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Péter Vida and Helmuts Āzacis

A Detail-Free Mediator

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Cardiff Business School
Cardiff University
Colum Drive
Cardiff CF10 3EU
United Kingdom
t: +44 (0)29 2087 4000
f: +44 (0)29 2087 4419
www.cardiff.ac.uk/carbs

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A Detail-Free Mediator*

Péter Vida† and Helmuts Āzacis‡

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Abstract

We present an extension to any finite complete information game with two players. In the extension, players are allowed to communicate directly and, additionally, send private messages to a simple, detail-free mediator, which, in turn, makes public announcements as a deterministic function of the private messages. The extension captures situations in which people engage in face-to-face communication and can observe the opponent's face during the conversation before choosing actions in some underlying game. We prove that the set of Nash equilibrium payoffs of the extended game approximately coincides with the set of correlated equilibrium payoffs of any underlying game.

Keywords: Correlated equilibrium, detail-free mechanism, mediated pre-play communication.

JEL Classification Numbers: C72

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†University of Vienna, Department of Economics. E-mail: peter.vida@univie.ac.at

‡Cardiff University, Cardiff Business School. E-mail: helmuts.azacis@cf.ac.uk

1 Introduction

In games of complete information, with the help of a mediator, 2 players can achieve payoffs, which give them higher welfare than any Nash equilibrium. As an example, consider the well known chicken game, introduced by Robert Aumann and interpreted by Cavaliere (2001) as the provision of discrete public good with a positive externality.[1] Figure 1 shows the payoff matrix of our example.

Player 2

		0	1
Player 1	0	$6, 6$	$2, 7$
	1	$7, 2$	$0, 0$

Figure 1: The chicken game

We refer to the above defined game as Γ. The players would like to play the action profile $(0, 0)$, which gives them the highest utilitarian welfare, but it is not a Nash equilibrium of the game. In the mixed strategy Nash equilibrium, $(0, 0)$ has probability $4/9$ but the profile $(1, 1)$ is also played with probability $1/9$. Hence, the bad outcome cannot be avoided in Nash equilibrium if the players wish to play $(0, 0)$ with a positive probability.

The Nash equilibrium outcomes can be improved if one extends Γ to a two stage game Γ^μ, where in the first stage a trusted third party, called mediator, randomizes over the action profiles according to some distribution μ, called correlation device, and sends private messages to the players. These private messages can be interpreted as suggested actions that the players should take in Γ. An example of μ is provided in Figure 2.

	0	1
0	$1/3$	$1/3$
1	$1/3$	0

Figure 2: The distribution μ.

In the second stage, the players choose actions in Γ. If it is a Nash equilibrium of Γ^μ that players obediently follow the suggestions received according to μ then μ is called a correlated equilibrium distribution of Γ (Aumann, 1974). Let $CE(\Gamma) =$

[1] Indeed, suppose there is a public good project to be implemented and two players can decide whether to contribute (action 0) or not to contribute (action 1) to the project. For example, in Section 4.1, we interpret the players as students who are working on a joint project to be submitted to the university. The project is carried out successfully if at least one of the players has contributed. The payoffs in Figure 1 reflect the contributor's cost and benefit from the public good compared to those of the free-rider. The payoffs enjoyed by the players when both contribute can be interpreted as a positive externality from the joint contribution to the project's quality.

$\cup_\mu NE(\Gamma^\mu)$ be the set of correlated equilibrium payoffs, where NE stands for the set of Nash equilibrium payoffs of some game. $CE(\Gamma)$ is the largest set of non-cooperative outcomes achievable when arbitrary means of communication are available to the players. For many games, $NE(\Gamma) \subsetneq CE(\Gamma)$.

Our main concern is that the correlation device, μ has to be tailor-made to the game and to the desired outcome. More precisely, it can be that:

1. μ is a correlated equilibrium distribution for Γ but not for Γ'.

2. A payoff profile in $CE(\Gamma)$ is achievable with μ but not with μ'.

Thus, the mediator in charge of the correlation device must essentially know the underlying game Γ and the distribution μ, which, in certain circumstances, can be a very strong requirement.

Instead, we may think that, prior to taking actions in Γ, players are explicitly communicating with each other. This communication is carried out through some communication technology that defines what messages the players can send and receive. To avoid the criticism mentioned in the previous paragraph, this communication technology must not vary with the underlying game Γ and distribution μ, in which case we say that the communication technology is detail free. However, by varying their strategies, the players may still induce different distributions over the action profiles in Γ. Furthermore, one would normally prefer a communication technology that employs simple input and output messages. We are interested in the following question: *Does there exist a simple, detail-free communication technology, with which the players can generate any correlation device in a Nash equilibrium of extended game just by varying their strategies?*

The posed question has been answered positively for games with more than two players. See the constructions in Bárány (1992), Ben-Porath (2003), and Gerardi (2004). It is shown that it is sufficient if any two players can directly communicate with each other in such a way that a third player cannot eavesdrop their conversation. In the two player case, however, having only direct communication is clearly not sufficient because it cannot generate uncertainty about the opponent's action, which is an inherent feature of most of the correlation devices. For example, given μ in Figure 2, if the action 0 is recommended to a player, he is uncertain what action has been recommended to the other player. It is well known that in two players games, only payoffs in the convex hull of Nash equilibria can be generated with direct communication.[2]

To generate the necessary uncertainty, we consider the following mediated communication, which is a natural extension to direct communication: besides talking to each other directly, players may also observe each other's face during

[2]See Urbano and Vila (2002, 2004) on implementation of the entire set of CE with unmediated pre-play communication when players are computationally restricted, that is, they cannot perform certain hard operations. Another solution is if players are allowed to use physical devices such as urns with balls or envelopes as in Ben-Porath (1998) or in Krishna (2004).

the conversation. We call it *face-to-face communication.* The uncertainty is generated in the following way. The players may or may not look in each other's face during their conversation. If a player looks in his opponent's face, then he knows whether the other has looked in his face or not. However, if a player does not look in his opponent's face, then he is uncertain whether the other has observed his face or not.

More specifically, the ability to observe opponent's face can be modeled with the following communication device:

	0	1
0	0	0
1	0	1

Figure 3: The *AND* communication device

The *AND* communication device in Figure 3 receives private inputs of 0 (do not look) or 1 (do look) from each of the two players, and produces public output of 1 if both inputs were 1, and public output of 0 otherwise.[3] An attractive feature of the *AND* device is that the output messages are a deterministic function of input messages and, additionally, these output messages are public. This could be important, for example, if the players wished to check if the communication device (more generally, the mediator) behaves as intended.

As mentioned above, direct communication cannot always generate all correlated equilibrium distributions of a game. However, a similar observation also applies to the *AND* device. Gossner and Vieille (2001) have shown that any communication strategy involving a repeated use of the *AND* device *alone* either generates no correlation or the strategy is not secure in the sense that there exists a game in which at least one of the players has incentives to deviate from the given communication strategy, even if the distribution generated by that strategy is a correlated equilibrium distribution of the game. Thus, the set of possible payoffs that the players can achieve in equilibrium using only the *AND* device is, in general, strictly included in the convex hull of Nash equilibria. Therefore, we allow for both direct communication and the communication through the *AND* device.

