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Non-unitary fusion categories and their doubles via endomorphisms

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ABSTRACT

We realise non-unitary fusion categories using subfactor-like methods, and compute their quantum doubles and modular data. For concreteness we focus on generalising the Haagerup–Izumi family of Q-systems. For example, we construct endomorphism realisations of the (non-unitary) Yang–Lee model, and non-unitary analogues of one of the even subsystems of the Haagerup subfactor and of the Grossman–Snyder system. We supplement Izumi's equations for identifying the half-braidings, which were incomplete even in his Q-system setting. We conjecture a remarkably simple form for the modular S and T matrices of the doubles of these fusion categories. We would expect all of these doubles to be realised as the category of modules of a rational VOA and conformal net of factors. We expect our approach will also suffice to realise the non-semisimple tensor categories arising in logarithmic conformal field theories.

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Contents

1. Introduction

The chiral part of a *unitary* rational conformal field theory (CFT) can be represented as either a completely rational conformal net of factors on a circle or a rational vertex operator algebra (VOA). Whilst conformal nets and subfactors theory focus and exploit the analytic aspects, vertex operator algebras focus on the algebraic aspects. The relation between these approaches is studied in [5]; at the simplest level, they both must give rise to the same modular tensor category (MTC) if they are to correspond to the same CFT.

Conformal nets of factors are a particularly rich framework, with connections with twisted equivariant K-theory and non-commutative geometry. Subfactor methods have proved to be much more effective than VOA methods in many ways. For example, structure theorems such as rationality of orbifolds or cosets is much easier in the conformal nets of subfactors picture (see e.g. [32]) than in the VOA picture. Also, the factor setting captures in a natural way the *full* CFT as an inclusion of (local) nets [3,31].

However, the VOA setting for the chiral CFT is apparently more flexible in allowing non-unitary examples. For example, the Virasoro minimal models are parametrised by pairs $p > q$ of coprime numbers; they are unitary if and only if $p = q + 1$. The simplest of these is the Yang–Lee model $V(2, 5)$ (see e.g. section 7.4.1 of [8]), which Cardy [4] showed arises as the Yang–Lee edge singularity in the Ising model in an imaginary magnetic field. Other non-unitary statistical mechanical examples are the scaling limit of critical dense polymers, and critical perculation, both with central charge $c = -2$. An unrelated non-unitary example crucial to string theory is the (super-)ghost CFT; what must be unitary is space where the physical states lie, namely the BRST cohomology of the ghosts coupled to a matter CFT. Wess–Zumino–Witten models on Lie supergroups provide other non-unitary examples important to string theory. In the VOA setting, realising non-unitary CFTs presents no special problems, whereas subfactors and nets of factors have unitarity built in.

A fundamental question is whether there are any rational CFTs beyond those constructed from loop groups or quantum groups, using standard methods such as orbifolds and cosets (see e.g. [32] for a discussion on this point). It is known that all unitary fusion categories, hence all unitary MTCs, can be realised by endomorphisms on a factor. These methods have produced countless 'exotic' examples of *unitary* MTCs [11,12]. Indeed, the relative abundance of these examples suggests that most modular tensor categories may be 'exotic'. Finding conformal net and VOA realisations of these 'exotic' MTCs is an important but difficult challenge — we expect most or all of them to have such realisations. The situation for the (double of the) Haagerup subfactor is discussed in detail in [11]. In any case, the effectiveness of these subfactor methods in constructing new *unitary* MTCs provides another compelling reason for extending these methods to the non-unitary setting.

The main purpose of this paper is to provide a broader context, dropping the requirement of unitarity, in which the subfactor methods can be applied. After all, most rational CFT are non-unitary, and one would like to exploit the powerful methods of subfactors and nets of factors in the general case.

In the remainder of the Introduction we sketch in more detail some of the terms used earlier, as well as the content of the paper.

The sectors of a rational CFT, or modules of a rational VOA, give rise to a tensor category of a very special type, namely an MTC. More generally, we are interested in *fusion categories*, which roughly speaking are MTCs without the braiding (we review their definition in section 3). Given a fusion category, the *double* or *centre construction* canonically associates an MTC. Unitarity in a category can be defined as follows. A $*$-operation on a $\mathbb{C}$-linear category $\mathcal{C}$ is a conjugate-linear involution $\mathrm{Hom}(X,Y) \to \mathrm{Hom}(Y,X)$ satisfying $(fg)^* = g^* f^*$ for all $f \in \mathrm{Hom}(X,Y)$, $g \in \mathrm{Hom}(Z,Y)$. If the category is tensor (and strict), we also require $(f \otimes g)^* = f^* \otimes g^*$ for all $f \in \mathrm{Hom}(X,Y)$, $g \in \mathrm{Hom}(Z,W)$. A $*$-operation is called positive if $f^* f = 0$ implies $f = 0$. A category equipped with a (positive) $*$-operation is called *hermitian* (resp. *unitary*).

Associated to an MTC is a representation of $\mathrm{SL}_2(\mathbb{Z})$ called *modular data*. It is generated by a symmetric unitary matrix S which gives the fusion coefficients (structure constants of the Grothendieck ring of the category) through Verlinde's formula, together with a diagonal matrix T of finite order. Some column of S must be strictly positive — e.g. in a *unitary* MTC that Perron–Frobenius column corresponds to the unit. In a rational CFT, the characters $\chi_M(\tau) = q^{h_M - c/24} \sum_{n=0}^{\infty} \dim M_n \, q^n$ of the irreducible modules $M = \coprod_n M_n$ form a vector-valued modular function for $\mathrm{SL}_2(\mathbb{Z})$ with modular data as its multiplier. The minimal conformal weight h_M corresponds to the positive column of S. The conformal weights and central charge c must be rational, but in a unitary theory they will also be non-negative. For more comparisons between the modular data of non-unitary versus unitary theories, see [15].

A very convenient realisation of tensor categories is through endomorphisms on an algebra, where objects are algebra endomorphisms and morphisms are intertwiners. The tensor product of objects corresponds to composition and of morphisms to the (twisted) product in the underlying algebra. However, it is awkward to realise other properties in the category, such as additivity or rigidity, without assuming special structures on the algebra. When the underlying algebra is a C^*-algebra such as the Cuntz algebra, these other properties arise naturally. Indeed, any *unitary* fusion category can be realised as a category of endomorphisms on a hyperfinite von Neumann algebra (see section 7 of [19]).

A natural question is, can we find systematic realisations by endomorphisms of *non-unitary* fusion categories? We will see that the answer is yes.

Our approach was influenced by recent work of Phillips [29], who studies non-unitary analogues of the Cuntz algebra. But all of our calculations are within a polynomial algebra (the *Leavitt algebra*). Rather than completing that algebra as studied by Phillips, we have found it sufficient to work exclusively within the Leavitt algebra itself.

For concreteness we focus on the *Haagerup–Izumi family of fusion rings*, but our method works more generally. Let G be any finite abelian group. The (isomorphism classes of) simple objects in these fusion rings are $[\alpha_g]$ and $[\alpha_g\rho]$ as g ranges over G. The fusions are given by

$$[\alpha_g][\alpha_h] = [\alpha_{g+h}]\,,\ [\alpha_g][\alpha_h\rho] = [\alpha_{g+h}\rho] = [\alpha_h\rho][\alpha_{-g}]\,,$$
$$[\alpha_g\rho][\alpha_h\rho] = [\alpha_{g-h}] + \sum\nolimits_k [\alpha_k\rho]\,. \qquad (1.1)$$

In the following sections we explain explicitly how to construct, using endomorphisms on the Leavitt algebra, fusion categories (not necessarily unitary) which realise the Haagerup–Izumi fusions when G has odd order. We compute the corresponding tube algebras and from that obtain the modular data S, T of the double of the system. We give several examples and explicitly classify these systems for small G.

The (unitary) Haagerup–Izumi fusions (1.1) for $|G|$ odd was introduced by Izumi in [22]. His motivation was to construct the Haagerup subfactor [18,2], so he focused on the special class of systems of Cuntz algebra endomorphisms, called *Q-systems*, which arise as the even subsystem of a subfactor with canonical endomorphism $1+\rho$. Q-systems correspond to especially constrained ρ; their fusion categories are always unitary. He showed that there was a unique Q-system satisfying (1.1) for the group $G = \mathbb{Z}_3$, and comparing indices observed that it must correspond to the Haagerup subfactor. Likewise, he showed that there is a unique Q-system for $G = \mathbb{Z}_5$. He also computed the modular data for the doubles of his systems (modulo a technicality discussed shortly). Evans–Gannon [11] pushed this further, finding Q-systems in this class for all G with $|G| \leq 19$ (including the complete lists for $|G| \leq 9$), and simplifying considerably Izumi's expressions for the modular data. Thanks to this work, it is now expected that there are subfactors (usually several) for each odd order, and they are all expected to correspond through their doubles to rational VOAs etc. Grossman–Snyder [17] found new systems of endomorphisms realising (1.1) (unitary but not Q-systems), for $G = \mathbb{Z}_3$ and $\mathbb{Z}_5$, which are Morita equivalent to Izumi's systems (and thus have the same doubles). This treatment has been extended to even order G, and to all unitary systems (not only Q-systems) realising (1.1), by Evans–Gannon [14,13] and independently Izumi [23].

In this paper, as an illustration of our method, we characterise all realisations by endomorphisms (not necessarily Q-systems nor unitary) of the Haagerup–Izumi fusions (1.1) for $|G|$ odd (though to keep the accommodations demanded by nonunitarity as clear as possible, we impose a simplifying assumption (4.1) — see the discussion in the paragraph before Theorem 1). We show they all yield fusion categories. Like [22], our systems correspond to solutions of finitely many equations in finitely many variables, but unlike [22] our equations are all polynomials (those of [22] involve complex conjugates). In broad strokes the method we use is analogous to that of [22], but the absence of unitarity introduces several complications and our argument is required to be much more subtle. We find the doubles and modular data of our systems.

For example, we find precisely 2,4,4 inequivalent fusion categories realised by endomorphisms, of Haagerup–Izumi type for $G = \mathbb{Z}_1, \mathbb{Z}_3, \mathbb{Z}_5$ respectively (of course we recover all of them). Precisely 1,2,2 of these, respectively, are unitary: 1,1,1 are Q-systems, and 0,1,1 are the aforementioned Grossman–Snyder systems. The Yang–Lee system is the unique non-unitary one corresponding to $G = \mathbb{Z}_1$.

Every fusion category $\mathcal{C}$ is defined over some number field [10]. An automorphism σ of that field acts on the quantities of that category in the natural way, defining a new fusion category $\mathcal{C}^\sigma$. These categories may or may not be equivalent — e.g. a Galois associate of a unitary fusion category may not be unitary. In general, $\mathcal{C}$ and $\mathcal{C}^\sigma$ will have identical fusion rings, but their modular data for example will be Galois associates. Our construction, unlike that of e.g. Izumi, is closed under this Galois action.

It turns out that all 5 non-unitary fusion categories we have found for $G = \mathbb{Z}_1, \mathbb{Z}_3, \mathbb{Z}_5$ are Galois associates of unitary categories. We expect though that this is an accident of small G. Our system of equations involve twice as many variables as in the unitary case, and approximately the same number of equations. For these reasons, we would expect typically many more non-unitary categories than unitary ones.

In any case, it is easy to construct non-unitary fusion categories, all of whose Galois associates are also non-unitary. A simple example is the tensor product of affine G_2 at level 1 (a unitary MTC) with the Yang–Lee model (a non-unitary one).

Actually, the equations in [22] are not sufficient to determine the half-braidings, and hence the modular data, for most odd abelian G, even in the Q-system case. In section 6 below we supply additional equations which are both necessary and sufficient.

Incidentally, another interesting class of CFTs and VOAs are the so-called *logarithmic* or *C_2-cofinite non-rational* ones [6], for example the symplectic fermions [7]. Unlike the rational CFTs, their category of modules will not be semisimple and so direct (sub)factor realisations of them wouldn't be possible. Logarithmic theories appear to be intimately connected with non-unitarity: all known ones are conformal embeddings of non-unitary rational VOAs (with states of negative conformal weight). In any case, although we address in this paper only fusion categories (which are semisimple by definition), modelling non-semisimple systems is also possible by our methods and we would expect we could realise with endomorphisms these logarithmic theories. The 'logarithmic' analogue of the fusion category is the *finite tensor category* of [9], and the analogue of the modular version thereof, in particular regarding mapping class group representations including modular data, is explored in [25].

2. The Yang–Lee model

This section illustrates the ideas developed in the following sections, with the simplest non-unitary example: the Yang–Lee model (this CFT is described e.g. in section 7.4.1 of [8]). It consists of two simple objects 1 and ρ, which obey the fusion rule

$$[\rho][\rho] = [1] + [\rho]\,. \tag{2.1}$$

Let us try to realise (2.1) as a system of algebra endomorphisms on some algebra $\mathcal{A}$. To motivate our solution though, let's reverse the logic and derive the consequences of such a realisation. It would require the relation

$$\rho(\rho(x)) = sxs' + t\rho(x)t'\,, \tag{2.2}$$

where $s, s', t, t' \in \mathcal{A}$ satisfy the Leavitt–Cuntz relations

$$ss' + tt' = 1\,, \quad s's = t't = 1\,, \quad s't = t's = 0\,. \tag{2.3}$$

More precisely, these relations say that (2.2) expresses $\rho \circ \rho$ as a direct sum of objects id and ρ in the category $\mathcal{END}(\mathcal{A})$ (we describe this category in detail next section). These elements s, s', t, t' generate by definition a copy of the Leavitt algebra $\mathcal{L}_2$ inside $\mathcal{A}$; we will see shortly that ρ restricts to an endomorphism of $\mathcal{L}_2$. In order to identify the restriction of ρ to $\mathcal{L}_2$, it is necessary and sufficient to determine the values $\rho(s)$, $\rho(s')$, $\rho(t)$ and $\rho(t')$ of ρ on the generators. For $*$-maps, we would have $\rho(s') = \rho(s)'$ etc, but we cannot require that here if we hope to realise the Yang–Lee model.

We require that both endomorphisms id and ρ be simple, equivalently that the intertwiner spaces $\mathrm{Hom}(\mathrm{id}, \mathrm{id})$ and $\mathrm{Hom}(\rho, \rho)$ in the algebra $\mathcal{A}$ be $\mathbb{C}1$, and that $\mathrm{Hom}(\rho, \mathrm{id}) = \mathrm{Hom}(\mathrm{id}, \rho) = 0$. (The definition of intertwiners is given next section.) From (2.2) we obtain

$$\rho^2(x)s = sx\,, \quad \rho^2(x)t = t\rho(x)\,, \quad s'\rho^2(x) = xs'\,, \quad t'\rho^2(x) = \rho(x)t'\,. \tag{2.4}$$

The first means $s \in \mathrm{Hom}(\mathrm{id}, \rho^2)$. Conversely, suppose $r \in \mathrm{Hom}(\mathrm{id}, \rho^2)$, i.e. $rx = \rho^2(x)r$ for all x. Then $s'rx = s'\rho^2(x)r = xs'r$ and $t'rx = t'\rho^2(x)r = \rho(x)t'r$. Thus by simplicity of id and ρ we have $s'r \in \mathbb{C}$ and $t'r = 0$, so $r = (ss' + tt')r = ss'r \in \mathbb{C}s$ using the Leavitt–Cuntz relation $ss' + tt' = 1$. We have shown $\mathrm{Hom}(\mathrm{id}, \rho^2) = \mathbb{C}s$. In the same way (see Lemma 3 below for details and the generalisation), we can identify the intertwiner spaces $\mathbb{C}t = \mathrm{Hom}(\rho, \rho^2)$, $\mathbb{C}s' = \mathrm{Hom}(\rho^2, \mathrm{id})$ and $\mathbb{C}t' = \mathrm{Hom}(\rho^2, \rho)$. These observations are crucial for what follows.

Note, using (2.4), that

$$s'\rho(s)\rho(x) = s'\rho(sx) = s'\rho(\rho^2(x)s) = s'\rho^2(\rho(x))\rho(s) = \rho(x)s'\rho(s)\,. \tag{2.5}$$

In other words, $s'\rho(s) \in \mathrm{Hom}(\rho,\rho) = \mathbb{C}$, so $s'\rho(s)$ equals some complex number a. Likewise, $t'\rho(s) \in \mathrm{Hom}(\rho,\rho^2)$ so $t'\rho(s) = bt$ for some $b \in \mathbb{C}$. The point is that

$$\rho(s) = (ss' + tt')\rho(s) = s(s'\rho(s)) + t(t'\rho(s)) = as + btt\,. \tag{2.6}$$

Similar calculations (see section 4 for details and the generalisation) give

$$\rho(s') = a's' + b't't'\,,\quad \rho(t) = cst' + dtss' + ettt'\,,\quad \rho(t') = c'ts' + d'ss't' + e'tt't'\,, \tag{2.7}$$

for some $a', b', c, c', d, d', e, e' \in \mathbb{C}$. Because ρ sends the generators of $\mathcal{L}_2$ into $\mathcal{L}_2$, this means ρ is actually an endomorphism of $\mathcal{L}_2$. If we required ρ to be a $*$-map, then we would have $a' = \overline{a}$ etc, but again we shouldn't do that if we are to recover Yang–Lee.

We can now use the constraints on ρ to solve for those 10 parameters. First, ρ is required to be an algebra endomorphism, so it must respect the Leavitt–Cuntz relations (2.3). One relation requires $1 = \rho(s')\rho(s)$, i.e.

$$1 = (a's' + b't't')(as + btt) = a'a + bb'\,. \tag{2.8}$$

Similarly, $\rho(s)\rho(s') + \rho(t)\rho(t') = 1$ gives the identities $1 = aa' + cc'$ (hence $b'b = c'c$), $aa' + cc' = dd'$ (hence $d'd = 1$), and $ab' = -ce'$, amongst others. More precisely, Lemma 1 below gives a unique form for any element of a Leavitt algebra, so once we expand out $\rho(s)\rho(s') + \rho(t)\rho(t') = 1$ and put it into reduced form (e.g. replacing ss' by $1 - tt'$), the identities fall out by comparing corresponding coefficients.

We also require that ρ satisfy (2.2). It implies for instance that $s'\rho^2(s) = ss'$. We can compute $s'\rho(\rho(s))$ directly from (2.6), (2.7), and we find

$$s'\rho^2(s) = as'\rho(s) + bs'\rho(t)\rho(t) = a^2 + bct'(cst' + dtss' + ettt') = a^2 + bcdss' + bcett'\,.$$

This must equal ss', which (using $1 = ss' + tt'$) gives $1 = a^2 + bcd$ and $1 = bcd - bce$ (hence $a^2 = -bce$). Likewise, $\rho^2(s')s = ss'$ gives $1 = a'^2 + b'c'd'$ and $a'^2 = -b'c'e'$. Similarly, (2.2) implies $t'\rho^2(s) = \rho(s)t'$; its t and st' coefficients give $ab = -bde$ and $a = bcd$ respectively. Likewise, $\rho^2(s')t = t\rho(s')$ gives $a'b' = -b'd'e'$ and $a' = b'c'd'$.

Plugging $a = bcd$ into $1 = a^2 + bcd$ (and likewise for the primed quantities) gives $1 = a^2 + a = a'^2 + a'$, which means $a, a' \in \{(-1 \pm \sqrt{5})/2\}$. Note that if $a \neq a'$ then $aa' = 1$ — we will use this shortly. Since $a = bcd, a' = b'c'd'$ are both non-zero, so are all b, c, d, b', c', d'. Note that we are free to rescale s by $\lambda \in \mathbb{C}^\times$ (hence s' by $1/\lambda$) without affecting (2.2) nor the Leavitt–Cuntz relations. Choosing λ appropriately we can simultaneously force $b = c$ and also $0 \leq \mathrm{Arg}(b) < \pi$, and then $bb' = cc'$ also gives $b' = c'$. Comparing $a^2 = -bce$, $a = bcd$, and $ab = -bde$ give $e = -da$ and $d \in \{\pm 1\}$ (and likewise $e' = -d'a'$ and $d' \in \{\pm 1\}$). But we knew $dd' = 1$, so we have $d' = d$. Putting $aa' = (bcd)(b'c'd') = b^2b'^2$ into (2.8) gives $bb' \in \{(-1 \pm \sqrt{5})/2\}$. In particular, aa' cannot be 1, so we must have $a = a'$ and thus $b = \sqrt{da}$ and $b' = db$.

We eliminate the possibility that $d = -1$ by considering the $st't'$ coefficient of $t'\rho^2(t) = \rho(t)t'$, which gives $c = ab'd^2 + cdee'$. So we have determined that $a = a' = -e = -e' = (-1 \pm \sqrt{5})/2$, $b = b' = c = c' = \sqrt{a}$, and $d = d' = 1$, where we can take the square-root for b so that $b \in \mathbb{R}_{>0} \cup i\mathbb{R}_{>0}$. So we have 2 possible solutions, corresponding to the choice of signs in $a = (-1 \pm \sqrt{5})/2$. In section 4 we generalise this argument to arbitrary odd order abelian G in (1.1).

Conversely, given either solution $a = (-1 \pm \sqrt{5})/2$, we can define ρ on the generators s, s', t, t' of $\mathcal{L}_2$ by (2.6)–(2.7). Using Corollary 1 below, this choice extends to an algebra endomorphism ρ on $\mathcal{L}_2$ iff it respects the Leavitt–Cuntz relations: i.e. $1 = \rho(s')\rho(s) = \rho(t')\rho(t) = \rho(s)\rho(s') + \rho(t)\rho(t')$ and $0 = \rho(s')\rho(t) = \rho(t')\rho(s)$. It is straightforward to verify this (this is done in full generality in section 5). To show ρ satisfies (2.2), note that both sides of (2.2) are manifestly endomorphisms, so it suffices to verify it for each of the four generators $x \in \{s, s', t, t'\}$. If we can show $s'\rho^2(x) = xs'$ and $t'\rho^2(x) = \rho(x)t'$ for $x = s, t$ (these must hold if (2.2) is to hold), then $\rho^2(x) = (ss' + tt')\rho^2(x)$ shows (2.2) holds for $x = s, t$. Likewise, if $\rho^2(y)s = sy$ and $\rho^2(y)t = t\rho(y)$ for $y = s', t'$, then (2.2) holds for $x = s', t'$. Again, the details are given in full generality in section 5. Thus ρ defined by (2.7) obeys the Yang–Lee fusions (2.1). Finally, we can confirm that the endomorphism ρ we have just constructed is indeed simple, i.e. $\mathrm{Hom}(\rho, \rho) = \mathbb{C}$ as well as $\mathrm{Hom}(\mathrm{id}, \mathrm{id}) = \mathbb{C}$ and $\mathrm{Hom}(\rho, \mathrm{id}) = \mathrm{Hom}(\mathrm{id}, \rho) = 0$ (this is done in full generality in Proposition 1 below).

