CARDIFF BUSINESS SCHOOL
WORKING PAPER SERIES

Cardiff Economics
Working Papers

Paulo Brito and Huw Dixon

Entry and the accumulation of capital: a two state-variable extension to the Ramsey model

E2007/16

Cardiff Business School
Cardiff University
Colum Drive
Cardiff CF10 3EU
United Kingdom
t: +44 (0)29 2087 4000
f: +44 (0)29 2087 4419
www.cardiff.ac.uk/carbs

ISSN 1749-6101
June 2007, updated October 2007

Entry and the accumulation of capital: a two state-variable extension to the Ramsey model.*

Paulo Brito[†] and Huw Dixon[‡]

October 12th, 2007.

Abstract

In this paper we consider the entry and exit of firms in a dynamic general equilibrium model with capital. At the firm level, there is a fixed cost combined with increasing marginal cost, which gives a standard U-shaped cost curve with optimal firm size. Entry is determined by a free entry condition such that the costs of entry are equal to the present value of incumbent firms, the cost of entry (exit) depends on the flow of entry (exit). Then equilibrium is saddle-point stable and the stable manifold is two-dimensional. Transitional dynamics can, under certain circumstances, be non-monotonic.

JEL Classification: D92, C62, E32, O41
Keywords: Entry, dynamics, Ramsey.

*Financial support by FCT is gratefully acknowledge. This article is part of project POCTI/ECO/46580/2002 which is co-funded by ERDF. This paper was presented at the 1st Meeting of the Portuguese Economic Journal (University of the Azores, June 2007), ASSET 2007, Porto and Cardiff. We would like to thank Luis Costa, Joao Correia da Silva, Partha Sen Aditya Goenka and Basant Kapur for their comments on earlier drafts. Faults remain our own.

[†]ISEG (School of Economics and Management) of the Technical University of Lisbon, Rua do Quelhas, 6, 1200-781 Lisboa, Portugal, and UECE (Research Unit on Complexity and Economics), Rua Miguel Lupi, 20, 1249-078 Lisboa, Portugal. pbrito@iseg.utl.pt.

[‡]Cardiff Business School, Aberconwy Building, Colum Drive, Cardiff CF12 3EU UK. dixonh@Cardiff.ac.uk

1 Introduction

The Marshallian approach to perfect competition focusses on a world where firms have U-shaped average cost curves and where the long-run competitive equilibrium was one in which all firms were producing at efficient scale. As pointed out by Novshek and Sonnenschein (1987), this contrasts with the Arrow-Debreu setting where firms production sets are convex, so that both average and marginal costs are non-decreasing[1]. Furthermore, the number of firms is taken as given in general equilibrium analysis (Smith (1974)). Turning to explicitly dynamic macroeconomic models, the firm is often ignored unless there is imperfect competition. Where the number of firms is endogenous, although entry and exit are essentially dynamic phenomena, macroeconomists have made them into non-intertemporal phenomena. One approach is to have instantaneous free entry, so that the number of firms is that which ensures zero actual profits (see for example Devereux et al. (1996), Heijdra (1988), Coto-Martinez and Dixon (2003))[2] or zero expected profits (Hopenhayn (1992)). An alternative is to treat the number of firms as fixed over time determined by a non-dynamic long-run zero profit condition (Hornstein (1993)).

What this paper does is to introduce a rigorous treatment of the entry process and number of firms in a perfectly competitive dynamic general equilibrium economy and traces through the production process from the firm level to the aggregate This model is a generalization of the classic Ramsey model in which the firm level is not modelled and the representative household chooses consumption as a control variable and capital is the state variable. Here, there is an additional state-variable (the number of firms) and control (entry). This results in a four dimensional dynamic system, with a two-dimensional stable manifold. As we shall see, the related Ramsey model implicitly adopts two approaches: either there is a fixed number of firms, or else the number of firms adjusts instantaneously to the level of capital. However, these are too extreme: we allow for the case where the flow of entry is determined endogenously by an equilibrium entry model developed by Datta and Dixon (2002). The process of entry and the accumulation of capital interact in an explicitly dynamic setting. The entry model assumes that entry has a price at each instant in time, and this is increasing in the flow of entry and is zero when there is no entry (this is a special case of endogenous entry costs introduced by Das and Das (1996)). The dynamic equilibrium which results is one in which the cost of entry at each instant equals the net present value of an incumbent firm: as a consequence firms have no

[1] McKenzie (1959) took a quasi-Marshallian approach by assuming that an aggregate production function existed so that production sets are convex cones, this imposing the long-run constant returns of the Marshallian view point by the argument that the possibility of division and replication implies that the aggregate production function will have constant returns.

[2] This assumption is also made in the theory of Contestable markets, (see Baumol et al. (1982)).

incentive to delay or bring forward the time of entry/exit. In steady-state, there is zero entry and firms earn zero profits and we have the Marshallian long-run where average and marginal cost are equated. On trajectories towards steady-state, the flow of profits may be positive or negative and output per firm will differ from the efficient scale of production.

From the perspective of the representative household and the social planner, there are two ways of accumulating wealth: one is to set up new firms, the other is to accumulate capital. At all times, there is an arbitrage condition which ensures that the two assets have the same return. In steady-state the firms are at one level worthless, since they earn zero profits and have a zero marginal product. However, on another they are highly valuable: they enable the efficient organization of production in steady-state, where labour and capital are combined so that production occurs at the efficient scale and marginal cost equals average cost. Even though firms earn no profits in steady-state, firms will be set up (or closed down) on the way to steady-state equilibrium.

One of the most interesting findings of the paper is that for a wide range of initial conditions we can have a non-monotonic trajectory in one of the state-variables (but not both, because the roots are all real) [3] . This happens because of the interactions between the two state-variables: the number of firms influences the marginal product of capital, and the stock of capital influences the profitability of firms. For example, even if the number of firms is above its steady-state value, if there is a large capital stock this will boost firm profitability and lead to entry on the initial part of the trajectory. Likewise, a large number of firms boosts the marginal product of capital which may lead to initial capital being accumulated even though in the long-run capital is decumulated.

We study two types of technology shocks: one is a productivity variable that alters the marginal product of labour and capital; the other is the flow fixed costs per firm. We prove that productivity shocks will expand the economy by generating an increase in the number of firms while the firm-size will be asymptotically invariant. An increase in the fixed cost will imply an increase in firm size and a reduction in the number of firms asymptotically. An increase in both size and the number of firms occurs if the two shocks are positively correlated. We derive critical values for the correlation of shocks that result in non-monotonic trajectories in capital or the number of firms.

Existing papers that adopt a genuine dynamic entry model have assumed that there is only one state variable by removing capital. Thus Aloi and Dixon (2002) adopt the same entry model as this paper, but assume imperfect competition and

[3]This complements findings in a non-general equilibrium context with non-monotonic entry dynamics Gort and Klepper (1982) and Das and Das (1996).

labour as the only input. Bilbiie et al (2005) assume that there is an exogenous fixed entry cost and that the entrant evaluates the expected net present value of an incumbent. Free entry means that in equilibrium the net present value of an incumbent is equal to the entry cost. Again, this model assumes imperfect competition and no capital. Smith (1974) develops a genuine dynamic model of entry in a perfectly competitive economy. What determines the flow of entry in Smith's model is the opportunity cost of current consumption, since setting up a firm requires a one-off fixed labour input. Hence, if more firms are set up (or dismantled), then there is less labour available to provide for current consumption.

The paper is organized as follows: in section 2 the model is presented, section 3 studies the dynamics and section 4 reports the comparative dynamics for productivity and fixed costs shocks (uncorrelated and correlated).

2 The model

There is an infinitely lived household and at any time t a continuum of firms $i \in [0, n(t)]$. Households offer a fixed labour supply to firms and invest in their equity. Firms produce a single final product which is used for consumption and investment. Firms and households are price-takers in all markets. We now turn to the optimization programmes of firms and households in more detail.

2.1 Household

Households consume and collect income from investments in financial assets (equity) and labour income. They choose the trajectory of consumption $\{C(t), t \geq 0\}$ to maximize lifetime utility U:

$$U = \int_0^{\infty} u(C(t))e^{-\rho t}dt$$

where $u' > 0 > u''$, $u'(0) = +\infty$; $\lim_{C \to +\infty} u'(c) = 0$. The accumulation equation for financial assets is the instantaneous budget constraint

$$\dot{V} = r(t)V(t) + w(t) - C(t). \tag{1}$$

where r and w are the real rate of return on equity and the real wage rate, respectively, and the labour supply is normalized to unity. The initial level of wealth $V(0)$ is given and the no-Ponzi-game condition $\lim_{t\to\infty} e^{-\int_0^t r(s)ds}V(t) = 0$ holds.

In the set of the admissible consumption and wealth accumulation strategies, the optimal path of $(C(t), V(t))$ satisfies the Euler equation

$$\dot{C} = \frac{C}{\sigma(C)}(r(t) - \rho) \tag{2}$$

where $\sigma \equiv -u^{''}(C)C/u^{'}(C)$, and the transversality condition which is

$$\lim_{t\to\infty} e^{-\rho t}u^{'}(C(t))V(t) = 0. \tag{3}$$

2.2 Firms

There is a continuum of firms, $i \in [0, n(t)]$, where $n(t)$ is the measure of firms operating at instant t. Firms are price takers in all the markets in which they participate: they hire labour and capital to produce output which they sell to households.

At every moment in time there is entry. That is, at instant t consider that the number of firms will pass in the interval between t to $t+\epsilon$ from $n(t)$ to $n(t+\epsilon)$. If $n(t) < n(t+\epsilon)$ there is entry and if $n(t) > n(t+\epsilon)$ there is exit. The rate of entry is $\frac{n(t+\epsilon)-n(t)}{\epsilon}$. If the interval shrinks to zero, then the instantaneous rate of entry is $\dot{n}(t) = \lim_{\epsilon\to 0} \frac{n(t+\epsilon)-n(t)}{\epsilon}$. We define:

$$e(t) = \dot{n}. \tag{4}$$

Conceptually, we can divide the decisions made by firms into two parts. First, there is the *intra*-temporal decision about how much output to produce and correspondingly how much labour and capital to employ. This decision is made by all the $n(t)$ incumbent firms at time t. This will depend only on the output and input prices prevailing at instant t. Second, there is the inherently dynamic *entry* decision. At any instant t, non-incumbent firms have to decide whether to enter now or later (or never); incumbent firms have to decide whether to exit now or later (or never). We will look at these two decisions in turn.

2.2.1 Production.

We start with the problem for incumbent firms $i \in [0, n(t)]$ in instant t. Each firm employs capital and labour according to the following technology:

$$y(i,t) = AF(k(i,t), l(i,t)) - \phi, \tag{5}$$

where $A > 0$ is a productivity parameter and $\phi > 0$ represents a fixed overhead in terms of final output. F is strictly concave, homogeneous of degree $\nu < 1$ in capital and labour[4]. The Inada conditions hold for the marginal products of capital and labour. Since the function F is homogenous, the dual cost-function corresponding to (5) can be written as

$$B(w,r,y) = b(w,r).\left[\frac{y+\phi}{A}\right]^{1/\nu}, \tag{6}$$

where w and r are the rental cost of labour and capital respectively and $b(w,r)$ is an increasing convex function of (w,r). The average cost function corresponding to (5) is of the standard $U-$shaped variety: marginal cost is increasing since $\nu < 1$, average cost is initially decreasing and then increasing because of the overhead element ($\phi > 0$). This implies that there is an optimal scale to the firm, where average cost is minimized. From (6), for any $(w,r) >> 0$ average cost AC is minimized at the efficient firm size y^e

$$y^e = \frac{\phi\nu}{1-\nu}. \tag{7}$$

Figure 1

The optimal capital and labour corresponding this can then be obtained using Shepherd's Lemma ($k^e = B_r(w,r,y^e); l^e = B_w(w,r,y^e)$). It is useful to note that A does not affect the efficient scale, although it does reduce optimal factor inputs. A decrease in ϕ reduces both efficient scale and factor inputs. As firms have the same technology, from now on we set $k(i,t) = k(t)$, $l(i,t) = l(t)$ and $y(i,t) = y(t)$ for any $i \in [0, n(t)]$. We define *firm size* by output $y(t)$.

We can define the supernormal profit of the firm π as the surplus when each factor is priced at its marginal product:

$$\pi(t) = y(t) - A\left(F_k k(t) + F_l l(t)\right) = (1-\nu)AF - \phi. \tag{8}$$

The zero-profit condition is thus:

$$AF(k(t), l(t)) = \frac{\phi}{1-\nu}. \tag{9}$$

Note that this condition is equivalent to (7): hence the zero profit condition implies technical efficiency when factors are priced at their marginal products (hence $P = MC = AC$). If output is above y^e profits are strictly positive (since $P = MC > AC$) and negative if below y^e.

[4]Recall that homogeneity of degree ν for F implies the following relationships between $F(x,y)$ and its derivatives: $F_x x + F_y y = \nu F$, $(x,y) \cdot \frac{\partial^2 F(x,y)}{\partial (x,y)^2} \cdot (x,y) = \nu(\nu-1)F$, $xF_{xx} + yF_{xy} = (\nu-1)F_x$ and $xF_{xy} + yF_{yy} = (\nu-1)F_y$.

2.2.2 Entry and exit

The model of entry is based on Datta and Dixon (2002). Potential entrants (and quitters) evaluate the net present value NPV of incumbency,

$$NPV(t) = \int_t^\infty \pi(s) e^{-\int_t^s r(\tau)d\tau} ds. \tag{10}$$

We assume that there is a congestion effect which makes the cost of entry (exit) q rise with the flow of entry (and exit): in particular we assume that they are proportional:

$$q(t) = \frac{\gamma}{2} e(t), \tag{11}$$

which implies a total cost of entry in terms of output used to set up (dismantle) firms is

$$Z(t) = q(t).e(t) = \gamma \frac{e(t)^2}{2} \tag{12}$$

where γ is a parameter measuring the *dynamic barriers to entry (DBE)*. The congestion effect might arise from the setting up of firms at the same instant of time stretching some finite resource: the technology for the setting up of new firms has diminishing returns.

The free entry condition means that the flow of entry equates the cost of entry q to the net present value of incumbency:

$$q(t) = \int_t^\infty \pi(s) e^{-\int_t^s r(\tau)d\tau} ds \tag{13}$$

If we time-differentiate equation (13) we obtain

$$\dot{q} = -\pi(t) + r(t)q(t) \tag{14}$$

Re-arranging (14), we can see that this is an arbitrage equation, equating the rate of returns on investment and setting up a new firms:

$$r(t) = \frac{\pi(t)}{q(t)} + \frac{\dot{q}(t)}{q(t)}. \tag{15}$$

There are two elements to the RHS of (15): the profit earned by entering now and the change in the cost of entry. If entry costs are rising (falling) it means that there is an incentive to bring forward (delay) the moment of entry. Substituting (11) into (14) we obtain the dynamic equation for entry:

$$\dot{e} = r(t)e(t) - \pi(t)/\gamma. \tag{16}$$

The entry decision is inherently intertemporal. The entrant looks over the future and decides whether or not to pay the entry cost now. An important implication of equation (10) is that entry can be non-zero when current profits are zero. Since it is the NPV of profits that matters, the entrant evaluates the flow of profits along the entire trajectory: thus for example, firms may enter even when current profits are negative if profits eventually become positive. As we shall see, this is exactly what happens along some equilibrium trajectories in this economy. This contrasts with the dynamic entry model of Howrey and Quandt (1968) where the flow of entry is related solely to the instantaneous profits[5]. Bilbiie et al (2005) present a genuine dynamic model of entry in which the entrant evaluates the expected net present value of being an incumbent: however, the model is not solved analytically and there is an exogenous fixed entry cost which means that there is not a unique steady state which makes it inconsistent with the long-run competitive equilibrium (in our paper, the entry cost is zero in steady-state).

In equilibrium the entrant is indifferent between entering and not entering. Since this holds at each point in time, the potential entrant is also indifferent as to the timing of entry. For example, if the firm delays entering when the cost of entry is falling, it will find that the lower entry cost is exactly offset by the lower NPV of profits when it finally enters. This dynamic model of entry yields a dynamic zero-profit condition. The presence of entry costs means that the incumbents can earn strictly non-zero profits (losses) on the path to steady-state: the flow of entry adjusts so that the entry (exit) cost just balances the NPV of profits (losses) to be made. In the long-run steady-state, the cost of entry is zero and both the NPV of incumbents and the flow of profits π are zero.

2.3 Aggregation

Let us denote the aggregate capital and labour available at time t as $K(t)$ and $L(t)$. For a given number of firms, the optimal allocation across firms is to have equal capital and labour in each firm. This follows from the fact that marginal cost is everywhere increasing at the firm level. This is the outcome of decentralized factor markets where all firms face the same factor prices. Hence

$$\begin{aligned} K(t) &= \int_0^{n(t)} k(i,t)di = \int_0^{n(t)} k(t)di = n(t)k(t), \\ L(t) &= \int_0^{n(t)} l(i,t)di = \int_0^{n(t)} l(t)di = n(t)l(t). \end{aligned}$$

[5] See also Meyers and Weintraub (1971) and Okuguchi (1972).

For simplicity, we assume that $L(t) = 1$, so that $l(t) = 1/n(t)$ and the firm size and profits are:

$$
\begin{aligned}
y(t) &= AF\left(\frac{K(t)}{n(t)}, \frac{1}{n(t)}\right) - \phi, \qquad (17)\\
\pi(t) &= (1-\nu)AF\left(\frac{K(t)}{n(t)}, \frac{1}{n(t)}\right) - \phi.
\end{aligned}
$$

Hence aggregate output in the economy is:

$$
Y(t) = \int_0^{n(t)} y(t)di = n(t)y(t),
$$

to yield the aggregate output as a function of (K, n)

$$
Y(K, n, A, \phi) = n\left[AF\left(\frac{K}{n}, \frac{1}{n}\right) - \phi\right].
$$

For analyzing the dynamics of the system, it is best to represent the aggregate production technology $Y(K, n)$ in terms of the marginal product of K and n, the two state variables. Hence, the aggregate and the firm level marginal productivity of capital are equal, and there are decreasing returns at the aggregate level:

$$
Y_K = AF_k > 0, Y_{KK} = \frac{A}{n}F_{kk} < 0.
$$

The derivative of aggregate output with respect to the number of firms is equal to the profit per firm,

$$
Y_n = -A\left[F_k\frac{K}{n} + F_l\frac{1}{n}\right] + AF - \phi = (1-\nu)AF - \phi = \pi. \qquad (18)
$$

Hence, a zero profit equilibrium maximizes output $Y_n = 0$ if $\pi = 0$. Also, entry boosts the marginal product of labour and capital:

$$
Y_{Kn} = -\frac{A}{n}\left[F_{kk}\frac{K}{n} + F_{kl}\frac{1}{n}\right] = \frac{A}{n}(1-\nu)F_k > 0,
$$

more firms means less inputs per firm so that the marginal products increase. Although there are diminishing returns at the firm level, the aggregate production function $Y(K, L, n)$ is homogeneous of degree 1 in (K, L, n). Hence, if you double capital and labour and also double the number of firms, productivity and output at the firm level are unaffected, so that aggregate output also doubles.