Finally, we assume that the private messages that players are sending to the *AND* communication device can be recorded and, if necessary, recalled. For

[3]One can attach various interpretations to the *AND* device. For example, an internet community offers its members the possibility to establish links for reasons of common interests, common friends etc. Each member has the possibility to accept or reject the link formation. The link forms if and only if both parties consent it. More generally, one can think of mediated negotiation process where the mediator only makes an announcement in the case of mutual (dis-)agreement. See another example in Ponsati, Jarque, and Sakovics (2003) for a mediated bargaining game, where the mediator announces agreement if and only if the offers are compatible.

example, imagine that their face-to-face conversation is recorded with a video camera! Later, prior to taking actions, this conversation can be replayed. We refer to this property of the communication technology as *revelation of past messages*. It plays an important role in detecting deviations. While communicating through the *AND*, if a player sends 1, he learns what message the other player has sent. Therefore, the player might have incentives to send the message 1, when 0 should have been sent, and as a result, learn something more about the opponent. The revelation of past messages allows to detect and punish such deviations. The assumption that past messages can be revealed is not new and has been already used in the literature by Bárány (1992) and Ben-Porath (2003). An alternative approach entails restricting the space of permissible messages as it is done in Lehrer (1996).[4]

In the benchmark game, for simplicity, we assume that it is nature that decides whether or not to reveal the past messages of players. To justify it, we could think of the players as politicians who are scrutinized by media. If politicians lie then with some probability journalists catch them in their lies and report on that. Or alternatively, the politicians can be involved in a lengthy negotiation process, and news editors decide to report on some parts of negotiations but ignore other parts. In either case, the politicians may decide to peg their decisions on communication that has *not* been reported in the news. Later we argue that the assumption of nature is not necessary and instead the players themselves can decide on the revelation of messages through the so-called jointly controlled lottery.

Given the described communication technology, we now outline the two-player extended game that we are going to study. Prior to choosing actions in Γ, two players communicate. The communication proceeds in rounds. During each round, the players first repeatedly communicate through the *AND* device, and next make direct simultaneous announcements. At the end of each communication round, nature randomly decides whether or not to reveal the private messages sent to the *AND*. After that, the players decide whether to carry out another round of communication or instead to choose actions in Γ.

The main result of the paper is as follows: *For any finite two-player game Γ with complete information*[5] *and any correlated equilibrium μ of Γ with rational entries that gives strictly individually rational payoffs to both players, there exists a Nash equilibrium strategy profile in the extended game that generates the same distribution μ over the actions.* Consequently, any strictly individually rational

[4] We elaborate on this more in Section 6.

[5] Our equilibrium construction is also useful for games of incomplete information. Vida and Forges (2012) have shown that almost any communication equilibrium payoff can be implemented in correlated equilibrium of an extended game where players, before choosing actions in the underlying game, engage in possibly infinitely long direct communication. Hence, using our construction, the players can dispense not only of the correlation devices but also of canonical communication devices and obtain any communication equilibrium payoff with face-to-face communication.

correlated equilibrium payoffs can be obtained in some Nash equilibrium of the game with face-to-face communication and the revelation of past messages.

The qualification that both players must expect strictly individually rational payoffs is necessary for our equilibrium construction. Some deviations during the communication cannot be detected with probability 1 but only arbitrarily close to it. To prevent such deviations, the players are threatened with their minmax payoffs if caught cheating. But this threat will only be effective if the players expect strictly individually rational payoffs when sticking to the given strategy as shown by examples in Section 6.3.

The paper is structured as follows. In Section 2 we introduce notation and provide necessary definitions. Section 3 formally describes the extended game that the players are playing, and states the main result of the paper. Section 4 contains a detailed example. It also gives an interpretation different from face-to-face communication to our extended game. The proof of theorem is in Section 5. Section 6 concludes the paper with a discussion. The proof of lemma is relegated to the Appendix.

2 Preliminaries

Consider a finite normal form game with 2 players, named 1 (the row player) and 2 (the column player), and complete information, $\Gamma = (g, A)$, where $A = A^1 \times A^2$ is a finite set of action profiles, and $g^i : A \to \mathbb{R}$ is a payoff function of player $i \in I = \{1, 2\}$. The opponent of player i is denoted by $-i$. The action spaces of players 1 and 2 are defined as $A^1 = \{0, 1, \ldots, a^1, \ldots, N_1 - 1\}$ and $A^2 = \{0, 1, \ldots, a^2, \ldots, N_2 - 1\}$. Sometimes we use k instead of a^1 and l instead of a^2 to denote generic actions of the players. Given a finite set E, denote by ΔE the set of probability distributions over the set E. For $\nu \in \Delta E$, let $\operatorname{supp} \nu = \{e \in E | \nu(e) > 0\}$ be the support of ν. We say that ν is rational if for all $e \in E$, $\nu(e)$ is a rational number. We extend linearly g^i for $\rho \in \Delta A$ as $g^i(\rho) = \mathbb{E}_\rho g^i(a)$. For any set X denote by $id_X : X \to X$ the identity map.

Definition 1 *An* information structure *on a finite set $E = E^1 \times E^2$ is a probability distribution ν over E. If an element $e = (e^1, e^2) \in E$ is chosen with probability $\nu(e)$, then player i is informed of the component e^i.*

Consider the following extended game Γ^ν for $\nu \in \Delta E$:

1. An element $e \in E$ is chosen with probability $\nu(e)$.

2. Player i is informed of the component e^i.

3. Player i chooses an action from A^i.

4. Payoffs are realized as in Γ.

A pure strategy for player i in Γ^ν is a function $r^i : E^i \to A^i$ and let $r = (r^1, r^2)$.

Definition 2 *(ν, r) is a* correlated equilibrium *of Γ if and only if r is a Nash equilibrium of Γ^ν.*

By the revelation principle we can restrict attention to information structures $\mu \in \Delta A$. Denote the set of correlated equilibrium payoffs by $CE(\Gamma) = \cup_{\mu \in \Delta A} NE(\Gamma^\mu) \subseteq \mathbb{R}^2$, where $NE(.)$ denotes the set of Nash equilibrium payoffs of an arbitrary game. Let $\mathcal{G}(\mu) = \{\Gamma | (\mu, id_A) \textit{ is correlated equilibrium of } \Gamma\}$. Sometimes we will write simply μ when referring to (μ, id_A).

Definition 3 *A payoff of Γ generated by some $\mu \in \Delta A$ is* (strictly) individually rational *if and only if*

$$g^i(\mu) \geq (>) \underline{w}^i = \min_{\rho^{-i} \in \Delta A^{-i}} \max_{\rho^i \in \Delta A^i} g^i(\rho^i, \rho^{-i})$$

for all $i \in I$.

Let $(S)IR(\Gamma)$ be the set of (strictly) individually rational payoffs of Γ. Notice that $CE(\Gamma) \subseteq IR(\Gamma)$ for any Γ. Also note that player i's expected payoff after learning his action is still in $IR(\Gamma)$ if μ is a correlated equilibrium distribution.

3 The Game with Communication

In this section we describe a game, in which before choosing actions in some underlying game Γ, players are allowed to communicate. The players can send private messages repeatedly to the *AND* communication device, which in turn produces public messages according to the table in Figure 3.

Consider the extension of Γ, denoted by $\Gamma(p, z)$, which unfolds as follows:

1. The players communicate repeatedly for p stages using the *AND* communication device, that is, at each stage the players send private messages of 0 or 1 to the *AND*, which in turn produces a public message of 0 or 1.

2. In stage $p+1$, the players simultaneously and publicly announce numbers from the set $\mathbb{N}_0$.

3. With probability $1-z$ nature reveals all the private messages that have been sent in the stages of point 1. With probability z nature reveals nothing.

4. After the move of nature, the players decide whether to continue to communicate or not.

5. If at least one of the players decides to stop the communication, then players choose actions in Γ and receive their payoffs accordingly.

6. Otherwise, the extended game continues as in point 1.

We do not define strategies $(\tilde{\sigma}, \tilde{r})$ in $\Gamma(p, z)$ formally; the strategy space must be clear from the timing above. We only observe that any strategy consists of two parts:

$\tilde{\sigma}^i$: The communication strategy, according to which the players communicate and decide whether to stop or continue communication.

$\tilde{r}^i$: The decision rule, which determines how to choose actions in Γ given the communication history.

We are ready to state our main result:

Theorem 1 *Given any finite 2 player game Γ with complete information and any correlated equilibrium (μ, id_A) of Γ such that $g(\mu) \in SIR(\Gamma)$ and μ is rational, there exist a pair (p, z) and a Nash equilibrium $(\tilde{\sigma}, \tilde{r})$ of $\Gamma(p, z)$ such that players' equilibrium payoffs are equal to $g(\mu)$. In equilibrium, the players stop communication with probability 1.*

From the construction of the equilibrium strategies it will be clear that the players not only expect the payoffs $g(\mu)$ but in fact the induced distribution on the action profiles will coincide with μ. Other variants of the result will be discussed in Section 6.