Much more delicate is to associate a (strict) fusion category to both of these ρ. The biggest challenge here for arbitrary G is to define arbitrary (but finite) sums of endomorphisms using the Leavitt algebra, in the sense of the right-side of (2.2). We are lucky here with the Yang–Lee: because its Leavitt algebra has 2×2 generators, we can capture arbitrary sums — e.g. $\rho \oplus \rho^3 \oplus \rho^5$ can be written $s\rho s' + ts\rho^3 s't' + tt\rho^5 t't'$, to choose a random example. The resulting fusion category for the solution with $a = (-1 + \sqrt{5})/2$ is the unitary category associated to e.g. the integrable modules of the affine G_2 algebra at level 1, whilst for $a = (-1 - \sqrt{5})/2$, we obtain the Yang–Lee fusion category. These two fusion categories are inequivalent even though they share the same fusions (2.1) — indeed, it can be shown that the categorical dimension of ρ (defined next section) is $1/a = (1 \pm \sqrt{5})/2$, so is positive in one and negative in the other. Nevertheless they are clearly related by the Galois automorphism interchanging $a = (-1 \pm \sqrt{5})/2$.

To realise the fusions (1.1) for general G, we will need a Leavitt algebra $\mathcal{L}$ with $(1 + |G|) \times 2$ generators (one pair for each term on the right of (1.1)), but for such an algebra only direct sums with $n \equiv 1 \pmod{|G|}$ terms can be realised. When ρ is a $*$-map (e.g. the case studied in [22,11]), we can extend ρ to an endomorphism of an infinite von Neumann factor N [22]; semisimplicity is then automatic, since N contains copies of the Leavitt algebras of arbitrary rank, so arbitrary sums of endomorphisms can be made. On the other hand, when ρ is not a $*$-map, we obtain semisimplicity by first forming the idempotent completion. In section 5 we show that any solution to the various consistency equations yields a (usually non-unitary) fusion category.

MTC structures can be placed on both of the $G = 1$ fusion categories constructed in this section, though in more than one way — e.g. the $a = (-1 + \sqrt{5})/2$ category is realised by both affine G_2 level 1 and affine F_4 level 1, which which are inequivalent as MTC since they have different central charges mod 8. This behaviour too is special to $G = 1$: the fusion categories for larger G never come with a braiding (this is clear from (1.1), as $[\alpha_g][\rho] \neq [\rho][\alpha_g]$ when $|G|$ is odd and > 1). For these other G, we realise in section 6 the associated MTC through the centre of the tube algebra. Incidentally, this construction applied to e.g. the fusion category of affine G_2 at level 1, would yield the MTC of affine $G_2 \oplus F_4$ at level $(1, 1)$.

Although the fusion (or modular tensor) categories of Yang–Lee and affine G_2 or F_4 at level 1 are merely related by a Galois automorphism, the corresponding VOAs do not seem related in any simple way. For example, the characters of Yang–Lee are

$$q^{11/60}(1 + q^2 + q^3 + q^4 + q^5 + 2q^6 + \cdots),$$
$$q^{-1/60}(1 + q + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + \cdots)$$

with modular data

$$S = \frac{1}{\sqrt{5}}\begin{pmatrix} -\sin(2\pi/5) & \sin(\pi/5) \\ \sin(\pi/5) & \sin(2\pi/5) \end{pmatrix}, \quad T = \begin{pmatrix} e^{2\pi i 11/60} & 0 \\ 0 & e^{-2\pi i/60} \end{pmatrix},$$

while those for affine G_2 at level 1 are

$$q^{-7/60}(1 + 14q + 42q^2 + 140q^3 + 350q^4 + 840q^5 + \cdots),$$
$$q^{17/60}(7 + 34q + 119q^2 + 322q^3 + 819q^4 + 1862q^5 + \cdots)$$

with modular data

$$S = \frac{1}{\sqrt{5}}\begin{pmatrix} \sin(\pi/5) & \sin(2\pi/5) \\ \sin(2\pi/5) & -\sin(\pi/5) \end{pmatrix}, \quad T = \begin{pmatrix} e^{-2\pi i 7/60} & 0 \\ 0 & e^{2\pi i 17/60} \end{pmatrix}$$

and those for affine F_4 at level 1 are

$$q^{-13/60}(1 + 52q + 377q^2 + 1976q^3 + 7852q^4 + \cdots),$$
$$q^{23/60}(26 + 299q + 1702q^2 + 7475q^3 + 27300q^4 + \cdots)$$

with modular data

$$S = \frac{1}{\sqrt{5}}\begin{pmatrix} \sin(\pi/5) & \sin(2\pi/5) \\ \sin(2\pi/5) & -\sin(\pi/5) \end{pmatrix}, \quad T = \begin{pmatrix} e^{-2\pi i 13/60} & 0 \\ 0 & e^{2\pi i 23/60} \end{pmatrix}.$$

In these cases, the first character given is that of the VOA $\mathcal{V} = \coprod_{n=0}^{\infty} \mathcal{V}_n$ itself, and so lists the dimensions of its graded spaces $\mathcal{V}_n$, so we see that there appears little relation between the Yang–Lee VOA and that of say the G_2 one. On the other hand, the naive inner product of the G_2 and F_4 character vectors is $j(\tau)^{1/3}$, reflecting the fact that the VOA $\mathcal{V}(G_2,1)\otimes\mathcal{V}(F_4,1)$ is a conformal subalgebra of the E_8 lattice VOA. Note also that the first column of the matrix S is strictly positive for the VOAs $\mathcal{V}(G_2,1)$ and $\mathcal{V}(F_4,1)$ (as it must be for unitary VOAs), and isn't for the Yang–Lee (as is typical for non-unitary VOAs).

3. Leavitt algebras and categories of endomorphisms

For each $n>1$ define the *Leavitt algebra* $\mathcal{L}_n$ to be the associative $*$-algebra freely generated over $\mathbb{C}$ by $x_1,\dots,x_n,x_1',\dots,x_n'$, modulo the *Leavitt–Cuntz relations*

$$x_i'x_j=\delta_{i,j}\,,\quad \sum_{i=1}^n x_ix_i'=1\,. \tag{3.1}$$

The elements of $\mathcal{L}_n$ are polynomials in the non-commuting variables x_i,x_j'. The $*$-operation sends $x_i\mapsto x_i'$, $x_i'\mapsto x_i$, and obeys $(cyz)'=\overline{c}z'y'$ for all $c\in\mathbb{C}$ and $x,y\in\mathcal{L}_n$. It has an obvious grading by $\mathbb{Z}^n$. The Leavitt algebra $\mathcal{L}_n$ can be regarded as the polynomial part of the Cuntz algebra $\mathcal{O}_n$, its C^*-algebra completion.

The Leavitt algebras $\mathcal{L}_n$ are all non-isomorphic for $n=2,3,4,\dots$, since the inclusion of $\mathcal{L}_n$ in $\mathcal{O}_n$ induces an isomorphism on K-theory with the cyclic group $\mathbb{Z}_{n-1}$ [1]. The only obstruction to embedding $\mathcal{L}_m$ unitally in $\mathcal{L}_n$ is given by the K-theory [30]. More precisely $\mathcal{L}_m$ embeds unitally in $\mathcal{L}_n$ if and only if $m-1$ divides $n-1$. In the Cuntz framework of Izumi [20,22,23] and Evans–Gannon [11–13], one constructs endomorphisms on a fixed Cuntz algebra $\mathcal{O}_n$, with prescribed fusion rules and then extends these to a completion as an infinite von Neumann factor N. Any Cuntz algebra $\mathcal{O}_m$ can be unitally embedded in the factor N for any m, even though usually it cannot be unitally embedded in $\mathcal{O}_n$. The fusion category will then be realised as a system of endomorphisms of N, since addition of any number m of endomorphisms can be expressed in N.

We will realise fusion categories through endomorphisms of $\mathcal{L}_n$. But we do not require that our endomorphisms be $*$-maps, so they need not extend to the completion, the Cuntz algebra $\mathcal{O}_n$ or the Banach algebras of Phillips [29].

Note that if ρ is any algebra endomorphism on $\mathcal{L}_n$, then so is $\tilde\rho$ defined by

$$\tilde\rho(y)=\rho(y')'\,. \tag{3.2}$$

Throughout this paper we distinguish an algebra endomorphism from a $*$-algebra endomorphism. The latter must obey $f(y)'=f(y')$ (equivalently $\tilde\rho=\rho$) while the former may not.

There is a canonical way to write any element of $\mathcal{L}_n$. Call any monomial in the generators x_i,x_j' *reduced* if no primed variable appears to the left of any unprimed variable, and x_1 is not adjacent to x_1' in the monomial. Call any linear combination over $\mathbb{C}$ of finitely many distinct reduced monomials, a *reduced sum.*

Lemma 1. *([24]) Any $y \in \mathcal{L}_n$ can be written in one and only one way as a reduced sum.*

This simple observation has several easy consequences, as we'll see. It easily implies the centre of $\mathcal{L}_n$ is trivial [24]. Moreover:

Corollary 1. *An algebra endomorphism ρ on $\mathcal{L}_n$ is uniquely defined by its values $\rho(x_i), \rho(x'_j)$ on the generators, and these can be assigned arbitrarily provided they respect the Leavitt–Cuntz relations (3.1).*

There are several complications caused by avoiding the completion and working exclusively with $\mathcal{L}_n$. In particular, two serious challenges are how to add endomorphisms, and how to get rigidity. We accomplish the former through the idempotent completion (described below), and the latter by hand.

Recall that because $\mathcal{L}_n$ is a unital algebra over $\mathbb{C}$, by general nonsense its algebra endomorphisms define a $\mathbb{C}$-linear preadditive strict tensor category $\mathcal{END}(\mathcal{L}_n)$. More precisely, the objects in $\mathcal{END}(\mathcal{L}_n)$ are algebra (but not $*$-algebra) endomorphisms of $\mathcal{L}_n$. The morphisms $r \in \mathrm{Hom}(\beta, \gamma)$ are intertwiners, i.e. $r \in \mathcal{L}_n$ for which $r\beta(x) = \gamma(x)r$ for all $x \in \mathcal{L}_n$; composition of morphisms is multiplication in $\mathcal{L}_n$. $\mathcal{END}(\mathcal{L}_n)$ is $\mathbb{C}$-linear, i.e. each $\mathrm{Hom}(\beta, \gamma)$ is a vector space over $\mathbb{C}$; it is also preadditive, i.e. composition of morphisms is bilinear. The tensor product of objects is composition: $\beta \otimes \gamma = \beta \circ \gamma$, whilst of morphisms is: $r \otimes s = r\beta(s) = \gamma(s)r \in \mathrm{Hom}(\beta \circ \rho, \gamma \circ \sigma)$ when $r \in \mathrm{Hom}(\beta, \gamma)$, $s \in \mathrm{Hom}(\rho, \sigma)$.

A *fusion category* [10] is a $\mathbb{C}$-linear semisimple rigid tensor category with finitely many isomorphism classes of simple objects and finite dimensional spaces of morphisms, such that the unit object 1 is simple. A simple object X is one with $\mathrm{End}(X) = \mathbb{C}\,\mathrm{id}_X$; amongst other things, every object in a semisimple category is a direct sum of simple ones. We say object X has a right-dual $X^\vee$ iff there is a pair of morphisms *evaluation* $e_X \in \mathrm{Hom}(X^\vee \otimes X, 1)$ and *co-evaluation* $b_X \in \mathrm{Hom}(1, X \otimes X^\vee)$ for which

$$(\mathrm{id}_X \otimes e_X) \circ (b_X \otimes \mathrm{id}_X) = \mathrm{id}_X\,, \quad (e_X \otimes \mathrm{id}_{X^\vee}) \circ (\mathrm{id}_{X^\vee} \otimes b_X) = \mathrm{id}_{X^\vee} \tag{3.3}$$

(where we assume the category is strict, for convenience). Left-dual ${}^\vee X$ is defined similarly. In particular in $\mathcal{END}(\mathcal{L})$, an object $\beta \in \mathrm{End}(\mathcal{L})$ has a right-dual $\beta^\vee \in \mathrm{End}(\mathcal{L})$ if there are elements $e_\beta \in \mathrm{Hom}(\beta \circ \beta^\vee, \mathrm{id})$ and $b_\beta \in \mathrm{Hom}(\mathrm{id}, \beta^\vee \circ \beta)$ in $\mathcal{L}$ such that

$$\beta(e_\beta)b_\beta = 1 = e_\beta \beta^\vee(b_\beta)\,. \tag{3.4}$$

A tensor category is called *rigid* if every object X has a right- and left-dual.

In a (strict) rigid category, we can define the right-dual $f^\vee \in \mathrm{Hom}(Y^\vee, X^\vee)$ of a morphism $f \in \mathrm{Hom}(X, Y)$ by

$$f^\vee = (e_Y \otimes \mathrm{id}_{X^\vee}) \circ (\mathrm{id}_{Y^\vee} \otimes f \otimes \mathrm{id}_{X^\vee}) \circ (\mathrm{id}_{Y^\vee} \otimes b_X)\,. \tag{3.5}$$

In particular in $\mathcal{END}(\mathcal{L})$, the right-dual of $r \in \mathrm{Hom}(\alpha, \beta)$ is defined by

$$r^\vee = e_\beta \, \beta^\vee(r \, b_\alpha) \,. \tag{3.6}$$

Then $(f \circ g)^\vee = g^\vee \circ f^\vee$ when the composition is defined. Left-dual ${}^\vee f$ is defined similarly. A rigid tensor category is *pivotal* if it is equipped with a natural monoidal isomorphism from the identity functor to the double-dual functor $X \mapsto X^{\vee\vee}$. In a pivotal category we can take ${}^\vee X = X^\vee$. In a rigid category the (left-)dimension of object X is $e_{X^\vee} b_X$; a semisimple pivotal category is called *spherical* if X and $X^\vee$ have the same dimension for all objects X (it suffices to check this for simple X). See e.g. [28] for the remaining terminology not explained here.

Let $\mathcal{E}$ be a collection of algebra endomorphisms of $\mathcal{L}_n$ closed under composition. We require the identity to be in $\mathcal{E}$. Let $\mathcal{C}(\mathcal{E})$ denote the subcategory of $\mathcal{END}(\mathcal{L}_n)$ restricted to $\mathcal{E}$. Then like $\mathcal{END}(\mathcal{L}_n)$, $\mathcal{C}(\mathcal{E})$ is a $\mathbb{C}$-linear tensor category, and the endomorphism algebra of the unit object 1 is $\mathbb{C}$. By its *idempotent completion* we mean the category $\overline{\mathcal{C}(\mathcal{E})}$ whose objects consist of pairs (p, β) where $\beta \in \mathcal{E}$ and $p \in \mathrm{End}(\beta)$ is an idempotent, i.e. $p^2 = p$, and whose morphism spaces are $\mathrm{Hom}((p, \beta), (q, \gamma)) = q\mathrm{Hom}(\beta, \gamma)p$ with composition again given by multiplication. $\overline{\mathcal{C}(\mathcal{E})}$ is a tensor category using $(p, \beta) \otimes (q, \gamma) := (p \otimes q, \beta \otimes \gamma) = (p\beta(q), \beta \circ \gamma)$, and the tensor product of $qrp \in \mathrm{Hom}((p, \beta), (q, \gamma))$ with $q'r'p' \in \mathrm{Hom}((p', \beta'), (q', \gamma'))$ is $(qrp) \otimes (q'r'p') = qrp\beta(q'r'p')$. We can introduce direct sums into $\overline{\mathcal{C}(\mathcal{E})}$ as follows. Objects in this new category consist of ordered n-tuples $((p_1, \beta_1), \ldots, (p_n, \beta_n)) = (p_1, \beta_1) \oplus \cdots \oplus (p_n, \beta_n)$, and the morphism spaces are

$$\begin{aligned} &\mathrm{Hom}(((p_1, \beta_1), \ldots, (p_n, \beta_n)),((q_1, \gamma_1), \ldots, (q_m, \gamma_m))) \\ &\qquad = \begin{pmatrix} q_1\mathrm{Hom}(\beta_1, \gamma_1)p_1 & \cdots & q_1\mathrm{Hom}(\beta_n, \gamma_1)p_n \\ \vdots & \ddots & \vdots \\ q_m\mathrm{Hom}(\beta_1, \gamma_m)p_1 & \cdots & q_m\mathrm{Hom}(\beta_n, \gamma_m)p_n \end{pmatrix} . \end{aligned} \tag{3.7}$$

Composition is matrix multiplication. Then $((p_1, \beta_1), \ldots, (p_n, \beta_n)) \otimes ((q_1, \gamma_1), \ldots, (q_m, \gamma_m))$ is the direct sum of $(p_i, \beta_i) \otimes (q_j, \gamma_j)$, while

$$\begin{pmatrix} q_1 r_{11} p_1 & \cdots & q_1 r_{n1} p_n \\ \vdots & \ddots & \vdots \\ q_m r_{1m} p_1 & \cdots & q_m r_{nm} p_n \end{pmatrix} \otimes \begin{pmatrix} q'_1 r'_{11} p'_1 & \cdots & q'_1 r'_{n'1} p'_{n'} \\ \vdots & \ddots & \vdots \\ q'_{m'} r'_{1m'} p'_1 & \cdots & q'_{m'} r'_{n'm'} p'_{n'} \end{pmatrix}$$

is the Kronecker product with $(ij, i'j')$-entry $q_i r_{ji} p_j \otimes q'_{i'} r'_{j'i'} p'_{j'}$. We will write $\overline{\mathcal{C}(\mathcal{E})}^{ds}$ for the idempotent completion $\overline{\mathcal{C}(\mathcal{E})}$ extended by direct sums in this way.

Lemma 2. *Let $\mathcal{E}$ be a collection of $\mathcal{L}_n$-endomorphisms as above, and recall (3.2). Suppose Hom$(\beta, \gamma) = \mathrm{Hom}(\tilde{\beta}, \tilde{\gamma})$ in $\mathcal{L}_n$ for all $\beta, \gamma \in \mathcal{E}$, and that these are all finite-dimensional. Then $\overline{\mathcal{C}(\mathcal{E})}^{ds}$, the idempotent completion extended by direct sums, is a semisimple strict*

$\mathbb{C}$-linear tensor category with finite-dimensional hom-spaces. If $\mathcal{C}(\mathcal{E})$ is rigid, then so is $\overline{\mathcal{C}(\mathcal{E})}^{ds}$.

Proof. The category $\overline{\mathcal{C}(\mathcal{E})}^{ds}$ is manifestly $\mathbb{C}$-linear and strict. Since all $\mathrm{Hom}(\beta,\gamma)$ are finite-dimensional, so are all Hom-spaces (3.7) in $\overline{\mathcal{C}(\mathcal{E})}^{ds}$. Since the anti-linear involution $x \mapsto x'$ sends $\mathrm{Hom}(\tilde\beta,\tilde\gamma)$ to $\mathrm{Hom}(\gamma,\beta)$, and $\mathrm{Hom}(\tilde\beta,\tilde\gamma) = \mathrm{Hom}(\beta,\gamma)$ by hypothesis, then $x \mapsto x'$ bijectively maps $\mathrm{Hom}(\beta,\gamma)$ to $\mathrm{Hom}(\gamma,\beta)$. This implies that the (finite-dimensional) algebra $\mathrm{End}(((p_1,\beta_1),\ldots,(p_n,\beta_n)))$ is a $*$-algebra, and hence is semisimple. Then Corollary 2.3 of [33] tells us $\overline{\mathcal{C}(\mathcal{E})}^{ds}$ is a semisimple category.

Moreover, suppose $\mathcal{C}(\mathcal{E})$ is rigid. Then Lemma 3.1 of [33] says that its idempotent completion $\overline{\mathcal{C}(\mathcal{E})}$ is also rigid: e.g. $(p,\beta)^\vee = (p^\vee,\beta^\vee)$ where the dual morphism $p^\vee$ is defined in (3.6), and (co-)evaluation is $e_{p,\beta} = p^\vee\beta^\vee(p)\, e_\beta$ and $b_{(p,\beta)} = p\beta(p^\vee)\, b_\beta$. Hence $\overline{\mathcal{C}(\mathcal{E})}^{ds}$ is also rigid: take $((p_1,\beta_1),\ldots,(p_n,\beta_n))^\vee = ((p_1,\beta_1)^\vee,\ldots,(p_n,\beta_n)^\vee)$ with diagonal (co-)evaluations $e_{(\ldots,(p_i,\beta_i),\ldots)} = \mathrm{diag}(e_{(p_i,\beta_i)})$ etc. □

This condition $\mathrm{Hom}(\beta,\gamma) = \mathrm{Hom}(\widetilde{\beta},\widetilde{\gamma})$ is crucial for extending the (unitary) Cuntz algebra methods to the (not necessarily unitary) Leavitt setting. We show near the end of section 5 that this condition holds for the Haagerup–Izumi systems considered here, and the same argument should work for the near-group systems constructed in [12]. Nevertheless, Lemma 2 emphasises that semisimplicity in the Leavitt picture is *not* automatic, and this is very good: it means our context should be flexible enough to include non-semisimple examples such as those corresponding to the logarithmic CFTs discussed in the Introduction.

4. Non-unitary Haagerup–Izumi: deconstruction

Let G be any abelian group of odd order $\nu = 2n+1$, and define $\delta_\pm = (\nu \pm \sqrt{\nu^2+4})/2$, the two roots of $x^2 = 1+\nu x$. Recall the Haagerup–Izumi fusions (1.1). A main result (Theorem 1) of this paper associates to any system of algebra endomorphisms realising these fusions, a set of numerical invariants. The converse, which associates a system of endomorphisms and a fusion category to these same numerical invariants, is given next section.