The real interest rate is the real rate of return on capital:

$$r(K,n,A,\phi) = Y_K(K,n,A,\phi),$$

which is decreasing in the stock of capital and is increasing in the number of firms and the productivity parameter: $r_K = Y_{KK} < 0$ and $r_n = Y_{Kn} > 0$, $r_A = Y_{KA} > 0; r_\phi = 0$.

From (18) we can write profits per firm as a function of aggregate variables:

$$\pi\,(K,n,A,\phi) = Y_n(K,n,A,\phi),$$

π is increasing in the stock of capital and the productivity and is decreasing in the number of firms and in the fixed cost: $\pi_K = Y_{Kn} > 0$, $\pi_n = Y_{nn} = -\nu(1-\nu)AF/n < 0$. By Young's Theorem, $\pi_K = r_n = Y_{Kn}$. Total operating profits in the economy are $\Pi(k,n,A,\phi) = n\pi$.

We have described a general firm level technology in which there is a clearly defined optimal scale of production for firms which depends on the underlying technology and firms have the text-book cost curves with rising marginal costs and a $U-$shaped average cost curve. The role of firms is to define the way factor inputs are divided up, and hence how efficiently they are combined. More firms means that capital and labour are divided up into smaller units, with the effect that their marginal products will increase but the additional fixed overheads may reduce or increase total output.

2.4 General Equilibrium.

The balance sheet for households equates the value of equity holdings to the total value of firms in terms of their assets (capital) plus the NPV of future profits:

$$V(t) = \int_0^{n(t)} [k(t) + q(t)]\, di = K(t) + n(t)q(t),$$

so that we can we can characterize the change in the value of equity as

$$\dot{V} = \dot{K} + n(t)\dot{q} + \dot{n}q(t) - Z(t),$$

where $\dot{n}q(t) - Z(t)$ reflects the impact of entry/exit on total equity value. Hence

$$\dot{K} = \dot{V} - n(t)\dot{q} - \dot{n}q(t) + Z(t). \tag{19}$$

From equations (1), (14), (4), (12), we have

$$\dot{K} = rK + w + n\pi - \frac{\gamma}{2}e^2 - C,$$

and the aggregate accumulation equation for capital is:

$$\dot{K} = n\left[AF\left(\frac{K}{n},\frac{1}{n}\right) - \phi\right] - C - \gamma\frac{e^2}{2}, \tag{20}$$

which is equivalent to the product market equilibrium equation

$$Y(t) = C(t) + I(t) + Z(t),$$

where $I(t) = \dot{K}$. That is, output is equal to consumption plus investment in capital plus "investment" $Z(t)$ in setting up or dismantling firms.

The household's transversality condition (3) can be related to $(K(t), n(t))$ by noting that as , $\lim_{t\to\infty} q(t) = 0$. Since $(K(t), n(t))$ are strictly positive we therefore have

$$\lim_{t\to\infty} e^{-\rho t}U'(C(t))\,n(t)q(t) = \lim_{t\to\infty} e^{-\rho t}U'(C(t))\,K(t) = 0. \tag{21}$$

Definition: Equilibrium. . *The general equilibrium is defined by the aggregate variables* $(C(t), K(t), Y(t), V(t))$*, factor prices* $(r(t), w(t))$*, and the number and rate of change of firms* $(n(t), e(t))$*, for* $0 \le t < \infty$ *such that:*

1. *households determine* $C(t)$ *and* $V(t)$ *by maximizing lifetime utility subject to* $(1, 3)$ *given the factor prices;*

2. *incumbent firms choose* $(k(t),\ l(t),\ y(t))$ *by maximizing profits, given factor prices;*

3. *the flow of entry (exit)* $e(t) = \dot{n}(t)$ *equates the cost of entry (exit) with the NPV of an incumbent firm;*

4. *Trajectories* $(K(t), n(t))$ satisfy the transversality condition (21)

5. *the factor prices ensure goods and factor markets clear.*

The aggregate variables $(C(t), K(t), e(t), n(t))$ are determined jointly from the four ODE's (2), (16), (20) and (4), the initial conditions $K(0)$ and $n(0)$ and the transversality conditions. From these we can determine firm level variables $k(t)$ and $l(t)$, and firm size $y(t)$.

The general equilibrium model is equivalent to a centralized economy model in which a social planner maximizes the intertemporal utility function by choosing the path of aggregate consumption and entry (C, e) subject to the economy wide constraints, equations (20) and (4). The social planner's problem is described in Appendix 1.

2.5 Steady-State Equilibrium.

If we restrict the equilibria to the set of positive levels for the capital stock and the number of firms, K and n, there will be a unique stationary equilibrium point such that the transversality conditions holds. The steady-state aggregate capital stock and the number of firms, (K^*, n^*), will be determined by equating the marginal product of capital to the discount rate and setting profits equal to zero:

$$\begin{aligned} \rho &= r(K^*, n^*, A), \quad (22) \\ 0 &= \pi(K^*, n^*, A, \phi). \quad (23) \end{aligned}$$

Hence, there will be no entry

$$e^* = 0. \quad (24)$$

and aggregate consumption C^* will be equal to aggregate output

$$C^* = Y^* = n^* y^*. \quad (25)$$

Since profits are zero, firm size will be equal to the efficient level $y^* = y^e$, with the corresponding levels of inputs.

The steady state has intuitively simple properties. Since consumption is constant, the real rate of interest is equal to the consumers' rate of time preference (22); long run profits and the entry flow are zero (23,24) which implies that firms are producing at minimum average-cost; consumption is constant and equal to aggregate output (25). Given the Inada property of the production function the steady state exists and, given global concavity, is unique.

Observe that the steady state depends on the technology parameters (A, ϕ), the rate of time preference ρ but not on the elasticity of intertemporal substitution (σ) and DBE (γ). Changes in the last two parameters only influence the adjustment dynamics since they determine *intertemporal* arbitrages (postponement of consumption and of entry).

Figure 2

To solve for the steady state in levels, note that the equations are recursive. The first two equations can be solved for (K, n), which then determines consumption by (25). We can make a geometrical projection of the phase diagram into (K, n)-space as in figure 2. The two steady-state equilibrium conditions $(22, 23)$are invariant to the control variables (C, e) and the parameter γ. Their slopes are

$$\left.\frac{dn}{dK}\right|_{r=\rho} = -\frac{r_K}{r_n} > 0, \ \left.\frac{dn}{dK}\right|_{\pi=0} = -\frac{\pi_K}{\pi_n} > 0, \quad (26)$$

hence:

$$\left.\frac{dn}{dK}\right|_{\pi=0} - \left.\frac{dn}{dK}\right|_{r=\rho} = \frac{\pi_n r_K - \pi_K r_n}{r_n \pi_n} < 0.$$

That is, they are both upward sloping but the $r = \rho$ line is steeper [6] . These two curves act as fixed reference points to which we can relate the elements defining the trajectories projected in (K, n). They are also coincident with the projections of the isoclines $\dot{C} = 0$ and $\dot{e} = 0$ if all the variables are at their steady state levels [7].

In figure 2, to the left of the $r = \rho$ line $r > \rho$ and hence $\dot{C} > 0$ to the right $r < \rho$ and $\dot{C} < 0$. Also, above the $\pi = 0$ line we have $\pi < 0$ and below it $\pi > 0$. Recall, from (10) we cannot infer the flow of entry from the instantaneous flow of profits π, since entry/exit is determined by the NPV for the subsequent trajectory.

3 Aggregate dynamics

Next, we will characterize qualitatively the local dynamics properties in a neighbourhood of the steady state, by studying the solution of the linearized system

$$\begin{pmatrix} \dot{C} \\ \dot{e} \\ \dot{K} \\ \dot{n} \end{pmatrix} = \begin{pmatrix} 0 & 0 & Cr_K/\sigma & Cr_n/\sigma \\ 0 & \rho & \pi_K/\gamma & \pi_n/\gamma \\ -1 & 0 & \rho & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} C(t) - C^* \\ e(t) \\ K(t) - K^* \\ n(t) - n^* \end{pmatrix} \tag{27}$$

given the initial conditions $(n_{0,}K_0)$ and the transversality conditions (21), where the Jacobian matrix is denoted by J.

3.1 The stable manifold.

Our first step is to determine the eigenvalues.

Proposition 1 *The eigenvalues of the Jacobian matrix of system (27) are given by*

$$\lambda_1^j = \frac{\rho}{2} \mp \left[\left(\frac{\rho}{2}\right)^2 - \frac{1}{2}\left(\frac{Cr_K}{\sigma} + \frac{\pi_n}{\gamma}\right) - \frac{1}{2}\Delta^{\frac{1}{2}} \right]^{\frac{1}{2}} \qquad j = s, u \tag{28}$$

$$\lambda_2^j = \frac{\rho}{2} \mp \left[\left(\frac{\rho}{2}\right)^2 - \frac{1}{2}\left(\frac{Cr_K}{\sigma} + \frac{\pi_n}{\gamma}\right) + \frac{1}{2}\Delta^{\frac{1}{2}} \right]^{\frac{1}{2}} \qquad j = s, u \tag{29}$$

[6] Observe that $\frac{\pi_n r_K - \pi_K r_n}{r_n \pi_n} = -\frac{F_{kk}F_{ll} - F_{kl}^2}{\nu(1-\nu)^2 F F_k}$.

[7] These isoclines will "slide" in (K, n) space when variables deviate from their steady-state values.

where the discriminant is $\Delta = \left(\frac{Cr_K}{\sigma} - \frac{\pi_n}{\gamma}\right)^2 + 4\frac{Cr_n}{\sigma}\frac{\pi_K}{\gamma} > 0$, *and the superscripts* u, s *refer to stable and unstable eigenvalues. Hence we have:*

$$\lambda_2^s < \lambda_1^s < 0 < \lambda_1^u < \lambda_2^u, \tag{30}$$

All proofs are in Appendix 2. Note that $\lambda_1^s + \lambda_1^u = \lambda_2^s + \lambda_2^u = \rho$. It is sometimes useful to deal with the *products* of the eigenvalues: let $l_i \equiv \lambda_i^s \lambda_i^u$ for $i = 1, 2$, then

$$l_1 = \frac{1}{2}\left(\frac{Cr_K}{\sigma} + \frac{\pi_n}{\gamma} + \Delta^{\frac{1}{2}}\right),\ l_2 = \frac{1}{2}\left(\frac{Cr_K}{\sigma} + \frac{\pi_n}{\gamma} - \Delta^{\frac{1}{2}}\right), \tag{31}$$

then $0 > l_1 > l_2$, $l_1 + l_2 = \frac{Cr_K}{\sigma} + \frac{\pi_n}{\gamma} < 0$ and $l_1 l_2 = \det(J) > 0$.

For subsequent analysis, it is useful to examine what happens to the negative eigenvalues as the dynamic barriers to entry (DBE) γ vary:

Corollary 1 λ_2^s *and* λ_1^s *are increasing functions of the dynamic barriers to entry,* γ : *for* λ_1^s *we have the limits*

$$\lim_{\gamma \to 0^+} \lambda_1^s = \frac{1}{2}\left[\rho - \left(\rho^2 + 4\frac{C}{\sigma}\left(\frac{r_n \pi_k - r_K \pi_n}{\pi_n}\right)\right)^{1/2}\right] < 0 < \lim_{\gamma \to \infty} \lambda_1^s = 0,$$

and for λ_2^s

$$\lim_{\gamma \to 0^+} \lambda_2^s = -\infty < \lim_{\gamma \to \infty} \lambda_2^s = \frac{1}{2}\left[\rho - \left(\rho^2 - 4\frac{Cr_K}{\sigma}\right)^{1/2}\right] < 0.$$

In terms of l_i *we have*

$$\begin{aligned} \lim_{\gamma \to 0^+} l_1 &= \frac{C}{\sigma}\left(\frac{r_n \pi_k - r_K \pi_n}{\pi_n}\right) < 0 < \lim_{\gamma \to \infty} l_1 = 0, \\ \lim_{\gamma \to 0^+} l_2 &= -\infty < \lim_{\gamma \to \infty} l_2 = \frac{Cr_K}{\sigma} < 0. \end{aligned}$$

In the limiting cases of no entry ($\gamma = \infty$) and costless entry ($\gamma = 0$) the dimension of the stable manifold collapses to 1: with $\gamma = \infty$ we have a zero eigenvalue, and when $\gamma = 0$ we have only one state variable. The speed of convergence is decreasing in γ for $0 < \gamma < \infty$.

We have four distinct real eigenvalues: the two negative eigenvalues $\lambda_2^s < \lambda_1^s < 0$ will determine the dynamics for the transversality condition to hold. In our model though the dynamics are saddle-path stable, the stable manifold is two-dimensional.

Since the dimension of the stable manifold equals the number of the predetermined variables the equilibrium is determinate.

The meaning of a two-dimensional stable manifold is that there are two independent sources of stability. If we look at the terms in the formula for the eigenvalues, we see parameters (ρ, σ) which reflect consumer time preference, γ which relates the flow of entry to the cost of entry, and (r_K, r_n, π_n) the effect of the state variables on the marginal products (which depend only on technological parameters). The diminishing marginal returns of capital and decreasing profits to entry are the main forces for stability in this economy. It should be noted that they also interact: an increase in the number of firms increases the marginal productivity of capital; more firms means less capital per firm and hence a higher marginal product since $\nu < 1$. An increase in the capital stock makes firms more profitable which affects entry. As we shall see, these interactions can lead to non-monotonic trajectories.

In the Appendix 2, Lemma A2, we show that the dynamics of the system along the linearized stable manifold take the form:

$$\begin{pmatrix} C(t) - C^* \\ e(t) - 0 \\ K(t) - K^* \\ n(t) - n^* \end{pmatrix} = P_1^s w_1^s e^{\lambda_1^s t} + P_2^s w_2^s e^{\lambda_2^s t} \tag{32}$$

where w_i^s are the weights (determined by the initial conditions for the state variables, $K(0)$ and $n(0)$), and P_i^s are the eigenvectors associated to the negative eigenvalues, λ_i^s and are obtained in Lemma A1.

The linear space tangent to the stable manifold E^s is a 2-dimensional linear space in (C, e, K, n) given by

$$E^s \equiv \{ \ (C, e, K, n) : C - C^* = h_C(K - K^*, n - n^*), e = h_e(K - K^*, n - n^*) \ \}$$

where h_C and h_e are linear functions,

$$\begin{aligned} h_C &= \left(\frac{\lambda_2^u - \lambda_1^u}{l_2 - l_1} \right) \left[\left(\frac{C r_K}{\sigma} - \lambda_1^u \lambda_2^u \right) (K - K^*) + \frac{C r_n}{\sigma} (n - n^*) \right], \\ h_e &= \left(\frac{\lambda_2^s - \lambda_1^s}{l_2 - l_1} \right) \left[\left(\frac{\pi_n}{\gamma} - \lambda_1^s \lambda_2^s \right) (K - K^*) + \frac{\pi_K}{\gamma} \ (n - n^*) \right], \end{aligned}$$

where the coefficients for K are both positive and for n are both negative. This means that, all effects considered, consumption and entry are positively related to the stock of physical capital and negatively related with the number of firms, in the transition to the steady state.

3.2 The Phase Diagram in (K, n)

The fact that the system is 4-dimensional, and the stable manifold is 2-dimensional, poses some challenges to a qualitative understanding of the dynamics of transition. However, we can understand it intuitively by concentrating on the projection onto the state-space (K, n), although there are insights to be gained by looking at the more familiar "Ramsey" projection (C, K).

We denote by E_1^s and E_2^s the lines in the four dimensional space (C, e, K, n) which have the slope given by P_1^s and P_2^s, respectively, and pass through the equilibrium point. The 1-dimensional lines E_1^s and E_2^s span the space which is tangent to the stable manifold at the steady state equilibrium, E^s.

In order to determine E_1^s (E_2^s) we set $w_2^s = 0$ $(w_1^s = 0)$ which implies that the dynamics is solely driven by λ_1^s (λ_2^s). Hence:

$$E_1^s := \{(C, e, K, n) : \frac{C - C^*}{n - n^*} = \frac{\lambda_1^u - Cr_n}{\sigma l_1 - Cr_K}, \frac{e}{n - n^*} = \lambda_1^s, \frac{K - K^*}{n - n^*} = \frac{Cr_n}{\sigma l_1 - Cr_K}\},$$

and

$$E_2^s := \{(C, e, K, n) : \frac{C - C^*}{n - n^*} = \frac{\lambda_2^u - Cr_n}{\sigma l_2 - Cr_K}, \frac{e}{n - n^*} = \lambda_2^s, \frac{K - K^*}{n - n^*} = \frac{Cr_n}{\sigma l_2 - Cr_K}\},$$

evaluated in a neighborhood of the steady state. The dynamics of the system are driven by these two lines. When the system is close to the steady-state, the equilibrium trajectories are asymptotically tanget to E_1^s and when further away the trajectories are parallel to E_2^s. This corresponds to the intuitive notion that the dynamics are driven at first more by the negative eigenvalue which is larger in absolute value, but that since this dies away more quickly the smaller eigenvalue predominates as you approach steady state.

