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Paulo Brito, Bipasa Datta and Huw Dixon

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Cardiff Business School
Aberconway Building
Colum Drive
Cardiff CF10 3EU
United Kingdom
t: +44 (0)29 2087 4000
f: +44 (0)29 2087 4419
business.cardiff.ac.uk

The evolution of mixed conjectures in the rent-extraction game*

Paulo Brito[†], Bipasa Datta [‡], and Huw Dixon[§]

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Abstract

This paper adopts an evolutionary perspective on the rent-extraction model with conjectural variations (CV) allowing for mixed-strategies. We analyze the dynamics of the model with n CVs under the replicator equation. We find that the end points of the evolutionary dynamics include the pure-strategy consistent CVs. However, there are also mixed-strategy equilibria that occur: these are on the boundaries between the basins of attraction of the pure-strategy sinks. Further, we develop a more general notion of consistency which applies to mixed-strategy equilibria. In a three conjecture example, by conducting a global dynamics analysis, we prove that in contrast to the pure-strategy equilibria, the mixed-strategy equilibria are not ESS: under the replicator dynamics, there are three or four mixed equilibria that may either be totally unstable (both eigenvalues positive), or saddle-stable (one stable eigenvalue). There also exist heteroclinic orbits that link equilibria together. Whilst only the pure-strategies can be fully consistent, we find a lower bound for the probability that mixed strategy conjectures will be *ex post* consistent.

JEL: D03, L15, H0.

KEYWORDS: Rent-extraction, evolutionary dynamics, consistent conjectures, global dynamics, mixed-strategy.

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†Universidade Técnica de Lisboa, ISEG, UECE (Research Unit on Complexity and Economics), Rua Miguel Lupi, 20, 1249-078 Lisboa, Portugal. Email: pbrito@iseg.utl.pt. Funding by FCT (*Fundação para a Ciência e Tecnologia*), under project PTDC/EGE-ECO/104659/2008, is gratefully acknowledged.

‡Department of Economics, University of York, York YO10 5DD, UK, Email: bipasa.datta@york.ac.uk

§Cardiff Business School, Aberconway Building, Column Drive, Cardiff, CF10 3EU. Email: DixonH@cardiff.ac.uk. Corresponding author.

1 Introduction.

In this paper, we adopt a dynamic approach to analyze the evolution of beliefs underlying agents' behavior in the context of a rent-extraction game à la Tullock (1980). The idea is that the boundedly-rational agents employ decision rules, such as reaction functions, based on certain beliefs about other players' behavior. But how are these beliefs formed? Recently, some authors have adopted an evolutionary approach to explaining such beliefs using Maynard-Smith's notion of evolutionary stable strategies (ESS)[1]. The idea is that the belief can be treated as a *meme*, and that beliefs that result in higher profits become more common. ESS is however a *local* stability condition: it considers the effects on payoffs of a small deviation in the make-up of the population. In this paper we broaden the focus to consider the *global* dynamics of an explicit evolutionary process - the replicator equation. We apply this dynamic evolutionary approach to explain belief formation in the context of a rent-seeking game (Tullock (1967, 1980, 1987), Posner (1975)) where agents spend resources to dispute over rents or some prizes. Agents' beliefs about other players' behavior are particularly important in such models as it can directly impact the magnitude of the rent extracted by altering the success function. Importantly, rent-seeking models have many applications in economics and politics e.g. in elections where resources allocated to campaigning directly affect the candidate's probability of success and where the allocation itself is done based on the agent's belief about his opponent's behavior. Menezes and Quiggin (2010) have provided several different interpretations of such rent-extraction models and have argued that they should be viewed as oligopsonistic markets for influence.

A decision rule in this context can be thought of as a reaction function (RF) which specifies the choice of action as a function of other agents' actions. Whilst there are various ways of parametrizing such decision rules, the one we adopt in this paper is the concept of *Conjectural Variations*. The notion of conjectures has maintained a long history in the Industrial Organization theory ever since the introduction of *Conjectural Variations Equilibria* by Bowley (1924) and Frisch (1951 [1933])[2]. Not only are conjectural variations (henceforth CV) models able to capture a range of behavioral outcomes - from competitive to cooperative, but also they have one parameter which has a simple economic interpretation. CV models have also been found quite useful in the empirical analysis of firm behavior in the sense that they provide a more general description of firms' behavior than the standard Nash equilibrium (Slade (1995)). The concept of CVs has also been seen as useful in anti-trust policy[3].

[1]See, e.g. Dixon and Somma (2003), Müller and Normann (2005), Possajennikov (2009). See Jean-Marie and Tidball (2006) for a non-evolutionary approach to formation of conjectures in a dynamic context.

[2]See Giocoli (2005) for a detailed account of the role of conjectural variations in the history of oligopoly games. Frisch paramterized the CV in terms of an elasticity rather than a derivative. Hicks (1935) survey is probably responsible for making the concept of CVs well known.

[3]See for example the recent Office of Fair Trading (2011) report.

In this context, the concept of *consistent* conjectures was developed by a number of authors in the 1980s (see Bresnahan (1981), Boyer and Moreaux (1983), Klemperer and Meyer (1988)) and has been widely applied ever since in a variety of circumstances such as public goods (Cornes and Sandler (1984)[4], Itaya and Okamura (2003), strategic investment models (Dixon (1986)), export subsidies (Tanaka (1991)), natural resource extraction (Quérou and Tidball (2009)). In Public Economics, Michaels (1989) applied this concept in the context of Tullock's rent-seeking game to show that the fraction of rents dissipated by seekers depends upon the type of CV assumed. In games with quadratic payoffs where the best-response functions with CVs are linear, the natural formulation for consistent conjectures is that the CV of one player equals the actual slope of the other player's RF. However, in games where the payoffs are not quadratic and therefore the RFs are non-linear (such as the ones in rent-seeking models with CVs), the notion of consistency can accordingly be adapted: consistency should imply that CVs are equal to the slopes of RFs at the *equilibrium* point.

Recently, the link between *consistency* and *evolutionary stability* has been made within the CV framework. One can think of economic agents' behavior being summarized by the CV term. One can imagine a population consisting of firms with different CVs which will earn different payoffs (on average) and a process of "natural selection" or social learning takes place (the CV is a *meme*). Firms with particular CVs do better than those with others: a process of imitation or adaption leads agents to switch from less successful CVs to more successful CVs. Dixon and Somma (2003) established that in a standard oligopoly setting with a quadratic payoff function[5], the consistent conjectures are the unique Nash equilibrium in a hypothetical "conjecture game": firms choose their CVs given the CVs of the other firms so as to maximize their payoffs in the output game. This Nash equilibrium in the conjecture game was the *consistent conjecture*. This enabled the link to be made with evolutionary stable strategies (ESS). In the case where there is a strict-Nash equilibrium in the conjecture game, the resultant consistent conjecture will be ESS. Müller and Normann (2005) generalized this result to a wider class of oligopoly models[6]. Both Dixon and Somma (2003) and Müller and Normann (2005) were in the class of quadratic payoff models. Possajennikov (2009) showed that the link between ESS models and consistent conjectures extends to some non-quadratic payoff models, including the rent-seeking model (such as the one considered by Michaels (1989)).

However, all of the above studies were limited in that they focussed exclusively on *pure-strategy* equilibria and that they only studied *local stability* using the ESS condition. In contrast, the main contribution of this paper is to extend the focus to analyze the *global* evolutionary

[4]See also Cornes and Sandler (1985) and Sugden (1985).
[5]Specifically, they consider a homogeneous good Cournot oligopoly with linear demand and quadratic costs.
[6]Specifically, differentiated oligopoly with linear demands.

dynamics in the context of *mixed-strategies*. We do indeed find that in addition to the pure-strategy equilibria, mixed-strategy equilibria will exist in a finite version of the conjecture game where we restrict the set of permissible CVs to a finite set of n distinct conjectures. We provide a bifurcation analysis and show that in addition to the pure-strategy equilibria, there will in general exist many mixed-strategy equilibria.

Further, we define a new concept of consistency that is applicable to the case of mixed-strategy equilibria. This is the notion of the probability that the conjectures will be consistent *ex post*. In the case of a pure-strategy equilibrium, the standard consistent conjectures are 100% consistent *ex post*. With mixed strategy equilibria, the conjectures will only be consistent a certain proportion of the time. Hence, whilst the link between consistency and equilibrium in the conjecture game still exists, it is weaker in the case of mixed-strategies than for pure-strategy equilibria.

Our main results about the dynamics are as follows. We are first able to determine some results which hold for the case where there are n conjectures. All n pure-strategy equilibria (except the Bertrand) are sinks (Proposition 2). We were also able to characterize the properties of mixed stationary points involving just two or three of the n strategies: some of these stationary points will be mixed Nash equilibria, which will have an $n-1$ dimensional stable manifold; others will not be Nash equilibria and will have stable manifolds with a lower dimension than $n-1$ (Propositions 3 and 4).

We are able to determine fully the dynamics in the $n = 3$ conjecture case which can be depicted on the two dimensional simplex. Proposition 5 and 6 summarize the local dynamics: the pure strategy-equilibria are sinks (the eigenvalues of the Jacobian are all negative), whilst the strictly-mixed stationary points can either be saddle-path stable (one negative and one positive eigenvalue) or unstable sources (all eigenvalues are positive). For the global dynamics, in Proposition 7 we find that there is a network of *heteroclinic* orbits[7] that connect equilibria. The heteroclinic orbits connecting these mixed-strategy stationary points with each other and the pure-strategy sinks constitute the boundaries of the basins of attraction for the pure-strategy sinks. There are two generic phase diagrams which describe the exact pattern of equilibria: in particular, if the most competitive conjecture is competitive enough we can have an internal mixed-equilibrium (with all three conjectures with strictly positive shares) which is a source. Otherwise, we have the more general case where there are three stationary points involving only two conjectures with strictly positive probabilities: two of these stationary points are Nash equilibria (and saddle-path stable) with the third being a non-Nash equilibrium unstable source.

We can use the global dynamics as a guide to equilibrium selection. The most cooperative

[7]An *heterclinic orbit* is an equilibrium path that connects two (or more) stationary points. This contrasts to *homoclinic orbits* which have only one stationary point at both end-points.

pure-strategy equilibrium is Pareto-dominant (from the point of view of the rent seekers) and involves the least rent dissipation and highest payoff. However, we do not find that in general the most cooperative conjecture has the biggest basin. Indeed, in the three conjecture case we might expect the intermediate conjecture to have the bigger basin. The reason is that in the rent-extraction model, the intermediate CV can do quite well against the two extremes, whilst the two extremes do badly against each other. Moderation can pay. This means that the intermediate conjecture can end up with a share of 1 even if it starts from a share of almost zero. In contrast, the two extreme conjectures require an initial base which is bounded well away from zero if they are to be selected. Whilst we cannot in general rank the most cooperative and the intermediate conjecture, we can in general say that the most competitive equilibrium will have a smaller basin than the most cooperative. Indeed, in the extreme case of a "Bertrand" CV of -1, the basin of attraction shrinks to zero.

The notion of evolutionary dynamics (such as the replicator) is not unproblematic: if one takes a literal view of the equations, they are based on random matching with the game played repeatedly in continuous time. However, one can think of this more as an *evolutionary metaphor*: over time, more successful strategies become more common. There are a variety of ways this can happen in social learning models. However, to explore the dynamics without recourse to simulating simple models we need to use a specific evolutionary process: the replicator equation is a robust framework that can stand for a wider class of payoff-monotone dynamics.

The organization of the paper is as follows. In section 2, we outline the basic rent-seeking model, which can also be thought of as a Cournot Oligopoly game, where we treat the conjectures as given. In section 3, we consider the underlying conjecture game and pure-strategy equilibria in the case where the strategy sets are a closed convex subset of the real line, and mixed-strategy equilibria where the strategy sets are a finite subset of the pure-strategy case. In section 4, we consider the relation between consistency and the equilibria in the conjecture game. In section 5, we analyze the evolutionary process of the model using the replicator equation. Section 6 concludes. All proofs are in the appendix.

2 The model.

We consider the following game where two firms X and Y choose actions $(x,\ y)$ independently with payoff functions given as follows:

$$\begin{aligned} U^X(x,y) &= \frac{x}{x+y} - x \\ U^Y(x,y) &= \frac{y}{x+y} - y \end{aligned}$$

This can be thought of as a simple rent-seeking game *à la* Tullock (1980) where players choose actions (e.g. effort or investment) to win a prize of fixed value (which is unity in the above formulation), where the first term in the payoff function denotes the probability of player i's winning the contest, $i = X, Y$, and the second term denotes constant unit cost of the action. Alternatively, this game can also be thought of as a homogeneous good Cournot duopoly[8] with unit elastic demand and constant unit cost where the market price is given by

$$P = \frac{1}{x+y}$$

so that total revenue equals 1, each firm receives a share of that revenue equal to its share of output[9], and the total cost of player i equals player i's output. For economically meaningful outcomes, we can restrict our attention to the strategy-space:

$$S = \{(x, y) : x \geq 0, y \geq 0 \text{ and } x + y \leq 1\}$$

The above payoff-function is strictly concave for $(x, y) \in S \cap (0, 1)^2$. The corresponding iso-payoff sets for X are characterized by

$$\bar{U}^X = \{(x, y) : U^X(x, y) = \bar{U}\}$$

and have slopes given by

$$\left.\frac{dy}{dx}\right|_{\bar{U}^x} = \frac{y - x^2 - 2xy - y^2}{x}$$

For $\bar{U} \in (0, 1)$, the iso-payoff curve intersects the $x-$ axis at $\left(1 - \bar{U}, 0\right)$. However, all iso-payoff sets with $\bar{U} \in (0, 1)$ originate from $(0, 0)$. The payoff function is undefined for $x = y = 0$. However, in order to convert the joint profit supremum into a maximum, we adopt the definition $U^X(0, 0) = U^Y(0, 0) = 0.5$. In the event of neither player doing anything, the prize is split.

2.1 Conjectural variation (CV) output game.

Each firm has a *conjecture* about the response of the other firm to variations in its own output. $\phi_x = \partial y/\partial x$ and $\phi_y = \partial x/\partial y$ denote such conjectures held by firms X and Y respectively where $\phi_i \in [-1, +1]$, $i = x, y$. This gives the reaction functions (RFs) defined by the following

[8] It has been shown that a standard Tullock contest of the above type is strategically equivalent to a Cournot oligopoly game, and that the same strategic equivalence applies also with a more general success function in the original Tullock game (see Okuguchi (1995), Szidarovsky and Okuguchi (1997).

[9] Henceforth, we will refer to x and y as 'outputs'.

first-order conditions:

$$\begin{aligned}\frac{1}{(x+y)} - \frac{x}{(x+y)^2}(1+\phi_x) - 1 &= 0 \\ \frac{1}{(x+y)} - \frac{y}{(x+y)^2}(1+\phi_y) - 1 &= 0\end{aligned}$$

From above, we get the reaction functions in the following form:

$$x = R(y,\phi_x) = -\frac{1}{2}\phi_x - y + \frac{1}{2}\sqrt[+]{\phi_x^2 + 4\phi_x y + 4y} \quad (1)$$

$$y = R(x,\phi_y) = -\frac{1}{2}\phi_y - x + \frac{1}{2}\sqrt[+]{\phi_y^2 + 4\phi_y x + 4x} \quad (2)$$

For $\{(\phi_x, \phi_y) \in [-1,1]^2 \text{ and } 1 - \phi_y\phi_x > 0\}$, the equilibrium values of output are given by:

$$x(\phi_x,\phi_y) = \frac{(1+\phi_y)(1-\phi_y\phi_x)}{(2+\phi_y+\phi_x)^2} \quad (3)$$

$$y(\phi_x,\phi_y) = \frac{(1+\phi_x)(1-\phi_y\phi_x)}{(2+\phi_y+\phi_x)^2} \quad (4)$$

In the cases where $\phi_y\phi_x = 1$, we set $x(1,1) = 0$ and $x(-1,-1) = \frac{1}{2}$ and likewise for y, these being the limiting values[10]. In case of *symmetric* conjectures ($\phi_x = \phi_y = \phi$), equilibrium outputs will be given by

$$x(\phi,\phi) = y(\phi,\phi) = \frac{1-\phi}{4} \quad (5)$$

We can consider the following special cases:

(i) Cournot-Nash conjectures: $\phi_x = \phi_y = 0$

(1) and (2) then yield

$$\begin{aligned}x &= -y + \sqrt{y} \\ y &= -x + \sqrt{x}\end{aligned}$$

so that Cournot-Equilibrium values are

$$\begin{aligned}x^c &= y^c = \frac{1}{4} \text{ and} \\ U^X &= U^Y = U|_{\text{Cournot}} = \frac{1}{4}\end{aligned}$$

[10]Alternatively, one can restrict the strategy set to $[-1+\varepsilon, 1-\varepsilon]$ for some arbitrarily small $\varepsilon > 0$.

(ii) Bertrand-Nash conjectures: $\phi_x = \phi_y = -1$

(1) and (2) then yield

$$\begin{aligned} x &= 1 - y \\ y &= 1 - x \end{aligned}$$

which has the set of solutions $x + y = 1$, with the symmetric solution being at $x = y = \frac{1}{2}$ with corresponding equilibrium payoffs $U|_{\text{Bertrand}} = 0$.

(iii) Fully collusive conjectures: $\phi_x = \phi_y = 1$

In this case, (3) and (4) imply $x = y = 0$.This is the joint profit maximum.

3 The Conjecture Game.

In order to analyze the evolutionary properties of conjectures, following Dixon and Somma (2003), we consider a further stage of the game where firms are choosing their conjectures.[11] We will first analyze this hypothetical "conjecture game" in terms of pure-strategies, where the strategy sets are intervals on the real line. We will then consider the case of finite strategy sets in order to analyze the possible existence of mixed-strategy equilibria where more than one strategy is played with a positive probability.