4 Example

Suppose the players face the chicken game given in Figure 1. Consider its extension as described in the previous section. We wish to construct a Nash equilibrium for the extended game, such that the equilibrium payoffs coincide with those of the correlated equilibrium given in Figure 2.

Let $p = 4$ and $z \leq \frac{3}{5}$. In order to specify the equilibrium strategies, we start by defining several auxiliary tables. First, we multiply the entries in the table of Figure 2 with the common denominator and rewrite the table in the following form:

	0	1
0	Q_{00}	Q_{01}
1	Q_{10}	Q_{11}

where $Q_{00} = Q_{01} = Q_{10} = 1$ and $Q_{11} = 0$.[6] The following table is constructed by

[6] More generally, Q_{kl} is a square matrix whose elements are 0's and 1's, and the ratio of 1's in Q_{kl} to the total number of 1's in the above table is equal to the probability that a correlating device assigns to action pair (k, l), and the number of 0's and 1's in each row and column of Q_{kl} is the same. The general construction for any rational μ is given in Lehrer (1991) and it is recalled in the proof of Lemma 1.

row and column concatenations of cyclical permutations of the rows and columns of the above table:

	0,0	0,1	1,0	1,1
0,0	Q_{00}	Q_{01}	Q_{10}	Q_{11}
0,1	Q_{10}	Q_{11}	Q_{00}	Q_{01}
1,0	Q_{01}	Q_{00}	Q_{11}	Q_{10}
1,1	Q_{11}	Q_{10}	Q_{01}	Q_{00}

Finally, let us arbitrarily index all 0's in the above table.

	0,0	0,1	1,0	1,1
0,0	Q_{00}	Q_{01}	Q_{10}	0_1
0,1	Q_{10}	0_2	Q_{00}	Q_{01}
1,0	Q_{01}	Q_{00}	0_3	Q_{10}
1,1	0_4	Q_{10}	Q_{01}	Q_{00}

Figure 4: Auxiliary table $\mathcal{Q}$.

Define player i's strategy as follows:

1. Let $t = -1$.
2. $t \doteq t + 1$.
3. Choose a row or column with equal probabilities in Figure 4. Let denote the choice of player i by (n_{-i}, n'_i).
4. For $s = 1, ..., 4$ send message 1 in stage $5t + s$ if 0_s is in the chosen row or column; send 0 otherwise.
5. In stage $5t + 5$ announce the number n_{-i}.
6. If nature reveals the private messages sent in stages $5t + 1$ to $5t + 4$ then:
   (a) if player $-i$'s revealed messages contain exactly one 1, and these messages and number n_i announced by player $-i$ in stage $5t + 5$ are compatible (according to Figure 4) then continue communication, and
      i. if player $-i$ also chose to continue communication then go to step 2;
      ii. otherwise punish player $-i$ by taking action 1;
   (b) otherwise stop communication and punish player $-i$ by taking action 1.
7. If nature does not reveal the private messages sent in stages $5t+1$ to $5t+4$ then:

(a) if all public messages were 0 in stages $5t + 1$ to $5t + 4$ then stop communication and choose action $(n_i + n'_i) \mod 2$ given that player $-i$'s announced number was n_i in stage $5t + 5$;

(b) otherwise continue communication, and

   i. if player $-i$ also chose to continue communication then go to step 2;

   ii. otherwise punish player $-i$ by taking action 1.

We will refer to stages $5t+1$ to $5t+5$ for $t = 0, 1, \ldots$ jointly as communication round t. Note that for each chosen row or column, step 4 defines a unique sequence of 0's and 1's of length 4 and a unique number n_{-i} for player i in stage $5(t + 1)$. Also observe that the actions chosen in Γ only depend on the messages sent and received during the round of communication after which the communication is terminated.

First, suppose that the players follow the prescribed strategies. Then the communication will be terminated in the first round in which nature does not reveal the private messages sent by the players and all public messages made by the *AND* are 0. The latter will occur if the cell corresponding to chosen row (n_2, n'_1) and column (n_1, n'_2) does not contain 0. We refer to this round as a *successful round*. In this case, player i selects action $(n_i + n'_i) \mod 2$. Suppose that the entry in row (n_2, n'_1) and column (n_1, n'_2) is Q_{kl} in the table of Figure 4. One can verify that the strategy tells the players to play exactly action pair (k, l), that is, $k = n_1 + n'_1$ and $l = n_2 + n'_2$.

As an example, suppose that in step 3, players 1 and 2 have chosen, respectively, row $(1, 0)$ and column $(1, 1)$ in Figure 4. Then, in step 4, players 1 and 2 send messages $0, 0, 1, 0$ and $1, 0, 0, 0$, respectively. As a result, all four public announcements by the *AND* are 0. In step 5, both players publicly announce number 1. If nature does not reveal their private messages to the *AND* and the players stop communication, then in step 7a players 1 and 2 choose actions $(0+1) \mod 2 = 1$ and $(1 + 1) \mod 2 = 0$, respectively. These actions correspond to the subscript of Q_{10}, which appears in the cell of row $(1, 0)$ and column $(1, 1)$ in Figure 4.

Hence, if the players follow the specified strategies, it is clear that each of the action pairs $(0, 0)$, $(0, 1)$, and $(1, 0)$ is indeed selected with a probability of one third. Furthermore, conditional on information acquired during the successful round, player i's belief about the opponent's action coincides with the one that would be derived from μ by conditioning on action $(n_i + n'_i) \mod 2$. This is achieved through the construction of the table in Figure 4.

For example, suppose player 1 has chosen row $(1, 0)$. If at the end of the successful round player 1 learns that he must take action 1, then it must be that player 2's announcement in stage $5t+5$ was 1. Since player 1 has sent the message 1 in stage $5t + 3$ and the round was successful, he can infer that player 2 has chosen column $(1, 1)$ and that player 2 will take action 0. If instead player 1 learns that he must take action 0 then it must be that 2's announcement in stage $5t+5$ was 0. Again, since player 1 has sent the message 1 only in stage $5t + 3$ and the round was successful, he can infer that player 2 has chosen either column $(0, 1)$ or $(0, 0)$, and attaches probability 0.5 to each of these events. Correspondingly, player 1 is unsure whether player 2 will take action 0 or 1, and he assigns equal probabilities to both of these actions of player 2.

Since μ given in Figure 2 is a correlated equilibrium, no player has incentives to change his action from the calculated one at the end of successful communication round if the player sent messages in that round according to the above strategy. It remains to argue that no player has incentives to deviate from his strategy during any of the communication rounds. Since the problem is symmetric, we focus on player 1.

First, we argue that player 1 is indifferent at the beginning of communication round which row to choose in Figure 4, as long as he sends messages as prescribed by his strategy once he has chosen a row. Given that the opponent randomizes uniformly between the columns, and given that each Q_{kl} appears exactly once in each row (and column) in Figure 4, if the round is successful, each of the action pairs $(0, 0)$, $(0, 1)$, and $(1, 0)$ is still selected with a probability of one third, and player 1 still has the same conditional beliefs about the action chosen by the opponent no matter which row player 1 has chosen. Even more, when selecting a row, player 1 cannot affect the probability that the round will be successful or not. Hence, player 1 is indeed willing to randomize uniformly between the rows.

Next, it is also easy to see that player 1 will not want to stop communication when the strategy calls for the continued communication. If player 1 stops communication, he will be punished and his payoff will be equal to 2 in this case. Instead if he continues to communicate, he expects a payoff of $g^1(\mu) = 5$, which is clearly higher.