Proposition 2 *Qualitative characterization of the orbits belonging to the stable manifold. Consider an initial non-steady-state point* $(C(0), e(0), n(0), K(0))$*. The transition dynamics along the stable manifold will be as follows :*

(a) If the initial position of the two state variables lies on E_i^s $i = 1, 2$*, then the control values jump to the corresponding values on* E_i^s *and the economy proceeds along* E_2^s *with* maximum *speed or* E_1^s *with* minimum *speed to the steady-state*

(b) If the initial position of the two state variables does not lie on either E_i^s*, then the economy will move in the direction of* E_1^s *intially parrallel to* E_2^s*. Asymptotically it will approach steady-state asymptotically tangent to* E_1^s*.*

Figure 3 *here*

We can now describe the two projections of $E_i^s, i = 1, 2$, in (K, n) and how they relate to our two reference curves (see figure 3):

Proposition 3 *The projections of E_1^s and E_2^s in (K, n):*
(a) *The projections have the opposite slopes*:

$$\left.\frac{dn}{dK}\right|_{E_1^s} = \frac{\sigma l_1 - Cr_K}{Cr_n} > 0,$$
$$\left.\frac{dn}{dK}\right|_{E_2^s} = \frac{\sigma l_2 - Cr_K}{Cr_n} < 0.$$

(b) *In general for* $\gamma \in (0, \infty)$

$$\left.\frac{dn}{dK}\right|_{E_2^s} < 0 < \left.\frac{dn}{dK}\right|_{\pi=0} < \left.\frac{dn}{dK}\right|_{E_1^s} < \left.\frac{dn}{dK}\right|_{r=\rho}.$$

(c) *if* $\gamma \to 0$ *then*

$$-\infty = \left.\frac{dn}{dK}\right|_{E_2^s} < 0 < \left.\frac{dn}{dK}\right|_{\pi=0} = \left.\frac{dn}{dK}\right|_{E_1^s} < \left.\frac{dn}{dK}\right|_{r=\rho}.$$

(d) *if* $\gamma \to \infty$ *then*

$$\left.\frac{dn}{dK}\right|_{E_2^s} = 0 < \left.\frac{dn}{dK}\right|_{\pi=0} < \left.\frac{dn}{dK}\right|_{E_1^s} = \left.\frac{dn}{dK}\right|_{r=\rho}.$$

That is, in general the projection of E_1^s lies in between the two curves $(r = \rho, \pi = 0)$, and the projection of E_2^s is negatively sloped. For $\gamma = 0$ (instantaneous free entry) E_2^s is vertical and E_1^s corresponds with the zero profit line. For $\gamma = \infty$ (fixed number of firms), E_1^s is coincident with the $r = \rho$ curve and E_2^s is horizontal.

Along with Proposition 2, Proposition 3 gives us the following simple dynamics for the two limiting cases. When $\gamma = 0$ (costless free entry as in Devereux et al. (1996), Heijdra (1988) and Coto-Martinez and Dixon (2003)), the number of firms n "jumps" down vertically (following E_1^s) to the $\pi = 0$ curve, and then (K, n) converge together along the $\pi = 0$ curve. With an eigenvalue of $\lambda_2^s = -\infty$ there is infinite speed in the adjustment of n, which in effect collapses the manifold to the one dimension of $\pi = 0$. When $\gamma = +\infty$, the number of firms does not change: Thus we move horizontally along with K (the slope of E_1^s) accumulating or decumulating as in the standard Ramsey story until we reach $r = \rho$. There is a zero eigenvalue here, so there are multiple equilibria: each point on $r = \rho$ is a possible equilibrium, which has as a basin of attraction the horizontal line passing through it. For a given n, there is a unique K such that $r = \rho$: capital is accumulated or decumulated to reach that point, with consumption moving up (down) with capital as in the Ramsey story. Again, for each equilibrium there is a one dimensional saddlepath in (C, n) and (C, K).

We now focus on the general case where $\gamma \in (0, \infty)$. To describe the dynamics in full, we need to introduce two new lines. These are the isoclines $\dot{K} = 0$ and $\dot{n} = 0$ linearized around the steady state. In general, these isoclines will depend upon (C, e) and will not be invariant in (K, n). However, close to steady state these depend only on (K, n) and hence can be used to characterize the equilibrium trajectories of the linearized system. Note that along the $\dot{n} = 0$ line $q = 0$. This is because $\dot{n} = e = 0$ if and only if $q = 0$ (the net present value of incumbency is zero).

Proposition 4 *The tangents to the isoclines $\dot{K} = 0$ and $\dot{n} = 0$ passing through the steady-state are*

$$
\begin{aligned}
(a)\ \left.\frac{dn}{dK}\right|_{\dot{n}=0} &= \frac{\pi_K}{\gamma\lambda_1^s\lambda_2^s - \pi_n} > 0, \\
(b)\ \left.\frac{dn}{dK}\right|_{\dot{K}=0} &= \frac{\sigma\lambda_1^s\lambda_2^s - Cr_K}{Cr_K} > 0.
\end{aligned}
$$

Figure 4 here

It is worth noting a few points about these isoclines. First, the $\dot{n} = 0$ has a positive slope that is less than the zero-profit isocline. The $\dot{K} = 0$ is positive in slope, and steeper than the $r = \rho$ line. The reason for this is that entry (exit) affects capital accumulation. The further South from steady state we are, the higher the flow of entry. From (20), higher entry means less is available for investment and consumption, leading to less investment. Hence capital accumulation may stop even though the marginal product of capital is above ρ. The converse happens North of the steady state. The $\dot{n} = q = 0$ line is flatter than the $\pi = 0$ line. This is because the NPV depends on the whole path of trajectories, not just the current flow of profit. As we will see, entry can occur even when $\pi < 0$, because the firm anticipates future profits. Exit can also occur when current profits are strictly positive, because firms get out now to avoid future losses.

Fig 5

Now we have the isoclines, we can divide up the (K, n) projection of the stable-manifold into regions depending on the types of trajectories. We can see that any trajectory which cuts the $\dot{K} = 0$ isocline must be vertical: any trajectory that cuts the $\dot{n} = 0$ must be horizontal. Any trajectory to the left of the $\dot{K} = 0$ must have capital increasing; any to the right decreasing. Any trajectory above the $\dot{n} = 0$ must have the number of firms decreasing and any below must have the number increasing. This enables us to intuitively draw the phase diagram in (K, n).

If we start in a region *inbetween* the iso-clines, then we have monotonic dynamics in both variables. In region A we are to the right of the $\dot{K} = 0$ and above $\dot{n} = 0$, so that both variables are declining. In region B, we are to the left of the $\dot{K} = 0$ and below $\dot{n} = 0$, so that both variables are increasing. As the trajectories get close to the steady-state, their slope will converge to that of E_1^s. We can treat the isoclines themselves as part of A and B, since trajectories starting from the isoclines also share the monotonicity.

In the regions outside A or B, we will in general get non-monotonic behaviour in one variable: since the eigenvalues are real, we cannot have non-monotonic behaviour in both variables along the same trajectory. If we are outside A and B, the only instance in which both variables are monotonic is when the initial position lies on E_2^s : in this case the trajectory travels straight down E_2^s to the steady state, and the two state variables move in different directions (depending on whether they start above or below steady-state). These two cases correspond to trajectories in which only the larger negative eigenvalue 2 is active.

If we start from the regions strictly between $\dot{K} = 0$ and E_2^s, we will observe non-monotonic behaviour in K. North West of the steady state we have the region N_k, there are initially too many firms $n > n^*$, and capital may be above or below steady-state. However, even if capital is above steady-state, the large number of firms boosts the marginal product of capital and encourages capital accumulation. This continues until the trajectory hits the $\dot{K} = 0$ isocline, and thenceforth enters region A and both variables decline towards steady-state. In the region S_k, South/South-West the same story happens, but we have too few firms: capital will initially fall since the marginal product is low, until the $\dot{K} = 0$ is reached and the trajectory enters region B and both state-variables increase to their steady state. Note that the regions N_k and S_k are both open sets: they do not include their boundaries E_2^s and $\dot{K} = 0$.

If we start from the region between E_2^s and $q = \dot{n} = 0$, we will observe non-monotonic behaviour in entry and hence n. To the West of steady state W_n, there is too little capital. This means that firms have negative NPV (we are above the $q = 0$ line), so that there will be exit until the $q = 0$ line is reached and then both variables enter region B and increase to steady-state. To the East of steady-state E_n, there is too much capital: this boosts firms NPV and induces entry, until the $q = 0$ line is met and the trajectory enters region A and both variables decline to the steady-state.

Since the steady-state is almost always approached along E_1^s, almost all trajectories must either approach through region A or B where both are increasing/decreasing. The only exception is where the initial position happens to lie on E_2^s. Hence, if the initial position lies outside $A \cup B$, the trajectory will move towards $A \cup B$ with one variable decreasing and one increasing: once it enters $A \cup B$ the dynamics become monotonic and there is a positive correlation between the state variables around the steady state.

We can now see why the $q = 0$ line is flatter than the $\pi = 0$. When the trajectory reaches the $q = 0$ line on the edge of A and there is too much capital, $NPV = 0$ despite $\pi > 0$ since the trajectory will cross the $\pi = 0$ line and subsequently earn strictly negative profits until steady-state is reached. Along the $q = 0$ line, the profits prior to reaching $\pi = 0$ are exactly offset by the subsequent losses. The opposite holds true when the there is too little capital: the $q = 0$ line on the border of B is reached even though $\pi < 0$. This is because the trajectory will cross the $\pi = 0$ line and subsequently earn profits. Intuitively, we can think of π as representing "short-run" profits and q as "long-run" profits, and as we cross the $\pi = 0$ and $q = 0$ lines, the correlation between them changes.

We can now formally summarise the above insights which we prove in the appendix:

Proposition 5 *Monotonous and non-monotonous transitional dynamics: Consider the following two sub-sets*

$$N_k \equiv \left\{(K,n):\ n > n^* \text{ and } \frac{Cr_n}{\sigma l_2 - Cr_K} < \frac{K-K^*}{n-n^*} < \frac{Cr_n}{\sigma \lambda_1^s \lambda_2^s - Cr_K}\right\} \quad (33)$$

$$S_k \equiv \left\{(K,n): n < n^* \text{ and } \frac{Cr_n}{\sigma l_2 - Cr_K} > \frac{K-K^*}{n-n^*} > \frac{Cr_n}{\sigma \lambda_1^s \lambda_2^s - Cr_K}\right\}, \quad (34)$$

and

$$W_n \equiv \left\{(K,n):\ K < K^* \text{ and } \frac{\pi_K}{\gamma \lambda_1^s \lambda_2^s - \pi_n} < \frac{n-n^*}{K-K^*} < \frac{\sigma l_2 - Cr_K}{Cr_n}\right\} \quad (35)$$

$$E_n \equiv \left\{(K,n):\ K > K^* \text{ and } \frac{\pi_K}{\gamma \lambda_1^s \lambda_2^s - \pi_n} > \frac{n-n^*}{K-K^*} > \frac{\sigma l_2 - Cr_K}{Cr_n}\right\} \quad (36)$$

Then,

1. if $(K(0), n(0)) \in N_k \cup S_k$, then $K(t)$ will adjust non-monotonically*:* it will increase and then decrease if starting in N_k; it will decrease an then increase if starting in S_k;

2. if $(K(0), n(0)) \in W_n \cup R_n$, then $n(t)$ will adjust monotonically; if starting in W_n it will increase and then decrease; if starting in R_n it will increase and then decrease;

3. if $(K(0), n(0)) \notin N_k \cup S_k \cup W_n \cup R_n$ then both state variables have monotonic trajectories.

3.3 Dynamics in (C, K) and (e, n)

Whilst the most intuition is gained by projecting the 4 dimensional phase space onto (K, n), it is also illuminating to take a look at the conventional "Ramsey" projection onto (C, K) and also (e, n).

Proposition 6 *Projections of E_i^s onto (C, K) and (n, e) spaces.*

(a) *in (C, K)*

$$\left.\frac{dC}{dK}\right|_{E_2^s} = \lambda_2^u > \lambda_1^u = \left.\frac{dC}{dK}\right|_{E_1^s} > 0.$$

(b) *in (e, n)*

$$\left.\frac{de}{dn}\right|_{E_2^s} = \lambda_2^s < \lambda_1^s = \left.\frac{de}{dn}\right|_{E_1^s} < 0.$$

In (C, K) for $\gamma \in (0, \infty)$, both E_i^s are upward sloping, so that consumption and capital move together. From Corollary 1, in the limiting cases of $\gamma = 0$, the slope of E_2^s becomes vertical and the slope of E_1^s equals ρ. In (e, n), for $\gamma \in (0, \infty)$ the slopes are negative, with entry slowing as steady-state is approached. In the limiting case of $\gamma = \infty$, E_1^s becomes vertical and the slope of E_2^s negative. In this case the number of firms is fixed: n remains constant and $e = 0$ so there is no dynamics in (e, n) to see. In the case of $\gamma = 0$, E_2^s is vertical and E_1^s is negatively sloped. This means that the number of firms jumps to the zero-profit line given K, and then n moves towards steady-state.

Figure 6 here.

It is interesting to see what the dynamics of the non-monotonic trajectories look like in terms of (C, K) space. The positive slopes of E_i^s mean that in general consumption and capital move together. Hence, in the case of a trajectory in which capital is non-monotonic, the trajectory of consumption will also be non-monotonic. Let us take the case where the initial capital stock is at its steady-state value, but firms are well above steady-state. This will induce the capital stock to increase initially, and then decrease. Consumption will initially jump to a level below steady-state: this reflects the fact that it has level of total assets below steady-state (in terms of capital and firms the initial value is $V(0) = qn(0) + K(0)$ with $q < 0$) and the household wants to reduce the number of firms which it has. However, there is initially an increase in capital: the large number of firms causes the marginal product of capital to be high which leads to capital accumulation. Consumption and capital move together with slope E_2^s. Eventually capital accumulation stops: this is when the trajectory in (K, n) reaches the $\dot{K} = 0$ isocline. However, consumption continues to increase even whilst capital is falling (this represents the part of the trajectory in (K, n) between the $\dot{K} = 0$ isocline and the $r = \rho$ line). Eventually capital and consumption both fall together along a pth tangent to E_1^s. The non-monotonic trajectories with too many firms will all share this pattern: consumption starts low and initially increases with capital; there is a period when consumption continues to increase whilst capital falls; finally both consumption and capital fall back down to steady-state. In the case where there are too few firms, the opposite happens. Since $q > 0$, consumption jumps above steady state. Initially it declines with capital tangentially to E_2^s; capital turns around and starts to increase, for a period consumption continues to decline until both approach the steady-state from below along E_1^s.

Figure 7 here

The projection on (e, n) is negatively sloped: the flow of entry is less when the number of firms is larger. The slope of E_2^s is λ_2^s which is larger than E_1^s which is associated with the smaller eigenvalue. The standard monotone dynamic is for e to jump and then decline rapidly at first tangent to E_2^s and then more slowly converge along E_1^s to the steady-state. The interesting story in (e, n) is what happens when the trajectory of n is non-monotonic. Let us suppose that the number of firms starts off in steady-state, but there is too much capital. This boosts the profitability of firms and encourages entry: e jumps and the number of firms increases. The trajectory from the initial point is parallel to E_2^s. Eventually, due to the decline in capital stock and the effect of entry, q falls to zero and entry stops. Entry turns to exit and initially the flow of exit increases, but then entry declines so that the trajectory approaches steady-state along E_1^s.

3.4 Firm size dynamics.

Firm size dynamics along the stable manifold can be derived from the firm size equation, (17) if we substitute the dynamics for the aggregate capital stock and the number of firms (32). In the neighborhood of the stationary equilibrium, the local dynamics for the firm size is given as

$$y(t) - y^* = \frac{1}{1-\nu}\left(\pi_K(K(t) - K^*) + \pi_n(n(t) - n^*)\right).$$

Therefore, the loci in the diagram (K, n) such that the size of firms is invariant, i.e., $y(t) = y^*$ is given by

$$\left.\frac{dn}{dK}\right|_{\dot{y}=0} = -\frac{\pi_K}{\pi_n},$$

which is coincident with the the zero-profit line $\pi(K, n) = 0$ (see figures 2 to 5). Above that line we will have $\pi < 0$ and $y(t) < y^*$ and below the line $\pi > 0$ and $y(t) > y^*$ below. Thus the current flow of profits perfectly captures the size of the firm.

Using our previous analysis on the dynamics for the aggregate variables K and n we also can describe in which situations the dynamics of firm size can be non-monotonic. First, we define some extra subsets on the space (K, n):

$$W_y \equiv \left\{(K,n):\ K < K^* \text{ and } -\frac{\pi_K}{\pi_n} < \frac{n-n^*}{K-K^*} < \frac{\pi_K}{\gamma\lambda_1^s\lambda_2^s - \pi_n}\right\} \tag{37}$$

$$E_y \equiv \left\{(K,n):\ K > K^* \text{ and } -\frac{\pi_K}{\pi_n} > \frac{n-n^*}{K-K^*} > \frac{\pi_K}{\gamma\lambda_1^s\lambda_2^s - \pi_n}\right\} \tag{38}$$

Proposition 7 *Monotonous and non-monotonous transitional dynamics for the firm size:*

(a) *if $(K(0), n(0)) \in W_n \cup W_y \cup E_n \cup E_y$ then $y(t)$ will adjust non-monotonically: it will increase and then decrease if starting in $W_n \cup W_y$; it will decrease and then increase if starting in $E_n \cup E_y$.*

(b) *if $(K(0), n(0)) \notin W_n \cup W_y \cup E_n \cup E_y$ then $y(t)$ will adjust monotonically.*

Note that for $0 < \gamma < \infty$ the non-monotonous adjustment of n is a sufficient (but not necessary) condition for the non-monotonous adjustment of firm size. If (K, n) belongs to W_y (E_y) the transition path will cross line $\pi = 0$ in its way to being tangent to E_1^s. This means that, though all the other variables vary monotonically, the variation in the size of firms shifts direction from being (decreasing) increasing to being decreasing (increasing). The dynamics for the size of firms will be similar when (K, n) belongs to W_n (E_n). But in this case the number of firms will also adjust non-monotonically.

4 Application: Productivity and Fixed cost shocks.

In this section we derive the comparative dynamics results for an unanticipated permanent shock in productivity or in the fixed costs. In general, a technology shock would be a simultaneous shock to both parameters. However, we shall first consider each parameter on its own.

4.1 Shock to A.

A positive shock to A has two effects: it boosts the marginal productivity of capital and labour, and it raises the profitability of firms (and hence increases the marginal product of an additional firm). In terms of Figure 1, an increase in A shifts both the average and marginal cost downwards, but leaves their point of intersection the same (as A does not affect y^e),

Proposition 8 *Long-Run Multipliers for A. If there is a constant, permanent and unanticipated increase in productivity:*

$$\frac{\partial C^*}{\partial A} > 0; \frac{\partial K^*}{\partial A} > 0; \frac{\partial n^*}{\partial A} > 0$$

At the firm level we have

$$\frac{\partial l^*}{\partial A} < 0 = \frac{\partial y^*}{\partial A} < \frac{\partial k^*}{\partial A}$$

In the long-run, a positive productivity shock in A leads to an increase in capital, the number of firms and consumption. Entry is zero in steady-state by definition. We can explore the long-run multiplier in terms of (K, n) space. An increase in A affects both the marginal product of capital and the profitability of firms. However, from (26) and the expressions for r_K and r_n, we can see that the slope of $r = \rho$ is unaffected, so that there is a parallel shift to the right. Turning to the $\pi = 0$ locus, from (26) and the expressions for π_K and π_n the schedule $\pi = 0$ has a parallel shift upwards. This is intuitive: the slopes of the two curves depend on the ratios of marginal effects of (n, K) on (r, π): all these marginal effects are proportional to A so the effects cancel out. That leaves just the direct effect of A on the marginal product of capital and profits resulting in the intuitive shifts in the loci.