3.1 Pure-strategy equilibria.

Given the equilibrium outputs as a function of the conjectures, we can think of a *reduced* form game of the equilibrium given conjectures with each firm choosing its conjecture. For ease of notation and for the purpose of analysing the dynamics (see section 5), we will reparameterize the conjectures as $\varphi_i = (1 + \phi_i)$ for $i = x, y$ where $\varphi_i \in [0, 2]$. With this re-parameterisation then $\varphi_i = 1$ implies Cournot-Nash conjectures; $\varphi_i = 0$ implies Bertrand-Nash conjectures; and $\varphi_i = 2$ implies fully collusive conjeectures. The outputs and payoffs for the conjecture game, after simplification, are respectively:

[11] The entire game can equivalently be considered "as if" a two stage game where firms choose their conjectures in the first stage, and then given their choice of conjectures in the first stage, they choose outputs in the second stage.

$$
\begin{aligned}
x(\varphi_x,\varphi_y) &= \frac{(\varphi_y)(\varphi_x+\varphi_y-\varphi_y\varphi_x)}{(\varphi_y+\varphi_x)^2} && (6)\\
y(\varphi_x,\varphi_y) &= \frac{(\varphi_x)(\varphi_x+\varphi_y-\varphi_y\varphi_x)}{(\varphi_y+\varphi_x)^2} && (7)
\end{aligned}
$$

and

$$
\begin{aligned}
U^X(\varphi_x,\varphi_y) &= \frac{\varphi_x\varphi_y^2}{(\varphi_y+\varphi_x)^2} && (8)\\
U^Y(\varphi_x,\varphi_y) &= \frac{\varphi_y\varphi_x^2}{(\varphi_y+\varphi_x)^2} && (9)
\end{aligned}
$$

Firms' equilibrium choice of conjectures will then be obtained from the following first-order conditions for X (and conversely for Y):

$$
\frac{dU^X(\varphi_x,\varphi_y)}{d\varphi_x} = \frac{\varphi_y^2}{(\varphi_y+\varphi_x)^3}(\varphi_y-\varphi_x) = 0 \tag{10}
$$

This yields the following reaction functions in the conjecture game for X:

$$
R^X(\varphi_y) = \varphi_y
$$

That is, the best-response of firm is to choose the same conjecture as the other firm[12]. Thus, we have the following proposition (stated without proof):

Proposition 1. *Pure strategy Nash equilibrium conjectures are symmetric.*

Thus, there is a continuum of "strict" Nash equilibria, each parameterized by the symmetric conjecture $\varphi \in [0,2]$ with equilibrium output levels given by

$$
x(\varphi,\varphi) = y(\varphi,\varphi) = \frac{2-\varphi}{4} \tag{11}
$$

and symmetric payoffs given by:

$$
U(\varphi) = \frac{\varphi}{4} \tag{12}
$$

There is also a "Bertrand" Nash equilibrium which is not strict: if one firm sets $\varphi = 0$, then the other firm earns zero profits whatever conjecture it has. Clearly, the equilibria are Pareto-ranked: the higher the conjecture, the higher the profits, with the limiting profit being half the joint profit maximum $U(2) = \frac{1}{2}$ and the minimum being the Bertrand case $U(0) = 0$. The

[12]The second order conditions are clearly satisfied from (10).

structure of the conjecture game is similar to a coordination game, except that the "off-diagonal" elements vary with the conjectures.

3.2 Mixed-strategy Equilibria.

Mixed-strategy equilibria will also exist if we take a finite subset of conjectures. In this section, we provide an example, prior to a more general analysis when we model the evolutionary dynamics in section 5.

Consider a finite subset of conjectures φ taken from $[0,2]$, with $\#\varphi = n$, and index set $S_1 = \{1, 2, \ldots, n\}$ so that $\varphi = \{\varphi_i\}_{i \in S_1}$. This then gives us an $n \times n$ payoff matrix A :

$$\underset{n \times n}{\mathbf{A}} = \left[\pi_{ij} = U^I(\varphi_i, \varphi_j)\right]_{ij \in S_1 \times S_1}, I = X, Y. \tag{13}$$

where the row i gives the payoff to the firm playing each strategy i (conjecture) against j and the column j gives us the payoff of playing strategy j against each of the strategies i. Note that since the game is payoff-symmetric, we can use either firm's payoff function to define the payoff matrix.

Let $\mathbf{z} = (z_1, \ldots, z_n) \in \Delta^{n-1}$, where z_i is the probability that conjecture i will be played. Then, the expected payoff of strategy i is

$$u_i(\mathbf{z}) = (\mathbf{Az})_i = \sum_{j \in S_1} \pi_{ij} z_j, \; i \in S_1, \tag{14}$$

and the n-vector of expected payoffs for all strategies $\mathbf{u}$ is

$$\mathbf{u}(\mathbf{z}) = \mathbf{Az}. \tag{15}$$

If we consider the 3×3 payoff matrix generated by conjectures $\Phi = \{1, 1.5, 2\}$, we have:

$$A = \begin{pmatrix} 0.25 & 0.36 & 0.4444 \\ 0.24 & 0.375 & 0.4898 \\ 0.2222 & 0.3673 & 0.5 \end{pmatrix} \tag{16}$$

In addition to the 3 pure strategy equilibria, there are also 2 mixed equilibria. Adapting the notation slightly, so that $z(\varphi)$ is the probability that conjecture φ is played, the 2 mixed equilibria are given by:

- $z^*(2) = 0.4302$, $z^*(\frac{3}{2}) = 1 - z^*(2)$, $z^*(1) = 0$.
- $z^*(2) = 0$, $z^*(\frac{3}{2}) = 0.4$, $z^*(1) = 0.6$.

There is the following profile in which $z^*(\frac{3}{2}) = 0$, and the two conjectures $(2,1)$ earn equal payoffs:

$$z^*(2) = \frac{1}{3},\ z\left(\frac{3}{2}\right) = 0,\ z^*(1) = \frac{2}{3}$$

This is not an equilibrium, because the expected payoff from playing 1.5 exceeds the payoffs of the other two. Note that in this example, both mixed-equilibria involve only pairs of strategies being played with strictly positive probabilities, there being no equilibrium with all three strategies being played. As we show below, this is not a general property: strictly interior solutions in which all three probabilities are strictly positive may also exist.

4 Consistency of conjectures.

There are several definitions of consistency of conjectures available[13]. However, we use the one in the sense of Bresnahan (1981), that in the output game each firm's conjecture about the slope of the other firm's reaction function is correct at the equilibrium outputs. Unlike the quadratic payoff framework considered by Dixon and Somma (2003) and Müller and Normann (2005), the CV reaction functions are not linear in this model, so that consistency-correctness at equilibrium outputs does not imply correctness elsewhere. This has important implications for the evolutionary stability of equilibria as we shall see.

From (1), the slopes of the reaction functions written in terms of φ_i are:

$$\frac{dR(y,\varphi_x)}{dy} = -1 + \frac{\varphi_x}{\sqrt{(\varphi_x - 1)^2 + 4\varphi_x y}} \tag{17}$$

$$\frac{dR(x,\varphi_y)}{dx} = -1 + \frac{\varphi_y}{\sqrt{(\varphi_y - 1)^2 + 4\varphi_y y}} \tag{18}$$

Now, we can set the outputs (x, y) at their equilibrium values given (φ_x, φ_y) using (6), (7), and then consider whether or not the conjectures are consistent.

[13]See, e.g. Hahn (1977, 1978); Perry (1982); Kamien and Schwartz (1983); Boyer and Moreaux (1983).

4.1 Pure-strategy Equilibria and consistency.

From Proposition 1, we can focus attention only on the symmetric conjectures: $\varphi_y = \varphi_x = \varphi$. Equations (17) and (18) then simplify as:

$$\frac{dR(y,\varphi)}{dy} = \frac{dR(x,\varphi)}{dx} = -1 + \frac{\varphi}{\sqrt{(\varphi-1)^2 + 4\varphi y}} \tag{19}$$

Evaluating the above slopes at the equilibrium values of output given by (11) and simplifying, we find:

$$\frac{dR(y,\varphi)}{dy} = \frac{dR(x,\varphi)}{dx} = \varphi - 1 = \phi \tag{20}$$

Hence, all pure-strategy (symmetric) Nash equilibrium conjectures are consistent.[14] This is true for any $\varphi \in [0,2]$ so that:

Observation 1 The set of consistent conjectures equilibria is equivalent to the set of pure-strategy Nash equilibria in the conjecture game.

Further, we also observe that,

Observation 2 Unlike Bresnahan (1981), Cournot conjectures are consistent in this model.

To see that, note for $\varphi_x = 1$ ($\phi_x = 0$), the slope of firm X's RF from (19) is:

$$\frac{dR(y,1)}{dy} = -1 + \frac{1}{2\sqrt{y}}$$

which when evaluated at Cournot output level $y = 1/4$, yields $\frac{dR(y,1)}{dy} = 0$. Likewise for $\varphi_y = 1$.

However, if the conjectures are asymmetric i.e. $\varphi_x \neq \varphi_y$ (as is the case in mixed-strategy) then that will involve inconsistent conjectures (in the above sense) by one or both of the firms.

4.2 Mixed-strategy equilibria and consistency.

The existing definition of consistency has been developed purely for the pure-strategy case. Is there any sense in which a mixed-strategy equilibrium in the conjecture game can consistent?

[14] A similar result is also be found in Michaels (1989) who showed that there can be multiple equilibria in the standard symmetric form of the game where any CV can be consistent. Michaels however does not consider a conjecture stage of the game as we do in this paper.

In this paper, we develop the notion of *ex post* consistency[15]

Definition. *Ex post consistency* PC. *In equilibrium, there is a probability that both players will choose the same conjecture.*

If both players choose the same conjecture, their conjectures are "consistent" in the resultant game *ex post*. If they choose different conjectures, they will not be consistent. Hence, we can define the probability of *ex post* consistency:

$$PC(z^*) = \sum_{i=1}^{n} (z_i^*)^2$$

For example, in the two mixed Nash equilibria identified in the 3×3 example, we have:

$$\begin{aligned} PC(0.4302,\ 0.5698, 0) &= (0.430\,2)^2 + (\ 0.569\,8)^2 = 0.509\,8 \\ PC(0.6, 0.4, 0) &= 0.36 + 0.16 = 0.52 \end{aligned}$$

In the case of pure-strategy equilibria, of course $PC(1) = 1$: the conjecture is correct in equilibrium. However, when we have strictly mixed-strategies, the conjectures will only be correct a certain proportion of the time: in the three mixed equilibria in our game they are correct $51 - 52\%$ of the time.

In general the isoquants for PC are simply concentric circles measuring the distance from the center of the simplex: the minimum is

$$\min_{\mathbf{z} \in \Delta^{N-1}} PC(\mathbf{z}) = \frac{1}{N},$$

which occurs the center point and the maximum is $PC = 1$ which occurs at the vertices (pure strategies).

This can be seen in the three dimensional case as depicted in Figure 1. The unit circle touches the three vertices and represents $PC = 1$. The equilibria on the edges satisfy[16]:

$$PC \in \left[\frac{1}{2}, 1\right]$$

The $PC = 1/2$ circle touches the three edges at their midpoint: PC is increasing along the edges in both directions. In the three conjecture case the minimum of $PC = 1/3$.

[15] In an earlier version of the paper, we also proposed a possible *ex ante* definition, that the average conjecture equaled the expected slope. This is a more distant concept from the original consistency condition and therefore we do not pursue it here.

[16] We can see that the two mixed equilibria in our example both lie in this range.

Insert figure 1 here

Hence in a mixed-strategy equilibrium there is a probability of consistency *ex post* which is captured by PC. The interesting question is the link between stability (local and global) and the probability of consistency. It is to this that we next turn.

5 Evolutionary Dynamics.

In this section, we analyze the dynamics of the model using the replicator equation. Previous authors have focussed only on the *local stability* of consistent conjectures using Maynard Smith's notion of an evolutionary stable strategy (ESS). Analyzing global dynamics is important as it will enable us to understand how the "population" behaves from any given starting point, rather than assuming a small deviation from a proposed equilibrium. Furthermore, this is particularly important in our context because of the large number of equilibria and the possibility of the dynamics providing a criterion for equilibrium selection as we show below. In this section we derive some results on the number of equilibria for any finite number of strategies, n, and provide a comprehensive global analysis of the replicator dynamics for the 3 conjecture case (the two and the four coenjecture cases are described in the appendix).

From (15), the mean payoff across all strategies is:

$$\bar{u}(\mathbf{z}, \varphi) = \mathbf{z}^\top \mathbf{A}(\varphi)\mathbf{z} = \sum_{i\in S_1}\sum_{j\in S_1} z_i z_j \pi_{ij}(\varphi).$$

Whilst the payoff-matrix A is not symmetric[17], the following *transformed symmetry* relationships hold for $\varphi_i\varphi_j \neq 0$,

$$\frac{\pi_{ij}}{\varphi_j} = \frac{\pi_{ji}}{\varphi_i} = \frac{\varphi_i\varphi_j}{(\varphi_i+\varphi_j)^2}, \ \frac{\pi_{ii}}{\varphi_i} = \frac{1}{4}.$$

If $\varphi_1 = 0$ then $\pi_{ij} = 0$ for $i = 1$ or $j = 1$.

We let $\Phi = \{(\varphi_i)_{i\in S_1} \subset [0,2]^n : 0 \leq \varphi_1 < \varphi_2 < \varphi_i < \varphi_{i+1} < \ldots < \varphi_n \leq 2\}$ be the set of possible ordered strategies and assume, without loss of generality, that $\varphi \in \Phi$.

The replicator dynamics (henceforth RD) is given by the n-dimensional ordinary differential equation system [18]

$$\dot{z}_i = F_i(\mathbf{z}, \varphi) \equiv z_i\left(u_i(\mathbf{z}, \varphi) - \bar{u}(\mathbf{z}, \varphi)\right), \ i \in S_1 \tag{21}$$

where $\mathbf{z} \in \Delta^{n-1}$, i.e., such that $\sum_{i\in S_1} z_i = 1$ and $0 \leq z_i \leq 1$, for all $i \in S_1$.

[17] The asymmetry of A arises because when $\varphi_i \neq \varphi_j$, $\pi_{ij} \neq \pi_{ji}$. as in our previous 3 conjecture example (16)

[18] See Hofbauer and Sigmund (2003) and Sandholm (2010) for recent accounts of the properties of this type of evolutionary dynamics.

Let us denote by S_k the set of all combination of the indices of k strategy profiles where each index is drawn from S_1. The number of combinations without repetition of k strategies drawn from the set of n strategies is

$$C(n,k) = \frac{n!}{k!(n-k)!},\ k = 1,\ldots,n.$$

Then the set of pure strategies has the index set S_1 with cardinality $C(n,1) = n$, the set of combinations of two strategies has the index set $S_2 = \{\ ij : i,j \in S_1,\ j > i\}$ with cardinality $C(n,2)$; the set of combinations of three strategies has the index set $S_3 = \{\ ijk : i,j,k \in S_1,\ k > j > i\}$ with cardinality $C(n,3)$, and so on. Set S_n has only one element.

We introduce the following notation for the elements of simplex Δ^{n-1}. First, we denote the boundary by $\partial(\Delta^{n-1})$ and the interior by $\text{int}(\Delta^{n-1})$. In the boundary of the simplex we distinguish further the elements of the boundary which are vertices $\mathbf{e}_i = \{\ \mathbf{z} \in \partial\Delta^{n-1} : z_i = 1\}$ for $i \in S_1$, one-dimensional hyperplanes joining two vertices i and j, $\mathbf{e}_{ij} = \{\ \mathbf{z} \in \partial\Delta^{n-1} : z_i + z_j = 1\}$, for $ij \in S_2$, two-dimensional hyperplanes joining three vertices i, j and l, $\mathbf{e}_{ijk} = \{\ \mathbf{z} \in \partial\Delta^{n-1} : z_i + z_j + z_k = 1\}$, for $ijk \in S_3$, and so on.

Whilst a Nash-equilibrium is a stationary distribution for the RD equation (21), not all stationary distributions are Nash equilibria. A necessary condition for a stationary distribution is that it should be a fixed points for equation (21). The set of all possible fixed points for the RD is $\mathcal{Z} = \{\ \mathbf{z} \in \mathbb{R}^n :\ \mathbf{F}(\mathbf{z}) = \mathbf{0}\}$. The set of fixed-points $\mathcal{Z}$ within the simplex Δ^{n-1} is $\mathcal{Z}^* = \{\ \mathbf{z} \in \Delta^{n-1} : \mathbf{F}(\mathbf{z}) = \mathbf{0}\}$. A stationary distribution $\mathbf{z}^* \in \mathcal{Z}^*$, is Nash equilibrium if the condition $\mathbf{u}(\mathbf{z}^*) \leq \bar{u}(\mathbf{z}^*)$ holds: $\mathcal{Z}^*_{Nash} = \{\mathbf{z} \in \Delta^{n-1} : \mathbf{F}(\mathbf{z}) = \mathbf{0},\ \mathbf{u}(\mathbf{z}) \leq \bar{u}(\mathbf{z})\}$. Hence $\mathcal{Z}^* \subseteq \mathcal{Z}^* \subseteq \mathcal{Z}^*_{Nash}$. The difference between $\mathcal{Z}^*$ and $\mathcal{Z}^*_{Nash}$ arises because of the "no return" feature of the RD: once a conjecture is extinct ($z_i = 0$), it can never come back. Hence there are stationary points for which all of the active conjectures earn equal profits, but for which the "extinct" conjectures would earn above average profits were they to "return" and have a strictly positive share. Clearly, stationary points which are not Nash equilibria will be *fragile*: they are stationary only because the replicator dynamics we analyze are deterministic.

5.1 The model with n distinct conjectures.

This subsection gathers some results for the n-dimensional case. Although the dynamics generated by equation (21) cannot be completely characterized, we can derive some general results. We then illustrate how this works in the case of three conjectures $n = 3$ (the 2 and 4 conjecture cases are analyzed in the appendix).

The maximum number of stationary points for equation (21), $\mathbf{F}(\mathbf{z}, \varphi) = \mathbf{0}$, for $\varphi \in \Phi$, ruling out the trivial case $\mathbf{z} = \mathbf{0}$, is

$$\sum_{k=1}^{n} C(n,k) = \sum_{k=1}^{n} \frac{n!}{k!(n-k)!}.$$

This is the cardinality of set $\mathcal{Z} - \{\mathbf{0}\}$ and gives an upper bound for the maximum number of stationary distributions and for the number of Nash equilibria. Each term in the summation refers to stationary points with only k conjectures having non-zero probabilities $z_i > 0$. There can only exist at most one stationary point with $k = n$ non-zero probabilities. Clearly there are possibly a very large number of stationary points. For example, if $n = 10$ there are up to 252 equilibria with $k = 5$, 210 each for $k = 4$ and $k = 6$, 120 for $k = 3$ and $k = 7$, 45 for $k = 2$ and $k = 8$, 10 for $k = 1$ and $k = 9$, and 1 for $k = 10$. That is a total of up to 1023 stationary equilibria, of which 10 are pure strategy profiles and the rest are mixed strategy profiles.

In order to characterize the set $\mathcal{Z}^*_{Nash}$ we present and characterize the stationary profiles which are in the vertices, in the edges joining two vertices, and in the hyperplane joining three vertices. This allow us to make a conjecture on the existence of stationary mixed strategies in the interior of the simplex Δ^{n-1}.

First, we consider distributions in the vertices $\mathbf{e}_i$ for $i \in S_1$, corresponding to pure strategy profiles. If the CV game starts from a mixed strategy sufficiently close to any pure strategy distribution there will be asymptotic convergence to that pure strategy. The only exception is the Bertrand conjecture which is a Nash equilibrium, but not ESS. The Bertrand vertex is unstable and has no sink.

Proposition 2. *For any $\varphi \in \Phi$ there are n pure strategy distribution profiles, $\mathbf{z}^* = \mathbf{e}_i$ for all $i \in S_1$. If $\varphi_i > 0$ $\mathbf{e}_i$ is a Nash equilibrium and is locally a sink. If $\varphi_1 = 0$ then $\mathbf{e}_1$ is a Nash equilibrium and is a fold bifurcation point.*

Second, we consider the mixed strategy stationary equilibria located over the boundary of the simplex which is formed by the hyperplane (edges) joining any two vertices $\mathbf{e}_i$ and $\mathbf{e}_j$:

$$\mathbf{e}^*_{ij} = \left\{ z_i = \frac{\varphi_j}{\varphi_i + \varphi_j}, \quad z_j = \frac{\varphi_i}{\varphi_i + \varphi_j}, \ z_k = 0, \ k \neq i, j \in S_1 \right\} \ \in \mathbf{e}_{ij} \text{ for } ij \in S_2. \tag{22}$$

For a *given* pair of strategies $ij \in S_2$, define $k_{ij} = \{$ number of $k :$ $i < k < j$ holds for all $k \neq i, j \in S_1\}$ and $k_2 = \{$ number of pairs ij : $k_{ij} \neq 0$, for all $ij \in S_2\}$. Clearly $k_{ij} < n$ and $k_2 \leq C(n,2)$.

