phi_{L(1)}(\zeta)\} \tag{2.2.2}$$

where $\phi_{L(1)} := \phi_L$ is the characteristic exponent of L given by

$$\phi_L(\zeta) = ib\zeta - \frac{1}{2}A^2\zeta^2 + \int_{\mathbb{R}} (e^{i\zeta x} - 1 - i\zeta x \mathbb{I}_{|x| \leq 1})\nu(dx) \tag{2.2.3}$$

where $\nu(dx)$ is the Lévy measure, A the Gaussian part, and b is the drift. The triple (b, A, ν) uniquely determines the characteristic function and hence the law of $L(t)$.

When $\{L(t), t \geq 0\}$ is a homogeneous positive increasing Lévy process it is called a Lévy subordinator. For a subordinator $\nu((-\infty, 0)) = 0$, $A = 0$ and the Laplace transform is given by

$$\mathbb{E}[e^{-\zeta L(t)}] = e^{-t\Psi_{L(1)}(\zeta)}, \quad \zeta \geq 0, \tag{2.2.4}$$

where the Laplace exponent $\Psi_{L(1)}(\zeta) = \Psi_L$ with triple $(b^*, 0, \nu)$ is given by

$$\Psi_L(\zeta) = b^*\zeta - \int_{(0,\infty)} (e^{-\zeta x} - 1)\nu(dx), \tag{2.2.5}$$

and $b^* = b - \int_0^1 x\nu(dx) \geq 0$ is the drift and the Lévy measure ν on $(0, \infty)$ satisfies $\int_0^\infty (1 \wedge x)\nu(dx) < \infty$.

For any Lévy process we have the following general result from Papapantoleon (2008), proposition 10.1.

Lemma 2. *Let $\{L(t), t \geq 0\}$ be a Lévy process with Lévy triplet (b, A, ν) and assume that $\mathbb{E}[|L(t)|] < \infty$. L is a martingale if and only if $b = 0$.*

An infinitely divisible distribution F with characteristic function $\psi_L(\zeta)$ is self decomposable (s.d) if for every $c \in (0, 1)$, there exists another distribution, say F_c with characteristic function $\psi_{L^{(c)}}(\zeta)$, such that

$$\psi_L(\zeta) = \psi_L(c\zeta)\psi_{L^{(c)}}(\zeta), \qquad \zeta \in \mathbb{R}. \tag{2.2.6}$$

The class of self decomposable distributions is a subclass of the class of infinity divisible distributions, i.e. if F is self decomposable it is also infinity divisible.

2.3 Stable Lévy processes

The early theory for the class of stable probability laws was predominately developed by Paul Lévy and Aleksandr Khinchin in the 1920s and 1930s. For a rigorous modern study of stable laws and processes consult Stable Non-Gaussian Random Processes by Samorodnitsky and Taqqu (1994). A stable random variable D is completely defined through its characteristic exponent

$$\phi_D(\zeta) = -\varpi^\alpha |\zeta|^\alpha \left(1 - i\beta(\text{sign}(\zeta)) \tan\frac{\pi\alpha}{2}\right) + i\eta\zeta \tag{2.3.1}$$

where

$$\text{sign}(\zeta) = \begin{cases} 1, & \text{if } \zeta > 0 \\ 0, & \text{if } \zeta = 0 \\ -1, & \text{if } \zeta < 0 \end{cases}$$

with index of stability $\alpha \in (0,1) \cup (1,2]$, scale $\varpi > 0$, skewness $\beta \in [-1,1]$ and location $\eta \in \mathbb{R}$ parameters. For simplicity in the above definition we have excluded the case when $\alpha = 1$. The use of the symbols ϖ and η for the scale and location parameters is not standard notation in the literature, in most works, σ and μ are used, we refrain from using these symbols since they are used in this present work for activity time models.

To indicate that D follows the four parameter α-stable distribution we write in notation

$$D \sim S(x; \alpha, \varpi, \beta, \eta).$$

The stable law is infinity divisible and we say that the stable random variable $D := D(1)$, generates a stable Lévy process $\{D(t), t \geq 0\}$. We are looking to construct activity times which are positive, since a time process that can take negative values is not desirable. To this end we provide a parametrization of the stable distribution that is only defined for the positive side of the real axis. In the literature this is known as a positively skewed stable law and is achieved by setting the skewness parameter to $\beta = 1$. The parameterization we will be concerned with in the proceeding section is

$$D \sim S(x; \alpha, (\delta 2^{\alpha} \cos(\pi\alpha/2))^{1/\alpha}, 1, 0), \tag{2.3.2}$$

which has Lévy measure $\nu(dx)$ given by

$$\nu(dx) = \delta 2^{\alpha} \frac{\alpha}{\Gamma(1-\alpha)} x^{-1-\alpha} dx. \tag{2.3.3}$$

This positively skewed stable random variable generates a positive increasing stable Lévy process, i.e. a subordinator.

2.4 Tempered stable Lévy processes

As previously mentioned we differentiate our work to a certain extent, from existing constructions in the literature, by constructing fractal activity times with tempered stable laws. This section introduces and defines tempered stable distributions which will be used throughout the thesis.

The family of distributions referred to as tempered stable was first introduced by Tweedie (1984). Tempered stable distributions arise by exponentially tilting a stable random variable, by a tempering function $h(x) = e^{-\varrho x}$, with exponent $\varrho > 0$ and re-normalizing. There are of course many tempering functions that could be used to tilt a stable law and in fact many parameterizations of the stable law that could be tilted. For an extensive study on tempering stable distributions, consult Rosinski (2007).

Our concern is with the tempered stable law proposed by Barndorff-Nielsen and Shephard (2002), introduced as follows.

Definition 2. *A tempered stable random variable* L^{TS} *is completely defined through its characteristic exponent*

$$\phi_{L^{TS}}(\zeta) = \delta(\gamma^{1/\alpha} - 2i\zeta)^{\alpha} - \delta\gamma, \tag{2.4.1}$$

with index of stability $\alpha \in (0,1)$*, scale* $\delta > 0$ *and tempering* $\gamma > 0$ *parameters.*

To indicate that the random variable L^{TS} follows the three parameter tempered stable distribution we shall write,

$$L^{TS} \sim TS(\alpha, \delta, \gamma). \tag{2.4.2}$$

Clearly the tempered stable law is infinity divisible and the random variable L^{TS} generates the tempered stable Lévy subordinator $\{L^{TS}(t), t \geq 0\}$. The tempered stable distribution arises by tilting a stable $S(x; \alpha, (\delta 2^{\alpha}\cos(\pi\alpha/2))^{1/\alpha}, 1, 0)$ random variable with a tempering exponent $\varrho = \frac{1}{2}\gamma^{1/\alpha}$ where $\gamma > 0$.

The probability density $f_{TS}(x)$ of L^{TS}, see figure 2.1, can be expressed in terms of the stable density $f_S(x)$ as follows

$$f_{TS}(x) = e^{\delta\gamma} f_{S_{x;\alpha,\delta}}(x) e^{-\frac{1}{2}\gamma^{1/\alpha}x} \qquad x > 0, \alpha \in (0,1), \delta > 0, \gamma > 0,$$

from which the Laplace exponent Ψ can be computed

$$\Psi_{L^{TS}}(\zeta) = \delta\gamma - \delta(2\zeta + \gamma^{1/\alpha})^{\alpha}. \tag{2.4.3}$$

The mean and variance are given by

$$\mathbb{E}[L^{TS}] = 2\delta\alpha\gamma^{\frac{\alpha-1}{\alpha}} \qquad \text{and} \qquad \mathbb{V}\mathrm{ar}[L^{TS}] = 4\delta\alpha(1-\alpha)\gamma^{\frac{\alpha-2}{\alpha}}. \tag{2.4.4}$$

The Lévy measure $\nu(x)$ of a $TS(\alpha,\delta,\gamma)$ random variable is given by

$$\nu(dx) = \frac{\delta 2^{\alpha}\frac{\alpha^2}{\Gamma(1-\alpha)}}{x^{\alpha+1}} \exp\{-\frac{1}{2}\gamma^{1/\alpha}x\}\mathbb{I}_{x>0}dx. \tag{2.4.5}$$

Thus it can be seen that exponentially tempering the density of a stable distribution is equivalent to exponential tempering of the Lévy measure, that is, tilt equation (2.3.3) to obtain (2.4.5).

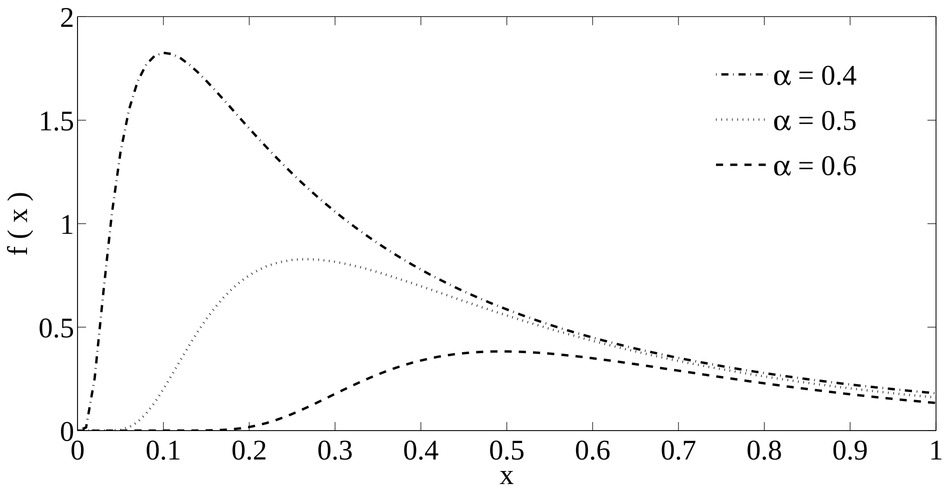

Figure 2.1: Tempered stable density[1]

Random variables distributed as $S(\alpha,\delta)$ can be simulated (see Kawaii and Masuda (2012)) through the equality in law

$$D \stackrel{d}{=} \left(\frac{\delta\Gamma(1-\alpha)}{\alpha\cos(U)}\right)^{1/\alpha} \sin(\alpha(U+\pi/2)) \left(\frac{\cos(U-\alpha(U+\pi/2)}{V}\right)^{\frac{1-\alpha}{\alpha}} \tag{2.4.6}$$

where U is a uniform random variable on $(-\pi/2, \pi/2)$ and V is a standard exponential random variable, $V \sim \exp(1)$. From Baeumer and Meerschaert (2009) we have the following algorithm to simulate a $TS(\alpha,\delta,\gamma)$ random variable.

[1]The Matlab program for the stable distribution is available for download from J. Nolan's website at *academic2.american.edu/jpnolan*.

i. Generate U as uniform on $[0, 1]$.

ii. Generate W as stable $S(\alpha, (\delta 2^{\alpha} \cos(\pi\alpha/2))^{1/\alpha})$.

iii. If $U \leq \exp(-\frac{1}{2}\gamma^{1/\alpha}W)$ return W otherwise return to step 1.

Note that for $\alpha = \frac{1}{2}$, $\psi_{TS} = \psi_{IG}$ where ψ_{IG} is the characteristic function of an inverse Gaussian random variable, $IG(\delta, \gamma)$. Also when $\delta = \frac{\delta_1}{\alpha}$, $\gamma = (2\gamma_1)^{\alpha}$ and $\alpha \to 0$, ψ_{TS} converges point-wise to ψ_{Γ}, the characteristic function of a gamma random variable, $\Gamma(\delta_1, \gamma_1)$. In this sense the tempered stable law is a justifiable choice for fractal activity times, since both the inverse Gaussian and gamma are commonly used probability laws for activity times.

2.5 Ornstein-Uhlenbeck type processes

This section gives known results and definitions on Ornstein-Uhlenbeck processes, for complete details see Sato (1999), Barndorff-Nielsen, Jensen, and Sorensen (1998) and references therein.

Ornstein-Uhlenbeck processes will be used to construct a fractal activity time process in later sections. Here we follow the approach of Barndorff-Nielsen et al. (1998) in discussion of the requirements for stationarity of such processes and provide the autocorrelation function, a key feature that enables processes with richer dependence structures to be built via superpositions.

Named after Leonard Ornstein and George Eugene Uhlenbeck, a non-Gaussian Ornstein-Uhlenbeck (OU) process $\{Y(t), t \geq 0\}$ taking values in $\mathbb{R}$ satisfies the stochastic differential equation

$$dY(t) = -\lambda Y(t)dt + dZ(\lambda t), \qquad Y(0) = y(0), \tag{2.5.1}$$

where $\{Z(t), t \geq 0\}$ is a Lévy process with Lévy triplet $(b, 0, \nu)$ and is referred to as the *background driving Lévy process* (BDLP). In this thesis we will be concerned with positive OU processes, taking values in $\mathbb{R}^+$, in such a case the driving Lévy process will be a subordinator.

The timing λt is chosen for the BDLP such that the marginal distributions of $Y(t)$ do not depend on the parameter λ, often referred to as the mean reversion parameter. Moreover λ effects the memory of the process, with the autocorrelation function decaying slower as λ decreases, see figure 2.3 in section 2.6.

The SDE (2.5.1) can be interpreted in the sense of the integral equation

$$Y(t) = Y(0) + Z(\lambda t) - \lambda \int_0^t Y(s)ds, \tag{2.5.2}$$

for which the solution is given by

$$Y(t) = e^{-\lambda t}Y(0) + \int_0^t e^{-\lambda(t-s)}dZ(\lambda s). \tag{2.5.3}$$

This can be seen as follows since starting with equation (2.5.3) as the solution, implies

$$\begin{aligned}\lambda \int_0^t Y(s)ds &= \lambda \int_0^t e^{-\lambda s}Y(0)ds + \lambda \int_0^t \int_0^s e^{-\lambda(s-u)}dZ(\lambda u)ds \\ &= Y(0) - e^{-\lambda t}Y(0) + \int_0^t dZ(\lambda u) - \int_0^t e^{-\lambda(t-u)}dZ(\lambda u)\end{aligned}$$

and rearranging

$$e^{-\lambda t}Y(0) + \int_0^t e^{-\lambda(t-u)}dZ(\lambda u) = Y(0) - \lambda \int_0^t Y(s)ds + \int_0^t dZ(\lambda u)$$

which is exactly the expression of equation (2.5.2). This solution is in fact the unique strong solution, see Sato (1999) page 104.

To investigate stationarity of this process we wish to compare the distributions of $Y(t)$ and $Y(t+u)$ where $u, t > 0$. First notice that for the latter we have,

$$Y(t+u) = e^{-\lambda(t+u)}Y(0) + e^{-\lambda(t+u)}\int_0^{t+u} e^{\lambda s}dZ(\lambda s),$$

which has equality in distribution

$$\begin{aligned} Y(t+u) &\stackrel{d}{=} e^{-\lambda(t+u)}Y(0) + e^{-\lambda(t+u)}\int_0^{t} e^{\lambda s}dZ(\lambda s) + e^{-\lambda u}\int_0^{u} e^{\lambda s}dZ(\lambda s) \\ &= e^{-\lambda u}Y(t) + \int_0^{u} e^{-\lambda(u-s)}dZ(\lambda s). \end{aligned} \tag{2.5.4}$$

In light of the second expression on the right hand side of (2.5.4), it will be useful to define a new random variable, denoted by $Y^{(c)}(t)$ with characteristic function

$$\psi_{Y^{(c)}(t)}(\zeta) := \mathbb{E}\Big[e^{i\zeta\int_0^u e^{-\lambda(u-s)}dZ(\lambda s)}\Big] = \exp\left\{\int_0^u \phi_{Z(1)}(\zeta e^{-\lambda(u-s)})ds\right\}, \tag{2.5.5}$$

where the last equality holds since $\{Z(t), t \geq 0\}$ has independent increments and $e^{-\lambda(t-s)}$ is continuous on $[0,t]$, see Lukacs (1969).

Returning to our investigation of stationarity, the stochastic process $\{Y(t), t \geq 0\}$ is stationary if we have equality in distribution for the characteristic functions

$$\psi_{Y(t)}(\zeta) \stackrel{d}{=} \psi_{Y(u+t)}(\zeta). \tag{2.5.6}$$

Since Y and Z are independent, by use of equation (2.5.4) the condition (2.5.6) can be restated in terms of our new random variable $Y^{(c)}(t)$ as

$$\psi_{Y(t)}(\zeta) \stackrel{d}{=} \psi_{Y(t)}(\zeta e^{-\lambda u})\psi_{Y^{(c)}(t)}(\zeta). \tag{2.5.7}$$

Then notice that by the definition of self decomposability, if $Y(t)$ is stationary it will also be self decomposable. To be precise, take $c \in (0,1)$ as $c = e^{\lambda u}$ for all $e^{\lambda u} \in (0,1)$ in equation (2.5.7) to satisfy the definition of self decomposability given by equation (2.2.6).

We follow the discussion in Barndorff-Nielsen et al. (1998) to investigate when the equality in distribution holds for equation (2.5.7). Using the characteristic function of $Y^{(c)}(t)$ given in (2.5.5) we require

$$\psi_{Y(t)}(\zeta) \stackrel{d}{=} \psi_{Y(t)}(\zeta e^{-\lambda u}) \exp\Big\{ \int_0^u \phi_{Z(1)}(\zeta e^{-\lambda(u-s)}) ds \Big\}.$$

Take $w = \zeta e^{-\lambda(u-s)}$ so

$$\frac{\psi_{Y(t)}(\zeta)}{\psi_{Y(t)}(\zeta e^{-\lambda u})} = \exp\{\int_{\zeta e^{-\lambda u}}^{\zeta} \phi_{Z(1)}(w) w^{-1} dw\}. \tag{2.5.8}$$

then as $u \to \infty$

$$\psi_{Y(t)}(\zeta) = \exp\{\int_0^{\zeta} \phi_{Z(1)}(w) w^{-1} dw\}. \tag{2.5.9}$$

Thus $Y(t)$ can be stationary if

$$\int_0^{\zeta} \phi_{Z(1)}(w) w^{-1} dw < \infty \tag{2.5.10}$$

According to Wolfe (1982) this is equivalent to the condition

$$\mathbb{E}[1 + \log |Z(1)|] < \infty. \tag{2.5.11}$$

Furthermore from Theorem 3.6.6. in Jurek and Mason (1993) equation (2.5.11) is also equivalent to the condition stated in Sato (1999), namely

$$\int_{|x|>2} \log |x| \nu(dx) < \infty \tag{2.5.12}$$

where ν is the Lévy measure of the BDLP $\{Z(t), t \geq 0\}$.

An important property of OU processes is their dependence structure, it is easy to show that the autocorrelation function of an OU process is given by

$$\rho(u) = \exp\{-\lambda |u|\}, \tag{2.5.13}$$

and the process exhibits short range dependence.

2.6 Tempered stable OU processes

Our goal is to construct an activity time model with a fractal activity time whose incremental process is tempered stable in law. In this section we specialize OU processes to have given marginals of tempered stable type.

Let $\{Y(t), t \geq 0\}$ be an OU process with driving Lévy process $\{Z(t), t \geq 0\}$ and assume

$$\mathbb{E}[1 + \log |Z(1)|] < \infty,$$

so $\{Y(t), t \geq 0\}$ can be a stationary process. We would like the OU process to have given marginals of tempered stable law, i.e. $Y(t) \overset{d}{=} Y \sim TS(\alpha, \delta, \gamma)$. It turns out that there is a unique choice for the BDLP $\{Z(t), t \geq 0\}$ such that $Y \sim TS$, as introduced in Barndorff-Nielsen and Shephard (2002).

To see this, since Y and Z are independent the characteristic function of $Y(t)$ is

$$\begin{aligned}\psi_{Y(t)}(\zeta) &= \mathbb{E}[e^{i(\zeta e^{-\lambda t})Y(0)}]\mathbb{E}[e^{i\zeta \int_0^t e^{-\lambda(t-s)} dZ(\lambda s)}] \\ &= \mathbb{E}[e^{i(\zeta e^{-\lambda t})Y(0)}] \exp\left\{\lambda \int_0^t \phi_{Z(1)}(\zeta e^{-\lambda(t-s)}) ds\right\} \end{aligned} \tag{2.6.1}$$

where $\phi_{Z(1)}(\zeta)$ is the characteristic exponent of $Z(1)$. By substitution of $w = \zeta e^{-\lambda(t-s)}$ equation (2.6.1) can be written as

$$\psi_{Y(t)}(\zeta) = \psi_{Y(0)}(\zeta e^{-\lambda t}) \exp\left\{ \int_{\zeta e^{-\lambda t}}^{\zeta} \phi_{Z(1)}(w) w^{-1} dw \right\}. \tag{2.6.2}$$

The last equation tells us how to choose $\phi_{Z(1)}$ to achieve given marginal distributions for $\{Y(t), t \geq 0\}$. Take for example the random variable L^{TS} with tempered stable law, say we choose

$$\phi_{Z(1)}(\zeta) = \zeta \frac{d}{d\zeta} \phi_{L^{TS}}(\zeta) \tag{2.6.3}$$

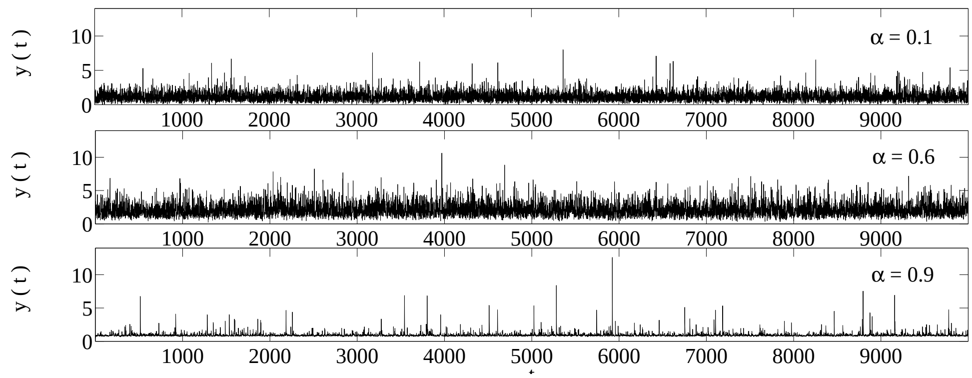

Figure 2.2: Simulated TS-OU process

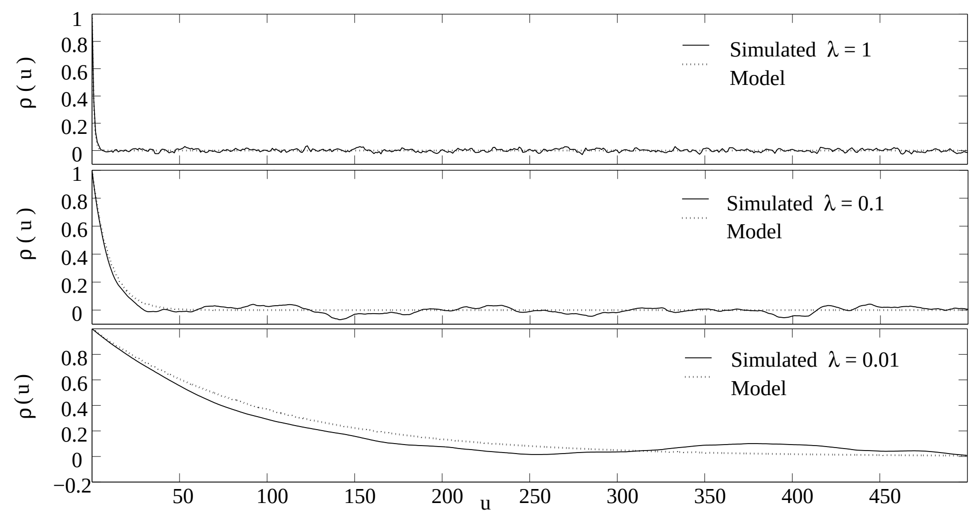

Figure 2.3: Autocorrelation plots for simulated data

then

$$\psi_{Y(t)}(\zeta) = \psi_{Y(0)}(\zeta e^{-\lambda t}) \exp\left\{ \int_{\zeta e^{-\lambda t}}^{\zeta} \frac{d}{dw}\phi_{L^{TS}}(w)dw \right\}$$
$$= \psi_{Y(0)}(\zeta e^{-\lambda t}) \exp\left\{ \phi_{L^{TS}}(\zeta) - \phi_{L^{TS}}(\zeta e^{-\lambda t}) \right\}$$

Then by stationarity of $\{Y(t), t \geq 0\}$ we have

$$\frac{\psi_{Y}(\zeta)}{\psi_{Y}(\zeta e^{-\lambda t})} = \frac{\psi_{L^{TS}}(\zeta)}{\psi_{L^{TS}}(\zeta e^{-\lambda t})}$$

So necessarily we must have $Y \sim L^{TS}$, i.e. $Y(t) \sim TS(\alpha, \delta, \gamma)$.

Lemma 3. *Let $Y(t)$ be an OU process with BDLP $\{Z(t), t \geq 0\}$ triplet $(b, 0, \nu)$ where*

$$\nu(dx) = 2^\alpha \delta \frac{\alpha}{\Gamma(1-\alpha)} \left(\frac{\alpha}{x} + \frac{\gamma^{1/\alpha}}{2} \right) x^{-\alpha} e^{-\frac{x\gamma^{1/\alpha}}{2}} dx. \tag{2.6.4}$$

Then $\{Y(t), t \geq 0\}$ is a stationary OU process with tempered stable marginals, $Y(t) \sim TS(\alpha, \delta, \gamma)$.

Proof: From equation (2.6.3) the explicit choice for the characteristic exponent of the BDLP $\{Z(t), t \geq 0\}$ is computed as

$$\begin{aligned} \phi_{Z(1)}(\zeta) = \zeta \frac{d}{d\zeta} \phi_{Y(t)}(\zeta) &= \zeta \frac{d}{d\zeta} \left(\delta(\gamma^{1/\alpha} - 2i\zeta)^\alpha - \delta\gamma \right) \\ &= 2i\zeta\alpha\delta(\gamma^{1/\alpha} - 2i\zeta)^{\alpha - 1}. \end{aligned}$$

This is equivalent to specification of the background driving Lévy process $\{Z(t), t \geq 0\}$ with Lévy measure ν of $Z(1)$ given by equation 2.6.4. □

This choice for the BDLP $\{Z(t), t \geq 0\}$ is valid since the TS laws are self decomposable and the condition

$$\mathbb{E}[1 + \log |Z(1)|] < \infty, \tag{2.6.5}$$

is satisfied and as such the tempered stable Ornstein-Uhlenbeck (TS-OU) process $\{Y(t), t \geq 0\}$ is stationary.

Simulation of tempered stable Ornstein-Uhlenbeck processes has been studied in Kawaii and Masuda (2011). Using the algorithms presented in the aforementioned paper, we simulate a TS-OU processes for $\lambda = 1$ and various α, we choose δ and γ for of each α such that the marginals have unit mean and variance, see figure 2.2. It can be seen from the plots that as α increases the tails of the distribution become heavier. Furthermore we simulated data for $\lambda = 0.01, 0.1, 1$ with $\alpha = 0.6$ fixed and the computed the sample autocorrelation (acf) function alongside the model acf given by equation (2.5.13), see figure 2.3.

2.7 Superpositions of TS-OU processes

This section introduces superpositions of Ornstein-Uhlenbeck (Sup-OU) processes as proposed in Barndorff-Nielsen (2001), see also Barndorff-Nielson and Leonenko (2005). A Sup-OU process is a weighted sum of independent OU process $\{Y_j(t), t \geq 1\}$, with the number of summations $j = 1, 2, \ldots$ either finite or infinite. Note that these OU processes although independent, do not necessarily have to be identically distributed.

Definition 3. *A finite superposition of Ornstein-Uhlenbeck processes (finite sup-OU process) is a stochastic process* $\{Y^{(m)}(t), t \geq 0\}$ *defined by*

$$Y^{(m)}(t) = \sum_{j=1}^{m} w_j Y_j(t) \tag{2.7.1}$$

where for each j, Y_j *is an OU process, for* $i \neq j$ *we assume* Y_j *is independent of* Y_i *and* w_j *are weights that sum to one,* $\sum_{j=1}^{m} w_j = 1$.