The new steady state involves a different ratio of K to n. The increase in the number of firms implies that given the inelastic labour force, the employment per firm *decreases*. Since the efficient output y^e does not change (9), an increase in A implies an increase in the steady-state capital per firm $k = K/n$. Graphically, the steady-state after the increase in A lies on a ray from the origin that is steeper: indeed as we vary A from 0 to ∞, the zero-profit capital per firm varies from ∞ to zero. These two equations are every powerful: they are independent of the dynamics of the system but tie down the steady-state. The initial steady state must lie to the left of the new $r = \rho$ line and below the $\pi = 0$ line, hence it is in the region of monotonic trajectories: both capital and the number of firms will increase towards the new steady state.

Next, for the case of the two control variables, we can consider the difference between the impact effect and the long-run effect and the transition dynamics. For consumption, we need to define critical values for the parameter pair (γ, σ) according to the function:

$$z(\gamma, \sigma) \equiv \gamma\lambda_1^u\lambda_2^u(\sigma\left(\lambda_1^u + \lambda_2^u\right) - \rho\left(\sigma + \nu\right)) > 0.$$

From the eigenvalues stated in Proposition 1, z is increasing in both σ and γ : furthermore, for both γ and σ, as they tend to zero so does z; as they tend to infinity so does z. Hence, for any $\Psi \in \mathbb{R}_{++}$, we can define the upper-countour set

$$\mathbb{Q}\left(\Psi\right) = \{(\gamma, \sigma) : z\left(\gamma, \sigma\right) \geq \Psi\}$$

The upper-contour sets are closed and strictly convex in (γ, σ).

Proposition 9 *Impact-Multipliers on the two controls (C, e).*

(a)

$$\frac{\partial c^*}{\partial A} > \frac{\partial c(0)}{\partial A} = \frac{n\phi}{\left(1-\nu\right)\gamma\sigma\lambda_1^u\lambda_2^u(\lambda_1^u + \lambda_2^u - \rho)}\left[z\left(\gamma, \sigma\right) - \frac{\delta\nu\phi}{n}\right]$$

(b)

$$sgn\left[\frac{\partial C(0)}{\partial A}\right] = sgn\left[z\left(\gamma, \sigma\right) - \frac{\rho\nu\phi}{n}\right]$$

(c)

$$\frac{\partial e(0)}{\partial A} > 0$$

$e(t) > 0$ *for all t and tends to zero monotonically.*

The result (a) means that given the values $\rho\nu\phi n^{-1}$, we can partition (γ, σ) into three sets in which consumption jumps up, jumps down or stays the same. Intuitively there are two things going on behind this result. First, there is the trajectory of $r - \rho$ over time: from (2) the bigger this gap is, the more the household will want to tilt consumption to sacrifice current for future consumption. This effect is smaller when σ is larger. Secondly, there is entry (exit) which influences how quickly r changes. If γ is larger, then entry is slower and r will converge to ρ more quickly (due to the effect of n on r). Hence, if γ and/or σ are large, the household has less incentive to tilt consumption away from the present and so consumption can increase on impact.

When viewed in (C, K) space, the effect of entry is to shift up the projection of the production-function $Y\left(K, n, A\right)$. Thus on impact there is an initial shift in the production function, as in the Ramsey model. But, in our model, the impact increase in aggregate output is a result of the increase in the size of firms, $y(0)$, for a given number of firms. However, this one-off productivity shock also triggers a flow of entry resulting in subsequent upward shifts in the projection of $Y(K, n, A)$ as n converges to its new steady state value. For a given time-path of K, the marginal product of capital declines more slowly as the number of firms increases alongside it. At the firm level, there is a reduction in size and a change in the factor composition, with an increase in the average capital stock and a reduction in the average labour input.

4.2 Shock to ϕ

Next we turn to a shock to the overhead ϕ. This operates in a very different way to A, since it does not affect marginal productivity and MC at all, but operates only on average productivity and cost. An increase in ϕ causes the average cost curve to shift upwards, and hence raises the efficient scale of production (see (7)). This means that for a given capital stock, there will be fewer firms if each produces at efficient scale. In terms of (n, K), the change in ϕ has no effect on the $r = \rho$ locus, since it has no direct effect on the marginal product of capital. An increase in ϕ leads to a downwards shift in the $\pi = 0$ line: for a given K, profits per-firm decline and so the zero-profits occurs with fewer firms.

An increase in ϕ causes a long run increase in the size of firms, because both capital and employment per firm will increase. However, there is no effect on firm size on impact (since k, l cannot vary).

Proposition 10 *Long-Run Multipliers for ϕ. If there is a constant, permanent and unanticipated increase in the overhead fixed cost:*

(a)

$$\frac{\partial C^*}{\partial \phi} < 0; \frac{\partial K^*}{\partial \phi} < 0; \frac{\partial n^*}{\partial \phi} < 0.$$

(b) *At the firm level we have*

$$\frac{\partial y^*}{\partial \phi}, \frac{\partial k^*}{\partial \phi}, \frac{\partial l}{\partial \phi} > 0.$$

Proposition 11 *The impact multipliers for ϕ.*

(a)

$$\frac{\partial C(0)}{\partial \phi} = \frac{n}{z(\gamma, \sigma)} \left(\frac{C(1-\nu)\rho}{n^2} - z(\gamma, \sigma) \right) > \frac{\partial C^*}{\partial \phi}.$$

(b)

$$sgn \left[\frac{\partial C(0)}{\partial A} \right] = sgn \left[z(\gamma, \sigma) - \frac{C(1-\nu)\rho}{n^2} \right].$$

(c)

$$\frac{\partial e(0)}{\partial \phi} < 0.$$

The fact that an increase in ϕ can cause an ambiguous impact on consumption deserves some comment. Less output is produced (more is used up in overheads) which tends to reduce consumption. However, the household also wants to decumulate capital which boosts consumption: if σ is low enough there will be an increase in consumption. The household also wants to dismantle firms: this uses up resources and tends to reduce consumption possibiltities. If γ is high (it is going to cost a lot to dismantle the firms) then consumption will tend to fall; if γ is small enough then it will boost consumption.

4.3 Correlated shocks in (A, ϕ).

If we assume that the two technology parameters as uncorrelated, each one taken on its own leads to a monotonic response of capital and the number of firms (the resulting dynamics belonging to the regions A or B). However, if we take the two parameters together, they span the whole space: given an initial steady-state, we can create a new steady state in any direction by choosing an appropriate combination of changes to (ϕ, A). Hence the whole range of dynamic responses is possible, including the cases where one of the variables (K, n) is non-monotonic. It is thus possible for two technological shocks which on their own provide for monotonic dynamics when combined will give rise to non-monotonous behaviour. In this section we show that if the two parameters are correlated (as in Rotemberg and Woodford (1991)) we can determine when we get monotonic and non-monotinic dynamics.

Assume that the shocks to ϕ are proportional to the shocks in A, such that $d\phi = \eta dA$[8]. The long run multipliers for the stock of capital and the number of firms are:

$$\begin{aligned}
\frac{dK^*}{dA} &= \frac{\partial K^*}{\partial A} + \eta\frac{\partial K^*}{\partial \phi} = \frac{\alpha n^*}{\rho\beta}\left(\frac{\phi}{A(1-\nu)} - \eta\right), \\
\frac{dn^*}{dA} &= \frac{\partial n^*}{\partial A} + \eta\frac{\partial n^*}{\partial \phi} = \frac{n^*}{\beta}\left(\frac{1}{A} - \eta\frac{(1-\alpha)}{\phi}\right).
\end{aligned}$$

Let us concentrate on the cases in which there is a positive correlation between ϕ and A: $\eta > 0$. From Propositions 8 and 10, the long-run multipliers have opposite signs:$\frac{\partial K^*}{\partial A} > 0$, $\frac{\partial n^*}{\partial A} > 0$, $\frac{\partial K^*}{\partial \phi} < 0$ and $\frac{\partial n^*}{\partial \phi} < 0$. Then, $\frac{dK^*}{dA}$ and $\frac{dn^*}{dA}$ are ambiguous and depend on the magnitude of η.

[8]The case for which $\eta = 0$ corresponds to the uncorrelated shocks in A and ϕ which were studied in the previous section.

We assign K^* and n^* to the steady state after the shock and $K(0)$ and $n(0)$ the steady state before the shock and set $dK = K^* - K(0)$ and $dn = n^* - n(0)$, next we compute

$$\frac{dn}{dK}(\eta) = \frac{dn^*/dA}{dK^*/dA} \tag{39}$$

and draw upon our study on local dynamics to determine critical values of η and relate them to the projection of the phase diagram in the space (K, n): we can have dynamics related to the regions A, B, W_y, W_n, N_k in figure 4.

From the values of η we can infer both the value of the long run multipliers and the type of transitional dynamics. We can get lots of results (see Lemma 3 in Appendix 2), but the following seems to be the most pertinent:

Proposition 12 *There are values for η, $\eta_0 < \eta < \eta_n$ such that there will be an increase in the aggregate capital, in the number of firms and in the size of firms. Within this interval, y will adjust non-monotonically, and n will adjust monotonically (non-monotonically) if $\eta_0 < \eta < \eta_1$ (if $\eta_1 < \eta < \eta_n$).*

Example In order to illustrate this and other cases we consider particular functions: a Cobb-Douglas production function $F(k, l) = F(K/n, 1/n) = (K/n)^\alpha (1/n)^\beta$ where $\nu = \alpha + \beta$ and $0 < \nu < 1$, and a CRRA utility function $u(C) = \frac{C^{1-\theta}-1}{1-\theta}$ where $\theta > 0$. In this case, the equilibrium number of firms is $n^* = L\left(A(\alpha/\rho)^\alpha(\phi/(1-\nu))^{\alpha-1}\right)^{1/\beta}$ and we find that $\eta_0 = \frac{\phi}{A}$, $\eta_1 = \frac{\phi}{A}\left(\frac{n^*\gamma\lambda_1^s\lambda_2^s+\beta\phi}{n^*\gamma\lambda_1^s\lambda_2^s(1-\alpha)+\beta\phi}\right)$, $\eta_n = \frac{\phi}{A(1-\alpha)}$, $\eta_2 = \frac{\phi}{A(1-\nu)}\left(\frac{\rho^2\beta\nu+\alpha\sigma l_2}{\alpha\sigma l_2}\right)$, $\eta_K = \frac{\phi}{A(1-\nu)}$, and $\eta_3 = \frac{\phi}{A}\left(\frac{\alpha\sigma\lambda_1^s\lambda_2^s+\rho^2\beta\nu}{\alpha\sigma\lambda_1^s\lambda_2^s(1-\nu)}\right)$.

Figures 8-14 here

Figures 8 to 15 display the trajectories for aggregate variables C, e, K and n and firm level variables K/n and y that were built with the following benchmark parameter values: $L = 1$, $\alpha = 0.4$, $\beta = 0.5$, $\rho = 0.03$, $\sigma = 2$, $A = 1$, $\gamma = 1$ and an initial value for $\phi = 0.03$. In Figure 8 we consider a 10% increase in A;in Figure 9 a 10% increase in ϕ. Figures 10-15 deal with the correlated case, where there is a 10% increase in A and a corresponding change in ϕ: the critical correlations (see Lemma A3, proof of Proposition 12) are $\eta_0 = 0.03$, $\eta_1 \approx 0.0375$, $\eta_n = .05$, $\eta_2 \approx 0.1345$, $\eta_K = 0.3$ and $\eta_3 \approx 0.9866$. As we can see, these provide examples of all types of trajectories: all variables monotonic (Figure 11, 12), non-monotonic in n (Figure 10), non-monotonic in K with a drop in consumption on impact followed by overshooting (Figure 13,14), and without overshootong (Figure 15).

5 Conclusion

In this paper, we extend the Ramsey model and generalize existing approaches to entry by allowing for an explicit and fully transparent treatment of both the number and output per firm at the micro level along with the behaviour of aggregate output, consumption and investment. We do this by allowing for an explicit treatment of two state variables: capital and the number of firms. This contrasts with existing models which try to keep the number of state variables to one: either by allowing for the number of firms to vary but with no capital, or having instantaneous free entry, or a fixed number of firms. The reward for this additional complexity is that we can have a richer dynamic behavior. In particular, we can get non-monotonic behavior in either one of the state variables (but not both since we have only real eigenvalues) resulting form the interaction of the state variables on each others' marginal product. We consider the dynamics induced by two types of technology shocks in this model (productivity and fixed cost) and show how they can generate non-monotonic responses if they are correlated.

We believe that entry and exit have long been the Cinderella of dynamic general equilibrium analysis. This has largely been due to the technical difficulty of making the number of firms endogenous in a non-trivial way. We show how by adopting the dynamic entry model of Datta and Dixon (2002), it is possible to develop an intuitive and tractable dynamic general equilibrium model with the two state variables. Furthermore, although the model is inherently a four dimensional system, the model can be represented graphically in two dimensions by projecting it onto the 2-dimensional subspace of the state variables.

There are several ways to develop the model in this paper. Most obviously, we can allow for imperfect competition, so that the long-run equilibrium is no longer optimal: in steady state the zero profit condition will imply that average cost is greater than marginal cost, so that we have the standard Chamberlin-Robinson excess capacity result.

References

Aloi M and Dixon H (2002). Entry Dynamics, Capacity Utilisation and productivity in a dynamic open economy, University of York Discussion Paper 02/05.

Baumol, W. J., Panzar, J. C., and Willig, R. D. (1982). *Contestable Markets and the Theory of Industry Structure*. Harcourt Brace Jovanovich.

Bilbiie, F. O., Ghironi, F., and Melitz, M. J. (2005). Business cycles and firm dynamics.

Coto-Martinez, J. and Dixon, H. (2003). Profits, markups and entry: fiscal policy in an open economy. *Journal of Economics Dynamics and Control*, 27(4):573–597.

Das, S. and Das, S. (1996). Dynamics of entry and exit of firms in the presence of entry adjustment costs. *International Journal of Industrial Organization*, 15:217–241.

Datta, B. and Dixon, H. (2002). Technological change, entry, and stock market dynamics: an analysis of transition in a monopolistic economy. *American Economic Review*, 92(2):231–235.

Devereux, M. B., Head, A. C., and Lapham, B. M. (1996). Monopolistic competition, increasing returns, and the effects of government spending. *Journal of Money, Credit and Banking*, 28(2):233–254.

Gort, M. and Klepper, S. (1982). Time paths in the diffusion of product innovation. *Economic Journal*, 92:630–653.

Guckenheimer, J. and Holmes, P. (1990). *Nonlinear Oscillations and Bifurcations of Vector Fields*. Springer-Verlag, 2nd edition.

Heijdra, B. (1998). Fiscal policy multipliers: the role of monopolistic competition, scale economies, and intertemporal substitution in labour supply. *International Economic Review*, 39(3):659–96.

Hirsch, M. and Smale, S. (1974). *Differential Equations, Dynamical Systems, and Linear Algebra*. Academic Press.

Hopenhayn, H. A. (1992). Entry, exit and firm dynamics in long run equilibrium. *Econometrica*, 60(5):1127–1150.

Hornstein, A. (1993). Monopolistic competition, increasing returns to scale, and the importance of productivity shocks. *Journal of Monetary Economics*, 31:299–316.

Howrey E. P. and Quandt, R. E. (1968). The dynamics of the number of firms in an industry. *Review of Economic Studies*, 35:349–353.

Leonard D, Van Long, N (1992). *Optimal Control Theory and Static Optimization in Economics*, Cambridge University Press.

McKenzie, L. W. (1959). On the existence of general equilibrium for a competitive market. *Econometrica*, 27:54–71.

Myers M.G. and Weintraub E.R (1971): A Dynamic model of Firm entry, *Review of Economic Studies*, 38, 127-9.

Novshek, W. and Sonnenschein, H. (1987). General equilibrium with free entry: a synthetic approach of the theory of perfect competition. *Journal of Economic Literature*, 25(3):1281–1306.

Okuguchi K (1972): A Dynamic Model of Firm Entry;' Comment. Review of Economic studies, 39, 521-522.

Rotemberg, J. J. and Woodford, M. (1991). Markups and the Business Cycle. In Blanchard, O. J. and Fischer, S., editors, *NBER Macroeconomics Annual 1991*, volume 6, pages 63–129. MIT Press.

Ryder H.E (1969): Optimal Accumulation in a two-sector neoclassical economy wiht non-shiftable capital, *Journal of Political Economy*, 87, 665-683.

Smith V. (1974). Optimal Costly Form Entry in General Equilibrium, *Journal of Economic Theory*, 9, 397-417.

6 Appendix 1: The Social Planner's Problem.

The social planner's problem is to choose $\{(C(t), e(t)), t \in [0, \infty)\}$ to solve

$$\max_{\{C,e\}} \int_0^{+\infty} U(C)e^{-\rho t} dt \tag{40}$$

subject to

$$\begin{aligned} \dot{K} &= n\left(AF(\frac{K}{n}, \frac{1}{n}) - \phi\right) - C - \gamma\frac{e^2}{2} &\quad (41)\\ \dot{n} &= e &\quad (42) \end{aligned}$$

As both the utility function and the constraints of the problem are concave functions of the controls, then (if the transversality conditions hold) the Pontriyagin maximum principle will give us necessary and sufficient conditions for optimality. The current value Hamiltonian is,

$$H \equiv U(C) + p_1\left(n\left(AF(\frac{K}{n}, \frac{1}{n}) - \phi\right) - C - \gamma\frac{e^2}{2}\right) + p_2 e$$

which is maximized by the optimal values for consumption and entry ($\hat{C}$, $\hat{e}$), such that

$$\begin{aligned} H_c &= U'(\hat{C}) - p_1 = 0 &\quad (43)\\ H_e &= -\gamma\hat{e}p_1 + p_2 = 0. &\quad (44) \end{aligned}$$

Then the optimal policy functions are

$$\begin{aligned} \hat{C} &= C(p_1), \quad C' < 0\\ \hat{e} &= \frac{p_2}{\gamma p_1}. \end{aligned}$$

The canonical equations are

$$\begin{aligned} \dot{p}_1 &= \rho p_1 - H_K = p_1(\rho - AF_K) &\quad (45)\\ \dot{p}_2 &= \rho p_2 - H_n = p_2\rho - p_1\left[AF(1-\nu) - \phi\right]. &\quad (46) \end{aligned}$$

If we time-differentiate (43), using (45), and time-differentiate the log of equation (44), we obtain an equivalent modified Hamiltonian dynamic system (after dropping the hats for notational simplicity):

$$\dot{C} = -\frac{C}{\sigma(C)}(\rho - r(K,n,A)) \tag{47}$$

$$\dot{e} = er(K,n,A) - \frac{\pi(K,n,A,\phi)}{\gamma} \tag{48}$$

$$\dot{K} = n\left(AF(\frac{K}{n},\frac{1}{n}) - \phi\right) - C - \gamma\frac{e^2}{2} \tag{49}$$

$$\dot{n} = e \tag{50}$$

given $K(0) = K_0$, $n(0) = n_0$ and the transversality conditions. These equations are exactly the same as the decentralised equilibrium outlined in the text. The only non-obvious translation is for NPV of firms: define $q \equiv \frac{p_2}{p_1}$ (from 44), hence (48) becomes (14).