Finally, suppose player 1 deviates in round t by sending messages different from the ones prescribed by the above strategy. That is, there is no possible choice of row by player 1, which would justify the messages he sends. Let v^1 be the expected payoff of player 1 from such a deviation. Note that as long as the player does not deviate, his expected payoff is $g^1(\mu) = 5$ at the beginning of all 5 stages of round t. His expectation might only change after stage $5t + 5$ when he learns his possible action but then there is no more possibility for player 1 to deviate in that round. Hence, the deviation is not profitable if $v^1 \leq 5$. Three outcomes can happen. First, if nature reveals messages sent by the players at the end of round t, then player 1 will be punished, in which case the payoff of player 1 will be equal to 2. Second, the round is successful and, consequently, player 2 chooses action $(n_2 + n_2') \mod 2$. The highest payoff that player 1 can achieve in this case is 7 corresponding to action profile $(1, 0)$. (For example, player 1 can choose row $(1, 0)$ in Figure 4 and send private messages $0, 1, 1, 1$, followed by direct public announcement of 1. Then the round can only be successful if player 2 has chosen column $(1,1)$. In this case, player 1 will know that player 2 will play action 0, and he himself can choose action 1 to maximize his payoff.) Third, the *AND* produced an output message of 1 at one of the stages $5t+1$ to $5t+4$. In this case, the players will proceed to the communication round $t+1$. If player 1 has a profitable deviation in round t, he also has one in round $t+1$. Therefore, the expected payoff of player 1 if the third outcome happens is again v^1. Therefore, the expected payoff of player 1 from deviating in round t is given by

$$v^1 = (1-z)\cdot 2 + z\cdot(x\cdot 7 + (1-x)\cdot v^1),$$

where x is the probability that all public messages by the *AND* were 0 in round t. Since $v^1 \leq 7$ the above expression can be upper bounded by

$$v_1 \leq 2(1-z) + 7z$$

and if $z \leq \frac{3}{5}$, which indeed holds, the deviation is not profitable because

$$v_1 \leq 2(1-z) + 7z \leq 5.$$

Hence, the constructed strategy profile indeed forms a Nash equilibrium.

4.1 An Alternative Interpretation

To provide additional motivation to the game we study, we now discuss an alternative interpretation of our example. The two players are students that are required to work together on a project. Suppose that the project consists of four tasks. A professor wants, first, that all four tasks are completed and, second, that each student works on three of these tasks in order to promote an interaction by the students. To facilitate the division of tasks, the professor acts as a mediator. He asks each student to privately submit in a single message a list of tasks that the student wants to work on. For example, if student 1 (the row player) tells that he wishes to work on tasks 1, 2, and 4 it is equivalent to choosing the third row in Figure 4 and sending a message sequence $0,0,1,0$ to the *AND* in $\Gamma(p,z)$. We assume that the message spaces of students are restricted. That is, a student cannot tell to the professor that he wants to work, for example, only on tasks 1 and 2 (send message sequence $0,0,1,1$). If the students' choices are such that each task is selected by at least one of the students, then the professor tells them to go ahead with the project, meaning that the round of communication has been successful. Otherwise, they are asked to submit new lists of tasks they wish to work on.

Suppose now that before embarking on the project, the students tell each other simultaneously, which two of the tasks they have selected. Student 1 announces one of the following: he is going to work for sure either on tasks 1 and 2 (announce $n_2 = 0$ at stage 5) or on tasks 3 and 4 (announce $n_2 = 1$). Student 2 announces one of the following: he is going to work for sure either on tasks 1 and 3 (announce $n_1 = 1$) or on tasks 2 and 4 (announce $n_1 = 0$). These announcements need not to be compatible with the ones sent to the professor. However, below we show that the students will not lie about the messages that they have sent to the professor.

Finally, we assume that each student must work on the tasks that he has submitted to the professor, but he has a choice between exerting high effort (action 0 in Figure 1) or low effort (action 1). Depending on their chosen effort levels, the students realize payoffs as in Figure 1. One can verify that this game has a Nash equilibrium, in which each student chooses uniformly 3 out of 4 tasks to perform, announces honestly to the other student about the 2 tasks that he is going to work on, and then decides on the effort level as follows. If a student can infer what are the 3 tasks that the other student works on, then he exerts low effort, and otherwise he exerts high effort.

Compared to the game $\Gamma(p, z)$, here we have dispensed of nature that reveals the messages sent by the players. We can do it because we have restricted the message spaces of the players. Note, however, that the public announcement by a student needs not to be compatible with his private message submitted to the professor. We now argue that the students will make honest announcements to each other.[7] Without loss of generality, suppose student 1 has chosen tasks 1, 2, and 4, and the professor has told them to proceed with the project. If student 1 reports honestly that he is going to work on tasks 1 and 2 then, as before, his expected payoff is 5. Suppose he lies and tells that he is going to work on tasks 3 and 4. It is equivalent to announcing $n_2 = 1$ instead of $n_2 = 0$ in $\Gamma(p, z)$. Hence, student 2 is going to take the action that is opposite to the one that he would have taken if student 1 did not lie. Thus, if under honest announcement, student 1 had learned that student 2 exerts high effort, then now he would know that student 2 exerts low effort. As a result, student 1 is better off to exert high effort and receive a payoff of 2 if he has lied. If instead under honest announcement, student 1 was unsure whether student 2 exerts high or low effort, then now he would still be unsure about the opponent's effort. Consequently, he still prefers to exert high effort and expect a payoff of 4. Hence, in ex ante terms, student 1 expects $\frac{10}{3}$ if he makes dishonest public announcement. Thus, we conclude that student 1 wants to make an honest announcement to the other student.

5 Proof of the Theorem

Pick a finite two-player game Γ and a distribution μ with rational values such that $\Gamma \in \mathcal{G}(\mu)$ and $g(\mu) \in SIR(\Gamma)$. Consider now the game $\Gamma(p, z)$, where the values of p and z will be specified later. We first construct a strategy of player i,

[7] This might not be true for another game Γ' or another correlated equilibrium distribution μ'.

$(\tilde{\sigma}^i, \tilde{r}^i)$, for $i = 1, 2$ in $\Gamma(p, z)$. Next we show that the defined strategy profile is a Nash equilibrium of $\Gamma(p, z)$ and the players expect $g(\mu)$ in the equilibrium.

As demonstrated in the example, players' communication strategies are defined to be the same mixed strategies in each round of communication and players' action choices on the equilibrium path only depend on the communication history of the round, in which the communication has been terminated by the players. To define these "round"-strategies and state some of their properties, which will be useful when constructing and proving equilibrium in $\Gamma(p, z)$, we consider a simple finite horizon game associated with $\Gamma(p, z)$.

Let us denote by $m^i \in M^i = \{0, 1\}^p$ the actual private messages sent by player i in stages $1, \dots, p$ of $\Gamma(p, z)$ and by $h \in H = \{0, 1\}^p$ the public messages received by the players from the *AND* device in stages $1, \dots, p$ of $\Gamma(p, z)$. Let $n = (n^1, n^2) \in \mathbb{N}_0^2$ be the message profile sent by the players directly, hence publicly to each other in stage $p + 1$ of $\Gamma(p, z)$. We have seen in the example that in equilibrium the players decide to terminate communication after a round if nature does not reveal the private messages of that round and that round is successful in the sense that all public messages produced by the *AND* device are 0. Let $h_0 = (0, \dots, 0) \in H$ denote the p-long sequence of output messages where all coordinates are 0.

5.1 The game $\Gamma(p, \mu, h_0)$

To facilitate the proof of the theorem, we now define an auxiliary, finite horizon game $\Gamma(p, \mu, h_0)$, which is as follows:

1. The players communicate repeatedly for p stages using the *AND* communication device.

2. In stage $p + 1$, the players simultaneously and publicly announce numbers from the set $\mathbb{N}_0$.

3. If $h = h_0$ then the players choose actions in Γ and receive their payoffs accordingly.

4. If $h \neq h_0$ then instead of the players, nature chooses an action profile a according to μ and the players receive payoffs of $g(a)$.