Definition 4. *A infinite superposition of Ornstein-Uhlenbeck processes (infinite sup-OU process) is a stochastic process* $\{Y^{(\infty)}(t), t \geq 0\}$ *defined by*

$$Y^{(\infty)}(t) = \sum_{j=1}^{\infty} w_j Y_j(t) \tag{2.7.2}$$

where for each j, Y_j *is an OU process, for* $i \neq j$ *we assume* Y_j *is independent of* Y_i *and* w_j *are weights that sum to one,* $\sum_{j=1}^{\infty} w_j = 1$.

Superpositions of OU processes creates a new process with a richer dependence structure since the correlation function of a sup-OU process will be

$$\mathbb{C}\mathrm{orr}[Y^{(m)}(t), Y^{(m)}(t+u)] = w_1 e^{-\lambda_1|u|} + \cdots + w_m e^{-\lambda_m|u|}. \tag{2.7.3}$$

By letting $m \to \infty$ and choosing appropriate weights w_j and memory parameters λ_j it is possible to obtain a process with long range dependence, as we will see in the next section.

The use of superpositions to construct the process $\{Y^{(m)}(t), t \geq 0\}$ with given tempered stable marginals and tractable dependence structure is possible since the tempered stable distribution has the additivity property in one of the parameters. If independent random variables $Y(1)$ and $Y(2)$ have $TS(\alpha, \delta_1, \gamma)$ and $TS(\alpha, \delta_2, \gamma)$ distributions respectively, then $Y(1) + Y(2)$ has $TS(\alpha, \delta_1 + \delta_2, \gamma)$ distribution. Moreover we will use the parameter δ to represent the weights in the proof of Theorem 1 for the infinite sup-OU case. Further, the variance of a tempered stable distribution is proportional to the parameter in which the additivity property holds. The construction of superpositions in the absence of these two properties is possible (see Bibby et al. (2013)), however the explicit distributions of the terms in the superpositions may be lost.

2.8 Long range dependent sup-TS-OU processes

Let us know introduce some new results building on the theory of the previous sections. For our activity time models of chapter 1, we required a process $\{T(t), t \geq 0\}$ whose sequence of unit increments exhibit dependence. In this section we construct the incremental process $\tau(t) = T(t) - T(t-1)$

through the following theorem.

Theorem 1. *Let m be an integer, and $\lambda^{(1)}, \dots, \lambda^{(k)} > 0$. There exists a finite sup-OU stationary process as in definition 3 denoted here by $\{\tau^m(t), t \geq 0\}$, with marginal $TS(\alpha, \delta, \gamma)$ distribution and short range dependence with covariance function*

$$\mathbb{C}\mathrm{ov}[\tau^m(s), \tau^m(t+s)] = \sum_{k=1}^{m} 4\alpha(1-\alpha)\delta_k \gamma^{\frac{\alpha-2}{\alpha}} e^{-\lambda^{(k)} t}.$$

There exists a infinite sup-OU stationary process as in definition 4 denoted here by $\{\tau^\infty(t), t \geq 0\}$, with marginal $TS(\alpha, \delta, \gamma)$ distribution and long range dependence with correlation function

$$\mathbb{C}\mathrm{orr}[\tau^\infty(t), \tau^\infty(t+h)] = \frac{R(h)}{h^{2(1-H)}},$$

where R is a slowly varying at infinity function and $H \in (0, 1)$.

Proof: From Lemma 3 we know that there exists a stationary OU process with tempered stable marginals with the BDLP $\{Z(t), t \geq 0\}$ having Lévy measure given by equation (2.6.4). Next we use a discrete version of superposition as described in the previous section. Let $\{\tau^{(k)}(t),\ k \geq 1\}$ be the sequence of independent processes such that each $\tau^{(k)}(t)$ is a solution of the equation

$$d\tau^{(k)}(t) = -\lambda^{(k)}\tau^{(k)}(t)dt + dZ^{(k)}(\lambda^{(k)}t), \quad t \geq 0, \tag{2.8.1}$$

in which the Lévy processes $\{Z^{(k)}(t), t \geq 0\}$ are independent and are such that the distribution of $\tau^{(k)}$ is $TS(\alpha, \delta_k, \gamma)$. In other words, the processes $\tau^{(k)}(t)$ are of OU type with given marginals. For a fixed integer m, define the process $\{\tau^m(t), t \geq 0\}$ using a finite superposition of OU processes

$$\tau^m(t) = \sum_{k=1}^{m} \tau^{(k)}(t), \qquad \tau^m(0) = 0.$$

The marginal distribution of $\tau^m(t)$ is $TS(\alpha, \sum_{k=1}^{m} \delta_k, \gamma)$, and to obtain the specified $TS(\alpha, \delta, \gamma)$ marginal distribution for the finite superposition, we choose $\delta = \sum_{k=1}^{m} \delta_k$.

The correlation function of the process $\{\tau^{(k)}(t), t \geq 0\}$ that solves the SDE (2.8.1) is

$$\mathbb{C}\mathrm{orr}[\tau^{(k)}(t), \tau^{(k)}(t+u)] = e^{-\lambda^{(k)}|u|}, \qquad u \geq 0. \tag{2.8.2}$$

For a finite superposition, the covariance function is

$$\mathbb{C}\mathrm{ov}[\tau^m(t), \tau^m(t+u)] = \sum_{k=1}^{m} 4\alpha(1-\alpha)\delta_k \gamma^{\frac{\alpha-2}{\alpha}} e^{-\lambda^{(k)}|u|}.$$

and correlation function

$$\mathbb{C}\mathrm{orr}[\tau^m(t), \tau^m(t+u)] = \sum_{k=1}^{m} e^{-\lambda^{(k)}|u|}. \tag{2.8.3}$$

To prove the existence of the process with long range dependence, consider the same setup with an infinite superposition

$$\tau_t^\infty = \sum_{k=1}^{\infty} \tau^{(k)}(t).$$

The construction with infinite superposition is well-defined in the sense of mean-square or almost-sure convergence provided that $\sum_{k=1}^{\infty} \delta_k < \infty$.

Choose and $\lambda^{(k)} = 1/k$, and

$$\delta_k = \frac{\delta}{k^{1+2(1-H)}\zeta(1+2(1-H))},$$

where $H \in (0,1)$ and

$$\zeta(j) = \sum_{n=1}^{\infty} \frac{1}{n^j}$$

is the Riemann zeta-function. Then the marginal distribution of $\tau^\infty(t)$ is

$TS(\alpha,\delta,\gamma)$, and the covariance function

$$\mathbb{C}\mathrm{ov}[\tau^{\infty}(t),\tau^{\infty}(t+h)] = \frac{4\delta\alpha(1-\alpha)\gamma^{(\alpha-2)/\alpha}}{\zeta(1+2(1-H))}\sum_{k=1}^{\infty}\frac{1}{k^{1+2(1-H)}}e^{-h/k}, \tag{2.8.4}$$

thus the correlation function can be written as

$$\mathbb{C}\mathrm{orr}[\tau^{\infty}(t),\tau^{\infty}(t+h)] = \frac{R(h)}{h^{2(1-H)}},$$

where R is a slowly varying at infinity function (see Leonenko et al. (2012) for proof). □

2.9 Fractal activity time - type I

This section introduces a new fractal activity time process with dependent tempered stable increments and continuous sample paths. We construct such a process by definition using superpositions of tempered stable Ornstein-Uhlenbeck process as described in the previous section.

Definition 5. *A fractal activity time process of type I $\{T(t), t \geq 0\}$ is defined by*

$$T(t) = \sum_{i=1}^{[t]}\tau^{m}(i) + (t-[t])\tau^{m}([t]+1), \qquad T(0)=0. \tag{2.9.1}$$

where $\tau^m(t)$ is either a finite ($m<\infty$) or infinite ($m=\infty$) superposition of TS-OU processes as constructed in Theorem 1.

We now show that the activity time process $T(t)$ constructed using our approach is asymptotically self-similar in the case of finite superposition. This property provides a way to obtain an approximation for the marginal distribution of $T(t)$. We will use the notation that $\mathcal{C}[0,1]$ is the space of continuous functions with supremum norm.

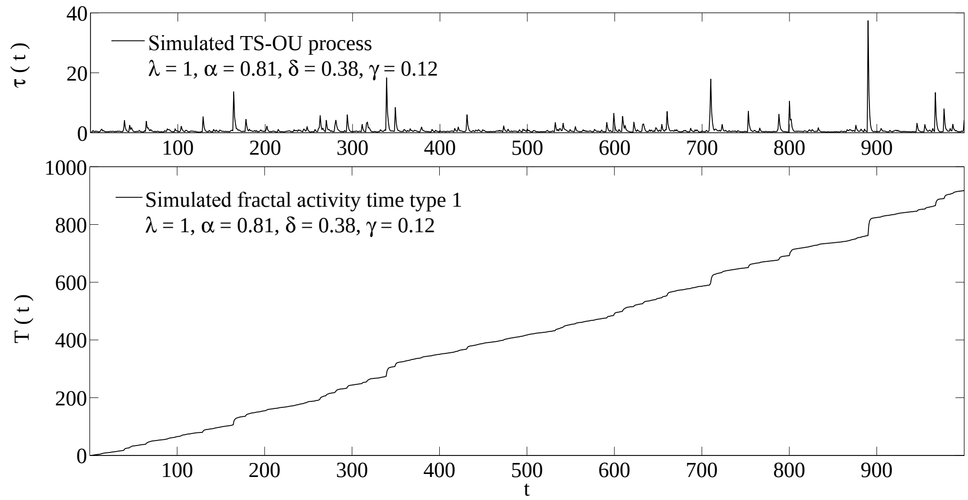

Figure 2.4: Simulated path of fractal activity time type I

Theorem 2. [2] *For a fixed $m < \infty$ (finite superposition)*

$$\frac{1}{c_m N^{1/2}}\left(T([Nt]) - \mathbb{E}[T([Nt])]\right) \Rightarrow B(t), \quad t \in [0,1],$$

as $N \to \infty$ in the sense of weak convergence in $C[0,1]$. The process $B(t)$ is Brownian motion, and the norming constant c_m is given by

$$c_m = \left(\sum_{k=1}^{m} \mathbb{V}\mathrm{ar}[\tau^{(k)}(m)]\frac{1-e^{-\lambda^{(k)}}}{1+e^{-\lambda^{(k)}}}\right)^{1/2},$$

where $\mathbb{V}\mathrm{ar}[\tau^{(k)}(m)] = 4\alpha(1-\alpha)\delta_k\gamma^{\frac{\alpha-2}{\alpha}}$.

Proof. We have

$$\frac{1}{c_m N^{1/2}}\left(T([Nt]) - \mathbb{E}[T([Nt])]\right)$$
$$= \frac{1}{c_m N^{1/2}}\left(\sum_{i=1}^{[Nt]}(\tau^m(i) - \mathbb{E}[\tau^m(i)])\right)$$

[2]The proof of this theorem was provided in collaboration with A. Sikorskii, Michigan State University and is not the sole work of the author of this thesis.

$$+\frac{Nt-[Nt]}{c_m N^{1/2}}\Big(\tau([Nt]+1)-\mathbb{E}[\tau([Nt]+1)]\Big).$$

Since $Nt-[Nt]<1$, the last term converges to zero in the mean square, and the weak limit is determined by the first term. The proof of the weak convergence of the first term to Brownian motion is the same as in Leonenko et al. (2011b). □

The stochastic sequence of unit increments $\{\tau(t), t = 1, 2\ldots\}$ of fractal activity time type I, will have $TS(\alpha,\delta,\gamma)$ marginal distributions by Theorem 1. This feature will be used in the next section to construct activity time models with normal tempered stable log returns. However it will be of use to compute, at least approximately, the marginal distributions of $\{T(t), t\geq 0\}$. Below we provide two approaches for obtaining $f_{T(t)}(x) := \mathbb{P}(T(t)\leq x)$, firstly based on the exact distribution of $T(t)$ and secondly based on the asymptotic distribution.

For the first approach, we assume that t is an integer (e.g., number of days). The density of $T(t)$ can then be computed as a convolution of densities of $\sum_{i=1}^{t}\tau^{(k)}(i)$. Each $\tau^{(k)}$ is a time-homogeneous Markov process, therefore the distribution function of $\sum_{i=1}^{t}\tau^{(k)}(i)$ is determined by the initial density $f^{(k)}$, which is $TS(\alpha,\delta_k,\gamma)$, and the transition probability $\mathbb{P}^{(k)}(t)(x,B)$ of the process $\tau^{(k)}$ from point x at time 0 to a set B at time t. Namely,

$$\begin{aligned}&\mathbb{P}\left(\sum_{i=1}^{t}\tau^{(k)}(i)\leq x\right)\\&\quad=\int_{x_1+x_2+\ldots+x_t\leq x} f^{(k)}(x_1;k)dx_1 \qquad (2.9.2)\\&\quad\quad\times\mathbb{P}^{(k)}(1,dx_2;x_1)\mathbb{P}^{(k)}(1,dx_3;x_2)\ldots\mathbb{P}^{(k)}(1,dx_t;x_{t-1}).\end{aligned}$$

The transition probability $\mathbb{P}^{(k)}(t)(x,B)$ has been derived in Zhang and Zhang

(2009), where it has been shown that the conditional distribution of $\tau^{(k)}(t)$ given $\tau^{(k)}(0) = x$ coincides with the distribution of the sum of a constant, a TS random variable, and a compound Poisson random variable, that is,

$$\tau_t^{(k)}|_{\tau^{(k)}(0)=x} \stackrel{d}{=} e^{-\lambda^{(k)}t}x + W_{k(0)}^t + \sum_{i=1}^{N^t(k)} W_{k(i)}^t, \tag{2.9.3}$$

where $W_{k(0)}^t$ is distributed $TS(\alpha, \delta_k(1-e^{-\alpha\lambda^{(k)}t}), \gamma)$, the random variable N_k^t has a Poisson distribution of intensity $\delta_k\gamma(1-e^{-\alpha\lambda^{(k)}t})$, and $W_{k(1)}^t, W_{k(2)}^t, ...$ are independent random variables having a common specified density function

$$f_{W_k^t}(w) = \frac{2^\alpha \alpha\gamma^{-1}}{\Gamma(1-\alpha)}(e^{\alpha\lambda^{(k)}t} - 1)w^{-\alpha-1} \tag{2.9.4}$$
$$\left(exp\left\{-\frac{1}{2}\gamma^{1/\alpha}w\right\} - \exp\left\{-\frac{1}{2}\gamma^{1/\alpha}we^{\lambda t}\right\}\right)\mathbf{1}_{\{w>0\}}.$$

Furthermore, for each k, $\{W_{k(0)}^t\}, \{W_{k(1)}^t, W_{k(2)}^t, ...\}$, and $\{N_k^t\}$ are independent. In Zhang and Zhang (2009), the exact simulation method for the computation of the transition probability is discussed. It is also shown that the computation of the transition density can be implemented via exact simulation method using the acceptance-rejection sampling technique.

The second approach to computing prices is based on the asymptotic self-similarity of $T(t)$ as suggested in Heyde and Leonenko (2005). Based on Theorem 2, the density $f_{T(t)}$ can be taken as approximately the density of $t\mathbb{E}[\tau(1)] + \sqrt{t}(T(1) - \mathbb{E}[\tau(1)])$, where $\mathbb{E}[\tau(1)] = 2\alpha\sum_{k=1}^m \delta_k\gamma^{(\alpha-1)/\alpha}$. The distribution of $T(1)$ is $TS(\alpha, \sum_{k=1}^m \delta_k, \gamma)$. Therefore an approximation to $f_{T(t)}(u)$, one can use $t^{-1/2}f_{TS}((u+\mathbb{E}[\tau(1)](\sqrt{t}-t))/\sqrt{t})$ with the appropriate parameters.

2.10 Normal tempered stable activity time model

This section builds an activity time model described by definition 1 in chapter 1, by using fractal activity time type I as described in the previous section.

Theorem 3. *Let $B(t)$ be a $\mathcal{F}_t$-adapted standard Brownian motion and $T(t)$ a $\mathcal{F}_t$-adapted, fractal activity time of type I. Let $\mu \in \mathbb{R}$, $\theta \in \mathbb{R}$ and $\sigma > 0$ be constants. Let $\{P(t), t \geq 0\}$ satisfy the SDE*

$$dP(t) = \mu P(t)dt + (\theta + \tfrac{1}{2}\sigma^2)P(t)dT(t) + \sigma P(t)dB(T(t)), \qquad (2.10.1)$$

then the log returns $\{X(t), t = 1, 2, \ldots\}$ form a stationary sequence with the exact normal tempered stable marginal distribution with moment generating function

$$\mathbb{E}[e^{\zeta X(t)}] = \exp\left\{\mu\zeta + \delta\gamma - \delta\left(\gamma^{1/\alpha} + \frac{\theta^2}{\sigma^2} - \sigma^2(\zeta + \frac{\theta}{\sigma^2})^2\right)^{\alpha}\right\}. \qquad (2.10.2)$$

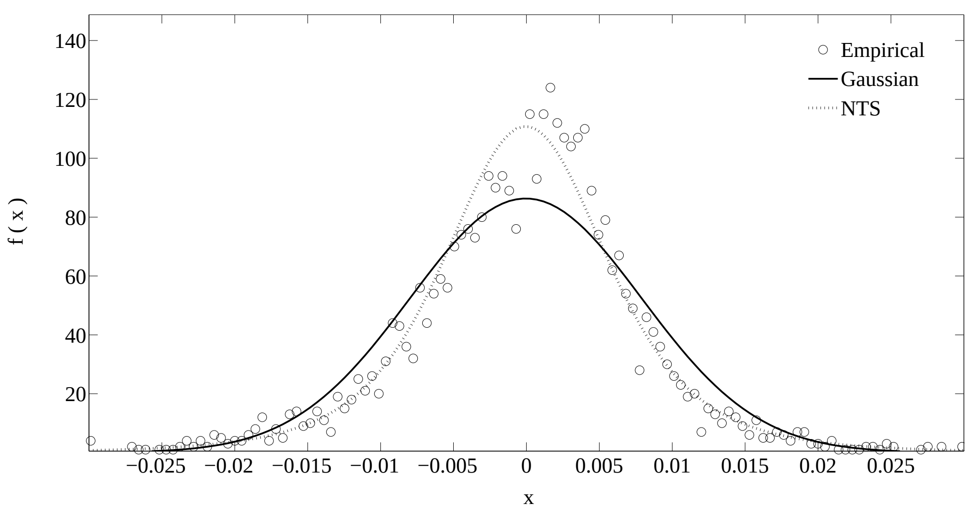

Figure 2.5: Empirical and model probability density for Yen to Euro

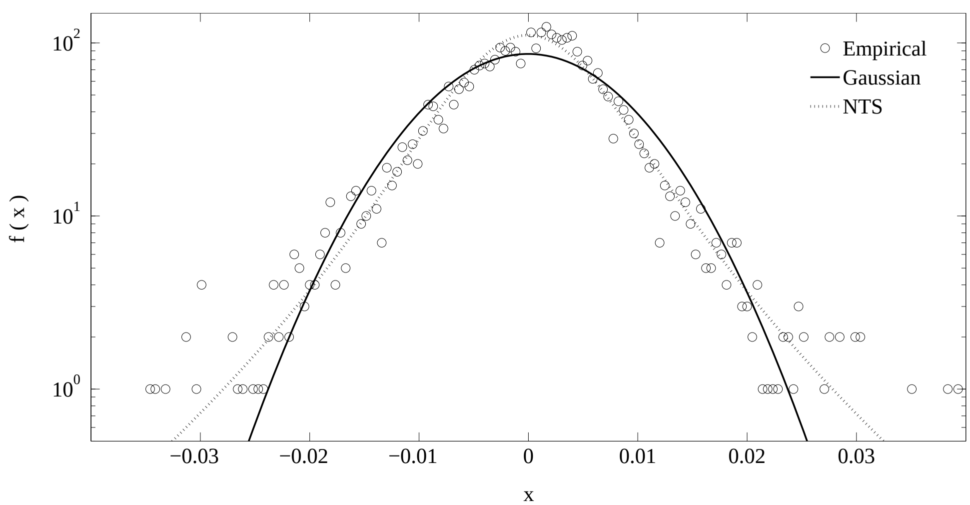

Figure 2.6: Empirical and model log probability density for Yen to Euro

Proof: Clearly from equation (2.9.1) the activity time process $\{T(t), t \geq 0\}$ is a continuous process, i.e. the map $t \mapsto T(t,\omega)$ is a continuous function of t for all paths ω, then the unique strong solution to the SDE (2.10.1) is due to Lemma 1, in chapter 1 and is given by

$$P(t) = P(0)\exp\Big\{\mu t + \theta T(t) + \sigma B(T(t))\Big\}. \tag{2.10.3}$$

Consider the log return sequence representing the increments of the logarithm of the price process. From Theorem 1, $\{\tau(t), t = 1, 2, \ldots\}$ is stationary with distribution $TS(\alpha, \delta, \gamma)$. By properties 1 in chapter 1, the log returns have equality in law

$$X(t) = \frac{\log P(t)}{\log P(t-1)} \stackrel{d}{=} \mu + \theta\tau(t) + \sigma\sqrt{\tau(t)}\xi(t) \tag{2.10.4}$$

for constants $\mu, \theta \in \mathbb{R}$, $\sigma > 0$ and where $\xi(t)$ is a sequence of standard normal random variables independent of $\tau(t)$. Since $\tau(t)$ has tempered stable marginals the moment generating function is

$$\mathbb{E}[e^{\zeta\tau(t)}] = \exp\Big\{-\delta\gamma + \delta(2\zeta + \gamma^{1/\alpha})^{\alpha}\Big\}. \tag{2.10.5}$$

Then conditioning on τ yields

$$\mathbb{E}[e^{\zeta X(t)}] = e^{\zeta\mu}\mathbb{E}\Big[\mathbb{E}[e^{\zeta(\theta\tau(t)+\sigma\sqrt{\tau(t)}\xi(t))}|\tau(t)]\Big]. \tag{2.10.6}$$

In other words, the conditional distribution of $\theta\tau(t) + \sigma\sqrt{\tau(t)}\xi(t)$ when $\tau(t)$ is fixed is normal with mean $\theta\tau(t)$ and variance $\sigma^2\tau(t)$. Since $\xi(t)$ and $\tau(t)$ are independent,

$$\mathbb{E}[e^{\zeta X(t)}] = e^{\zeta\mu}\mathbb{E}[e^{\zeta\theta\tau(t)+\frac{\zeta^2}{2}\sigma^2\tau(t)}] = e^{\zeta\mu}\mathbb{E}\Big[\exp\{\tau(t)(\zeta\theta + \frac{\zeta^2}{2}\sigma^2)\}\Big]. \tag{2.10.7}$$

We rewrite

$$\begin{aligned} z = \zeta\theta + \frac{\zeta^2}{2}\sigma^2 &= \frac{1}{2}\sigma^2\Big[\zeta^2 + 2\zeta\frac{\theta}{\sigma^2} + (\frac{\theta}{\sigma^2})^2 - \Big(\frac{\theta}{\sigma^2}\Big)^2\Big] \\ &= \frac{1}{2}\sigma^2\Big[\zeta + \frac{\theta}{\sigma^2}\Big]^2 - \frac{1}{2}\sigma^2\Big(\frac{\theta}{\sigma^2}\Big)^2. \end{aligned} \tag{2.10.8}$$

By (2.10.4)

$$e^{\zeta\mu}\mathbb{E}\Big[\exp\Big\{\tau(t)(\zeta\theta + \frac{\zeta^2}{2}\sigma^2)\Big\}\Big] = e^{\zeta\mu}\mathbb{E}\Big[\exp\{z\tau(t)\}\Big] \tag{2.10.9}$$

and so

$$\mathbb{E}[e^{\zeta X(t)}] = e^{\zeta\mu}\exp\Big\{\delta\gamma - \delta\Big(\gamma^{1/\alpha} + \frac{\theta^2}{\sigma^2} - \sigma^2\Big(\zeta + \frac{\theta}{\sigma^2}\Big)^2\Big)^\alpha\Big\}. \tag{2.10.10}$$

□

We say that the log returns have marginal law of normal tempered stable distribution, in notation to indicate this we write

$$X(t) \sim NTS(\alpha, \delta, \gamma, \mu, \theta, \sigma).$$

The empirical probability density for the Japanese Yen traded against the Euro and a calibrated normal tempered stable probability density plot is displayed in figure 2.5 and 2.6. It is clear that the normal tempered stable distribution allows for a more realistic fit to the observed data. See table 3.1

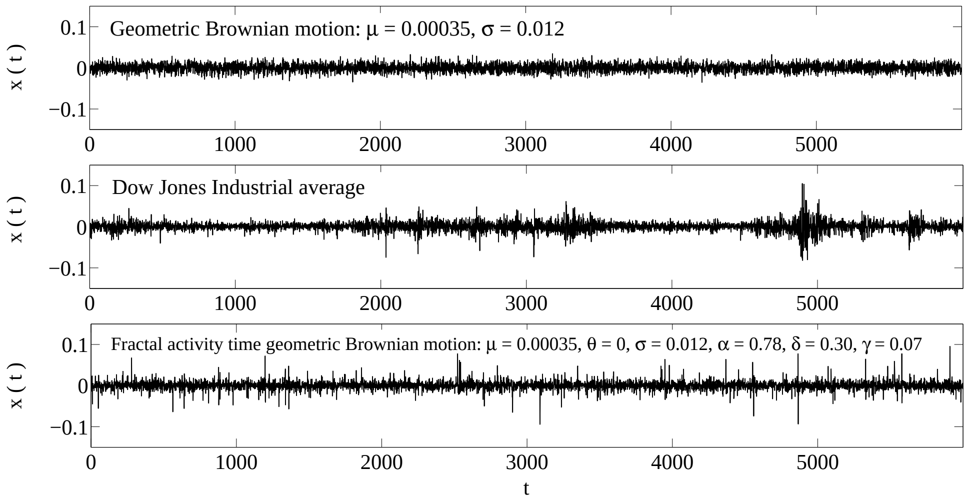

Figure 2.7: Simulated log return sequence of fractal activity time model, geometric Brownian motion and empirical Dow Jones

of the next chapter for the parameter estimates. The log return sequence for a simulated activity time model with parameters set to that of the Dow Jones industrial index (see table 3.1 in the next chapter) alongside a simulation of the classic geometric Brownian motion model can be compared visually with the empirically observed log return sequence, see figure 2.7. It can be seen that the model does allow for sudden shocks to the market when high magnitude log returns occur. However it can be seen that the idea of volatility clustering, when large shocks cluster together, as the market suggests, is not as well represented in the model. This is a drawback that could be addressed in further research not undertaken in this current work.