Proposition 3. *Let $\varphi \in \Phi$. There are $C(n,2) = n(n-1)/2$ mixed strategy stationary equilibria in the edges joining two vertices of the simplex Δ^{n-1}, $\mathbf{z}^* = \mathbf{e}^*_{ij} \in \mathbf{e}_{ij}$ for all $ij \in S_2$. Associated with φ are the corresponding k_2 and k_{ij} numbers. There is an associated multiplicity of stationary equilibrium distributions in which there are $\{$ $n(n-1)/2 - s_2\}$ Nash equilibria and s_2 non-Nash* *equilibria, for some $s_2 \in \{0,\ldots,k_2\}$. Nash equilibria are generalized saddle points in which the local saddle manifold is of dimension $n-1$. Stationary profiles which are non Nash equilibria, have local stable manifolds of dimensions $\{n-1-s_{ij}\}$, for some $s_{ij} \in \{0,\ldots,k_{ij}\}$.*

The number of stationary mixed strategies combining two pure strategies, $\mathbf{e}^*_{ij}$, that may not Nash equilibria is $s_2 \leq k_2$, and have a local unstable manifold with dimension $s_{ij}+1 \leq k_{ij}+1$. We can briefly explain this result.

For a given conjecture profile $\varphi \in \Phi$, we define the *transformed payoff difference* from playing strategy i against strategy j:

$$m_{ij} \equiv \frac{\pi_{ii}-\pi_{ij}}{\varphi_i} = \frac{1}{4} - \frac{\varphi_i\varphi_j}{(\varphi_i+\varphi_j)^2} = \left(\frac{\varphi_i-\varphi_j}{2(\varphi_i+\varphi_j)}\right)^2 \in (0,1/4]\,,\ \text{for } ij \in S_2.$$

It then follows immediately that $m_{ij}=m_{ji}$ and $m_{ii}=0$. We can define the *relative profitability difference* of two strategie, i and j, relative to a third strategy, k, by

$$m_{ij(k)} = m_{ij} - m_{ik} - m_{jk},\ ij \in S_2 \tag{23}$$

which is the difference between the profitability difference between strategies i and j, m_{ij}, relative to the sum of the profit differences of both strategies i and j against a third strategy k. As the ordering of conjectures is the same as the indexes in S_1, e.g. $\varphi_i < \varphi_j = \varphi_{i+1}$, the signs of $m_{ij(k)}$ depend on the order of k relative to i and j. In general we have:

$$m_{ij(k)} \begin{cases} <0 & \text{, if } i<j<k \text{ or } k<i<j \\ \gtreqless 0 & \text{, if } i<k<j. \end{cases}$$

The equilibrium $\mathbf{e}^*_{ij}$ for $ij \in S_2$ is a Nash equilibrium only if **all** differences $m_{ij(k)}$, for all $k \in S_1$ excluding i and j, are non-positive.

We can associate three types of counting to the number of $m_{ij(k)}$ differentials: (i) their total number is the same as the number of combinations of ijk, that is $C(n,3)$ (which is the cardinality of S_3); (ii) k_{ij} is the maximum number of these differentials $m_{ij(k)}$ which may be non-negative, for a *given* pair of strategies $ij \in S_2$; and (iii) k_2 is the maximum number of non-negative differentials for *all* $ij \in S_2$. The key point here is that the last two numbers are just associated with the ordering of indexes, but, nevertheless, give us a the maximum number of pairs ij which may have a non-negative relative differential and hence may not be Nash equilibria.

We denote further by $s_{ij} \in \{0,\ldots,k_{ij}\}$, the *actual* number of differentials which are positive, for a given edge of index $ij \in S_2$ and the total number of positive differentials by $s_2 \in \{0,\ldots,k_2\}$, for all indices $ij \in S_2$. That is, s_2 is the number of pairs ij which have at least one positive differential and so cannot be a Nash equilibrium, and an equilibrium in the edge $\mathbf{e}_{ij}$ has unstable manifold of dimension $s_{ij}+1$ if $s_{ij} < k_{ij}$ this means that the equilibrium point $\mathbf{e}^*_{ij}$ is a local bifurcation point where the center manifold has maximum dimension $k_{ij} - s_{ij}$.

Proposition 3 introduces, for any possible combination of two conjectures drawn from a set of possible n conjectures $\varphi \in \Phi$, two numbers, s_2 and k_2, and a partition over Φ, $\{\Phi^2_0, \dots, \Phi^2_{s_2}, \dots, \Phi^2_{k_2}\}$, such that $\Phi = \cup_{s_2=0}^{k_2} \Phi^2_{s_2}$, associated to the number equilibria $\mathbf{e}^*_{ij}$ which are not Nash equilibria. Clearly, some subsets of the partition may be empty: Φ^2_0 is the subset of Φ for which all $m_{ij(k)}(\varphi)$ are negative and hence all stationary distributions are Nash equilibria. If $\Phi^2_0 = \Phi$ it means that $s_2 = k_2 = 0$, that is all the relative profitability differentials are negative; Φ^2_1 is the subset of Φ such that $s_2 = 1$ and there is only one pair ij with at least one profitability difference $m_{ij(k)}(\varphi)$ which is strictly positive, and all the others are non-positive, and there is one stationary equilibria which is not Nash.

Proposition 3 also states that we can perform a further partition over every set $\Phi^2_{s_2}$ which is non-empty, and is related to the *number* of differentials which are non-negative and are counted by s_{ij}. This partition is related to the dimension of the local stable manifold.

The strategy space $\mathbf{\Phi}$ is partioned into subsets $\Phi^2_{s_2}$: depending on its location, φ will have an associated s_2 and s_{ij} which will determine the number of Nash equilibria and the dimension of the local stable manifolds of the non-Nash stationary profiles locarted on the edges joinging the verticese. Intuitively, if the initial population of CVs starts sufficiently closed to a mixed strategy located at any one of the edges $\mathbf{e}_{ij}$ it will converge to $\mathbf{e}^*_{ij}$ only if is located exactly over the local stable manifold passing through $\mathbf{e}^*_{ij}$. However, the local stable manifold is a set of measure zero. Generically, if the CV game starts close to $\mathbf{e}_{ij}$ the conjecture game solution will diverge away and converge asymptotically to one of the two pure strategies $\mathbf{e}_i$ or $\mathbf{e}_j$. Proposition 3 states that there is a close association between the partition of the space of conjectures Φ, which is related to the number of probability profiles in which the Nash property does not hold, $\{\Phi^2_{s_2}\}_{s_2=0}^{k_2}$, and a bifurcation analysis associated to the dimension of the local stable manifold. Local bifurcations are associated to the boundaries of two subsets in which the number of distributions which verify the Nash property varies.

Third, we can also derive some general results for the mixed strategy stationary equilibria located on the boundary of the simplex which is formed by the hyperplane (edges) joining any three vertices, $\mathbf{e}_{ijk}$. $C(n,3)$ is the number of combinations $i<k<j : i \neq j \neq k$. Then $C(n,3)$ also counts the number, considering all the combinations, of relative profitability differences $m_{ij(k)}$ that can be non-negative, for all coefficients ijk such that they are all different. A new partition over set Φ can be performed, $\{\Phi^2_{s_3}\}_{s_3=0}^{C(n,3)}$ associated to the *total* number of ambiguously signed relative profitability differences. Observe that partition $\{\Phi^2_{s_2}\}$ involves unions of subsets in the partition $\{\Phi^2_{s_3}\}$.

We define a new magnitude involving relative profitability differences

$$m_{ijk} = m_{ij}m_{ij(k)}m_{kl} + m_{ik}m_{il(j)}m_{jl} + m_{jk}m_{jk(i)}m_{il} + 2m_{ij}m_{ik}m_{kl},\ l \neq i,k,l \in S_1, \tag{24}$$

and a new partition over set Φ

$$\Phi^3_{s_3} = \{\ \varphi\ \in \Phi : \text{there is at least one } m_{ijl}(\varphi) > 0,\ l \neq i \neq j \in S_1\},\ s_3 \in \{0, C(n,3)\}. \tag{25}$$

Proposition 4. *If there is a partition of Φ by non-empty sets $\{\Phi^2_{s_3}\}_{s_3=0}^{C(n,3)}$ then there is an associated multiplicity of stationary equilibrium distributions in which there are $s_3 \in 0,\ldots,C(n,3)$ distributions of type $\mathbf{z}^* = \mathbf{e}^*_{ijk}$ on the edges $\mathbf{e}_{ijk} \in \Delta^{n-1}$. A stationary equilibrium distribution $\mathbf{e}^*_{ijk}$ is a Nash equilibrium if the associated coefficients m_{ijl} for $l \neq i \neq j \neq k \in S_1$ are all negative. If there is a partition of Φ by non-empty sets $\{\Phi^3_{s_3}\}_{s_3=0}^{C(n,3)}$ then the maximum number of Nash equilibria is $\{C(n,3) - s_3\}_{s_3=0}^{C(n,3)}$.*

In Propositions 3 and 4 we found that here is a close relationship between the number of stationary distributions in the edges $\mathbf{e}_{ij}$, $\mathbf{e}^*_{ij}$, *which are Nash equilibria*, the *dimension of the local stable manifold* for stationary distribution in $\mathbf{e}_{ij}$ and the *number of stationary distributions on edges* $\mathbf{e}_{ijk}$. This type of relationship holds further between the number of stationary distributions on the edges $\mathbf{e}_{ijk}$ which are Nash equilibria, the dimension of the local stable manifold at $\mathbf{e}^*_{ijk}$ [19] and the number of stationary distributions on edges $\mathbf{e}_{ijkl}$. In this case a fixed point of $\mathbf{F}(\mathbf{z}) = \mathbf{0}$ will be of type

$$\mathbf{z} = \Bigg(\ z_i = \frac{\varphi_j\varphi_k\varphi_l m_{jkl}}{d_{ijkl}},\quad z_j = \frac{\varphi_i\varphi_k\varphi_l m_{ikl}}{d_{ijkl}}, z_k = \frac{\varphi_i\varphi_j\varphi_l m_{ijl}}{d_{ijkl}} \\ z_l = \frac{\varphi_i\varphi_j\varphi_k m_{ijk}}{d_{ijkl}}, z_p = 0,\ p\ \neq i,j,k,l \in S_1\Bigg) \tag{26}$$

where

$$d_{ijkl} = \varphi_i\varphi_j\varphi_k m_{ijk} + \varphi_i\varphi_j\varphi_l m_{ijl} + \varphi_i\varphi_k\varphi_l m_{ikl} + \varphi_j\varphi_k\varphi_l \frac{m_{jkl}}{d_{ijl}}$$

for all $ijkl \in S_4$, that has $C(n,4)$ components. Again, $\mathbf{z} \in \mathbf{e}_{ijkl}$ if all the components of type m_{ijk} are negative.

Although we cannot go much beyond edges of type $\mathbf{e}_{ijk}$ a similar reasoning applies for members of the boundary of Δ^{n-1} in which there are $n-4$, $n-5$, $n-1$ zero components of $\mathbf{z}$.

Since n is finite this suggests a conjecture over the existence of stationary distributions belonging to the interior of Δ^{n-1}:

[19] This will be clear in the $n = 4$ case outlined in the appendix.

Conjecture. *There is one interior stationary distribution* $\hat{\mathbf{z}}$ *only if there is a non-empty subset of* Φ *such that there are stationary distributions belonging to all the* $C(n, n-1) = n$ *edges of type* $\mathbf{e}_{12\ldots i}$, $i \in S_1$, $\{\mathbf{e}^*_{12\ldots j}\}_{j=1}^n$ *and they are all Nash equilibria.*

An interior distribution, if it exists, is a Nash equilibrium because $\mathbf{u}(\hat{\mathbf{z}}) = \overline{u}(\hat{\mathbf{z}})$.

Then the set of stationary equilibria of the replicator dynamics is

$$\mathcal{Z}^* = \{\{\mathbf{e}_i\}_{i\in S_1}, \{\mathbf{e}^*_{ij}\}_{ij\in S_2}, \ldots, \{\mathbf{e}^*_{ijk}\}_{ijk\in S_3}, \ldots, \{\mathbf{e}^*_{123\ldots i}\}_{123\ldots i\in S_{n-1}}, \hat{\mathbf{z}}\},$$

if n is finite and the sequences up until $n-1$ have only Nash equilibria on all the edges. If there is not an equilibrium point for every edge, or there is an equilibrium point which is not a Nash equilibrium, for edges with indices S_i, then there are no stationary distributions over edges indexed S_{i+1}, for $i = 2, \ldots, n-1$. In this case stationary equilibrium set is

$$\mathcal{Z}^* = \{\{\mathbf{e}_i\}_{i\in S_1}, \ldots, \{\mathbf{e}^*_{i_1 1_2\ldots i_N}\}_{i_1 1_2\ldots i_N\in S_N}\},$$

where N is the maximum number of vertices which as connected by the edges in which there is a fixed point $\{\mathbf{z} \in \partial(\Delta^{n-1}) : \mathbf{F}(\mathbf{z}) = \mathbf{0}\}$.

Therefore, stationary distributions are always multiple. Given an initial conjecture at time $t = 0$, $\mathbf{z}(0)$, the dynamics of the conjecture game will generate convergence to a unique asymptotic distribution, $\mathbf{z}^* \in \mathcal{Z}^*$. If the initial conjecture does not belong to a particular set of measure zero (i.e., if it is not a bifurcation point) there will be asymptotic convergence to one of the pure strategy profiles $\mathbf{e}_i$ depending on the specific value of vector φ. We also show there is a close connection between the global dynamic properties (i.e., the basin of attraction of $\mathbf{e}_i$) and the probability of asymptotic convergence to $\mathbf{e}_i$, for any given initial mixed conjecture $\mathbf{z}(0)$.

5.2 *Ex post* consistency in the n conjecture game

Propositions 3 and 4 describe the set of all possible Nash equilibria in the conjecture game. Since most of these are mixed-strategy equilibria (for n larger than 3), what can we say about the *ex post* consistency of these possible equilibria? The first point to note is that insofar as mixed equilibria are on the edges of the simplex, they involve subsets of k conjectures with strictly positive probabilities and the complementary $(n-k)$ conjectures being played with zero probability. This enables us to place a lower bound on the probability of consistency:

Observation. With n distinct conjectures, if z^* is a Nash-equilibrium and there are are k strategies with strictly positive probabilties, then

$$PC\,(z^*) \in [\frac{1}{k}, 1)$$

Clearly, only the pure-strategy equilibria can be fully consistent with $PC = 1$.

5.3 The 3 conjecture case.

In section 3 we considered a specific 3 conjecture example: here, we consider the general 3 conjecture case. It provides an example of the n conjecture case and is also sufficiently simple for us to undertake a full characterization of all equilibria. With $n = 3$, the indices sets are $S_1 = \{1, 2, 3\}$ and $S_2 = \{12, 13, 23\}$, and the conjecture space is $\Phi = \{\varphi_1, \varphi_2, \varphi_3\}$, with $0 \leq \varphi_1 < \varphi_2 < \varphi_3 \leq 2$. The candidate probability profiles are $\mathbf{z} = \Delta^2$. Probability profiles associated to pure strategies belong to set $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. The three boundary profiles located at one of the three edges of the simplex, excluding the vertices, are $\{\mathbf{e}_{12}, \mathbf{e}_{13}, \mathbf{e}_{23}\}$, where $\mathbf{e}_{12} = \{\mathbf{z} \in \Delta^2 : z_3 = 0\}$, $\mathbf{e}_{13} = \{\mathbf{z} \in \Delta^2 : z_2 = 0\}$ and $\mathbf{e}_{23} = \{\mathbf{z} \in \Delta^2 : z_1 = 0\}$, correspond to boundary mixed strategies which are distinguished from the interior mixed strategies $\mathbf{z} \in \mathrm{Int}(\Delta^2)$.

Applying equation (23) we have relative profitability differences $m_{12(3)} < 0$, $m_{23(1)} < 0$ and

$$m_{13(2)} = m_{13(2)}(\varphi_1, \varphi_2, \varphi_3) \equiv m_{13} - (m_{12} + m_{23}), \tag{27}$$

which has an ambiguous sign. Using the previous idea for the partition of Φ into sets $\Phi^2_{s_2}$, we have the subset of Φ in which all the three profitability differences are non-positive $\Phi_0 = \{\ \varphi \in \Phi : m_{13(2)}(\varphi) \leq 0\}$, and the subset in which there is one positive profitability difference $\Phi_1 = \{\ \varphi \in \Phi : m_{13(2)}(\varphi) > 0\}$. Observe that

$$m_{13(2)} \leq 0 \Leftrightarrow \varphi_3^3(\varphi_2-\varphi_1)-\varphi_1^3(\varphi_3-\varphi_2)-9\varphi_1\varphi_2\varphi_3(\varphi_1+\varphi_3)-6\varphi_1\varphi_3(\varphi_3\varphi_1+\varphi_2^2)-\varphi_2^2(\varphi_1^2+\varphi_3^2) \geq 0$$

where the first term is positive and all the others are negative.

Function $m_{13(2)}$ defined over $(\varphi_1, \varphi_2, \varphi_3)$, determines the number of stationary states and whether or not they are Nash equilibria as given by the following proposition [20]

Proposition 5 (Stationary profiles). *(a) If $\varphi \in \Phi_1$ then there are six stationary probability profiles*

$$\mathcal{Z}^* = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}^*_{12}, \mathbf{e}^*_{13}, \mathbf{e}^*_{23}\},$$

where

$$\mathbf{e}^*_{12} \equiv \left(\frac{\varphi_2}{\varphi_1 + \varphi_2}, \frac{\varphi_1}{\varphi_1 + \varphi_2}, 0\right), \mathbf{e}^*_{13} \equiv \left(\frac{\varphi_3}{\varphi_1 + \varphi_3}, 0, \frac{\varphi_1}{\varphi_1 + \varphi_3}\right), \mathbf{e}^*_{23} \equiv \left(0, \frac{\varphi_3}{\varphi_2 + \varphi_3}, \frac{\varphi_2}{\varphi_2 + \varphi_3}\right)$$

which are all Nash equilibria, except for $\mathbf{e}^*_{13}$;

[20] Observe that $\varphi_1 = 0$ (i.e. the Bertrand case) implies $m_{13(2)} < 0$.