7 Appendix 2: Proofs.

7.1 Proof of Proposition 1

The characteristic polynomial of the jacobian matrix is $c(\lambda) = \lambda^4 - 2\rho\lambda^3 + M_2\lambda^2 - \rho(M_2 - \rho^2)\lambda + M_4$, where M_2 and M_4 are the sum of the principal minors of order 2 and 4 of the jacobian J, where $M_2 = \rho^2 + \frac{Cr_K}{\sigma} + \frac{\pi_n}{\gamma}$ and $M_4 = \det(J) = \frac{C}{\sigma\gamma}(r_K\pi_n - r_n\pi_K)$. The characteristic polynomial can be equivalently written as $c(\lambda) = \left(\frac{\rho}{2}\right)^4(w^2 + a_1w + a_0)$, where

$$w \equiv \left(\lambda - \frac{\rho}{2}\right)^2\left(\frac{\rho}{2}\right)^{-2}, \tag{51}$$

and $a_1 \equiv \left(\frac{\rho}{2}\right)^{-2}(M_2 - \rho^2) - 2$ and $a_0 \equiv -a_1 + \left(\frac{\rho}{2}\right)^{-4}M_4 - 1$. Then $c(\lambda) = 0$ if and only if $w^2 + a_1w + a_0 = 0$. The roots of this polynomial on w are $w_{1,2} = -\frac{a_1}{2} \mp \left(\left(\frac{a_1}{2}\right)^2 - a_0\right)^{\frac{1}{2}}$. If we substitute the expressions for the coefficients a_1 and a_0 we get

$$\left(\frac{\rho}{2}\right)w_{1,2} = \left(\frac{\rho}{2}\right)^2 - \frac{1}{2}\left(\frac{Cr_K}{\sigma} + \frac{\pi_n}{\gamma}\right) \mp \frac{1}{2}\left[\left(\frac{Cr_K}{\sigma} - \frac{\pi_n}{\gamma}\right)^2 + 4\frac{Cr_n}{\sigma}\frac{\pi_K}{\gamma}\right]^{\frac{1}{2}}.$$

Then, solving equation (51) for λ we get the eigenvalues as $\left(\lambda - \frac{\rho}{2}\right)^{s,u}_{1,2} = \mp\left[\left(\frac{\rho}{2}\right)w_{1,2}\right]^{\frac{1}{2}}$ which is equivalent to equations (28)-(29).

Next we demonstrate that the eigenvalues are real and satisfy (30). Recall that $r_K < 0$, $r_n > 0$, $\pi_n < 0$ and $\pi_K > 0$. Then $\frac{Cr_K}{\sigma} + \frac{\pi_m}{\gamma} < 0$. The determinant of the Jacobian J, which is equal to the sum of the principal minors of order 4, M_4, is positive, as $\det(J) = \frac{\bar{c}}{\sigma\gamma}\left(\pi_n r_K - \pi_K r_n\right) = \frac{C}{\sigma\gamma}\left(\frac{A}{\bar{n}}\right)^2\left(\frac{F_{KK}F_{LL}-F_{KL}^2}{\bar{n}^2}\right) > 0$. Additionally the discriminant is positive, as $\Delta = \left(\frac{Cr_K}{\sigma} + \frac{\pi_m}{\gamma}\right)^2 - 4\det(J) = \left(\frac{Cr_K}{\sigma} - \frac{\pi_n}{\gamma}\right)^2 + 4\frac{cC}{\sigma\gamma}\pi_K r_n > 0$, which implies that $\Delta^{\frac{1}{2}}$ is real and positive. It also implies that $\frac{1}{2}\left(\frac{cr_K}{\sigma} + \frac{\pi_n}{\gamma}\right) + \frac{1}{2}\Delta^{\frac{1}{2}} < 0$ and that $\left(\frac{\rho}{2}\right)^2 - \frac{1}{2}\left(\frac{cr_K}{\sigma} + \frac{\pi_n}{\gamma}\right) - \frac{1}{2}\Delta^{\frac{1}{2}} > \left(\frac{\rho}{2}\right)^2 > 0$. Then $\left[\left(\frac{\rho}{2}\right)^2 - \frac{1}{2}\left(\frac{cr_K}{\sigma} + \frac{\pi_n}{\gamma}\right) - \frac{1}{2}\Delta^{\frac{1}{2}}\right]^{\frac{1}{2}}$ is positive and real and larger than $\frac{\rho}{2}$. Therefore the eigenvalues, given in equations (28)-(29), are real and verify equation (30). QED.

7.2 Proof of Proposition 2.

This proof starts with two Lemmas, A1 and A2.

Lemma A1 *The (transposed) eigenvectors associated to matrix J have the generic form*

$$P_i^j = \left[\frac{(\rho - \lambda_i^j)\frac{Cr_n}{\sigma}}{l_i - \frac{Cr_K}{\sigma}}, \lambda_i^j, \frac{\frac{Cr_n}{\sigma}}{l_i - \frac{Cr_K}{\sigma}}, 1\right]^T, \quad j = s, u, \; i = 1, 2 \tag{52}$$

where $l_1 - \frac{Cr_K}{\sigma} > 0$ and $l_2 - \frac{Cr_K}{\sigma} < 0$, for any value of the parameters.

Proof Eigenvector P_i^j in equation (52) is obtained in the usual way, as the non-zero solution of the homogeneous system $(J - \lambda_i^j I_4)P_i^j = 0$ for $j = s, u$ and $i = 1, 2$. As

$$\begin{aligned}
l_1 - \frac{Cr_K}{\sigma} &= -\frac{1}{2}\left(\frac{Cr_K}{\sigma} - \frac{\pi_n}{\gamma} - \left[\left(\frac{Cr_K}{\sigma} - \frac{\pi_n}{\gamma}\right)^2 + 4\frac{C}{\sigma\gamma}\pi_K r_n\right]^{\frac{1}{2}}\right) \\
&> -\frac{1}{2}\left(\frac{Cr_K}{\sigma} - \frac{\pi_n}{\gamma} - \left[\left(\frac{Cr_K}{\sigma} - \frac{\pi_n}{\gamma}\right)^2\right]^{\frac{1}{2}}\right) \\
&= -\frac{1}{2}\left(\frac{Cr_K}{\sigma} - \frac{\pi_n}{\gamma} - \left|\frac{Cr_K}{\sigma} - \frac{\pi_n}{\gamma}\right|\right) \\
&= \begin{cases} 0 & \text{if } \; 0 > \frac{Cr_K}{\sigma} > \frac{\pi_n}{\gamma} \\ \frac{\pi_n}{\gamma} - \frac{Cr_K}{\sigma} \geq 0 & \text{if } \; 0 > \frac{\pi_n}{\gamma} \geq \frac{Cr_K}{\sigma} \end{cases},
\end{aligned}$$

and

$$
\begin{aligned}
l_2 - \frac{Cr_K}{\sigma} &= -\frac{1}{2}\left(\frac{Cr_K}{\sigma} - \frac{\pi_n}{\gamma} + \left[\left(\frac{Cr_K}{\sigma} - \frac{\pi_n}{\gamma}\right)^2 + 4\frac{C}{\sigma\gamma}\pi_K r_n\right]^{\frac{1}{2}}\right) \\
&< -\frac{1}{2}\left(\frac{Cr_K}{\sigma} - \frac{\pi_n}{\gamma} + \left[\left(\frac{Cr_K}{\sigma} - \frac{\pi_n}{\gamma}\right)^2\right]^{\frac{1}{2}}\right) \\
&= -\frac{1}{2}\left(\frac{Cr_K}{\sigma} - \frac{\pi_n}{\gamma} + \left|\frac{Cr_K}{\sigma} - \frac{\pi_n}{\gamma}\right|\right) \\
&= \begin{cases} \frac{\pi_n}{\gamma} - \frac{Cr_K}{\sigma} \leq 0 & \text{if} \quad 0 > \frac{Cr_K}{\sigma} \geq \frac{\pi_n}{\gamma} \\ 0 & \text{if} \quad 0 > \frac{\pi_n}{\gamma} > \frac{Cr_K}{\sigma} \end{cases},
\end{aligned}
$$

then $l_1 - \frac{Cr_K}{\sigma} > 0$ and $l_2 - \frac{Cr_K}{\sigma} < 0$.

QED

Corollary A1 *The eigenvectors associated with the negative eigenvalues λ_1^s and λ_2^s are*

$$
P_1^s = \left[\frac{\lambda_1^u Cr_n}{\sigma l_1 - Cr_K}, \lambda_1^s, \frac{Cr_n}{\sigma l_1 - Cr_K}, 1\right]^T,
$$

$$
P_2^s = \left[\frac{\lambda_2^u Cr_n}{\sigma l_2 - Cr_K}, \lambda_2^s, \frac{Cr_n}{\sigma l_2 - Cr_K}, 1\right]^T.
$$

Lemma A2 *The orbits, belonging to the space tangent to the stable manifold, are determined from:*

$$
\begin{pmatrix} C(t) - C^* \\ e(t) - 0 \\ K(t) - K^* \\ n(t) - n^* \end{pmatrix} = P_1^s w_1^s e^{\lambda_1^s t} + P_2^s w_2^s e^{\lambda_2^s t} \tag{53}
$$

where

$$
w_1^s = \frac{l_1 - \frac{Cr_K}{\sigma}}{l_2 - l_1}\left[\left(\frac{l_2 - \frac{Cr_K}{\sigma}}{\frac{Cr_n}{\sigma}}\right)(K(0) - K^*) - (n(0) - n^*)\right], \tag{54}
$$

$$
w_2^s = \frac{l_2 - \frac{Cr_K}{\sigma}}{l_2 - l_1}\left[-\left(\frac{l_1 - \frac{Cr_K}{\sigma}}{\frac{Cr_n}{\sigma}}\right)(K(0) - K^*) + (n(0) - n^*)\right]. \tag{55}
$$

Proof. Let $x = (x_1(t), x_2(t), x_3(t), x_4(t))^T \equiv (C(t) - C^*, e(t), K(t) - K^*, n(t) - n^*)^T$. Then, from the Hartman-Grobman theorem, (Guckenheimer and Holmes, 1990, p. 13), the local dynamics is topologically equivalent to the solution of the linear system $\dot{x}(t) = Jx(t)$. The eigenvalues and the eigenvector matrix associated to J where already determined. Let $P = [P_2^u P_1^u P_1^s P_2^s]$ where Consider the vector $q = (q_2^u, q_1^u, q_1^s, q_2^2)$ such that $x = Pq$. It is well know that, if we take the time derivatives of w and substitute $\dot{x}$ we get the product system $\dot{q}(t) = \Lambda q(t)$, where $\Lambda = P^{-1}JP$ is the Jordan matrix of J. Therefore, $\Lambda = \text{diag}(\lambda_2^u, \lambda_1^u, \lambda_1^s, \lambda_2^s)$. This system has the solution $q_i^u(t) = w_i^u e^{\lambda_i^u t}$, $q_i^s(t) = w_i^s e^{\lambda_i^s t}$ for $i = 1, 2$, where w_i^s and w_i^u are arbitrary constants. There is conditional convergence towards the steady state, if we set $w_1^u = w_2^u = 0$. If we perform the inverse transformation (which is always possible because the eigenvector matrix is non singular) we get the trajectories along the saddle, given by equation (53).

If we assume that we know $K(t) - K^*$ and $n(t) - n^*$ at time t, then we get

$$w_1^s e^{\lambda_1^s t} = \frac{K(t) - K^* - P_2^s(3)(n(t) - n^*)}{P_1^s(3) - P_2^s(3)} \tag{56}$$

$$w_2^s e^{\lambda_2^s t} = \frac{-(K(t) - K^*) + P_1^s(3)(n(t) - n^*)}{P_1^s(3) - P_2^s(3)}. \tag{57}$$

As we usually know the initial values of the state variables, at time $t = 0$, then we get equations (54) and (55), if we substitute the appropriate elements of P_1^s and P_2^s. QED.

Corollary A2 *The linear space tangent to the stable manifold E^s is a 2-dimensional linear space in (C, e, K, n) given by*

$$E^s \equiv \{\ (C, e, K, n) : C - C^* = h_C(K - K^*, n - n^*), e = h_e(K - K^*, n - n^*)\ \}$$

where h_C and h_e are linear functions,

$$h_C = \left(\frac{\lambda_2^u - \lambda_1^u}{l_2 - l_1}\right)\left[\left(\frac{Cr_K}{\sigma} - \lambda_1^u\lambda_2^u\right)(K - K^*) + \frac{Cr_n}{\sigma}(n - n^*)\right],$$

$$h_e = \left(\frac{\lambda_2^s - \lambda_1^s}{l_2 - l_1}\right)\left[\left(\frac{\pi_n}{\gamma} - \lambda_1^s\lambda_2^s\right)(K - K^*) + \frac{\pi_n}{\gamma}\ (n - n^*)\right],$$

Proof. If we substitute equation (53) for the non-predetermined variables C and e the expressions for $w_1^s e^{\lambda_1^s t}$ and $w_2^s e^{\lambda_2^s t}$ given by (56) and (57) then we get a 2-dimensional linear manifold which expresses the functional dependence with the state variables, along the stable manifold, E^s. QED

7.3 Proof of Proposition 3.

From Proposition 2, we can use the definition of the schedules $w_1^s = 0$ and $w_2^s = 0$ to characterize the dynamics in the space (C, e, K, n) relative to the lines E_1^s and E_2^s. From Lemma A2, we get

$$\left.\frac{dn^*}{dK}\right|_{E_1^s} == \frac{1}{P_1^s(3)} = \frac{\sigma l_1 - Cr_K}{Cr_n} > 0.$$

and

$$\left.\frac{dn}{dK}\right|_{E_2^s} = \frac{1}{P_2^s(3)} = \frac{\sigma l_2 - Cr_K}{Cr_n} < 0$$

As E_1^s is associated with $w_2^s = 0$ and E_2^s with $w_1^s = 0$ QED

7.4 Proof of Proposition 4

If we consider equations (20) and (14) we see that the isoclines $\dot{K} = 0$ and $\dot{n} = 0$ depend on the control variables C and e, which depend on the K and n, and therefore their projections in (K, n) are always shifting. But, we can determine loci in the space (K, n) which are analogous to the isoclines in a two-dimensional model: we can determine the loci where the trajectory belonging to the approximation to the stable manifold change direction, that is the slope of $(n - n*)/(K - K^*)$ such that $d(K(t) - K^*)/dt = 0$ and the slope of $(n - n^*)/(K - K^*)$ such that $d(n(t) - n^*)/dt = 0$.

Consider equation (53): then $d(n(t) - n^*)/dt = 0$ if and only if

$$\lambda_1^s w_1^s e^{\lambda_1^s t} + \lambda_2^s w_2^s e^{\lambda_2^s t} = 0.$$

As $e^{\lambda_1^s t} \geq 0$ and $e^{\lambda_2^s t} \geq 0$ for any $0 \leq t < \infty$, then a necessary condition is that $\text{sign}(w_1^s) \neq \text{sign}(w_2^s)$. Then $e^{(\lambda_2^s - \lambda_1^s)t} = -\frac{\lambda_1^s w_1^s}{\lambda_2^s w_2^s}$ or, equivalently, we can find a critical time $t_n = \frac{1}{\lambda_2^s - \lambda_1^s} \ln\left(-\frac{\lambda_1^s w_1^s}{\lambda_2^s w_2^s}\right)$. If we substitute one of these expressions in equations (53) we get

$$\begin{aligned} n(t) - n^* &= \left(\frac{\lambda_2^s - \lambda_s^1}{\lambda_2^s}\right) w_1^s e^{\lambda_1^s t}, \\ K(t) - K^* &= \frac{Cr_n w_1^s}{\sigma \lambda_2^s}\left(\frac{\lambda_2^s}{l_1 - \frac{Cr_n}{\sigma}} - \right) e^{\lambda_1^s t}, \end{aligned}$$

Substituting in equations (54) and (55) and determining the ratio we get

$$\frac{n(t) - n^*}{K(t) - K^*} = -\frac{(\lambda_2^s - \lambda_1^s)(l_1 - Cr_n/\sigma)(l_2 - Cr_n/\sigma)}{\frac{Cr_n}{\sigma}\left(\lambda_2^s(l_1 - Cr_n/\sigma) - \lambda_1^s(l_2 - Cr_n/\sigma)\right)}.$$

As $(l_1 - Cr_n/\sigma)(l_2 - Cr_n/\sigma) = -Cr_n\pi_K/\sigma\gamma$ and $\lambda_2^s(l_1 - Cr_n/\sigma) - \lambda_1^s(l_2 - Cr_n/\sigma) = (\lambda_1^s - \lambda_2^s)(\pi_n/\gamma - \lambda_1^s\lambda_2^s)$ then we get equation (a).