Suppose α_g, ρ are algebra endomorphisms of an algebra $\mathcal{A}$ which realise the Haagerup–Izumi fusions. More precisely, this means

$$\alpha_g \circ \alpha_h = \alpha_{g+h}\,,\ \ \alpha_g \circ \rho = \rho \circ \alpha_{-g}\,, \tag{4.1}$$

$$\rho(\rho(x)) = sxs' + \sum\nolimits_g t_g \alpha_g(\rho(x)) t'_g\,, \tag{4.2}$$

where $s, s', t_g, t'_g \in \mathcal{A}$ satisfy $s's = 1$, $s't_g = t'_g s = 0$, $t'_g t_h = \delta_{g,h}$, and $1 = ss' + \sum_g t_g t'_g$. We do not assume $\mathcal{A}$ is a $*$-algebra. Equation (4.1) implies that each α_g is invertible. Note that we have the freedom to rescale the $\nu+1$ elements s, t_g arbitrarily and independently, provided we then rescale s', t'_g inversely. We also require α_g and $\alpha_g\rho = \alpha_g \circ \rho$ to be simple, i.e. that their intertwiners in $\mathcal{A}$ are $\mathrm{Hom}(\alpha_g\rho, \alpha_h\rho) = \mathrm{Hom}(\alpha_g, \alpha_h) = \mathbb{C}\delta_{g,h}$ and $\mathrm{Hom}(\alpha_g, \alpha_h\rho) = \mathrm{Hom}(\alpha_g\rho, \alpha_h) = 0$. This implies for instance that $\mathcal{A}$ has trivial centre, and that the representation $g \mapsto \alpha_g$ of G is faithful.

Unless G is cyclic (in which case $H^2_G(\mathrm{pt}; \mathbb{T}) = 1$), (4.1) can be generalised by twisting by 2-cocycles $\xi \in Z^2_G(\mathrm{pt}; \mathbb{T})$ and (1.1) will still hold, as explained e.g. in the proof of Theorem 1 in [12]. We will ignore this generalisation, as it is conceptually straightforward and merely makes the arithmetic a little messier, and our primary purpose with this paper is to explain how to capture non-unitary fusion categories by endomorphisms. Izumi [22] also ignored these cocycles, but the unpublished notes [23] introduces them (though of course in the unitary setting).

Equation (4.1) can be generalised to $\alpha_g \circ \alpha_h = ad(U_{g,h})\alpha_{g+h}$ for invertible $U_{g,h}$. When $\mathcal{A}$ is a C^*-algebra, these U's can be absorbed into the α's, but there is no reason to expect this to hold for more general $\mathcal{A}$. We will also ignore this generalisation of (4.1) here, for the same reasons as given last paragraph. But we return to this possibility in Section 7.1.

Theorem 1. *Let G, α_g, ρ and $s, s', t_g, t'_g \in \mathcal{A}$ be as above. Then*

$$\rho(s) = \delta_\pm^{-1} s + b \sum\nolimits_g t_g t_g \,, \quad \rho(s') = \delta_\pm^{-1} s' + \omega b \sum\nolimits_g t'_g t'_g \,, \tag{4.3}$$

$$\rho(t_g) = bst'_{-g} + \omega t_{-g} ss' + \sum\nolimits_{h,k} A_{h+g,k+g} t_h t_{h+k+g} t'_k \,, \tag{4.4}$$

$$\rho(t'_g) = \omega b t_{-g} s' + \overline{\omega} s s' t'_{-g} + \sum\nolimits_{h,k} A_{k+g,h+g} t_k t'_{g+h+k} t'_h \,, \tag{4.5}$$

$$\alpha_g(s) = s \,, \quad \alpha_g(s') = s' \,, \quad \alpha_g(t_h) = t_{h+2g} \,, \quad \alpha_g(t'_h) = t'_{h+2g} \,, \tag{4.6}$$

for some fixed sign $\pm$, where $b \in \{1/\sqrt{\omega\delta_\pm}\}$ and $\omega^3 = 1$. In particular, α_g and ρ restrict to algebra endomorphisms of the Leavitt algebra $\mathcal{L} = \mathcal{L}_{\nu+1}$ with generators s, s', t_g, t'_g. Moreover, $A_{g,h} \in \mathbb{C}$ satisfy

$$A_{g,h} = \omega A_{-h,g-h} = \overline{\omega} A_{h-g,-g} \,, \tag{4.7}$$

$$\sum\nolimits_h A_{h,0} = -\overline{\omega}\delta_\pm^{-1} \,, \tag{4.8}$$

$$\sum\nolimits_g A_{h+g,k} A_{k,g} = \delta_{h,0} - \delta_\pm^{-1}\delta_{k,0} \,, \tag{4.9}$$

$$\sum\nolimits_{l,m} A_{l,m} A_{l+g,h} A_{h+m,l+i} A_{i,k+m} = A_{h-g,i-g}\delta_{k,g} - \overline{\omega}\delta_\pm^{-1}\delta_{h,0} A_{i,k} - \omega\delta_\pm^{-1} A_{g,h}\delta_{i,0} \,. \tag{4.10}$$

We will show in Proposition 2 below that in fact

$$\overline{\omega} \sum\nolimits_m A_{m,g+h} A_{g,m+k} A_{h,m+l} = A_{g+l,k} A_{h+k,l} - \delta_\pm^{-1}\delta_{g,0}\delta_{h,0} \,, \tag{4.11}$$

for all $g, h, k, l \in G$. We expect that this can be used to derive the more complicated (4.10), but we haven't established this yet.

According to Izumi [22], a (unitary) Q-system corresponds to the special case of Theorem 1 with $\omega = 1$, $\delta_\pm = \delta_+$, $\overline{A_{g,h}} = A_{h,g}$, $A_{g,0} = \delta_{g,0} - 1/(\delta_+ - 1)$. In this case the quartic identity (4.10) can be replaced with the cubic identity (4.11). This special case corresponds to fusion categories coming from one of the even subsystems of a finite depth finite index subfactor.

Incidentally, it doesn't matter which square-root is chosen for b in Theorem 1: replacing $s \mapsto -s$, $s' \mapsto -s'$ shows b is equivalent to $-b$. This means that we can require without loss of generality that b lies on the positive halves of the real or imaginary axes. Which triples $(\pm, \omega, A)$ yield isomorphic fusion categories is answered below in Theorem 2, as is the question of unitarity.

Lemma 3. *Let ρ be any algebra endomorphism on $\mathcal{A}$ satisfying (4.1) and (4.2), and assume α_g and $\alpha_g\rho$ are all simple. Then $Hom(\alpha_g, \rho^2) = \mathbb{C}s\delta_{g,0}$, $Hom(\alpha_g\rho, \rho^2) = \mathbb{C}t_g$, $Hom(\rho^2, \alpha_g) = \mathbb{C}s'\delta_{g,0}$, and $Hom(\rho^2, \alpha_g\rho) = \mathbb{C}t'_g$. Moreover, $Hom(\rho^2, \alpha_g\rho^2) = \mathbb{C}ss'\delta_{g,0} +$ $\mathrm{span}_h\{t_{h+g}t'_h\}$.*

Proof. Directly from (4.2) we find $\rho^2(x)s = sx$, $\rho^2(x)t_g = t_g\alpha_g\rho(x)$, $s'\rho^2(x) = xs'$, and $t'_g\rho^2(x) = \alpha_g\rho(x)t'_g$. In other words, $s \in \mathrm{Hom}(\mathrm{id}, \rho^2)$, $t_g \in \mathrm{Hom}(\alpha_g\rho, \rho^2)$, $s' \in \mathrm{Hom}(\rho^2, \mathrm{id})$, and $t'_g \in \mathrm{Hom}(\rho^2, \alpha_g\rho)$.

Now suppose $r \in \mathrm{Hom}(\alpha_g, \rho^2)$. Then $s' \in \mathrm{Hom}(\rho^2, \mathrm{id})$ and $t'_h \in \mathrm{Hom}(\rho^2, \alpha_h\rho)$ immediately imply $s'r \in \mathrm{Hom}(\alpha_g, \mathrm{id}) = \mathbb{C}\delta_{g,0}$ and $t'_hr \in \mathrm{Hom}(\alpha_g, \alpha_h\rho) = 0$ by simplicity. Therefore $r = ss'r + \sum_h t_ht'_hr \in \mathbb{C}s\delta_{g,0}$, hence $\mathrm{Hom}(\alpha_g, \rho^2) = \mathbb{C}s\delta_{g,0}$.

Next, suppose $r \in \mathrm{Hom}(\rho^2, \alpha_g)$. Then $rs \in \mathrm{Hom}(\mathrm{id}, \alpha_g) = \mathbb{C}\delta_{g,0}$ and $rt_h \in \mathrm{Hom}(\alpha_h\rho, \alpha_g) = 0$, which forces $r \in \mathbb{C}s'\delta_{g,0}$ as before, and thus $\mathrm{Hom}(\rho^2, \alpha_g) = \mathbb{C}s'\delta_{g,0}$.

Now consider $r \in \mathrm{Hom}(\alpha_g\rho, \rho^2)$. Then $s'r \in \mathrm{Hom}(\alpha_g\rho, \mathrm{id}) = 0$ and $t'_hr \in \mathrm{Hom}(\alpha_g\rho, \alpha_h\rho) = \mathbb{C}\delta_{g,h}$, and thus $\mathrm{Hom}(\alpha_g\rho, \rho^2) = \mathbb{C}t_g$.

Similarly, let $r \in \mathrm{Hom}(\rho^2, \alpha_g\rho)$. Then $rs \in \mathrm{Hom}(\mathrm{id}, \alpha_g\rho) = 0$ and $rt_h \in \mathrm{Hom}(\alpha_h\rho, \alpha_g\rho) = \delta_{h,g}\mathbb{C}$, which gives us $\mathrm{Hom}(\rho^2, \alpha_g\rho) = \mathbb{C}t'_g$.

Finally, suppose $r \in \mathrm{Hom}(\rho^2, \alpha_g\rho^2)$. Then, using the invertibility of α and the calculation $\alpha_g\rho^2 = \rho\alpha_{-g}\rho = \rho^2\alpha_g$, we get $rs \in \mathrm{Hom}(\mathrm{id}, \alpha_g\rho^2) = \mathrm{Hom}(\alpha_g, \rho^2\alpha_g) = \mathbb{C}s\delta_{g,0}$. Similarly, $rt_h \in \mathrm{Hom}(\alpha_h\rho, \alpha_g\rho^2) = \mathrm{Hom}(\alpha_{h+g}\rho\alpha_g, \rho^2\alpha_g) = \mathbb{C}t_{h+g}$. This suffices to identify $\mathrm{Hom}(\rho^2, \alpha_g\rho^2)$ in the usual way. □

Note that because α_g is an algebra endomorphism and $s \in \mathrm{Hom}(\mathrm{id}, \rho^2)$, $\alpha_g(s) \in \mathrm{Hom}(\alpha_g, \alpha_g\rho^2)$. But $\mathrm{Hom}(\alpha_g, \alpha_g\rho^2) = \mathrm{Hom}(\alpha_g, \rho^2\alpha_g) = \mathrm{Hom}(\mathrm{id}, \rho^2)$ since α_g is invertible. By Lemma 3 this means $\alpha_g(s) = \psi(2g)s$ for some $\psi(2g) \in \mathbb{C}$ (the 2 is introduced for later convenience; because the order of G is odd, 2 is invertible). Because $\alpha_g\alpha_h = \alpha_{g+h}$, we see $\psi \in \widehat{G}$. From the Leavitt–Cuntz relation $s's = 1$, we obtain $\alpha_g(s') = \psi(-2g)s'$. Likewise, $\alpha_h(t_g) \in \mathrm{Hom}(\alpha_{h+g}\rho, \alpha_h\rho^2) = \mathrm{Hom}(\alpha_{g+2h}\rho, \rho^2) = \mathbb{C}t_{g+2h}$, and hence

$$\alpha_h(t_g) = \epsilon_h(g)t_{g+2h}$$

for some $\epsilon_h(g) \in \mathbb{C}$. Again, $\alpha_g\alpha_h = \alpha_{g+h}$ implies these numbers $\epsilon_h(g)$ are non-zero and satisfy

$$\epsilon_{h+k}(g) = \epsilon_h(g)\epsilon_k(g+2h)\,. \tag{4.12}$$

We can rescale $t_1, \ldots, t_{\nu-1}$ so that $\epsilon_h(0) = 1$ for all h. But from (4.12) with $g = 0$ this implies $\epsilon_k(2h) = 1$ for all $h, k \in G$, and invertibility of 2 then implies all $\epsilon_k(h) = 1$. From $t'_g t_g = 1$ we likewise get $\alpha_h(t'_g) = t'_{g+2h}$. Thus we know all α_g restrict to endomorphisms of the Leavitt algebra $\mathcal{L}_{\nu+1}$ generated by the s, s', t_g, t'_g.

Since $s \in \mathrm{Hom}(\mathrm{id}, \rho^2)$ and ρ is an endomorphism, $\rho(s) \in \mathrm{Hom}(\rho, \rho^3)$. Hence $s'\rho(s) \in \mathrm{Hom}(\rho, \rho) = \mathbb{C}$ and $t'_0\rho(s) \in \mathbb{C}t_0$. Write $s'\rho(s) = a$ and $t'_0\rho(s) = bt_0$ for some $a, b \in \mathbb{C}$. Hitting the latter equation with α_h, we get $t'_{2h}\alpha_h(\rho(s)) = bt_{2h}$, i.e. $t'_g\rho(s) = \psi(2g)bt_g$. Likewise, $\rho(s')s = a'$ and $\rho(s')t_g = b't'_g$ for some $a', b' \in \mathbb{C}$. We thus obtain from $\rho(s) = ss'\rho(s) + \sum_g t_g t'_g \rho(s)$ that

$$\rho(s) = as + b\sum\nolimits_g \psi(g)t_g t_g\,, \quad \rho(s') = a's' + b'\sum\nolimits_g \psi(-g)\,t'_g t'_g\,. \tag{4.13}$$

The computation of $\rho(t_g)$ is similar. First note that $\rho(t_0) \in \mathrm{Hom}(\rho^2, \rho^3)$, so $s'\rho(t_0) \in \mathrm{Hom}(\rho^2, \rho) = \mathbb{C}t'_0$ and $t'_h\rho(t_0) \in \mathrm{Hom}(\rho^2, \alpha_h\rho^2) = \mathrm{span}\{\delta_{h,0}ss', t_k t'_{k-h}\}$, using Lemma 3. Write $s'\rho(t_0) = ct'_0$ and $t'_h\rho(t_0) = \delta_{h,0}dss' + \sum_k A_{h,k}t_{h+k}t'_k$, for complex numbers $c, d, A_{h,k}$. Then $\rho(t_0) = cst'_0 + dt_0ss' + \sum_{h,k} A_{h,k}t_h t_{h+k}t'_k$. The calculation for $\rho(t'_0)t_h$ is identical, and involves complex numbers $c', d', A'_{h,k}$. Hitting these with $\alpha_{-g/2}$ yields

$$\rho(t_g) = \psi(-g)cst'_{-g} + dt_{-g}ss' + \sum\nolimits_{h,k} A_{h+g,k+g}t_h t_{g+h+k}t'_k\,, \tag{4.14}$$

$$\rho(t'_g) = \psi(g)c't_{-g}s' + d'ss't'_{-g} + \sum\nolimits_{h,k} A'_{h+g,k+g}t_k t'_{g+h+k}t'_h\,. \tag{4.15}$$

Thus we also know ρ restricts to an endomorphism of the Leavitt algebra $\mathcal{L}_{\nu+1}$ generated by the s, s', t_g, t'_g.

Thus the $\mathcal{A}$-endomorphism ρ is determined from the $2\nu^2+8$ parameters $a, a', b, b', c, c', d, d', A_{h,k}, A'_{h,k}$, as well as the character $\psi \in \widehat{G}$. However there are several consistency conditions, coming from (4.2) and also the fact that ρ being an endomorphism must preserve the Leavitt–Cuntz relations. To compute various expressions in $\mathcal{L}_{\nu+1}$, it is convenient to collect our equations

$$s'\rho(s) = a\,,\ t'_g\rho(s) = b\psi(g)t_g\,, \tag{4.16}$$

$$s'\rho(t_g) = \psi(-g)ct'_{-g}\,,\ t'_g\rho(t_h) = d\delta_{g,-h}ss' + \sum\nolimits_k A_{g+h,k+h}t_{g+h+k}t'_k\,, \tag{4.17}$$

$$s'\rho(t'_g) = d's't'_{-g}\,,\ t'_h\rho(t'_g) = \psi(g)c'\delta_{g,-h}s' + \sum\nolimits_k A'_{k+g,h+g}t'_{g+h+k}t'_k\,. \tag{4.18}$$

Implicit in the following is Lemma 1, which permits us to compare corresponding coefficients of an expression in $\mathcal{L}_{\nu+1}$ in reduced form (i.e. replace any occurrence of ss' with $1 - \sum_g t_g t'_g$).

Because ρ satisfies (4.2), we must have $s'\rho(\rho(x)) = xs'$. But if instead we compute $s'\rho(\rho(s)) = ss'$ directly from (4.13) and (4.14), using (4.16) and (4.17), we obtain

$$s'\rho(\rho(s)) = a^2 + bc\left(\nu dss' + \sum\nolimits_g A_{0,g} \sum\nolimits_k t_k t'_k\right) .$$

Comparing these expressions for $s'\rho^2(s)$, and performing the analogous calculation for $\rho(\rho(s'))s = ss'$, we obtain

$$bc\sum\nolimits_g A_{0,g} = -a^2 = \nu bcd - 1\,, \quad b'c'\sum\nolimits_g A'_{0,g} = -a'^2 = \nu b'c'd' - 1\,. \tag{4.19}$$

Likewise, the st'_0 coefficient of $t'_0\rho(\rho(s)) = \rho(s)t'_0$ becomes $a = bcd$ (and similarly we get $a' = b'c'd'$). Substituting this into (4.19), we obtain $-a^2 = \nu a - 1$ and so $a \in \{1/\delta_\pm\}$ (similarly for a').

In particular, $a, a' \neq 0$, so also $b, b', c, c', d, d' \neq 0$. Hitting $a = s'\rho(s)$ with α_g, we obtain

$$a = \alpha_g(s')\rho(\alpha_{-g}(s)) = \psi(-2g)s'\rho(\psi(-2g)s) = \psi(-4g)a$$

for all $g \in G$. Thus, since the order ν of G is odd, we have that ψ is identically 1. We thus recover (4.6).

Other coefficients of $t'_0\rho(\rho(s)) = \rho(s)t'_0$ we need now give

$$d\sum\nolimits_h A_{h,0} = -a\,, \quad d'\sum\nolimits_h A'_{h,0} = -a'\,, \tag{4.20}$$

$$\sum\nolimits_h A_{h,k}A_{k,k'+h} - d\delta_{k,0}\sum\nolimits_h A_{h,0} = \delta_{k',0}\,. \tag{4.21}$$

From $\rho(s')\rho(s) = 1$ we obtain $1 = (a's' + b'\sum_g t'_g t'_g)(as + b\sum_g t_g t_g)$, i.e.

$$aa' + \nu bb' = 1\,. \tag{4.22}$$

The $st'_g t'_g$, $t_g t_g s'$, constants, $t_g t'_g$, and $t_h t_{h+k} t'_{l+k} t'_l$ terms of the Leavitt–Cuntz relation $1 = \rho(s)\rho(s') + \sum_g \rho(t_g)\rho(t'_g)$ give respectively

$$c\sum\nolimits_h A'_{h,0} = -ab'\,, \quad c'\sum\nolimits_h A_{h,0} = -ba'\,, \tag{4.23}$$

$$aa' + cc'\nu = 1\,, \quad dd' = 1\,, \tag{4.24}$$

$$\sum\nolimits_g A_{h+g,k}A'_{g,k} = \delta_{h,0} - bb'\delta_{k,0}\,. \tag{4.25}$$

Note that we still have the freedom to rescale $s \mapsto \lambda s$ and $s' \mapsto s/\lambda$; choose λ so that $c = b$. Then $bb' = cc'$ (obtained by comparing (4.22) with (4.24)) implies $b' = c'$. Now, $aa' = (bcd)(b'c'd') = (bb')^2$, so (4.22) implies $bb' \in \{1/\delta_\pm\}$. However, if $a \neq a'$, then $aa' = 1/(\delta_+\delta_-) = -1$, contradicting our value for bb'. Thus $a = a' = bb'$. Moreover, comparing (4.23) and (4.20) gives $b' = bd$.

Multiplying (4.25) by $A_{k,h+m}$ and summing over h using (4.21) gives

$$A'_{g,h}+\delta_{k,0}d\left(\sum{}_g A'_{g,0}\right)\left(\sum{}_h A_{h,0}\right)=A_{h,g}-\delta_{k,0}bb'\sum{}_h A_{0,h}\,. \tag{4.26}$$

But the terms proportional to $\delta_{k,0}$ are $d\,(-a'/d')(-a/d)=da^2$ and $-bb'\,(-a^2/bc)=a^2b'/b$, which we now know are equal. Thus $A'_{g,h}=A_{h,g}$ for all $g,h\in G$.

The $st'_{h-g}t'_h$ coefficient of $t'_0\rho^2(t_g)=\rho(t_g)t'_0$ is

$$c\delta_{h,0}=d^2ab'\delta_{0,g}+cd\sum{}_k A_{g,k+g}A'_{h+k,-g}\,. \tag{4.27}$$

Multiplying (4.27) by $A_{h+l,-g}$ and summing over h using (4.25) collapses to $A_{l,-g}=dA_{g,l+g}$, which recovers (4.7); because the permutation $(l,-g)\mapsto(g,l+g)$ is order 3, d must be a 3rd root ω of 1.

We obtain (4.8) and (4.21) from (4.20) and (4.9). Finally, (4.10) arises from the $t_ht_{i-g+h}t'_{k+i}t'_k$ coefficient of $t'_0\rho(\rho(t_{-g}))=\rho(t_{-g})t'_0$. This completes our derivation of Theorem 1.

5. Non-unitary Haagerup–Izumi: reconstruction

This section is devoted to a proof of the following theorem, another main result of our paper. Recall $\delta_\pm=(\nu\pm\sqrt{\nu^2+4})/2$.

Theorem 2. *Choose any finite abelian group G of odd order ν.*

- **(a)** *Let $b\in\{1/\sqrt{\omega\delta_\pm}\}$ and $\omega^3=1$, and choose any solution $A_{g,h}$ to (4.7)–(4.10). Define the values of ρ and α_g on the generators s,s',t_g,t'_g by (4.3)–(4.6). Then these extend to algebra endomorphisms ρ,α_g on the Leavitt algebra $\mathcal{L}$ generated by s,s',t_g,t'_g. Then $\overline{\mathcal{C}(\{\alpha_g\rho^n\})}^{ds}$, the idempotent completion extended by direct sums as described in section 3, is a strict spherical fusion category we'll denote by $\mathcal{C}(G;\pm,\omega,A)$. The simple objects of this category are $\alpha_g=(1,\alpha_g)$ and $\alpha_g\rho=(1,\alpha_g\rho)$ up to equivalence, and they satisfy the Haagerup–Izumi fusions (1.1). The categorical dimensions of α_g are 1 and of $\alpha_g\rho$ are $\delta_\pm$.*
- **(b)** *Two such fusion categories $\mathcal{C}(G^{(i)};\pm^{(i)},\omega^{(i)},A^{(i)})$ are equivalent as tensor categories iff $\pm^{(1)}=\pm^{(2)}$, $\omega^{(1)}=\omega^{(2)}$ and there is a group isomorphism $\pi:G^{(1)}\to G^{(2)}$ such that $A^{(1)}_{g,h}=A^{(2)}_{\pi g,\pi h}$ for all $g,h\in G^{(1)}$.*
- **(c)** *$\mathcal{C}(G;\pm,\omega,A)$ is unitary iff $\pm=+$ and A is a hermitian matrix: $A_{g,h}=\overline{A_{h,g}}$ for all $g,h\in G$. $\mathcal{C}(G;\pm,\omega,A)$ is hermitian iff $A_{g,h}$ is hermitian.*

We will learn below that the simple objects are all of the form $(uu',\alpha_g\rho^n)$ or $(vv',\alpha_g\rho^n)$ for certain monomials $u=u^{g,n}_{h,i},v=v^{g,n}_{h,j}$ recursively constructed below. The modular data S,T associated to the double of $\mathcal{C}(G;\pm,\omega,A)$ is computed next section.