(b) if $\varphi \in int(\Phi_0)$ and $\varphi_1 > 0$ then there are seven stationary population profiles:

$$\mathcal{Z}^* = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_{12}^*, \mathbf{e}_{13}^*, \mathbf{e}_{23}^*, \hat{\mathbf{z}}\},$$

where

$$\mathbf{z}^* = \hat{\mathbf{z}} \equiv \left(\frac{\varphi_2\varphi_3 m_{23} m_{23(1)}}{d_{123}}, \frac{\varphi_1\varphi_3 m_{13} m_{13(2)}}{d_{123}}, \frac{\varphi_1\varphi_2 m_{12} m_{12(3)}}{d_{123}} \right) \in int(\Delta^2) \tag{28}$$

where $d_{123} \equiv \varphi_1\varphi_2 m_{12} m_{12(3)} + \varphi_1\varphi_3 m_{13} m_{13(2)} + \varphi_2\varphi_3 m_{23} m_{23(2)} < 0$. All stationary probability profiles are Nash equilibria. If $\varphi_1 = 0$ then $\mathcal{Z}^* = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_{23}^*\}$ and there are all Nash equilibria;
(c) if $\varphi \in \partial(\Phi_0)$ then there are six stationary profiles as described in (a) and they are all Nash equilibria.

`Figure 2 here`

Clearly, the precise value of $m_{13(2)}$ is crucial in determining whether we have 1, 3 or 4 mixed equilibria. We can take (31), and assume the three strategies are equally spaced, by setting $\varphi_2 = 1$, and plot a bifurcation diagram in the space of conjectures (φ_1, φ_3) in Figure 2. If we set different values for ϕ the diagram will not change qualitatively. There are two bifurcation loci $\{(\varphi_1, \varphi_3) : \varphi_1 = 0\}$ and $\{(\varphi_1, \varphi_3) : m_{13(2)}(\varphi_1, \varphi_3) = 0\}$. The last set divides the conjecture space into two: there is a small area where φ_1 is less than 0.066, for which $m_{13(2)} < 0$. Most of the parameter space results in $m_{13(2)} > 0$. This means that in the 3×3 example the vast majority of combinations of conjectures will yield only two boundary mixed equilibria with a third mixed non-Nash boundary stationary point. In this sense, the interior mixed equilibrium is a rarity, and requires one firm to have a very competitive conjecture ($\varphi_1 < 0.066$). We can now see that the example in section 3.2 where $\varphi_1 = 1$ and $\varphi_3 = 2$ is firmly in the region where $m_{13(2)} > 0$, so that there are only three stationary points on the edges and no interior equilibria..

`Figure 3 here`

We can think about the strategy profiles in terms of the unit-simplexes, depicted in Figure 3 [21] for the cases not corresponding to bifurcations. The pure-strategy equilibria are on the vertices: the most competitive is in the bottom right corner ($z_1 = 1$), the least competitive at the top ($z_3 = 1$). All those equilibria are sinks.

[21] We have used Dynamo by Sandholm et al. (2010), to draw the phase diagrams.

When $m_{13(2)} > 0$ we have the generic simplex as depicted in Figure 3(a). There are three partially mixed stationary states: one on each of the edges between the three vertices. There are two stationary profiles $\mathbf{e}_{12}^*$ and $\mathbf{e}_{23}^*$ that involve conjecture φ_2 with each of the other two conjectures: these are both Nash equilibria and are saddle-points with the stable manifold belonging to the interior of the simplex. Note that $\mathbf{e}_{12}^*$ is closer to $\mathbf{e}_1$ than $\mathbf{e}_2$: this follows because to equate the payoffs, the more competitive conjecture needs a higher probability of meeting itself. Likewise, $\mathbf{e}_{23}^*$ is closer to $\mathbf{e}_2$ than $\mathbf{e}_3$. There is a third stationary state that is not a Nash-equilibrium, which is a mixed profile with $z_3 = 0$, and is a source.

When $m_{13(2)} < 0$ we have the simplex as depicted in Figure 3(b). In this case, there are two differences: first, the stationary mixed profile with $z_2 = 0$ becomes a saddle-point stable Nash-equilibrium, and secondly an additional interior mixed stationary state emerges, which is also a Nash-equilibrium but is a source. Again the stable manifold associated to boundary equilibria for $z_2 = 0$ belongs to the interior of the simplex. When $m_{13(2)} \downarrow 0$, the mixed equilibria gets closer to the interior mixed equilibrium in $\mathbf{e}_{13}$, and when $m_{13(2)} = 0$ the two merge. In this case, the boundary mixed equilibrium is a Nash-equilibrium. This property however does not show up when $m_{13(2)} \uparrow 0$. This corresponds to a local bifurcation point of the fold type.

The next proposition formally assert that the local dynamics at the stationary points displayed at the two phase diagrams hold generically:

Proposition 6 (Local dynamics). *The pure strategy Nash equilibria,* $\mathbf{e}_1$, $\mathbf{e}_2$ *and* $\mathbf{e}_3$, *are always sinks, and the two boundary mixed Nash equilibria* $\mathbf{e}_{12}^*$ *and* $\mathbf{e}_{23}^*$ *are always saddle points. In addition:*
(a) if $m_{13(2)} > 0$ *then the boundary non-Nash stationary state* $\mathbf{e}_{13}^*$ *is a saddle point with a one-dimensional stable manifold.*
(b) If $m_{13(2)} < 0$ *two cases can occur: if* $\varphi_1 > 0$ *then the boundary mixed Nash equilibrium* $\mathbf{e}_{13}^*$ *is a saddle point and the interior mixed Nash equilibrium* $\hat{\mathbf{z}}$ *is a source. If* $\varphi_1 = 0$, *then* $\mathbf{e}_{12}^*$, $\mathbf{e}_{13}^*$ *and* $\hat{\mathbf{z}}$ *merge with* $\mathbf{e}_1$, *which is a fold bifurcation;*
(c) if $m_{13(2)} = 0$ *then there is a local fold bifurcation at equilibrium point* $\mathbf{z}^* = \mathbf{e}_{13}^* = \hat{\mathbf{z}}$.

The dimension of the stable manifold reduces by one dimension if we consider the reduced two-dimensional ODE equation.

Since a stationary point can only be an ESS if it is a sink, (Taylor and Jonker, 1978, p. 150), it follows that all of the mixed equilibria are not ESS and the probability of consistency is strictly less than 1 : for the edge equilibria e_{12}^* and e_{23}^* and $PC \in [\frac{1}{2}, 1)$. If $\varphi_1 > 0$, then all three pure equilbria are ESS and $PC = 1$. Hence we can conlcude that $PC = 1$ only for equilibria with $k = 1$.

5.4 Global dynamics in the 3 conjecture case.

Phase diagrams in Figure 3 displays not only local dynamics but also global dynamics, for the two generic cases. It shows there is a heteroclinic network which is joining all the stationary points of the replicator dynamics. Heteroclinic orbits exist in the intersection of the stable manifold associated to one equilibrium point to the unstable manifold associated to another equilibrium point. Therefore, there are heteroclinic orbits linking sinks to saddle points, in the interior of the simplex, and saddle points to sinks, in the boundaries of the simplex. This implies that the heteroclinic orbits in the interior of the simplex separate the basins of attractions of the three pure strategy Nash equilibria

$$\mathcal{B}_i \equiv \left\{ \mathbf{y} \in \Delta^2 : \lim_{t\to\infty} \mathbf{z}(t,\mathbf{y}) = \mathbf{e}_i \right\} \ , \ i = 1, 2, 3.$$

Proposition 7 (Global dynamics). *(a) Let $m_{13(2)} > 0$. Then there is a heteroclinic network composed of 8 heteroclinic orbits: six heteroclinic orbits join the boundary mixed equilibria to the pure strategy equilibria, and two heteroclinic orbits join the steady state on edge $\mathbf{e}_{13}$ to the boundary mixed equilibria on the edges $\mathbf{e}_{12}$ and $\mathbf{e}_{23}$. These two heteroclinics separate the boundaries for the basins of attraction $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$ associated to the three pure strategies equilibria $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$.*
(b) Let $m_{13(2)} < 0$. If $\varphi_1 > 0$, then there is a heteroclinic network composed of 9 heteroclinic orbits, six heteroclinic orbits join the boundary mixed equilibria to the pure strategy equilibria, and three heteroclinic orbits join the interior mixed equilibrium, $\hat{\mathbf{z}}$ to the boundary mixed equilibrium on the edges $\mathbf{e}_{13}$, $\mathbf{e}_{12}$ and $\mathbf{e}_{23}$. These three heteroclinics separate the basins of attraction $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$ associated to the three pure strategy equilibria $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$. If $\varphi_1 = 0$, then there is a heteroclinic network composed of 5 heteroclinic orbits, three heteroclinic orbits joining $\mathbf{e}_1$ to $\mathbf{e}_2$, $\mathbf{e}_3$ and $\mathbf{e}_{23}$, and two heteroclinic orbits joining $\mathbf{e}_{23}$ to $\mathbf{e}_2$ and $\mathbf{e}_3$. The heteroclinic orbit between $\mathbf{e}_1$ and $\mathbf{e}_{23}$ separates the basins of attraction $\mathcal{B}_2$ and $\mathcal{B}_3$. Basin $\mathcal{B}_1$ is empty.

From the above proposition, we can see that:

(a) Let $\varphi_1 > 0$. The three pure-strategy equilibria are asymptotically stable, have fully consistent conjectures ($PC = 1$) and are ESS. These properties hold for φ_2 and φ_3 even when $\varphi_1 = 0$

(b) None of the non-pure strategy fixed-points are asymptotically stable, ESS or have fully consistent conjectures. The probability of consistency is at least 0.5 and strictly less than 1.

(c) The non-pure strategy fixed points are either unstable sources or saddle-stable with a stable manifold of dimension 1.

We can see that the non-pure-strategy stationary states are on the borders of the basins of attraction of the three pure-strategy equilibrium conjectures. The boundaries of the basins are heteroclinic orbits which connect the "mixed" stationary states with each other and with the pure strategy equilibria. Hence, there is a sense in which the non-pure strategy stationary points are "fragile": the replicator dynamics on the two dimensional simplex results in a stable manifold of at most one dimension. This means that these stationary states are not locally stable, since a small deviation will almost always lead away to one of the three pure-strategy sinks. Whilst they are fragile in this sense, they are also *essential* to the model, as with their heteroclinic orbits they define the boundaries between the basins of attraction of the pure-strategy sinks.

5.4.1 Equilibrium Selection with three conjectures.

Clearly, the evolutionary dynamics imply that the initial position determines which equilibrium comes about in the long-run. However, what can we say about the size of the basins of attraction? In particular, what determines the size of the basins of attraction? Does the Pareto dominant equilibrium have a larger basin of attraction? If we consider each point in the unit simplex to be equally likely, we can interpret the size of the basin as the *probability* of the corresponding equilibrium. In the general case of $m_{13(2)} > 0$, we are able to approximate each basin under the assumption that the heteroclinic orbits are all linear, so that the three basins can be broken down into triangles using Proposition 8 [22]. Let us call $\mathsf{P}(\mathbf{e}_i)$ the (approximate) probability of asymptotic convergence to pure strategy $\mathbf{e}_i$. Our approximations are:

$$\begin{aligned} \mathsf{P}(\mathbf{e}_1) &= \frac{\varphi_1^2}{(\varphi_1+\varphi_2)(\varphi_1+\varphi_3)}, \\ \mathsf{P}(\mathbf{e}_2) &= \frac{\varphi_2}{\varphi_1+\varphi_3}\left(\frac{\varphi_1}{\varphi_1+\varphi_2}+\frac{\varphi_3}{\varphi_2+\varphi_3}\right), \\ \mathsf{P}(\mathbf{e}_3) &= \frac{\varphi_3^2}{(\varphi_1+\varphi_3)(\varphi_2+\varphi_3)}. \end{aligned}$$

Since $\varphi_3 > \varphi_1$ we can see that the basin of attraction of the Pareto dominant equilibrium $\mathbf{e}_3$ is larger than that of the most competitive equilibrium $\mathbf{e}_1$: $\mathsf{P}(\mathbf{e}_3)/\mathsf{P}(\mathbf{e}_1) = \varphi_3^2/\varphi_1^2 > 1 \Rightarrow \mathsf{P}(\mathbf{e}_3) > P(\mathbf{e}_1)$. However, the relative size of $\mathsf{P}(\mathbf{e}_2)$ is more complicated to understand. To take the simplest case, if $\varphi_1 = 0$ (Bertrand), then the exact probabilities are

[22] We show in the proof of Proposition 8 that there is not an analytic first integral for the replicator dynamic system and therefore the separatices of the basins of attraction cannot be determined analytically. However, we also prove that they will be close to the straight lines connecting equilibria, which allows us to approximate of the dimension of basins of attraction for the pure strategies.

$$\begin{aligned}
\mathsf{P}(\mathbf{e}_3) &= \frac{\varphi_3}{(\varphi_2+\varphi_3)} \\
\mathsf{P}(\mathbf{e}_2) &= \frac{\varphi_2}{(\varphi_2+\varphi_3)}
\end{aligned}$$

and so we have the unambiguous ranking $\mathsf{P}(\mathbf{e}_3) > \mathsf{P}(\mathbf{e}_2)$. In general, however, it is more than possible to have $\mathsf{P}(\mathbf{e}_3) < \mathsf{P}(\mathbf{e}_2)$. In particular, as $\varphi_1 \to \varphi_3$, then $\mathsf{P}(\mathbf{e}_1)$ and $\mathsf{P}(\mathbf{e}_3)$ both tend to 1/4 whilst $\mathsf{P}(\mathbf{e}_2)$ tends to 1/2. If we take another example with $\varphi_1 = 1$ (Cournot) and $\varphi_3 = 2$ (Joint profit maximization), then $\mathsf{P}(\mathbf{e}_2) > \mathsf{P}(\mathbf{e}_3)$ for $\varphi_2 > 1.155$.

Hence we cannot claim that the Pareto dominant equilibrium will have the largest basin. The reason for this is due to the payoff function of the rent-extraction game. The most competitive CV will do worst. The middle conjecture does better than the most cooperative when both are played against the most competitive. Likewise, the middle conjecture does better than the most competitive when both are played against the most cooperative. This was why (for $m_{13(2)} > 0$) the stationary point with only the most and least cooperative conjectures is not a Nash equilibrium and is an unstable source. The result is that the basin of attraction for the intermediate conjecture φ_2 is often (although certainly not always) larger than the most cooperative conjecture φ_3.

There is also a key difference between the intermediate conjecture and the most cooperative. If we look at Figure 3(a), we can see that for $m_{13(2)} > 0$, if the population starts close to $\mathbf{e}_{23}$ but in $\mathcal{B}_2$, there are equilibrium paths that start with an almost zero share for φ_2 but tend asymptotically to $\mathbf{e}_2$ where the share is 1. The cooperative conjecture φ_3 however requires a minimum share to start off with. The lowest starting share for φ_3 occurs on the boundary of $\mathcal{B}_3$ at stationary point $\mathbf{e}_{13}$: its initial share must be just above that at $\mathbf{e}_{13}$ for it to be able to get to $\mathbf{e}_1$. So long as $\varphi_1 > 0$, this is bounded away from zero.

6 Conclusion.

In this paper, we have taken the rent-extraction model with conjectural variations and applied a social learning model to it in the form of the evolutionary replicator dynamics. CVs become more (less) common as their average payoffs are above (below) average. The endpoints of this evolutionary process can be both pure-strategy equilibria and mixed-strategy equilibria. However, the mixed-equilibria are either unstable, or have limited saddle-path stability and hence are not ESS. The pure-strategy equilibria have large basins of attraction, and their boundaries are separated by heteroclinic orbits that connect the mixed-equilbria. Whilst all the pure-strategy equilibrium conjectures are consistent conjectures, the standard definition of consistency does not apply to mixed equilibria.

We develop the concept of the *ex post* probability of consistency PC which generalizes the conventional notion of consistency ($PC = 1$) to apply to mixed-strategies. Whilst only pure-strategy equilibria can be consistent, we are able to find a simple lower bound for the *ex-post* probability of consistency for mixed equilibria which is the reciprocal of the number of strategies played with a strictly positive probability.

In our analysis of the rent-extraction game, we do not find a tendency for all of the rent to be extracted in the evolutionary long-run. The rent is only fully dissipated when there are competitive (Bertrand) conjectures, which are not ESS and will have no basin of attraction. The Pareto-optimum of zero-rent dissipation is not only possible, but also has a significant basin of attraction which in certain cases may be the biggest. However, in the three conjecture case we have analyzed, the intermediate conjecture may well have the larger basin of attraction than the Pareto-optimum. In the general n conjecture case, we find that there are two types if stationary equilibria other than the pure-strategy "corner" equilibria: there may be at most one "interior" stationary point which will be a Nash-equilibrium in which all conjectures are played with a strictly positive probability, which is a source. There are then many "edge" stationary points with less than n strategies played with a strictly positive probability. These can be either Nash-equilibria with a stable manifold of dimension

There are very many shortcomings to using simple evolutionary dynamics: they certainly are not a literal real-time representation of how agents behave. However, the long-run dynamics give us a guide as to what social institutions and individual strategies might emerge over time. In the case of the rent-extraction model they have given us an insight into what types of behavior and associated beliefs will succeed in earning above average payoffs, and in so doing become more common.

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A The 2 conjecture case

In this case $\mathbf{z} = (z_1, z_2)$ and $\Phi = (\varphi_1, \varphi_2)$ for $0 \leq \varphi_1 < \varphi_2 \leq 2$. There ODE (21) has three stationary equilibria, all belonging to the simplex Δ: two equilibria in the vertices $\mathbf{z}_1^* = \mathbf{e}_1 = (1,0)$, and $\mathbf{z}_2^* = \mathbf{e}_2 = (0,1)$ one interior equilibrium $\mathbf{z}_3^* = \hat{\mathbf{z}} = (\varphi_2/(\varphi_1+\varphi_2), \varphi_1/(\varphi_1+\varphi_2))$. They all belong to the simplex Δ and verify the Nash property: $\mathbf{u}(\mathbf{e}_1) - \overline{u}(\mathbf{e}_1) = (0, -\varphi_1 m_{12}) < \mathbf{0}$, $\mathbf{u}(\mathbf{e}_2) - \overline{u}(\mathbf{e}_2) = (-\varphi_2 m_{12}, 0) < \mathbf{0}$, and $\mathbf{u}(\hat{\mathbf{z}}) - \overline{u}(\hat{\mathbf{z}}) = \mathbf{0}$.

The spectra for the Jacobian $\mathbf{F}(\mathbf{z}^*)$, evaluated at the three stationary equilibria are:

$$\sigma(\mathbf{e}_1) = \left\{ \ -\frac{\varphi_1}{4} \ , -\varphi_1 m_{12} \right\} \ , \ \sigma(\mathbf{e}_2) = \left\{ \ -\frac{\varphi_2}{4} \ , -\varphi_2 m_{12} \right\}$$

and

$$\sigma(\hat{\mathbf{z}}) = \left\{ \frac{\varphi_1+\varphi_2}{\varphi_1\varphi_2} \ m_{12}, -\frac{\varphi_1+\varphi_2}{\varphi_1\varphi_2} \ \left(\frac{1}{2} \ - m_{12} \right) \right\}$$

Then equilibria $\mathbf{e}_1$ and $\mathbf{e}_2$ are sinks and equilibrium $\hat{\mathbf{z}}$ is a saddle point.