Remark 1. *The correlation function of $X(t)$ is given by*

$$\mathbb{C}orr[X(t), X(t+k)] = \theta^2 \mathbb{C}orr[\tau(t), \tau(t+k)], \tag{2.10.11}$$

and if $\theta \neq 0$, the short or long range range dependence in log returns is

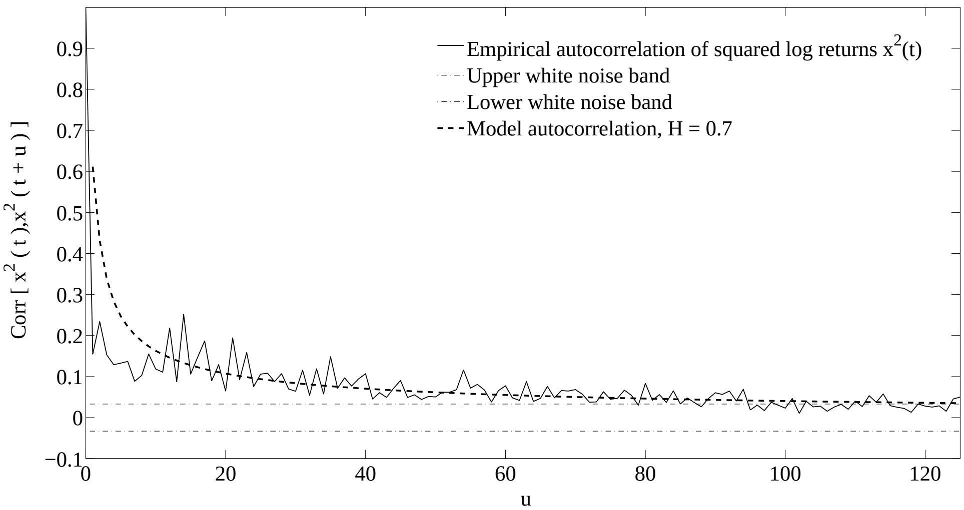

Figure 2.8: Empirical and model autocorrelation function for Yen to Euro

present when short or long range dependence is present in the process τ. If $\theta = 0$, then the log returns are uncorrelated, but the correlation persists in squared log returns:

$$\mathbb{C}orr[X^2(t), X^2(t+k)] = \sigma^4 \mathbb{C}orr[\tau(t), \tau(t+k)]. \tag{2.10.12}$$

To make use of equation (2.10.12) above, the correlation function of the unit increments of the fractal activity time $\mathbb{C}\text{orr}[\tau(t), \tau(t+k)]$ given by equation (2.8.3) in the case of finite superpositions and equation (2.8.4) for the case of infinite superpositions can be used. For an graphical illustration of the model fit in terms of dependence structure for the infinite superpositions see figure 2.8. In this present work we do not investigate techniques to estimate the memory parameter H, which appears in equation (2.8.4), this will be left for future research. For the purpose of illustration we have chosen $H = 0.7$ in figure 2.8 for the computed model correlation structure.

2.11 Convoluted subordinators

We now move on to define our second construction of fractal activity time, for which we shall refer to as type II. Before doing so in this section and the next we will introduce some known theory which will be needed. This section details some known results on *convoluted subordinators* as introduced by Bender and Marquardt (2009).

A convoluted subordinator $\{T(t), t \geq 0\}$ is defined by

$$T(t) = \int_0^t k(t,s)L(ds) \tag{2.11.1}$$

where $\{L(t), t \geq 0\}$ is a strictly increasing Lévy process (subordinator) and $k(s,t)$ with $t, s > 0$, a deterministic function $k(s,t) : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ with properties

1. $k(s,t) = 0$ when $s > t$.

2. The mapping $t \mapsto k(s,t)$ is continuous and strictly increasing.

3. The mapping $s \mapsto k(s,t)$ is integrable for a fixed t.

The choice of the driving Lévy process provides flexibility to incorporate distributional properties such as heavy tails. More generally, the second order structure (and hence the memory) of a convoluted subordinator is encoded in the choice of the kernel.

For a fixed trajectory (sample path), the process $\{T(t), t \geq 0\}$ is continuous and strictly increasing a.s. (see Bender and Marquardt (2009) Proposition 1).

In this general setup a convoluted subordinator $T(t)$ has mean given by

$$\mathbb{E}[T(t)] = \mathbb{E}[L(1)] \int_0^t k(t,s)\,ds, \tag{2.11.2}$$

variance

$$\mathbb{V}\text{ar}[T(t)] = \mathbb{V}\text{ar}[L(1)] \int_0^t k(t,s)^2\,ds \tag{2.11.3}$$

and covariance

$$\mathbb{C}\text{ov}[T(t),T(u)] = \mathbb{V}\text{ar}[L(1)] \int_0^{t\wedge u} k(t,s)k(u,s)\,ds. \tag{2.11.4}$$

Note that the last two equations correct an error present in Bender and Marquardt (2009), corollary 1, part (ii). The pre-factor before the integral has to be the variance of $L(1)$ and not the second moment.

In general given some kernel $k(s,t)$ and some driving Lévy subordinator $\{L(t), t \geq 0\}$, computation of the resulting distribution for the convoluted subordinator will be non-trivial. However certain choices for the kernel and driving subordinator do exists such that exact marginal distributions can be obtained. One such possible choice already investigated is

$$k(s,t) = e^{-\lambda(t-s)} \tag{2.11.5}$$

and the convoluted subordinator is then a positive OU process. Another possible route that allows computation of marginal distributions is by quantile kernels.

2.12 Quantile clocks

This section details some known results on *quantile clocks* as introduced by James and Zhang (2011).

A quantile clock is a stochastic process $\{T(t), t \geq 0\}$ that has a convoluted subordinator representation given by

$$T(t) = \int_0^t Q_R\Big(\Big(1 - \frac{s}{t}\Big)_+\Big) dL(s) \tag{2.12.1}$$

where $\{L(t), t \geq 0\}$ is a Lévy subordinator. Here the kernel $k(s,t)$ is expressed via a quantile function $Q_R(\cdot)$, defined as the inverse of a strictly increasing cumulative distribution function F_R of some non negative continuous random variable R, namely

$$Q_R(u) = \inf\{t : F_R(t) \geq u\}. \tag{2.12.2}$$

The Laplace transform of $T(t)$ will be given by

$$\begin{aligned}\mathbb{E}[e^{-\zeta T(t)}] &:= \mathbb{E}\Big[e^{-\zeta \int_0^t Q_R\left(\left(1-\frac{s}{t}\right)_+\right) dL(s)}\Big] \\ &= \exp\Big\{ - \int_0^t \Psi_{L(1)}\Big(\zeta Q_R\Big(\Big(1 - \frac{s}{t}\Big)_+\Big)\Big) ds\Big\},\end{aligned}$$

for which exact laws of $T(t)$ may be computed if the random variable R has the equality in distribution

$$RY \stackrel{d}{=} U^{1/b}.$$

Where Y is some other random variable, U is a uniform random variable on $[0, 1]$ and $b > 0$. Then computation of marginals of $T(t)$ is possible with an appropriate choice of driving Lévy subordinator.

To see this consider the simplest case when Y is degenerate, say $Y = 1$, then $R \stackrel{d}{=} U^{1/b}$. Since $F_U(x) = \mathbb{P}(U \leq x) = x$ is the cdf of U then the cdf of the random variable R is

$$\begin{aligned}F_{U^{1/b}}(x) &= \mathbb{P}(U^{1/b} \leq x) = \mathbb{P}(U \leq x^b) \\ &= F_U(x^b) = x^b = F_R(x)\end{aligned}$$

and the probability density function f_R of R is

$$f_R(x) := \frac{dF_R(x)}{dx} = bx^{b-1}.$$

The expectation of R is

$$\begin{aligned}\mathbb{E}[R] &= \int_0^1 x dF_R(x) = \int_0^1 x f_R(x) dx \\ &= \int_0^1 bx^b dx = \frac{b}{b+1}.\end{aligned}$$

For the quantile function Q_R there is a closed form given by

$$\begin{aligned}Q_R(u) &= \inf\{x : F_R(x) \geq u\} = \inf\{x : x^b \geq u\} \\ &= \inf\{x : x \geq u^{1/b}\} = u^{1/b}.\end{aligned}$$

By noticing the integral

$$\int_0^1 Q_R(u) du = \int_0^1 u^{1/b} du = \frac{b}{1+b}, \tag{2.12.3}$$

we conclude the well known result that

$$\int_0^1 Q_R(u) du = \mathbb{E}[R]. \tag{2.12.4}$$

The Laplace transform of $T(t)$ of a quantile clock can then be expressed as

$$\begin{aligned}\mathbb{E}[e^{-\zeta T(t)}] &= \exp\left\{ -\int_0^t \Psi_{L(1)}\Big(\zeta Q_R\Big(\Big(1-\frac{s}{t}\Big)_+\Big)\Big) ds\right\} \\ &= \exp\left\{ t\int_0^1 \Psi_{L(1)}\Big(\zeta Q_R(u)\Big) du\right\} \\ &= \exp\Big\{ t\mathbb{E}[\Psi_{L(1)}(\zeta R)]\Big\}. \qquad (2.12.5)\end{aligned}$$

Then since $R \overset{d}{=} U^{1/b}$ we have

$$\begin{aligned}\mathbb{E}[e^{-\zeta T(t)}] &= \exp\Big\{ t\mathbb{E}[\Psi_{L(1)}(\zeta U^{1/b})]\Big\} \\ &= \exp\left\{ t\int_0^1 \Psi_{L(1)}(\zeta x^{1/b}) f_U(x) dx\right\} \\ &= \exp\left\{ t\int_0^1 \Psi_{L(1)}(\zeta x^{1/b}) dx\right\}\end{aligned}$$

$$= \exp\left\{t\zeta^{-b}\int_0^\zeta \Psi_{L(1)}(w)bw^{b-1}dw\right\}. \tag{2.12.6}$$

The last expression tells us how we can choose $\Psi_{L(1)}(\zeta)$ so that the integral can be solved. This can be seen as follows, with the choice

$$\Psi_{L(1)}(\zeta) = \Psi_{L^{TS}}(\zeta) + \zeta b\Psi'_{L^{TS}}(\zeta) \tag{2.12.7}$$

then

$$\begin{aligned}\mathbb{E}[e^{-\zeta T(t)}] &= \exp\left\{t\zeta^{-b}\int_0^\zeta \Big(\Psi_{L^{TS}}(w) + \frac{w}{b}\Psi'_{L^{TS}}(w)\Big)bw^{b-1}dw\right\}\\ &= \exp\left\{t\zeta^{-b}\Big(\int_0^\zeta bw^{b-1}\Psi_{L^{TS}}(w)dw + \int_0^\zeta w^b\Psi'_{L^{TS}}(w)dw\right\}\\ &= \exp\left\{t\zeta^{-b}\Big[w^b\Psi_{L^{TS}}(w)\Big]_0^\zeta\right\}\\ &= \exp\left\{t\Psi_{L^{TS}}(\zeta)\right\}. \end{aligned} \tag{2.12.8}$$

Then with the choice for $\Psi_{L(1)}(\zeta)$ given by (2.12.7) we necessarily have

$$T(1) \stackrel{d}{=} L^{TS} \qquad \text{or} \qquad T(t) \sim TS(\alpha, t\delta, \gamma). \tag{2.12.9}$$

That is we can compute the exact marginal distributions of the convoluted subordinator $\{T(t), t \geq 0\}$. This will be used in the following section to construct an activity time process $T(t)$ with long range dependence.

2.13 Fractal activity time - type II

This section builds a activity time process $\{T(t), t \geq 0\}$ which has the representation of a Holmgren-Leuville fractional integral, this construction is essentially new to the literature on activity time models.

Definition 6. *A fractal activity time of type II is a stochastic process* $\{T(t), t \geq 0\}$ *defined by*

$$T(t) = \int_0^t (1 - s/t)^{H-1/2} L(ds), \tag{2.13.1}$$

for $1/2 < H < 1$ *with* $\{L(t), t \geq 0\}$ *a Lévy process given by the sum of independent components:*

$$L(t) = \upsilon(t) + (H - 1/2)Z(t), \tag{2.13.2}$$

where $\{\upsilon(t), t \geq 0\}$ *is a tempered stable Lévy process and* $\{Z(t), t \geq 0\}$ *the BDLP of a TS-OU process.*

Theorem 4. *A fractal activity time of type II has exact tempered stable* $TS(\alpha, t\delta, \gamma)$ *marginal distributions.*

Proof: First, we choose the process L, a Lévy subordinator, so that $T(t)$ has tempered stable marginal distribution. We follow the approach from James and Zhang (2011), where the kernel

$$k(t, s) = (1 - s/t)_+^{H-1/2} = Q_R(1 - s/t)_+ \tag{2.13.3}$$

is expressed using the function Q_R, which is a quantile function of Beta $(a, 1)$ distribution with $a = 1/(H - 1/2)$. Here $x_+ = \max(x, 0)$. Using this notation

$$T(t) = T_R(t) = \int_0^t Q_R\Big((1 - s/t)_+\Big)L(ds). \tag{2.13.4}$$

From equation (2.12.7), the Laplace exponent of L is given by

$$\Psi_L(\zeta) = \Psi_\upsilon(\zeta) + \frac{1}{a}\zeta\Psi'_\upsilon(\zeta), \tag{2.13.5}$$

holds for all $\zeta = u + iv$ with $v \in \mathbb{R}$, $u \leq 0$. Consider a Lévy process $Z(t)$, which is BDLP for the stationary TS-OU process $\tilde{\upsilon}$:

$$d\tilde{\upsilon}(t) = -\lambda\tilde{\upsilon}(t)dt + dZ(\lambda t). \tag{2.13.6}$$

Here a stationary TS-OU type process $\tilde{\upsilon}$ has the same marginal distribution as the distribution of $\upsilon(1)$, $TS(\alpha, \delta, \gamma)$. Then the Laplace exponent of Z is related to the Laplace exponent of $\tilde{v}$ (see equation 2.6.3) according to

$$\Psi_Z(\zeta) = \zeta \Psi'_{\tilde{v}}(\zeta), \tag{2.13.7}$$

which (up to a factor of $1/a$) is the second term in (2.13.5). This means that process L can be represented as the sum of two independent components: a TS Lévy process v, and a Lévy process $(H-1/2)Z(t)$. The explicit expression for the Laplace exponent of process L is

$$\Psi_L(\zeta) = \delta\gamma - \delta(\gamma^{1/\alpha} + 2\zeta)^\alpha - 2(H-1/2)\zeta\delta\alpha(\gamma^{1/\alpha} + 2\zeta)^{\alpha-1}. \tag{2.13.8}$$

This completes the proof for our second construction of the activity time. □

We could have also proved the preceding Theorem directly by computation by using the Laplace exponent of process L given by equation (2.13.8). To see this

$$\Psi_{T(t)}(\zeta) = \int_0^t \Psi_L(\zeta k(t,s))ds = \int_0^t \Psi_L\left(\zeta\left(1-\frac{s}{t}\right)_+^{H-1/2}\right)ds$$

and after a substitution to $v = \zeta(1-s/t)^{H-1/2}$ we obtain

$$\begin{aligned}
&t\int_0^\zeta \zeta^{-1/(H-1/2)}(H-\frac{1}{2})^{-1}v^{(3/2-H)(H-1/2)^{-1}}\Psi_L(v)dv \\
&= t\zeta^{-1/(H-\frac{1}{2})}\Big[\int_0^\zeta (H-\frac{1}{2})^{-1}v^{\frac{\frac{3}{2}-H}{H-\frac{1}{2}}}(\delta\gamma - \delta(\gamma^{1/\alpha}+2v)^\alpha)dv \\
&\quad + \int_0^\zeta v^{\frac{\frac{3}{2}-H}{H-\frac{1}{2}}}\left(-2v\delta\alpha(\gamma^{1/\alpha}+2v)^{\alpha-1}\right)dv\Big] \\
&= t\zeta^{-1/(H-1/2)}\left[v^{\frac{\frac{3}{2}-H}{H-\frac{1}{2}}}(\delta\gamma - \delta(\gamma^{1/\alpha}+2v)^\alpha)\right]_0^\zeta \\
&= t\delta\gamma - t\delta(\gamma^{1/\alpha}+2\zeta)^\alpha.
\end{aligned}$$

Let us now look at the first and second order properties of fractal activity time type II. Firstly note that the distribution of $T(t)$ is infinity divisible and from Bender and Marquardt (2009) Theorem 1, we have

$$\mathbb{E}[e^{\zeta T(t)}] = \exp\{\Psi(\zeta)\} \tag{2.13.9}$$

where

$$\Psi(\zeta) = \zeta b^* \int_0^t (1 - s/t)^{H-1/2} ds + \int_0^t \int_0^\infty (e^{\zeta(1-s/t)^{H-1/2}x} - 1)\nu(dx)ds. \tag{2.13.10}$$

Corollary 4. *A fractal activity time of type II has expectation*

$$\mathbb{E}[T(t)] = 2\delta\alpha\gamma^{(\alpha-1)/\alpha}t, \tag{2.13.11}$$

variance

$$\mathbb{V}\text{ar}[T(t)] = 4\delta\alpha(1-\alpha)\gamma^{(\alpha-2)/\alpha}t, \tag{2.13.12}$$

and covariance

$$\mathbb{C}\text{ov}[T(t), T(u)] = 8H\delta\alpha(1-\alpha)\gamma^{(\alpha-2)/\alpha} \times \int_0^{t\wedge u} (1 - s/t)_+^{H-1/2}(1 - s/u)_+^{H-1/2} ds. \tag{2.13.13}$$

Proof: Follows directly from equations (2.11.2), (2.11.3) and (2.11.4). □

The next corollary gives the asymptotic behavior of the correlations between $T(t)$ and $T(t+u)$ for t, $u > 0$ when $t/u \to \infty$ and when $t/u \to 0$, the directions considered in Marinucci and Robinson (1999). We use notation $f \sim g$ for $\lim_{t\to\infty} f(t)/g(t) = 1$.

Corollary 5. [3] *For $u > 0$, $t > 0$, $t/u \to 0$*

$$\mathbb{C}\text{ov}[T(t), T(t+u)] \sim \mathbb{V}\text{ar}[L(1)]t/(H + 1/2)$$

[3]The proof of this theorem was provided by A. Sikorskii, in Michigan State University and is not the sole work of the author of this thesis.

and

$$\mathbb{C}\text{orr}[T(t),T(t+u)] \sim \frac{2H}{H+1/2}\sqrt{\frac{t}{t+u}}.$$

When $t/u \to \infty$

$$\mathbb{C}\text{ov}[T(t),T(t+u)] \sim \mathbb{V}\text{ar}[L(1)]\frac{t}{2H},$$

and

$$\mathbb{C}\text{orr}[T(t),T(t+u)] \sim \sqrt{\frac{t}{t+u}}.$$

The mean and the variance of the increments of the process T *are asymptotically homogeneous,*

$$\mathbb{E}[(T(t+u)-T(t)] = \mathbb{E}[L(1)]\frac{u}{H+1/2},$$

and when $t/u \to \infty$ *or when* $t/u \to 0$

$$\mathbb{V}\text{ar}[T(t+u)-T(t)] \sim \mathbb{V}\text{ar}[L(1)]\frac{u}{2H}.$$

For the unit increment process $\tau(t) = T(t) - T(t-1)$*, when* $k \to \infty$ *and* $t/k \to 0$

$$\mathbb{C}\text{orr}[\tau(t),\tau(t+k)] \sim H(2H-1)B(H+1/2,2)\frac{2t-1}{k^2},$$

where B *is the Beta-function.*

Proof: From Corollary 4

$$\begin{aligned}
&\mathbb{C}\text{ov}[T(t),T(t+u)] \\
&\quad= \mathbb{V}\text{ar}[L(1)]\int_0^t (1-s/t)_+^{H-1/2}(1-s/(t+u))_+^{H-1/2}\,ds \\
&\quad= \mathbb{V}\text{ar}[L(1)]t\int_0^1 (1-\tau)^{H-1/2}\Big(1-\frac{t}{t+u}\tau\Big)^{H-1/2}\,d\tau,
\end{aligned}$$

and the asymptotic behavior follows. As for the increments of the activity time, we have

$$\begin{aligned}\mathbb{V}\mathrm{ar}[T(t+u)-T(t)] = \mathbb{V}\mathrm{ar}[L(1)]\Big(&\frac{t+u}{2H}+\frac{t}{2H}\\ &-2\int_0^t (1-s/t)_+^{H-1/2}(1-s/(t+u))_+^{H-1/2}\,ds\Big),\end{aligned}$$

and the asymptotic behavior follows from that of the covariance.

The covariance function for the process $\tau(t)=T(t)-T(t-1)$ is

$$\begin{aligned}\mathbb{C}\mathrm{ov}[\tau(t),\tau(t+k)] = \frac{1}{2}[&\mathbb{E}(T(t)-T(t-1))^2\\ &+\mathbb{E}(T(t+k)-T(t+k-1))^2\\ &-\mathbb{E}(T(t)-T(t-1)-T(t+k)+T(t+k-1))^2]\\ &-\mathbb{E}(T(t)-T(t-1))\mathbb{E}(T(t+k)-T(t+k-1)).\end{aligned}$$

Substitute the expressions obtained for the second moments of the increments to complete the calculation:

$$\begin{aligned}&\mathbb{C}\mathrm{ov}[\tau(t),\tau(t+k)]\\ &\quad=\mathbb{V}\mathrm{ar}[L(1)]\int_0^t\Big(1-\frac{s}{t}\Big)^{H-1/2}\Big(\Big(1-\frac{s}{t+k}\Big)^{H-1/2}\\ &\qquad-\Big(1-\frac{s}{t+k-1}\Big)^{H-1/2}\Big)\,ds\\ &\qquad-\mathbb{V}\mathrm{ar}[L(1)]\int_0^{t-1}\Big(1-\frac{s}{t-1}\Big)^{H-1/2}\Big(\Big(1-\frac{s}{t+k}\Big)^{H-1/2}\\ &\qquad-\Big(1-\frac{s}{t+k-1}\Big)^{H-1/2}\Big)\,ds.\end{aligned}$$

From the last equation, we derive the asymptotic behavior of the covariances. Change the variables $s/t=u$ in the first integral, and $s/(t-1)=u$ in the second integral to get

$$\begin{aligned}&\mathbb{C}\mathrm{ov}[\tau(t),\tau(t+k)]\\ &\quad=\mathbb{V}\mathrm{ar}[L(1)]t\int_0^1(1-u)^{H-1/2}\Big(\Big(1-\frac{t}{t+k}u\Big)^{H-1/2}\end{aligned}$$

$$- \left(1 - \frac{t}{t+k-1}u\right)^{H-1/2}\Bigg) du$$
$$- \mathbb{V}\mathrm{ar}[L(1)](t-1)\int_0^1 (1-u)^{H-1/2}\Bigg(\left(1 - \frac{t-1}{t+k}u\right)^{H-1/2}$$
$$- \left(1 - \frac{t-1}{t+k-1}u\right)^{H-1/2}\Bigg) du.$$

When $t/k \to 0$, use Taylor formula to evaluate the difference terms in each integral:

$$\mathbb{C}\mathrm{ov}[\tau(t), \tau(t+k)] \sim \mathbb{V}\mathrm{ar}[L(1)]\frac{(H-1/2)(2t-1)}{(t+k-1)(t+k)} \times \int_0^1 (1-u)^{H-1/2} u \, du.$$

Therefore when $k \to \infty$ and $t/k \to 0$

$$\mathbb{C}\mathrm{orr}[\tau(t), \tau(t+k)] \sim H(2H-1)B(H+1/2, 2)\frac{2t-1}{k^2},$$

where B is the Beta-function. □

Let us now make some remarks on activity time models described by definition 1 in chapter 1, using a fractal activity time of type II.

Remark 2. *Let $P(t)$ satisfy the SDE*

$$dP(t) = \mu P(t)dt + (\theta + \tfrac{1}{2}\sigma^2)P(t)dT(t) - \sigma P(t)dB(T(t)) \qquad (2.13.14)$$

where $\{T(t), t \geq 0\}$ is fractal activity time of type II. Then the logarithm of the stock price has a normal tempered stable marginal distribution with the Laplace exponent

$$\Psi_{\log P(t)}(\zeta) = (\log p(0) + \mu t)\zeta + t\Big(\delta\gamma - \delta\Big[\gamma^{1/\alpha} + \frac{\theta^2}{\sigma^2} - \sigma^2\Big(\zeta + \frac{\theta}{\sigma^2}\Big)^2\Big]\Big). \qquad (2.13.15)$$

Remark 3. *The slow decay of correlations can be interpreted as a long-range dependence for non-stationary process $T(t)$. Since with this construction of* *the activity time,*

$$\mathbb{C}\text{ov}[\log P(t), \log P(u)] = \theta^2 \mathbb{C}\text{ov}[T(t), T(u)]$$

this long-range dependence is also present in the logarithm of the price.

2.14 Inverse stable subordinators

Stochastic processes known as *inverse subordinators* are defined as the first hitting time of a Lévy subordinator. An example we will rely upon is the so called *inverse stable subordinator*, defined as follows. Let $\{D(t), t \geq 0\}$ be a standard stable Lévy subordinator with Laplace exponent $\Psi_D(\zeta) = -\zeta^\alpha$, $\zeta > 0$, $t \geq 0$ with $\alpha \in (0, 1)$. The inverse stable subordinator $E(t)$ is defined as the inverse of the stable subordinator $D(t)$, that is

$$E(t) = \inf\{u \geq 0 : D(u) > t\}, \quad t \geq 0,$$

see, for example, Bingham (1971) or Meerschaert and Sikorskii (2012). Note the stochastic process $\{E(t), t \geq 0\}$ is non-Markovian with non-stationary and non-independent increments. Both processes $\{D(t),\ t \geq 0\}$ and $\{E(t),\ t \geq 0\}$ are self-similar

$$\frac{D(at)}{a^{1/\alpha}} \overset{d}{=} D(t), \qquad \frac{E(at)}{a^\alpha} \overset{d}{=} E(t), \qquad a > 0. \tag{2.14.1}$$

From Bondesson et al. (1996), the moments are

$$\mathbb{E}[E^k(t)] = \frac{t^{\alpha k} k!}{\Gamma(\alpha k + 1)}. \tag{2.14.2}$$

The Laplace transform of the inverse stable subordinator is

$$\mathbb{E}[e^{-\zeta E(t)}] = \mathcal{E}_\alpha(-\zeta t^\alpha), \quad \zeta > 0,\ t \geq 0,\ \alpha \in (0, 1) \tag{2.14.3}$$

where

$$\mathcal{E}_\alpha(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + 1)} \qquad z \in \mathbb{C},\ \alpha \in (0,1) \tag{2.14.4}$$

is the one-parameter Mittag-Leffler function, see for example Mainardi and Gorenflo (2000) and Haubold, Mathai, and Saxena (2011).

The covariance function of this process is computed in Veillette and Taqqu (2010):

$$\mathbb{C}\text{ov}[E(t), E(s)] = \frac{1}{\Gamma(1+\alpha)\Gamma(\alpha)} \int_0^{\min(t,s)} \Big((t-\tau)^\alpha + (s-\tau)^\alpha\Big)\tau^{\alpha-1} d\tau - \frac{(st)^\alpha}{\Gamma(1+\alpha)}, \quad t,\ s \geq 0. \tag{2.14.5}$$

From Leonenko, Meerschaert, Sikorskii, and Schilling equations (10) and (11) it follows that the correlation function is approximately

$$\mathbb{C}\text{orr}[E(t), E(s)] \approx \left(\frac{s}{t}\right)^\alpha \left[2 - \frac{\Gamma(2\alpha+1)}{\Gamma(1+\alpha)^2}\right]^{-1} \qquad \text{as} \quad t \to \infty. \tag{2.14.6}$$

This power law decay of the correlation function can be viewed as a long range dependence for the inverse stable subordinator $E(t)$, since the correlation function is not integrable at infinity.

2.15 Fractal activity time - type III

The third construction is by definition an inverse stable subordinator. Although this appears more simple than fractal activity time of type I and II, it does however produce an activity time model which in terms of the limit when appropriately normed is related to models in chapter 4.