By Corollary 1, it is trivial that the α_g defined by (4.6) are algebra endomorphisms of $\mathcal{L}$. Similarly, to show that ρ satisfying (4.3)–(4.5) extends to an algebra endomorphism of $\mathcal{L}$, it suffices to verify that the values of $\rho(s)$ etc preserve the Leavitt–Cuntz relations. It is readily verified that these all reduce to the identities $b^4 + \nu\omega b^2 = 1$, (4.8), (4.9), and

$$\sum\nolimits_g A_{0,g} = -\omega\delta_{\pm}^{-1} \tag{5.1}$$

(the latter follows from (4.8) and (4.7)). Thus ρ is an algebra endomorphism.

To verify that $\alpha_g\rho = \rho\alpha_{-g}$, we need to show that $\alpha_g(\rho(x)) = \rho(\alpha_{-g}(x))$ for $x = s, s', t_h, t'_h$. This is trivial to verify: e.g.

$$\alpha_g(\rho(t'_l)) = bt_{-l+2g}s' + ss't'_{-l+2g} + \sum_{h,k} A_{k+l,h+l}t_{k+2g}t'_{l+h+k+2g}t'_{h+2g} = \rho(t'_{l-2g})\,. \tag{5.2}$$

To see that ρ satisfies (4.2), it suffices to verify that $s'\rho(\rho(x)) = xs'$, $t'_g\rho(\rho(x)) = \alpha_g\rho(x)t'_g$, $\rho(\rho(y))s = sy$ and $\rho(\rho(y))t_g = t_g\alpha_g\rho(y)$ for all $g \in G$, $x \in \{s, t_h\}$ and $y \in \{s', t'_h\}$. This is because those equations imply using $\rho^2(x) = (ss' + \sum_g t_g t'_g)\rho^2(x) = \rho^2(x)(ss' + \sum_g t_g t'_g)$ that (4.2) holds when x is any generator, and this suffices to prove (4.2) for all x because both sides of (4.2) are manifestly endomorphisms. In fact, by α_g-equivariance, it suffices to establish these for $g = 0$. All of these equations reduce to $b^4 + \nu\omega b^2 = 1$, (4.8), (4.9), and (5.1), except for the following.

The equation $s'\rho(\rho(t_g)) = t_g s'$ yields the equations

$$1 = 2\varpi b^4 + \omega b^2 \sum\nolimits_{h,k} A_{h,k}A_{k,h}\,, \tag{5.3}$$

$$-\omega b^2 A_{h,k} - \varpi b^2\delta_{h,0} = \sum\nolimits_{\ell,m} A_{\ell,m}A_{m,\ell+h}A_{h,k+m}\,. \tag{5.4}$$

The former follows from $\sum_{h,k} A_{h,k}A_{k,h} = \nu - \nu b^2$, which in turn follows from (4.9). The latter follows directly from (4.9). The equation $t'_0\rho(\rho(t_g)) = \rho(t_g)t'_0$ gives (4.10) as well as

$$\sum\nolimits_{l,m} A_{l,m}A_{l+g,k}A_{k+m,l} = -b^2\delta_{k,0} - \omega b^2 A_{g,k}\,, \tag{5.5}$$

$$\sum\nolimits_k A_{k+g,h}A_{k,-h} = \omega\delta_{h,g} - \varpi b^2\delta_{h,0}\,, \tag{5.6}$$

$$\sum\nolimits_m A_{g,m+g}A_{-g,m+k} = \varpi\delta_{k,0} - b^2\delta_{0,g}\,, \tag{5.7}$$

which follow from (4.9) and (4.7).

The simplicity of ρ etc is established by the following proposition.

Proposition 1. *Let ρ be as above. Then for each $g, h \in G$,* $\mathrm{Hom}(\alpha_g\rho, \alpha_h\rho) = \mathrm{Hom}(\alpha_g, \alpha_h) = \mathbb{C}\delta_{g,h}$ *and* $\mathrm{Hom}(\alpha_g, \alpha_h\rho) = \mathrm{Hom}(\alpha_g\rho, \alpha_h) = 0$.

Proof. Write $a = \delta_{\pm}^{-1}$. Choose any $x \in \mathcal{L}$ commuting with $\rho(s) = as + b\sum_g t_g t_g$. We will begin by proving that such an x must be a polynomial in $\rho(s)$. Write x in reduced form (recall Lemma 1). We can assume without loss of generality that no term in x is a scalar times a power of s, i.e. cs^l, since otherwise we could replace x with $x - c(\rho(s)/a)^l$ (the result will still lie in $\mathcal{L}$ and commute with $\rho(s)$, and will be in $\mathbb{C}[\rho(s)]$ iff x is). Suppose for contradiction that $x \neq 0$.

Assume first that not all terms in x begin with s'. Amongst those terms, let $w = s^l w' \neq 0$ be the sum of all terms with the maximal leading string of s's (l may be 0). Then $\rho(s)x$ contains the terms $asw = as^{l+1}w'$, and these are reduced and have longer leading strings of s's than any other terms in $\rho(s)x - x\rho(s)$ (since no term can be a pure power of s). Being reduced, these terms asw cannot cancel anything, contradicting $\rho(s)x = x\rho(s)$.

It remains to consider $x = s'x'$. Then every term in $x' \neq 0$ involves only s''s and t'_g's (since x is reduced). Then $\rho(s)x - x\rho(s)$ when reduced contains terms $-a\sum_h t_h t'_h s' x'$ with leading factors $t_h t'_h$. Again, these terms cannot cancel, which contradicts $\rho(s)x = x\rho(s)$.

These contradictions mean $x = 0$. Thus any $x \in \mathcal{L}$ commuting with $\rho(s)$ must be a polynomial in $\rho(s)$, and hence can contain no s', t'_k. Likewise, any $x \in \mathcal{L}$ commuting with $\rho(s')$ must be a polynomial in $\rho(s')$, and thus contains no s, t_k. Together, they tell us that any x commuting with both $\rho(s)$ and $\rho(s')$ must be a scalar.

Now suppose $x\alpha_g\rho(y) = \alpha_h\rho(y)x$ for all y. Then taking $y = s$ tells us $x\rho(s) = \rho(s)x$, since $\alpha_g\rho(s) = \rho(\alpha_{-g}s) = \rho(s)$, while taking $y = s'$ tells us $x\rho(s') = \rho(s')x$. Therefore $x \in \mathrm{Hom}(\alpha_g\rho, \alpha_h\rho)$ must again be a scalar $\lambda \in \mathbb{C}$. Now, for $\lambda \neq 0$, $\lambda\alpha_g\rho(t_0) = \alpha_h\rho(t_0)\lambda$ iff $\lambda\rho(t_{-2g}) = \lambda\rho(t_{-2h})$, iff $g = h$ (since $b \neq 0$). Thus $\mathrm{Hom}(\alpha_g\rho, \alpha_h\rho) = \delta_{g,h}\mathbb{C}$.

Now turn to $x \in \mathrm{Hom}(\alpha_g, \alpha_h)$, i.e. $x\alpha_g(y) = \alpha_h(y)x$ for all $y \in \mathcal{L}$. In particular, $xs = sx$ and $xs' = s'x$. By the identical argument as above, the former requires $x \in \mathbb{C}[s]$ while the latter requires $x \in \mathbb{C}[s']$, and thus x is a scalar $\lambda \in \mathbb{C}$. Of course, $\lambda \neq 0$ intertwines α_g and α_h iff $g = h$, by evaluating at $y = t_0$. Hence $\mathrm{Hom}(\alpha_g, \alpha_h)$.

Finally, suppose $x \in \mathrm{Hom}(\alpha_g\rho, \alpha_h)$ and $x \neq 0$ is reduced. Then e.g. $x\rho(s) = sx$. Assume first that at least one term in x does not begin with s'. Amongst those terms, let y be one with a maximal string of leading s's (this string may be empty, if no term in x begins with s). Then sy will be a reduced term in sx, and the only reduced terms in $x\rho(s)$ with a leading string of s's of similar length are those which are pure monomials in s. So $y = rs^n$ for some $n \geq 0$ and some non-zero scalar r. But even those y won't work: the reduced terms in $sx - x\rho(s)$ corresponding to y are $rs^{n+1} - ars^{n+1}$, which can never vanish because $a \neq 1$. If instead all terms in x begin with an s', then none of them end with an s, so repeat this argument with $x\rho(s') = s'x$. The proof that $\mathrm{Hom}(\alpha_g, \alpha_h\rho) = 0$ is identical. □ to Proposition 1

Recall the category $\mathcal{END}(\mathcal{L})$ defined in section 3. Let $\mathcal{E}$ consist of all monomials of the form $\alpha_g\rho^n$. Since $(\alpha_g\rho^m)(\alpha_h\rho^n) = \alpha_{g\pm h}\rho^{m+n}$, the set $\mathcal{E}$ is closed under composition. Let $\mathcal{C}(\mathcal{E})$ be the subcategory of $\mathcal{END}(\mathcal{L})$ with objects $\alpha_g\rho^n$. We want to show $\mathcal{C}(\mathcal{E})$ is rigid. Define $(\alpha_g\rho^{2k+1})^\vee = \alpha_g\rho^{2k+1}$ and $(\alpha_g\rho^{2k})^\vee = \alpha_{-g}\rho^{2k}$. Then $(\alpha_g\rho^n)^\vee(\alpha_g\rho^n) = (\alpha_g\rho^n)(\alpha_g\rho^n)^\vee = \rho^{2n}$ for all $g \in G, n \geq 0$. Define $e_{\alpha_g\rho^n} = \omega^n b^{-n} s'\rho(s')\cdots\rho^{n-1}(s')$ and $b_{\alpha_g\rho^n} = \omega^n b^{-n}\rho^{n-1}(s)\cdots\rho(s)s$. Since $s' \in \mathrm{Hom}(\rho^{k+2},\rho^k)$ (this is a special case of $s'\rho^2(x) = xs'$), $\rho^m(s') \in \mathrm{Hom}(\rho^{n+2},\rho^n)$ for any $m \leq n$ follows because ρ is an endomorphism. Therefore, $e_{\alpha_g\rho^n} \in \mathrm{Hom}(\rho^{2n},\mathrm{id})$ as required. Likewise, $b_{\alpha_g\rho^n} \in \mathrm{Hom}(\mathrm{id},\rho^{2n})$. To see that $e_{\alpha_g\rho^n}, b_{\alpha_g\rho^n}$ satisfy (3.4), first note that for any $k \geq l$,

$$\rho^k(s')\rho^l(s) = \rho^l(\rho^{k-l}(s')s) = \begin{cases} 1 & k = l \\ \omega b^2 & k = l+1 \\ \rho^l(s)\rho^{k-2}(s') & k \geq l+2 \end{cases}. \tag{5.8}$$

Using this, it is easy to see that for any $n \geq 2$, we have

$$\begin{aligned}&\rho^n(s')\rho^{n+1}(s')\cdots\rho^{2n-1}(s')\rho^{n-1}(s)\cdots\rho(s)s \\ &\quad = \omega b^2\rho^{n-1}(s')\rho^n(s')\cdots\rho^{2n-3}(s')\rho^{n-2}(s)\cdots\rho(s)s\,,\end{aligned}$$

which by an easy induction on n gives the first equation of (3.4). The second equation in (3.4) is handled analogously. Thus $\mathcal{C}(\mathcal{E})$ is rigid, with (co)evaluations e, b.

We want to apply Lemma 2. That means we must verify first that $\mathrm{Hom}(\alpha_g\rho^m,\alpha_h\rho^n) = \mathrm{Hom}(\widetilde{\alpha_g\rho^m},\widetilde{\alpha_h\rho^n})$ in $\mathcal{END}(\mathcal{L})$, where $\tilde\beta(x) = \beta(x')'$ is defined by (3.2). Note that $\tilde{\alpha_g} = \alpha_g$ (i.e. α_g is a $*$-map), but $\tilde\rho$ is defined by (4.3)–(4.5) using the adjoint $\overline{A}_{h,g}$ in place of $A_{g,h}$, $\overline{b}$ in place of b, and $\overline{\omega}$ in place of ω. It is manifest that $\tilde\rho$ is an endomorphism of $\mathcal{L}$ satisfying (4.2).

We have $\widetilde{\alpha_g\rho^n} = \alpha_g\tilde\rho^n$. An easy induction from (4.2) (replacing x there with $\rho^{n-2}(x)$ and hitting with α_g) verifies

$$\alpha_g\rho^n(x) = \sum\nolimits_{h,i} u^{g,n}_{h,i}\alpha_h(x)u^{g,n}_{h,i}{}' + \sum\nolimits_{k,j} v^{g,n}_{k,j}\alpha_k\rho(x)v^{g,n}_{k,j}{}'\,, \tag{5.9}$$

$$\alpha_g\tilde\rho^n(x) = \sum\nolimits_{h,i} u^{g,n}_{h,i}\alpha_h(x)u^{g,n}_{h,i}{}' + \sum\nolimits_{k,j} v^{g,n}_{k,j}\alpha_k\tilde\rho(x)v^{g,n}_{k,j}{}'\,, \tag{5.10}$$

where $u^{g,n}_{h,i} \in \mathrm{Hom}(\alpha_h,\alpha_g\rho^n) \cap \mathrm{Hom}(\alpha_h,\alpha_g\tilde\rho^n)$ and $v^{g,n}_{k,j} \in \mathrm{Hom}(\alpha_k\rho,\alpha_g\rho^n) \cap \mathrm{Hom}(\alpha_k\tilde\rho,\alpha_g\tilde\rho^n)$ are (finitely many) monomials in the Leavitt generators s, t_l and (for each fixed pair g,n) the collection $\{u^{g,n}_{h,i}, u^{g,n}_{h,i}{}', v^{g,n}_{k,j}, v^{g,n}_{k,j}{}'\}$ together satisfy the Leavitt–Cuntz relations $u^{g,n}_{h,i}{}'u^{g,n}_{k,j} = \delta_{i,j}\delta_{h,k}$ etc. More precisely, $\{u^{g,n+1}_{h,\star}\} = \{v^{g,n}_{h,i}\}_i$ and $\{v^{g,n+1}_{k,\star}\} = \{u^{g,n}_{k,i}\}_i \cup \{v^{g,n}_{h,j}t_{k-h}\}_{j,h}$.

Certainly $\mathrm{Hom}(\alpha_g\rho^n,\alpha_{g'}\rho^{n'})$ contains all $u^{g',n'}_{h,i'}u^{g,n}_{h,i}{}'$ and $v^{g',n'}_{k,j'}v^{g,n}_{k,j}{}'$. In fact, we will show now using simplicity (Proposition 1) that together they span that Hom-space. To see this, choose any $x \in \mathrm{Hom}(\alpha_g\rho^n,\alpha_{g'}\rho^{n'})$. Then $u^{g,n}_{h,i}{}'xu^{g',n'}_{h',i'} \in \mathrm{Hom}(\alpha_h,\alpha_{h'}) = \mathbb{C}\delta_{h,h'}$; when $h = h'$, call this number $q_{h;i,i'}$. Likewise, $v^{g,n}_{h,i}{}'xv^{g',n'}_{h',i'} \in \mathrm{Hom}(\alpha_h\rho,\alpha_{h'}\rho) = \mathbb{C}\delta_{h,h'}$; when $h = h'$, call this number $r_{h;i,i'}$. Moreover, $u^{g,n}_{h,i}{}'xv^{g',n'}_{h',i'} = v^{g,n}_{h,i}{}'xu^{g',n'}_{h',i'} = 0$ since $\mathrm{Hom}(\alpha_h,\alpha_{h'}\rho) = \mathrm{Hom}(\alpha_h\rho,\alpha_{h'}) = 0$. Thus $x = (\sum_{h,i} u^{g',n'}_{h,i}u^{g,n}_{h,i}{}' + v^{g',n'}_{h,i}v^{g,n}_{h,i}{}')x$ $(\sum_{h',i'} u^{g',n'}_{h',i'}u^{g,n}_{h',i'}{}' + v^{g',n'}_{h',i'}v^{g,n}_{h',i'}{}') = \sum_{h,i,i'}(q_{h;i',i}u^{g',n'}_{h,i}u^{g,n}_{h,i}{}' + r_{h;i',i}v^{g',n'}_{h,i}v^{g,n}_{h,i}{}')$. Thus

$\mathrm{Hom}(\alpha_g\rho^n,\alpha_{g'}\rho^{n'}) = \mathrm{span}_{h,i,i'}\{u^{g',n'}_{h,i'}u^{g,n\,\prime}_{h,i}, v^{g',n'}_{h,i'}v^{g,n\,\prime}_{h,i}\}$. The identical argument shows $\mathrm{Hom}(\alpha_g\tilde\rho^n,\alpha_{g'}\tilde\rho^{n'})$ is also spanned by the same elements, and so those Hom-spaces are identical (and finite-dimensional). Thus Lemma 2 applies.

Recall $\overline{\mathcal{C}(\mathcal{E})}^{ds}$, the idempotent completion of $\mathcal{C}(\mathcal{E})$ extended by direct sums. Note that all $u^{g,n}_{h,i}u^{g,n\,\prime}_{h,i}, v^{g,n}_{k,j}v^{g,n\,\prime}_{k,j}$ are idempotents in $\mathrm{End}(\alpha_g\rho^n)$, thanks to the Leavitt–Cuntz relations. Enumerate these $p_1,\dots,p_N$. Then $p_i\mathrm{End}(\alpha_g\rho^n)p_j = \delta_{ij}\mathbb{C}p_i$ and $\sum_i p_i = \mathrm{id}$, using the above spanning set (in fact basis) for $\mathrm{End}(\alpha_g\rho^n)$, so the p_i form a complete set of minimal idempotents in $\mathrm{End}(\alpha_g\rho^n)$. All $(p_i,\alpha_g\rho^n)$ are objects in $\overline{\mathcal{C}(\mathcal{E})}^{ds}$. Since $\mathrm{End}(p_i,\alpha_g\rho^n) := p_i\mathrm{End}(\alpha_g\rho^n)p_i = \mathbb{C}p_i$ is 1-dimensional, the $(p_i,\alpha_g\rho^n)$ are simple in $\overline{\mathcal{C}(\mathcal{E})}^{ds}$. These $(p_i,\alpha_g\rho^n)$ (as i,g,n vary) exhaust all simple objects in $\overline{\mathcal{C}(\mathcal{E})}^{ds}$, as any other idempotent in $\mathrm{End}(\alpha_g\rho^n)$ is a disjoint sum of the p_i. Moreover, $(u^{g,n}_{h,i}u^{g,n\,\prime}_{h,i},\alpha_g\rho^n)$ and $(1,\alpha_h)$ are isomorphic, with isomorphism $u^{g,n}_{h,i}$ and inverse $u^{g,n\,\prime}_{h,i}$, since $u^{g,n}_{h,i}u^{g,n\,\prime}_{h,i}$ is the identity in $\mathrm{End}((u^{g,n}_{h,i}u^{g,n\,\prime}_{h,i},\alpha_g\rho^n))$. Likewise, $(v^{g,n}_{k,j}v^{g,n\,\prime}_{k,j},\alpha_g\rho^n)$ and $(1,\alpha_k\rho)$ are isomorphic. We thus get a fusion category, because there are only finitely many isomorphism classes of simple objects, namely the $[(1,\alpha_g)],[(1,\alpha_g\rho)]$.

To show that $\overline{\mathcal{C}(\mathcal{E})}^{ds}$ is pivotal, note first that $(\alpha_g\rho^n)^{\vee\vee} = \alpha_g\rho^n$. We want to show also that the double-dual on all intertwiner spaces $\mathrm{Hom}(\alpha_g\rho^n,\alpha_{g'}\rho^{n'})$ is also the identity map. We must be careful here (and elsewhere) to keep track of the Hom-space we are working in by writing $(\xi|x|\eta)$ for $x\in\mathrm{Hom}(\xi,\eta)$. For convenience abbreviate $1_\xi = (\xi|1|\xi)$, $s = (\mathrm{id}|s|\rho^2)$, $s' = (\rho^2|s'|\mathrm{id})$, $t_g = (\alpha_g\rho|t_g|\rho^2)$ and $t'_g = (\rho^2|t'_g|\alpha_g\rho)$. We can compute directly from (3.6) that $(1_\xi)^\vee = 1_{\xi^\vee}$, $s^\vee = s'$, $t_g^\vee = t'_g$, $s'^\vee = s$, and $t_g'^\vee = t_g$, and so the double-dual leaves unchanged all of these. But the double-dual is a monoidal functor, so it will also leave unchanged the morphisms $(\alpha_k\rho^l|s|\alpha_k\rho^{l+2}) = s\otimes 1_{\alpha_k\rho^l}$, $(\alpha_{k+h}\rho^{l+1}|t_{k+2h}|\alpha_h\rho^{l+2}) = t_{k+2h}\otimes 1_{\alpha_h\rho^l}$, $(\alpha_k\rho^{l+2}|s'|\alpha_k\rho^{l+2}) = s'\otimes 1_{\alpha_k\rho^l}$, and $(\alpha_h\rho^{l+2}|t'_{k+2h}|\alpha_{k+h}\rho^{l+1}) = t'_{k+2h}\otimes 1_{\alpha_h\rho^l}$. By writing $u^{g,n}_{h,i}\in\mathrm{Hom}(\alpha_h,\alpha_g\rho^n)$ and $v^{g,n}_{h,i}\in\mathrm{Hom}(\alpha_h\rho,\alpha_g\rho^n)$ as monomials in s,t_k, they can be written as a sequence of compositions of these morphisms $(\alpha_k\rho^l|s|\alpha_k\rho^{l+2})$ and $(\alpha_{k+h}\rho^{l+1}|t_{k+2h}|\alpha_h\rho^{l+2})$ (this is manifest in the recursions given earlier). Hence the double-dual also leaves unchanged $u^{g,n}_{h,i}$ and $v^{g,n}_{h,i}$. Identical conclusions applies to $u^{g,n\prime}_{h,i}$ and $v^{g,n\prime}_{h,i}$, and hence to the compositions $u^{g',n'}_{h,i'}u^{g,n\prime}_{h,i}$ and $v^{g',n'}_{h,i'}v^{g,n\prime}_{h,i}$. But those compositions span $\mathrm{Hom}(\alpha_g\rho^n,\alpha_{g'}\rho^{n'})$. Thus the double-dual fixes every morphism $r\in\mathrm{Hom}(\alpha_g\rho^n,\alpha_{g'}\rho^{n'})$. From this we get that the double-dual functor $X\mapsto X^{\vee\vee}$ is the identity functor on $\overline{\mathcal{C}(\mathcal{E})}^{ds}$, and so $\overline{\mathcal{C}(\mathcal{E})}^{ds}$ is pivotal. The dimension calculation is now trivial: $e_{\alpha_g}b_{\alpha_g^\vee} = 1$ and $e_{(\alpha_g\rho^n)^\vee}b_{\alpha_g\rho^n} = \overline{\varpi}^n b^{-2n} = \delta^n_\pm$, from which we read off that X and $X^\vee$ have the same dimension for any simple X. This means that $\mathcal{C}$ is spherical.