Let (σ, r) denote a strategy profile in this game, where $\sigma = (\sigma^1, \sigma^2)$ are possibly mixed communication strategies of the players and $r = (r^1, r^2)$ are pure decision rules, that is, $r^i : M^i \times \mathbb{N}_0^2 \to A^i$ and $\sigma^i = (\sigma^i_s)_{1 \leq s \leq p+1}$, where $\sigma^i_s : \{0, 1\}^{s-1} \times \{0, 1\}^{s-1} \to \Delta\{0, 1\}$ for $1 \leq s \leq p$ and $\sigma^i_{p+1} : M^i \times H \to \Delta\mathbb{N}_0$. Clearly, a strategy profile induces a distribution r_σ on $M \times H \times \mathbb{N}_0^2 \times A$, where $M = M^1 \times M^2$. Let $c_\sigma \in \Delta(M \times H \times \mathbb{N}_0^2)$ be the marginal of r_σ.

Given a communication strategy σ^i in $\Gamma(p, \mu, h_0)$, let us define another communication strategy, which ignores the public messages announced by the *AND*

and instead specifies for player i to send a message in stage s according to σ^i assuming that all the public announcements up to stage d by the AND were 0. It is easy to see that this strategy is equivalent with choosing an element of $M^i \times \mathbb{N}_0$ according to some distribution $\dot{\sigma}^i \in \Delta(M^i \times \mathbb{N}_0)$, which can be derived from σ^i. It is clear that $\dot{\sigma}^i$ and σ^i are equivalent strategies in the game $\Gamma(p, \mu, h_0)$ in the sense that $r_\sigma(\cdot|h_0) = r_{(\dot{\sigma}^i,\sigma^{-i})}(\cdot|h_0)$ for all r and σ. (Note, however, that $r_\sigma \neq r_{(\dot{\sigma}^i,\sigma^{-i})}$ in general.) Hence, without loss of generality, we can identify the set of pure communication strategies for player i in $\Gamma(p, \mu, h_0)$ with the set $M^i \times \mathbb{N}_0$, and when we write σ, we mean the associated $\dot{\sigma}$. Also, we write $\operatorname{supp}\sigma$ instead of $\operatorname{supp} c_\sigma(\cdot|h_0)$.

To facilitate the exposition of the proof, next we define certain properties of strategies.

Definition 4 (σ, r) mimics μ *for player i if and only if for all $a \in A$, all $a^{-i} \in A^{-i}$, and all $(m^i, n) \in \operatorname{supp}\sigma$:*

1. $r_\sigma(a|m^i, h_0) = \mu(a)$,
2. $r_\sigma(a^{-i}|m^i, h_0, n) = \mu(a^{-i}|r^i(m^i, n))$.

(σ, r) mimics μ *if it mimics μ for both players.*

In words, the first condition says that the generated distribution on A conditional on h_0 and on any $m^i \in \operatorname{supp}\sigma$ equals to μ. As a consequence, if the players communicate according to σ, play according to r and given that $h = h_0$ then in *every* stage $1, \ldots, p$ prior to exchanging their last messages in stage $p+1$, the players expect a payoff of $g(\mu)$. The second conditions says that $r^i(m^i, n)$ is a sufficient statistic for a^{-i}. Thus, if the players have followed σ then player i's information about the other's action given (m^i, h_0, n) is the same as had he received the recommendation to take action $r^i(m^i, n)$ from μ.

The statement of the following remark follows simply from the definition of correlated equilibrium and from the definition of mimicking.

Remark 1 *Let $\nu = c_\sigma(\cdot|h_0) \in \Delta(M \times \mathbb{N}_0^2)$ and let $E^i = M^i \times \mathbb{N}_0^2$ be player i's private information. Then, $(E^1 \times E^2, \nu)$ defines an information structure. If (σ, r) mimics μ then for all $\Gamma \in \mathcal{G}(\mu)$, (ν, r) is a correlated equilibrium of* Γ.[8]

Or alternatively, if (σ, r) mimics μ then for all $\Gamma \in \mathcal{G}(\mu)$, for any player i and for any profitable deviation $(\sigma^{i\prime}, r^{i\prime})$ in $\Gamma(p, \mu, h_0)$ it must be that $(\sigma^{i\prime}, \sigma^{-i}, r)$ does not mimic μ. That is, to have a profitable deviation, player i must change his communication strategy.

[8]The remark also follows from Gossner (2000). By comparing the information structures (A, μ) and (E, ν), it is easy to see that (E, ν) is richer than (A, μ) and that r is a faithful interpretation from (E, ν) to (A, μ).

Definition 5 (σ, r) simulates μ *if and only if for all* i *and for any* $(m^i, n^i) \in \operatorname{supp} \sigma^i$, $((m^i, n^i), \sigma^{-i}, r)$ *mimics* μ *for player* i.

Definition 5 together with Remark 1 lead to the following observation.

Remark 2 *If* (σ, r) *simulates* μ *then for all* i *and for any profitable deviation* $((m^i, n^i), r^{i\prime})$ *in the game* $\Gamma(p, \mu, h_0)$, $(m^i, n^i) \notin \operatorname{supp} \sigma^i$.

Lemma 1 *For any* μ *with rational entries there is a value of* p *and a strategy profile* (σ, r) *in* $\Gamma(p, \mu, h_0)$, *which simulates* μ.

The proof of Lemma 1 is relegated to the Appendix.

We now define Nash equilibrium strategies for the game $\Gamma(p, z)$. Let (σ, r) be a strategy profile in $\Gamma(p, \mu, h_0)$, which simulates μ. The strategy of player i, $(\tilde{\sigma}^i, \tilde{r}^i)$, in $\Gamma(p, z)$ is as follows:

1. Let $t = -1$.
2. $t \doteq t + 1$.
3. Play according to σ^i in communication round t (consisting of $p+1$ stages).
4. If nature reveals the private messages $m = (m^1, m^2)$ sent in round t then:
   (a) if $(m^{-i}, n^{-i}) \in \operatorname{supp} \sigma^{-i}$ then continue communication, and
      i. if player $-i$ also chose to continue communication then go to step 2;
      ii. otherwise punish player $-i$;
   (b) otherwise, if $(m^{-i}, n^{-i}) \notin \operatorname{supp} \sigma^{-i}$ then stop communication and punish player $-i$.
5. If nature does not reveal the private messages sent in round t then:
   (a) if $h = h_0$ then stop communication and choose action in Γ according to r^i;
   (b) otherwise, if $h \neq h_0$ then continue communication, and
      i. if player $-i$ also chose to continue communication then go to step 2;
      ii. otherwise punish player $-i$.

Given that the players follow the strategy profile $(\tilde{\sigma}^i, \tilde{r}^i)$, the communication will be terminated in the first round, in which all the public messages announced by the *AND* are 0, that is, $h = h_0$ and nature does not reveal the private messages of that round. Hence, by Lemma 1, and condition 1 of Definition 4 in all stages

$t(p+1)+1,\dots,t(p+1)+p$ the players expect the payoff of $g(\mu)$ given that round t is reached. In particular, the players expect the payoff of $g(\mu)$ at the start of the game.

We verify that no profitable deviation from $(\tilde{\sigma},\tilde{r})$ exists for either player. First of all, we observe that no player will stop communication when the strategy calls for the continued communication. If player i stops communication, he will be punished and his payoff will be equal to $\underline{w}^i$. If he continues to communicate, he expects a higher payoff of $g^i(\mu) > \underline{w}^i$. Hence, in the continuation we only consider deviation strategies, in which the players choose to continue communication when $(\tilde{\sigma},\tilde{r})$ calls for the continued communication.

Suppose there exists a profitable deviation for player i. Then there must necessarily exist a communication round t such that the communication will be terminated at the end of that round with a positive probability, and conditional on that event, the expected payoff of player i strictly exceeds $g^i(\mu)$. This payoff only depends on the round-communication strategy (m^i,n^i) and the round-decision rule $r^{i'}$ that player i follows in round t. That is, player i's deviations from $(\tilde{\sigma}^i,\tilde{r}^i)$ in rounds other than round t will not matter, once round t is reached. It means that we can construct another profitable deviation for player i, denoted $(\hat{\sigma}^i,\hat{r}^i)$, whereby in round 0 he communicates according to (m^i,n^i) and takes an action according to $r^{i'}$ if $h=h_0$ and nature does not reveal his messages to the AND, and follows the original strategy in all other rounds. Since $(\hat{\sigma}^i,\hat{\rho}^i)$ is a profitable deviation for player i in $\Gamma(p,z)$, it must be that $((m^i,n^i),r^{i'})$ is also a profitable deviation from (σ^i,r^i) in $\Gamma(p,\mu,h_0)$ when the opponent plays (σ^{-i},r^{-i}). By Remark 2, it follows that $(m^i,n^i)\notin \operatorname{supp}\sigma^i$.