Definition 7. *A fractal activity time of type III is a stochastic process*

$\{T(t), t \geq 0\}$ *defined by the inverse of stable subordinator* $D(t)$*, that is*

$$T(t) = \inf\{u \geq 0 : D(u) > t\}.$$

To construct activity time models with fractal activity time of type III we define $P(t)$ to satisfy the SDE

$$dP(t) = (\theta + \tfrac{1}{2}\sigma^2)P(t)dT(t) + \sigma P(t)dB(T(t)) \tag{2.15.1}$$

where $\{T(t), t \geq 0\}$ is an inverse stable subordinator. The solution to the SDE by Lemma 1 is given as

$$P(t) = P(0)\exp\Big\{\theta T(t) + \sigma B(T(t)\Big\}. \tag{2.15.2}$$

Due to the self-similarity of the fractal activity time, see equation 2.14.1, we have

$$T(t) \overset{d}{=} t^{\alpha}T(1) \qquad \text{and} \qquad T(t-1) \overset{d}{=} (t-1)^{\alpha}T(1).$$

The log return sequence $X(t) = \log P(t) - \log P(t-1)$ will have Laplace transform

$$\begin{aligned}
\mathbb{E}\Big[e^{-\zeta X(t)}\Big] &= \mathbb{E}\Big[e^{-\zeta(\theta(T(t)-T(t-1))+\sigma\sqrt{T(t)-T(t-1)}B(1))}\Big] \\
&= \mathbb{E}\Big[\mathbb{E}[e^{-\zeta(\theta u+\sigma\sqrt{u}B(1))}]\Big|T(t)-T(t-1)=u\Big] \\
&= \mathbb{E}\Big[e^{-(\zeta\theta+\frac{1}{2}\zeta^2\sigma^2)(T(t)-T(t-1))}\Big] \\
&= \mathbb{E}\Big[e^{-(\zeta\theta+\frac{1}{2}\zeta^2\sigma^2)(t^{\alpha}-(t-1)^{\alpha})T(1)}\Big] \\
&= \mathcal{E}_{\alpha}\Big(-\Big(\zeta\theta+\frac{1}{2}\zeta^2\sigma^2\Big)\Big(t^{\alpha}-(t-1)^{\alpha}\Big)\Big).
\end{aligned}$$

We say the log returns are normal inverse stable and to indicate this we shall write in notation,

$$X \sim NIS(\theta, \sigma, \alpha).$$

Activity time process given by equation (2.15.1) with inverse stable activity times have been studied in Hahn, Kobayshi, and Sabir (2011) and Kobayashi (2011) where the governing fractional differential equations for the probability density function are given. Furthermore martingale properties are given by Theorem 2 in Magdziarz (2009).

2.16 Concluding remarks

It has been shown that starting with a Lévy process as a driving noise, more complex activity time processes exhibiting dependence can be built. This was accomplished by convolution of a memory kernel and a Lévy process. In the OU case the kernel induced SRD and we used superpositions to create LRD, whereas in the second type the kernel directly induced LRD onto the time process. A key point has been that in either case we are able to compute approximate or exact probabilities, useful for option pricing in the next chapter. The activity times were then used as the time change for the risky asset model inducing memory and distributional qualities onto the price process, as desired. If no memory was induced we would be left with the model of Clark (1973) suggesting an exponential Lévy process for the log returns. Our activity times had memory thus our processes were not Lévy and we remained in the class of FATGBM models. Chapter one defined such models whilst this chapter constructed the time process that drives the model, the next chapter will assume the model is given and develop applications in finance, namely a pricing formula for the valuation of European option.

Chapter 3

Option pricing

3.1 Introduction

This chapter details and investigates an application in finance for activity time models, namely the valuation of options. An option is a contract entered into between two parties which gives the holder the right but not the obligation to buy or sell (to exercise) a set amount of the asset at or before a future point in time (the maturity date) at a price (the strike) agreed upon today. In this chapter we will be using the fractal activity time model for a risky asset presented in section 2.10, namely the TS-OU type activity time model.

Firstly we show how to fit the model to real world data, in the form of observed risky asset prices, which we call the statistical fit. This is accomplished by estimating the parameters of the model using method of moments to historical records of log returns. Thus we say this is the statistical fit in the sense that the probability distribution matches what has been observed over the lifespan of the asset. Option valuation techniques are then described for so called *European options*, which have the feature that they may only be exercised at the maturity date and not before.

We then move on to discuss risk neutral fit where the parameters are obtained by calibrating the option pricing formula to market option prices. The model then values options at the same price that the market does (at the time of calibration). Surprisingly these are not the same parameters one obtains under a statistical fit. We conclude the chapter with some remarks on the performance of the pricing of options under activity time models in comparison to the more classical Black-Scholes formula.

3.2 Statistical parameter estimation

Two common approaches to estimate the parameters of a distribution are maximum likelihood (ML) and method of moments (MOM). The first method ML assumes independence and since we have built models with either short or long range dependence, we reject ML as a viable approach. For the method of moments approach we look to see if the moment equations can be solved, to simplify things we consider the cumulants of the distribution. From the normal tempered stable characteristic exponent given by equation (2.10.2) the cumulants of all orders may be obtained by

$$C_j(X) = \frac{d^j}{d\zeta^j}\Phi_X(\zeta)\bigg|_{\zeta=0}. \tag{3.2.1}$$

The first cumulant equals the mean and is given by

$$C_1(X) = \mathbb{E}[X] = \mu + 2\delta\alpha\theta\Big(\gamma^{\frac{1}{\alpha}}\Big)^{\alpha-1}. \tag{3.2.2}$$

The second cumulant of X equals the variance,

$$C_2(X) = \mathbb{V}\mathrm{ar}[X] = 2\delta\alpha\sigma^2\Big(\gamma^{\frac{1}{\alpha}}\Big)^{\alpha-1} - 4\delta\alpha\theta^2(\alpha-1)\Big(\gamma^{\frac{1}{\alpha}}\Big)^{\alpha-2}. \tag{3.2.3}$$

The third cumulant is

$$C_3(X) = 8\delta\alpha\theta^3(\alpha-1)(\alpha-2)\Big(\gamma^{\frac{1}{\alpha}}\Big)^{\alpha-3} - 12\delta\alpha\theta\sigma^2(\alpha-1)\Big(\gamma^{\frac{1}{\alpha}}\Big)^{\alpha-2} \tag{3.2.4}$$

and the forth

$$\begin{aligned} C_4(X) = &-16\delta\alpha\theta^4(\alpha-1)(\alpha-2)(\alpha-3)\Big(\gamma^{\frac{1}{\alpha}}\Big)^{\alpha-4} \\ &+ 48\delta\alpha\theta^2\sigma^2(\alpha-1)(\alpha-2)\Big(\gamma^{\frac{1}{\alpha}}\Big)^{\alpha-3} \\ &- 12\delta\alpha\sigma^4(\alpha-1)\Big(\gamma^{\frac{1}{\alpha}}\Big)^{\alpha-2}. \end{aligned} \tag{3.2.5}$$

Furthermore the 5th and 6th cumulants of the random variable X can be computed as

$$\begin{aligned} C_5(X) = &\,32\delta\alpha\theta^5(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)\Big(\gamma^{\frac{1}{\alpha}}\Big)^{\alpha-5} \\ &- 160\delta\alpha\theta^3\sigma^2(\alpha-1)(\alpha-2)(\alpha-3)\Big(\gamma^{\frac{1}{\alpha}}\Big)^{\alpha-4} \\ &+ 120\delta\alpha\theta\sigma^4(\alpha-1)(\alpha-2)\Big(\gamma^{\frac{1}{\alpha}}\Big)^{\alpha-3} \end{aligned} \tag{3.2.6}$$

and

$$\begin{aligned} C_6(X) = &\,480\delta\alpha\theta^4\sigma^2(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)\Big(\gamma^{\frac{1}{\alpha}}\Big)^{\alpha-5} \\ &- 64\delta\alpha\theta^6(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)(\alpha-5)\Big(\gamma^{\frac{1}{\alpha}}\Big)^{\alpha-6} \\ &- 720\delta\alpha\theta^2\sigma^4(\alpha-1)(\alpha-2)(\alpha-3)\Big(\gamma^{\frac{1}{\alpha}}\Big)^{\alpha-4} \\ &+ 120\delta\alpha\sigma^6(\alpha-1)(\alpha-2)\Big(\gamma^{\frac{1}{\alpha}}\Big)^{\alpha-3}. \end{aligned} \tag{3.2.7}$$

It does not seem possible to explicitly solve the above system of equations. However for the symmetric case, when $\theta = 0$ with the additional assumption

$$\mathbb{E}[\tau] = 1, \tag{3.2.8}$$

the cumulant equations can be solved and yields the following MOM parameter estimates. This assumption is justifiable in the sense that it leads to an unbiased reflection of calendar time, since for the increments of the activity time process $\{T(t), t \geq 0\}$ given by $\tau(t) = T(t) - T(t-1)$, we have $\mathbb{E}[\tau(t)] = t$, time over the long run moves only as fast as calender time. To achieve this we restrict the δ parameter of the unit increments of the fractal activity time $\tau(t) \sim TS(\alpha, \delta, \gamma)$ to be given by equation (3.2.13) below. We proceed with giving method of moment estimators as follows.

Lemma 5. *Let $X \sim NTS(\mu, 0, \sigma, \alpha, \delta, \gamma)$ be a normal tempered stable random variable, then the method of moment estimators are given by*

$$\hat{\mu} = \hat{C}_1 \tag{3.2.9}$$

$$\hat{\alpha} = \frac{10\hat{C}_4^2 - 3\hat{C}_6\hat{C}_2}{5\hat{C}_4^2 - 3\hat{C}_6\hat{C}_2} \tag{3.2.10}$$

$$\hat{\gamma} = \left(\frac{6\hat{C}_2^2(1-\alpha)}{\hat{C}_4}\right)^{\hat{\alpha}} \tag{3.2.11}$$

$$\hat{\sigma} = \sqrt{\frac{\hat{C}_4\hat{\gamma}^{1/\hat{\alpha}}}{6\hat{C}_2(1-\hat{\alpha})}} \tag{3.2.12}$$

$$\hat{\delta} = \frac{1}{2\hat{\alpha}\hat{\gamma}^{(\hat{\alpha}-1))/\hat{\alpha}}} \tag{3.2.13}$$

where $\hat{C}_j$ is the j empirical cumulant.

For an observed sample $x(1), x(2), \ldots, x(n)$ of size n define the empirical mean as

$$\hat{m}_1 = \sum_{q=1}^{n} \frac{x(q)}{n} \tag{3.2.14}$$

and the j-th empirical central moment as

$$\hat{m}_j = \sum_{q=1}^{n} \frac{(x(q) - \hat{m}_1)^j}{n} \tag{3.2.15}$$

then the empirical estimators for the first six cumulants $\hat{C}_j(X)$, $j = 1, \ldots, 6$ can be computed by

$$\begin{aligned}
\hat{C}_1 &= \hat{m}_1 \\
\hat{C}_2 &= \hat{m}_2 \\
\hat{C}_3 &= \hat{m}_3 \\
\hat{C}_4 &= \hat{m}_4 - 3\hat{m}_1^2 \\
\hat{C}_5 &= \hat{m}_5 - 10\hat{m}_3\hat{m}_2 \\
\hat{C}_6 &= \hat{m}_6 - 15\hat{m}_4\hat{m}2 - 10\hat{m}_3^2 + 30\hat{m}_2^3
\end{aligned} \tag{3.2.16}$$

We compute parameters using the method of moment estimators for a cross section of empirical samples of risky assets for indexes, currencies, stocks and commodities, see table 3.1. Empirical and model probability density functions are displayed in figures 3.1 and 3.2.

When $\theta \neq 0$ it does not seem possible to solve the system of equations, however a generalized method of moment (GMM) approach can be taken (see Carrasco and Florens (2002)). The GMM estimator $\hat{\Theta}$ for the parameter set $\Theta = (\mu, \theta, \sigma, \alpha, \delta, \gamma)$, satisfies

$$\arg\min_{\Theta} \int_{\mathbb{R}} w(u)(\psi_n(u) - \psi_\Theta(u))du \tag{3.2.17}$$

where $\psi_\Theta(u)$ is the model characteristic function and $\psi_n(u)$ its empirical

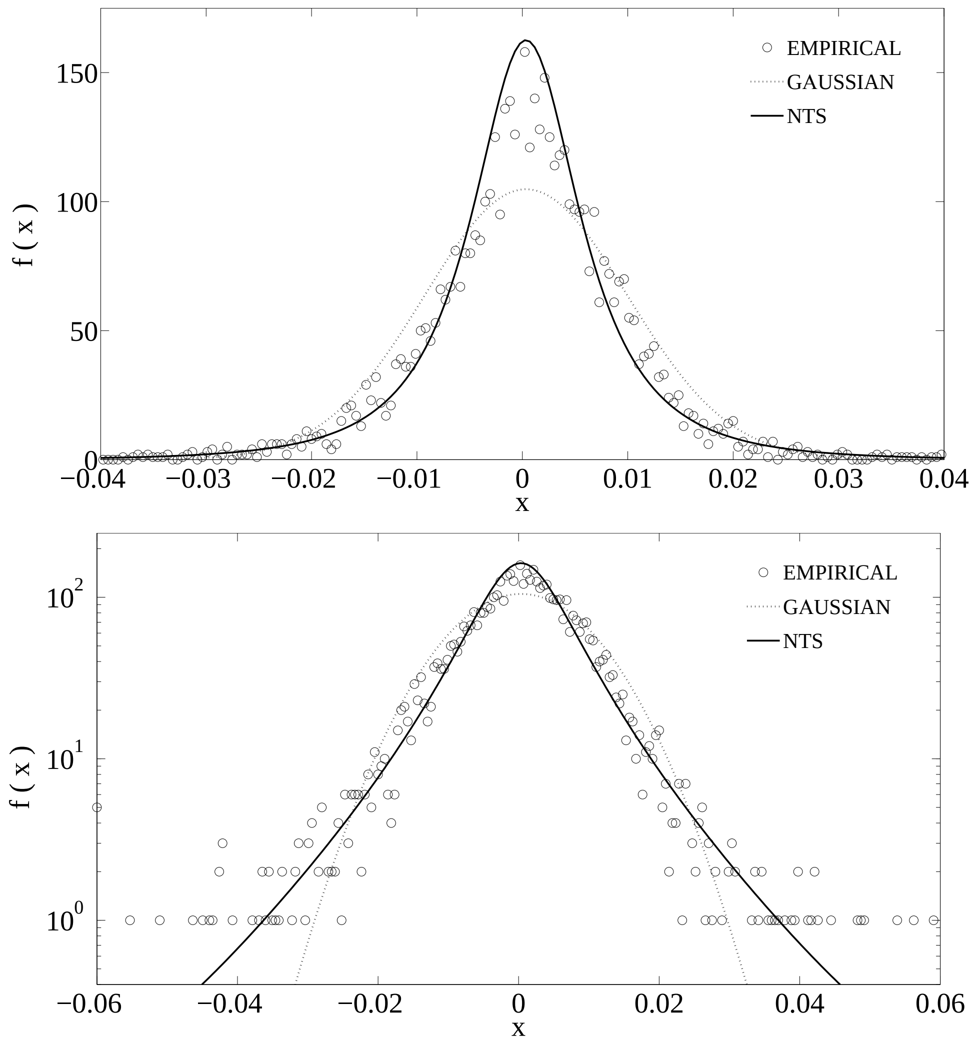

Figure 3.1: Australian Share Index

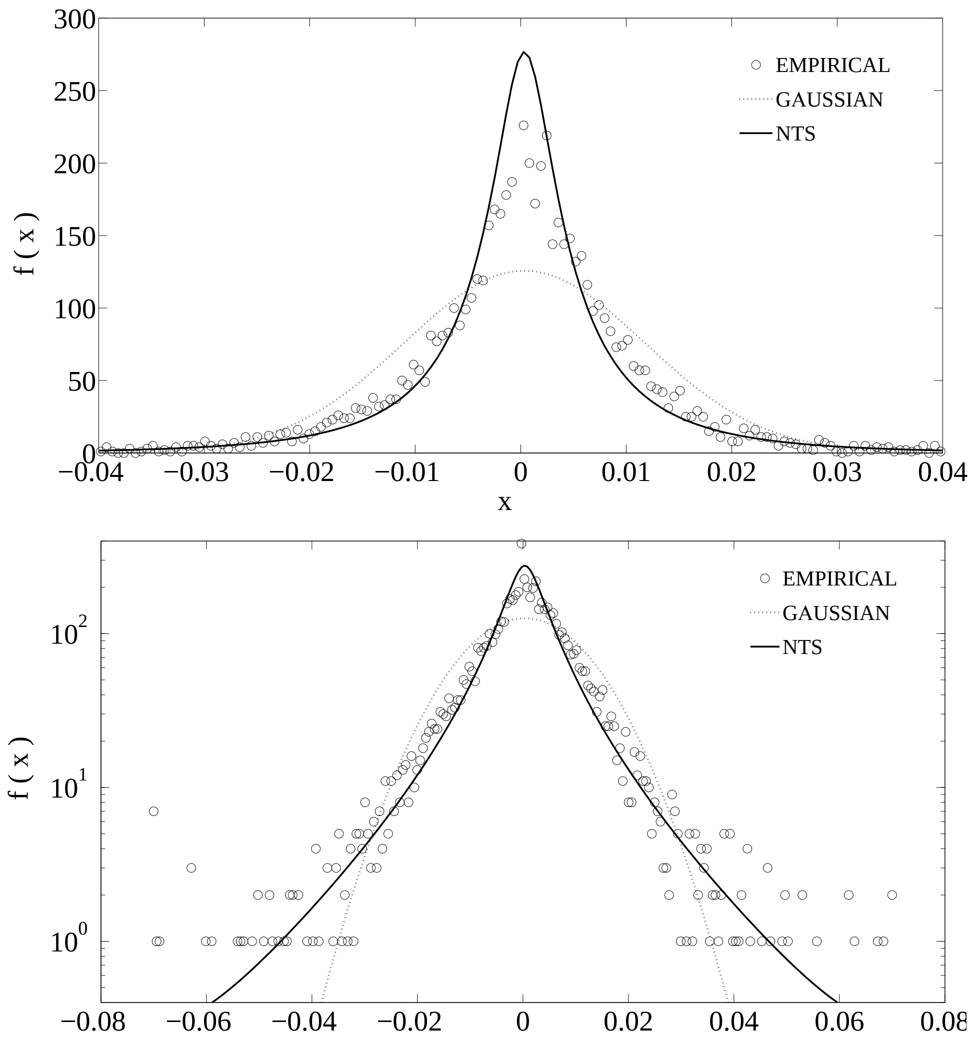

Figure 3.2: Standard and Poor Share Index

Risky asset	$\hat{\mu}$	$\hat{\theta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\delta}$	$\hat{\gamma}$
FTSE 100	0.0003517	0	0.01125	0.60487	0.485752	0.44314
ASX 200	0.0003452	0	0.00962	0.50128	0.694365	0.69485
DOW JONES	0.0003468	0	0.01161	0.77829	0.296764	0.06645
NASDAQ 100	0.0003250	0	0.01429	0.43155	0.744702	0.71494
HANG SENG	0.0004858	0	0.01631	0.50235	0.554435	0.55397
USD:EUR	-1.9526e-05	0	0.00510	0.34640	1.483233	1.01453
GBP:EUR	2.3843e-05	0	0.00457	0.57119	0.624828	0.63819
YEN:EUR	-4.1393e-05	0	0.00795	0.65824	0.578738	0.59228
USD:GBP	-4.3365e-05	0	0.00591	0.40301	1.116129	0.93108
GOLDBLN	0.0001755	0	0.01051	0.49320	0.595843	0.59618
CRUDOIL	1.8357e-04	0	0.02525	0.81247	0.377379	0.12018
SLVCASH	0.0002591	0	0.02342	0.44164	0.309902	0.35888
GLAXOSMITHKLINE	0.0002714	0	0.00714	0.70249	0.443341	0.32600
HSBC	0.0002302	0	0.00805	0.64590	0.470256	0.40285
WAL MART STORES	0.0003029	0	0.00844	0.47020	0.808938	0.78449
GENERAL ELECTRIC	0.0001625	0	0.00741	0.60510	0.499415	0.46229
SP 500 COMPOSITE	0.0003532	0	0.01137	0.37828	0.763830	0.71630
PFIZER	0.0001877	0	0.00769	0.78170	0.494130	0.39685
FTSE ALL SHARE	0.0003593	0	0.01049	0.57308	0.488157	0.45863

Table 3.1: MOM parameter estimates

counterpart, defined by

$$\psi_\Theta(\zeta) := \int_{\mathbb{R}} e^{i\zeta x} dF_\Theta(x) \quad \text{and} \quad \psi_n(\zeta) = \tfrac{1}{n}\sum_{j=1}^{n} e^{i\zeta x(j)}. \tag{3.2.18}$$

A simple approximation can be made by taking a discrete version of the integral in (3.2.17) but two choices must be made, first the truncation of the real line from $(-\infty, \infty)$ to some interval $[-u, u]$ and secondly the mesh size that partitions this interval. There is also the choice of the weighting function $w(u)$ to be decided upon. Finally, starting values for the minimization are required and since we do not know the true parameter values the choice of starting values may influence the estimated parameter values. We suggest the following choices

i. Truncation of the integral at $[-9, 9]$.

ii. A mesh size of one (a smaller mesh size may result in a singular covariance matrix, see Carrasco and Florens (2002), Carrasco and Florens (2000)).

iii. For the weight function use the probability density of a standard normal random variable (following Carrasco and Florens (2002) in regards to this choice).

iv. For the starting values the symmetric parameters ($\theta = 0$) for which the moments equations can be solved explicitly (see Lemma 5) could be used.

We do not go on to estimate parameters in a non symmetric normal tempered stable distribution since the above choices may not be optimal and the topic of accurately fitting all six parameters is a question for further research not undertaken in this thesis.

3.3 Option pricing

Pricing formula for related models was obtained in Heyde and Leonenko (2005); Nicolato and Venardos (2003); Barndorff-Nielsen et al. (2002). The approaches to deriving pricing formula varied depending upon the models considered and specific assumptions such as independence of increments, see Carr et al. (1998); Finlay and Seneta (2008b); Carr and Madan (1999); Lee (2004). In our model, the log returns are not independent, therefore we consider approaches to deriving pricing formula that are not based on the assumption of independence.

Activity time models are part of the wider class of *incomplete market models*. Incomplete markets are those in which perfect risk transfer is not possible, by this we mean that there is more than one risk neutral measure that can be used to price options, whereas in the classical Black-Scholes model the market is complete. This is because calibration of the Black-Scholes model to option data requires finding the value of the parameter σ such that the model prices match the market prices and there can only be one such value of σ that make this true. Whereas in activity time models the additional parameters whilst being able to provide a better fit to option data over different strikes, results in an array of parameter values that calibrate well to market data and the model is said to be incomplete.

We begin with the real world model and assume that a risk-neutral model has the same form, namely

$$dP(t) = \mu P(t)dt + (\theta + \tfrac{1}{2}\sigma^2)P(t)dT(t) + \sigma P(t)dB(T(t)), \tag{3.3.1}$$

but with different parameter values. To price options, we impose the parameter restrictions to ensure that the discount stock price process $e^{-rt}P(t)$ is a martingale with respect to the filtration $\mathcal{F}_t = \sigma(\{B(u), u \leq T(t)\}, \{T(u), u \leq t\})$. Here r is the interest rate. We follow the approach to obtaining a skew correcting martingale proposed in Heyde and Leonenko (2005) and used in Finlay and Seneta (2006). With this approach, parameter restrictions $\mu = r$ and $\theta = -\frac{1}{2}\sigma^2$ are imposed in the identity

$$\mathbb{E}(e^{-rt}P(t)|\mathcal{F}_s) = e^{-rs}P(s)e^{(\mu-r)(t-s)}\mathbb{E}(e^{(\theta+\frac{1}{2}\sigma^2)(T(t)-T(s))}|\mathcal{F}_s),$$

so that $\mathbb{E}(e^{-rt}P(t)|\mathcal{F}_s) = e^{-rs}P(s)$. Since parameter θ controls the skewness of the distribution of the returns, this approach to obtaining a risk neutral measure is called a skew correcting martingale approach. First note that

$$ae^{-\frac{1}{2}c^2+cZ} > b \Longleftrightarrow Z > \frac{1}{2}c - \frac{1}{c}\log\frac{a}{b} \qquad a, b, c > 0.$$

We now go on to derive the valuation formula for a European option which mature at time Y and have be written on a strike K, under a skew correcting martingale measure $\mathbb{Q}$ with $\mu = r$ and $\theta = -\frac{1}{2}\sigma^2$

$$\begin{aligned}
C(Y,K) &= e^{-rY}\mathbb{E}\left[(P(Y)-K)^+\right] \\
&= \mathbb{E}\left[(P(0)e^{-\frac{1}{2}\sigma^2 T(Y)+\sigma B_{T(Y)}} - Ke^{-rY})^+\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[(P(0)e^{-\frac{1}{2}\sigma^2 T(Y)+\sigma\sqrt{T(Y)}Z} - Ke^{-rY})\mathbb{I}_{Z>-\tilde{d}_2}\right] \middle| T(Y)\right] \\
&= \mathbb{E}\Big[P(0)\mathbb{E}\Big[\mathbb{I}_{Z>-\tilde{d}_1}\Big| T(Y)\Big] - Ke^{-rY}\mathbb{E}\Big[\mathbb{I}_{Z>-\tilde{d}_2}\Big| T(Y)\Big]\Big] \\
&= \mathbb{E}\Big[P(0)\mathbb{E}\left[\mathbb{I}_{Z<\tilde{d}_1}\Big| T(Y)\right] - Ke^{-rY}\mathbb{E}\Big[\mathbb{I}_{Z<\tilde{d}_2}\Big| T(Y)\Big]\Big] \\
&= \mathbb{E}\Big[P(0)\phi(\tilde{d}_1) - Ke^{-rY}\phi(\tilde{d}_2)\Big]
\end{aligned}$$

where $\phi(\cdot)$ is the cdf of $N(0,1)$, with notation $(z)^+ = \max(0,z)$ and

$$\tilde{d}_1 = \frac{\log\frac{P(0)}{K} + rY + \frac{1}{2}\sigma^2 T(Y)}{\sigma\sqrt{T(Y)}}, \qquad \tilde{d}_2 = \frac{\log\frac{P(0)}{K} + rY - \frac{1}{2}\sigma^2 T(Y)}{\sigma\sqrt{T(Y)}}.$$

Above we used the fact for $\mathbb{E}[F(Z)] < \infty$

$$\begin{aligned}
\mathbb{E}[F(Z+c)] &= \int_{-\infty}^{\infty} F(Z+c)\phi(Z)dZ \\
&= \int_{-\infty}^{\infty} F(y)\phi(y-c)dy \\
&= \int_{-\infty}^{\infty} e^{-\frac{1}{2}c^2+cy} F(y)\phi(y)dy \\
&= \mathbb{E}\left[e^{-\frac{1}{2}c^2+cZ} F(Z)\right]
\end{aligned}$$

in our case

$$F(Z) = \mathbb{I}_{Z > -\tilde{d}_2}$$

and

$$\mathbb{E}[F(Z)] = \int_{-\infty}^{\infty} F(-Z)\phi(Z)dZ.$$

By specifying a distribution for $T(Y)$ with pdf $F_{T(Y)}(t)$, then the call price becomes

$$C(Y,K) = \int_0^{\infty} \Big(P(0)\phi\Big(\tilde{d}_1(t)\Big) - Ke^{-rY}\phi\Big(\tilde{d}_2(t)\Big)\Big) f_{T(Y)}(t)dt. \tag{3.3.2}$$

In Heyde and Leonenko (2005), this pricing formula was used for a different fractal activity time, however this formula is valid for any construction of the activity time in the FATGBM model. Also, if $T(Y) = t$, then (3.3.2) reduces to the Black-Scholes formula.