Global dynamics properties are easier to obtain if we observe that, because the system should lie on the manifold $z_1 + z_2 = 1$, the planar ODE (21) has an equivalent dynamic behavior as a reduced scalar ODE

$$\dot{z}_1 = z_1 \left(u_1(z_1, 1-z_1) - \overline{u}(z_1, 1-z_1) \right) = z_1(1-z_1) \left(z_1 - \frac{\varphi_2}{\varphi_1+\varphi_2} \right) m_{12}$$

together with $z_2 = 1 - z_1$. We readily conclude that three types of dynamics can occur, if the initial conjecture profile $\mathbf{z}(0) = (z_1(0), z_2(0))$ is a mixed conjecture: (1) if $1/2 < \varphi_2/(\varphi_1+\varphi_2) < z_1(0) < 1$ then $\lim_{t\to\infty} \mathbf{z}(t) = \mathbf{e}_1$, (2) if $\varphi_2/(\varphi_1+\varphi_2) < z_1(0) < 1$ then $\lim_{t\to\infty} \mathbf{z}(t) = \mathbf{e}_2$ and (3) if $z_1(0) = \varphi_2/(\varphi_1+\varphi_2)$ then $\mathbf{z}(t) = \hat{\mathbf{z}}$ for any $t \in [0, \infty)$. Then the relative dimensions of the the basins of attraction of the two pure strategy profiles, allows us to determine the probabilities of convergence to each one of them, given any initial conjecture: $\mathsf{P}(\mathbf{e}_1) = \varphi_1/(\varphi_1+\varphi_2) < 1/2$ and $\mathsf{P}(\mathbf{e}_2) = \varphi_2/(\varphi_1+\varphi_2) < 1/2$. As $\varphi_2 > \varphi_1$ then $\mathsf{P}(\mathbf{e}_2) > \mathsf{P}(\mathbf{e}_1)$.

B The 4 conjecture case

Now the set of conjectures is four dimensional $\varphi = \{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$ where we assume that $0 < \varphi_1 < \varphi_2 < \varphi_3 < \varphi_4 \leq 4$ and $\mathbf{z} = (z_1, z_2, z_3, z_4)$[23].

In this case we have $S_2 = 6$ profitability differentials of type $m_{ij} \in [0, 1/4)$ and 12 relative differentials of type $m_{ij(k)}$, which are negative, with the exception of four, $m_{13(2)}$, $m_{14(2)}$, $m_{14(3)}$ and $m_{24(3)}$. Then $k_2 = 4$. We also have $S_3 = 4$ functions of type m_{ijk} (see equation (24)) which

[23] We do not consider the case $\varphi_1 = 0$, which should be obvious.

have all ambiguous signs. We can define two partitions over Φ, $\{\ \Phi^2_0, \dots, \Phi^2_4\}$ and $\{\ \Phi^3_0, \dots, \Phi^3_4\}$ such that $\cup_{s_2=0}^4 \Phi^2_{s_2} = \cup_{s_3=0}^4 \Phi^3_{s_3} = \Phi$ where $\Phi^2_{s_2}$ ($\Phi^3_{s_3}$) is the set of values of φ such that there are s_2 (s_3) relative differentials $m_{ij(k)}$ (m_{ijk}) which are positive. As we already saw, stationary equilibrium profiles and their characteristics as regards the multiplicity of equilibria, Nash property and local stability properties, depend on the local intersection of sets $\Phi^2_{s_2}$ and $\Phi^2_{s_3}$.

In this section we apply proposition 5 specifically we derive analytical conditions for the existence of equilibria in edges $\mathbf{e}_{ijk}$ and present a bifurcation analysis for the case in which we have Nash equilibria.

Proposition 8 (**Stationary profiles**). *The conjecture space can be partitioned into as much as 32 (possibly empty) subsets. For a given set of conjectures $\varphi \in \Phi$ there is a set of stationary equilibrium distributions, $\mathcal{Z}^*$. Every $\mathcal{Z}^*$ has the following general properties: (a) it contain between 10 and 15 stationary elements, $\mathbf{z}^*$; (b) every $\mathcal{Z}^*$ contains all S_1 vertices $\mathbf{e}_i$, everyone of each verifies the Nash property; (c) every $\mathcal{Z}^*$ contains points in all S_2 edges $\mathbf{e}_{ij}$; equilibria $\mathbf{e}^*_{12}$, $\mathbf{e}^*_{23}$, $\mathbf{e}^*_{23}$ and $\mathbf{e}^*_{34}$ always verify the Nash property and $\mathbf{e}^*_{13}$, $\mathbf{e}^*_{14}$ and $\mathbf{e}^*_{24}$ only verify the Nash property if there are particular equilibria in edges $\mathbf{e}_{ijk}$: $\mathbf{e}^*_{13}$ is a Nash equilibrium if equilibrium $\mathbf{e}^*_{123}$ exists, $\mathbf{e}^*_{14}$ is a Nash equilibrium if equilibria $\mathbf{e}^*_{124}$ and $\mathbf{e}^*_{134}$ exist, and $\mathbf{e}^*_{24}$ is a Nash equilibrium if equilibrium $\mathbf{e}^*_{234}$ exists; (d) if $\mathcal{Z}^*$ contains equilibria in any of the edges, $\mathbf{e}*_{ijk} \in \mathbf{e}_{ijk}$) then this equilibrium verifies the Nash property; (e) half of the 32 steady state sets, if $\varphi \in \Phi^3_0 \cup \Phi^3_4$, contain the interior steady state $\mathbf{z}$ which is always a Nash. equilibrium*

That is, we have 32 different combinations of multiple steady states, combining 10 to 15 stationary distributions. Just to illustrate we consider the two extreme cases [24]. If $\varphi \in \Phi^2_4 \cap \cup_{s_3=1}^3 \Phi^3_{s_3}$ then $\mathcal{Z}^* = \left\{\ \{\mathbf{e}_i\}_{i\in S_1}, \{\mathbf{e}^*_{ij}\}_{ij\in S_2}\right\}$ has 10 elements which are Nash equilibria, with the exception of $\mathbf{e}^*_{13}$, $\mathbf{e}^*_{14}$ and $\mathbf{e}^*_{24}$. If $\varphi \in \Phi^2_4\ \cap\ \Phi^3_0 \cup \Phi^3_4$ then $\mathcal{Z}^* = \left\{\ \{\mathbf{e}_i\}_{i\in S_1}, \{\mathbf{e}^*_{ij}\}_{ij\in S_2}, \hat{\mathbf{z}}\right\}$. If $\varphi \in \Phi^2_0\ \cap\ \Phi^3_0 \cup \Phi^3_4$ then $\mathcal{Z}^* = \left\{\ \{\mathbf{e}_i\}_{i\in S_1}, \{\mathbf{e}^*_{ij}\}_{ij\in S_2}, \{\mathbf{e}^*_{ijk}\}_{ijk\in S_3}, \hat{\mathbf{z}}\right\}$ has 15 elements and they are all Nash equilibria.

`Figure 4 here`

There is an very large number of combinations, therefore we took $\phi_1 = -0.99$ and $\phi_4 = 0.9$ and consider variations $\phi_2 \in (-0.99, \phi_3)$ and $\phi_3 \in (\phi_2, 0.9)$ just for illustration purposes, which is depicted in Figure 4. This figure has three panels: the left diagram refers to partition $\{\Phi^2_{s_2}\}$, the center panel to partition $\{\Phi^3_{s_3}\}$ and in the right panel there is a superimposition of the other two, with a legend indicating which stationary profiles would exist. As $\phi_2 < \phi_3$ the top left part of the graph is the only relevant.

[24]See the Appendix for a complete presentation of all the stationary distributions.

On the left panel we observe that all sets Φ^2_0 to Φ^2_4 are non-empty. These sets are divided by combinations of the parameters such that the ambiguously signed differences $m_{ij(k)}$ are equal to zero. The subsets associated with the maximum and minimum number of stationary distributions on edges $\mathbf{e}_{ijk}$ are smaller and are located in the extremes of the graph: subset Φ^2_0, associated with four distributions on those edges is on the left, four values of ϕ_2 very close to -1 and subset Φ^2_4, associated with zero distributions on those edges is on the right, four values of ϕ_2 close to 1.

On the center panel we have subsets $\Phi^3_{s_3}$. As, with the values of parameters in this example we always have $m_{123} < 0$ and $m_{234} < 0$ then sets Φ^3_4 and Φ^3_3 are empty. Set Φ^3_0 is again very small and holds for values of ϕ_2 close to -1.

Then, in the right panel we present the multiple stationary distributions $\mathcal{Z}^*$ that exist: $\mathcal{Z}_1$, $\mathcal{Z}_4$, $\mathcal{Z}_{11}$ (in this case $\mathbf{e}^*_{13}$ is Nash), $\mathcal{Z}_{14}$ ($\mathbf{e}^*_{14}$ is Nash), $\mathcal{Z}_{20}$ ($\mathbf{e}^*_{14}$ is Nash), $\mathcal{Z}_{23}$ ($\mathbf{e}^*_{13}$ and $\mathbf{e}^*_{14}$ are Nash), $\mathcal{Z}_{26}$ ($\mathbf{e}^*_{23}$ and $\mathbf{e}^*_{14}$ are Nash), $\mathcal{Z}_{27}$ ($\mathbf{e}^*_{13}$ and $\mathbf{e}^*_{14}$ are Nash), $\mathcal{Z}_{30}$ ($\mathbf{e}^*_{23}$ and $\mathbf{e}^*_{14}$ are Nash), $\mathcal{Z}_{32}$ (all Nash).

Proposition 9 (Local dynamics). *We assume that $\varphi_1 > 0$. For any value of $\varphi \in \Phi$, the pure strategy Nash equilibria, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and $\mathbf{e}_4$, are always sinks, with a four-dimensional local stable manifold, and boundary mixed Nash equilibria $\mathbf{e}^*_{12}$, $\mathbf{e}^*_{23}$ and $\mathbf{e}^*_{34}$ are saddle points, in which the local stable manifold is of dimension three. In addition:*
(a) if $\varphi \in \Phi^2_0$ then $\mathbf{e}^*_{13}$, $\mathbf{e}^*_{14}$ and $\mathbf{e}^*_{24}$ are also saddle points with three-dimensional local stable manifolds; if $\varphi \in \Phi^2_4$ then $\mathbf{e}^*_{13}$, and $\mathbf{e}^*_{24}$ have two-dimensional local stable manifolds and $\mathbf{e}^*_{14}$ has a one-dimensional stable manifold; if $\varphi \in \Phi^2_1 \cup \Phi^2_2 \cup \Phi^2_3$ the local stable manifolds of $\mathbf{e}^*_{13}$, and $\mathbf{e}^*_{24}$ can be three or two-dimensional and the local stable manifolds of $\mathbf{e}^*_{14}$, can be three, two or one-dimensional;
(b) if $\varphi \in \Phi^3_0$ then $\mathbf{e}^*_{123}$, $\mathbf{e}^*_{124}$, $\mathbf{e}^*_{134}$, and $\mathbf{e}^*_{234}$ are also saddle points with two-dimensional local stable manifolds; if $\varphi \in \Phi^3_4$ their stable manifolds are all of dimension one; if $\varphi \in \Phi^3_1 \cup \Phi^3_2 \cup \Phi^3_3$ their local stable manifolds can be of dimension one or two. (c) if there is an interior distribution profile, $\mathbf{z}$, that is, if $\varphi \in \Phi^3_0$, its local stable manifold is of dimension one.

Again there is a close connection between the Nash property and stability properties for distributions $\mathbf{e}^*_{ij}$ and existence of probabilities $\mathbf{e}^*_{ijk}$, related to the signs of $m_{ij(k)}$, and Nash property and stability properties for $\mathbf{e}_{ijk}$ and existence of probabilities $\hat{\mathbf{z}}$ related to the signs of m_{ijk}.

For example, if there is an equilibrium $\mathbf{e}^*_{13}$ profile in $\mathcal{Z}^*$ if it is not Nash then the local stable manifold is two-dimensional and there is not an equilibrium profile $\mathbf{e}^*_{123}$. However, if it is a Nash equilibrium, then its local stable manifold is three-dimensional and there is an equilibrium profile $\mathbf{e}^*_{123}$ as well.

Again equilibria in the edges or in the interior belong to the boundaries between the basins of attraction for the equilibria in the vertices. There is a generic convergence for one pure strategy equilibrium, depending on the initial guess $\mathbf{z}(0)$. As in the 3×3 case, the difference between cases in which the set of steady states $\mathcal{Z}^*$ does not contain equilibrium profiles in the edges $\mathbf{e}_{ijk}$, as regards , cases in which it contains contains distributions in those edges or in the interior, is that in the later case some vertices will become unreachable. In particular, the vertex associated to the more competitive strategy $\mathbf{e}_1$ will have a larger basin of attraction in the latter case. As we see in Figure 3 this case occurs only if the value for ϕ_1 is very close to -1 and ϕ_2 is also very close to $\phi_{1.}$.

C proofs.

C.1 Proof of Proposition 2

Proof. As $\mathbf{F}(\mathbf{e}_i) = \mathbf{0}$ and $\mathbf{e}_i^\top \mathbf{1} = 1$ then $\mathbf{z}^* = \mathbf{e}_i \in \mathcal{Z}^*$, for any $i \in S_1$. As $u_i(\mathbf{e}_i) = \overline{u}(\mathbf{e}_i)$ and $u_j(\mathbf{e}_i) - \overline{u}(\mathbf{e}_i) = -\varphi_i m_{ij} < 0$ for any $j \neq i \neq 1 \in S_1$, and $u_j(\mathbf{e}_1) - \overline{u}(\mathbf{e}_1) \leq 0$. Then every $\mathbf{e}_i$ is a Nash equilibrium. The spectrum associated to the Jacobian $\mathbf{F}^{'}(\mathbf{z})$ of equation (21), evaluated locally at $\mathbf{z} = \mathbf{e}_i$, is

$$\sigma(\mathbf{e}_i) \equiv \sigma(\mathbf{F}^{'}(\mathbf{e}_i)) = \{ \; -\varphi_i m_{ij} \}_{j \in S_1} \; , \; i \in S_1,$$

where $m_{ii} = 1/4$. For every $i \neq 1 \in S_1$, or if $i = 1$ and $\varphi_1 > 0$, all the eigenvalues are negative, and if $i = 1$ and $\varphi_1 = 0$ then the spectrum $\sigma(\mathbf{e}_1)$ is equal to zero. This means that all stationary distributions in the vertices of Δ^{n-1} are locally sinks (the eigenvalues are real and the local stable manifold has dimension n), except for the case in which $\varphi_1 = 0$. In this case $\mathbf{e}_1$ is a fold bifurcation point. □

C.2 Proof of Proposition 3

Proof. By direct calculation, we determine the fixed points, located at hyperplanes $\mathbf{e}_{ij}$ belonging to $\partial \Delta^{n-1}$:

$$\mathbf{z}^* = \mathbf{e}^*_{ij} = \left\{ z_i = \frac{\varphi_j}{\varphi_i + \varphi_j}, \;\; z_j = \frac{\varphi_i}{\varphi_i + \varphi_j}, \; z_k = 0, \; k \neq i, j \in S_1 \right\} \; \in \mathbf{e}_{ij} \; ij \in S_2.$$

(22)

Their number is equal to the cardinality of S_2, $C(n, 2)$.

Every stationary equilibrium distribution $\mathbf{e}^*_{ij}$ verifies:

$$u_k(\mathbf{e}^*_{ij}) - \overline{u}(\mathbf{e}^*_{ij}) = \begin{cases} 0, & \text{if } k = i, \text{ or } k = j \in S_1 \\ \frac{\varphi_i \varphi_j}{\varphi_i + \varphi_j} m_{ij(k)}, & \text{if } k \neq i, j \ \in S_1. \end{cases}$$

The Nash property holds for a particular pair $ij \in S_2$ if all differences $m_{ij(k)}$, for k running along S_1, are non-positive. Using our previous definitions and notations, we have potentially k_{ij} positive $m_{ij(k)}$ for every $ij \in S_2$ and k_2 pairs ij in which the Nash property does not hold (i.e., there are k_2 pairs such that $m_{ij(k)} > 0$). Consider our definition of sets $\{\Phi^2_{s_2}\}^{k_2}_{s_2=0}$. If Φ_0 is non-empty, then there is a subset of measure different from zero of Φ such that all $n(n-1)/2$ equilibria $\mathbf{e}^*_{ij}$ are Nash. If Φ_1 is non-empty then there is a subset of parameter values such that there are $n(n-1)/2 - 1$ Nash equilibria and one non-Nash equilibrium. If Φ_2 is non-empty then there is a subset of parameter values such that there are $n(n-1)/2 - 2$ Nash equilibria and two non-Nash equilibrium. Generally, if set Φ_{s_2}, for $s_2 = 0, \ldots, k_2$ is non-empty there are $n(n-1)/2 - s_2$ Nash equilibria and s_2 non-Nash equilibrium. In the boundary between two sets $\Phi^2_i \cap \Phi^2_j$ there is a profitability differential which is equal to zero.

The spectrum of the Jacobian of equation (21) evaluated at $\mathbf{e}^*_{ij}$ is

$$\sigma(\mathbf{e}^*_{ij}) = \left\{ \frac{\varphi_i \varphi_j}{\varphi_i + \varphi_j} \ m_{ij}, \frac{\varphi_i \varphi_j}{\varphi_i + \varphi_j} \ \left(m_{ij} - \frac{1}{2} \right), \frac{\varphi_i \varphi_j}{\varphi_i + \varphi_j} \ m_{ij(k)}, \right. \\ \left. k \neq i, j \in S_1 \right\}, \text{ for all } ij \in S_2 \quad (29)$$

All the eigenvalues are real numbers, the first eigenvalue is always positive, the second is always negative and among the other $n-2$ eigenvalues there are k_{ij} eigenvalues which may be potentially positive for any $\varphi \in \Phi$. Therefore $\mathbf{e}^*_{ij}$ is locally a generalized saddle point in which the dimension of the stable manifold is equal to $n-1$ or is smaller. Observe that the expressions for $u_k(\mathbf{e}^*_{ij}) - \overline{u}(\mathbf{e}^*_{ij})$ for $k \neq i, j \in S_1$ and for the last $n-2$ eigenvalues are exactly the same. Then the same partition $\{\Phi^2_{s_2}\}$ is an unfolding of parameter space associated to the dimension of the stable manifold: if $\varphi \in \text{int}(\Phi^2_{s_2})$ then the local stable manifold associated to $\mathbf{e}^*_{ij}$ is of dimension $n - 1 - s_{ij}$, for $s_{ij} \in \{0, \ldots, k_{ij}\}$. In the boundary between two sets $\Phi^2_i \cap \Phi^2_j$ there is a fold bifurcation and the spectrum $\sigma(\mathbf{e}^*_{ij})$ contain at least one zero eigenvalue. $\square$

C.3 Proof of Proposition 4

Proof. If we set $n-3$ elements of the vector $\mathbf{z}$ equal to zero, there are fixed points $n(n-1)(n-2)/6$ of $\mathbf{F}(\mathbf{z}) = \mathbf{0}$ of the form

$$\mathbf{z} = \left(\ z_i = \frac{\varphi_j\varphi_k m_{jk} m_{jk(i)}}{d_{ijk}}, \quad z_j = \frac{\varphi_i\varphi_k m_{ik} m_{ik(j)}}{d_{ijk}}, z_k = \frac{\varphi_j\varphi_j m_{ij} m_{ij(k)}}{d_{ijk}}, z_l = 0, \ l \ \neq i, j, k \in S_1 \right)$$

where $d_{ijk} = \varphi_i\varphi_j m_{ij} m_{ij(k)} + \varphi_i\varphi_k m_{ik} m_{ik(j)} + \varphi_j\varphi_j m_{jk} m_{jk(i)}$ These vectors clearly verify the summing up condition for a distribution $\mathbf{z}^\top \mathbf{1}_{S_2} = 0$. Another condition for $\mathbf{z} \in \mathbf{e}_{ijk}$, is that all the three components of $\mathbf{z}$ are positive, which in this case holds if all $m_{ij(k)}$ have the same sign. However, given the combinations of the indices involved, there is is always one term $m_{ij(k)}$ which has an ambiguous sign, and all the other are always negative. This means that in general $\mathbf{z}$ is not a distribution. As there are, overall $C(n,3)$ relative differences in which the sign of $m_{ij(k)}$ are ambiguous (for all combinations without repetitions of ijk), then we can define a partition $\{\ \Phi^2_{s_3}\}_{s_3=0}^{C(n,3)}$ over the set Φ such that there ate $s_3 = 0, \dots C(n,3)$ profitability differences $m_{ij(k)}$ which are non-positive. If the set $\Phi^2_{s_3}$ is non-empty, then there are $C(n,3) - s_3$ stationary equilibrium distributions $\mathbf{z}^* = \mathbf{e}^*_{ijk}$, belonging to the edges linking vertices i, j and k.