Now turn to the proof of part (b) of Theorem 2. Suppose there is a tensor category equivalence between $\mathcal{C}(G^{(i)};\pm^{(i)},\omega^{(i)},A^{(i)})$. Because $\alpha^{(1)} = (1,\alpha^{(1)})$ is simple, the equivalence must send $\alpha^{(1)}_g$ to $(p^{(2)},\alpha^{(2)}_h\rho^{(2)\,m})$ for some (minimal) idempotent $p^{(2)}$ and some $\alpha^{(2)}_h\rho^{(2)\,m}$. Then $\mathrm{id}^{(1)} = \alpha^{(1)\,\nu}_g\mapsto(x,\alpha^{(2)}_k\rho^{(2)\,m\nu})$ for some $x\in\mathcal{L}^{(2)}, k\in G^{(2)}$. But if $m>0$, this can never equal $\mathrm{id}^{(2)} = (1,\mathrm{id}^{(2)})$. Similarly, if $\rho^{(1)}\mapsto(p,\alpha^{(2)}_h\rho^{(2)\,m})$ for some $m>1$, then no object in $\mathcal{C}(G^{(1)};\pm^{(1)},\omega^{(1)},A^{(1)})$ can get sent to $\rho^{(2)} = (1,\rho^{(2)})$.

So our tensor equivalence defines a bijection $\pi : G^{(1)} \to G^{(2)}$ and an element $r \in G^{(2)}$ by $\alpha_g^{(1)} \mapsto \alpha_{\pi(g)}^{(2)}$ and $\rho^{(1)} \mapsto \alpha_r^{(2)}\rho^{(2)}$. Thanks to the fusion rules, π must be a group isomorphism, and the tensor equivalence must send $\alpha_g^{(1)}\rho^{(1)} \mapsto \alpha_{\pi(g)+r}^{(2)}\rho^{(2)}$.

Although the tensor equivalence will map Hom-spaces to Hom-spaces, we don't know *a priori* whether it lifts to a well-defined algebra homomorphism between the Leavitt algebras, so as above we will be careful to keep track of the Hom-space we are working in by using the $(\xi|x|\eta)$ notation. For convenience abbreviate $1_\xi^{(i)} = (\xi|1^{(i)}|\xi)$, $s^{(i)} = (\mathrm{id}|s^{(i)}|\rho^{(i)\,2})$, and $s'^{(i)}$, $t_g^{(i)}$, $t_g'^{(i)}$ similarly. Note if the tensor equivalence sends object ξ to object ξ', then it must take the identity $1_\xi^{(1)}$ in $\mathrm{End}(\xi)$ to the identity $1_\xi^{(2)}$ in $\mathrm{End}(\xi')$.

By simplicity (Proposition 1), we know $s^{(1)}$ (which spans $\mathrm{Hom}(\mathrm{id}^{(1)}, \rho^{(1)\,2})$) is sent to $\lambda s^{(2)}$ (which spans $\mathrm{Hom}(\mathrm{id}^{(2)}, (\alpha_r^{(2)}\rho^{(2)})^2) = \mathrm{Hom}(\mathrm{id}^{(2)}, \rho^{(2)\,2})$) and likewise $t_g^{(1)} \in \mathrm{Hom}(\alpha_g^{(1)}\rho^{(1)}, \rho^{(1)\,2})$ to $\mu_g t_{r+\pi g}^{(2)} \in \mathrm{Hom}(\alpha_{\pi g+r}^{(2)}\rho^{(2)}, \rho^{(2)\,2})$ for some non-zero $\lambda, \mu_g \in \mathbb{C}$. Since $1_\xi^{(1)}$ is sent to $1_{\xi'}^{(2)}$, the relations $s' \circ s = 1_{\mathrm{id}}$ and $t_g' \circ t_h = 1_{\alpha_g\rho}\delta_{g,h}$ give $s'^{\,(1)} \mapsto \lambda^{-1}s'^{\,(2)}$ and $t_g'^{\,(1)} \mapsto \mu_g^{-1}t_{r+\pi g}'^{\,(2)}$. From $1_{\alpha_k} \otimes t_h = (\alpha_{k+h}\rho|\alpha_k(t_h)|\alpha_k\rho^2)$ and $t'_{2k+h} \otimes 1_{\alpha_k} = (\alpha_k\rho^2|t'_{2k+h}|\alpha_{k+h}\rho)$ we obtain

$$(t'_{2k+h} \otimes 1_{\alpha_k}) \circ (1_{\alpha_k} \otimes t_h) = 1_{\alpha_{h+k}\rho}\,; \tag{5.11}$$

hence $1^{(1)}_{\alpha^{(1)}_{h+k}\rho^{(1)}}$ gets sent to both $1^{(2)}_{\alpha^{(2)}_{\pi(h+k)+r}\rho^{(2)}}$ and $\mu_{2k+h}^{-1}\mu_h 1^{(2)}_{\alpha^{(2)}_{\pi(h+k)+r}\rho^{(2)}}$. Thus $\mu_g = \mu$ is independent of g. Comparing dimensions of $\rho^{(1)}$ and $\alpha_r^{(2)}\rho^{(2)}$, we get $\omega^{(1)}b^{(1)\,2} = \omega^{(2)}b^{(2)\,2}$, i.e. we must have $b^{(1)} = \pm b^{(2)}$ (hence $b^{(1)} = b^{(2)}$ and the signs $\pm^{(1)}$ and $\pm^{(2)}$ are equal) and $\omega^{(1)} = \omega^{(2)}$. The calculation

$$t_0' \circ (t_0' \otimes \rho) \circ (1_\rho \otimes s) = t_0' \circ (\rho^3|t_0'|\rho^2) \circ (\rho|\rho(s)|\rho^3) = b\,1_\rho \tag{5.12}$$

means, computing the image of the tensor equivalence in two ways, $b^{(1)} = \mu^{-2}\lambda b^{(2)}$, which fixes the value of λ. Similarly, the calculation

$$(t'_{h+k} \otimes 1_{\alpha_h})(t'_h \otimes 1_\rho)(1_\rho \otimes t_0)t_k = (\alpha_h\rho^2|t'_{h+k}|\alpha_k)(\rho^3|t'_h|\alpha_h\rho^2)(\rho^2|\rho(t_0)|\rho^3)t_k = A_{h,k}1_{\alpha_k\rho} \tag{5.13}$$

gives $A_{h,k} = A_{\pi h,\pi k}$.

Note that s, s', t_g, t_g' obey the Leavitt–Cuntz relations (3.1), iff $\pm\mu^2 s, \pm\mu^{-2}s', \mu t_{g+r}, \mu t'_{g+r}$ do, for any sign $\pm$, $\mu \in \mathbb{C}^\times$ and $r \in G$. These choices leave unchanged the algebra $\mathcal{L}_{\nu+1}$ and its endomorphisms ρ, α_g. Part (b) follows.

Finally, let us turn to part (c) of Theorem 2. Suppose A is hermitian. Define a conjugate-linear map on $\mathcal{L}$ by $s^* = \pm s'$, $s'^{\,*} = \pm s$, $t_g^* = t_g'$ and $t_g'^{\,*} = t_g$, extended so that $(cxy)^* = \bar{c}y^*x^*$ for all $c \in \mathbb{C}$ and $x, y \in \mathcal{L}$, where the sign in these expressions is as in $\omega^2 b^{-2} = \delta_\pm$. Then $\overline{b} = \pm\omega b$ so $(\rho(x))^* = \rho(x^*)$ for the $2 + 2\nu$ generators x, and hence that relation holds for all $x \in \mathcal{L}$. It is easy to see that this determines a $*$-operation on $\mathcal{C}(G; \pm, \omega, A)$, in the sense defined in the introduction. If in addition $\pm = +$, then this conjugate linear map is the usual $*$-operation on $\mathcal{L}$, and so taking completions we get a system of endomorphisms on the Cuntz algebra which extend to the infinite factor N and thus we possess a unitary category.

Conversely, suppose $\mathcal{C}(G;\pm,\omega,A)$ possesses a $*$-operation. Again, we don't know *a priori* whether the $*$-operation (which by definition is defined only on individual Hom-spaces) lifts to a well-defined $*$-operation on the Leavitt algebra, so again write $(\xi|x|\eta)$ for $x \in \mathrm{Hom}(\xi,\eta)$ as before. Note that the $*$-operation must take the identity $(\xi|1|\xi)$ in $\mathrm{End}(\xi)$ to itself. From simplicity (Proposition 1), we may write $t_g^* = \beta_g t'_g$ and $t'^*_g = \beta'_g t_g$ for some non-zero $\beta_g, \beta'_g \in \mathbb{C}$. Then taking $*$ of (5.11) gives $\mu_h \mu'_{2k+h} = 1$, i.e. that $\mu_g = \mu'^{-1}_h = \mu$ is independent of $g, h \in G$. Now taking $*$ of (5.13), we get $\overline{A_{h,k}} = A_{k,h}$, and we see that for the category to be hermitian, the matrix A must be hermitian.

Finally, in a unitary category the categorical dimensions must all be positive. But $d_\rho = \delta_\pm$, and $\delta_- < 0$. This concludes the proof of Theorem 2.

6. The tube algebra and modular data

6.1. The tube algebra and its centre

We will now determine the quantum double or centre of our categories $\mathcal{C}(G;\pm,\omega,A)$ using the tube algebra approach of [21]. That approach assumes unitarity, but [27] categorises the method, generalising it beyond the context we need, and all of our equations come from there.

Let $\Delta = \{\alpha_g, \alpha_h\rho\}_{g,h\in G}$ be a finite system of endomorphisms associated to a solution of our equations (4.7)–(4.10). Write $\Sigma\Delta$ for the objects in $\mathcal{C}(G;\pm,\omega,A)$, and write $[\sigma]$ for the sector or equivalence class of an object (where the conjugation now need not be by a unitary). The categorical dimension $d_\sigma = d_{[\sigma]}$ of any object $\sigma \in \Sigma\Delta$ was computed last section. We found there the dimensions $d_{[\alpha_g]} = 1$ and $d_{[\alpha_g\rho]} = \delta_\pm$ for the simple objects (note that $\delta_+ > 0 > \delta_-$, so these dimensions can be negative). The global dimension is then $\lambda_\pm = \nu(1+\delta_\pm^2) = 2\nu + \nu^2\delta_\pm$, which is strictly positive as it must be.

The tube algebra Tube Δ is a finite-dimensional algebra over $\mathbb{C}$, defined as a vector space by

$$\mathrm{Tube}\,\Delta = \oplus_{\xi,\eta,\zeta\in\Delta}\mathrm{Hom}(\xi\zeta,\zeta\eta)\,. \tag{6.1}$$

It will be semisimple even if the fusion category is non-unitary [27]. As in section 5, given an element X of $\mathrm{Tube}\,\Delta$, it is convenient to write $(\xi\zeta|X|\zeta\eta)$ for the restriction to $\mathrm{Hom}(\xi\zeta,\zeta\eta)$, since the same operator may belong to distinct intertwiner spaces. For readability we will often write g and $g\rho$ for α_g and $\alpha_g\rho$, respectively. In our case the intertwiner spaces are computed by Lemma 3. Then a basis for Tube Δ consists of $\mathcal{A}_{gh} = (g,h|1|h,g)$, $\mathcal{B}_{gh} = (g,h\rho|1|h\rho,-g)$, $\mathcal{C}_{gh} = (g\rho,(g-h)/2|1|(g-h)/2,h\rho)$, $\mathcal{D}_{gkh} = (g,k\rho|t_{2k+g-h}|k\rho,h\rho)$, $\mathcal{E}_{gkh} = (g\rho,k\rho|t'_{g-h}|k\rho,h)$, $\mathcal{F}_{gh} = (g\rho,(g+h)/2\rho|ss'|(g+h)/2\rho,h\rho)$, and $\mathcal{G}^{kl}_{gh} = (g\rho,k\rho|t_{l-h+k}t'_{l+g-k}|k\rho,h\rho)$ (note that the vector space structure of Tube Δ given at the bottom of p. 655 of [22] is incomplete). Thus Tube Δ is $\nu^4+2\nu^3+4\nu^2$-dimensional.

The multiplicative structure of Tube Δ is given by

$$(\xi\zeta|X|\zeta\eta)(\bar{\xi}\bar{\zeta}|Y|\bar{\zeta}\bar{\eta})=\delta_{\eta,\bar{\xi}}\sum_{\nu\prec\zeta\bar{\zeta}}(\xi\nu|T(\nu)'\zeta(Y)X\xi(T(\nu))|\nu\bar{\eta})\,, \tag{6.2}$$

where we continue to write $\alpha_g\rho=g\rho$ and g for α_g, and $T(\nu)$ denotes whichever $1,s,t_l$ lies in $\mathrm{Hom}(\nu,\zeta\,\bar{\zeta})$. In particular, we obtain: $\mathcal{A}_{gh}\mathcal{A}_{gl}=\mathcal{A}_{g,h+l}$, $\mathcal{A}_{gh}\mathcal{B}_{gl}=\mathcal{B}_{gl}\mathcal{A}_{-g,-h}=\mathcal{B}_{g,h+l}$, $\mathcal{B}_{gh}\mathcal{B}_{-gl}=\mathcal{A}_{g,h-l}+\delta_{g0}\sum_m\mathcal{B}_{0m}$, $\mathcal{C}_{gh}\mathcal{C}_{hk}=\mathcal{C}_{gk}$, $\mathcal{A}_{gh}\mathcal{D}_{gkh'}=\mathcal{D}_{g,h+k,h'}$, $\mathcal{B}_{g0}\mathcal{D}_{-gk0}=\sum_l A_{l+k-g,l+k+g}\mathcal{D}_{g,l,0}$, $\mathcal{D}_{gkh}\mathcal{C}_{hk'}=\mathcal{D}_{g,k+(k'-h)/2,k'}$, $\mathcal{E}_{gkh}\mathcal{A}_{hl}=\mathcal{E}_{g,k-l,h}$, $\mathcal{E}_{0kh}\mathcal{B}_{h0}=\sum_m A_{m+k-h,m+k+h}\mathcal{E}_{0m,-h}$, $\mathcal{C}_{gh}\mathcal{E}_{hkg'}=\mathcal{E}_{g,(g-h+2k)/2,g'}$, $\mathcal{D}_{g0h}\mathcal{E}_{h0l}=\delta_{gl}\bar{\omega}\mathcal{A}_{g,0}+\delta_{g,-l}\sum_m A_{m+g+h,2g}\mathcal{B}_{gm}$,

$$\mathcal{E}_{0kh}\mathcal{D}_{hl0}=\bar{\omega}\delta_{\pm}^{-1}\delta_{l,k}\mathcal{C}_{00}+\bar{\omega}\delta_{k,l+h}\mathcal{F}_{00}+\sum\nolimits_{g,m}A_{m-k+l+h,g-k+l}A_{m-h+k-l,g+k-l}\mathcal{G}_{00}^{mg}\,,$$

$\mathcal{C}_{gh}\mathcal{F}_{hk}=\mathcal{F}_{gk}$, $\mathcal{F}_{gh}\mathcal{C}_{hk}=\mathcal{F}_{gk}$, $\mathcal{D}_{k0h}\mathcal{F}_{h0}=\delta_{\pm}^{-1}\mathcal{D}_{k,-k-h/2,0}$, $\mathcal{F}_{0h}\mathcal{E}_{h0l}=\delta_{\pm}^{-1}\mathcal{E}_{0,l-h/2,l}$, $\mathcal{F}_{0h}\mathcal{F}_{h0}=\delta_{\pm}^{-3}\mathcal{C}_{00}+\omega\delta_{\pm}^{-2}\sum_l\mathcal{G}_{00}^{l0}$, $\mathcal{C}_{gh}\mathcal{G}_{hh'}^{kl}=\mathcal{G}_{gh'}^{(g-h+2k)/2,(2l+g-h)/2}$, $\mathcal{G}_{hh'}^{kl}\mathcal{C}_{h'g'}=\mathcal{G}_{hg'}^{(2k+g'-h')/2,(2l+g'-h')/2}$, $\mathcal{D}_{k0h}\mathcal{G}_{h0}^{k'l}=\sum_m A_{m+l,m+k+k'}A_{m+k+k',m+l}\mathcal{D}_{km0}$, $\mathcal{G}_{0g}^{k'l}\mathcal{E}_{g0k}=\sum_m A_{m+g-k-k',l-k-k'}A_{l+m,m-k+k'}\mathcal{E}_{0mk}$,

$$\mathcal{G}_{0g}^{kl}\mathcal{F}_{g0}=\delta_{\pm}^{-2}\delta_{2k,g}\mathcal{C}_{00}+\delta_{\pm}^{-1}\sum\nolimits_m A_{l+m-g/2,2k-g}\mathcal{G}_{00}^{m,k-g/2}\,,$$

$\mathcal{F}_{0k}\mathcal{G}_{k0}^{k'l}=\delta_{\pm}^{-2}\delta_{2k',k}\mathcal{C}_{00}+\delta_{\pm}^{-1}\sum_m A_{m+l-k/2,2k'-k}\mathcal{G}_{00}^{m,k'-k/2}$, and

$$\begin{aligned}\mathcal{G}_{0h}^{kl}\mathcal{G}_{h0}^{k'l'}=&\bar{\omega}\delta_{\pm}^{-1}\delta_{k,k'}A_{l+l'-h,2k-h}\mathcal{C}_{00}+\delta_{kl'}\omega\delta_{k'l}\mathcal{F}_{00}\\&+\sum\nolimits_{m,m'}A_{m+l'-k,m'+k'-k}A_{m'-k'+h-k,l'-k'-k+l}A_{m+l-k',m'+k-k'}\mathcal{G}_{00}^{mm'}\,,\end{aligned}$$

where we've only written the non-zero products. Note that we have G actions, by multiplying by $\mathcal{A}$ or $\mathcal{C}$, so for simplicity we restrict to subscripts equal to 0 when this G-action can yield the other values.

Unless A is hermitian, we can't expect Tube Δ to have a natural structure as a $*$-algebra.

Let $\sigma\in\Sigma\Delta$. A half-braiding for σ is a choice of invertible $\mathcal{E}_\sigma(\xi)\in\mathrm{Hom}(\sigma\xi,\xi\sigma)$ for each $\xi\in\Delta$, such that for every $\eta,\zeta\in\Delta$ and any $X\in\mathrm{Hom}(\zeta,\xi\eta)$,

$$X\mathcal{E}_\sigma(\zeta)=\xi(\mathcal{E}_\sigma(\eta))\,\mathcal{E}_\sigma(\xi)\,\sigma(X)\,. \tag{6.3}$$

In general, σ will be a formal direct sum of $\mathcal{L}$-endomorphisms, so the values $\mathcal{E}_\sigma(\xi)$ will be matrices with entries in $\mathcal{L}$. In this case, by $\sigma(X)$ in (6.3) we mean the diagonal matrix with entries $\eta(X)$ as η runs over all simples in σ, with multiplicities, and by $\xi(\mathcal{E}_\sigma(\zeta))$ we mean to evaluate each entry of the matrix $\mathcal{E}_\sigma(\zeta)$ by ξ. This equation makes sense as the morphisms (matrices over $\mathcal{L}$) on the left side are intertwiners for $\sigma\zeta \to \zeta\sigma \to \xi\eta\sigma$ while the right side intertwines $\sigma\zeta \to \sigma\xi\eta \to \xi\sigma\eta \to \xi\eta\sigma$ — composition is just matrix multiplication. Invertibility of $\mathcal{E}_\sigma(\xi)$ is equivalent to $\mathcal{E}_\sigma(\mathrm{id}) = 1$, the identity matrix. There may be more than one half-braiding associated to a given σ; in that case we denote them by $\mathcal{E}_\sigma^j$.

The quantum double or centre of the fusion category $\mathcal{C} = \mathcal{C}(G; \pm, \omega, A)$ is a strict modular tensor category (MTC) with objects $(\sigma, \mathcal{E}_\sigma)$ where $\sigma \in \Sigma\Delta$ and $\mathcal{E}_\sigma$ is a half-braiding. The morphisms $x \in \mathrm{Hom}((\sigma, \mathcal{E}_\sigma), (\tau, \mathcal{E}_\tau))$ are $x \in \mathrm{Hom}_\mathcal{C}(\sigma, \tau)$ satisfying $\sigma(x)\mathcal{E}_\sigma(\zeta) = \mathcal{E}_\tau(\zeta)x \ \forall\zeta \in \Sigma\Delta$; composition is as in $\mathcal{C}$. The tensor product of objects is given by $(\sigma, \mathcal{E}_\sigma) \otimes (\tau, \mathcal{E}_\tau) = (\sigma\tau, \mathcal{E}_{\sigma\tau})$ where $\mathcal{E}_{\sigma\tau}(\zeta) = \mathcal{E}_\sigma(\zeta)\mathcal{E}_\tau(\zeta)$; the tensor product of morphisms is multiplication as in $\mathcal{C}$. The unit is $(\mathrm{id}, 1)$. The braiding is $c_{(\sigma,\mathcal{E}_\sigma),(\tau,\mathcal{E}_\tau)} = \mathcal{E}_\sigma(\tau)$. Duals are $(\sigma, \mathcal{E}_\sigma)^\vee = (\sigma^\vee, \mathcal{E}_{\sigma^\vee})$ where $\mathcal{E}_{\sigma^\vee}(\zeta) = e_{\sigma^\vee} \otimes \mathrm{id}_{\zeta\sigma^\vee}(\sigma^\vee(\mathcal{E}_\sigma(\zeta)^{-1})\sigma^\vee\zeta(b_\sigma))$; (co-)evaluation is as in $\mathcal{C}$. If $\mathcal{C}$ is hermitian (resp. unitary), one should require the $\mathcal{E}_\sigma(\xi)$ to be unitary and not merely invertible, in which case the resulting category will be a hermitian (resp. unitary) MTC. See [27] for details.