Note that using the relationship

$$\tilde{d}_2 = \tilde{d}_1 - \sigma\sqrt{T(Y)},$$

it can easily be shown that

$$\tilde{d}_1 = e^{\tilde{d}_1\sigma\sqrt{T(Y)} - rY - \frac{1}{2}\sigma^2 T(Y)} = \frac{P(0)}{K}$$

and

$$Ke^{-rY}N'(\tilde{d}_2) = P(0)N'(\tilde{d}_1).$$

We now derive some derivatives or the so called sensitivities of the pricing formula, first reported by Roger Gay in the working paper *Fractals and Contingent Claims* which seems to be no longer be available. The rate of change of the pricing formula with respect to the price of the risky asset can be computed as

$$\begin{aligned}\frac{\partial C(Y,K)}{\partial P(0)} &= \mathbb{E}\left[\frac{\partial}{\partial P(0)}P(0)N(\tilde{d}_1) - \frac{\partial}{\partial P(0)}Ke^{-rY}N(\tilde{d}_2)\right] \\ &= \mathbb{E}\left[N(\tilde{d}_1) + P(0)N'(\tilde{d}_1)\frac{\partial \tilde{d}_1}{\partial P(0)} - Ke^{-rY}N'(\tilde{d}_2)\frac{\partial \tilde{d}_2}{\partial P(0)}\right] \\ &= \mathbb{E}\left[N(\tilde{d}_1)\right].\end{aligned}$$

The fractal activity time $T(t)$ is self-similar, that is

$$T_{ct} - ct \stackrel{d}{=} c^H(T(t) - t)$$

from which we can see that if we let $t = 1$ and $c = Y$ we can show,

$$T(1) \stackrel{d}{=} Y^{-H}(T(Y) - Y) + 1.$$

Using this fact we can see that

$$\begin{aligned}\frac{\partial(\sigma\sqrt{T(Y)})}{\partial Y} &= \frac{\partial(\sigma\sqrt{Y^H(T(1)-1)+Y})}{\partial Y} \\ &= \frac{1}{2}\sigma\frac{HY^{H-1}(Y^{-H}(T(Y)-Y)+1) - HY^{H-1}+1}{\sqrt{T(Y)}} \\ &= \frac{1}{2}\sigma\frac{T(Y)HY^{-1} - H + 1}{\sqrt{T(Y)}}.\end{aligned}$$

Another useful quantity is known as the theta,

$$-\frac{\partial C(Y,K)}{\partial Y} =$$

$$
\begin{aligned}
&- \mathbb{E}\left[P(0)\frac{\partial N(\tilde{d}_1)}{\partial \tilde{d}_1}\frac{\partial \tilde{d}_1}{\partial Y} + rKe^{-rY}N(\tilde{d}_2) - Ke^{-rY}\frac{\partial N(\tilde{d}_2)}{\partial \tilde{d}_2}\frac{\partial \tilde{d}_2}{\partial Y}\right] \\
&= -\mathbb{E}\left[P(0)N'(\tilde{d}_1)\left(\frac{\partial \tilde{d}_1}{\partial Y} - \frac{\partial \tilde{d}_2}{\partial Y}\right) + rKe^{-rY}N(\tilde{d}_2)\right] \\
&= -\mathbb{E}\left[P(0)N'(\tilde{d}_1)\left(\frac{\partial \tilde{d}_1}{\partial Y} - \frac{\partial(\tilde{d}_1 - \sigma\sqrt{T(Y)})}{\partial Y}\right) + rKe^{-rY}N(\tilde{d}_2)\right] \\
&= -\mathbb{E}\left[\frac{1}{2}P(0)N'(\tilde{d}_1)\sigma\frac{T(Y)HY^{-1} - H + 1}{\sqrt{T(Y)}} + rKe^{-rY}N(\tilde{d}_2)\right].
\end{aligned}
$$

Similar computation yield the Gamma

$$\frac{\partial^2 C(Y,K)}{\partial P(0)^2} = \mathbb{E}\left[N'(\tilde{d}_1)\frac{1}{\sigma P(0)\sqrt{T(Y)}}\right],$$

Vega

$$\frac{\partial C(Y,K)}{\partial \sigma} = \mathbb{E}\left[P(0)N'(\tilde{d}_1)\sqrt{T(Y)}\right]$$

and Rho

$$\frac{\partial C(Y,K)}{\partial r} = \mathbb{E}\left[KYe^{-rY}N(\tilde{d}_2)\right].$$

3.4 Risk neutral parameter estimation

This empirical section will look at implementation and calibration of the TS-OU activity time construction leading to normal tempered stable log returns in the case of finite superpositions as described in chapter two.

The data set contains transaction records for the September 2011 Eurofx futures and option contracts traded on the Chicago Mercantile Exchange (CME) over a three month horizon from the 22nd June until expiration on the 22nd September 2011. These records have been obtained directly from the CME. The Eurofx is a collection of derivative products where the underlying asset for the futures contract is the USD:EUR foreign exchange (FX) spot rate and for the options the underlying is the futures contract with the corresponding same expiration date. The spot rate is predominantly traded inter bank via electronic systems. The spot rate has been obtained from Thompson Datastream as previously mentioned in the empirical study in chapter one. For the risk free rate r we have obtained the US LIBOR rates also from Thompson Datastream.

Options are priced using formulae (3.3.2) under an equivalent martingale measure (EMM), termed the risk neutral measure $\mathbb{Q}$ with risk neutral parameters $\Theta^{\mathbb{Q}} = (\mu, \delta, \gamma, \alpha, \theta, \sigma)$ for the NTS case and when $T(Y) = t$ in equation (3.3.2), with parameters $\Theta^{\mathbb{Q}} = (\mu, \sigma)$ referred to as the geometric Brownian motion (GBM) case. Here r is the risk free rate. For the skew correcting martingale approach described earlier, $\mu = r$ and $\theta = -\frac{1}{2}\sigma^2$ under $\mathbb{Q}$ for the NTS case and $\mu = r$ and $\theta = 0$ for the GBM case. The risk neutral parameters that do not have martingale restrictions imposed on them, i.e. $(\delta, \gamma, \alpha, \sigma)$ for the NTS case and σ for the GBM case, could be set to their statistical estimates. In general though, the valuation of options based on these statistical parameters will not result in prices that match market prices. Instead the model is calibrated to options data and *implied* risk neutral parameters are returned (see for example Carr et al. (1998); Bates (1991)).

In the NTS case the stability and scale parameters δ and α should not change (see Carr et al. (2002); Kassberger and Liebmann (2011)) under EMM, and under the risk neutral measure they should be fixed to their statistical counterparts. However it may turn out that changing one or

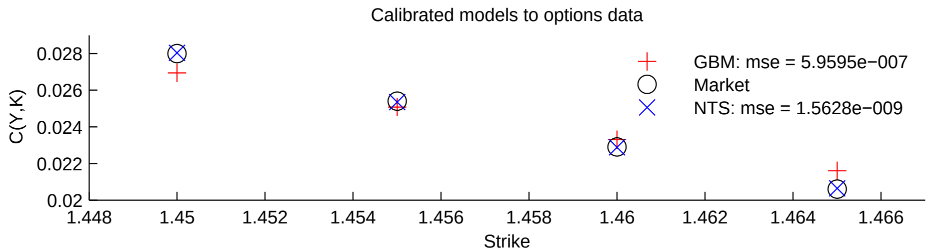

Figure 3.3: Comparison of option pricing models on 3.15am, August 12th

more of the parameters, allowing them to float during calibration to option values produces a better fit. Therefore as reasoned in Carr et al. (2002), during calibration allowing δ and α to change under the EMM, may suggest that the statistical parameters are not reliable as estimators when moving to the risk neutral measure.

To aid computation the expectation of the activity time is set equal to one, this ensures stability through time for the distribution of the activity time, thus $\delta = 1/(2\alpha\gamma^{(\alpha-1)/\alpha})$ and furthermore α is fixed at 0.3460 (see table 3.1) in all computations. Risk neutral implied parameter estimates are then computed, γ and σ for the NTS case and σ for the GBM case by matching the models to fit market data in the mean square sense, i.e. compute in both cases

$$\arg\min_{\Theta^{\mathbb{Q}}} \sqrt{\frac{1}{N}\sum_{i=1}^{N}(C_i(Y,K) - C_i(\tilde{Y},K))^2}$$

where C_i is the market price for the option and $\tilde{C}_i$ is the model implied price. As can be seen in Figure 3.3, the additional parameter γ allows a closer calibration to the observed market prices for the NTS model with a MSE of 1.5628e-9 against the GBM model with MSE 5.9595e-7.

3.5 Model performance

In order to compare option pricing models we follow the regression approach described in Carr et al. (1998) and in references therein. This approach assumes time variable parameters, so fitting in-sample and testing out-of-sample may not be that useful. Instead the errors of the in-sample model fit are investigated in terms of how predictable they are. This follows the notion that estimating risk neutral parameters in one period and using them in a subsequent period will perform poorly. Indeed Bakshi et al. (1997) results suggest as the GBM case has only one parameter to calibrate, it is more stable through time compared to a model with more parameters, where a better fit may be obtained in sample but parameters have little stability out of sample.

To investigate model performance, as in Carr et al. (1998), the options data set is searched for fixed points in time when the futures and a series of option strikes trade in synchronization. Each time the futures trade, we look to see whether the first four *out of the money* (OTM) call strikes have traded in a proceeding five second period. This leaves cross sectional data at fixed points in time for option prices, the corresponding futures price, the time until expiration and the risk free rate on that day. On average this resulted in 28 cross sectional data sets for each trading day at irregular spaced time intervals often grouped together in relatively short periods. As the option pricing formulae involves numerical integration as does evaluation of $f_{T(Y)}(t)$ which is computationally expensive, we reduce the data set further. This is achieved by extracting each day the first cross sectional set during the periods 3am-5am and 8am-10am. This relates to the open of European and US markets when the most transactions are recorded. The result of these filters is a set of 89 cross sections from 22nd of June until 11th of September, giving a total of 356 option trade prices. At each time a cross section of market observed option prices is used and the calibration aims to find parameters that fit across all strikes in a mean squared sense.

	GBM	NTS
	in-sample	*in-sample*
constant η_0	0.144	0.001
t-stat	(2.4087)	(0.365)
moneyness η_1	-0.143	-0.001
t-stat	(-40.7)	(-1.823)
R^2	0.824	0.0088
$F-statistic$	831.6	1.72
OBS	356	356

Table 3.2: Comparison of option pricing

The regression approach reasons that the option pricing errors should not exhibit any predictable patterns, and as such we carry out the regression, $PE_i = \eta_0 + \eta_1 M_i + \epsilon_i$ where M_i, is the moneyness of the i^{th} option respectively and ϵ a random noise term. The moneyness is the ratio of the option strike price and the current price of the underlying asset. The estimated coefficients η_0 and η_1, their corresponding t-values and the R^2 and F statistic are reported in table 3.2.

The results from the regression analysis show by a large t-statistic for the GBM case that moneyness can be used to explain the pricing errors, and with a R^2 of 82% as a measure of the percentage reduction in mean-squared-error that the regression model achieves through the explanatory moneyness variable. In contrast, for the NTS model the t-statistic indicates that moneyness is of no help in the predictability of the pricing errors, with an R^2 of only 0.1% very little reduction in MSE is obtained. These results are similar to those found in Carr et al. (1998).

3.6 Concluding remarks

An option valuation equation has been developed which can be interpreted as a weighted Black-Scholes formula, moreover the requirement for pricing is knowledge of the probability density for the activity time. Thus our models are tractable since we can compute activity time probabilities (see sections 2.9 and 2.13 in the previous chapter). We have seen that the statistical parameters obtained from the history of the assets price are not the parameter estimates that produce a realistic option valuation and risk neutral parameters were discussed to price options according to current market valuation. Furthermore it is clear that the additional structure of a process with more parameters can provide a better fit to options data in the sense of mean squared error of model prices to market prices. Although comparative MSE is small it would have significant impact when trading large volumes of option contracts where a small price discrepancy would translate into a substantial profit or loss for the option trader.

The next chapter is not a continuation, in the sense that we now take a few steps back and look at asset returns not over days but over short periods of a few seconds. Models are constructed, empirical properties are investigated and results on option pricing given.

Chapter 4

Integer valued models

4.1 Introduction

We now turn our attention to risky assets measured at *high frequency*. The advent of high frequency financial data has spurred some new modeling techniques to describe characteristics of trade by trade data. Recent literature on the subject includes Barndorff-Nielsen, Shephard, and Pollard (2011), Carr (2011), Bacry, Delattre, Hoffmann, and Muzy (2013b) and Bacry et al. (2013) where models based on the difference of two point processes are proposed. The difference of Poisson processes is considered in Barndorff-Nielsen et al. (2011) and Carr (2011), known as the Skellam process. The difference of two Hawkes processes are discussed in Bacry et al. (2013b) and Bacry et al. (2013).

A drawback of the existing models, which may be at odds with empirical facts, is exponential inter-arrival time, or time between trades. Mainardi, Gorenflo, and Scalas (2004) studied the fractional Poisson process, where the exponential waiting time distribution is replaced by a Mittag-Leffler distribution, see also Beghin and Orsingher (2009), Laskin (2003), Repin and Saichev (2000) and Uchaikin et al. (2008). The goal of this chapter is to extend the Skellam models of Barndorff-Nielsen et al. (2011) with their exponential inter-arrival times to the fractional setting and the more flexible Mittag-Leffler inter-arrival times.

Throughout this chapter we will be modeling the so called futures price $F(t)$, which is directly linked to a risky assets spot price $P(t)$. The forward price depends on a forward contract upon which it is written. A forward contract between two parties is an agreement to buy or sell a risky asset at a specified future time T, at a price agreed upon today. The price of the forward contract is the so called *futures price* denoted by $F(t)$, $t \in [0, T]$. With r the interest rate applicable to the underlying risky asset, the futures price is related by no arbitrage arguments, to the spot price through the formula,

$$F(t) = P(t)e^{rt}. \tag{4.1.1}$$

When looked at over very small time horizons, risky assets such as futures, no longer trade in a diffusive manner. This is an inherit property from the exchanges for which these assets trade upon. The exchange allows participant to trade such assets but only at discrete evenly spaced values, which are multiples of the tick value. See as an example, figure 4.1 for an empirical path of the Eurofx futures price over the short interval of 600 seconds.

To capture this phenomenon, Barndorff-Nielsen et al. (2011) proposed the

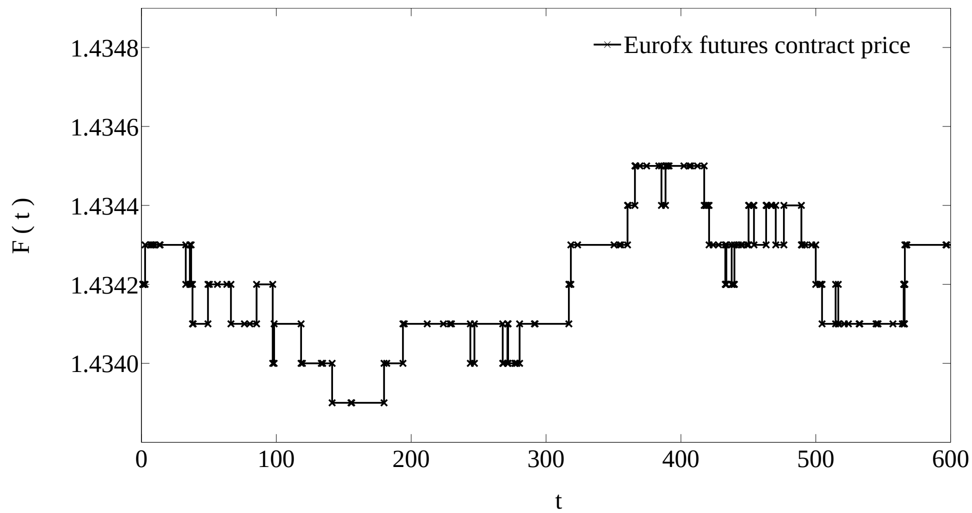

Figure 4.1: Eurofx empirical price path

following arithmetic model for the forward price,

$$F(t) = F(0) + aS(t) \tag{4.1.2}$$

with $F(0) > 0$, a some positive constant and $\{S(t), t \geq 0\}$ an integer valued stochastic process. The forward price has jumps up or down by a magnitude of size a, mimicking reality when trade by trade prices jump up in scaler values of the tick size, indeed a is the tick size. The above model also appears in Carr (2011) and is extended so that the process remains positive, to the geometric case

$$F(t) = F(0)\exp\{gS(t)\}, \tag{4.1.3}$$

again with initial value $F(0) > 0$ and g some positive constant. In this case the up jumps have size $F(t-)(e^{g} - 1)$ and the down jumps have size $F(t-)(e^{-g} - 1)$.

The rest of this chapter is organized as follows. Section 4.2 details integer valued Lévy process before defining the Poisson process in section 4.3. In section 4.4 Skellam Lévy processes are defined, along with associated differential equations and distributional properties. Section 4.5 gives martingale properties for Skellam processes and the timed changed Skellam inverse Gaussian process is introduced in section 4.6. Section 4.7 contains known option valuation techniques for integer valued models. We then proceed to generalize to a wider *fractional* setting in the sense that our processes have associated fractional differential equations. First in section 4.8 the fractional Poisson process is defined. We then define and give details for fractional Skellam processes of type I and type II in sections 4.9 and 4.10 respectively. These processes generalize the Skellam process to where the inter arrival times have Mittag-Leffler distribution. Martingale properties of fractional Skellam processes of type II are then discussed in section 4.11. An empirical investigation in section 4.12 shows a more realistic fit to observed waiting times between trades at a high frequency scale. section 4.13 gives known details on continuous time random walks and section 4.14 gives the CTRW representation of the fractional Skellam process type II and discusses the convergence to the third type of activity model in chapter 2, section 4.15 introduces fractional Skellam tempered stable processes and section 4.16 introduces delta fractional negative binomial processes, both of which are further generalizations.

4.2 Integer valued Lévy processes

From hereon in, for our investigations for the most part, we will be working with integer valued Lévy processes $\{L(t), t \geq 0\}$, i.e. processes that take values on the integer set $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$,

$$L(t) : [0, \infty] \times \Omega \to \mathbb{Z}.$$

We shall use the same notation previously used for Lévy processes see section 2.2 chapter 2, for details. As detailed in Barndorff-Nielsen et al. (2011), an integer valued Lévy process $\{L(t), t \geq 0\}$ can be decomposed into its positive $\{L_1(t), t \geq 0\}$ and negative $\{L_2(t), t \geq 0\}$ parts simply by summing the positive and negative jumps of L separately, i.e.

$$L(t) = L_1(t) - L_2(t).$$

Thus both $\{L_1(t), t \geq 0\}$ and $\{L_2(t), t \geq 0\}$ are discrete Lévy subordinators with triplets (b_1, A_1, ν_1) and (b_2, A_2, ν_2) respectively. Furthermore since a subordinator has no Gaussian part $A_1 = A_2 = 0$, the Laplace transform is given by

$$\mathbb{E}[e^{-\zeta L_i(t)}] = e^{-t\Psi_{L_i(1)}(\zeta)}, \quad \zeta \geq 0, \quad i = 1, 2. \tag{4.2.1}$$

Where the Laplace exponent $\Psi_{L_i(1)}(\zeta) = \Psi_{L_i}$, $i = 1, 2$ with triple $(b_i^*, 0, \nu_i)$ is given by

$$\Psi_{L_i}(\zeta) = b_i^* \zeta - \int_{(0,\infty)} (e^{-\zeta x} - 1)\nu_i(dx) \tag{4.2.2}$$

here $b_i^* = b_i - \int_0^1 x\nu_i(dx) \geq 0$ and the Lévy measures are restrictions of ν to the positive and negative half axes, i.e.

$$\nu_1((-\infty, 0)) = 0 \qquad \nu_1((0, \infty)) = \nu((0, \infty))$$
$$\nu_2((-\infty, 0)) = 0 \qquad \nu_2((0, \infty)) = \nu((-\infty, 0))$$

Suppose L is a integer-valued Lévy process, then the Lévy measure ν of L is concentrated on $\mathbb{Z} \setminus 0$ and has finite mass, see Barndorff-Nielsen et al.

(2011), proposition 1. Since $A_1 = A_2 = 0$ and $\nu_1(\mathbb{R}), \nu_1(\mathbb{R}) < \infty$ from Sato (1999) theorem 12.2, both $\{L_1(t), t \geq 0\}$ and $\{L_2(t), t \geq 0\}$ are piecewise constant a.s. if $b_i = 0$.

4.3 Poisson processes

This short section introduces the well known Poisson process, which is the simplest case of a positive integer valued Lévy process.

We say an increasing sequence of random variables $\tau(1), \tau(2), \ldots$, called arrival times (or sometimes epochs) form an arrival process. The arrival times represent some repeating event occurring. The arrival process starts at time zero and multiple arrivals can't occur simultaneously.

An arrival process can also be viewed as a counting process $\{N(t), t \geq 0\}$, where for each $t \geq 0$, the random variable $N(t)$ is the number of arrivals up to and including time t. For any given integer $n > 1$ and time $t > 0$, the nth arrival time, $\tau(n)$, and the counting random variable, $N(t)$, are related by

$$\{\tau(n) > t\} = \{N(t) < n\}. \tag{4.3.1}$$

Definition 8. *A renewal process is an arrival process for which the sequence of inter-arrival times is a sequence of independent and identically distributed random variables.*

Definition 9. *A Poisson process $\{N(t), t \geq 0\}$ is a renewal process in which the inter-arrival intervals have an exponential distribution function; i.e. for some real $\lambda > 0$, each $\tau(i)$ has the density $f(x) = \lambda \exp(-\lambda x)$ for $x > 0$.*

The process is named after the French mathematician Siméon Denis Poisson.

For a Poisson process with rate λ the probability mass function $p(n,t) = \mathbb{P}(N(t) = n)$ at time $t \geq 0$ has the well known form

$$p(n,t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}. \tag{4.3.2}$$

The Poisson process is in fact a counting process, which is a stochastic process with values that are positive, integer, and increasing. Moreover the Poisson process is the only counting process with independent and stationary increments. A Poisson process is a subordinator, i.e. an increasing Lévy process.

4.4 Skellam Lévy processes

This section introduces the Skellam distribution and the associated Skellam Lévy process which has been used as the integer valued random process in the arithmetic and geometric models as described by equations (4.1.2) and (4.1.3) in the introduction to this chapter. The Skellam distribution has been introduced in Skellam (1946) and Irwin (1937), and the corresponding Lévy processes are considered in Barndorff-Nielsen et al. (2011).

Definition 10. *A Skellam Lévy process* $\{S(t), t \geq 0\}$ *is defined as*

$$S(t) = N_1(t) - N_2(t), \qquad t \geq 0,$$

where $\{N_1(t),\, t \geq 0\}$ *and* $\{N_2(t),\, t \geq 0\}$ *are two independent homogeneous Poisson processes with intensities* $\lambda_1 > 0$ *and* $\lambda_2 > 0$ *respectively.*

We write in notation

$$S \sim Sk(\lambda_1, \lambda_2)$$

and for the marginal distributions of the Skellam Lévy process, we write

$S(t) \sim Sk(t\lambda_1, t\lambda_2)$. The Lévy triplet (b, A, ν) is given by

$$b = \lambda_1 + \lambda_2, \qquad A = 0, \qquad \nu(dx) = \lambda_1 \delta_{\{1\}} dx + \lambda_2 \delta_{\{-1\}} dx \tag{4.4.1}$$

where $\delta_{\{z\}}$ is the Dirac delta function with point mass at z. Since $A = 0$ then $\{S(t), t \geq 0\}$ is a pure jump process. The moment generating function Φ of $S(t)$ is given by

$$\Phi_{S(t)}(\zeta) := \mathbb{E}[e^{\zeta S(t)}] = \exp\Big\{t\Big(\lambda_1(e^{\zeta} - 1) + \lambda_2(e^{-\zeta} - 1)\Big)\Big\} \tag{4.4.2}$$

The probability mass function $s_k(t)$ for $k \in \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$, of the random variable $S(t)$ is

$$s_k(t) = \mathbb{P}(S(t) = k) = e^{-t(\lambda_1 + \lambda_2)} \left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{k}{2}} I_{|k|}\left(2t\sqrt{\lambda_1 \lambda_2}\right), \tag{4.4.3}$$

where I_k is the modified Bessel function of the first kind (Sneddon, 1956, p. 114)

$$I_k(z) = \sum_{n=0}^{\infty} \frac{(z/2)^{2n+k}}{n!(n+k)!}.$$

The mean and the variance are

$$\mathbb{E}[S(t)] = (\lambda_1 - \lambda_2)t, \qquad \mathbb{V}\mathrm{ar}[S(t)] = (\lambda_1 + \lambda_2)t, \tag{4.4.4}$$

and the covariance function

$$\mathbb{C}\mathrm{ov}(S(t), S(s)) = (\lambda_1 + \lambda_2)\min(t, s), \ t, \ s > 0. \tag{4.4.5}$$

The next result on the Skellam processes is straightforward, but to the best of our knowledge, it has not appeared in the literature.

Lemma 6. *The Skellam process is a stochastic solution of the following system of differential equations:*

$$\frac{d}{dt} s_k(t) = \lambda_1(s_{k-1}(t) - s_k(t)) - \lambda_2(s_k(t) - s_{k+1}(t)), \quad k \in \mathbb{Z} \tag{4.4.6}$$

with the initial conditions $s_0(0) = 1$ *and* $s_k(0) = 0$ *for* $k \neq 0$. *The moment* *generating function* Φ *of the Skellam process solves the differential equation*

$$\frac{d\Phi_{S(t)}(\zeta)}{dt} = \Phi_{S(t)}(\zeta)(\lambda_1(e^{\zeta} - 1) + \lambda_2(e^{-\zeta} - 1)) \tag{4.4.7}$$

with the initial condition $\Phi_{S(0)}(\zeta) = 1$.

Proof: Using the properties of modified Bessel function (Sneddon, 1956, p. 115)

$$I_k(x) = I_{-k}(x) \tag{4.4.8}$$

for any integer k and all x, and

$$\frac{dI_\nu(z)}{dz} = \frac{1}{2}\left(I_{\nu-1}(z) + I_{\nu+1}(z)\right), \tag{4.4.9}$$

differentiate both sides of equation (4.4.3) to get

$$\frac{d}{dt}s_k(t) = \lambda_1(s_{k-1} - s_k) - \lambda_2(s_k - s_{k+1}), \quad k \in \mathbb{Z}. \tag{4.4.10}$$

Thus (4.4.6) holds. Now multiply both sides of (4.4.6) by $e^{\zeta k}$ and sum over k to get

$$\frac{d\Phi_{S(t)}(\zeta)}{dt} = \Phi_{S(t)}(\zeta)(\lambda_1(e^{\zeta} - 1) + \lambda_2(e^{-\zeta} - 1)) \tag{4.4.11}$$

with the initial condition $\Phi_{S(0)}(\zeta) = 1$, thus equation (4.4.7) holds. This equation clearly has solution

$$\Phi_{S(t)}(\zeta) = e^{-t(\lambda_1+\lambda_2-\lambda_1 e^{\zeta}-\lambda_2 e^{-\zeta})}, \qquad \zeta \in \mathbb{R} \tag{4.4.12}$$

which agrees with equation (4.4.2) above and completes the proof. □

4.5 Martingale properties of Skellam processes

This section investigates martingale properties of Skellam processes, martingales have considerable interest in finance for the pricing of options in section 4.7.

From Lemma 2 in section 2.2 chapter 2, the Skellam process is not a martingale since in general the drift b of the Lévy triplet of $S(t)$ is $b = \lambda_1 + \lambda_2 \neq 0$. However there are some specific cases when a Skellam process it is a martingale which we refer to as the symmetric, standard and compensated types.

Definition 11. *Let $\{S(t), t \geq 0\}$ be a Skellam process, set*

$$\lambda_1 = \lambda_2 := \lambda > 0$$

then $\{S(t), t \geq 0\}$ is referred to as a symmetric Skellam process and we write $S(t) \sim Sk(t\lambda, t\lambda)$.

The characteristic exponent of the symmetric Skellam process is

$$\phi_S(\zeta) = \lambda(e^{i\zeta} + e^{-i\zeta} - 2), \tag{4.5.1}$$

generated from the Lévy triplet $(0, 0, \nu)$, with

$$\nu(dx) = \lambda\delta_{\{1\}}dx + \lambda\delta_{\{-1\}}dx, \tag{4.5.2}$$

hence by Lemma 2 the symmetric Skellam process is a martingale since $b = 0$.