We proceed in an analogous way to determine equation (b). From equation (53): then $d(K(t) - K^*)/dt = 0$ if and only if

$$\lambda_1^s w_1^s \left(\frac{Cr_n/\sigma}{l_1 - Cr_K/\sigma}\right) e^{\lambda_1^s t} + \lambda_2^s w_2^s \left(\frac{Cr_n/\sigma}{l_2 - Cr_K/\sigma}\right) e^{\lambda_2^s t} = 0.$$

Then $e^{(\lambda_2^s - \lambda_1^s)t} = -\frac{\lambda_1^s w_1^s (l_2 - Cr_K/\sigma)}{\lambda_2^s w_2^s (l_1 - Cr_K/\sigma)}$, so that

$$\frac{n(t) - n^*}{K(t) - K^*} = \frac{1}{\lambda_2^s - \lambda_1^s}\left(\frac{\lambda_2^s(l_1 - Cr_K/\sigma)}{Cr_n/\sigma} - \frac{\lambda_1^s(l_2 - Cr_K/\sigma)}{Cr_n/\sigma}\right),$$

which, after some algebra leads to the result stated. QED

7.5 Proof of Proposition 5.

(1) Consider Lemma A2: as $e^{\lambda_1^s t} \geq 0$ and $e^{\lambda_2^s t} \geq 0$ for any $t \in \mathbb{R}_+$, if $\text{sign}(w_1^s) = \text{sign}(w_2^s)$ then $n(t)$ will converge monotonically towards the steady state n^*. If they are positive (negative) $n(t)$ will decrease (increase). If $\text{sign}(w_1^s) \neq \text{sign}(w_2^s)$ then $n(t) - n^*$ may change sign along the transition and, therefore, have a non-monotonous behavior. If this is the case then there will be a combination of parameters, of the initial data such that $\frac{d(n(t)-n^*)}{dt} = 0$ for a particular time t_n. If this is the case then $e^{(\lambda_2^s - \lambda_1^s)t} = -\frac{\lambda_1^s w_1^s}{\lambda_2^s w_2^s}$ or, equivalently $t = \frac{1}{\lambda_2^s - \lambda_1^s}\ln\left(-\frac{\lambda_1^s w_1^s}{\lambda_2^s w_2^s}\right)$. If there are conditions under which $t \geq 0$ then we denote it by t_n. As $\lambda_2^s - \lambda_1^s < 0$ then a critical time exists if $\frac{\lambda_1^s w_1^s}{\lambda_2^s w_2^s} \geq -1$, which is equivalent to

$$\begin{cases} \lambda_1^s w_1^s + \lambda_2^s w_2^s > 0 & \text{and} \quad w_2^s < 0, w_1^s > 0 \\ \lambda_1^s w_1^s + \lambda_2^s w_2^s < 0 & \text{and} \quad w_2^s > 0, w_1^s < 0 \end{cases}.$$

After some algebra we get the equivalent condition

$$\begin{cases} \frac{n(0)-n^*}{K(0)-K^*} < \frac{\pi_K}{\gamma\lambda_1^s\lambda_2^s - \pi_n} & \text{and} \quad w_2^s < 0, w_1^s > 0 \\ \frac{n(0)-n^*}{K(0)-K^*} > \frac{\pi_K}{\gamma\lambda_1^s\lambda_2^s - \pi_n} & \text{and} \quad w_2^s > 0, w_1^s < 0. \end{cases} \tag{58}$$

This gives only a necessary condition. That is, if the conditions in equation (58) hold then there is a critical time

$$t_n = \frac{1}{\lambda_2^s - \lambda_1^s}\ln\left(-\frac{\lambda_1^s w_1^s}{\lambda_2^s w_2^s}\right) \geq 0,$$

such that $\frac{d}{dt}(n(t)-n^*)\big|_{t=t_n}=0$. If not, then $\frac{d}{dt}(n(t)-n^*)\neq 0$ for any $0\leq t<\infty$.

At time $t=t_n\geq 0$ we get the values of the state variables as

$$\begin{aligned} n(t_n)-n^* &= \left(\frac{\lambda_2^s-\lambda_1^s}{\lambda_2^s}\right) w_1^s e^{\lambda_1^s t_n} \\ K(t_n)-K^* &= \left(\frac{\gamma\lambda_1^s\lambda_2^s-\pi_n}{\pi_K}\right)\left(\frac{\lambda_2^s-\lambda_1^s}{\lambda_2^s}\right) w_1^s e^{\lambda_1^s t_n} \end{aligned}$$

by substituting $e^{(\lambda_2^s-\lambda_1^s)t}=-\frac{\lambda_1^s w_1^s}{\lambda_2^s w_2^s}$ into equations (54) and (55). In the (K,n)-space the locus where $\frac{d}{dt}(n(t)-n^*)=0$, is then given by

$$\frac{n-n^*}{K-K^*}=\frac{n(t_n)-n^*}{K(t_n)-K^*}=\frac{\pi_K}{\gamma\lambda_1^s\lambda_2^s-\pi_n}>0 \tag{59}$$

Which is of course the $\dot{n}=0$ isocline defined in Proposition 4.

Therefore, if $w_1^s<0$, if $w_2^s>0$ (i.e., in the W_n) and if $\frac{n(0)-n^*}{K(0)-K^*}>\frac{\pi_K}{\gamma\lambda_1^s\lambda_2^s-\pi_n}$ then the number of firms will fall until $t=t_n$, where it crosses the $\frac{d}{dt}(n(t)-n^*)=0$ line and increases afterwards. It will "cut" with a positive slope curve $\pi(K,n)=0$ and will be asymptotically tangent to the E_1^s projection. If $w_1^s<0$, if $w_2^s>0$ (that is, in the same quadrant) but if $\frac{n(0)-n^*}{K(0)-K^*}<\frac{\pi_K}{\gamma\lambda_1^s\lambda_2^s-\pi_n}$ then $(n(t)-n^*)<0$, $\frac{d}{dt}(n(t)-n^*)>0$ for any value of $0\leq t<\infty$ and $\lim_{t\to\infty}\frac{d}{dt}(n(t)-n^*)=0$ and n will monotonously increase towards n^*. We can make the same reasoning as regarding the quadrant E_n: (a) If $w_1^s>0$, if $w_2^s<0$ and $\frac{n(0)-n^*}{K(0)-K^*}<\frac{\pi_K}{\gamma\lambda_1^s\lambda_2^s-\pi_n}$ then the number of firms will increase until $t=t_n$, where it crosses the $\frac{d}{dt}(n(t)-n^*)=0$ line and decreases afterwards, crossing with a positive slope the curve $\pi(K,n)=0$ and becoming tangent asymptotically to the E_1^s projection. If $w_1^s>0$, if $w_2^s<0$ but if $\frac{n(0)-n^*}{K(0)-K^*}>\frac{\pi_K}{\gamma\lambda_1^s\lambda_2^s-\pi_n}$ then $(n(t)-n^*)>0$, $\frac{d}{dt}(n(t)-n^*)<0$ for any value of $0\leq t<\infty$ $\lim_{t\to\infty}\frac{\partial}{\partial t}(n(t)-n^*)=0$ and n will monotonously decrease towards n^*.

(2). Consider equation (53): as $e^{\lambda_1^s t}\geq 0$ and $e^{\lambda_2^s t}\geq 0$ for any $t\in\mathbb{R}_+$, and as $\frac{Cr_n}{\sigma l_1-Cr_K}>0$ and $\frac{Cr_n}{\sigma l_2-Cr_K}<0$ then $\text{sign}(w_1^s)=\text{sign}(w_2^s)$ then $K(t)$ will converge monotonically towards the steady state K^*. If $w_1^s>0$ and $w_2^s<0$ ($w_1^s<0$ and $w_2^s>0$) then $K(t)$ will decrease (increase) monotonically towards K^*. This is what happens in quadrants W_n and E_n. If we are in quadrants N_k or S_k, where $\text{sign}(w_1^s)=\text{sign}(w_2^s)$, then $K(t)-K^*$ may change sign along the transition and, therefore, have a non-monotonous behavior. In order to determine the conditions and the loci in which we may have $\frac{d}{dt}(K(t)-K^*)=0$, we follow the same procedure as in the previous case.

If

$$\begin{cases} \frac{n(0)-n^*}{K(0)-K^*}<\frac{\sigma\lambda_1^s\lambda_2^s-Cr_K}{Cr_n} & \text{and} \quad w_1^s<0, w_2^s<0 \\ \frac{n(0)-n^*}{K(0)-K^*}>\frac{\sigma\lambda_1^s\lambda_2^s-Cr_K}{Cr_n} & \text{and} \quad w_1^s>0, w_2^s>0 \end{cases}, \tag{60}$$

then there is a critical time

$$t_K = \frac{1}{\lambda_2^s - \lambda_1^s} \ln \left(-\frac{\lambda_1^s w_1^s(\sigma l_2 - C r_K)}{\lambda_2^s w_2^s(\sigma l_1 - C r_K)} \right) \geq 0,$$

such that $\frac{d}{dt}(K(t) - K^*)\big|_{t=t_K} = 0$, and substituting the critical time in equations (54,55) we get the set of values for (K, n) such that this condition holds,

$$\frac{n - n^*}{K - K^*} = \frac{n(t_K) - n^*}{K(t_K) - K^*} = \frac{\sigma \lambda_1^s \lambda_2^s - C r_K}{C r_n}. \tag{61}$$

We can make the same reasoning as for the case of n to see that, if the initial and steady state values for the state variables $(K(0), n(0))$ and (K^*, n^*) verify the conditions given by equation (60) then the saddle path will cut the line given by equation (61) at time t_K, changing the direction of evolution of the K variable, after a while it will cross line $r(K, n) = \rho$ and will converge asymptotically to (K^*, n^*). If conditions given by equation (60) do not hold, then $\frac{d}{dt}(K(t) - K^*) \neq 0$ for any $0 \leq t < \infty$ and the adjustment of K will be monotonic. Those conditions hold in the areas N_k and S_k given analytically in equations (33) and (34): in the first case K will decrease until it reaches the $\dot{K} = 0$ line and increases afterwards, and in the second case it has the opposite evolution. QED

7.6 Proofs of Propositions 6-9

Proof of proposition 6 If $w_1^s = 0$, then from (54) and Lemma's A1-A2 the projections of E_2^s are:

$$\left.\frac{dC}{dK}\right|_{E_2^s} = \frac{P_2^s(1)}{P_2^s(3)}, \quad \left.\frac{de}{dn}\right|_{E_2^s} = \frac{P_2^s(2)}{P_2^s(4)},$$

yielding the values reported. Likewise for E_1^s. QED

Proof of Proposition 7. Analogous to Proposition 5.

Proof of Proposition 8

The long run multipliers, with all the variables evaluated at the steady state values are

$$\begin{aligned}
\frac{\partial K^*}{\partial A} &= -\frac{C(r_A \pi_n - \pi_A r_n)}{\sigma\gamma \det(J)} = \frac{\rho C \phi}{\sigma\gamma n A \det(J)} > 0, \\
\frac{\partial n^*}{\partial A} &= \frac{C(r_A \pi_K - \pi_A r_K)}{\sigma\gamma \det(J)} = -\frac{C(1-\nu)A(F F_{kk} - (F_k)^2)}{\sigma\gamma n \det(J)} > 0, \\
\frac{\partial C^*}{\partial A} &= \rho \frac{\partial K^*}{\partial A} + nF = \frac{\rho^2 C \phi}{\sigma\gamma n A \det(J)} + \frac{n\phi}{A(1-\nu)} > 0.
\end{aligned}$$

As $l = 1/n$ then $\partial l/\partial A < 0$ and as $y^* = y^e = \phi\nu/(1-\nu)$ then $\partial y/\partial A = 0$, which means that $\partial k/\partial A > 0$. QED

Proof of Proposition 9

We have

$$\frac{\partial C(t)}{\partial A} = \frac{\partial C^*}{\partial A} + \frac{\lambda_1^u}{\sigma l_1 - Cr_k}\omega_{1,A}^s e^{\lambda_1^s t} + \frac{\lambda_2^u}{\sigma l_2 - Cr_k}\omega_{2,A}^s e^{\lambda_2^s t}$$

for any $t \geq 0$ where

$$\omega_{1,A}^s = \frac{\partial \omega_1^s}{\partial A} = \frac{\pi_K}{\gamma(l_2 - l_1)}\left(\frac{\partial K^*}{\partial A} - \frac{Cr_n}{\sigma l_2 - Cr_K}\frac{\partial n^*}{\partial A}\right) < 0 \quad (62)$$

$$\omega_{2,A}^s = \frac{\partial \omega_2^s}{\partial A} = \frac{\pi_K}{\gamma(l_2 - l_1)}\left(\frac{\partial K^*}{\partial A} - \frac{Cr_n}{\sigma l_1 - Cr_K}\frac{\partial n^*}{\partial A}\right). \quad (63)$$

For time $t = 0$ we have

$$\begin{aligned}\frac{\partial C(0)}{\partial A} &= \frac{\partial C^*}{\partial A} + \frac{\lambda_1^u}{\sigma l_1 - Cr_K}\omega_{1,A}^s + \frac{\lambda_2^u}{\sigma l_2 - Cr_K}\omega_{2,A}^s = \\ &= \frac{\partial C^*}{\partial A} + \frac{\lambda_2^u - \lambda_1^u}{l_2 - l_1}\left(\lambda_1^u\lambda_2^u\frac{\partial K^*}{\partial A} + \frac{C}{\sigma}r_A\right)\end{aligned}$$

using equation (62,63) and the fact that $(\sigma l_1 - Cr_K)(\sigma l_2 - Cr_K) = -\sigma Cr_n\pi_K/\gamma$. The sign of the expression is ambiguous but as the second term is negative, we readily conclude that $\frac{\partial C(0)}{\partial A} < \frac{\partial C(\infty)}{\partial A} = \frac{\partial C^*}{\partial A}$. In order to sign $\frac{\partial C(0)}{\partial A}$ we further substitute $\frac{\partial C^*}{\partial A}$ to get

$$\begin{aligned}\frac{\partial C(0)}{\partial A} &= \frac{1}{x(\sigma,\gamma)A}\left[\gamma\lambda_1^u\lambda_2^u\left(-\rho C + (\lambda_1^u + \lambda_2^u - \rho)\frac{\sigma\phi n}{1-\nu}\right) - \frac{C\rho\phi}{n}\right] = \\ &= \frac{n\phi}{(1-\nu)x(\sigma,\gamma)A}\left(z(\sigma,\gamma) - \frac{\rho\nu\phi}{n}\right)\end{aligned}$$

where

$$z(\gamma,\sigma) \equiv \gamma\lambda_1^u\lambda_2^u\left(\sigma(\lambda_1^u + \lambda_2^u) - \rho(\sigma + \nu)\right). \quad (64)$$

As $\frac{\rho\nu\phi}{n} > 0$ and as $\lim_{\gamma\to 0} z = 0$ and $\lim_{\gamma\to\infty} z = \infty$ for any finite and positive σ, then there is a critical value of γ, call it γ_A, depending on σ such that $x(\gamma_A,\sigma) = \frac{\rho\nu\phi}{n} > 0$. In this case we have $\frac{\partial c(0)}{\partial A} = 0$ The same reasoning can be made as regards σ.

For the case of entry,

$$\begin{aligned}\frac{\partial e(t)}{\partial A} &= \lambda_1^s\omega_{1,A}^s e^{\lambda_1^s t} + \lambda_2^s\omega_{2,A}^s e^{\lambda_2^s t} \\ &= \frac{\pi_K}{\gamma(l_2 - l_1)}\left[\lambda_1^s\left(\frac{\partial K^*}{\partial A} - \frac{Cr_n}{\sigma l_2 - Cr_K}\frac{\partial n^*}{\partial A}\right)e^{\lambda_1^s t} - \lambda_2^s\left(\frac{\partial K^*}{\partial A} - \frac{Cr_n}{\sigma l_1 - Cr_K}\frac{\partial n^*}{\partial A}\right)e^{\lambda_2^s t}\right],\end{aligned}$$

which is ambiguous for all $t \in [0, \infty)$ and $\frac{\partial e(\infty)}{\partial \phi} = \frac{\partial e^*}{\partial \phi} = 0$. Evaluating at time $t = 0$, we get

$$\begin{aligned} \frac{\partial e(0)}{\partial A} &= \frac{\pi_K}{\gamma(l_2 - l_1)} \left[\lambda_1^s \left(\frac{\partial K^*}{\partial A} - \frac{C r_n}{\sigma l_2 - C r_K} \frac{\partial n^*}{\partial A} \right) - \lambda_2^s \left(\frac{\partial K^*}{\partial A} - \frac{C r_n}{\sigma l_1 - C r_K} \frac{\partial n^*}{\partial A} \right) \right] = \\ &= \frac{\lambda_2^s - \lambda_s^1}{l_2 - l_1} \left(\frac{\pi_A}{\gamma} + \lambda_s^1 \lambda_s^2 \frac{\partial n^*}{\partial A} \right) > 0. \end{aligned}$$

As this result is equivalent to $\lambda_1^s \omega_{1,A}^s > \lambda_2^s \omega_{2,A}^s$ and as $e^{\lambda_1^s t} > e^{\lambda_2^s t}$ for $0 < t < \infty$ then $e(t) > 0$ for $0 \leq t < \infty$. QED

7.7 Proof of Proposition 10.

(a) The long run multipliers, with all the variables evaluated at the steady state values, for the state variables are:

$$\frac{\partial K^*}{\partial \phi} = \frac{C r_n \pi_\phi}{\sigma \gamma \det(J)} = -\frac{\rho C (1 - \nu) \rho}{\sigma \gamma n \det(J)} < 0 \tag{65}$$

and

$$\frac{\partial n^*}{\partial \phi} = -\frac{r_K}{r_n} \frac{\partial K^*}{\partial \phi} < 0.$$

For consumption we get

$$\frac{\partial C^*}{\partial \phi} = \rho \frac{\partial K^*}{\partial \phi} - n < 0 \tag{66}$$

(b) Note that $y^* = y^e$ defined in (7) which is increasing in ϕ. Capital per firm can be derived from part (a).

QED

7.8 Proof of Proposition 11

For consumption we have

$$\frac{\partial C(t)}{\partial \phi} = \frac{\partial C^*}{\partial \phi} + \frac{\lambda_1^u}{\sigma l_1 - C r_K} \omega_{1,\phi}^s e^{\lambda_1^s t} + \frac{\lambda_2^u}{\sigma l_2 - C r_K} \omega_{2,\phi}^s e^{\lambda_2^s t}$$

with an ambiguous sign because the first and third are negative and the second is positive, as

$$\omega_{1,\phi}^s = \frac{\sigma \pi_K l_2}{\gamma(l_2 - l_1)(\sigma l_2 - C r_K)} \frac{\partial K^*}{\partial \phi} > 0 \tag{67}$$

$$\omega_{2,\phi}^s = -\frac{\sigma \pi_K l_1}{\gamma(l_2 - l_1)(\sigma l_1 - C r_K)} \frac{\partial K^*}{\partial \phi} > 0 \tag{68}$$

where $\frac{\partial K^*}{\partial \phi} < 0$. For time $t = 0$ we get, substituting the expressions from equations equations (67) and (68) and after some algebra

$$\frac{\partial C(0)}{\partial \phi} = \frac{\partial C^*}{\partial \phi} + \frac{\lambda_1^u \lambda_2^u}{\rho - \lambda_1^u - \lambda_2^u} \frac{\partial K^*}{\partial \phi}$$

which is ambiguous as the first term is negative and the second is positive. We may immediately conclude that $\frac{\partial C(0)}{\partial \phi} > \frac{\partial C^*}{\partial \phi}$. In order to determine the sign for the shock in $C(0)$, we use equations (66) and (65) and the expression for r_n to get

$$\frac{\partial C(0)}{\partial \phi} = \frac{n}{z(\gamma, \sigma)} \left(\frac{C(1-\nu)\rho}{n^2} - z(\gamma, \sigma) \right) \tag{69}$$

where $z(\gamma, \sigma)$ is defined in (64). Observe that while λ_1^u and λ_2^u depend on σ and γ, all the other elements entering in equation (69) do not depend on those parameters. Also, if we assume that σ has a positive and finite value, as $\lim_{\gamma \to 0} z(\gamma, .) = 0$ and $\lim_{\gamma \to \infty} z(\gamma, .) = \infty$, there is at least one critical value γ_c, such that $z(\gamma_c, \sigma) = \frac{C(1-\nu)\rho}{n^2} > 0$. In this case we have $\frac{\partial C(0)}{\partial \phi} = 0$ The same reasoning can be made as regards σ.