Tube Δ, being a finite-dimensional semisimple algebra over $\mathbb{C}$, decomposes into a direct sum $\oplus_i M_{k_i\times k_i}(\mathbb{C})$ of matrix algebras. The (indecomposable) half-braidings $\mathcal{E}_\sigma^j$ make this explicit. Decompose the sector $[\sigma]$ into a sum $\sum_{i=1}^{k'}[g_i] + \sum_{i=1}^{k''}[h_i\rho]$ of simples, repetitions allowed. In $\mathcal{C}$, σ is the formal direct sum

$$\sigma = ((1, g_1), \dots, (1, g_{k'}), (1, h_1\rho), \dots, (1, h_{k''}\rho)),$$

where the 1's denote the identity idempotent (and will be dropped for readability). Let $k = k' + k''$. Then by (3.7), for each simple ξ $\mathcal{E}_\sigma^j(\xi)$ will be a $k \times k$ matrix with entries $\mathcal{E}_\sigma^j(\xi)_{\eta,\bar\eta} \in \mathrm{Hom}(\eta\xi, \xi\bar\eta) \subset \mathcal{L}$, as $\eta, \bar\eta$ run over all simples $\{g_i, h_i\rho\}$ in σ, repetitions included. The resulting $k \times k$ matrix algebra $\{\mathcal{E}_\sigma^j(\xi)\}$ (with entries contained in $\mathcal{L}$) is isomorphic as a $\mathbb{C}$-algebra to an irreducible summand of Tube Δ, and all irreducible summands are of that form.

We will determine the possible half-braidings $\mathcal{E}_\sigma^j$, by determining the *matrix units* in Tube Δ of the corresponding simple summand $M_{k\times k}$. Matrix units $e_{i,j}$ of $M_{k\times k}$ are a basis satisfying $e_{i,j}e_{m,l} = \delta_{j,m}e_{i,l}$. The relation between the matrix units and the corresponding half-braidings is [21]

$$e(\sigma^j)_{\eta,\bar\eta} = \frac{d_\sigma}{\lambda_\pm\sqrt{d_\eta\, d_{\bar\eta}}} \sum\nolimits_\xi d_\xi\, (\eta\xi|\mathcal{E}_\sigma^j(\xi)_{\eta,\bar\eta}|\xi\bar\eta)\,, \tag{6.4}$$

where the sum is over $\xi \in \Delta$, and again $\eta, \bar\eta$ run through the simples $\{g_i, h_i\rho\}$ in σ. The corresponding central projection (the unit of that simple summand) is then $z(\sigma^j) = \sum_\eta e(\sigma^j)_{\eta,\eta}$. Our primary interest this section is in determining the modular data of the double, and for this purpose the diagonal matrix units are all that we need.

As a $\mathbb{C}$-algebra, Tube Δ decomposes as a direct sum

$$\mathrm{Tube}\,\Delta \cong M_{1\times1} \oplus M_{\nu+1\times\nu+1} \oplus \frac{\nu-1}{2} M_{\nu+2\times\nu+2} \oplus \frac{\nu^2-\nu}{2} M_{\nu+2\times\nu+2} \oplus \frac{\nu^2+3}{2} M_{\nu\times\nu} \tag{6.5}$$

corresponding to half-braidings $\mathcal{E}_{[\mathrm{id}]}$, $\mathcal{E}_{[\mathrm{id}]+\Sigma_g[g\rho]}$, $\mathcal{E}^{\psi}_{2[\mathrm{id}]+\Sigma_g[g\rho]}$, $\mathcal{E}^{\phi}_{[h]+[-h]+\Sigma_g[g\rho]}$, and $\mathcal{E}^{l}_{\Sigma_g[g\rho]}$ respectively, where $\psi,\phi\in\widehat{G}$ but ψ is non-trivial and $\psi,\overline{\psi}$ give the same half-braiding, $h\in G$ but h is non-trivial and $\pm h$ give same half-braiding, and $1\le l\le\frac{\nu^2+3}{2}$ is some parameter to be interpreted later.

The proof of (6.5) for $\mathcal{C}(G;\pm,\omega,A)$, in particular the determination of the associated matrix units, follows the analysis in section 8 of [22], which does this for the (unitary) Q-systems. The main differences are the presence of ω and the absence of a $*$-structure. The central projection of the unique half-braiding of $\sigma=\mathrm{id}$ is again given by $z^i=\lambda_\pm^{-1}\sum\mathcal{A}_{0g}+\delta\lambda_\pm^{-1}\sum\mathcal{B}_{0g}$, and so

$$\mathcal{E}^i(g)_{0,0}=\mathcal{E}^i(g\rho)_{0,0}=1\,. \tag{6.6}$$

The matrix units corresponding to the second summand of (6.5) are

$$\begin{aligned} e^{ii}_{0,0}&=\frac{\delta_\pm}{\lambda_\pm}(\delta_\pm\sum\nolimits_g\mathcal{A}_{0,g}-\sum\nolimits_g\mathcal{B}_{0,g})\,,\\ e^{ii}_{0,g\rho}&=\frac{\overline{\omega}\delta_\pm}{\nu\sqrt{\nu\delta_\pm+2}}\sum\nolimits_k\mathcal{D}_{0kg}\,,\quad e^{ii}_{g\rho,0}=\frac{\overline{\omega}\delta_\pm}{\nu\sqrt{\nu\delta_\pm+2}}\sum\nolimits_k\mathcal{E}_{gk0}\,,\\ e^{ii}_{g\rho,h\rho}&=\frac{\delta_\pm}{\lambda_\pm}\left(\mathcal{C}_{gh}+\delta_\pm\mathcal{F}_{gh}+\omega\delta_\pm\sum_{k,l,m}A_{k+m,l+m}A_{k-m,l-m}\mathcal{G}^{k+g/2+h/2,l+g/2+h/2}_{gh}\right). \end{aligned} \tag{6.7}$$

Compare with Proposition 8.2(2) of [22]. This corresponds to

$$\begin{aligned} &\mathcal{E}^{ii}(h)_{0,0}=1\,,\ \mathcal{E}^{ii}(h)_{g\rho,g\rho}=\delta_{h,0}\,,\ \mathcal{E}^{ii}(h\rho)_{0,0}=-\delta_\pm^{-2}\,,\\ &\mathcal{E}^{ii}((k+g)\rho)_{g\rho,g\rho}=\delta_{k,g}ss'+\omega\sum\nolimits_{l,m}A_{k+m,l+m}A_{k-m,l-m}t_{g+k+l}t'_{g-k+l}\,. \end{aligned} \tag{6.8}$$

The third class of half-braidings is parametrised by pairs $\{\psi,\overline{\psi}\}$ of non-trivial characters $\psi\in\widehat{G}$, and has diagonal matrix units

$$\begin{aligned} &e^{iii;\psi}_{0,0}=\nu^{-1}\sum\nolimits_g\psi(g)\mathcal{A}_{0,g}\,,\ e^{iii;\psi}_{0',0'}=\nu^{-1}\sum\nolimits_g\overline{\psi(g)}\mathcal{A}_{0,g}\,,\\ &e^{iii;\psi}_{g\rho,g\rho}=\frac{1}{\nu\delta_\pm}\left(\mathcal{C}_{gg}+\delta_\pm\mathcal{F}_{gg}+\omega\delta_\pm\sum_{k,l,m}\psi(m)A_{k+m,l+m}A_{k-m,l-m}\mathcal{G}^{k+g,l+g}_{gg}\right). \end{aligned} \tag{6.9}$$

This corresponds to

$$\begin{aligned} &\mathcal{E}^{iii;\psi}(g)_{0,0}=\psi(g)\,,\ \mathcal{E}^{iii;\psi}(g)_{0',0'}=\overline{\psi(g)}\,,\ \mathcal{E}^{iii;\psi}(g)_{k\rho,k\rho}=\delta_{g,0}\,,\\ &\mathcal{E}^{iii;\psi}(g\rho)_{0,0}=\mathcal{E}^{iii;\psi}(g\rho)_{0',0'}=0\,,\\ &\mathcal{E}^{iii;\psi}((g+k)\rho)_{k\rho,k\rho}=\delta_{g,0}ss'+\omega\sum\nolimits_{l,m}\psi(m)A_{k+m,l+m}A_{k-m,l-m}t_{k+l+g}t'_{k+l-g}\,. \end{aligned} \tag{6.10}$$

The fourth class of half-braidings is parametrised by all characters $\psi \in \widehat{G}$ and non-trivial pairs $\pm h \in G$, and has diagonal matrix units

$$e^{iv;h,\psi}_{h,h} = \nu^{-1}\sum_g \psi(g)\mathcal{A}_{h,g}\,,\ e^{iv;h,\psi}_{-h,-h} = \nu^{-1}\sum_g \overline{\psi(g)}\mathcal{A}_{-h,g}\,, \tag{6.11}$$

$$e^{iv;h,\psi}_{g\rho,g\rho} = \frac{1}{\nu\delta_\pm}\left(\mathcal{C}_{gg} + \delta_\pm\overline{\psi(h)}\mathcal{F}_{gg} + \omega\delta_\pm\sum_{k,l,m}\psi(m)A_{k+h+m,l+m}A_{k-h-m,l-m}\mathcal{G}^{k+g,l+g}_{gg}\right).$$

Compare with Proposition 8.2 of [22] (where there is a minor typo there for $e^{iv;h,\psi}_{-h,-h}$). This corresponds to

$$\begin{aligned}\mathcal{E}^{iv;h,\psi}(g)_{h,h} = \psi(g)\,,\ \mathcal{E}^{iv;h,\psi}(g)_{-h,-h} = \overline{\psi(g)}\,,\ \mathcal{E}^{iv;h,\psi}(g)_{k\rho,k\rho} = \delta_{g,0}\,,\\ \mathcal{E}^{iv;h,\psi}(g\rho)_{h,h} = \mathcal{E}^{iv;h,\psi}(g\rho)_{-h,-h} = 0\,,\\ \mathcal{E}^{iv;h,\psi}((g+k)\rho)_{k\rho,k\rho} = \delta_{g,0}\overline{\psi(h)}ss'\\ + \omega\sum_{l,m}\psi(m)A_{k+h+m,l+m}A_{k-h-m,l-m}t_{k+l+g}t'_{k+l-g}\,.\end{aligned} \tag{6.12}$$

The matrix units for the final summand of (6.5) are addressed next subsection.

6.2. The half-braidings for $\sigma = \sum \alpha_h\rho$

Define n, μ, m by $\nu = 2n+1$ and $\mu = \nu^2 + 4 = 2m+1$. The analysis of [22] is not complete in determining the matrix units for the final summand of (6.5), even if one is only interested in unitary Q-systems as in [22]. Because of this, it is not possible to determine the modular S, T matrices in general. (To be fair, [22] was mainly interested in the solution for $\nu = 3$ corresponding to the Haagerup subfactor, and for this solution his equations do uniquely determine the matrix units.) In this subsection we supplement the equations given in [22], Lemma 8.3.

Generalising Lemma 8.3 of [22] to our context, we learn that the matrix units corresponding to the final summand, i.e. to the half-braidings with $[\sigma] = \sum_h[\alpha_h\rho]$, are of the form

$$e^{v;j}_{g\rho,h\rho} = \frac{\nu}{\lambda_\pm}\left(\mathcal{C}_{gh} + \overline{w_j}\delta_\pm\mathcal{F}_{gh} + \delta_\pm\sum_{k,l}C^j_{k,l}\mathcal{G}^{k+g/2+h/2,l+g/2+h/2}_{gh}\right), \tag{6.13}$$

for $1 \le j \le (\nu^3+3)/2$, where the $m(\nu^2+1)$ variables $w_j, C^j_{k,l} \in \mathbb{C}$ satisfy the $m(\nu^2+1)$ equations:

$$\sum_g C^j_{0,g} = w_j - \overline{w_j}\,\delta_\pm^{-1}\,,\ w_j\,C^j_{g,h} - \sum_k A_{g+k,2h}C^j_{h,k} = \delta_{h,0}\,\omega\overline{w_j}\,\delta_\pm^{-1}\,, \tag{6.14}$$

for all $g, h, k \in G$. This half-braiding corresponds to

$$\mathcal{E}^{v;j}(k)_{g\rho,g\rho} = \delta_{k,0}\ \text{ and }\ \mathcal{E}^{v;j}((k+g)\rho)_{g\rho,g\rho} = \delta_{k,0}\overline{w_j}ss' + \sum_l C^j_{k,l}t_{l+k+g}t'_{l-k+g}\,, \tag{6.15}$$

for all $g, k \in G$.

The w_j are the corresponding diagonal entries of the modular matrix T, and so must be roots of unity. Some solutions to (6.14), occurring for w_j of small order, are redundant (i.e. correspond to the previous summands of (6.5)) and should be dropped.

Note that, when there is more than one half-braiding $\mathcal{E}^j_{\Sigma\, g\rho}$ with the same value of w_j, say $w_j = w_{j'}$, then there will be infinitely many different solutions to (6.14) with w_j, namely $tC^j_{g,h} + (1-t)C^{j'}_{g,h}$ for any $t \in \mathbb{C}$. This is because (6.14) are linear, for fixed w_j. Such w_j can indeed occur — in [11], 6 fusion categories (in fact Q-systems) $\mathcal{C}(\mathbb{Z}_\nu; +, 1, A)$ were found with w_j of higher multiplicities. Those examples correspond to $\nu = 9, 11, 19$; for reasons explained in [11], we expect there to be higher multiplicities, and hence ambiguities, whenever μ is composite. Whenever we cannot uniquely determine the $C^j_{g,h}$, we cannot uniquely determine e.g. the modular S, T matrices.

The situation will only get worse as we generalise the context beyond Q-systems to not-necessarily-unitary fusion categories. For this reason, we need to supplement Izumi's (6.14) with non-linear constraints. This is done next.

Proposition 2. *Let $\mathcal{C}(G; \pm, \omega, A)$ be any category in Theorem 2. Then (4.11) holds. Moreover, in addition to (6.14), $w_j, C^j_{g,h}$ must satisfy:*

$$\omega w_j C^j_{p,s} C^j_{h,r} \delta_\pm = \delta_{s,h}\delta_{r,p} + \overline{w_j} A_{p+h,2s}\delta_{r,s} \tag{6.16}$$

$$+ \delta_\pm \sum\nolimits_{k,l} C^j_{k,l} A_{h+l-s,r+k-s} A_{r-k-s,l-k-s+p} A_{h+p-k,r+s-k}\,, \tag{6.17}$$

$$\frac{\lambda_\pm}{\nu}\delta_{j,j'} = 1 + \overline{w}_{j'} w_j + \delta_\pm \omega w_j \sum\nolimits_{t,q} C^{j'}_{t,q} C^j_{q,t}\,, \tag{6.18}$$

$$0 = 1 + \overline{\psi(g)} w_j + \delta_\pm \omega w_j \sum_{t,q,m} \psi(m)\, C^j_{q,t} A_{t+m+g,q+m} A_{t-m-g,q-m}\,, \tag{6.19}$$

for all $\psi \in \hat{G}$, $g \in G$. Conversely, these equations uniquely determine $C^j_{g,h}$ and w_j.

Proof. Consider the subalgebra $\mathcal{A}_\rho = \mathrm{span}\{\mathcal{C}_{00}, \mathcal{F}_{00}, \mathcal{G}^{gh}_{00}\}$ of Tube Δ. From the products calculated in the previous subsection, we find that $\mathcal{A}_\rho$ is commutative with unit $\mathcal{C}_{00}$. Now, the diagonal matrix unit $e^j := e^{v;j}_{\rho,\rho}$ in (6.13) is a minimal projection in $\mathcal{A}_\rho$, and hence for any $P \in \mathcal{A}_\rho$, $e^j P e^j$ (which equals Pe^j by commutativity) must be a scalar multiple of e^j. Write $e^j \mathcal{G}^{gh}_{00} = x_{g,h} e^j$ for scalars $x_{g,h} \in \mathbb{C}$. We compute

$$\frac{\lambda_\pm}{\nu} x_{s,p} e^j = \mathcal{G}^{sp}_{00} e^j = \mathcal{G}^{sp}_{00} + \overline{w_j}\left(\delta_{s,0}\frac{\mathcal{C}_{00}}{\delta_\pm} + \sum\nolimits_k A_{p+k,2s}\mathcal{G}^{ks}_{00}\right) + \delta_\pm \sum\nolimits_{k,l} C^j_{k,l}$$
$$\times \left(\bar{\omega}\delta_{s,k} A_{p+l,2s}\frac{\mathcal{C}_{00}}{\delta_\pm} + \omega\delta_{s,l}\delta_{k,p}\mathcal{F}_{00}\right.$$
$$\left. + \sum\nolimits_{h,r} A_{h+l-s,r+k-s} A_{r-k-s,l-k-s+p} A_{h+p-k,r+s-k}\mathcal{G}^{hr}_{00}\right).$$

Therefore $x_{s,p} = \omega w_j C^j_{p,s}$ and we recover (6.17).

Similarly, we compute $e^j\mathcal{C}_{00} = e^j$ and $e^j\mathcal{F}_{00} = \delta_\pm^{-1}w_je^j$. (6.18) and (6.20) now immediately follow from $e^je^{j'} = \delta_{jj'}e^{j'}$ and $e^je^{iii;\psi}_{\rho,\rho} = e^je^{iv;g,\psi}_{\rho,\rho} = 0$ respectively. Comparing the $\mathcal{G}^{mh}_{00}$ coefficients of the associativity of $\mathcal{F}_{00}\mathcal{F}_{00}\mathcal{G}^{kl}_{00}$ gives

$$A_{h+l,2k}A_{m+k,2h} = \frac{\delta_{k,m}\delta_{l,h}}{\delta_\pm} + \sum\nolimits_r A_{m-k,h+r-k}A_{h-r-k,l-r-k}A_{m+l-r,h+k-r}\,, \tag{6.20}$$

which is equivalent to (4.11) using (4.7).

Conversely, the matrix units e^j are uniquely determined by their orthogonality to those for the other half-braidings, as well as the relations $e^je^{j'} = \delta_{jj'}e^j$. These are equivalent to (6.18) and (6.20), once we know $e^j\mathcal{C}_{00} = e^j$, $e^j\mathcal{F}_{00} = \delta_\pm^{-1}w_je^j$, $e^j\mathcal{G}^{gh}_{00} = \omega w_j C^j_{h,g}e^j$. These latter equations follow from (6.17) and (6.14). □

Curiously, the right-side of (6.18) isn't manifestly symmetric in $j \leftrightarrow j'$, even though the left-side is. We know we have a complete list of identities satisfied by A, ω and $\delta_\pm$, so (6.20) (equivalently (4.11)) is redundant, but it doesn't seem to be trivially redundant. Conversely, we expect that it, in conjunction with (4.7)–(4.9), implies the more complicated (4.10), and so can replace it in Theorems 1 and 2, but we haven't verified this yet.

6.3. Modular data for the double of $\mathcal{C}(G;\pm,\omega,A)$

Definition 1. *Modular data* consists of a pair S, T of unitary matrices satisfying:

(i) S is symmetric (i.e. $S^t = S$) and T is diagonal and of finite order ($T^N = I$);
(ii) S^2 is a permutation matrix of order ≤ 2, and $(ST)^3 = S^2$;
(iii) $S_{1i} \in \mathbb{R} \setminus \{0\}$ and some index $1'$ has $S_{1'i} > 0$, for all i;
(iv) for each i, j, k, the numbers N^k_{ij} defined by

$$N^k_{ij} := \sum_l \frac{S_{il}S_{jl}\overline{S_{kl}}}{S_{1l}} \tag{6.21}$$

are nonnegative integers.

Any MTC has modular data. The index i parametrises the simple objects (primaries) X_i. The entries $T_{i,i}$ of the diagonal matrix T (up to normalisation) are eigenvalues of the twist $\theta_{X_i} = (\mathrm{tr}_{X_i} \otimes \mathrm{id}_{\mathrm{End}\,X_i})(c_{X_i,X_i}) \in \mathbb{C}\,\mathrm{id}_{X_i}$ while those of the symmetric matrix S are associated to the Hopf link: up to normalisation, $S_{i,j} = \mathrm{tr}_{X_i\otimes X_j}(c_{X_i,X_j} \circ c_{X_j,X_i})$. By Proposition 2.12 of [10], the matrices S and T will be unitary in any MTC, even when the category is not unitary (or even hermitian). '1' corresponds to the tensor identity X_1 and the permutation S^2 sends i to $i^\vee$, where $[X_i^\vee] = [X_{i^\vee}]$. (6.21) is called Verlinde's formula, and the numbers N^k_{ij} are the structure constants $[X_i \otimes X_j] = \sum_j N^k_{ij}[X_k]$ of the Grothendieck ring of the MTC.

Ignoring the normalisation, those matrices S and T in a MTC define through $\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)\mapsto S$, $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\mapsto T$ a projective representation of the modular group $\mathrm{SL}_2(\mathbb{Z})=\langle\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right),\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\rangle$, but it is always possible to choose a normalisation so that it defines a linear (i.e. true) representation of $\mathrm{SL}_2(\mathbb{Z})$. This choice uniquely determines S up to a sign and then T up to a third root of 1. Property (iv) says the S matrix diagonalises the fusion coefficients N_{ij}^k, so some column of S (a Perron–Frobenius eigenvector) will have constant phase. We require that column (which we call the $1'$th) to be strictly positive, as this is necessary for the existence of a character vector, as explained in section 7.3. This will be the case e.g. in a rational CFT.

From the point of view of modular data, there is little difference between unitary and non-unitary MTCs. In a unitary category, $1'$ must equal 1.