Definition 12. *Let $\{S(t), t \geq 0\}$ be a Skellam process, set*

$$\lambda_1 = \lambda_2 = \tfrac{1}{2}$$

then $\{S(t), t \geq 0\}$ is referred to as a standard Skellam process and we write $S(t) \sim Sk(t\frac{1}{2}, t\frac{1}{2})$.

The characteristic function of the standard Skellam process has the trigonometric form

$$\psi_{S(t)}(\zeta) = \exp\Big\{ - t(1 - \cos(\zeta))\Big\}, \tag{4.5.3}$$

Clearly the Lévy triplet has drift component $b = 0$, hence the standard Skellam process is a martingale.

Definition 13. *Let $\{S(t), t \geq 0\}$ be a Skellam process. Define*

$$S^*(t) := S(t) - (\lambda_1 + \lambda_2)t, \qquad t \geq 0$$

then $\{S^*(t), t \geq 0\}$ is referred to as a compensated Skellam process and we write $S^*(t) \sim cSk(t\lambda_1, t\lambda_2)$.

Again since the Lévy triplet has drift component $b = 0$ so the compensated Skellam process is also a martingale.

Remark 4. *Both the symmetric and standard Skellam are integer valued processes, however the compensated Skellam process is not integer valued.*

4.6 Skellam inverse Gaussian Lévy processes

Although the Skellam process when used as the random component in the arithmetic and geometric models as described by equations (4.1.2) and (4.1.3) is useful, it does however have some drawbacks. Firstly it assumes inter-arrival times are exponentially distributed which may not be the case, this will be discussed further and a new process will be proposed in sections 4.9 and 4.10 to counter this. The second inherent flaw is that the Skellam process can only jump up or down by a magnitude of one. For a heavily traded product such as the futures series on the Euro to the US dollar this may not be a problem, since high liquidity means the price will rarely jump more than a single tick. However for assets that do not have such high trading volumes, the price may jump by several ticks from one trade to the next. This section aims to provide a generalization of the Skellam process to allow for larger than unit jumps. By subordination of a Skellam process by a Lévy subordinator, the process then has properties similar to a compound Poisson process, in fact it can be viewed as a compound Skellam process.

Let us introduce the following Lévy process, which is certainly not an integer valued process. The inverse Gaussian (IG) distribution is infinitely divisible and the corresponding inverse Gaussian subordinator $\{T(t), t \geq 0\}$ is distributed as $T(t) \sim IG(t\delta, \gamma)$. The Laplace exponent is

$$\Psi_{T(1)}(\zeta) = \delta((\gamma^2 + 2\zeta)^{1/2}) - \gamma), \qquad \delta, \gamma > 0,$$

and the marginal probability density function $r_x(t) := \mathbb{P}(T(t) \leq x)$ is given by

$$r_x(t) = \frac{1}{\sqrt{2\pi}} \delta t e^{\delta\gamma t} x^{-3/2} e^{-\frac{1}{2}(t^2\delta^2 x^{-1} + \gamma^2 x)}$$

Definition 14. *Let $\{S(t), t \geq 0\}$ be a standard Skellam process $S(t) \sim Sk(t\frac{1}{2}, t\frac{1}{2})$ and $\{T(t), \geq 0\}$ be a inverse Gaussian subordinator $T(t) \sim IG(t\delta, \gamma)$. Further assume that $S(t)$ is independent of $T(t)$. The stochastic process $\{Y(t), t \geq 0\}$ defined by*

$$Y(t) = S(T(t)),$$

is called a Skellam inverse Gaussian process.

Theorem 6. *Let $\{Y(t), t \geq 0\}$ be a Skellam inverse Gaussian process. The marginal distribution function $y_k(t) := \mathbb{P}(Y(t) = k)$, $k \in \mathbb{Z}$ is given by*

$$y_k(t) = \frac{1}{\sqrt{2\pi}} \delta t e^{\delta\gamma t} \int_0^\infty e^{-\frac{1}{2}(t^2\delta^2 u^{-1} + (\gamma^2+2)u)} I_{|k|}(u) u^{-3/2} du.$$

The characteristic exponent is

$$\phi_Y(\zeta) = \delta\gamma - \delta\sqrt{\gamma^2 + 2(1 - \cos(\zeta))}.$$

The probability mass function $y_k(t)$ solves the differential equation

$$\frac{\partial^2}{\partial t^2}y_k(t) - 2\delta\gamma\frac{\partial}{\partial t}y_k(t) = y_{k-1}(t) + y_{k+1}(t) + 2y_k(t).$$

Proof: The probability mass function is computed from

$$y_k(t) := y(k,t) = \int_0^\infty s(x,u)r(u,t)du \tag{4.6.1}$$

where $s(x,u)$ is the pmf of the standard Skellam process and $r(u,t)$ the pdf of the inverse Gaussian subordinator. The characteristic exponent can be computed as

$$\begin{aligned}\phi_Y(\zeta) &= \int e^{i\zeta k}y(k,t)dk = \int_0^\infty \left(\int e^{i\zeta k}s(k,u)dk\right)r(u,t)du\\ &= \int_0^\infty \psi_{S(u)}(\zeta)r(u,t)du = \int_0^\infty e^{-u(1-\cos(\zeta))}r(u,t)du\end{aligned}$$

and the result follows. To show the differential equation note that $\psi_{Y(0)}(\zeta) = 1$ and

$$\frac{\partial}{\partial t}\psi_{Y(t)}(\zeta) = -\delta\Big(\sqrt{\gamma^2+2(1-\cos(\zeta))}-\gamma\Big)\psi_{Y(t)}(\zeta) \tag{4.6.2}$$

let $\bar{\psi}_{Y(s)}(\zeta) := \int e^{-st}\psi_{Y(t)}(\zeta)dt$ then taking Laplace transforms of both sides of equation (4.6.2) we find

$$s\bar{\psi}_{Y(s)}(\zeta) - \psi_{Y(0)}(\zeta) = -\delta\Big(\sqrt{\gamma^2+2(1-\cos(\zeta))}-\gamma\Big)\bar{\psi}_{Y(s)}(\zeta)$$

rearrange to see

$$\begin{aligned}\bar{\psi}_{Y(s)}(\zeta) = &\frac{1}{(s-\delta\gamma)+\delta\sqrt{\gamma^2+2(1-\cos(\zeta))}}\\ &\times\frac{(s-\delta\gamma)-\delta\sqrt{\gamma^2+2(1-\cos(\zeta))}}{(s-\delta\gamma)-\delta\sqrt{\gamma^2+2(1-\cos(\zeta))}}\end{aligned}$$

which can be written as

$$\begin{aligned}&s^2\bar{\psi}_{Y(s)}(\zeta) - s\psi_{Y(0)}(\zeta) + \delta\Big(\sqrt{\gamma^2+2(1-\cos(\zeta))}-\gamma\Big)\psi_{Y(0)}(\zeta)\\ &-2\delta\gamma(s\bar{\psi}_{Y(s)}(\zeta)-\psi_{Y(0)}(\zeta)) = 2(1-\cos(\zeta))\bar{\psi}_{Y(s)}(\zeta)\end{aligned}$$

using equation (4.6.2) we have

$$s\Big(s\bar{\psi}_{Y(s)}(\zeta) - \psi_{Y(0)}(\zeta)\Big) + \frac{\partial}{\partial t}\psi_{Y(0)}(\zeta) - 2\delta\gamma\Big(s\bar{\psi}_{Y(s)}(\zeta) - \psi_{Y(0)}(\zeta)\Big)$$
$$= 2(1-\cos(\zeta))\bar{\psi}_{Y(s)}(\zeta).$$

Computing the Laplace inversion gives

$$\frac{\partial^2}{\partial t^2}\psi_{Y(t)}(\zeta) - 2\delta\gamma\frac{\partial}{\partial t}\psi_{Y(t)}(\zeta) = 2(1-\cos(\zeta))\psi_{Y(t)}(\zeta)$$

and since $e^{ik}y_k(\zeta)$ is the Fourier transform of $y_{k-1}(t)$ we have after inverting again

$$\frac{\partial^2}{\partial t^2}y_k(t) - 2\delta\gamma\frac{\partial}{\partial t}y_k(t) = y_{k-1}(t) + y_{k+1}(t) - 2y_k(t)$$

as desired. □

4.7 Option pricing

This section details some known results on option pricing under integer valued processes, see Barndorff-Nielsen et al. (2011) and Carr (2011).

Firstly some notation. The no arbitrage price of a standard put and call options at time $t \in [0,T]$ with strike K are defined by

$$V_{PUT}(K) = e^{-rT}\mathbb{E}[(K - F(T))^+] \tag{4.7.1}$$

and

$$V_{CALL}(K) = e^{-rT}\mathbb{E}[(F(T) - K)^+] \tag{4.7.2}$$

respectively. In the above the expectation is with respect to some risk neutral measure that makes the forward price process $\{F(t), t \in [0,T]\}$ a martingale, see section 4.5.

Consider the arithmetic model, then we can compute call option prices as follows.

$$\begin{aligned} V_{CALL}(K) &= e^{-rT}\mathbb{E}[(F(T)-K)^+] \\ &= e^{-rT}\mathbb{E}\Big[\Big(F(T)-F(0)-(K-F(0))\Big)^+\Big]. \end{aligned}$$

Let $K^* = K - F(0)$ be the distance between the strike and the forward value at time zero. Let $F^*(T) = F(T) - F(0)$ denote the shifted forward price with probability density $p(n,t) := \mathbb{P}(F^*(t) = n)$, then

$$\begin{aligned} V_{CALL}(K) &= e^{-rT}\mathbb{E}[(F^*(T)-K^*)^+] \\ &= e^{-rT}\sum_{j=1}^{\infty} jp(j+K^*,T). \end{aligned}$$

Notice that the call price only exists when $F(0)$ is an integer so the delta $\partial V_{CALL}/\partial F(0)$ and gamma $\partial^2 V_{CALL}/\partial F(0)^2$ do not exist.

For the symmetric Skellam case we have

$$V_{CALL}(K) = e^{-(r+2\lambda)T}\sum_{j=1}^{\infty} I_{|j+K-F(0)|}(2T\lambda) \tag{4.7.3}$$

where $I_k(x)$ is the modified Bessel function of the third kind.

Next consider the arithmetic model, let $Y(u) := F(u) - F(t)$ and choose the Skellam process to be either of standard or symmetric type, i.e. a martingale. Since the distribution of $Y(u)$ is symmetric, therefore we have

$$\begin{aligned} V_{CALL}(t, F(t)+c) &= e^{-r(T-t)}\mathbb{E}[F(T)-(F(t)+c))^+] \\ &= e^{-r(T-t)}\mathbb{E}[(Y(T)-c))^+] \\ &= e^{-r(T-t)}\mathbb{E}[(-Y(T)-c))^+] \\ &= e^{-r(T-t)}\mathbb{E}[(F(t)-c-F(T))^+] \\ &= V_{PUT}(t, F(t)-c) \end{aligned}$$

which is known as the the arithmetic put call symmetry, see Carr (2011).

Let us now we proceed as in Carr (2011) and discuss pricing and hedging of barrier options. A barrier option is an exotic derivative typically an option on the underlying asset whose price reaching the pre-set barrier level either springs the option into existence or extinguishes an already existing option. In our case the barrier option under consideration is a contract that promises to pay the holder the sum of one dollar if the price of the security hits or crosses the barrier level $L > 0$, which is some preset value that does not necessarily have to be an integer.

The payoff of a one touch barrier option at time T is $1(\min_{t\in[0,T]} F(t) \leq L)$ where $L \in (0, F(0))$ is the lower barrier. The barrier may not be at a level where the forward price can trade, due to the discreteness of assumed values of the forward price, when it does cross or touch L from above it must be equal to

$$L^a := F(0) + a \left\lfloor \frac{L - F(0)}{a} \right\rfloor, \tag{4.7.4}$$

where $\lfloor\cdot\rfloor$ denotes the integer part. As in Carr (2011), consider the payoff of holding[1] $\frac{1}{2a}$ vertical put spreads with strikes $L^a \pm a$

$$\begin{aligned}&\frac{1}{2a}[(L^a + a - F(T))^+ - (L^a - a - F(T))^+]\\&= 1(F(T) < L^a) + 0.5 \times 1(F(T) = L^a)\end{aligned}$$

or holding $\frac{1}{2a}$ vertical call spreads with strikes $L^a \pm a$

$$\begin{aligned}&\frac{1}{2a}[(F(T) - (L^a - a))^+ - (F(T) - (L^a + a))^+]\\&= 1(F(T) > L^a) + 0.5 \times 1(F(T) = L^a).\end{aligned}$$

The no arbitrage value $V(t, L)$ of the one touch barrier option (derived in

[1]We point out a mistake in Carr (2011) which writes $\frac{1}{a}$, this should in fact be $\frac{1}{2a}$ in order to get a vertical spread of width a.

Carr (2011)) is given by

$$V(0, L) = \frac{1}{a}[P(0, L^a + a) - P(0, L^a - a)]. \tag{4.7.5}$$

To see this consider the following hedge when writing a single one touch barrier option.

1. At time zero, buy two vertical put spreads at strikes $L^a \pm a$.
2. If $F > L$ for all t then both the one touch and vertical put spreads expire worthless.
3. If F touches or falls below L then sell the long $\frac{1}{2a}$ put struck at $L^a + a$ and buy $\frac{1}{2a}$ calls struck at $L^a - a$. Also sell $\frac{1}{2a}$ calls at strike $L^a + a$ and buy back the $\frac{1}{2a}$ puts with strike $L^a - a$.
4. Regardless of future movements in the forward price the combination of standard puts and calls that is left, $\frac{1}{2a}$ units of vertical put and call spreads, has exactly unit payoff which is enough to match the obligation of the sold one touch.

The hedge is called semi-static in the sense that after the initial trade is done, at most only one further re-balancing of the hedge is required.

4.8 Fractional Poisson process

This section introduces fractional Poisson processes which will be used through out the remainder of this thesis. We will rely upon such processes in the next section to generalize the Skellam process to its fractional counterpart.

The fractional Poisson process $\{N_\alpha(t), t \geq 0\}$, $\alpha \in (0, 1)$ can be obtained as a renewal process with Mittag-Leffler waiting times $\{T(n), n \geq 1\}$ between events (see Mainardi, Gorenflo, and Scalas (2004)):

$$N_\alpha(t) = \max\{n \geq 0 : T(1) + \ldots + T(n) \leq t\}, \tag{4.8.1}$$

where $T(j)$, $j \geq 1$ are independent identically distributed random variables with common Mittag-Leffler distribution function

$$F_\alpha(x) = \mathbb{P}(T(j) \leq x) = 1 - \mathcal{E}_\alpha(-\lambda x^\alpha), \quad x \geq 0, \ \alpha \in (0,1) \tag{4.8.2}$$

and $F_\alpha(x) = 0$ for $x < 0$. The probability density function of the Mittag-Leffler distribution is

$$f(x) = \frac{d}{dx} F_\alpha(x) = \lambda x^{\alpha-1} \mathcal{E}_{\alpha,\alpha}(-\lambda x^\alpha) \qquad x \geq 0,$$

where

$$\mathcal{E}_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + \beta)}, \qquad z \in \mathbb{C}, \alpha > 0, \beta > 0$$

is two parameter Mittag-Leffler function, see Haubold, Mathai, and Saxena (2011).

The three parameter Mittag-Leffler function is defined as (see for example Haubold, Mathai, and Saxena (2011))

$$\mathcal{E}^\gamma_{\alpha,\beta}(z) = \sum_{r=0}^{\infty} \frac{(\gamma)_r z^r}{r! \Gamma(\alpha r + \beta)},$$

$$\alpha, \beta, \gamma \in \mathbb{C}, \ Re(\alpha) > 0, \ Re(\beta) > 0, \ Re(\gamma) > 0,$$

where

$$(\gamma)_r = \frac{\Gamma(\gamma + r)}{\Gamma(\gamma)}$$

whenever the Gamma function Γ is defined, and $(\gamma)_0 = 1$ for $\gamma \neq 0$.

From (Mainardi et al., 2004, Equation (3.10)) (see also Beghin and Orsingher

(2009), the probability mass function of the fractional Poisson process is

$$q_k^\alpha(t) := \mathbb{P}(N_\alpha(t) = n) = \frac{(\lambda t^\alpha)^n}{n!}\sum_{r=0}^{\infty}\frac{(r+n)!}{r!}\frac{(-\lambda t^\alpha)^r}{\Gamma(\alpha(n+r)+1)}$$
$$= (\lambda t^\alpha)^n \mathcal{E}^{n+1}_{\alpha,\alpha n+1}(-\lambda t^\alpha) = \frac{(\lambda t^\alpha)^n}{n!}\mathcal{E}^{(n)}_\alpha(-\lambda t^\alpha) \tag{4.8.3}$$

where $\mathcal{E}^{(n)}_\alpha$ is the n^{th} derivative of the one-parameter Mittag-Leffler function.

It was shown in Beghin and Orsingher (2009) that the probability mass function of the fractional Poisson process satisfies the system of fractional differential equations

$$ {}_tD^\alpha_* p_0(t) = -\lambda p_0(t)$$
$$ {}_tD^\alpha_* p_n(t) = \lambda(p_{n-1}(t) - p_n(t))$$

with $p_{-1}(t) = p(-1,t) = 0$ and initial condition

$$p(n,0) = p_n(0) = \begin{cases} 1 & n = 0 \\ 0 & n \geq 1. \end{cases}$$

Here ${}_tD^\alpha_*$ is the Caputo fractional derivative defined as

$$ {}_tD^\alpha_* f(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{df(\tau)}{d\tau}\frac{1}{(t-\tau)^\alpha}d\tau. \tag{4.8.4}$$

It is also proven in Meerschaert, Nane, and Vellaisamy (2011) that the definition of a fractional Poisson process as a renewal process with Mittag-Leffler distribution of inter-arrival times is equivalent to the following time change definition:

$$N_\alpha(t) = N_1(E(t)), \tag{4.8.5}$$

where $N_1(t)$, $t \geq 0$ is a homogeneous Poisson process with parameter $\lambda > 0$ and $E(t)$, $t \geq 0$ is the inverse stable subordinator independent of $N_1(t)$.

From Beghin and Orsingher (2009), the mean and variance are given by

$$\mathbb{E}[N_\alpha(t)] = \frac{\lambda t^\alpha}{\Gamma(\alpha+1)},$$

$$\mathbb{V}\text{ar}[N_\alpha(t)] = \frac{t^\alpha \lambda}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}\lambda^2}{\alpha}\left(\frac{1}{\Gamma(2\alpha)} - \frac{1}{\alpha\Gamma(\alpha)^2}\right).$$

From Leonenko et al., the covariance function of the fractional Poisson process is

$$\mathbb{C}\text{ov}[N_\alpha(s), N_\alpha(t)] = \frac{\lambda(\min(t,s))^\alpha}{\Gamma(1+\alpha)} + \lambda^2\mathbb{C}\text{ov}[E(s), E(t)]$$

where $\mathbb{C}\text{ov}[E(s), E(t)]$ is given by equation (2.14.5) so that for $0 \le s \le t$

$$\begin{aligned}\mathbb{C}\text{ov}[N_\alpha(s), N_\alpha(t)] &= \frac{\lambda s^\alpha}{\Gamma(1+\alpha)} \\ &+ \lambda^2\Big(\frac{\alpha t^{2\alpha}}{\Gamma^2(1+\alpha)}B(\alpha+1,\alpha;s/t) \\ &+ \frac{\alpha s^{2\alpha}}{\Gamma^2(1+\alpha)}B(\alpha+1,\alpha) - \frac{ts^\alpha}{\Gamma^2(1+\alpha)}\Big),\end{aligned} \tag{4.8.6}$$

where B is the Beta function, and $B(\alpha,\beta;\cdot)$ is an incomplete Beta function.

4.9 Fractional Skellam type I processes

We now introduce a generalization of the Skellam process into a setting where the inter-arrival times are no longer exponential but instead are of Mittag-Leffler type. We will be using the inverse stable subordinator throughout the remainder of this thesis. Recall that $D(t), t \ge 0$ is a standard stable Lévy subordinator with Laplace exponent $\Psi_D(\zeta) = -\zeta^\alpha$, $\zeta > 0, t \ge 0$, $\alpha \in (0,1)$. The inverse stable subordinator $E(t)$ is defined as the inverse of the stable subordinator $D(t)$, that is

$$E(t) = \inf\{u \ge 0 : D(u) > t\}, \quad t \ge 0,$$

see section 2.14 for more details on the inverse of the stable subordinator. Let us now go on to generalize the Skellam Lévy process to its fractional counterpart.

Definition 15. *Let $N_1(t)$ and $N_2(t)$ be two independent homogeneous Poisson processes with intensities $\lambda_1 > 0$ and $\lambda_2 > 0$. Let $E_1(t)$ and $E_2(t)$ be two independent inverse stable subordinators with indices $\alpha_1 \in (0,1)$ and $\alpha_2 \in (0,1)$ respectively, which are also independent of the two Poisson processes. The stochastic process*

$$X(t) = N_1(E_1(t)) - N_2(E_2(t))$$

is called a fractional Skellam process of type I.

A fractional Skellam process of type I $X(t)$ has marginal laws of fractional Skellam type I denoted by $X(t) \sim fSk(k, t; \lambda_1, \alpha_1, \lambda_2, \alpha_2)$, which is a new four parameter distribution.

Theorem 7. *Let $X(t)$ be a fractional Skellam process of type I, the probability mass function is given by*

$$\mathbb{P}(X(t) = k) = \left(\lambda_1 t^{\alpha_1}\right)^k \sum_{n=0}^{\infty} \left(\lambda_1 \lambda_2 t^{\alpha_1+\alpha_2}\right)^n \times \mathcal{E}^{n+k+1}_{\alpha_1,\alpha_1(n+k)+1}\Big(-\lambda_1 t^{\alpha_1}\Big)\mathcal{E}^{n+1}_{\alpha_2,\alpha_2 n+1}\Big(-\lambda_2 t^{\alpha_2}\Big)$$

for $k \in \mathbb{Z}$, $k \geq 0$ and when $k < 0$

$$\mathbb{P}(X(t) = k) = \left(\lambda_2 t^{\alpha_2}\right)^{|k|} \sum_{n=0}^{\infty} \left(\lambda_1 \lambda_2 t^{\alpha_1+\alpha_2}\right)^n \times \mathcal{E}^{n+|k|+1}_{\alpha_2,\alpha_2(n+|k|)+1}\Big(-\lambda_2 t^{\alpha_2}\Big)\mathcal{E}^{n+1}_{\alpha_1,\alpha_1 n+1}\Big(-\lambda_1 t^{\alpha_1}\Big).$$

The moment generating function is

$$\mathbb{E}[e^{\zeta X(t)}] = \mathcal{E}_{\alpha_1}\left(\lambda_1 t^{\alpha_1}(e^{\zeta} - 1)\right)\mathcal{E}_{\alpha_2}\left(\lambda_2 t^{\alpha_2}(e^{-\zeta} - 1)\right), \ \zeta \in \mathbb{R}. \qquad (4.9.1)$$

Proof: Since $N_1(E_1(t))$ and $N_2(E_2(t))$ are independent,

$$\mathbb{P}(X(t)=k) = \sum_{n=0}^{\infty} \mathbb{P}(N_1(E_1(t))=n+k)\mathbb{P}(N_2(E_2(t))=n)\mathbb{I}_{k\geq 0} + \sum_{n=0}^{\infty} \mathbb{P}(N_1(E_1(t))=n)\mathbb{P}(N_2(E_2(t))=n+|k|)\mathbb{I}_{k<0}.$$

Now use the expression for the probability mass function of the fractional Poisson process given in equation (4.8.3) to complete the calculation. When $k>0$

$$\mathbb{P}(X(t)=k) = \Big(\lambda_1 t^{\alpha_1}\Big)^k \sum_{n=0}^{\infty} \Big(\lambda_1\lambda_2 t^{\alpha_1+\alpha_2}\Big)^n \times \mathcal{E}^{n+k+1}_{\alpha_1,\alpha_1(n+k)+1}\Big(-\lambda_1 t^{\alpha_1}\Big)\mathcal{E}^{n+1}_{\alpha_2,\alpha_2 n+1}\Big(-\lambda_2 t^{\alpha_2}\Big).$$

The case $k<0$ is treated similarly.

The moment generating function is computed using that of the fractional Poisson process. Denote by $h(\cdot,t)$ the density of $E(t)$, then

$$\mathbb{E}\left[e^{\zeta N_\alpha(t)}\right] = \int_0^\infty \mathbb{E}\left[e^{\zeta N(u)} h(u,t)du\right] = \int_0^\infty e^{\lambda u(e^\zeta-1)} h(u,t)du = \mathbb{E}\left[e^{\lambda(e^\zeta-1)E(t)}\right] = \mathcal{E}(\lambda(e^\zeta-1)t^\alpha),$$

using formula (2.14.3) for the Laplace transform of the inverse stable subordinator. Note that formula (2.14.3) remains true for all $\zeta\in\mathbb{R}$. This can be seen from the proof of Bondesson, Kristiansen, and Steutel (1996) Theorem 4.3 and (2.14.2):

$$\mathbb{E}\left[e^{\zeta E(t)}\right] = \mathbb{E}\left[\sum_{k=0}^{\infty}\frac{\zeta^k E^k(t)}{k!}\right] = \sum_{k=0}^{\infty}\frac{(\zeta t^\alpha)^k}{\Gamma(\alpha k+1)} = \mathcal{E}_\alpha(\zeta t^\alpha).$$

Therefore for the fractional Skellam process of type I

$$\mathbb{E}\left[e^{\zeta X(t)}\right] = \mathbb{E}\left[e^{\zeta N_1(E_1(t))}\right]\mathbb{E}\left[e^{-\zeta N_2(E_2(t))}\right] = \mathcal{E}_{\alpha_1}\left(\lambda_1 t^{\alpha_1}(e^\zeta-1)\right)\mathcal{E}_{\alpha_2}\left(\lambda_2 t^{\alpha_2}(e^{-\zeta}-1)\right).$$

□

Remark 5. *The moments of all orders can be obtained either from the moment generating function* (4.9.1) *or using the moments of the fractional Poisson processes. For example, the first moment of* $X(t) \sim fSk$ *is*

$$\mathbb{E}[X(t)] = \frac{\lambda_1 t^{\alpha_1}}{\Gamma(\alpha_1 + 1)} - \frac{\lambda_2 t^{\alpha_2}}{\Gamma(\alpha_2 + 1)}. \tag{4.9.2}$$

The variance is

$$\begin{aligned}\mathbb{V}ar[X(t)] = &\frac{t^{\alpha_1}\lambda_1}{\Gamma(1+\alpha_1)} + \frac{t^{2\alpha_1}\lambda_1^2}{\alpha_1}\left(\frac{1}{\Gamma(2\alpha_1)} - \frac{1}{\alpha_1\Gamma^2(\alpha_1)}\right)\\ &+ \frac{t^{\alpha_2}\lambda_2}{\Gamma(1+\alpha_2)} + \frac{t^{2\alpha_2}\lambda_2^2}{\alpha_2}\left(\frac{1}{\Gamma(2\alpha_2)} - \frac{1}{\alpha_2\Gamma^2(\alpha_2)}\right).\end{aligned} \tag{4.9.3}$$

A random variable X *is called* over dispersed *if* $\mathbb{V}ar[X] - \mathbb{E}[X] > 0$*. From inspection of equations (4.9.2) and (4.9.3) it is clear that the fractional Skellam law of type I has the property of over dispersion. Figure 4.2 displays the probability mass function for the fractional Skellam distribution with selected parameter values.*

The covariance function for the fractional Skellam process of type I can be computed by substituting the expression for the covariance function of the fractional Poisson process (4.8.6) into the equation below:

$$\begin{aligned}&\mathbb{C}\text{ov}[X(t), X(s)]\\ &\quad = \mathbb{C}\text{ov}[N_1(E_1(t)), N_1(E_1(s))] + \mathbb{C}\text{ov}[N_2(E_2(t)), N_2(E_2(s))].\end{aligned}$$

4.10 Fractional Skellam type II processes

This section introduces an alternative fractional Skellam process which we shall refer to as type II. An interesting property in this case is that we are able to deduce a system of fractional differential equations for which the

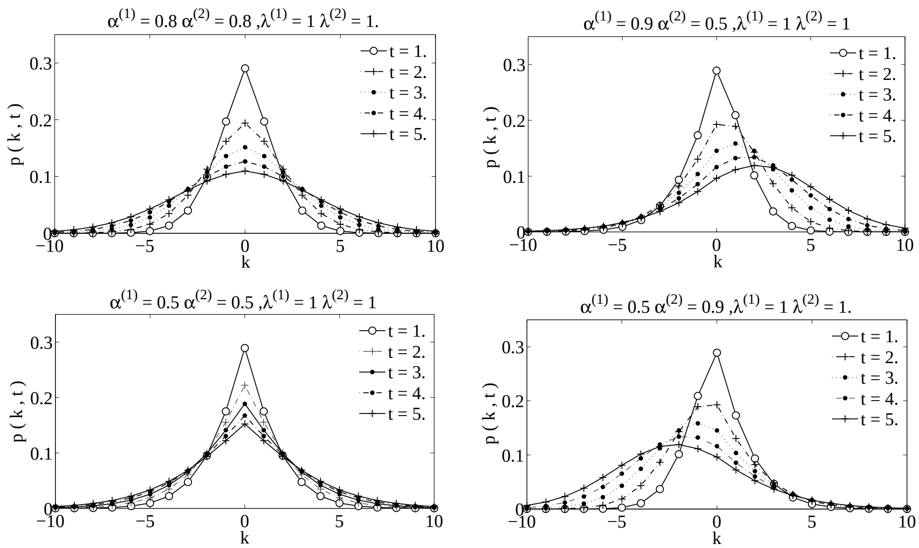

Figure 4.2: Probability mass function for the fractional Skellam distribution at times $t = 1, \cdots, 5$

marginal distributions satisfy. Let us now proceed and define this process.