From equations (67) and (68) it is clear that

$$\begin{aligned} \frac{\partial e(t)}{\partial \phi} &= \lambda_1^s \omega_{1,\phi}^s e^{\lambda_1^s t} + \lambda_2^s \omega_{2,\phi}^s e^{\lambda_2^s t} = \\ &= \frac{\sigma \pi_K}{\gamma(l_2 - l_1)} \left(\frac{\lambda_1^s l_2}{\sigma l_2 - C r_K} e^{\lambda_1^s t} - \frac{\lambda_2^s l_1}{\sigma l_1 - C r_K} e^{\lambda_2^s t} \right) \frac{\partial K^*}{\partial \phi} \leq 0, \end{aligned}$$

for all $t \in [0, \infty)$ and $\frac{\partial e(\infty)}{\partial \phi} = \frac{\partial e^*}{\partial \phi} = 0$. Therefore, $e(t)$ is negative at time $t = 0$ and increases monotonously (through negative values) towards zero. QED

7.9 Proof of Proposition 12.

We establish Lemma A3, which together with Propositions 5 and 7 establishes Proposition 12.

Lemma A3. Critical values for η. *Consider only the case in which there is a positive correlation between ϕ and A and the case in which there are positive shocks to those parameters. Then there is a sequence of values for η,*

$$0 < \eta_0 < \eta_1 < \eta_n < \eta_2 < \eta_K < \eta_3$$

such that immediately after the shock the initial values for (K, n) will be located: at the new $\pi(K, n)$ line if $\eta = \eta_0$; at the new $\dot{n} = 0$ line if $\eta = \eta_1$; at the new

E_2^s line if $\eta = \eta_2$; at the new $\dot{K} = 0$ line if $\eta = \eta_3$; for $\eta = \eta_n$ and $\eta = \eta_K$ it will be located at the horizontal and vertical lines passing through the new steady state so that (respectively) $dn^*/dA = 0$ and $dK^*/dA = 0$.

Proof If we equate equation (39) with the expressions for the projections of curves $\pi(K,n) = 0$, $\dot{n} = 0$, E_2^s and $\dot{K} = 0$ and solve for η, we find, respectively

$$\begin{aligned}
\eta_0 &= -\frac{\frac{\partial n^*}{\partial A} + \frac{\pi_K}{\pi_n}\frac{\partial K^*}{\partial A}}{\frac{\partial n^*}{\partial \phi} + \frac{\pi_K}{\pi_n}\frac{\partial K^*}{\partial \phi}} \\
\eta_1 &= -\frac{\frac{\partial n^*}{\partial A} - \frac{\pi_K}{\gamma\lambda_1^s\lambda_2^s - \pi_n}\frac{\partial K^*}{\partial A}}{\frac{\partial n^*}{\partial \phi} - \frac{\pi_K}{\gamma\lambda_1^s\lambda_2^s - \pi_n}\frac{\partial K^*}{\partial \phi}} \\
\eta_2 &\equiv -\frac{\frac{\partial n^*}{\partial A} - \frac{\sigma l_2 - Cr_K}{Cr_n}\frac{\partial K^*}{\partial A}}{\frac{\partial n^*}{\partial \phi} - \frac{\sigma l_2 - Cr_K}{Cr_n}\frac{\partial K^*}{\partial \phi}} \\
\eta_3 &\equiv -\frac{\frac{\partial n^*}{\partial A} - \frac{\sigma\lambda_1^s\lambda_2^s - Cr_K}{Cr_n}\frac{\partial K^*}{\partial A}}{\frac{\partial n^*}{\partial \phi} - \frac{\sigma\lambda_1^s\lambda_2^s - Cr_K}{Cr_n}\frac{\partial K^*}{\partial \phi}}
\end{aligned}$$

For values belonging to the interval $\eta_1 < \eta < \eta_2$ there is a value for η such that $dn^*/dA = 0$, $\eta_n = -\frac{\partial n^*}{\partial A}/\frac{\partial n^*}{\partial \phi}$. Also, in the interval $\eta_2 < \eta < \eta_3$ there is a value for η such that $dK^*/dA = 0$, $\eta_K = -\frac{\partial K^*}{\partial A}/\frac{\partial K^*}{\partial \phi}$. QED

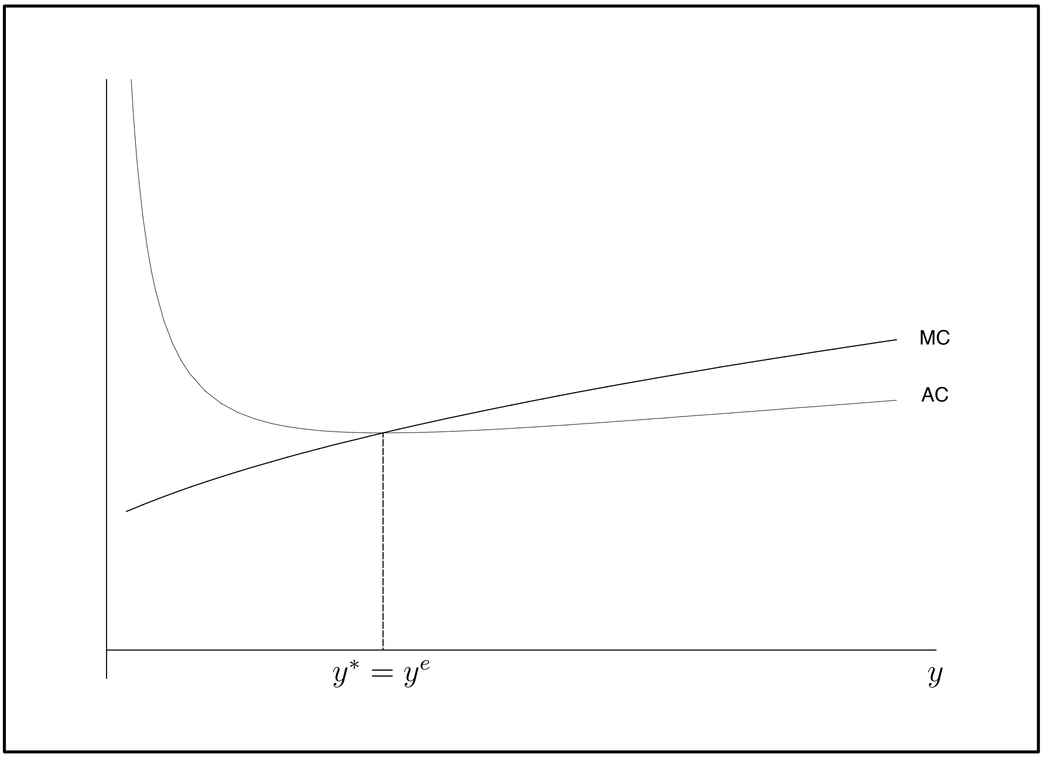

Figure 1: Cost functions and efficient production with $\pi = 0$

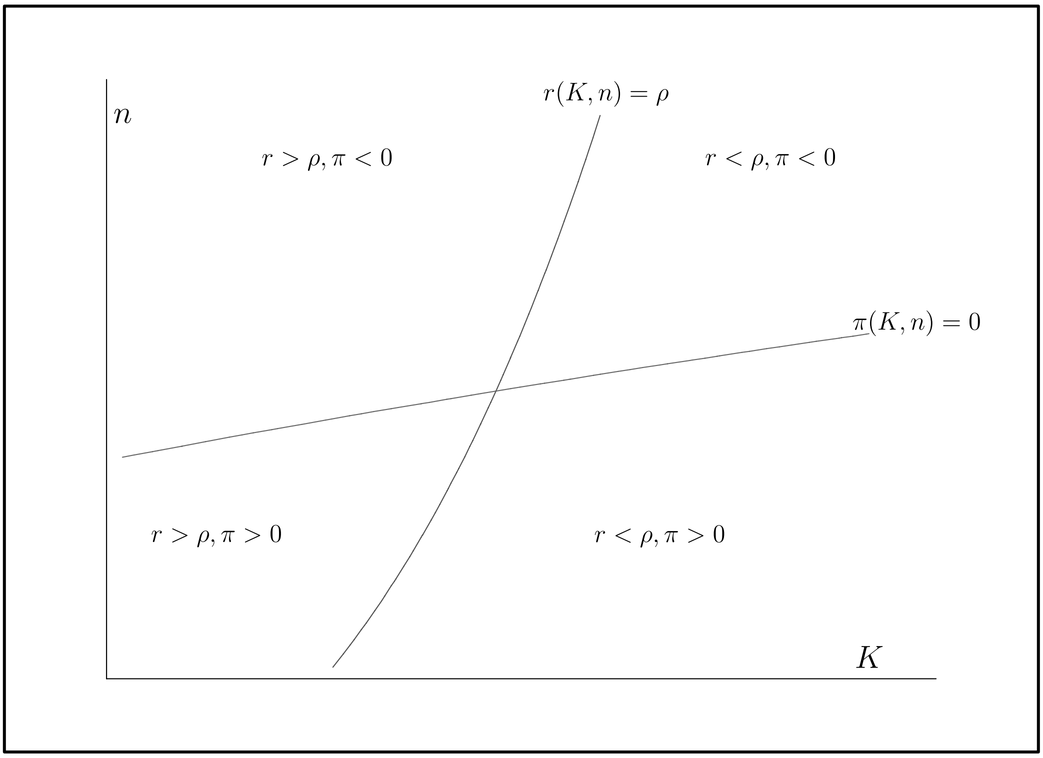

Figure 2: The reference invariant lines in (K, n)

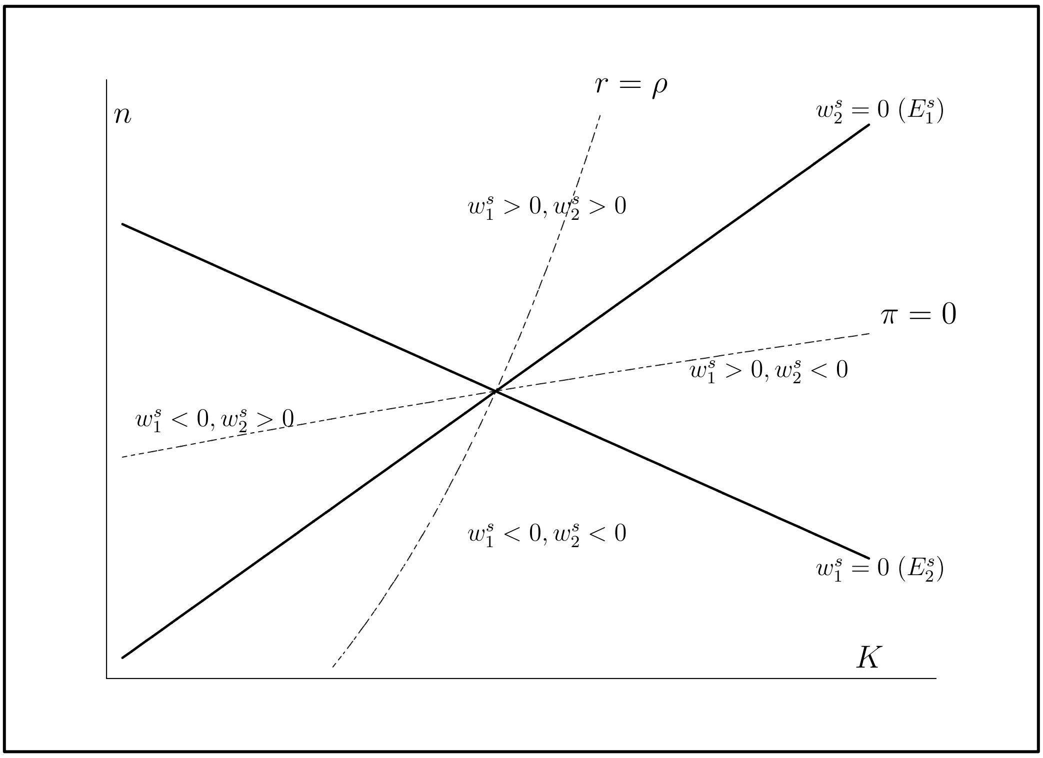

Figure 3: Reference lines E_i^s corresponding to the projections of eigenvectors which span the stable manifold