Now specialise to the MTCs which are the doubles of the fusion categories $\mathcal{C}(G;\pm,\omega,A)$ of Theorem 2. Write as before $\nu=|G|=2n+1$, $\mu=\nu^2+4=2m+1$, $\delta_\pm=(\nu\pm\sqrt{\mu})/2$ and $\lambda_\pm=2\nu+\nu^2\delta_\pm$. The main reason for introducing the tube algebra in section 6.1 is to construct its modular data. The simple objects of the MTC are in one-to-one correspondence with the simple summands in (6.5), or equivalently with the irreducible half-braidings. As mentioned earlier, in the tube algebra picture, the braidings are given by the half-braidings, and (co-)evaluations hence traces are as in $\mathcal{C}$. In particular, we obtain the normalised S,T matrices from the diagonal entries $\mathcal{E}_\sigma^j(\xi)_{\eta,\eta}$:

$$T_{\sigma^j,\sigma^j}=d_\xi\,\phi_\xi(\mathcal{E}_\sigma^j(\xi)_{\xi,\xi})\,, \tag{6.22}$$

$$\overline{S_{\sigma^i,\bar\sigma^j}}=\frac{d_\sigma}{\lambda_\pm}\sum{}_\xi\, d_\xi\,\phi_\xi(\mathcal{E}_{\bar\sigma}^j(\eta)_{\xi,\xi}\,\mathcal{E}_\sigma^i(\xi)_{\eta,\eta})\,, \tag{6.23}$$

for any j in (6.22) and (6.23), and any simple $\eta\prec\sigma$ in (6.23). In (6.22), ξ can be any simple in σ, and in (6.23) the sum is over all simple ξ in $\overline{\sigma}$ while η is any (fixed) simple in σ. The *standard left inverse* ϕ_ξ of the endomorphism ξ is $\phi_\xi(x)=R_\xi'\xi^\vee(x)\,R_\xi$, where $R_\zeta\in\mathrm{Hom}(1,\zeta^\vee\zeta)$ and $\overline{R}_\zeta\in\mathrm{Hom}(1,\zeta\zeta^\vee)$ are normalised by $\overline{R}_\zeta'\zeta(R_\zeta)=d_\zeta^{-1}=R_\zeta'\zeta^\vee(\overline{R}_\zeta)$. Note that for $x\in\mathrm{End}(\eta\xi)$, $\phi_\xi(x)\in\mathrm{End}(\eta)=\mathbb{C}1$. (6.22), (6.23) have the desired normalisation built in — as computed in section 5.3 of [27], the normalisation of T is trivial (i.e. $T_{1,1}=1$) for the double of any (not necessarily unitary) fusion category. The derivation of (6.22), (6.23) is as in [21], except that the complex conjugate in (6.23) replaces the $*$'s in his Lemma 5.3: his formula assumes ϕ_ξ is a $*$-map; equation (5.6) of [27] writes this as $S_{\sigma^{i\vee},\bar\sigma^j}$, which is equivalent to our complex conjugation.

In our case, $R_{\alpha_g}=\overline{R}_{\alpha_g}=1$ and $R_{\alpha_g\rho}=\overline{R}_{\alpha_g\rho}=s$, so $\phi_{\alpha_g}=\alpha_{-g}$ and $\phi_{\alpha_g\rho}(x)=s'\alpha_g(\rho(x))s$. We see from (6.22), (6.23) and the matrix units computed earlier this section that the modular data is formally identical to that of [22] (e.g. ω doesn't explicitly appear), except for a trivial dependence on the sign $\pm$. In particular, using (6.5), the primaries fall into four classes:

(i) two primaries, denoted $\mathbf{0}$ and $\mathfrak{b}$;
(ii) n primaries, denoted $\mathfrak{a}_\psi=\mathfrak{a}_{\overline{\psi}}$ for non-trivial $\psi\in\widehat{G}$;

(iii) $n\nu$ primaries, denoted $\mathfrak{c}_{h,\phi} = \mathfrak{c}_{-h,\phi}$ for $h \in G, h \neq 0$ and $\phi \in \widehat{G}$;
(iv) m primaries, denoted $\mathfrak{d}_l$.

Breaking S and T into 16 blocks, we get

$$T = \mathrm{diag}(1,1;1,\ldots,1;\phi(h);w_1,\ldots,w_m)\,,$$
$$S = \frac{1}{\nu}\begin{pmatrix} B & 1_{2\times n} & 1_{2\times n\nu} & C \\ 1_{n\times 2} & 2_{n\times n} & D & 0_{n\times m} \\ 1_{n\nu\times 2} & D^t & E & 0_{n\nu\times m} \\ C^t & 0_{m\times n} & 0_{m\times n\nu} & F \end{pmatrix}, \tag{6.24}$$

where $k_{a\times b}$ for any $k \in \mathbb{C}$ is the $a\times b$ matrix with constant entry k, $D_{\psi,(h,\phi)} = \psi(h)+\overline{\psi(h)}$, $E_{(h,\phi),(h',\phi')} = \phi'(h)\phi(h') + \overline{\phi'(h)\phi(h')}$,

$$B = \frac{1}{2}\begin{pmatrix} 1\mp y & 1\pm y \\ 1\pm y & 1\mp y \end{pmatrix} \text{ and } C = \pm y\begin{pmatrix} 1 & 1 & \cdots & 1 \\ -1 & -1 & \cdots & -1 \end{pmatrix}$$

for $y = \frac{\nu}{\sqrt{\mu}}$. We denote transpose with 't'.

Much more difficult is to identify the $m\times m$ matrix F and the phases w_l. Once the solutions $C^j_{k,l}$ and w_l to (6.14), (6.17)–(6.20) have been found, we conclude

$$F_{\mathfrak{d}_j,\mathfrak{d}_l} = \frac{\nu}{\lambda_\pm}\left(w_j w_l + \delta_\pm \sum{}_{g,p}\, \overline{C^j_{-g,p}}\,\overline{C^l_{g,p+g}}\right). \tag{6.25}$$

Incidentally, (6.17) gives an alternate expression for the diagonal entries of S:

$$S_{\mathfrak{d}_j,\mathfrak{d}_j} = \frac{1}{\lambda_\pm}\Big(\omega w_j n_3 + w_j^2(1-\delta_\pm) + \delta_\pm \omega w_j \times \sum_{g,h,k,l} \overline{C^j_{k,l}}\,\overline{A_{l-p-2g,k-g}}\,\overline{A_{-k-g,l-k-g}}\,\overline{A_{-k,2p+g-k}}\Big)$$

where n_3 is the number of $g \in G$ with order dividing 3.

We have thus identified the S and T matrices for any fusion category $\mathcal{C}(G;\pm,\omega,A)$, although the numbers w_j and the submatrix F seem at this point completely opaque. However, in the following section we list all known fusion categories (unitary or otherwise) of type $\mathcal{C}(G;\pm,\omega,A)$, and identify their modular data. We will find that the mysterious matrix F and phases w_l always seem to take a remarkably simple form. For this reason we conjecture:

Conjecture 1. *Choose any finite abelian group G of odd order ν, and choose any fusion category $\mathcal{C} = \mathcal{C}(G;\pm,\omega,A)$. Then there is an abelian group H of order $\mu = \nu^2+4$ and a nondegenerate bilinear form β on H, which determines the submatrix F and the phases*

w_j for the double of $\mathcal{C}$ explicitly. In particular, the $m = (\nu^2+3)/2$ primaries $\mathfrak{d}_l$ of class (iv) are parametrised by pairs $\pm l$ of elements in H, $l \neq 0$, and

$$w_l = \exp[2\pi \mathrm{i} m\, \beta(l,l)]\,, \tag{6.26}$$

$$F_{l,l'} = \mp \frac{2}{\sqrt{\mu}} \cos(2\pi\beta(l,l'))\,. \tag{6.27}$$

By a nondegenerate bilinear form β on H, we mean $\beta : H \times H \to \mathbb{Q}/\mathbb{Z}$ obeys $\beta(g + g', h + h') \equiv \beta(g,h) + \beta(g,h') + \beta(g',h) + \beta(g',h') \pmod 1$ for all $g, g', h, h' \in H$, and for any non-zero $g \in H$ there is an $h \in H$ such that $\beta(g,h) \not\equiv 0 \pmod 1$.

It is possible that not all G and H arise in Conjecture 1. For example, we know of no fusion categories of type $\mathcal{C}(\mathbb{Z}_3 \times \mathbb{Z}_3; \pm, \omega, A)$ ([11] showed there are no Q-systems for $\mathbb{Z}_3 \times \mathbb{Z}_3$), and we know of no fusion categories $\mathcal{C}(G; \pm, \omega, A)$ whose corresponding modular data has $H = \mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_5$ (it would necessarily have $G = \mathbb{Z}_{11}$). But in both cases, we haven't come close to an exhaustive search.

This conjecture fits into the grafting framework of section 3.3 of [11]. In particular, associated to H and β is a pointed modular tensor category $\mathcal{C}(H,\beta)$, and the role of the affine algebra $B_{r,2}$ in [11] could be played by a $\mathbb{Z}_2$-orbifold of $\mathcal{C}(H,\beta)$. The modular data of Conjecture 1 can be twisted by $H^3(G \rtimes \mathbb{Z}_2; \mathbb{T})$, as explained in section 3 of [11], and indeed as explained there in section 3.3, non-unitarity is the natural context for some of these twists. We have nothing more to add to this discussion. As mentioned earlier, the method of this paper can be generalised to even-order G [14,13,23], and a very small number of solutions are known at present. Although the corresponding elements of S and T also appear to be surprisingly simple, they do not fit into Conjecture 1, and we are not yet prepared to extend the conjecture to cover them.

7. Explicit solutions

7.1. The fusion category classification for small G

This subsection obtains all fusion categories $\mathcal{C}(G; \pm, \omega, A)$ for $|G| \le 5$. Recall $\delta_\pm = (\nu \pm \sqrt{\nu^2+4})/2$, where $|G| = \nu$, and Conjecture 1 from section 6.3.

Theorem 3. *The complete list of fusion categories $\mathcal{C}(G; \pm, \omega, A)$ appearing Theorem 2 for $G = \mathbb{Z}_1, \mathbb{Z}_3, \mathbb{Z}_5$ are (up to equivalence):*

(i) *for $G = \mathbb{Z}_1$: exactly one for either sign; $A = (-1/\delta_\pm)$; both have $\omega = 1$; their modular data has $H = \mathbb{Z}_5$ and $\beta(k,l) = kl/5$ (for '+'), $\beta(k,l) = 2kl/5$ (for '−');*

(ii) *for $G = \mathbb{Z}_3$: two inequivalent unitary ones with '+', and two inequivalent hermitian but non-unitary ones with '−'; all four have $\omega = 1$ and*

$$A = \begin{pmatrix} c & d & e \\ d & e & f \\ e & g & d \end{pmatrix}, \tag{7.1}$$

where the parameters for these four solutions are

$$+ : (c, d, e, f, g) = (c_1, d_1, d_2, f_5, f_5)\,,$$

$$+ : (c, d, e, f, g) = (c_2, d_5, d_5, f_1, f_2)\,,$$

$$- : (c, d, e, f, g) = (c_3, d_6, d_6, f_3, f_4)\,,$$

$$- : (c, d, e, f, g) = (c_4, d_3, d_4, f_6, f_6)\,,$$

for c_i, d_j, f_k explicitly defined below; the modular data for all four has $H = \mathbb{Z}_{13}$, and $\beta(k,l) = kl/13$ resp. $\beta(k,l) = 2kl/13$ for '+' resp. '−';

(iii) *for $G = \mathbb{Z}_5$: two inequivalent unitary ones with '+', and two inequivalent hermitian but non-unitary ones with '−'; all four have $\omega = 1$ and*

$$A = \begin{pmatrix} c & d & e & f & g \\ d & g & h & i & h \\ e & j & f & i & i \\ f & k & k & e & h \\ g & j & k & j & d \end{pmatrix} \tag{7.2}$$

where the parameters for these four solutions are

$$+ : (c, d, e, f, g, h, i, j, k) = (c_2, d_1, d_1, d_1, d_1, h_7, h_{11}, h_8, h_{10})\,,$$

$$+ : (c, d, e, f, g, h, i, j, k) = (c_4, d_4, d_3, d_6, d_5, h_4, h_2, h_4, h_2)\,,$$

$$- : (c, d, e, f, g, h, i, j, k) = (c_1, d_2, d_2, d_2, d_2, h_5, h_{12}, h_9, h_6)\,,$$

$$- : (c, d, e, f, g, h, i, j, k) = (c_3, d_7, d_{10}, d_9, d_8, h_3, h_1, h_3, h_1)\,,$$

for c_i, d_j, h_k explicitly defined below; the modular data for all four has $H = \mathbb{Z}_{29}$, and $\beta(k,l) = kl/29$ resp. $\beta(k,l) = 2kl/29$ for '+' resp. '−'.

The two fusion categories for $\nu = 1$ are realised by affine G_2 at level 1 ('+'), and Yang–Lee ('−'). The first two fusion categories for $\nu = 3$ are realised by an even subsystem of the Grossman–Snyder system H_3 [17] and an even subsystem of the Haagerup subfactor. The other two are their Galois associates. The first two fusion categories for $\nu = 5$ are realised by an even subsystem of the Haagerup–Izumi subfactor for $G = \mathbb{Z}_5$ found in [22], and to one of the even subsystems of the Grossman–Snyder system described in section 6.6 of [17]. The other two are their Galois associates.

As explained after Proposition 7.5 in [26], Ostrik constructed the two twisted Haagerup categories conjectured in Section 3.2 of [11] by de-equivariantising the two $\mathbb{Z}_9$ near-group categories constructed in [12]. Two others arise as their Galois associates. These four fusion categories possess Haagerup–Izumi fusions (1.1) for $G = \mathbb{Z}_3$, but don't appear in Theorem 3(ii) presumably because they involve the generalisation of (4.1) discussed before Theorem 1. Ostrik's construction emphasises the desirability of extending

Theorems 1–3 in this paper to that generalisation of (4.1). This extension should now be straightforward but perhaps messy.

Our proof of Theorem 3 uses Gröbner basis techniques as implemented in Maple 17.02. First, we find a basis for the ideal generated by the identities of Theorem 1. Using it, the eigenvalues are found corresponding to multiplication by each of the variables in the quotient of the polynomial ring by our ideal. The eigenvalues are the possible values of the variables. All of these steps are completed in a fraction of a second for $\nu = 3, 5$. We then have to determine (by trial and error) which eigenvalues go together to form solutions.

$\nu = 1$ was worked out in section 2, so turn to $G = \mathbb{Z}_3$. Consider first $\omega = 1$. The order-3 symmetry (4.7) gives us (7.1). These variables (c, d, e, f, g) satisfy (4.8)–(4.10). The Gröbner basis method tells us there are precisely 8 solutions. However by Theorem 2, two solutions $A^{(1)}$, $A^{(2)}$ yield equivalent fusion categories if they can be obtained from each other by the action of $\mathrm{Aut}(\mathbb{Z}_3) \cong \{\pm 1\}$, i.e. if $A^{(1)}_{i,j} = A^{(2)}_{-i,-j}$ for all $i, j \in \mathbb{Z}_3$. In other words, the 5-tuples (c, d, e, f, g) and (c, e, d, g, f) are equivalent. Up to this equivalence, we then get 4 solutions, as given in Theorem 3. There, $c_1 = (2 - \sqrt{13})/3$, $c_2 = (7 - \sqrt{13})/6$, $c_3 = (7 + \sqrt{13})/6$, $c_4 = (2 + \sqrt{13})/3$. $d_1, \dots, d_4 \approx -0.321, 0.554, 0.717 - 0.329i, 0.717 + 0.329i$ respectively are the roots of $9x^4 - 15x^3 + 7x^2 + x - 1$, while $d_5 = (1 - \sqrt{13})/6$ and $d_6 = (1 + \sqrt{13})/6$. Finally, $f_1, \dots, f_4 \approx 0.217 + 0.758i, 0.217 - 0.758i, -0.954, 0.186$ respectively are the roots of $9x^4 + 3x^3 + x^2 + 5x - 1$, and $f_5 = (1 + \sqrt{13})/6$, $f_6 = (1 - \sqrt{13})/6$.

Now consider $G = \mathbb{Z}_3$ with $\omega \neq 1$, a nontrivial third root of 1. Then (4.7) gives

$$A = \begin{pmatrix} 0 & d & e \\ \overline{\omega} d & \omega e & 0 \\ \overline{\omega} e & 0 & \omega d \end{pmatrix},$$

where the zeros arise for any entry of A fixed by the order-3 symmetry. The quadratic identities (4.9) give e.g. $\overline{\omega}(d^2 + e^2) = 1 - 1/\delta_\pm$ and $d^2 + e^2 = 1$, which are incompatible. Thus there are no solutions for $G = \mathbb{Z}_3$ with $\omega \neq 1$.

Now turn to $G = \mathbb{Z}_5$, with $\omega = 1$. (4.7) gives (7.2). The Gröbner basis method tells us (4.8)–(4.10) have exactly 16 solutions (as always, half with '+' and half with '−'). As before, we must identify solutions related by the action of $\mathrm{Aut}\, G \cong \mathbb{Z}_4$, which sends $(c, d, e, f, g, h, i, j, k) \mapsto (c, e, g, d, f, i, j, k, h)$. This yields the 4 inequivalent fusion categories given in Theorem 3. Explicitly, $c_1 = (13 + \sqrt{29})/10$, $c_2 = (13 - \sqrt{29})/10$, $c_3 = (7 + \sqrt{29})/5$, and $c_4 = (7 - \sqrt{29})/5$. Also, $d_1 = (3 - \sqrt{29})/10$, $d_2 = (3 + \sqrt{29})/10$, $d_3 \approx -0.537$, $d_4 \approx -0.426$, $d_5 \approx -0.032$, $d_6 \approx 0.480$, $d_7 \approx 0.400 - 0.282i$, $d_8 \approx 0.400 + 0.282i$, $d_9 \approx 0.957 - 0.983i$, $d_{10} \approx 0.957 + 0.983i$, where the final 8 of these d_i are the roots of the irreducible polynomial $625x^8 - 1375x^7 + 1275x^6 + 245x^5 - 654x^4 + 152x^3 + 75x^2 - 29x - 1$. Finally, $h_1 \approx -0.675$, $h_2 \approx 0.218$, $h_3 \approx 0.437$, $h_4 \approx 0.620$, $h_5 \approx -1.270$, $h_6 \approx -0.095$, $h_7 \approx 0.084 - 0.536i$, $h_8 \approx 0.084 + 0.536i$, $h_9 \approx 0.106$, $h_{10} \approx 0.534 - 0.099i$, $h_{11} \approx 0.534 + 0.099i$, $h_{12} \approx 1.420$, where h_1 to h_4 are solutions to the irreducible polynomial $25x^4 - 15x^3 - 9x^2 + 7x - 1$, while h_5 to h_{12} are solutions to the irreducible $625x^8 - 875x^7 - 525x^6 + 1110x^5 - 789x^4 + 402x^3 - 95x^2 - 3x + 1$.

Finally, turn to $G = \mathbb{Z}_5$ and $\omega \neq 1$ a nontrivial third root of 1. Write

$$A = \begin{pmatrix} 0 & d & e & f & g \\ \bar{\omega}d & \omega g & h & i & \bar{\omega}h \\ \bar{\omega}e & j & \omega f & \omega i & \bar{\omega}i \\ \bar{\omega}f & k & \bar{\omega}k & \omega e & \omega h \\ \bar{\omega}g & \omega j & \omega k & \bar{\omega}j & \omega d \end{pmatrix} .$$

Using the Gröbner basis method, it can be shown that (4.8) and (4.9) with $h = 0, 1$ are inconsistent. This concludes the proof of Theorem 3.

As is curious aside, the linear and quadratic identities (4.8), (4.9) suffice to fix A for $G = \mathbb{Z}_5$, but for $G = \mathbb{Z}_3$ there are 8 spurious solutions which run afoul of the quartic (4.10) (or cubic (4.11)) identities.

We know of no examples of fusion categories with $\omega \neq 1$.

Of course, the set of all fusion categories $\mathcal{C}(G; \pm, \omega, A)$ for fixed G is closed under Galois actions. Theorem 3 is disappointing, in that all fusion categories for $G = \mathbb{Z}_1, \mathbb{Z}_3, \mathbb{Z}_5$ are Galois associates of known unitary fusion categories. But we see no reason at all to expect this to continue for larger G, and expect it is an accident of small G.

7.2. Some Q-systems and their doubles

Q-systems are unitary fusion categories coming from an even part of a subfactor. After Theorem 1 we explained they correspond here to $\omega = 1$, '+', and A with specified values for $A_{0,g}, A_{g,0}, A_{g,g}$. Evans and Gannon [11] found several new Q-systems of type $\mathcal{C}(G; \pm, \omega, A)$, although was unable to identify the modular data of some of them. In this subsection we use Proposition 3 to explain how they all fit into Conjecture 1.

A convenient way to express the matrix A of a Q-system, for $G = \mathbb{Z}_\nu$, is in terms of numbers $j_2, j_3, \ldots, j_{n+1} \in \mathbb{R}$ (recall $\nu = 2n + 1$): for $0 < g < h < \nu$ we have

$$A_{g,h} = \overline{A_{h,g}} = \frac{\sqrt{\delta}}{\delta - 1} \exp[\mathrm{i}(j_h - j_g - j_{h-g})] ,$$

where $j_1 = 0$ and $j_{n+1+i} = j_{n+1} + j_n - j_{n-i}$ for $1 \leq i < n$ (see Lemma 7.3 of [22]). The Q-systems found in [11] correspond to

$$(j_2^{(7)}, j_3^{(7)}, j_4^{(7)}) \approx (2.471228, 0.51685555, 0.2137724) ;$$

$$(j_2^{(9)}, \ldots, j_5^{(9)}) \approx (2.396976693, 2.079251103, -0.2079168419, -2.508673987) ;$$

$$(j_2^{(9)\prime}, \ldots, j_5^{(9)\prime}) \approx (-2.364737070, 1.031057162, 1.569692175, 0.3383837765) ;$$

$$j^{(11)} \approx (0.9996507, 2.7258434, -0.5714203, -1.7797340, 1.2675985) ,$$

$$j^{(11)\prime} \approx (-2.6444397, -1.7629598, -2.6444440, 2.7572657, 0.1128260) ;$$

$$j^{(13)} \approx (-3.1050384, 0.5993399, -0.111708, -0.969766, 1.336848, 1.00483129) ;$$

$$j^{(15)} \approx (-1.0777623, -.7748018, -2.171863, -1.6068402, -0.257508, 2.092502, 0.72289565) ;$$

$$j^{(17)} \approx (-1.466074, .291489, 3.130735, -2.693185, 1.398153, -0.611938,$$
$$-1.667078, -1.754821)\,;$$
$$j^{(19)} \approx (-2.677465, 1.088972, -0.899442, 0.015448, -1.240928, -0.493394, 1.839879,$$
$$-1.525884, -2.084374)\,;$$
$$j^{(19)\prime} \approx (0.896858, -0.882585, -2.369855, -1.873294, -1.711620, -0.119360, 2.972018,$$
$$-2.460652, 0.041334)\,,$$

where the superscript (7) etc refers to the value of ν. These approximate values suffice to determine the exact algebraic values of the $A_{g,h}$, and to verify that these do indeed satisfy all the equations (4.8)–(4.10), using the method described in section 3.5 of [12]. (The solutions for $11 \le \nu \le 19$ were announced as conjectural in [11], but using [12] have now been shown to yield exact solutions.) This list constitutes the complete classification of Q-systems for $\mathbb{Z}_7, \mathbb{Z}_9$, up to equivalence. There is no Q-system solution for $G = \mathbb{Z}_3 \times \mathbb{Z}_3$ (more precisely, any such solution would require nontrivial 2-cocycle twists of (4.1)).