For the stationary equilibrium distribution belonging to $\partial(\Delta^{n-1})$, $\mathbf{z}^* = \mathbf{e}^*_{ijk}$ we can check if it is a Nash-equilibria by evaluating the sign of vector $\mathbf{u}(\mathbf{e}^*_{ijk}) - \overline{u}(\mathbf{e}^*_{ijk})$. For every $\mathbf{e}^*_{ijk}$ we obtain,

$$u_l(\mathbf{e}^*_{ijk}) - \overline{u}(\mathbf{e}^*_{ijk}) = \begin{cases} 0, & \text{if } l = i, \text{ or } l = j, \text{or } l = k \in S_1 \\ -\frac{\varphi_i\varphi_j\varphi_l}{\varphi_i+\varphi_j+\varphi_l}\frac{m_{ijl}}{d_{ijl}}, & \text{if } l \neq i, j, k \ \in S_1. \end{cases}$$

where $d_{ijk} < 0$ if $\mathbf{z}^* = \mathbf{e}^*_{ijk}$ (otherwise it will not be a distribution), and m_{ijk} is given by equation (24). This allows us to define a further partition over Φ, $\{\ \Phi^3_{s_3}\}_{s_3=0}^{C(n,3)}$ associated to the number of functions m_{ijk} which are positive. Therefore stationary equilibrium distributions which have the Nash property belong to the inclusions of related subsets $\Phi^2_{s_3}$ and $\Phi^3_{s_3}$ such that $\mathbf{z}^* = \mathbf{e}^*_{ijk}$ and all the associated m_{ijk} are non-positive. That is, there is a maximum of $C(n,3) - s_3$ stationary distributions , $\mathbf{e}^*_{ijk}$, which are Nash equilibria. □

C.4 Proof of Proposition 5

Proof. The ODE (21) has seven stationary equilibria where six equilibria are in the simplex Δ^2, for any value of the conjectures. There are three equilibria in the vertices of the simplex, $\mathbf{z}_1 = \mathbf{e}_1$, $\mathbf{z}_2 = \mathbf{e}_2$, $\mathbf{z}_3 = \mathbf{e}_3$, and other three equilibria in the edges $\mathbf{z}_4 = \mathbf{e}^*_{12} = (\varphi_2/(\varphi_1 + \varphi_2), \varphi_1/(\varphi_1 + \varphi_2), 0) \in \mathbf{e}_{12}$, $\mathbf{z}_5 = \mathbf{e}^*_{13} = (\varphi_3/(\varphi_1 + \varphi_3), 0, \varphi_1/(\varphi_1 + \varphi_3)) \in \mathbf{e}_{13}$, and $\mathbf{z}_6 = \mathbf{e}^*_{23} =$

$(0, \varphi_3/(\varphi_2+\varphi_3), \varphi_2/(\varphi_2+\varphi_3)) \in \mathbf{e}_{23}$. There is another fixed point

$$\mathbf{z}_7 = \left(\frac{\varphi_2\varphi_3}{\varphi_2+\varphi_3} \left(\frac{m_{23}m_{23(1)}}{\hat{d}} \right), \frac{\varphi_1\varphi_3}{\varphi_1+\varphi_3} \left(\frac{m_{13}m_{13(2)}}{\hat{d}} \right), \frac{\varphi_1\varphi_2}{\varphi_1+\varphi_2} \left(\frac{m_{12}m_{12(3)}}{\hat{d}} \right) \right)$$

where

$$\hat{d} \equiv \varphi_1\varphi_2 m_{12}m_{12(3)} + \varphi_1\varphi_3 m_{13}m_{13(2)} + \varphi_2\varphi_3 m_{23}m_{23(1)}.$$

We readily see that $\mathbf{z}_7^\top \mathbf{1} = 1$. However, applying equation (23) we have, for any value of the parameters in Φ,

$$m_{12(3)} \equiv m_{12} - m_{13} - m_{23} < 0 \tag{30}$$

$$m_{13(2)} \equiv m_{13} - m_{12} - m_{23} \gtreqless 0 \tag{31}$$

$$m_{23(1)} \equiv m_{23} - m_{12} - m_{13} < 0 \tag{32}$$

then, in general this stationary equilibria may not belong to the simplex. A necessary and sufficient condition for $\mathbf{z}_7$ to be in the unit simplex is that $m_{13(2)} \leq 0$. However, if $m_{13(2)} = 0$ then $\mathbf{z}_7 = \mathbf{e}_{13}$ which means that this is a singularity. If $m_{13(2)} < 0$ then $\mathbf{z}_7 = \hat{\mathbf{z}} \in \text{int}(\Delta^2)$ and $\mathbf{z}_7$ is a stationary equilibrium of the RD. If $m_{13(2)} > 0$ then $\mathbf{z}_7$ will not be a stationary RD equilibrium point (although it is a fixed point of $\mathbf{F}(\mathbf{z}) = \mathbf{0}$).

The Nash property for equilibria can be assessed from the vectors $\mathbf{u}(\mathbf{z}^*) - \overline{u}(\mathbf{z}^*)$ where z^* is an equilibrium of the RD. Then $\mathbf{u}(\mathbf{e}_1) - \overline{u}(\mathbf{e}_1) = (0, -\varphi_1 m_{12}, -\varphi_1 m_{13})$, $\mathbf{u}(\mathbf{e}_2) - \overline{u}(\mathbf{e}_2) = (-\varphi_2 m_{12}, 0, -\varphi_2 m_{23})$, $\mathbf{u}(\mathbf{e}_3) - \overline{u}(\mathbf{e}_3) = (-\varphi_3 m_{13}, -\varphi_3 m_{23}, 0)$ $\mathbf{u}(\mathbf{e}_{12}^*) - \overline{u}(\mathbf{e}_{12}^*) = (0, 0, \varphi_1\varphi_2 m_{12(3)}/(\varphi_1 + \varphi_2))$, $\mathbf{u}(\mathbf{e}_{13}^*) - \overline{u}(\mathbf{e}_{13}^*) = (0, \varphi_1\varphi_3 m_{13(2)}/(\varphi_1+\varphi_3), 0)$, $\mathbf{u}(\mathbf{e}_{23}^*) - \overline{u}(\mathbf{e}_{23}^*) = (\varphi_2\varphi_3 m_{23(1)}/(\varphi_2+\varphi_3))$, and $\mathbf{u}(\mathbf{z}_7) - \overline{u}(\mathbf{z}_7) = \mathbf{0}$. All the vectors are non-positive, except for the case of $\mathbf{e}_{13}^*$: if $m_{13(2)} \leq 0$ then it is Nash equilibrium, if $m_{13(2)} > 0$ it is not.

Then if $\varphi \in \text{int}(\Phi_0)$ then there are seven equilibrium points in the simplex, and all of them are Nash equilibria. If $\varphi \in \partial(\Phi_0)$ there are six equilibrium points all or them are Nash equilibrium, although equilibria $\mathbf{e}_{13}^*$ and $\mathbf{z}_7$ coalesce. $\varphi \in \Phi_1$ then there are six equilibrium points, all belonging to the edges and vertices of the simplex, in which all are Nash equilibria, except for the case of $\mathbf{e}_{13}^*$. □

C.5 Proof of Proposition 6

Proof. The spectra of the Jacobian of 3-dimensional ODE, (21), $\mathbf{F}'(\mathbf{z}^*)$, evaluated at the equilibrium points are:

1. for the equilibria in the vertices, the Jacobian

$$\sigma(\mathbf{e}_1) = \left\{ -\frac{\varphi_1}{4}, -\varphi_1 m_{12}, -\varphi_1 m_{13} \right\}$$

$$\sigma(\mathbf{e}_2) = \left\{ -\frac{\varphi_2}{4}, -\varphi_2 m_{12}, -\varphi_2 m_{23} \right\}$$

$$\sigma(\mathbf{e}_3) = \left\{ -\frac{\varphi_3}{4}, -\varphi_3 m_{13}, -\varphi_3 m_{23} \right\}.$$

are all negative and real, for every equilibria, then all the vertices are sinks;

2. for the equilibria located in the edges, the Jacobian has the eigenvalues

$$\sigma(\mathbf{e}^*_{12}) = \left\{ \frac{\varphi_1 \gamma_2}{\varphi_1 + \varphi_2} \left(m_{12} - \frac{1}{2} \right), \frac{\varphi_1 \varphi_2}{\varphi_1 + \varphi_2} m_{12}, \; \frac{\varphi_1 \varphi_2}{\varphi_1 + \varphi_2} m_{12(3)} \right\}$$

$$\sigma(\mathbf{e}^*_{13}) = \left\{ \frac{\varphi_1 \varphi_3}{\varphi_1 + \varphi_3} \left(m_{13} - \frac{1}{2} \right), \frac{\varphi_1 \varphi_3}{\varphi_1 + \varphi_3} m_{13}, \; \frac{\varphi_1 \varphi_3}{\varphi_1 + \varphi_3} m_{13(2)} \right\}$$

$$\sigma(\mathbf{e}_{23}) = \left\{ \frac{\varphi_2 \varphi_3}{\varphi_2 + \varphi_3} \left(m_{23} - \frac{1}{2} \right), \frac{\varphi_2 \varphi_3}{\varphi_2 + \varphi_3} m_{23}, \; \frac{\varphi_2 \varphi_3}{\varphi_2 + \varphi_3} m_{23(1)} \right\}$$

the first eigenvalue is negative and the second is positive for the three equilibria. However, while for for equilibria $\mathbf{e}^*_{12}$ and $\mathbf{e}^*_{23}$ the third eigenvalue is negative as well, the last eigenvalue for $\mathbf{e}^*_{13}$ has the same sign as $m_{13(2)}$. Then, all the equilibria in the edges are generalized saddle points: at $\mathbf{e}^*_{12}$ and $\mathbf{e}^*_{23}$ the local stable manifold has dimension two and at $\mathbf{e}^*_{13}$ the local stable manifold has dimension two if $\varphi \in \text{int}(\Phi_0)$ and has dimension one if $\varphi \in \Phi_1$, at it is a fold bifurcation point if $\varphi \in \partial\Phi_0$;

3. the spectrum of the Jacobian for the last equilibria are:

$$\sigma(\hat{\mathbf{z}}) = \left\{ -\frac{\varphi_1 \varphi_2 \varphi_3}{4\hat{d}} \left(m_{12} m_{12(3)} + m_{13} m_{13(2)} + m_{23} m_{23(1)} + 2 m_{12} m_{13} m_{23} \right), \right.$$
$$\left. -\frac{\varphi_1 \varphi_2 \varphi_3}{4\hat{d}} m_{12} m_{13} m_{23} \left(1 + \Delta^{1/2} \right), -\frac{\varphi_1 \varphi_2 \varphi_3}{4\hat{d}} m_{12} m_{13} m_{23} \left(1 - \Delta^{1/2} \right) \right\} \quad (33)$$

where the discriminant is $\Delta \equiv 1 + m_{12(3)} m_{13(2)} m_{23(1)} / m_{12} m_{13} m_{23}$. If $\varphi \in \text{int}(\Phi_0)$, which is the same condition for $\mathbf{z}_7 \in \text{int}(\Delta^2)$ then $\hat{d} < 0$ and the discriminant is positive, then all the three eigenvalues are real and and two are positive and one is negative, and the interior point is a saddle with a one-dimensional stable manifold.

□

C.6 Proof of Proposition 7.

Proof. Again, as in the 2×2 case it is convenient to study the 3-dimensional ODE, (21) on the simplex by the equivalent 2-dimensional projection of the dynamic system (21) into, v.g., the space (z_1, z_3) by the relation $z_2 = 1 - z_1 - z_3$,

$$\dot{z}_1 = z_1 \left[(1 - z_1 - z_3)\left(-\varphi_2 m_{12} + m_{12}(\varphi_1 + \varphi_2)z_1 + m_{23}(\varphi_2 + \varphi_3)z_3\right) + \right.$$
$$\left. + m_{13} z_3((\varphi_1 + \varphi_3)z_1 - \varphi_3)\right] \quad (34)$$

$$\dot{z}_3 = z_3 \left[(1 - z_1 - z_3)\left(-\varphi_2 m_{23} + m_{12}(\varphi_1 + \varphi_2)z_1 + m_{23}(\varphi_2 + \varphi_3)z_3\right) + \right.$$
$$\left. + m_{13} z_1((\varphi_1 + \varphi_3)z_3 - \varphi_1)\right]. \quad (35)$$

We obtain equivalent results if we study the local dynamics from the ones we derived in the proof of proposition 6 The steady states of this reduced system are obtained as $\mathbf{z}^* = (z_1^*, 1 - z_1^* - z_3^*, z_3^*)$, $\mathbf{J}(z_1^*, z_3^*)$. The spectra for the Jacobian evaluated at the different steady states, as the system is 2-dimensional, contains eigenvalues which are equal to the last two we have obtained in the proof of proposition 6 .

In order to characterize global dynamics we have to completely describe phase diagram. Zeeman (1980) presents a complete classification of the phase portraits of the replicator dynamics (RD) for the 3×3 case. They include the phase portraits in Figure 3. These phase portraits suggest there is a heteroclinic network which is not an heteroclinic cycle as in some RD games (e.g., the rock - scissor -paper RD game). The heteroclinic network consists of six heteroclinic orbits joining equilibria on the edges, $\mathbf{e}_{12}$, $\mathbf{e}_{13}$ and $\mathbf{e}_{23}$, to equilibria on the vertices of the simplex, $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$, and two interior heteroclinic orbits joining steady state $\mathbf{e}_{13}^*$ to steady states $\mathbf{e}_{12}^*$, and $\mathbf{e}_{23}^*$, respectively. Those two heteroclinic orbits separate the boundaries of the basins of attractions in the interior of Δ^2. Next we prove that the phase diagram in Figure **??**, for case $m_{13(2)} > 0$, is generic. The proof for case $m_{13(2)} < 0$ is similar.

Heteroclinic orbits lay along invariants of type $\{(z_1, z_2, z_3) : F(z_1, z_2, z_3) = \text{constant}\}$. The best way to prove that their layout as in Figure 3 is generic, is to determine a first integral of the RD system (21) explicitly. If we transform the 3-dimensional RD system (21) into a 2-dimensional Lotka-Volterra (LV) system, using a well known transformation (see (Hofbauer and Sigmund, 1998, p.77), and if we draw upon the relevant literature on the determination of the first integrals of the LV equation, e.g. Llibre and Valls (2007), we find that there is not an analytic first integral for the associated LV equation.

Therefore we resort to a heuristic proof by using equations (34)-(35).

The orbits along the edges of the simplex lay along invariants $\{(z_1, z_3) : z_1 = 0\}$, $\{(z_1, z_3) : 1 - z_1 - z_3 = 0\}$, and $\{(z_1, z_3) : z_3 = 0\}$. In the first case the dynamics is given by $\dot{z}_1 = 0$ and $\dot{z}_3 = z_3(1 - z_3)(z_3 - z_3(\mathbf{e}_{23}))m_{23}(\varphi_2 + \varphi_3)$, which means that $z_1(t) = 0$, for any $t \geq 0$ and if $0 < z_3(0) < z_3(\mathbf{e}^*_{23})$ ($1 > z_3(0) > z_3(\mathbf{e}^*_{23})$) then $z_3(t)$ will converge asymptotically to vertex $\mathbf{e}_2$ ($\mathbf{e}_3$). In the second case the dynamics is given by $\dot{z}_3 = -\dot{z}_1$ and $\dot{z}_1 = z_1(1 - z_1)(z_1 - z_1(\mathbf{e}_{13})m_{13}(\varphi_1 + \varphi_3)$, which means that $z_3(t) = 1 - z_1(t)$, for any $t \geq 0$, and if $0 < z_1(0) < z_1(\mathbf{e}^*_{13})$ ($1 > z_1(0) > z_1(\mathbf{e}^*_{13})$) then the trajectory $z_1(t)$ will converge asymptotically to vertex $\mathbf{e}_1$ ($\mathbf{e}_3$). In the last case, the dynamics is given by $\dot{z}_3 = 0$ and $\dot{z}_1 = z_1(1 - z_1)(z_1 - z_1(\mathbf{e}_{12})m_{12}(\varphi_1 + \varphi_2)$, which means that $z_3(t) = 0$, for $t \geq 0$, and if $0 < z_1(0) < z_1(\mathbf{e}^*_{12})$ ($1 > z_1(0) > z_1(\mathbf{e}^*_{12})$) then the trajectory $z_1(t)$ will converge asymptotically to vertex $\mathbf{e}_1$ ($\mathbf{e}_2$).

Next, we prove that, if $m_{13(2)} > 0$ there are two heteroclinic orbits inside a closed trapping area $\mathbb{T}$ which is bounded by equilibrium points $\mathbf{e}_2$, $\mathbf{e}^*_{12}$, $\mathbf{e}^*_{13}$, and $\mathbf{e}^*_{23}$:

$$\mathbb{T} = \left\{ (z_1, z_3) : \; z_1 \geq 0, \; z_3 \geq 0, \; \frac{-\varphi_2 + (\varphi_1 + \varphi_3) z_1}{\varphi_3 - \varphi_2} \leq z_3 \leq \frac{\varphi_2(1 - z_1) + \varphi_1}{\varphi_2 + \varphi_3} \right\}.$$

As we already saw, all the points belonging to segments of the edges $\mathbf{e}_2$-$\mathbf{e}_{12}$, and $\mathbf{e}_2$-$\mathbf{e}_{23}$, converge to the pure strategy steady state $\mathbf{e}_2$. By continuity, given any initial point close to the those edges, the replicator dynamics will also imply asymptotic convergence to $\mathbf{e}_2$. However, all the dynamics starting close to the straight line $\mathbf{e}_{12} - \mathbf{e}_{13}$, passing through points $\mathbf{e}^*_{13}$ and $\mathbf{e}^*_{12}$, will exit $\mathbb{T}$ and converge to vertex $\mathbf{e}_1$. Similarly, all the dynamics starting close to the straight line $\mathbf{e}_{13} - \mathbf{e}_{23}$, passing through points $\mathbf{e}^*_{13}$ and $\mathbf{e}^*_{23}$, will exit $\mathbb{T}$ and converge to vertex $\mathbf{e}_3$. This means that there are two separatrices belonging to the interior of $\mathbb{T}$: the first is in the intersection of the stable manifold associated to the saddle point $\mathbf{e}^*_{12}$ with the unstable manifold associated with the source $\mathbf{e}^*_{13}$, $W^s(\mathbf{e}^*_{12}) \cap W^u(\mathbf{e}^*_{13})$; and the second is in the intersection of the stable manifold associated to the saddle point $\mathbf{e}^*_{23}$ with the unstable manifold associated with the source $\mathbf{e}^*_{13}$, $W^s(\mathbf{e}^*_{23}) \cap W^u(\mathbf{e}^*_{13})$.