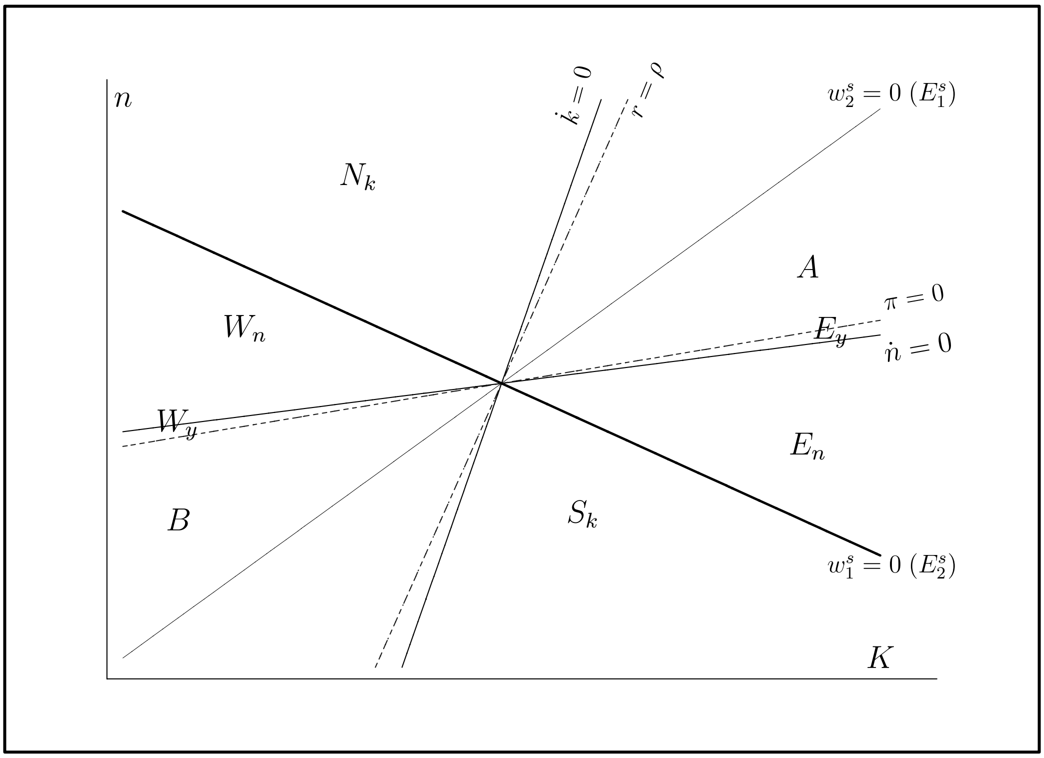

Figure 4: The projections of the linearized isoclines for K and n

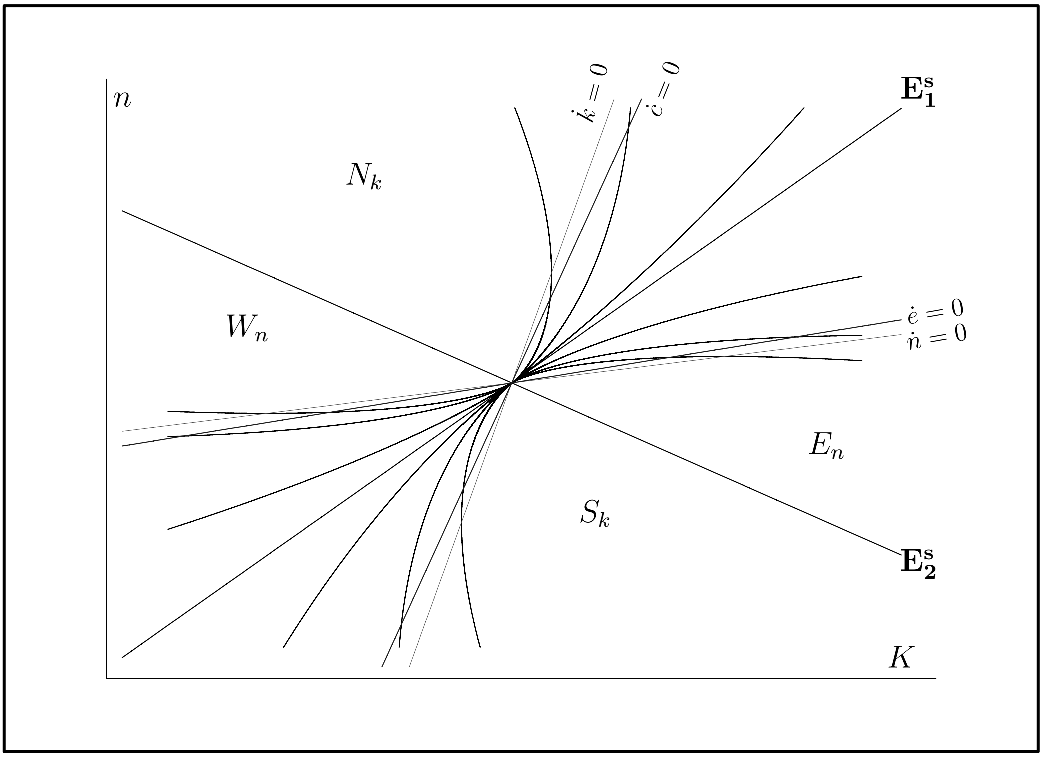

Figure 5: Representative trajectories for $0 < \gamma < \infty$

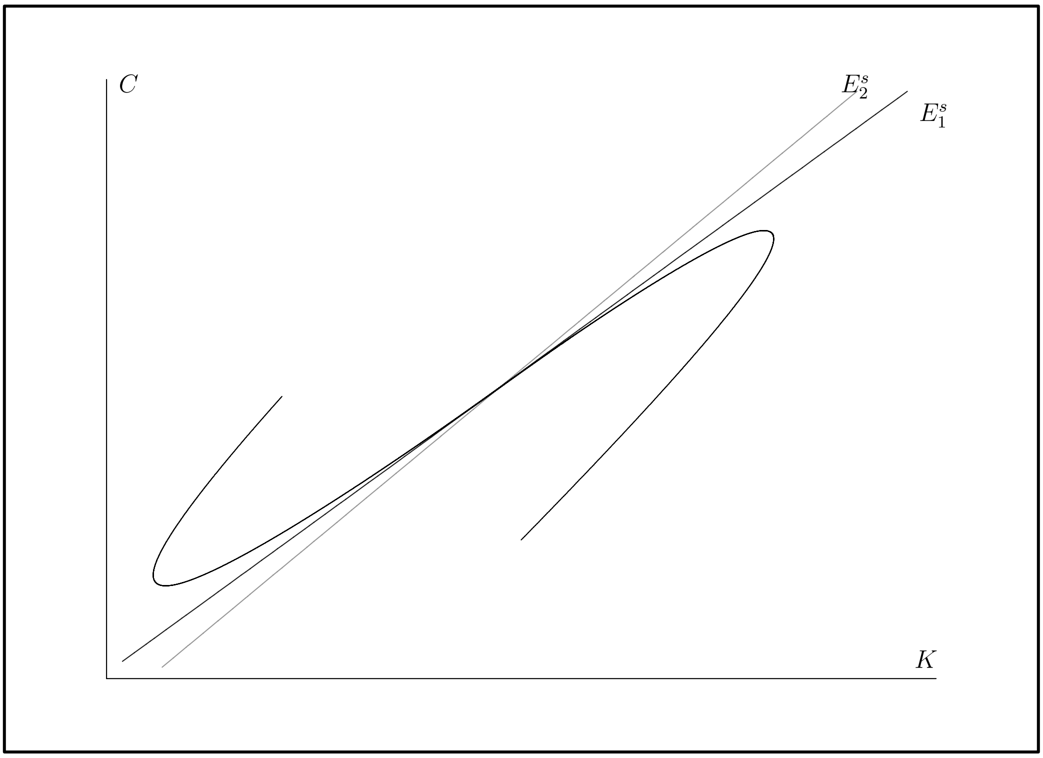

Figure 6: Trajectories where K is non-monotonic in (C, K)-space

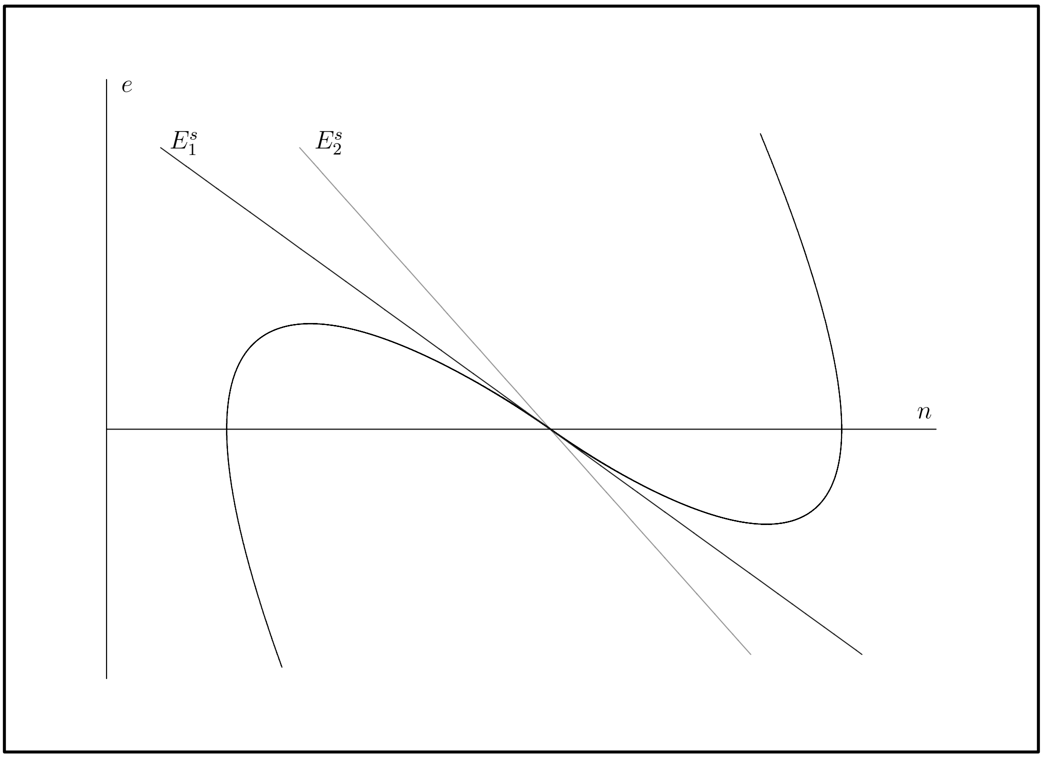

Figure 7: Trajectories where e is non-monotonic in (e, n)-space

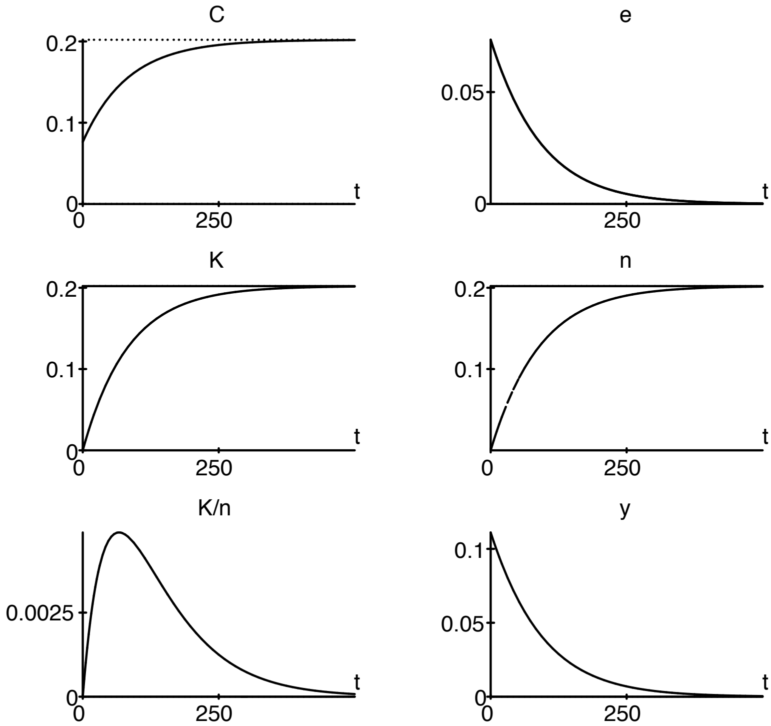

Figure 8: Permanent 10% increase in productivity. Income effect dominates.

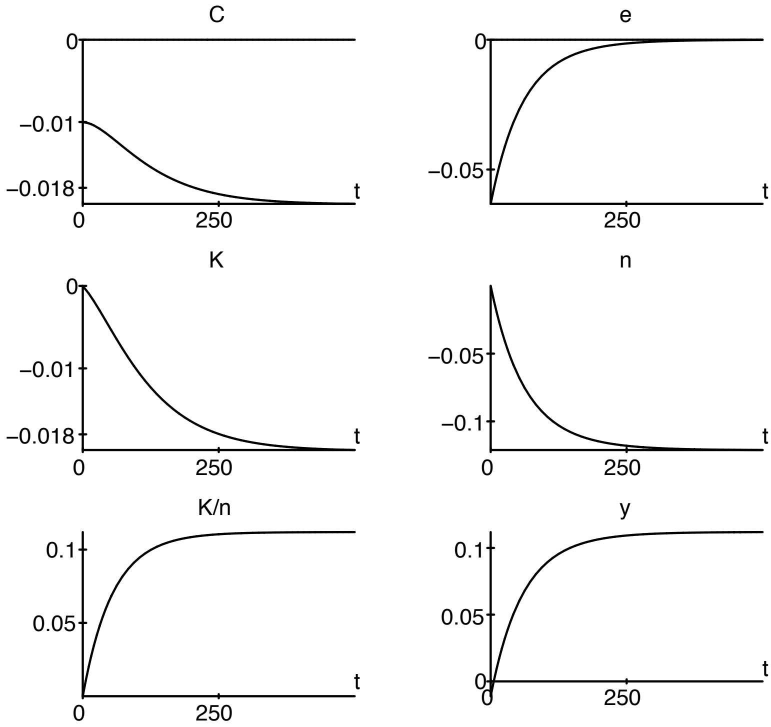

Figure 9: Permanent 10% increase in ϕ. Income effect dominates.

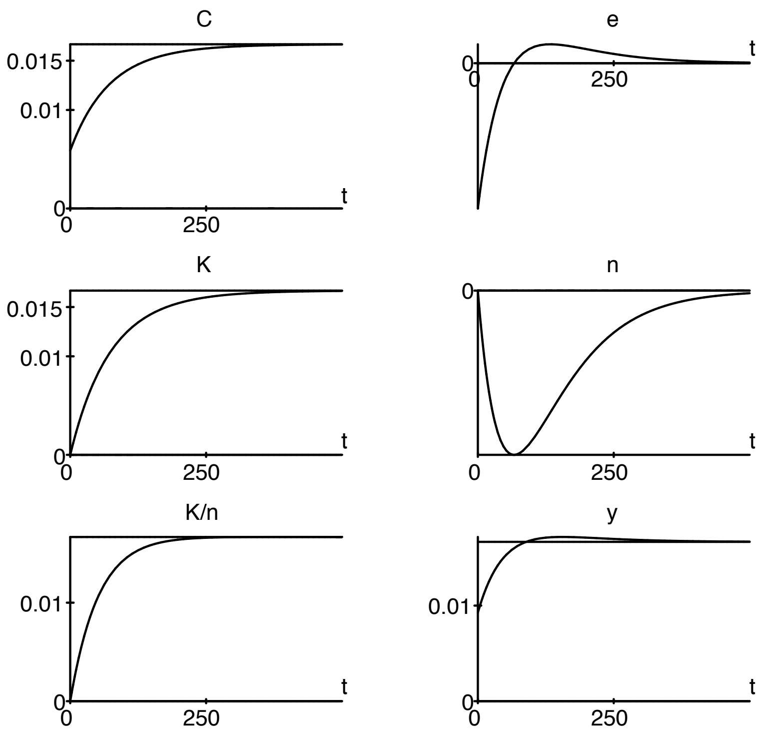

Figure 10: global shock for $\eta = \eta_n$

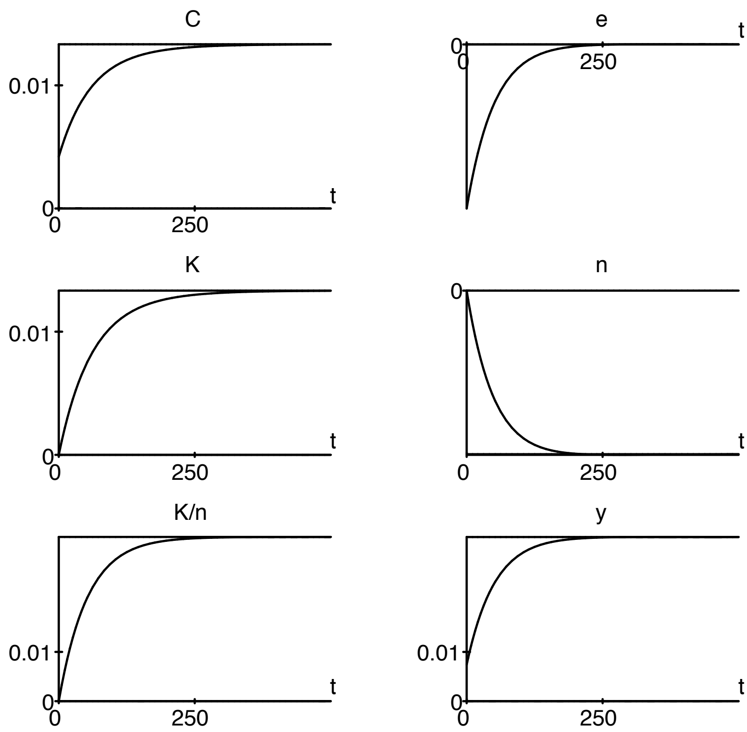

Figure 11: global shock for $\eta_n < \eta < \eta_2$

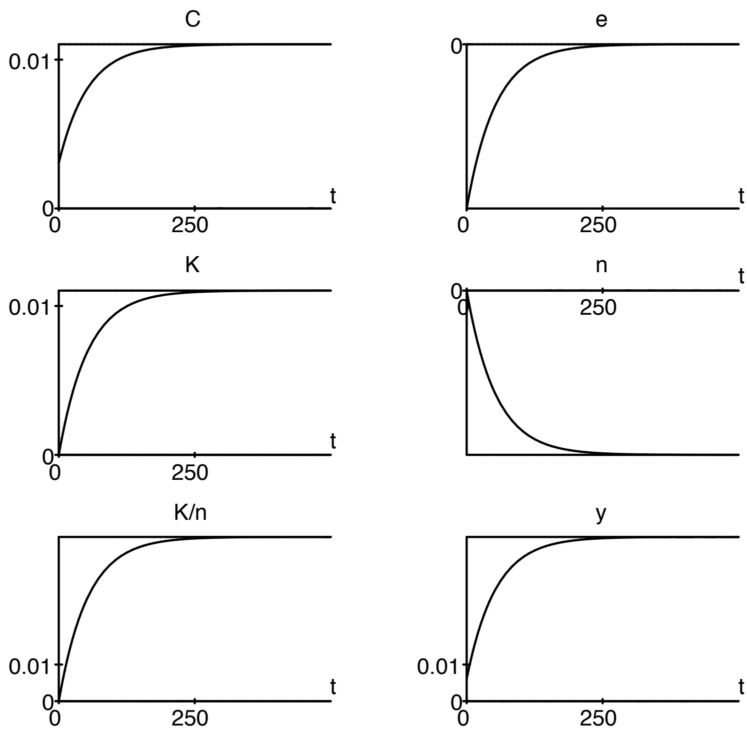

Figure 12: global shock for $\eta = \eta_2$

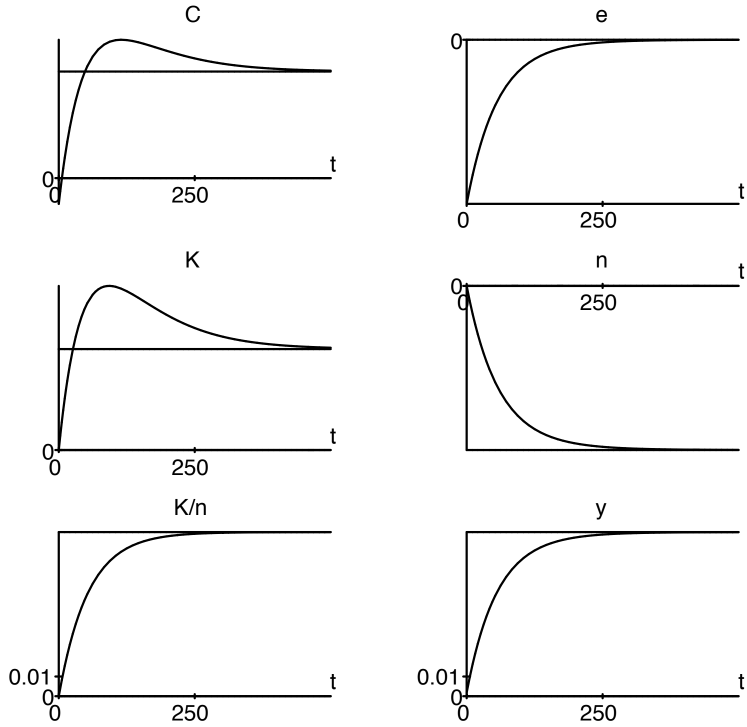

Figure 13: global shock for $\eta_2 < \eta < \eta_K$

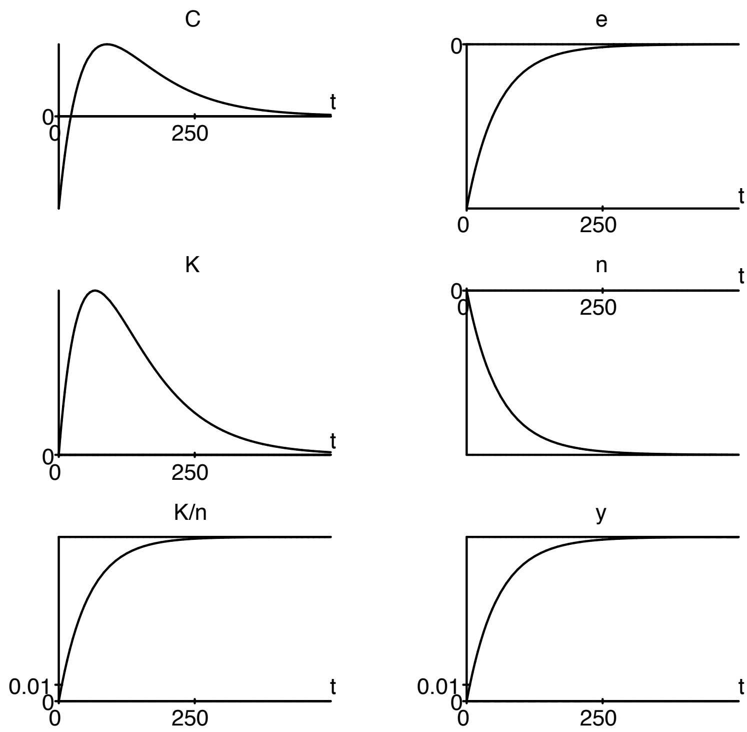

Figure 14: global shock for $\eta = \eta_K$

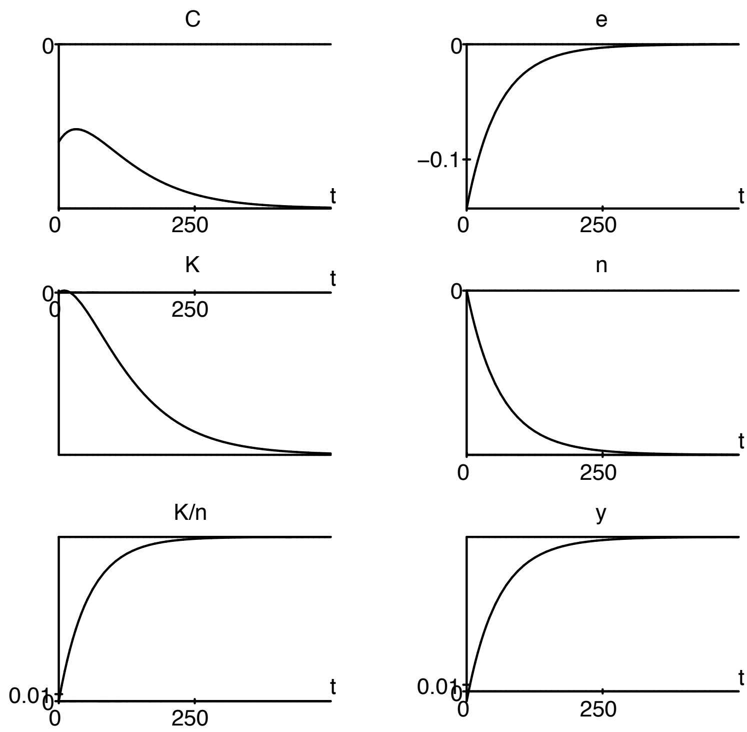

Figure 15: global shock for $\eta_K < \eta < \eta_3$