Three outcomes can happen. First, if nature reveals the messages sent by the players at the end of round 0, then player i will be punished, in which case his payoff will be equal to $\underline{w}^i$. Second, nature does not reveal the messages sent by the players and the output messages of round 0 are $h=h_0$. The highest payoff that player i can possibly attain is $\overline{w}^i=\max_{a\in A} g^i(a)$. Third, nature does not reveal the messages sent by the players and the output messages of round 0 are $h\neq h_0$. In this case, the players will proceed to the next communication round. Whatever payoff player i expects from continued communication, it can be upper-bounded by $\overline{w}^i$. Hence, the expected payoff of player i from the deviation $(\hat{\sigma}^i,\hat{r}^i)$ is

$$(1-z)\cdot\underline{w}^i + z\cdot\overline{w}^i.$$

If $z\le\min(\overline{z}^1,\overline{z}^2)$ where

$$\overline{z}^i=\frac{g^i(\mu)-\underline{w}^i}{\overline{w}^i-\underline{w}^i},$$

then the expected payoff of player i from deviation $(\hat{\sigma}^i,\hat{r}^i)$ is less than the expected payoff from strategy $(\tilde{\sigma}^i,\tilde{r}^i)$. Thus, we have obtained a contradiction: $(\hat{\sigma}^i,\hat{r}^i)$ is not a profitable deviation, which also means that player i does not have any profitable deviation from $(\tilde{\sigma}^i, \tilde{r}^i)$ given that the opponent follows the strategy $(\tilde{\sigma}^{-i}, \tilde{r}^{-i})$.

6 Discussion

We conclude the analysis by discussing several modifications and extensions of game $\Gamma(p, z)$, and how these would affect our results.

6.1 Replacing Nature's Random Choice

We have assumed in the game $\Gamma(p, z)$ that after each round t nature decides whether or not to reveal m, that is, the players' private messages from stages $t(p+1)+1, \ldots, t(p+1)+p$. The assigned probabilities are $(1-z, z)$ and z is chosen such that the players are deterred from deviating during any communication round t. We discuss now how we can dispense of nature and instead allow the players themselves securely to replicate nature's randomization. Basically, we show how the players can jointly toss a fair coin. It is known as the jointly controlled lottery due to Aumann, Maschler, and Stearns (1995).

6.1.1 Jointly controlled lottery

Here we follow the discussion in Aumann and Hart (2003, Section 4.3). Recall that in our example $z \leq 3/5$ must hold. Therefore, we can set $z = 1/2$. The players can easily replicate nature's randomization if they have access to a publicly observable fair coin. However, the players can also perfectly generate the desired randomness even if each player has only access to his own fair coin and he does not trust the fairness of the opponent's coin. The randomness is generated as follows. At the end of each round, the players each send either message 0 or 1 to the other player simultaneously. Suppose they agree (in equilibrium) that if the messages coincide then the private messages of the round are revealed, and if messages differ then the private messages are not revealed. Now, if each player sends either message 0 or 1 with equal probabilities of 0.5 then the probability that messages coincide is exactly 0.5 and the revelation of past private messages is induced with a probability of 0.5. Furthermore, if one of the players follows the prescribed randomization then the other player, by randomizing differently, cannot affect the probabilities of the events that the announced messages are the same or different. Hence, no player has a profitable deviation. Aumann and Hart (2003) also discuss in footnote 10 how to produce a lottery for any $(1 - z, z)$.

Remember that we can think of the revelation of past messages as the ability of players to replay the film that the video camera has recorded. In this case, it also does not matter whether the players press a "replay" button jointly or unilaterally. If a player replays the film when he should not, or to the contrary, a player does not replay the film when he should, the other player can stop the conversation and punish the deviator.

6.2 Binary Message Spaces

It has been assumed that the players publicly announce numbers in $\mathbb{N}_0$ in stages $t(p+1)$ for $t = 1, 2, \ldots$. We can reduce their message spaces by instead requiring the players to use simple binary messages as in Aumann and Hart (2003). Of course, in order to announce number n^i in binary messages, player i will need more than one stage in general. Therefore, it can happen that player i's expectation about his payoff can differ from $g(\mu)$ while (n^i, n^{-i}) are transmitted in public binary messages. As a result, player i might find it optimal to announce a number different from $n^i = n_{-i}$ (if the round has been successful). One can, however, argue that with a sufficiently low value of z such deviations can be deterred. To see it, suppose that the players still publicly announce numbers from the set $\mathbb{N}_0$ but they do it sequentially. In particular, suppose that player 1 has already sent $n^1 = n_2$ to player 2. Hence, player 2 learns his action $n_2' + n_2 \mod N_2$ and updates his belief about his expected payoff before he has announced n^2. In the worst case, player 2 expects a payoff of $\underline{w}^2$ if nature does not reveal their messages of that round. We check if player 2 will still send message $n^2 = n_1$ in this case. He will not deviate if

$$(1-z)g^2(\mu) + z\underline{w}^2 \geq (1-z)\underline{w}^2 + z\overline{w}^2.$$

The inequality holds if

$$z \leq \frac{g^2(\mu) - \underline{w}^2}{\overline{w}^2 + g^2(\mu) - 2\underline{w}^2}.$$

that is, if z is chosen to be sufficiently small.

6.3 A Universal Extension

Consider now another extended game of Γ, denoted by Γ^{ext}:

1. At each stage two players are allowed to send both private messages of 0 or 1 to the *AND* device and direct messages of 0 or 1 to each other simultaneously.

2. After each stage the players decide first whether or not to reveal the private messages of all previous stages and then whether or not to stop the communication.

3. If the communication is stopped, the players choose actions in Γ and receive their payoffs accordingly.

4. Otherwise, the players proceed to the next stage of communication.

Unlike $\Gamma(p,z)$, where p and z are chosen appropriately for Γ and μ, this extension of Γ neither depends on the underlying game itself nor on the distribution μ. Hence, in this sense it is universal (see Forges, 1990). Furthermore, by playing the game Γ^{ext}, players can always replicate game $\Gamma(p,z)$ as there is nothing "physical" about p and z. As discussed in Section 6.1, the players themselves can select z through a jointly controlled lottery. Similarly, the players can specify the length of each communication round and, in particular, the value of p through their strategies. Also, it is irrelevant whether the players send private and public messages simultaneously as in Γ^{ext} or sequentially as in $\Gamma(p,z)$. To see it, as an example, suppose that the players are required to communicate through the *AND* device at some stage in $\Gamma(p,z)$. Then any public messages made at the same stage can take the form of babbling and be ignored by the players. Even if one of the players deviates and tries to convey meaningful information through his public announcement, such deviation will not be profitable as the other player's strategy will not depend on this announcement.

Based on the above observations, one can easily establish the following result: for all finite 2 player games Γ with *rational payoffs* it is true that

$$CE(\Gamma) \cap SIR(\Gamma) \subseteq NE(\Gamma^{ext}).$$

Since Γ has rational payoffs then the extreme points of the correlated equilibrium distributions are rational as well. Then any correlated equilibrium distribution μ such that $g(\mu) \in SIR(\Gamma)$ can be obtained as a convex combination of some correlated equilibrium distributions μ_j such that $g(\mu_j) \in SIR(\Gamma)$ for all j. Hence, as a first step, let the players, using a jointly controlled lottery, select a μ_j with the probability equal to the weight of μ_j in the convex combination. Next, let them follow the strategies that are constructed in the proof of Theorem 1 to simulate μ_j with the only difference that instead of sending natural numbers in stages $t(p+1)$, the players send binary codes corresponding to messages (n^1, n^2) for sufficiently many stages. When the construction in the proof does not specify what private or public messages to send at some stage then the players are allowed to send arbitrary messages.