Proposition 3. *The modular data for the 10 Q-systems listed above, is given by Conjecture 1 with abelian group H and bilinear form β given by:*
$j^{(7)}$: $H = \mathbb{Z}_{53}$, $\beta(l, l') = ll'/53$;
$j^{(9)}$: $H = \mathbb{Z}_{85}$, $\beta(l, l') = ll'/85$;
$j^{(9)\prime}$: $H = \mathbb{Z}_{85}$, $\beta(l, l') = 12ll'/85$;
$j^{(11)}$: $H = \mathbb{Z}_{125}$, $\beta(l, l') = ll'/125$;
$j^{(11)\prime}$: $H = \mathbb{Z}_{25} \times \mathbb{Z}_5$, $\beta((l_1, l_2), (l_1', l_2')) = 2l_1l_1'/25 + 2l_2l_2'/5$;
$j^{(13)}$: $H = \mathbb{Z}_{173}$, $\beta(l, l') = ll'/173$;
$j^{(15)}$: $H = \mathbb{Z}_{229}$, $\beta(l, l') = ll'/229$;
$j^{(17)}$: $H = \mathbb{Z}_{293}$, $\beta(l, l') = ll'/293$;
$j^{(19)}$: $H = \mathbb{Z}_{365}$, $\beta(l, l') = ll'/365$;
$j^{(19)\prime}$: $H = \mathbb{Z}_{365}$, $\beta(l, l') = 22ll'/365$.

Given a nondegenerate bilinear form β on some abelian group of order $\nu^2 + 4$, let S^β, T^β denote the modular data described in Conjecture 1. Section 4.1 of [11] proved this proposition for these Q-systems at $\nu = 7, 13, 15, 17$, and conjectured the correct H and β for 5 of the 6 remaining. It was unable to determine the modular data for the 2+2+2 Q-systems at $G = \mathbb{Z}_9, \mathbb{Z}_{11}, \mathbb{Z}_{19}$, because of the ambiguity described in section 6.2 above. It had no guess for the modular data for $j^{(11)\prime}$ because it did not think of trying noncyclic H.

Our proof of Proposition 3 followed very closely what we used in [11], section 4.1. In particular, a floating point proof is possible and effective, since the integrality of the fusion coefficients N_{ij}^k in (6.21) serves as error-correction. More precisely, equation (1.3) of [11] shows S in modular data is uniquely determined from the fusion coefficients, T and the entries $S_{1,i}$. Our strategy here is to guess at a phase w_j consistent with

Conjecture 1, use the linear equations (6.14) and (6.20) to determine the corresponding $C^j_{g,h}$ up to a small number of parameters (for almost all choices of w_j, this linear system will be inconsistent and we can throw away that choice). For typical examples, the choice $w_j = e^{2\pi i 182/365}$ for solution $j^{(19)}$ identifies $C^j_{g,h}$ up to 1 parameter, while the choice $w_j = e^{2\pi i 2/5}$ for solution $j^{(11)\prime}$ needs 4 parameters. Then we chose at random some nonlinear equations from (6.17) to fix those parameters.

7.3. Character vectors

A natural question is to realise the doubles of these fusion categories by completely rational nets of factors and/or by rational vertex operator algebras (VOAs). As a first step, one should consider the corresponding character vectors. This is quite accessible, and provides considerable information.

Definition 2. Let ρ be a d-dimensional representation of $\mathrm{SL}_2(\mathbb{Z})$, such that $T := \rho\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ is diagonal. By a *character vector* $\mathbb{X}(\tau) = (\chi_1(\tau), \ldots, \chi_d(\tau))^t$ for ρ, we mean a holomorphic function $\mathbb{X}$ from the upper half-plane $\mathbb{H} = \{\tau \in \mathbb{C} \,|\, \mathrm{Im}\,\tau > 0\}$ to $\mathbb{C}^d$, which obeys

$$\mathbb{X}\left(\frac{a\tau+b}{c\tau+d}\right) = \rho\begin{pmatrix} a & b \\ c & d\end{pmatrix}\mathbb{X}(\tau) \tag{7.3}$$

for all $\tau \in \mathbb{H}$ and $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, and for which there exist exponents $\lambda_k \in \mathbb{R}$ and coefficients $\chi_{k;n} \in \mathbb{Z}_{\geq 0}$, such that

$$e^{-2\pi i\lambda_k\tau}\chi_k(\tau) = \sum_{n=0}^{\infty}\chi_{k;n}q^n \tag{7.4}$$

converges absolutely for $|q| < 1$, for $k = 1, \ldots, d$, where we write $q = e^{2\pi i\tau}$. We also require $\chi_{1;0} = 1$.

Choosing any λ'_k so that $T_{kk} = e^{2\pi i\lambda'_k}$, it is clear from holomorphicity and the transformation law (7.3) at $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$, that $e^{-2\pi i\lambda'_k\tau}\chi_k(\tau)$ is holomorphic in the punctured disc $0 < |q| < 1$ with an isolated singularity at $q = 0$, so (7.4) should be regarded as a meromorphicity condition at the so-called cusp $\tau = i\infty$. Any holomorphic $\mathbb{X} : \mathbb{H} \to \mathbb{C}^d$ obeying (7.3) and (7.4) is called a weakly-holomorphic vector-valued modular function (vvmf) for $\mathrm{SL}_2(\mathbb{Z})$ with multiplier ρ. The characters of the irreducible modules M_j for any completely rational conformal net of factors on S^1, or for any strongly rational VOA, form a character vector, where ρ is the modular data coming from the corresponding MTC. The label 1 is the vacuum module $M_1 = \mathcal{V}_1$ (the VOA or net itself), and $T_{11} = e^{-\pi ic/12}$ for a parameter c called the central charge. We can assume without loss of generality that all $\chi_{k;0} \neq 0$, in which case $h_k = \lambda_k + c/24$ is called the conformal weight of the module M_j. Because T is only determined by the MTC up to a third root of 1, the category determines the central charge only mod 8. For the doubles of fusion categories, as mentioned previously, the central charge c is known to be in $8\mathbb{Z}$.

The existence of a character vector is not at all automatic. For one thing, it requires that all $\lambda_j \in \mathbb{Q}$, but that holds in any MTC. Moreover, given any character vector $\mathbb{X}(\tau)$, the vector $\mathbf{v} := \mathbb{X}(i)$ exists and is strictly positive (since at $\tau = i$ we have $q = e^{-2\pi} > 0$); then (7.3) says $\mathbf{v} = S\mathbf{v}$ and hence S must have a strictly positive eigenvector with eigenvalue 1. But we know that in any modular data, some column (equivalently row, since $S = S^t$) of S, namely the common Perron–Frobenius eigenvector of the fusion matrices $N_i = (N_{ij}^k)$, must have constant phase. This is why we demanded that a column of S be strictly positive, in section 6.3.

When the MTC is unitary, we must have $c \geq 0$ and $h_k > h_1 = 0$ for $k \neq 1$. The only unitary VOA or net at $c = 0$ is the trivial theory. In the unitary case, the positive row of S must be the first (=vacuum) row. When unitarity is dropped, then $h_k \geq h_{1'}$ for all k. The quantity $c_{\text{eff}} = -24h_{1'}$ is called the effective central charge, and must be nonnegative. Again, $c_{\text{eff}} = 0$ can only occur for the trivial VOA and conformal net. To our knowledge, all known examples have $h_j > h_{1'}$ for $j \neq 1'$, but this is not yet a theorem.

The Hauptmodul $j(\tau) = q^{-1} + 744 + 196884q + \cdots$ of $\mathrm{SL}_2(\mathbb{Z})$ is a weakly-holomorphic modular function for the trivial multiplier. For any ρ in Definition 2, the space $\mathcal{M}^!(\rho)$ of weakly-holomorphic vvmfs is trivially a module over the polynomial ring $\mathbb{C}[j(\tau)]$. It turns out that this module is always free of rank d (Theorem 3.3(a) of [16]). Put another way, there is a $d \times d$ matrix

$$\Xi(\tau) = q^{\Lambda} \sum_{n=0} \Xi_n q^n ,$$

with coefficients $\Xi_n \in M_{n\times n}(\mathbb{C})$, with the property that $\mathbb{X}(\tau) \in \mathcal{M}^!(\rho)$ iff there is a vector-valued polynomial $p(x) \in \mathbb{C}^d[x]$ such that $\mathbb{X}(\tau) = \Xi(\tau)p(j(\tau))$. So knowing all weakly-holomorphic vvmfs for ρ is equivalent to knowing $\Xi(\tau)$. We can and will require $\Xi_0 = I_{d\times d}$. The matrix Λ will be diagonal, with entries satisfying $T_{kk} = e^{2\pi i \Lambda_{kk}}$. There is a recursion uniquely determining each Ξ_n from the complex matrices Λ and Ξ_1 (equation (36) of [16]). In short, knowing all weakly-holomorphic vvmfs for ρ is equivalent to knowing the exponents Λ and the first nontrivial coefficient matrix Ξ_1.

Once $\Xi(\tau)$ (or equivalently Λ, Ξ_1) are known, it is then just combinatorics to find all character vectors for a given effective central charge (since c_{eff} directly gives bounds for the degrees of all component polynomials p_k in $p(x) \in \mathbb{C}^d[x]$). In [11], this procedure was done for several doubles, including the double of the Haagerup fusion categories, for central charges $8, 16, 24$.

To illustrate this for a non-unitary example, in this subsection we give $\Xi(\tau)$ for the non-unitary cousin of the Haagerup ($G = \mathbb{Z}_3$). Its fusion category and MTC is a Galois associate of that of the Haagerup. By contrast, $\Xi(\tau)$ and hence the corresponding VOA or conformal net, are not at all related in an obvious way to those of the Haagerup, as we'll see.

The double of either of the unitary fusion categories for $G = \mathbb{Z}_3$, at any (effective) central charge $c = c_{\rm eff} \equiv 8 \pmod{24}$ (one of the three possibilities), was found in [11] to have Λ resp. Ξ_1 be

$$\mathrm{diag}(-1/3, -1/3, -1/3, -1/3, -1, -2/3, -34/39, -19/39, -5/39, -37/39, -31/39, -28/39)\,,$$
$$\begin{pmatrix} 6 & 80 & 81 & 81 & 8748 & 1215 & 3549 & 273 & 13 & 5538 & 2275 & 1378 \\ 80 & 6 & 81 & 81 & 8748 & 1215 & -3549 & -273 & -13 & -5538 & -2275 & -1378 \\ 81 & 81 & 167 & -81 & -8748 & -1215 & 0 & 0 & 0 & 0 & 0 & 0 \\ 81 & 81 & -81 & 167 & -8748 & -1215 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 3 & -3 & -3 & -12 & 18 & 0 & 0 & 0 & 0 & 0 & 0 \\ 27 & 27 & -27 & -27 & 1458 & -152 & 0 & 0 & 0 & 0 & 0 & 0 \\ 7 & -7 & 0 & 0 & 0 & 0 & -88 & -14 & -1 & 50 & 63 & 64 \\ 42 & -42 & 0 & 0 & 0 & 0 & -1484 & 92 & 16 & 2940 & -192 & -1041 \\ 119 & -119 & 0 & 0 & 0 & 0 & -2142 & 987 & 11 & -24990 & -6035 & 4641 \\ 5 & -5 & 0 & 0 & 0 & 0 & 17 & 13 & -3 & -2 & 35 & -14 \\ 13 & -13 & 0 & 0 & 0 & 0 & 174 & -1 & -5 & 294 & -147 & 51 \\ 14 & -14 & 0 & 0 & 0 & 0 & 448 & -77 & 7 & -343 & 125 & -24 \end{pmatrix}$$

(we are following the conventions of [16], which has Λ shifted by the identity from the Λ used in [11]). Here, the positive row of S is $1' = 1$, the vacuum 0. At (effective) central charge $c = 8$, the polynomial $p(x)$ will be $(\alpha, 0, 0, 0, 0, 0, 0, 0, \beta, 0, 0, 0)^t$ for constants $\alpha, \beta \in \mathbb{C}$ (otherwise λ_1 would not be the unique minimum). But $\alpha = 1$, since $\chi_{1;0} = 1$. Thus the only possible character vectors at central charge $c = 8$ are

$$\begin{pmatrix} \chi_0(\tau) \\ \chi_{\mathfrak{b}}(\tau) \\ \chi_{\mathfrak{a}}(\tau) = \chi_{\mathfrak{c}_0}(\tau) \\ \chi_{\mathfrak{c}_1}(\tau) \\ \chi_{\mathfrak{c}_2}(\tau) \\ \chi_{\mathfrak{d}_1}(\tau) \\ \chi_{\mathfrak{d}_2}(\tau) \\ \chi_{\mathfrak{d}_3}(\tau) \\ \chi_{\mathfrak{d}_4}(\tau) \\ \chi_{\mathfrak{d}_5}(\tau) \\ \chi_{\mathfrak{d}_6}(\tau) \end{pmatrix} = \Xi(\tau) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \beta \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} q^{-1/3}\left(1 + (6+13\beta)q + (120+78\beta)q^2 + (956+351\beta)q^3 + (6010+1235\beta)q^4 + \cdots\right) \\ q^{2/3}\left((80-13\beta) + (1250-78\beta)q + (10630-351\beta)q^2 + (65042-1235\beta)q^3 + \cdots\right) \\ q^{2/3}\left(81 + 1377q + 11583q^2 + 71037q^3 + \cdots\right) \\ 3 + 243q + 2916q^2 + 21870q^3 + \cdots \\ q^{1/3}\left(27 + 594q + 5967q^2 + 39852q^3 + \cdots\right) \\ q^{5/39}\left((7-\beta) + (292-6\beta)q + (3204-43\beta)q^2 + (23010-146\beta)q^3 + \cdots\right) \\ q^{20/39}\left((42+16\beta) + (777+121\beta)q + (7147+547\beta)q^2 + (45367+2000\beta)q^3 + \cdots\right) \\ q^{-7/39}\left(\beta + (11\beta+119)q + (73\beta+1623)q^2 + (300\beta+12996)q^3 + (76429+1063\beta)q^4 + \cdots\right) \\ q^{2/39}\left((5-3\beta) + (229-50\beta)q + (2738-252\beta)q^2 + (19942-1032\beta)q^3 + \cdots\right) \\ q^{8/39}\left((13-5\beta) + (347-37\beta)q + (3804-212\beta)q^2 + (26390-794\beta)q^3 + \cdots\right) \\ q^{11/39}\left((14+7\beta) + (441+61\beta)q + (4445+303\beta)q^2 + (30329+1167\beta)q^3 + \cdots\right) \end{pmatrix}$$

The first coefficient of $\chi_{\mathfrak{d}_3}(\tau)$ (i.e. $\chi_{9;0}$) tells us $\beta \in \mathbb{Z}_{\geq 0}$, while the first coefficient of $\chi_{\mathfrak{d}_4}(\tau)$ (i.e. $\chi_{10;0}$) then implies $\beta = 0, 1$. Thus there are only two possible character vectors for the Haagerup modular data at $c = 8$, as stated in [11].

The double of either of the non-unitary fusion categories for $G = \mathbb{Z}_3$, at effective central charge $c_{\text{eff}} \equiv 8 \pmod{24}$ (one of three possibilities), has Λ resp. Ξ_1 equal to

$$\mathrm{diag}(-1/3, -1/3, -1/3, -1/3, -1, -2/3, -16/39, -25/39, -40/39, -22/39, -49/39, -4/39)\,,$$

$$\begin{pmatrix} 110 & -24 & 81 & 81 & -4374 & 1215 & -390 & -1820 & -16770 & -910 & -53872 & 52 \\ -24 & 110 & 81 & 81 & -4374 & 1215 & 390 & 1820 & 16770 & 910 & 53872 & -52 \\ 81 & 81 & 167 & -81 & 4374 & -1215 & 0 & 0 & 0 & 0 & 0 & 0 \\ 81 & 81 & -81 & 167 & 4374 & -1215 & 0 & 0 & 0 & 0 & 0 & 0 \\ -6 & -6 & 6 & 6 & -12 & -36 & 0 & 0 & 0 & 0 & 0 & 0 \\ 27 & 27 & -27 & -27 & -729 & -152 & 0 & 0 & 0 & 0 & 0 & 0 \\ -28 & 28 & 0 & 0 & 0 & 0 & 143 & -405 & -9580 & -518 & 3654 & -1 \\ -1/2 & 1/2 & 0 & 0 & 0 & 0 & -81 & -262 & 1457 & 56 & -3832 & 26 \\ -1/2 & 1/2 & 0 & 0 & 0 & 0 & -7 & 7 & -12 & 6 & -7 & 1 \\ -28 & 28 & 0 & 0 & 0 & 0 & -35 & 120 & 1820 & -314 & 7224 & -27 \\ -1/2 & 1/2 & 0 & 0 & 0 & 0 & 2 & 2 & 1 & -1 & 0 & -1 \\ -57/2 & 57/2 & 0 & 0 & 0 & 0 & 399 & 2660 & 8436 & -854 & -204212 & 79 \end{pmatrix}$$

Here, the positive row of S is $l = 2$, the primary $\mathfrak{b}$. At effective central charge $c_{\text{eff}} = 8$ for this ρ, the polynomial $p(x)$ will be $(\alpha, \beta, \gamma, \delta, 0, 0, 0, 0, 0, 0, 0, \epsilon)^t$ for constants $\alpha, \beta, \gamma, \delta, \epsilon \in \mathbb{C}$ (otherwise λ_2 would not be the unique minimum). Thus the only possible character vectors at effective central charge $c_{\text{eff}} = 8$ are

$$\begin{pmatrix} q^{-1/3}\,(\alpha + (110\alpha + 52\epsilon - 24\beta + 81\gamma + 81\delta)q + (1589\alpha - 219\beta + 1377\gamma + 1377\delta + 650\epsilon)q^2 \\ + (12721\alpha - 1135\beta + 11583\gamma + 11583\delta + 4108\epsilon)q^3 + \cdots) \\ q^{-1/3}\,(\beta + (110\beta - 24\alpha + 81\gamma + 81\delta - 52\epsilon)q + (1589\beta + 1377\gamma + 1377\delta - 650\epsilon - 219\alpha)q^2 \\ + (12721\beta + 11583\gamma + 11583\delta - 4108\epsilon - 1135\alpha)q^3 + \cdots) \\ q^{-1/3}\,(\gamma + (167\gamma + 81\alpha - 81\delta + 81\beta)q + (2747\gamma - 1377\delta + 1377\alpha + 1377\beta)q^2 \\ + (23169\gamma - 11583\delta + 11583\alpha + 11583\beta)q^3 + \cdots) \\ q^{-1/3}\big(\delta + (167\delta + 81\alpha + 81\beta - 81\gamma)q + (2747\delta + 1377\alpha + 1377\beta - 1377\gamma)q^2 \\ + (23169\delta + 11583\alpha + 11583\beta - 11583\gamma)q^3 + \cdots) \\ -6\alpha - 6\beta + 6\gamma + 6\delta + (-486\alpha - 486\beta + 486\gamma + 486\delta)q + (-5832\alpha - 5832\beta + 5832\gamma + 5832\delta)q^2 + \cdots \\ q^{1/3}\,(27\alpha + 27\beta - 27\gamma - 27\delta + (594\alpha + 594\beta - 594\gamma - 594\delta)q \\ + (5967\alpha + 5967\beta - 5967\gamma - 5967\delta)q^2 + \cdots) \\ q^{23/39}\,(-28\alpha + 28\beta - \epsilon + (-1025\alpha/2 + 1025\beta/2 - 52\epsilon)q + (-4359\alpha + 4359\beta - 378\epsilon)q^2 + \cdots) \\ q^{14/39}\,(-\alpha/2 + \beta/2 + 26\epsilon + (-95\alpha + 95\beta + 352\epsilon)q + (-1416\alpha + 1416\beta + 2431\epsilon)q^2 + \cdots) \\ q^{-1/39}\,(-\alpha/2 + \beta/2 + \epsilon + (-67\alpha + 67\beta + 53\epsilon)q + (-932\alpha + 932\beta + 431\epsilon)q^2 + \cdots) \\ q^{17/39}\,(-28\alpha + 28\beta - 27\epsilon + (-512\alpha + 512\beta - 378\epsilon)q + (-8585\alpha/2 + 8585\beta/2 - 2510\epsilon)q^2 + \cdots) \\ q^{-10/39}\,(-\alpha/2 + \beta/2 - \epsilon + (-67\alpha + 67\beta - 53\epsilon)q + (-904\alpha + 904\beta - 457\epsilon)q^2 + \cdots) \\ q^{-4/39}\,(\epsilon + (79\epsilon - 57\alpha/2 + 57\beta/2)q + (756\epsilon - 579\alpha + 579\beta)q^2 + (4513\epsilon - 5196\alpha + 5196\beta)q^3 + \cdots) \end{pmatrix}$$

We see that $\alpha, \beta, \gamma, \delta, \epsilon \in \mathbb{Z}_{\geq 0}$ and $\alpha \equiv \beta \pmod 2$; in fact $\beta > 0$ since $c_{\text{eff}} = 8$. Comparing the leading terms of $\chi_5(\tau)$ and $\chi_6(\tau)$, we must have $\alpha + \beta = \gamma + \delta$ and hence also $\gamma \equiv \delta$

(mod 2). This means that the q, q^2, q^3 coefficients of $q^{1/3}\chi_1(\tau)$ are all even and thus cannot equal 1. Hence either $c \leq -88$ or $\alpha = 1$. Assume $\alpha = 1$. Then all coefficients of e.g. $\chi_5(\tau)$ up to at least q^4 vanish. We don't have a proof yet that there is no character vector with $c_{\text{eff}} = 8$ for this ρ, but it seems highly likely.

This calculation is meant to give further evidence that, even though these unitary and non-unitary fusion categories and hence MTCs are related simply by a Galois automorphism, the relation if any between corresponding VOAs or conformal nets will be far from straightforward.

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