Those separatrices partition $\mathbb{T}$ in three subsets, where there will be asymptotic convergence towards one and only one of the three vertices of the simplex. The subset associated to $\mathbf{e}_2$ is the basin os attraction of $\mathbf{e}_2$ and the other subsets of $\mathbb{T}$ belong to the basins of attraction of $\mathbf{e}_1$ or $\mathbf{e}_3$. The separatrices are invariants and contain all the heteroclinic orbits converging asymptotically to either $\mathbf{e}^*_{12}$ or $\mathbf{e}^*_{23}$.

To prove this formally, observe that the formal expression of line $\mathbf{e}_{13} - \mathbf{e}_{12}$ is

$$z_3 = -\frac{\varphi_2}{\varphi_3 - \varphi_2} + \frac{\varphi_1 + \varphi_3}{\varphi_3 - \varphi_2} z_1 : \; (z_1, z_3) \in \mathbb{T}$$

which is positively sloped. Evaluating equations (34)-(35) along that line we get

$$\dot{z}_1 = z_1 \frac{(\varphi_1+\varphi_2)(\varphi_1+\varphi_3)}{(\varphi_3-\varphi_2)^2} \, (z_1 - z_1(\mathbf{e}_{13}^*))\,(z_1 - z_1(\mathbf{e}_{12}^*))\left(\varphi_2 m_{12(3)} + \varphi_3 m_{13(2)}\right) > 0$$

$$\dot{z}_3 = \frac{(\varphi_1+\varphi_2)^2(\varphi_1+\varphi_3)}{(\varphi_3-\varphi_2)^3} \, (z_1 - z_1(\mathbf{e}_{13}^*))\,(z_1 - z_1(\mathbf{e}_{12}^*))\left[\left(\varphi_2 m_{12(3)} + \varphi_3 m_{13(2)}\right) z_1 + \frac{2\varphi_2\varphi_3 m_{23}}{\varphi_1+\varphi_2}\right] < 0.$$

Then the vector field is negatively sloped along line $\mathbf{e}_{13} - \mathbf{e}_{12}$ and, locally, z_1 is increasing and z_3 is decreasing towards $\mathbf{e}_1$. Therefore, the global dynamics involves exit from trapping area $\mathbb{T}$.

The formal expression of line $\mathbf{e}_{13} - \mathbf{e}_{23}$ is

$$z_3 = \frac{\varphi_2}{\varphi_2+\varphi_3} - \frac{\varphi_2-\varphi_1}{\varphi_2+\varphi_3} z_1 : \ (z_1, z_3) \in \mathbb{T}$$

which is negatively sloped. Evaluating equations (34)-(35) along that line we get

$$\dot{z}_1 = z_1\,(z_1 - z_1(\mathbf{e}_{13}^*))\,\frac{(\varphi_1+\varphi_3)}{(\varphi_2+\varphi_3)} \left(\varphi_2 m_{23(1)}(z_1-1) + \varphi_1 m_{13(2)} z_1\right) < 0$$

and

$$\dot{z}_3 = z_1\,(z_1 - z_1(\mathbf{e}_{13}^*))\,\frac{(\varphi_1+\varphi_3)}{(\varphi_2+\varphi_3)^2} \,\left(\varphi_2(1-z_1) + \varphi_1 z_1\right)\left(\varphi_2 m_{23(1)} + \varphi_1 m_{13(2)}\right) > 0$$

which implies that the slope of the vector field along the line $\mathbf{e}_{13} - \mathbf{e}_{23}$ is also negative. But the slope of the vector field is steeper than the slope of line $\mathbf{e}_{13} - \mathbf{e}_{23}$, because

$$\left.\frac{dz_3}{dz_1}\right|_{(\dot{z}_1,\dot{z}_3)} - \left.\frac{dz_3}{dz_1}\right|_{e_{13}-e_{12}} = \frac{2\varphi_1\varphi_2 m_{12}}{(\varphi_2+\varphi_3)\left(\varphi_2 m_{23(1)}(1-z_1) + \varphi_1 m_{13(2)} z_1\right)} < 0.$$

Then, locally, z_1 is decreasing and z_3 is increasing towards $\mathbf{e}_3$. Therefore, the global dynamics also involves exit from trapping area $\mathbb{T}$.

At last, we prove that the separatrices lay inside the trapping area $\mathbb{T}$. First, recall that the stable eigenspaces, $E^s(\mathbf{e}_{12}^*)$ and $E^s(\mathbf{e}_{23}^*)$, are tangent to the stable manifolds associated to the two boundary saddle points, $\mathbf{e}_{12}^*$ and $\mathbf{e}_{23}^*$. This means that the heteroclinic trajectories are asymptotically tangent to the stable eigenspaces. The stable eigenspace associated to $\mathbf{e}_{12}^*$ has slope

$$\left.\frac{dz_3}{dz_1}\right|_{E^s(e_{12}^*)} = \frac{(\varphi_1+\varphi_2)(m_{13}+m_{23})}{(m_{13}-m_{23})\varphi_3 - (m_{13}+m_{23})\varphi_2}$$

which is positive if $(\varphi_1^2+\varphi_3^2)\varphi_2 - 2\varphi_1\varphi_3^2 > 0$, and is negative or vertical otherwise. In the second case, the separatrix is clearly inside $\mathbb{T}$. However, the separatrix is also inside $\mathbb{T}$ when it is positively sloped, because it is steeper than line $\mathbf{e}_{13} - \mathbf{e}_{12}$, as, in this case,

$$\left.\frac{dz_3}{dz_1}\right|_{E^s(e_{12}^*)} - \left.\frac{dz_3}{dz_1}\right|_{e_{13}-e_{12}} = \frac{\varphi_3(\varphi_1+\varphi_2)(\varphi_1+\varphi_3)^2}{(\varphi_1^2+\varphi_3^2)\varphi_2 - 2\varphi_1\varphi_3^2} > 0$$

The stable eigenspace associated to $\mathbf{e}_{23}^*$ is also negatively sloped, because

$$\left.\frac{dz_3}{dz_1}\right|_{E^s(e_{23}^*)} = -\frac{(m_{12}-m_{13})\varphi_1 + (m_{12}+m_{13})\varphi_2}{(\varphi_2+\varphi_3)(m_{12}+m_{13})} < 0.$$

Again, it is inside $\mathbb{T}$ because it is steeper than line $\mathbf{e}_{13} - \mathbf{e}_{23}$ as

$$\left.\frac{dz_3}{dz_1}\right|_{E^s(e_{23}^*)} - \left.\frac{dz_3}{dz_1}\right|_{e_{13}-e_{23}} = -2\frac{\varphi_1 m_{12}}{(\varphi_1+\varphi_3)(m_{12}+m_{13})} > 0.$$

In the case of $m_{13(2)} < 0$, the proof is similar in the case of $\varphi_1 > 0$. If $\varphi_1 = 0$, we have the additional factor of the merging of equilibria (Proposition 6) and resultant fold bifurcation and disappearance of $\mathcal{B}_1$. □

C.7 Proof of Proposition 8.

Proof. The ODE system (21) for the case in which both $\mathbf{z}$ and φ are four dimensional, has fifteen stationary points, in which five may not be located in the simplex Δ^3 for any value of $\varphi \in \Phi$. First, there are ten stationary points which are always located in the vertices of the simplex $\mathbf{z}_1 = \mathbf{e}_1$, $\mathbf{z}_2 = \mathbf{e}_2$, $\mathbf{z}_3 = \mathbf{e}_3$, and $\mathbf{z}_4 = \mathbf{e}_4$ and six stationary points are located in the hyperplanes belonging to $\partial\Delta^4$ connecting two vertices, $\mathbf{z}_5 = \mathbf{e}_{12}^*$, $\mathbf{z}_6 = \mathbf{e}_{13}^*$, $\mathbf{z}_7 = \mathbf{e}_{14}^*$, $\mathbf{z}_8 = \mathbf{e}_{23}^*$, $\mathbf{z}_9 = \mathbf{e}_{24}^*$ and $\mathbf{z}_{10} = \mathbf{e}_{34}^*$, where

$$\begin{aligned}
\mathbf{e}_{12}^* &= \left(\frac{\varphi_2}{\varphi_1+\varphi_2}, \frac{\varphi_1}{\varphi_1+\varphi_2}, 0, 0\right) \in \mathbf{e}_{12}, &\qquad \mathbf{e}_{13}^* &= \left(\frac{\varphi_3}{\varphi_1+\varphi_3}, 0, \frac{\varphi_1}{\varphi_1+\varphi_3}, 0\right) \in \mathbf{e}_{13},\\
\mathbf{e}_{14}^* &= \left(\frac{\varphi_4}{\varphi_1+\varphi_4}, 0, 0, \frac{\varphi_1}{\varphi_1+\varphi_4}\right) \in \mathbf{e}_{14}, &\qquad \mathbf{e}_{23}^* &= \left(0, \frac{\varphi_3}{\varphi_2+\varphi_3}, \frac{\varphi_2}{\varphi_2+\varphi_3}, 0\right) \in \mathbf{e}_{23},\\
\mathbf{e}_{24}^* &= \left(0, \frac{\varphi_4}{\varphi_2+\varphi_4}, 0, \frac{\varphi_2}{\varphi_2+\varphi_4}\right) \in \mathbf{e}_{24}, &\qquad \mathbf{e}_{34}^* &= \left(0, 0, \frac{\varphi_4}{\varphi_3+\varphi_4}, \frac{\varphi_3}{\varphi_3+\varphi_4}\right) \in \mathbf{e}_{34}
\end{aligned}$$

Second, the next five stationary points may not be belong to the simplex. Among them, there are four that can be potentially located in $\partial\Delta^3$, in hyperplanes joining three vertices:

$$\begin{aligned}
\mathbf{z}_{11} &= \left(\frac{\varphi_2\varphi_3 m_{23}, m_{23(1)}}{d_{123}}, \frac{\varphi_1\varphi_3 m_{13}, m_{13(2)}}{d_{123}}, \frac{\varphi_1\varphi_2 m_{12}, m_{12(3)}}{d_{123}}, 0\right), \\
\mathbf{z}_{12} &= \left(\frac{\varphi_2\varphi_4 m_{24}, m_{24(1)}}{d_{124}}, \frac{\varphi_1\varphi_4 m_{14}, m_{14(2)}}{d_{124}}, 0, \frac{\varphi_1\varphi_2 m_{12}, m_{12(4)}}{d_{124}}\right), \\
\mathbf{z}_{13} &= \left(\frac{\varphi_3\varphi_4 m_{34}, m_{34(1)}}{d_{134}}, 0\frac{\varphi_1\varphi_4 m_{14}, m_{14(3)}}{d_{134}}, \frac{\varphi_1\varphi_3 m_{13}, m_{13(4)}}{d_{134}}\right), \\
\mathbf{z}_{14} &= \left(0, \frac{\varphi_3\varphi_4 m_{34}, m_{34(2)}}{d_{234}}, \frac{\varphi_2\varphi_4 m_{24}, m_{24(3)}}{d_{234}}, \frac{\varphi_2\varphi_3 m_{23}, m_{23(4)}}{d_{234}}\right),
\end{aligned}$$

where, using the expressions for $m_{12(3)}$, $m_{13(2)}$ and $m_{23(1)}$ and signs already derived in (30), (31) and (32), and the general rule presented in equation (23),

$$\begin{aligned}
&m_{12(4)} \equiv m_{12} - m_{14} - m_{24} < 0, && m_{14(2)} \equiv m_{14} - m_{12} - m_{24} \gtreqless 0 \\
&m_{24(1)} \equiv m_{24} - m_{12} - m_{14} < 0, && m_{13(4)} \equiv m_{13} - m_{14} - m_{34} < 0 \\
&m_{14(3)} \equiv m_{14} - m_{13} - m_{34} \gtreqless 0, && m_{34(1)} \equiv m_{34} - m_{13} - m_{14} < 0 \\
&m_{23(4)} \equiv m_{23} - m_{24} - m_{34} < 0, && m_{24(3)} \equiv m_{24} - m_{23} - m_{34} \gtreqless 0 \\
&m_{34(2)} \equiv m_{34} - m_{23} - m_{24} < 0,
\end{aligned}$$

and

$$\begin{aligned}
d_{123} &\equiv \varphi_1\varphi_2 m_{12} m_{12(3)} + \varphi_1\varphi_3 m_{13} m_{13(2)} + \varphi_2\varphi_3 m_{23} m_{23(1)} \\
d_{124} &\equiv \varphi_1\varphi_2 m_{12} m_{12(4)} + \varphi_1\varphi_4 m_{14} m_{14(2)} + \varphi_2\varphi_4 m_{24} m_{24(1)} \\
d_{134} &\equiv \varphi_1\varphi_3 m_{13} m_{13(4)} + \varphi_1\varphi_4 m_{14} m_{14(3)} + \varphi_3\varphi_4 m_{34} m_{34(1)} \\
d_{234} &\equiv \varphi_1\varphi_2 m_{12} m_{12(4)} + \varphi_1\varphi_4 m_{14} m_{14(2)} + \varphi_2\varphi_4 m_{24} m_{24(1)}
\end{aligned}$$

Then $\mathbf{z}_{11} = \mathbf{e}^*_{123}$ if $m_{13(2)} \leq 0$, $\mathbf{z}_{12} = \mathbf{e}^*_{124}$ if $m_{14(2)} \leq 0$, $\mathbf{z}_{13} = \mathbf{e}^*_{134}$ if $m_{14(3)} \leq 0$, and $\mathbf{z}_{14} = \mathbf{e}^*_{234}$ if $m_{24(3)} \leq 0$.

The last fixed point is

$$\mathbf{z}_{15} = \left(\frac{\varphi_2\varphi_3\varphi_4 m_{234}}{\hat{d}}, \frac{\varphi_1\varphi_3\varphi_4 m_{134}}{\hat{d}}, \frac{\varphi_1\varphi_2\varphi_4 m_{124}}{\hat{d}}, \frac{\varphi_1\varphi_2\varphi_3 m_{123}}{\hat{d}}\right)$$

where

$$\begin{aligned}
m_{234} &\equiv m_{23}m_{23(4)}m_{14} + m_{24}m_{24(3)}m_{13} + m_{34}m_{34(2)}m_{12} + 2m_{23}m_{24}m_{34} \\
m_{134} &\equiv m_{13}m_{13(4)}m_{24} + m_{14}m_{14(3)}m_{23} + m_{34}m_{34(1)}m_{12} + 2m_{13}m_{14}m_{34} \\
m_{124} &\equiv m_{12}m_{12(4)}m_{34} + m_{14}m_{14(2)}m_{23} + m_{24}m_{24(1)}m_{13} + 2m_{12}m_{14}m_{24} \\
m_{123} &\equiv m_{12}m_{12(3)}m_{34} + m_{13}m_{13(2)}m_{24} + m_{23}m_{23(1)}m_{14} + 2m_{12}m_{13}m_{23}
\end{aligned}$$

and

$$\hat{d} \equiv \varphi_2\varphi_3\varphi_4 m_{234} + \varphi_1\varphi_3\varphi_4 m_{134} + \varphi_1\varphi_2\varphi_4 m_{124} + \varphi_1\varphi_2\varphi_3 m_{123}.$$

Then $\mathbf{z}_{15} = \hat{\mathbf{z}}$ if $\text{sign}(m_{123}) = \text{sign}(m_{124}) = \text{sign}(m_{134}) = \text{sign}(m_{234})$.

The two partitions of Φ have the following subsets. The first partition involves $m_{ij(k)}$, in which we only report the cases in which $m_{ij(k)}$ are ambiguous (all the other $m_{ij(k)}$ are negative):

$$\Phi_4^2 \equiv \{\varphi \in \Phi : m_{13(2)} > 0, m_{14(2)} > 0, m_{14(3)} > 0, m_{24(3)} > 0\}$$

$$\begin{aligned}
\Phi_3^2 \equiv \big\{\varphi \in \Phi :\ & m_{13(2)} > 0, m_{14(2)} > 0, m_{14(3)} > 0, \text{ or } m_{13(2)} > 0, m_{14(2)} > 0, m_{24(3)} > 0, \text{ or} \\
& \text{or } m_{13(2)} > 0, m_{14(3)} > 0, m_{24(3)} > 0, \text{ or } m_{14(2)} > 0, m_{14(3)} > 0, m_{24(3)} > 0\big\}
\end{aligned} \tag{36}$$

$$\begin{aligned}
\Phi_2^2 \equiv \big\{\varphi \in \Phi :\ & m_{13(2)} > 0, m_{14(2)} > 0, \text{ or } m_{13(2)} > 0, m_{14(3)} > 0, \text{ or } m_{13(2)} > 0, m_{24(3)} > 0, \text{ or} \\
& \text{or } m_{14(2)} > 0, m_{14(3)} > 0, \text{ or } m_{14(2)} > 0, m_{24(3)} > 0, \text{ or } m_{14(3)} > 0, m_{24(3)} > 0\big\}
\end{aligned} \tag{37}$$

$$\Phi_1^2 = \big\{\varphi \in \Phi :\ m_{13(2)} > 0, \text{ or } m_{14(2)} > 0, \text{ or } m_{14(3)} > 0 \text{ or } m_{24(3)} > 0\big\}.$$

and

$$\Phi_0^2 = \big\{\varphi \in \Phi :\ m_{ij(k)} \leq 0 \text{ for } i \neq j \neq k \in S_1 \times S_1 \times S_1\big\}$$