One could hope to establish more general result of the form $CE(\Gamma) = NE(\Gamma^{ext})$. However, in our construction, profitable deviations cannot be detected with probability 1. Therefore, we need the equilibrium payoffs to be strictly individually rational as it allows to threaten a potential deviator with his minmax payoffs. Figure 5 provides an example of correlated equilibrium μ such that $g(\mu) \notin SIR(\Gamma)$ since player 1 receives his minmax payoff of 0. This player has incentives to manipulate his messages in a way that permits him to learn when player 2's action is 1, in which case player 1 will select action 2 and receive a payoff of 1. And he cannot be punished for this deviation since in the worst case, he still receives the same payoff as the one provided by μ.

The *distribution* μ in Figure 5 cannot be obtained in equilibrium with our construction. However, it could still be that the associated payoffs $g(\mu) = (0,5)$

Γ

	0	1
0	$0,6$	$0,7$
1	$0,2$	$0,0$
2	$-1,0$	$1,0$

μ

	0	1
0	1/3	1/3
1	1/3	0
2	0	0

Figure 5: Example of correlated equilibrium μ with $g(\mu) \notin SIR(\Gamma)$

might be obtained without the use of the *AND* device, say, as a convex combination of Nash equilibria. However, it is impossible because the highest payoff for player 2 in any Nash equilibrium of Γ is only 14/3. On the other hand, one can verify that $CE(\Gamma) \cap SIR(\Gamma)$ is not empty. This is illustrated in the graph on the left hand side of Figure 7 below.[9] Hence, we can get arbitrarily close to the payoffs $(0, 5)$ by constructing an equilibrium for a distribution resulting in payoffs of $(\varepsilon, 5)$. The previous observation is true in general: if $CE(\Gamma) \cap SIR(\Gamma) \neq \emptyset$ then

$$CE(\Gamma) = \overline{NE(\Gamma^{ext})},$$

where $\overline{E}$ denotes the closure of the set E. $CE(\Gamma) \cap SIR(\Gamma)$ can be empty, however, as illustrated by the following example:

Γ

	0	1	
0	$0,6$	$0,7$	$-1,7$
1	$0,2$	$0,0$	$0,1$
2	$-1,7$	$1,6$	$-1,7$

μ

	0	1	
0	1/3	1/3	0
1	1/3	0	0
2	0	0	0

Figure 6: Example of game with $CE(\Gamma) \cap SIR(\Gamma) = \emptyset$

One can see from the graph on the right hand side of Figure 7, that the payoff (0,5) cannot be approximated with correlated equilibrium payoffs in $SIR(\Gamma)$.

6.4 Revelation of Past Messages

The idea of recording players' private messages and revealing them under certain circumstances has been first introduced in Bárány (1992). In particular, the method of recording messages is formalized in footnote 5 in Bárány (1992). In Bárány's (and our) construction, players can push a "stop" bottom and then *all* the past messages of the players are publicly revealed. For comparison, Ben-Porath (2003) uses a different, more flexible revelation technology. He assumes that players can define any interval of stages and only the messages which are

[9]The Nash equilibria of the examples are computed using a program by Rahul Savani, and can be found at http://banach.lse.ac.uk/form.html. The graphs are generated by Matlab code of Iskander Karibzhanov, and can be found at http://www.mathworks.com/matlabcentral/fileexchange/25281.

$g(\mu) \notin SIR(\Gamma)$ $\qquad$ $CE(\Gamma) \cap SIR(\Gamma) = \emptyset$

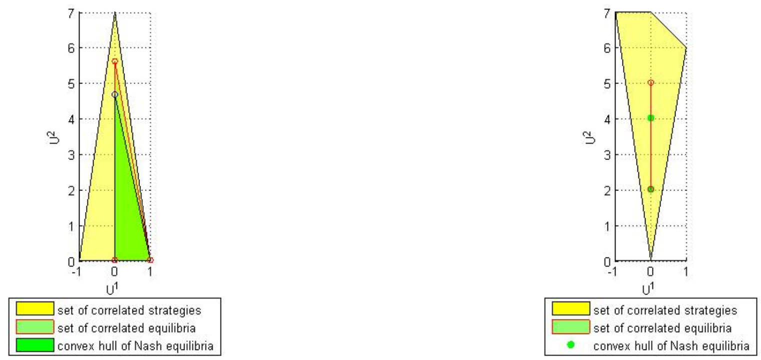

Figure 7: The set of feasible, correlated and convex hull of Nash equilibrium payoffs

sent in stages outside this interval are revealed. We have opted for the simpler technology that with a certain probability reveals all messages after each round of the communication. It can be shown that Ben-Porath's technology can be applied in our game only for correlated equilibria, where players' expected payoffs, given their actions, are still in $SIR(\Gamma)$.

One can also dispense of recording and revealing of past messages if the set of allowed messages is restricted. For example, in our students-professor story of Section 4.1, the students are only allowed to send private messages that contain a single 1. However, they are free to send direct messages to each other that are independent of their private messages. We proved that the students have no incentives to lie in this stage. We have pointed out that this is true only for the given game and distribution. Now assume that instead of sending the direct messages after the private ones, each student sends a single message containing two parts, one of which is the private message m^i to the professor and the other is the direct message n^i to the opponent. If the message space is restricted to messages $(m^i, n^i) \in \operatorname{supp} \sigma^i$ then there is no need for the revelation technology at all, no matter what the game and the distribution is. Of course, this extension is not universal as the restriction on message space depends on the distribution.

This approach has been taken by Lehrer (1996). The mediator in Lehrer's construction, similar to ours, also uses matrix $\mathcal{Q}$ (defined in the proof of Lemma 1) but instead of sequential communication through the *AND* device, players send a *single* private message to the mediator once they have chosen a row and a column of $\mathcal{Q}$. If the combination of messages are such that they "hit" a 0 in $\mathcal{Q}$ then the players are asked to send new messages.[10] If players' chosen row and

[10]Lehrer and Sorin (1997) provide a construction for one-shot communication; it ensures that players cannot "hit" a 0 and consequently it is enough with a single stage of communication,

column do not cross at a 0 in $\mathcal{Q}$, the mediator tells the players which sub-matrix $[\mathcal{Q}]_{n_2 n_1}$ they have chosen, by announcing different public messages for different sub-matrices. On contrary, in our case, players themselves jointly select a sub-matrix through their direct announcements (n_2, n_1) to each other in stage $p+1$. In this sense, Lehrer's mediator is somewhat more complicated.

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after which the players can choose actions.

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Appendix

Proof of Lemma 1: Fix a rational $\mu \in \Delta A$ and let $\mu(k,l) = \frac{c_{kl}}{d}$.

Lehrer (1991) suggests the following construction. Let $Q_{kl} \in \{0,1\}^{d\times d}$ for $0 \le k \le N_1 - 1$ and $0 \le l \le N_2 - 1$ be matrices such that in each row and each column there are exactly c_{kl} of ones and $d - c_{kl}$ of zeros. We have $N_1 N_2$ such matrices, which we put in a matrix Q as follows:

$$Q = \begin{pmatrix} Q_{00} & \dots & Q_{1(N_2-1)} \\ \vdots & \ddots & \vdots \\ Q_{(N_1-1)0} & \dots & Q_{(N_1-1)(N_2-1)} \end{pmatrix}.$$

Denote the entry in the kth row and lth column of a matrix by $[\cdot]_{kl}$. If entries are also matrices then it will be made clear from their definitions. For example, $[Q]_{kl} = Q_{kl}$. The number of zeros in Q is exactly $p' = d\sum_{kl}(d - c_{kl})$. Let $0_1, \dots, 0_{p'}$ be an arbitrary order of these zeros.

Define (σ, r) in $\Gamma(p', \mu, h_0)$ as follows:

★: The row player chooses a row of Q, that is, chooses a pair of numbers (n_1, d_1) from $A^1 \times \{1,\dots,d\}$ with equal probabilities; the column player chooses a column, that is, a pair (n_2, d_2) from $A^2 \times \{1,\dots,d\}$ with equal probabilities. The chosen row is $n_1 d + d_1$, and the chosen column is $n_2 d + d_2$.