Definition 16. *Let $S(t) = N_1(t) - N_2(t)$, $t \geq 0$ be a Skellam process. Let $E(t)$, $t \geq 0$ be an inverse stable subordinator of exponent $\alpha \in (0,1)$ independent of $N_1(t)$ and $N_2(t)$. The stochastic process*

$$Y(t) = S(E(t))$$

is called a fractional Skellam process of type II.

Fractional Skellam process of type II $Y(t)$ has marginal laws of fractional Skellam type II, for which we shall write

$$Y(t) \sim fSk(k, t; \lambda_1, \lambda_2, \alpha)$$

Theorem 8. *Let $Y(t) = S(E(t))$ be fractional Skellam process of type II,* *and let* $r_k(t) = P(Y(t) = k)$, $k \in \mathbb{Z}$. *The marginal distribution is given by*

$$r_k(t) = \frac{1}{t^\alpha}\left(\frac{\lambda_1}{\lambda_2}\right)^{k/2} \int_0^\infty e^{-u(\lambda_1+\lambda_2)} I_{|k|}\left(2u\sqrt{\lambda_1\lambda_2}\right)\Phi_\alpha\left(\frac{u}{t^\alpha}\right) du, \quad (4.10.1)$$

where

$$\Phi_\alpha(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(1-n\alpha-\alpha)}, \quad 0 < \alpha < 1$$

is the Wright function. The marginal distribution satisfies the following system of fractional differential equations:

$$D_t^\alpha r_k(t) = \lambda_1(r_{k-1}(t) - r_k(t)) - \lambda_2(r_k(t) - r_{k+1}(t)) \quad (4.10.2)$$

with the inital conditions $r_0(0) = 1$ *and* $r_k(0) = 0$ *for* $k \neq 0$.

The moment generating function $L(\zeta, t) = \mathbb{E}[e^{\zeta X(t)}]$ *is*

$$L(\zeta, t) = \mathcal{E}_\alpha(-(\lambda_1 + \lambda_2 - \lambda_1 e^\zeta - \lambda_2 e^{-\zeta})t^\alpha), \quad (4.10.3)$$

and for every $\zeta \in \mathbb{R}$ *it satisfies the fractional differential equation*

$$D_t^\alpha L(\zeta, t) = (\lambda_1(e^\zeta - 1) + \lambda_2(e^{-\zeta} - 1))L(\zeta, t) \quad (4.10.4)$$

with the initial condition $L(\zeta, 0) = 1$.

Proof: With $s_k(t) = \mathbb{P}(S(t) = k)$ as before in (4.4.3), use conditioning argument to write

$$r_k(t) = \int_0^\infty s_k(u)h(u,t)du \quad (4.10.5)$$

where $h(\cdot, t)$ is the density of $E(t)$. Using the expression for the probability mass function of the Skellam process (4.4.3) and the fact that

$$h(u,t) = \frac{1}{t^\alpha}\Phi_\alpha\left(\frac{u}{t^\beta}\right),$$

see Meerschaert, Schilling, and Sikorskii (2014) equation (3.7), equation (4.10.1) follows.

To derive the governing fractional differential equation, note that from (Meerschaert and Scheffler, 2008, Theorem 4.1), for $t > 0$, $u > 0$, $h(u,t)$ satisfies

$$\mathbb{D}_t^\alpha h(u,t) = -\frac{\partial}{\partial u} h(u,t),$$

where the Riemann-Leuville fractional derivative for $0 < \alpha < 1$ is

$$\mathbb{D}_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t f(t-s) s^{-\alpha} ds.$$

Then integration by parts yields

$$\begin{aligned} \mathbb{D}_t^\alpha r_k(t) &= \int_0^\infty s_k(u) \mathbb{D}_t^\alpha h(u,t) du = -\int_0^\infty s_k(u) \frac{\partial}{\partial u} h(u,t) du \\ &= \int_0^\infty h(u,t) \frac{\partial}{\partial u} s_k(u) du - s_k(0) h(0+,t), \end{aligned}$$

and $h(0+,t) = t^{-\alpha}/\Gamma(1-\alpha)$, see (Hahn et al., 2011, Lemma 2.1). Since $s_k(0) = 0$ for $k \neq 0$, the boundary term disappears except when $k = 0$. Also, from (4.10.5), $r_k(0) = s_k(0) = 1$. Since for $0 < \alpha < 1$ the Caputo and Riemann-Leuville derivatives are related by

$$D_t^\alpha r_k(t) = \mathbb{D}_t^\alpha r_k(t) - r_k(0) \frac{t^{-\alpha}}{\Gamma(1-\alpha)},$$

for both cases, $k = 0$ and $k \neq 0$, we have

$$D_t^\alpha r_k(t) = \int_0^\infty h(u,t) \frac{\partial}{\partial u} s_k(u) du.$$

Now apply(4.4.6) to get

$$D_t^\alpha r_k(t) = \int_0^\infty h(u,t) \left(\lambda_1 (s_{k-1}(u) - s_k(u)) - \lambda_2 (s_k(u) - s_{k+1}(u)) \right) du$$

and arrive at (4.10.2) using (4.10.5).

Through the use of conditioning and equation (2.14.3), the moment generating function

$$\mathbb{E}[e^{\zeta X(t)}] = \mathbb{E}[e^{\zeta S(E(t))}] = \int_0^\infty \mathbb{E}[e^{\zeta S(u)}] h(u,t) du \tag{4.10.6}$$

$$= \int_0^\infty e^{-u(\lambda_1+\lambda_2-\lambda_1 e^{\zeta}-\lambda_2 e^{-\zeta})} h(u,t)du$$
$$= \mathcal{E}_\alpha(-(\lambda_1+\lambda_2-\lambda_1 e^{\zeta}-\lambda_2 e^{-\zeta})t^\alpha).$$

Since the one-parameter Mittag-Leffler function is the eigenfunction for the Caputo derivative Mainardi and Gorenflo (2000) or Meerschaert et al. (2009), $D_t^\alpha \mathcal{E}_\alpha(-\lambda t^\alpha) = -\lambda \mathcal{E}_\alpha(-\lambda t^\alpha)$, and equation (4.10.4) follows.

Note that equation (4.10.4) can also be obtained by multiplying both sides of equation (4.10.2) by $e^{-\zeta k}$ and summing over $k \in \mathbb{Z}$ to get

$$D_t^\alpha L(\zeta,t) = (\lambda_1(e^{\zeta}-1)+\lambda_2(e^{-\zeta}-1))L(\zeta,t),$$

which has the solution (4.10.3). □

Remark 6. *The mean, variance and covariance functions for the fractional Skellam process of type II are obtained from Leonenko, Meerschaert, Sikorskii, and Schilling Theorem 2.1, moments of the Skellam process* (4.4.4) *and the time-change process:*

$$\mathbb{E}[Y(t)] = \frac{t^\alpha(\lambda_1-\lambda_2)}{\Gamma(1+\alpha)},$$

$$\mathbb{V}\text{ar}[Y(t)] = \frac{t^\alpha(\lambda_1+\lambda_2)}{\Gamma(1+\alpha)} + (\lambda_1-\lambda_2)^2 t^{2\alpha}\left[\frac{2}{\Gamma(2\alpha+1)} - \frac{1}{\Gamma^2(1+\alpha)}\right],$$

and for $0 \le s \le t$

$$\mathbb{C}\text{ov}[Y(t),Y(s)] = \frac{s^\alpha(\lambda_1+\lambda_2)}{\Gamma(1+\alpha)} + (\lambda_1-\lambda_2)^2\,\mathbb{C}\text{ov}[E(t),E(s)],$$

where the covariance function for the inverse stable subordinator is given by (4.8.6) *and Leonenko et al. equation (9). Fractional Skellam law of type II also has the property of over dispersion, as does fractional Skellam law of type I.*

4.11 Martingale properties of fractional Skellam processes

In analogue to the Skellam process let us now introduce the symmetric, standard and compensated versions.

Definition 17. *Let $\{Y(t), t \geq 0\}$ be a fractional Skellam process of type II, when $\lambda_1 = \lambda_2 := \lambda > 0$ then $\{Y(t), t \geq 0\}$ is referred to as a symmetric fractional Skellam process of type II and we write $Y(t) \sim Sk(t\lambda, t\lambda, \alpha)$.*

Definition 18. *Let $\{Y(t), t \geq 0\}$ be a fractional Skellam process of type II, set $\lambda_1 = \lambda_2 = \frac{1}{2}$ then $\{S(t), t \geq 0\}$ is the one parameter standard fractional Skellam process of type II and we write $Y(t) \sim Sk(t\frac{1}{2}, t\frac{1}{2}, \alpha)$.*

We can also define the compensated process as follows.

Definition 19. *A compensated fractional Skellam process of type II $\{Y^*(t), t \geq 0\}$ is defined by*

$$Y^*(t) := S(E(t)) - (\lambda_1 + \lambda_2)E(t)$$

where $\{S(t), t \geq 0\}$ is a Skellam process and $\{E(t), t \geq 0\}$ an inverse stable subordinator of exponent $\alpha \in (0, 1)$ independent of $\{S(t), t \geq 0\}$. We write in notation

$$Y^*(t) \sim cfSk(t\lambda_1, t\lambda_2, \alpha).$$

Theorem 9. *The compensated fractional Skellam process type II, namely $\{Y^*(t), t \geq 0\}$, is a $\mathcal{G}_t := \mathcal{F}_{E(t)}$ martingale.*

Proof: Define a new process $T(n)$, $n \geq 0$ as

$$T(n) = \inf\{u : |S(u) - u(\lambda_1 - \lambda_2)| = n\}$$

Then for each n we have that T is a stopping time, i.e. $\{T_n(\omega) \leq t\} \in \mathcal{F}_t$. Since the filtration is right continuous and

$$|S(T(n) \wedge t) - (\lambda_1 - \lambda_2)(T(n) \wedge t)| \leq n$$

the process $S(T(n) \wedge t) - (\lambda_1 - \lambda_2)(T(n) \wedge t)$ is a right continuous closed martingale. Therefore by Doob's optional sampling theorem

$$\begin{aligned}&\mathbb{E}\left[S(T(n) \wedge E(t)) - (\lambda_1 - \lambda_2)(T(n) \wedge E(t))\Big|\mathcal{F}_{E(s)}\right] \\ &\quad = S(T(n) \wedge E(s)) - (\lambda_1 - \lambda_2)(T(n) \wedge E(s)).\end{aligned}$$

First note that since $T(n) \to \infty$ as $n \to \infty$ and E is a finite time change

$$\begin{aligned}&S(T(n) \wedge E(t)) - (\lambda_1 - \lambda_2)(T(n) \wedge E(t)) \\ &\quad \to S(E(t)) - (\lambda_1 - \lambda_2)(E(t)), \qquad n \to \infty.\end{aligned}$$

Now the submartingale $\Big|S(T(n) \wedge E(t)) - (\lambda_1 - \lambda_2)(T(n) \wedge E(t))\Big|$ is dominated by the element $\sup_{0 \leq u \leq t} \left\{\Big|S(E(t)) - (\lambda_1 - \lambda_2)E(t)\Big|\right\}$ and

$$\mathbb{E}\left[\sup_{0 \leq u \leq t} \left\{\Big|S(E(t)) - (\lambda_1 - \lambda_2)E(t)\Big|\right\}\right] < \infty$$

by the maximal inequality. Then conditioning on the sigma algebra generated by $E(t)$, i.e. $\sigma(E(t))$, the dominated convergence Theorem tells us that

$$\begin{aligned}&\mathbb{E}\left[S(T(n) \wedge E(t)) - (\lambda_1 - \lambda_2)(T(n) \wedge E(t))\Big|\mathcal{F}_{E(s)}\right] \\ &\quad \to \mathbb{E}\left[S(E(t)) - (\lambda_1 - \lambda_2)(E(t))\Big|\mathcal{F}_{E(s)},\right]\end{aligned}$$

as $n \to \infty$. Then we have

$$\mathbb{E}\left[S(E(t)) - (\lambda_1 - \lambda_2)(E(t))\Big|\mathcal{F}_{E(s)}\right] = N(E(s)) - (\lambda_1 - \lambda_2)(E(s))$$

and the stochastic process $S(E(t)) - (\lambda_1 - \lambda_2)E(t)$ is a $\mathcal{G}_t$ martingale. $\square$

Using similar arguments the symmetric fractional Skellam process type II and the standard fractional Skellam process type II can also be shown to both be $\mathcal{G}_t$ martingales.

4.12 An empirical investigation of waiting times

We consider transaction records for the September 2011 Eurofx over a three month horizon from the 22nd June until expiration on the 22nd September 2011. The Eurofx is a type of forward asset known as a future, and the data set was obtained directly from the Chicago mercantile exchange. The market is open from 12pm Sunday evening until Friday at 5pm with a one hour close each day between 4pm and 5pm.

The observed price of the future at time t is denoted by $F(t)$, $t = 1, 2, ..., N$. For this period there are $N = 5,465,779$ timestamped transactions recorded over market opening hours. Of these records, 71% of transactions get completed at the previous trade price. No tick change from one trade to the next, and single tick price changes account for 98% of all transactions.

Close symmetry between negative and positive tick jumps of the same magnitude is seen. The count for jumps of three ticks up or down is 1,411 and 1,419 respectively a difference of only eight counts, with a similarly finding for jumps of a four ticks. The frequency of both positive and negative jumps in general decreases as the jump size increases but does not hold true for an absolute jump size of eight ticks, which has a higher frequency than both six and seven tick jumps.

The data contains the transacted price along with timestamps binned to the nearest second, when multiple trades occur during the same second interval, the trades are recorded in the order they are filled but with identical time stamps. Since we are interested in the inter arrival time between trades this rounding off in timestamps will cause a data loss.

A second issue is market micro structure noise in the form of the bid-ask bounce. The futures contract is very liquid and it is not uncommon to see strings of transactions occurring in rapid succession bouncing from the bid to the ask, a difference of a single tick. We note though, that our data set does not implicitly state the bid and ask prices we have only interpreted the price bounce to be such a spread. The bid price and the ask price have not changed but the transaction record details a series of positive and negative returns of a single tick.

We filter the series by only recording the transactions if they go outside the bid ask spread. The spread is fixed to a single tick of 0.0001 by setting $F(0)^{\text{bid}} = F(0)$ and $F(t)^{\text{ask}} = F(t)^{\text{bid}} + 0.0001$ and computing $F(t)^{\text{bid}}$ as

$$F(t)^{\text{bid}} = \begin{cases} F(t-1)^{\text{bid}} & \text{if } F(t-1)^{\text{bid}} \leq F(t) \leq F(t-1)^{\text{ask}} \\ F(t) & \text{if } F(t) < F(t-1)^{\text{bid}} \\ F(t) - 0.0001 & \text{if } F(t) > F(t-1)^{\text{ask}} \end{cases}$$

The resulting filtered transaction chain still contains $5,465,779$ records but we now deleted all entries where the bid price has not changed from the previous bid price, that is no up or down jump has occurred, leaving $682,550$ records.

Next we consider the up and down jump processes in two models for the spot prices. First is the model from Barndorff-Nielsen et al. (2011), where

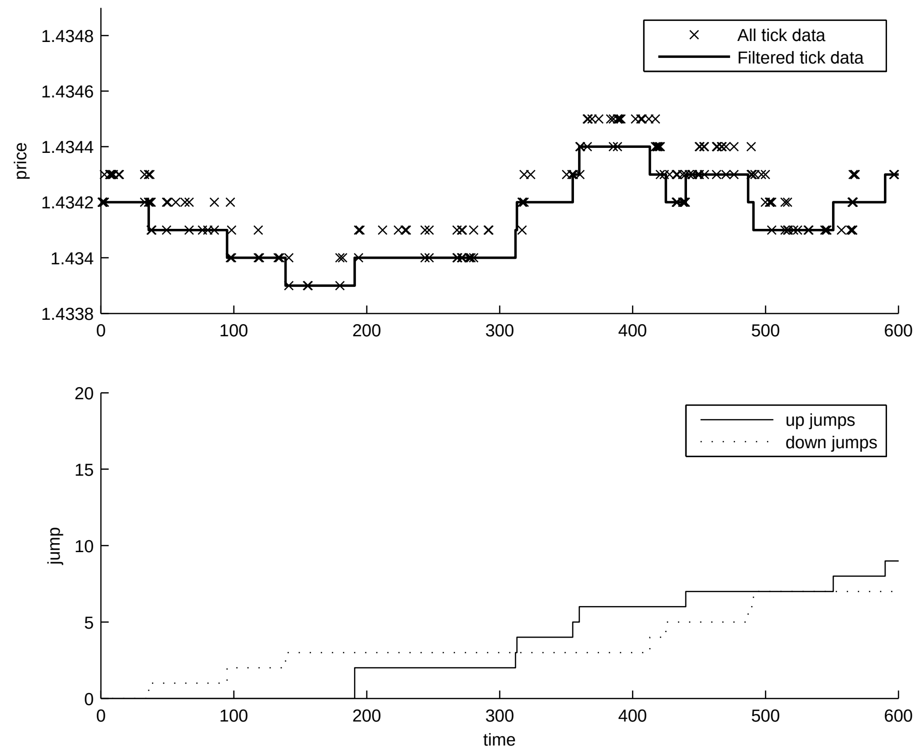

Figure 4.3: Price path plot of original data and filtered data

the price is modeled by the Skellam process. The second model is proposed by us and it uses fractional Skellam process of type I to model the price movements. In the empirical analysis of these models, we separate up and down jumps seen in Figure 4.3. In the case of Skellam processes, which is the difference of two independent Poisson processes, the absence of simultaneous jumps for the two processes follows from a general result: two independent Lévy processes have no common points of discontinuity almost surely (Meerschaert and Sikorskii (2012) page 106). As follows from the Lemma below, absence of simultaneous jumps also holds for two components in fractional Skellam process of type I.

Lemma 7. [2] *Let $X(t) = N_1(E_1(t)) - N_2(E_2(t))$ be the fractional Skellam process of type I. The processes $N_1(E_1(t))$ and $N_2(E_2(t))$ have no common points of discontinuity almost surely.*

Proof: We use the definition of Mainardi, Gorenflo, and Scalas (2004) of the fractional Poisson process as a renewal process. Since the sample paths of $E_1(t)$ and $E_2(t)$ are continuous almost surely, the discontinuities of the fractional Poisson process come from jumps of the outer Poisson process. Therefore

$$\mathbb{P}\Big[N_1(E_1(t+)) > N_1(E_1(t)) \text{ and } N_2(E_2(t+)) > N_2(E_2(t))$$
$$\text{for some t} > 0\Big] = \mathbb{P}\Big[\sum_{i=1}^{n} T_1(i) = \sum_{j=1}^{m} T_2(j) \text{ for some m, n } \in \mathbb{N}\Big]$$
$$\leq \sum_{m,n\in\mathbb{N}} \mathbb{P}\Big[\sum_{i=1}^{n} T_1(i) = \sum_{j=1}^{m} T_2(j)\Big],$$

where the independent random variables $T_1(i)$ and $T_2(j)$ are waiting times between events from (4.8.1). Since these random variables follow Mittag-Leffler distribution, the distribution of their sum has a density, and the probabilities of the events summed above all have probability zero. □

We now proceed with the data analyses by separating the up and down jump processes.

Up jump process: To construct the up jump time series we remove all trades with negative jumps leaving $317,212$ observations, all duplicate time stamps are removed leaving only the last recorded entry for each second. A time series of $253,092$ entries remain representing the positive jump process.

[2]The proof of this theorem was provided in by A. Sikorskii, in Michigan State University and is not the work of the author of this thesis.

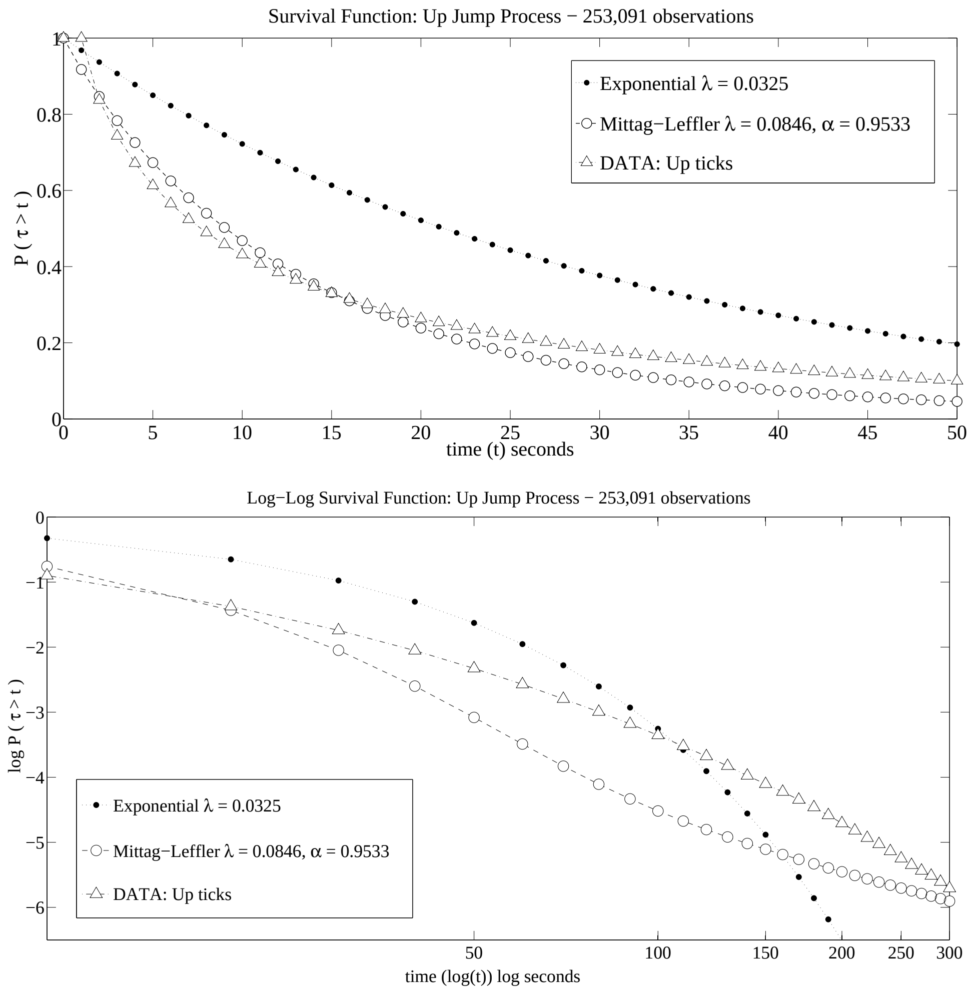

Figure 4.4: Survival function for the up jump process

Figure 4.4 clearly shows that the exponential distribution provides a poor fit to the data which can be quantified with the 95% confidence interval $(0.9512, 0.9554)$ for α_1 and so $\alpha_1 \neq 1$. The Mittag-Leffler provides a closer fit to the data and supports our generalization to a fractional process in this setting.

Down jump process: As with the up jump process to build the down jump series we remove the trades with negative jumps leaving $365,338$ observations, all duplicate time stamps are removed leaving only the last recorded entry for each second. A time series of $281,833$ observations is left representing the down jump process.

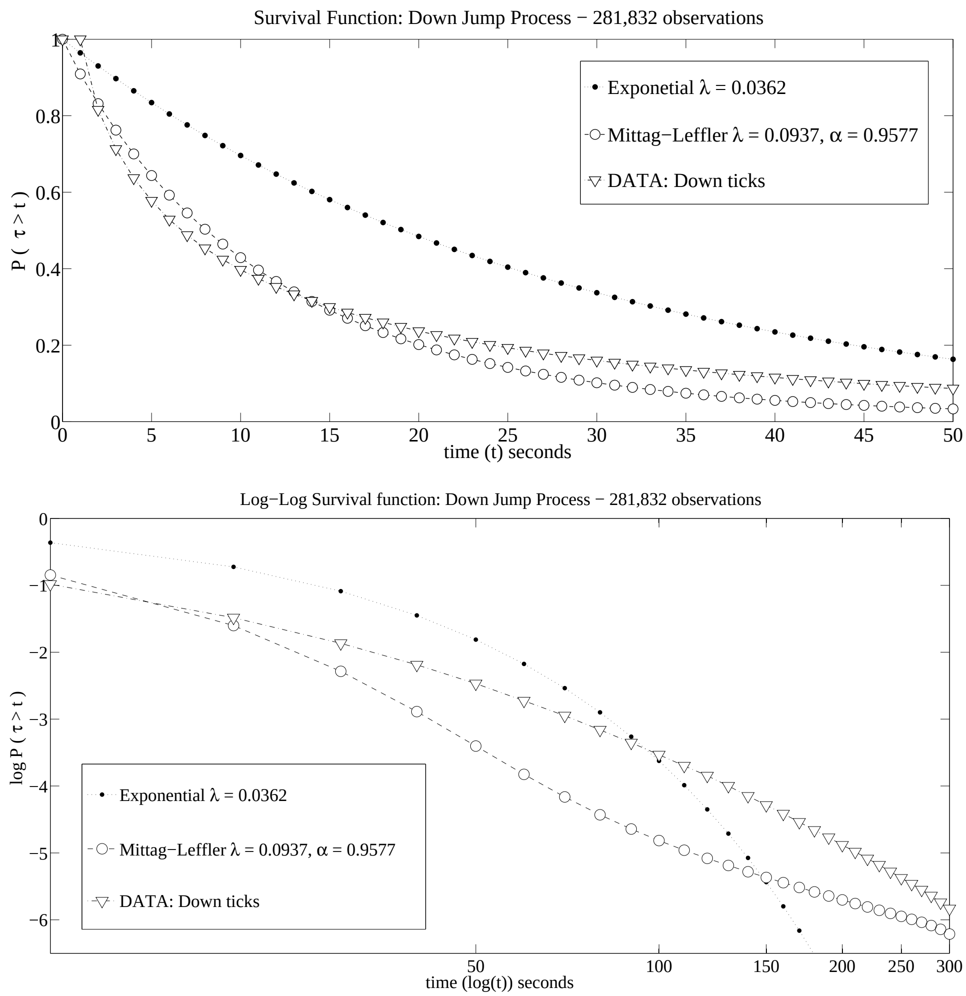

Figure 4.5: Survival function for the down jump process

Similar to the up jump process, Figure 4.5 shows the exponential distribution does not provide a realistic match to the empirically observed survival probabilities. This can be quantified through the 95% confidence interval for α_2 computed as $(0.9557, 0.9597)$, concluding that $\alpha_2 \neq 1$ as would be the case if inter-arrival times where exponential in law. It can also be seen that the Mittag-Leffler although producing a closer fit to the down jump waiting times is not perfect, as the data depict heavier tails than the Mittag-Leffer law can support, although there is a considerable improvement over the exponential.

In summary, we have shown that the inter-arrival times between the jumps in both the positive and negative jump processes are clearly not exponential. The Mittag-Leffler law provides a closer fit to the data, however the fit is not perfect and even with the added flexibility of an additional parameter, the Mittag-Leffler does not seem to provide tails that are as heavy as the market suggests. This is true for our data set and more empirical work would be needed to see if this is a common feature amongst different asset classes. Further, although the magnitude of ninety eight percent of jumps is a single tick, there is the case to extend the models to allow for jumps greater than one tick. It would then seem sensible to model the random component not as the difference between two fractional Poisson processes but instead as the difference of two compound fractional Poisson processes.

Appendix: Statistical analysis of the Mittag-Leffler distribution

The methods for parameter estimation are from Cahoy et al. (2013). Let T be a random variable with Mittag-Leffler distribution and $T_1, \ldots, T_n$ iid sample, then the moment estimators for the parameters,

$$\hat{\alpha} = \frac{2\pi}{\sqrt{2(6\mathbb{V}\mathrm{ar}[\widehat{\log(T)}] + \pi^2)}}, \quad \text{and} \quad \hat{\lambda} = \exp\{-\hat{\alpha}(\mathbb{E}[\widehat{\log(T)}] + \gamma)\}$$

where γ is Euler's constant and

$$\mathbb{E}[\widehat{\log(T)}] := \frac{1}{n}\sum_{i=1}^{n} \log T_i, \qquad \mathbb{V}\mathrm{ar}[\widehat{\log(T)}] := \frac{1}{n}\sum_{i=1}^{n}(\log T_i - \mathbb{E}[\widehat{\log(T)}])^2$$

For the above estimators to be of use we must have data where $\mathbb{V}\mathrm{ar}[\widehat{\log(T)}] > \pi^2/6 = 1.6449$ so that the standard deviation of $\widehat{\log(T)}$ is greater than 1.2825. The estimator for α is asymptotically normal as $n \to \infty$:

$$\sqrt{n}(\hat{\alpha} - \alpha) \longrightarrow N\left[0, \frac{\alpha^2(32 - 20\alpha_2 - \alpha^4)}{40}\right],$$

and we obtained an asymptotic $(1 - \epsilon)100\%$ confidence interval for α.

4.13 Continuous time random walks

In this section we give details on some definitions and known results for continuous time random walks (CTRW), see Meerschaert and Sikorskii (2012) for a complete discussion.

Firstly we define an integer valued random walk

$$W(n) = J(1) + \cdots + J(n)$$

where the integer jumps $J(n)$ are independent and identically distributed with the random variable J which takes integer values. Consider another random walk $T(n)$ of independent and identically distributed waiting times $\tau(i)$,

$$T(n) = \tau(1) + \cdots + \tau(n)$$

where $\tau(n)$ is independent and identically distributed with τ. Let

$$N(t) = \max\{n \geq 0, T(n) \geq t\}$$

denote the number of jumps by time $t \geq 0$, where $T(0) = 0$. Then a continuous time random walk is defined by

$$W(N(t)) = J(1) + \cdots + J(N(t)).$$

Next let us detail the limit process for a CTRW, see sections 4.3 and 4.4 of Meerschaert and Sikorskii (2012) and reference therein for a complete discussion. Suppose that Y is a random variable that is not degenerate, we want to know when

$$a_n(J(1) + \cdots + J(n)) - b_n \Rightarrow Y \tag{4.13.1}$$

for some $a_n > 0$ and $b_n \in \mathbb{R}$. We say that J belongs to the *domain of attraction* of Y, and we write $J \in \mathrm{DOA}(Y)$, if (4.13.1) holds. From Theorem 4.5 in Meerschaert and Sikorskii (2012) if $J \in \mathrm{DOA}(Y)$ the distribution of Y is either normal if and only if

$$\mathbb{E}[J^2 \mathbb{I}_{J \leq x}]$$

is slowly varying or is stable if and only if $\mathbb{P}(|J| > x)$ is regularly varying with index $-\alpha$ and

$$\lim_{x \to \infty} \frac{\mathbb{P}(J > x)}{\mathbb{P}(|J| > x)} = p, \qquad \text{for some} \quad 0 \leq p \leq 1.$$

The convergence (4.13.1) extends to random walks (see remark 4.17 in Meerschaert and Sikorskii (2012)) and we have

$$a_n(J(1) + \cdots + J([nt])) - \frac{[nt]}{n} b_n \Rightarrow Z(t).$$

The limit is a Lévy process $\{Z(t), t \geq 0\}$.

Remark 7. *If* $\mathbb{E}[J] = 0$ *then*

$$a_n W([nt]) \Rightarrow B(t)$$

where $B(t)$ *is a Brownian motion. Furthermore if* $\mathbb{E}[\tau]$ *exists, then by the* *renewal theorem* $N(t)/t \to \lambda = 1/\mathbb{E}[\tau]$*, then*

$$a_n W(N(nt)) \Rightarrow B(\lambda t)$$

The effect of the waiting times is just a change of scale.

Remark 8. *If* $\mathbb{E}[\tau] = \infty$*, the CTRW behaves quite differently from the previous remark. Let* A *be either normal or stable and* $E(t)$ *the inverse stable subordinator then from Meerschaert and Sikorskii (2012) pages 100-102 we have convergence in distribution*

$$(c^{-1/\alpha} W([nt]), c^{-\beta} N(ct)) \Rightarrow (A(t), E(t)) \tag{4.13.2}$$

Using the continuous mapping Theorem it can be seen that (4.13.2) also holds in the sense of finite dimensional distributions. Furthermore in the space $\mathbb{D}[0, \infty)$ *of cádlág functions with the Skorokhod* M_1 *topology we have convergence in stochastic process*

$$\{c^{-\beta/\alpha} W(N([nt])), t \geq 0\} \Rightarrow \{A(E(t)), t \geq 0\} \tag{4.13.3}$$

This is a hard result to prove, for complete details see Meerschaert and Sikorskii (2012) and references therein.

4.14 Fractional Skellam type II CTRW representation

In this section we show that the standard fractional Skellam of type II has a continuous time random walk representation which appropriately normed converges to the activity time model of section 2.15, thus providing a link between the models explored in this thesis.

Firstly the standard Skellam process $\{S(t), t \geq 0\}$, where $S(t) \sim Sk(\frac{1}{2}t, \frac{1}{2}t)$ has a continuous time random walk representation. To see this define an integer valued random walk

$$W(n) = J(1) + \cdots + J(n)$$

with integer jumps $J(n)$ which are independent and identically distributed with random variable $J : \Omega \to \mathbb{Z}$, whose probability mass function is given by

$$\mathbb{P}(J = k) = \frac{1}{2}\delta_{\{1\}} + \frac{1}{2}\delta_{\{-1\}}.$$

where δ is the Dirac delta function. Consider another random walk $T(n)$ of independent and identically distributed waiting times $\tau(i)$,

$$T(n) = \tau(1) + \cdots + \tau(n)$$

where $\tau(n)$ is independent and identically distributed with $\tau \sim \exp(1)$ an exponential random variable. Let

$$N(t) = \max\{n \geq 0 : T(n) \geq t\}$$

denote the number of jumps by time $t \geq 0$, where $T(0) = 0$. Then the continuous time random walk

$$W(N(t)) = J(1) + \cdots + J(N(t))$$

has equality in distribution to the standard Skellam process, i.e.

$$W(N(t)) \stackrel{d}{=} S(t).$$

The expectation of the jump J is

$$\mathbb{E}[J] := \sum_{k=-\infty}^{k=\infty} k\mathbb{P}(J = k) = 0$$

and the variance is

$$\mathbb{V}\text{ar}[J] := \sum_{k=-\infty}^{k=\infty} k^2\mathbb{P}(J = k) = 1.$$

Now instead of using the exponential distribution for the waiting times $\tau(i)$, we now use the Mittag-Leffler distribution (see equation 4.8.2), namely

$$\tau(i) \sim ML(\alpha, \lambda).$$

Then $\mathbb{E}[\tau] = \infty$ and the CTRW behaves as in Remark 8 and then by the results in Meerschaert and Sikorskii (2012) pages 100-106 we have the stochastic process convergence

$$c^{-\beta/\alpha}(\sigma W(N([nt])) - N([ct])(\theta + \tfrac{1}{2}\sigma^2)) + c^{-1}N([ct])(\theta + \tfrac{1}{2}\sigma^2)$$
$$\Rightarrow (\theta + \tfrac{1}{2}\sigma^2)E(t) + \sigma B(E(t)). \tag{4.14.1}$$

In other words the standard fractional Skellam of type II appropriately normed converges to the activity time model of section 2.15.

4.15 Fractional Skellam tempered stable process

We now go on to generalize a step further in analogue to section 4.6 where we extended the Skellam process to exhibit jumps greater than one. Here in this section we will be extending the fractional Skellam process of type II, to allow for greater than unit jumps.

Definition 20. *The fractional Skellam tempered stable process* $\{X(t), t \geq 0\}$ *is defined by*

$$X(t) := S\left(T\left(E(t)\right)\right).$$

Where $S(t) \sim Sk(\frac{1}{2}t, \frac{1}{2}t)$ *is a standard Skellam process,* $T(t) \sim TS(\kappa, t\delta, \gamma)$ *a tempered stable Lévy subordinator and* $E(t) \sim IS(\alpha)$ *an inverse stable subordinator. In notation we write*

$$X(t) \sim fSkTS(\alpha, \kappa, t\delta, \gamma).$$

Theorem 10. *Let* $X(t) \sim fSkTS$ *then the characteristic function is given* *by*

$$\psi_{X(t)}(\zeta) = \mathcal{E}_\alpha \left(-\left((\gamma^{1/\kappa} + 2(1 - cos(\zeta)))^\kappa - \delta\gamma\right) t^\alpha\right). \tag{4.15.1}$$

The probability mass function $f(x,t)$ satisfies the fractional differential equation

$$\begin{aligned} D_t^\alpha D_t^\alpha f(x,t) - 2\delta\gamma D_t^\alpha f(x,t) = 2^{2\kappa}\delta_2 \Bigg(\sum_{j=0}^{\infty} \binom{2\kappa}{j} (\tfrac{1}{2}\gamma^{1/\kappa} + 1)^{2\kappa - j} 2^j \\ \times \sum_{r=0}^{j} \binom{j}{r} f(x - j + 2r, t) \Bigg) - \delta_2\gamma_2 f(x,t) \end{aligned}$$

where D_t^α is the Caputo fractional derivative given in equation (4.8.4).

Proof: The characteristic function can be computed as follows

$$\begin{aligned} \psi_{X(t)}(\zeta) &= \int e^{i\zeta x} f(x,t) dx \\ &= \int e^{i\zeta x} \Big(\int_0^\infty \int_0^\infty p(x,z) r(z,u) h(u,t) du dz \Big) dx \end{aligned}$$

where $p(x,z) := \mathbb{P}(S(z) = x)$ is the pmf of the Skellam process, $r(z,u) := \mathbb{P}(T(u) \leq z)$ the pdf of the tempered stable process and $h(u,t) := \mathbb{P}(E(t) \leq u)$ is the probability density function of the inverse stable subordinator. Then

$$\begin{aligned} \psi_{X(t)}(\zeta) &= \int_0^\infty \int_0^\infty e^{-z\phi_S(\zeta)} r(z,u) h(u,t) du dz \\ &= \int_0^\infty e^{-u\phi_T(\phi_S(\zeta))} h(u,t) du \\ &= \mathcal{E}_\alpha \left(-\phi_T(\phi_S(\zeta)) t^\alpha\right) \end{aligned}$$

and (4.15.1) follows. For the second part since clearly

$$D_t^\alpha \psi_{X(t)}(\zeta) = -\phi_T(\phi_S(\zeta)) \psi_{X(t)}(\zeta)$$

let $\bar{\psi}_{X(t)}(\zeta) := \int e^{-st}\psi_{X(t)}(\zeta)dt$ then taking Laplace transforms of both sides yields

$$s^{\alpha}\bar{\psi}_{X(t)}(\zeta) - s^{\alpha-1}\bar{\psi}_{X(0)}(\zeta) = -\phi_T(\phi_S(\zeta))\bar{\psi}_{X(t)}(\zeta)$$

rearranging gives

$$\bar{\psi}_{X(t)}(\zeta) = \frac{s^{\alpha-1}}{s^{\alpha}+\phi_T(\phi_S(\zeta))}\frac{s^{\alpha}-\phi_T(\phi_S(\zeta))}{s^{\alpha}-\phi_T(\phi_S(\zeta))}$$

which can be written as

$$\begin{aligned}
&s^{\alpha}(s^{\alpha}\bar{\psi}_{X(t)}(\zeta) - \psi_{X(0)}(\zeta)) - \frac{\partial}{\partial t}\psi_{X(0)}(\zeta) - 2\delta\gamma(s^{\alpha}\bar{\psi}_{X(t)}(\zeta) - \psi_{X(0)}(\zeta)) \\
&= 2^{2\kappa}\delta_2(\frac{1}{2}\gamma^{1/\kappa} + 1 - \cos(\zeta))^{2\kappa}\bar{\psi}_{X(t)}(\zeta) - \delta_2\gamma_2\bar{\psi}_{X(t)}(\zeta).
\end{aligned}$$

Invert the Laplace transform to see

$$\begin{aligned}
&D_t^{\alpha}D_t^{\alpha}\psi_{X(t)}(\zeta) - 2\delta\gamma D_t^{\alpha}\psi_{X(t)}(\zeta) \\
&= 2^{2\kappa}\delta_2(\frac{1}{2}\gamma^{1/\kappa} + 1 - \cos(\zeta))^{2\kappa}\psi_{X(t)}(\zeta) - \delta\gamma\psi_{X(t)}(\zeta). \qquad (4.15.2)
\end{aligned}$$

Note that for the first expression on the right hand side can be expanded as

$$\begin{aligned}
\left(\frac{1}{2}\gamma^{1/\kappa} + 1 - \cos(\zeta)\right)^{2\kappa} &= \sum_{j=0}^{\infty}\binom{2\kappa}{j}(\frac{1}{2}\gamma^{1/\kappa} + 1)^{2\kappa-j}\cos^j(\zeta) \\
&= \sum_{j=0}^{\infty}\binom{2\kappa}{j}(\frac{1}{2}\gamma^{1/\kappa} + 1)^{2\kappa-j}\sum_{r=0}^{j}\binom{j}{r}e^{i\zeta(j-2r)}.
\end{aligned}$$

Since the Fourier transform of $f(x-j+2r,t)$ is $e^{i\zeta(j-2r)}\psi_{X(t)}(\zeta)$, then by Fourier inversion of (4.15.2) the result follows. □

4.16 Delta fractional negative binomial process

In the previous section we extended a fractional Skellam type II to allow for jumps greater than one. However it might be of interest in terms of econometrics to extend processes of fractional Skellam of type I. The logic here is that type I processes can be split into their negative and positive parts (the up and down jump process), this is important empirically as we are then able to calibrate the up and down jump processes separately, which may be of some use.

The fractional negative binomial process has recently been introduced by Beghin and Macci (2014). We consider the case of the difference between two fractional negative binomial processes, firstly let us introduce the well known logarithmic distribution with probability mass function

$$\mathbb{P}(Y(j) = n) = \frac{1}{|\log(1-p)|}\frac{p^n}{n}, \quad i = 1, 2, \quad p \in (0, 1).$$

The mean is given by

$$\mathbb{E}[Y] = \frac{1}{|\log(1-p)|}\frac{p}{1-p}$$

and variance

$$\mathbb{V}\mathrm{ar}[Y] = -p\frac{p + \log(1-p)}{(1-p)^2 \log^2(1-p)}.$$

The n-fold convolution density is known in closed form and is given by

$$\mathbb{P}(Y(1) + \cdots + Y(n) = k) = \frac{n!}{(-\log(1-p))^n}\frac{p^k |s(k,n)|}{k!}$$

where $|s(k, n)|$ are the unsigned Stirling numbers of the first kind.

Definition 21. *A delta fractional negative binomial process* $\{X(t), t \geq 0\}$ *is defined by*

$$X(t) = \sum_{j=1}^{N_1(E_1(t))} Y_1(j) - \sum_{j=1}^{N_2(E_2(t))} Y_2(j)$$

where for $i = 1, 2$, $N_i(t)$ *are two independent Poisson processes with intensities*

$$\lambda_i = \delta_i |\log(1 - p_i)|, \qquad p_i \in (0, 1), \quad i = 1, 2$$

and for $i = 1, 2$, $E_i(t)$ are two independent inverse stable subordinators with parameter $\alpha_i = p_i \in (0, 1)$ both independent from all other process. And the innovations follow the logarithmic distribution with parameter $p_i \in (0, 1)$ for $i = 1, 2$, which are again independent from all other processes.

If instead we set $\alpha = 1$ we arrive at the delta negative binomial process as introduced by Barndorff-Nielsen, Shephard, and Pollard (2011).

Theorem 11. *Let $\{X(t), t \geq 0\}$ be a delta fractional negative binomial process, then the marginal distribution of $X(t)$ will have point probabilities given by*

$$\begin{aligned}
&\mathbb{P}(X(t) = k) \qquad (4.16.1)\\
&= \sum_{n=0}^{\infty} \frac{(1-p_1)^{n+k}}{(n+k)!} \sum_{j=1}^{n+k} |s(n+k, j)| t^{p_1(n+k)} \mathcal{E}^{j+1}_{p_1, p_1 j+1}(\log(p_1) t^{p_1})\\
&\times \frac{(1-p_2)^n}{n!} \sum_{j=1}^{n} |s(n, j)| t^{p_2 n} \mathcal{E}^{j+1}_{p_2, p_2 j+1}(\log(p_2) t^{p_2})
\end{aligned}$$

the moment generating function of $X(t)$ has the form

$$\begin{aligned}
\mathbb{E}[e^{\zeta X(t)}] = &\mathcal{E}_{\alpha_1}\Big(\delta_1 \log\Big(1 - \frac{(e^{-\zeta} - 1)p_1}{1 - p_1}\Big) t^{p_1}\Big)\\
&\times \mathcal{E}_{\alpha_2}\Big(\delta_2 \log\Big(1 - \frac{(e^{\zeta} - 1)p_2}{1 - p_2}\Big) t^{p_2}\Big)
\end{aligned}$$

where for $i = 1, 2$, $\mathcal{E}_{\alpha_i}(\cdot)$ is the one-parameter Mittag-Leffler function given by equation (2.14.4).

We shall use the the notation

$$X(t) \sim \Delta f NB(t\delta_1, p_1, t\delta_2, p_2)$$

to indicate that $X(t)$ follows a delta fractional negative binomial distribution, which appears to be a new four parameter distribution.

Remark 9. *For $X(t) \sim \Delta fNB(t\delta_1, p_1, t\delta_2, p_2)$ the mean of $X(t)$ is*

$$\mathbb{E}[X(t)] = \frac{\delta_1 p_1 t^{\alpha_1}}{\Gamma(p_1+1)(1-p_1)} - \frac{\delta_2 p_2 t^{\alpha_2}}{\Gamma(p_2+1)(1-p_2)}$$

and the variance is

$$\begin{aligned}\mathbb{V}ar[X(t)] = {} & \frac{p_1 t^{p_1+1}\delta_1}{(1-p_1)\Gamma(1+p_1)}\Big(\frac{p_1 t^{p_1+1}\delta_1}{(1-p_1)\Gamma(1+p_1)} \\ & \times (2p_1 B(p_1+1,p_1)-1) + \frac{p_1}{1-p_1}+1\Big) \\ & + \frac{p_2 t^{p_2+1}\delta_2}{(1-p_2)\Gamma(1+p_2)}\Big(\frac{p_2 t^{p_2+1}\delta_2}{(1-p_2)\Gamma(1+p_2)} \\ & \times (2p_2 B(p_2+1,p_2)-1) + \frac{p_2}{1-p_2}+1\Big)\end{aligned}$$

For $t \geq s$ the covariance of the delta fractional negative binomial process is given by

$$\begin{aligned}&\mathbb{C}ov[X(t),X(s)] \\ &= t\delta_1\Big(\frac{p_1}{1-p_1} + \Big(\frac{p_1}{1-p_1}\Big)_2\Big)\frac{s^{p_1}}{\Gamma(1+p_1)} + \Big(\frac{t\delta_1 p_1}{1-p_1}\Big) \\ &\times \Big(\frac{p_1 t^{2p_1}}{\Gamma_2(1+p_1)}B(p_1+1,p_1;s/t) + \frac{p_1 s^{2p_1}}{\Gamma_2(1+p_1)}B(p_1+1,p_1) \\ &- \frac{t s^{p_1}}{\Gamma_2(1+p_1)}\Big) \\ &+ t\delta_2\Big(\frac{p_2}{1-p_2} + \Big(\frac{p_2}{1-p_2}\Big)_2\Big)\frac{s^{p_2}}{\Gamma(1+p_2)} + \Big(\frac{t\delta_2 p_2}{1-p_2}\Big) \\ &\times \Big(\frac{p_2 t^{2p_2}}{\Gamma_2(1+p_2)}B(p_2+1,p_2;s/t) + \frac{p_2 s^{2p_2}}{\Gamma_2(1+p_2)}B(p_2+1,p_2) \\ &- \frac{t s^{p_2}}{\Gamma_2(1+p_2)}\Big)\end{aligned}$$

We can also give a mixed representation

$$X(t) = N^{*(1)}_{\alpha_1}(L_1(t)) - N^{*(2)}_{\alpha_2}(L_2(t)), \qquad t \geq 0 \tag{4.16.2}$$

where for $i = 1, 2$ the Lévy subordinator is gamma distributed, that is $L_i(t) \sim Ga(t\delta_i, (1-p_i)/p_i)$ with density given by

$$g_i(x,t) = \frac{\left(\frac{p_i}{1-p_i}\right)^{t\delta_i}}{\Gamma(t\delta_i)} x^{t\delta_i - 1} e^{-\frac{p_i}{1-p_i}x} \mathbb{I}_{(0,\infty)}(x)$$

and mean

$$\mathbb{E}[L_i(t)] = \frac{t\delta_i p_i}{1-p_i}$$

variance

$$\mathbb{V}ar[L_i(t)] = t\delta_i \left(\frac{p_i}{1-p_i}\right)^2$$

and Laplace exponent

$$\Psi_{L_i(1)}(\zeta) = \delta_i \log\left(1 - \frac{\zeta p_i}{1-p_i}\right).$$

The process of mixed representation given by equation (4.16.2) with $L_i(t)$ gamma in law as described above will have the same probability mass function as the compound version, that is the pmf given by equation (4.16.1).

4.17 Concluding remarks

This chapter has developed some new fractional integer valued models motivated by the analysis of high frequency trade by trade data. The modeling focus was on the distribution of times between trades. Using high frequency data for the EuroFX currency product it was demonstrated that the Mittag-Leffler distribution provides a more realistic description of inter-arrival times between trades. These models are quite different from activity time models, however we proved a link back in the form of convergence of limiting behavior over long time periods. Finally we considered the situation when the price may jump up or down in multiple of the tick size and proposed suitable models for these cases.

Chapter 5

Conclusion

Contributions to activity time models have been made in the form of fractal activity time types I, II and III and their corresponding risky asset models. The models proposed provided a realistic fit to real world data, the normal tempered stable distribution is a suitable description for the probability empirical observed. The dependence properties of the models allow the practitioner to choose the memory parameter to match his beliefs going forward or look for ways to calibrate to empirical autocorrelations. However since we did not state methods for computation of the memory parameter H for dependent data, this would form future research. For the zero skew case, calibration by method of moments was possible, so there is tractability in the sense of model fit. Estimation of all six parameters with no restrictions will require further theory and a numerical investigation. The concept of volatility clustering is not directly displayed by our activity time models, it would be of some interest to extend to incorporate such a feature.

Our first construction is closely related to the Ornstein-Uhlenbeck constructions of Leonenko, Petherick, and Sikorskii (2011b) and Finlay and Seneta (2012), where inverse Gaussian and generalized inverse Gaussian were used. This present work described a Ornstein-Uhlenbeck construction given by fractal activity time type I with tempered stable laws and extends further by constructing such a process with continuous sample paths. This allows the alternative starting point for the model in the form of a stochastic differential equation, which is not possible in the above mentioned papers. The technique of superpositions was used to construct processes with long range dependence or short range dependence in the case of finite superpositions. An interesting furture research project would be to establish some procedures for estimation of the number of superpositions that should be used in the finite case.

The second construction is essentially new to the fractal activity time geometric Brownian motion literature. However like most theory it relies on the theory developed by others in the form of convoluted subordinators and quantile clocks. Fractal activity time type II is, to the best of the authors knowledge, the only fractional tempered stable motion with long range dependence where exact distributions can be obtained. An alternative fractional tempered stable motion was introduced in Houdré and Kawai (2006) with long range dependence, theoretically they showed that the process has tempered stable marginal distributions, however it does not seem possible to compute the exact parametrization of the resulting tempered stable law. Further research could construct fractional motions for other distributions such as the inverse Gaussian, gamma and generalized inverse Gaussian.

The third construction of the inverse stable subordinator has been introduced in Magdziarz (2009), although under a different name of a subdiffusive regime. These activity models were presented as a bridge, in terms of convergence to related integer valued models.

Option valuation under activity time models produced a good fit to market prices over different strikes. A broker can certainly use such models to compute option prices to a greater accuracy than the classic model. The hedging of options is an open question as under activity time models the dependence property presents a significant issue. Without dependence the practitioner looks to buy or sell stock for which the option is written upon. The amount of stock transacted to hedge is directly related to the rate of change of the option value with respect to time. In practice, this is done by taking derivatives of the pricing formula, known as the delta. For activity time models, derivatives for the pricing formula were presented. However since the price process has dependence by its construction, it is unclear if the derivatives will suffice as a hedging tool. Consider the classic model with no dependence, then the derivatives are computed and the hedge constructed, but with memory models the price is changing not only due to current instantaneous conditions but also due to the entire price history in the case of long range dependence. Therefore hedging strategies under activity time models would be a useful future research project from the viewpoint of their use in practice for writing and hedging options. We saw that parameter estimation under the symmetric model was possible using method of moments. For the asymmetric model GMM techniques were discusses and a numerical and theoretical investigation to estimate all parameters under GMM would be a useful work. Furthermore the case of multiple assets has yet to be addressed for either activity time models or integer valued models.

For integer valued models we have shown through an empirical investigation that fractional models with Mittag-Leffler waiting times provide a more realistic fit to inter-arrival times between trades at high frequency. We have seen how a one tick model, where the price jumps by single ticks, can be extended to integer models where the price may jump up or down by multiple ticks. It may be of some interest to look for an improvement to the fit of empirical waiting times, for which the three parameter Mittag-Leffler distribution may be of use. Furthermore we saw that the traditional hedging tools of the delta and gamma of the option pricing formula under integer valued process do not exist and an investigation into techniques that could be used for hedging European options may prove useful to practitioners. Future research could focus on even smaller time scales by obtaining nanosecond data for trade by trade dynamics, also empirically the covariance structure at such small time intervals may be investigated. However we feel our work generalizing integer valued models to their fractional counterparts to be a worthwhile exercise.

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