$\mathbf{m^i}$: Send message 1 at stage s for $1 \le s \le p'$ if 0_s is in the chosen row (column) of Q and send 0 otherwise.

$\mathbf{n^i}$: Send an uninformative message in stage $p'+1$, say, let $n^i = 0$.

$\mathbf{r^i}$: The row player plays n_1, the column player plays n_2 in Γ.

(σ, r) does not mimic μ although it satisfies condition 2 of Definition 4 and it is true that

$$r_\sigma(a|h_0) = \mu(a). \tag{L1}$$

Moreover, (σ, r) does not simulate μ since for any pure communication strategy (m^i, n^i) of player i we have that in general

$$r_{((m^i,n^i),\sigma^{-i})}(a|h_0) \neq \mu(a).$$

To fix Lehrer's construction in a way that it satisfies condition 1 of the definition of mimicking and the definition of simulation consider the following augmented matrix. Using the matrices Q_{kl} as entries, we build a Latin square $\mathcal{Q}$ of size $N_1N_2 \times N_1N_2$, such that in each row and each column there is exactly one instance of matrix Q_{kl} for all $(k,l) \in A$. Here, we look at $\mathcal{Q}$ as an $N_1N_2 \times N_1N_2$ matrix whose entries are matrices of size $d \times d$, that is, $\mathcal{Q} \in \left\{\{0,1\}^{d\times d}\right\}^{N_1N_2\times N_1N_2}$. Instead, $\mathcal{Q}$ can also be viewed as an element of $\left\{\left\{\{0,1\}^{d\times d}\right\}^{N_1\times N_2}\right\}^{N_2\times N_1}$. That is, the building blocks of $\mathcal{Q}$ are from $\left\{\{0,1\}^{d\times d}\right\}^{N_1\times N_2}$, meaning that $\mathcal{Q}$ has N_2 *rows* and N_1 *columns* and the entries are matrices of size $N_1 \times N_2$ having N_1 rows and N_2 columns. Denote such an entry in the n_2th *row* and n_1th *column* by $[\mathcal{Q}]_{n_2n_1}$.

We require that the Latin square $\mathcal{Q}$ additionally satisfies the following property: for all $0 \le n_2 \le N_2 - 1, 0 \le n_1 \le N_1 - 1$, $[\mathcal{Q}]_{n_2n_1}$ can be obtained by permuting the rows and the columns of Q. A Latin square satisfying this property clearly exists: it is a Sudoku-like construction. Below we construct this Latin square explicitly by specifying each $[\mathcal{Q}]_{n_2n_1}$. To this end we write $n_1 + k \doteq n_1 + k \mod N_1$ and $n_2 + l \doteq n_2 + l \mod N_2$. Define

$$[\mathcal{Q}]_{n_2n_1} = \begin{pmatrix} Q_{n_1n_2} & \cdots & Q_{n_1(n_2+l)} & \cdots & Q_{n_1(n_2+N_2-1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ Q_{(n_1+k)n_2} & \cdots & Q_{(n_1+k)(n_2+l)} & \cdots & Q_{(n_1+k)(n_2+N_2-1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ Q_{(n_1+N_1-1)n_2} & \cdots & Q_{(n_1+N_1-1)(n_2+l)} & \cdots & Q_{(n_1+N_1-1)(n_2+N_2-1)} \end{pmatrix}.$$

That is,

$$[[\mathcal{Q}]_{n_2n_1}]_{kl} = Q_{(n_1+k)(n_2+l)}$$

for $0 \le k \le N_1 - 1$ and $0 \le l \le N_2 - 1$. For example, $[\mathcal{Q}]_{00} = Q$.

Finally, we can also view $\mathcal{Q}$ as an element of $\{0,1\}^{N_1N_2d\times N_1N_2d}$. That is, $\mathcal{Q}$ has N_1N_2d rows and columns and the elements of this matrix are 0's and 1's. The total number of zeros in $\mathcal{Q}$ is exactly $p = N_1N_2p'$. Let $0_1,\dots,0_p$ be an arbitrary order of these zeros.

Define (σ, r) in $\Gamma(p,\mu,h_0)$ as:

★: The row player chooses a row of $\mathcal{Q}$, that is, a number between 1 and N_1N_2d by choosing (n_1', n_2, d_1) from $A^1\times A^2\times\{1,\dots,d\}$ with equal probabilities. The column player chooses a column of $\mathcal{Q}$, that is, a number between 1 and N_1N_2d by choosing (n_1, n_2', d_2) from $A^1\times A^2\times\{1,\dots,d\}$ with equal probabilities. The chosen row is then $n_2N_1d + n_1'd + d_1$ and the chosen column is $n_1N_2d + n_2'd + d_2$.

$\mathbf{m^i}$: Communicate for p stages through the *AND* by sending message 1 at stage s if 0_s is in the chosen row (column) of $\mathcal{Q}$ and sending 0 otherwise.

$\mathbf{n^i}$: In stage $p+1$ the row player announces $n^1 = n_2$, the column player announces $n^2 = n_1$.

$\mathbf{r^i}$: The row player chooses action in Γ according to:

$$r^1(m^1, n) = r^1(., n_1', n_2, n_1) = n_1 + n_1',$$

while the column player chooses action according to:

$$r^2(m^2, n) = r^2(., n_2', n_2, n_1) = n_2 + n_2'.$$

Notice that the selected actions are modulo N_1 and N_2, respectively. Through their public announcements $(n^1, n^2) = (n_2, n_1)$, the players jointly choose a matrix $[\mathcal{Q}]_{n_2n_1}$. Given this matrix, the row player plays according to the index k for which $Q_{k.} = [[\mathcal{Q}]_{n_2n_1}]_{n_1'.}$, that is, $Q_{k.}$ is in the n_1'th row of $[\mathcal{Q}]_{n_2n_1}$. The column player plays according to the index l for which $Q_{.l} = [[\mathcal{Q}]_{n_2n_1}]_{.n_2'}$, that is, $Q_{.l}$ is in the n_2'th column of $[\mathcal{Q}]_{n_2n_1}$. These decision rules are properly defined because $[\mathcal{Q}]_{n_2n_1}$ is obtained as row and column permutations of Q. Therefore, the first subindexes of Q_{kl} are constant in a row and the second subindexes are constant in a column of $[\mathcal{Q}]_{n_2n_1}$.

Once a player has chosen a row or column of $\mathcal{Q}$ in Lehrer's construction, he knows which action is to be taken if $h = h_0$. Here, instead, the players do not know their actions before the direct talk. For example, the choice of n_1' by player 1 (the row player) in step (★) does not determine his action if the communication is successful, meaning, $h = h_0$ since the action of player 1 also depends on the column players's random choice of n_1 from a uniform distribution. Hence, player 1's belief about his own action after stage p is $\mu(a^1)$ no matter which pure strategy he follows in the support of σ_1. And conversely, player 1's knowledge of n_2 gives no information to him about the action of player 2, since the latter also depends on the uniformly chosen n_2' which is unknown to player 1. Hence, player 1's belief after stage p about player 2's action is $\mu(a^2)$. Even more, not only player 1's belief about the marginal distributions stays the same after stage p, but also his belief about the joint distribution of actions, $\mu(a)$ remains the same, that is, condition 1 of mimicking is satisfied whichever row has been chosen in $\mathcal{Q}$. It follows from property (L1) of Lehrer's matrix, and the facts that each row of $\mathcal{Q}$ (when treated as being of size $N_1 N_2 \times N_1 N_2$) consists of "stretched-out" Lehrer' matrix, and that player 2 chooses a column of $\mathcal{Q}$ uniformly. Condition 2 of mimicking holds trivially: by announcing (n_2, n_1), the players choose a sub-matrix of $\mathcal{Q}$, which is obtained by row and column permutations of Lehrer's matrix but we already know that condition 2 holds for Lehrer's construction. ■