The second partition involves $m_{ij(k)}$, in which we only report the cases in which $m_{ij(k)}$ are ambiguous (all the other $m_{ij(k)}$ are negative):

$$\Phi_4^3 \equiv \{\varphi \in \Phi : m_{123} > 0, m_{124} > 0, m_{134} > 0, m_{234} > 0\}$$

$$\Phi_3^3 \equiv \{\varphi \in \Phi : m_{123} \leq 0, m_{124} > 0, m_{134} > 0, m_{234} > 0, \text{ or}$$
$$\text{or } m_{123} > 0, m_{124} \leq 0, m_{134} > 0, m_{234} > 0, \text{ or } m_{123} > 0, m_{124} > 0, m_{134} \leq 0, m_{234} > 0, \text{ or}$$
$$\text{or } m_{123} > 0, m_{124} > 0, m_{134} > 0, m_{234} \leq 0\}$$

$$\Phi_2^3 \equiv \{\varphi \in \Phi : m_{123} \leq 0, m_{124} \leq 0, m_{134} > 0, m_{234} > 0, \text{ or}$$
$$\text{or } m_{123} \leq 0, m_{124} > 0, m_{134} \leq 0, m_{234} > 0, \text{ or } m_{123} \leq 0, m_{124} > 0, m_{134} > 0, m_{234} \leq 0, \text{ or}$$
$$\text{or } m_{123} > 0, m_{124} \leq 0, m_{134} \leq 0, m_{234} > 0, \text{or } m_{123} > 0, m_{124} \leq 0, m_{134} > 0, m_{234} \leq 0,$$
$$\text{or } m_{123} > 0, m_{124} > 0, m_{134} \leq 0, m_{234} \leq 0\}$$

$$\Phi_1^3 \equiv \{\varphi \in \Phi : m_{123} \leq 0, m_{124} \leq 0, m_{134} \leq 0, m_{234} > 0, \text{ or}$$
$$\text{or } m_{123} \leq 0, m_{124} \leq 0, m_{134} > 0, m_{234} \leq 0, \text{ or } m_{123} \leq 0, m_{124} > 0, m_{134} \leq 0, m_{234} \leq 0, \text{ or}$$
$$\text{or } m_{123} > 0, m_{124} \leq 0, m_{134} \leq 0, m_{234} \leq 0\}$$

and

$$\Phi_0^3 \equiv \{\varphi \in \Phi : m_{123} \leq 0, m_{124} \leq 0, m_{134} \leq 0, m_{234} \leq 0\}.$$

Next we present all the possible 32 candidate steady states: $\mathcal{Z}^* = \{\mathbf{z} \in \Delta^2 : \ \mathbf{F}(\mathbf{z}) = \mathbf{0}\}$:

$$
\begin{aligned}
\mathcal{Z}_1 &\equiv \left\{\{\mathbf{e}_i\}_{i\in S_1}, \{\mathbf{e}^*_{ij}\}_{ij\in S_2}\right\},\\
\mathcal{Z}_2 &\equiv \left\{\{\mathbf{e}_i\}_{i\in S_1}, \{\mathbf{e}^*_{ij}\}_{ij\in S_2}, \hat{\mathbf{z}}\right\},\\
\mathcal{Z}_3 &= \left\{\{\mathbf{e}_i\}_{i\in S_1}, \{\mathbf{e}^*_{ij}\}_{ij\in S_2}, \mathbf{e}^*_{123}\right\},\\
\mathcal{Z}_4 &= \left\{\{\mathbf{e}_i\}_{i\in S_1}, \{\mathbf{e}^*_{ij}\}_{ij\in S_2}, \mathbf{e}^*_{124}\right\},\\
\mathcal{Z}_5 &= \left\{\{\mathbf{e}_i\}_{i\in S_1}, \{\mathbf{e}^*_{ij}\}_{ij\in S_2}, \mathbf{e}^*_{134}\right\},\\
\mathcal{Z}_6 &= \left\{\{\mathbf{e}_i\}_{i\in S_1}, \{\mathbf{e}^*_{ij}\}_{ij\in S_2}, \mathbf{e}^*_{234}\right\},\\
\mathcal{Z}_7 &= \left\{\{\mathbf{e}_i\}_{i\in S_1}, \{\mathbf{e}^*_{ij}\}_{ij\in S_2}, \mathbf{e}^*_{123}, \hat{\mathbf{z}}\right\},\\
\mathcal{Z}_8 &= \left\{\{\mathbf{e}_i\}_{i\in S_1}, \{\mathbf{e}^*_{ij}\}_{ij\in S_2}, \mathbf{e}^*_{124}, \hat{\mathbf{z}}\right\},\\
\mathcal{Z}_9 &= \left\{\{\mathbf{e}_i\}_{i\in S_1}, \{\mathbf{e}^*_{ij}\}_{ij\in S_2}, \mathbf{e}^*_{134}, \hat{\mathbf{z}}\right\},\\
\mathcal{Z}_{10} &= \left\{\{\mathbf{e}_i\}_{i\in S_1}, \{\mathbf{e}^*_{ij}\}_{ij\in S_2}, \mathbf{e}^*_{234}, \hat{\mathbf{z}}\right\}\\
\mathcal{Z}_{11} &= \left\{\{\mathbf{e}_i\}_{i\in S_1}, \{\mathbf{e}^*_{ij}\}_{ij\in S_2}, \mathbf{e}^*_{123}, \mathbf{e}^*_{124}\right\},\\
\mathcal{Z}_{12} &= \left\{\{\mathbf{e}_i\}_{i\in S_1}, \{\mathbf{e}^*_{ij}\}_{ij\in S_2}, \mathbf{e}^*_{123}, \mathbf{e}^*_{134}\right\},\\
\mathcal{Z}_{13} &= \left\{\{\mathbf{e}_i\}_{i\in S_1}, \{\mathbf{e}^*_{ij}\}_{ij\in S_2}, \mathbf{e}^*_{123}, \mathbf{e}^*_{234}\right\},\\
\mathcal{Z}_{14} &= \left\{\{\mathbf{e}_i\}_{i\in S_1}, \{\mathbf{e}^*_{ij}\}_{ij\in S_2}, \mathbf{e}^*_{124}, \mathbf{e}^*_{134}\right\},\\
\mathcal{Z}_{15} &= \left\{\{\mathbf{e}_i\}_{i\in S_1}, \{\mathbf{e}^*_{ij}\}_{ij\in S_2}, \mathbf{e}^*_{124}, \mathbf{e}^*_{234}\right\},\\
\mathcal{Z}_{16} &= \left\{\{\mathbf{e}_i\}_{i\in S_1}, \{\mathbf{e}^*_{ij}\}_{ij\in S_2}, \mathbf{e}^*_{134}, \mathbf{e}^*_{234}\right\},\\
\mathcal{Z}_{17} &= \left\{\{\mathbf{e}_i\}_{i\in S_1}, \{\mathbf{e}^*_{ij}\}_{ij\in S_2}, \mathbf{e}^*_{123}, \mathbf{e}^*_{124}, \hat{\mathbf{z}}\right\},\\
\mathcal{Z}_{18} &= \left\{\{\mathbf{e}_i\}_{i\in S_1}, \{\mathbf{e}^*_{ij}\}_{ij\in S_2}, \mathbf{e}^*_{123}, \mathbf{e}^*_{134}, \hat{\mathbf{z}}\right\},\\
\mathcal{Z}_{19} &= \left\{\{\mathbf{e}_i\}_{i\in S_1}, \{\mathbf{e}^*_{ij}\}_{ij\in S_2}, \mathbf{e}^*_{123}, \mathbf{e}^*_{234}, \hat{\mathbf{z}}\right\},\\
\mathcal{Z}_{20} &= \left\{\{\mathbf{e}_i\}_{i\in S_1}, \{\mathbf{e}^*_{ij}\}_{ij\in S_2}, \mathbf{e}^*_{124}, \mathbf{e}^*_{134}, \hat{\mathbf{z}}\right\},\\
\mathcal{Z}_{21} &= \left\{\{\mathbf{e}_i\}_{i\in S_1}, \{\mathbf{e}^*_{ij}\}_{ij\in S_2}, \mathbf{e}^*_{124}, \mathbf{e}^*_{234}, \hat{\mathbf{z}}\right\},\\
\mathcal{Z}_{22} &= \left\{\{\mathbf{e}_i\}_{i\in S_1}, \{\mathbf{e}^*_{ij}\}_{ij\in S_2}, \mathbf{e}^*_{134}, \mathbf{e}^*_{234}, \hat{\mathbf{z}}\right\},\\
\mathcal{Z}_{23} &= \left\{\{\mathbf{e}_i\}_{i\in S_1}, \{\mathbf{e}^*_{ij}\}_{ij\in S_2}, \mathbf{e}^*_{123}, \mathbf{e}^*_{124}, \mathbf{e}^*_{134}\right\},\\
\mathcal{Z}_{24} &= \left\{\{\mathbf{e}_i\}_{i\in S_1}, \{\mathbf{e}^*_{ij}\}_{ij\in S_2}, \mathbf{e}^*_{123}, \mathbf{e}^*_{124}, \mathbf{e}^*_{234}\right\},\\
\mathcal{Z}_{25} &= \left\{\{\mathbf{e}_i\}_{i\in S_1}, \{\mathbf{e}^*_{ij}\}_{ij\in S_2}, \mathbf{e}^*_{123}, \mathbf{e}^*_{134}, \mathbf{e}^*_{234}\right\},\\
\mathcal{Z}_{26} &= \left\{\{\mathbf{e}_i\}_{i\in S_1}, \{\mathbf{e}^*_{ij}\}_{ij\in S_2}, \mathbf{e}^*_{124}, \mathbf{e}^*_{134}, \mathbf{e}^*_{234}\right\},\\
\mathcal{Z}_{27} &= \left\{\{\mathbf{e}_i\}_{i\in S_1}, \{\mathbf{e}^*_{ij}\}_{ij\in S_2}, \mathbf{e}^*_{123}, \mathbf{e}^*_{124}, \mathbf{e}^*_{134}, \hat{\mathbf{z}}\right\},\\
\mathcal{Z}_{28} &= \left\{\{\mathbf{e}_i\}_{i\in S_1}, \{\mathbf{e}^*_{ij}\}_{ij\in S_2}, \mathbf{e}^*_{123}, \mathbf{e}^*_{124}, \mathbf{e}^*_{234}, \hat{\mathbf{z}}\right\},\\
\mathcal{Z}_{29} &= \left\{\{\mathbf{e}_i\}_{i\in S_1}, \{\mathbf{e}^*_{ij}\}_{ij\in S_2}, \mathbf{e}^*_{123}, \mathbf{e}^*_{134}, \mathbf{e}^*_{234}, \hat{\mathbf{z}}\right\},\\
\mathcal{Z}_{30} &= \left\{\{\mathbf{e}_i\}_{i\in S_1}, \{\mathbf{e}^*_{ij}\}_{ij\in S_2}, \mathbf{e}^*_{124}, \mathbf{e}^*_{134}, \mathbf{e}^*_{234}, \hat{\mathbf{z}}\right\},\\
\mathcal{Z}_{31} &= \left\{\{\mathbf{e}_i\}_{i\in S_1}, \{\mathbf{e}^*_{ij}\}_{ij\in S_2}, \{\mathbf{e}^*_{ijk}\}_{ijk\in S_3}\right\},\\
\mathcal{Z}_{32} &= \left\{\{\mathbf{e}_i\}_{i\in S_1}, \{\mathbf{e}^*_{ij}\}_{ij\in S_2}, \{\mathbf{e}^*_{ijk}\}_{ijk\in S_3}, \hat{\mathbf{z}}\right\}.
\end{aligned}
$$

Then, if the following subsets are non-empty then:

- if $\varphi \in \Phi_4^2 \cap (\cup_{s_3=1}^3 \Phi_{s_3}^3)$ then $\mathcal{Z}^* = \mathcal{Z}_1$;
- if $\varphi \in \Phi_4^2 \cap (\Phi_0^3 \cup \Phi_4^3)$ then $\mathcal{Z}^* = \mathcal{Z}_2$;
- if $\varphi \in \Phi_3^2 \cap (\cup_{s_3=1}^3 \Phi_{s_3}^3)$ then $\mathcal{Z}^* \in \{\mathcal{Z}_3, \dots \mathcal{Z}_6\}$;
- if $\varphi \in \Phi_3^2 \cap (\Phi_0^3 \cup \Phi_4^3)$ then $\mathcal{Z}^* \in \{\mathcal{Z}_7, \dots \mathcal{Z}_{10}\}$;
- if $\varphi \in \Phi_2^2 \cap (\cup_{s_3=1}^3 \Phi_{s_3}^3)$ then $\mathcal{Z}^* \in \{\mathcal{Z}_{11}, \dots \mathcal{Z}_{16}\}$;
- if $\varphi \in \Phi_2^2 \cap (\Phi_0^3 \cup \Phi_4^3)$ then $\mathcal{Z}^* \in \{\mathcal{Z}_{17}, \dots \mathcal{Z}_{22}\}$;
- if $\varphi \in \Phi_1^2 \cap (\cup_{s_3=1}^3 \Phi_{s_3}^3)$ then $\mathcal{Z}^* \in \{\mathcal{Z}_{23}, \dots \mathcal{Z}_{26}\}$;
- if $\varphi \in \Phi_1^2 \cap (\Phi_0^3 \cup \Phi_4^3)$ then $\mathcal{Z}^* \in \{\mathcal{Z}_{27}, \dots \mathcal{Z}_{30}\}$;
- $\varphi \in \Phi_0^2 \cap (\cup_{s_3=1}^3 \Phi_{s_3}^3)$ then $\mathcal{Z}^* = \mathcal{Z}_{31}$;
- if $\varphi \in \Phi_0^2 \cap (\Phi_0^3 \cup \Phi_4^3)$ then $\mathcal{Z}^* = \mathcal{Z}_{32}$.

Next, we evaluate the verification of Nash conditions: First, the stationary states in the vertices are all Nash equilibria, because $\mathbf{u}(\mathbf{e}_1) - \overline{u}(\mathbf{e}_1) = (0, -\varphi_1 m_{12}, -\varphi_1 m_{13}, -\varphi_1 m_{14})$, $\mathbf{u}(\mathbf{e}_2) - \overline{u}(\mathbf{e}_2) = (-\varphi_2 m_{12}, 0, -\varphi_2 m_{23}, -\varphi_2 m_{24})$, $\mathbf{u}(\mathbf{e}_3) - \overline{u}(\mathbf{e}_3) = (-\varphi_3 m_{13}, -\varphi_3 m_{23}, 0, -\varphi_3 m_{34})$, and $\mathbf{u}(\mathbf{e}_4) - \overline{u}(\mathbf{e}_4) = (-\varphi_4 m_{14}, -\varphi_4 m_{24}, -\varphi_4 m_{34}, 0)$. Also, if there is an interior steady state $\hat{\mathbf{z}}$ it is also a Nash equilibrium because $\mathbf{u}(\hat{\mathbf{z}}) = \overline{u}(\hat{\mathbf{z}})$.

Second, the stationary states in the edges $\mathbf{e}_{ij}$ verify:

$$\mathbf{u}(\mathbf{e}_{12}^*) - \overline{u}(\mathbf{e}_{12}^*) = \left(0, 0, \frac{\varphi_1 \varphi_2 m_{12(3)}}{\varphi_1 + \varphi_2}, \frac{\varphi_1 \varphi_2 m_{12(4)}}{\varphi_1 + \varphi_2}\right) \leq 0$$

$$\mathbf{u}(\mathbf{e}_{13}^*) - \overline{u}(\mathbf{e}_{13}^*) = \left(0, \frac{\varphi_1 \varphi_3 m_{13(2)}}{\varphi_1 + \varphi_3}, 0, \frac{\varphi_1 \varphi_3 m_{13(4)}}{\varphi_1 + \varphi_3}\right)$$

$$\mathbf{u}(\mathbf{e}_{14}^*) - \overline{u}(\mathbf{e}_{14}^*) = \left(0, \frac{\varphi_1 \varphi_4 m_{14(2)}}{\varphi_1 + \varphi_4}, \frac{\varphi_1 \varphi_4 m_{14(3)}}{\varphi_1 + \varphi_4}, 0\right)$$

$$\mathbf{u}(\mathbf{e}_{23}^*) - \overline{u}(\mathbf{e}_{23}^*) = \left(\frac{\varphi_2 \varphi_3 m_{23(1)}}{\varphi_2 + \varphi_3}, 0, 0, \frac{\varphi_2 \varphi_3 m_{23(4)}}{\varphi_2 + \varphi_3}\right) \leq 0$$

$$\mathbf{u}(\mathbf{e}_{24}^*) - \overline{u}(\mathbf{e}_{24}^*) = \left(\frac{\varphi_2 \varphi_4 m_{24(1)}}{\varphi_2 + \varphi_4}, 0, \frac{\varphi_2 \varphi_4 m_{24(3)}}{\varphi_2 + \varphi_4}, 0\right)$$

$$\mathbf{u}(\mathbf{e}^*_{34}) - \overline{u}(\mathbf{e}^*_{34}) = \left(\frac{\varphi_3\varphi_4 m_{34(1)}}{\varphi_3+\varphi_4}, \frac{\varphi_3\varphi_4 m_{34(2)}}{\varphi_3+\varphi_4}, 0, 0 \right) \leq 0$$

As $k_{12} = k_{34} = 0$ then, $\mathbf{e}^*_{12}$ and $\mathbf{e}^*_{34}$ are always Nash equilibria , and the other equilibria may not be Nash, because $k_{13} = k_{34} = k_{34} = 1$ and $k_{14} = 2$. However there is a close relationship between the Nash property for the other equilibria $\mathbf{e}^*_{ij}$ and the existence of equilibria in edges $\mathbf{e}_{ijk}$: if $\mathbf{z}_{11} = \mathbf{e}^*_{123}$ then $\mathbf{e}^*_{13}$ is a Nash equilibrium, if $\mathbf{z}_{12} = \mathbf{e}^*_{124}$ and $\mathbf{z}_{13} = \mathbf{e}^*_{134}$ then $\mathbf{e}^*_{14}$ is a Nash equilibrium and if $\mathbf{z}_{14} = \mathbf{e}^*_{234}$ then $\mathbf{e}^*_{34}$ is a Nash equilibrium. This means that unless all the the expressions $m_{ij(k)}$ are non-positive, there will always be at least one equilibrium of type $\mathbf{e}^*_{ij}$ which is not a Nash equilibrium.

For the equilibria $\mathbf{e}^*_{ijk}$, because we need to impose conditions on the parameters that ensure that they belong to the simplex, we readily see that they are all Nash equilibria:

$$\mathbf{u}(\mathbf{e}^*_{123}) - \overline{u}(\mathbf{e}^*_{123}) = \left(0, 0, 0, -\frac{\varphi_1\varphi_2\varphi_3 m_{123}}{d_{123}} \right) \leq 0$$

$$\mathbf{u}(\mathbf{e}^*_{124}) - \overline{u}(\mathbf{e}^*_{124}) = \left(0, 0, -\frac{\varphi_1\varphi_2\varphi_4 m_{124}}{d_{124}}, 0 \right) \leq 0$$

$$\mathbf{u}(\mathbf{e}^*_{134}) - \overline{u}(\mathbf{e}^*_{134}) = \left(0, -\frac{\varphi_1\varphi_3\varphi_4 m_{134}}{d_{134}}, 0, 0 \right) \leq 0$$

$$\mathbf{u}(\mathbf{e}^*_{234}) - \overline{u}(\mathbf{e}^*_{234}) = \left(-\frac{\varphi_2\varphi_3\varphi_4 m_{234}}{d_{234}}, 0, 0, 0 \right) \leq 0$$

□

C.8 Proof of Proposition 9.

Proof. We evaluate the spectra for the Jacobian $\mathbf{F}'(\mathbf{z}^*)$ at the stationary strategy profiles, such that $\mathbf{z}^* \in \Delta^3$. For the equilibria in the vertices, we have

$$\sigma(\mathbf{e}_1) = \left\{ -\frac{\varphi_1}{4}, -\varphi_1 m_{12}, -\varphi_1 m_{13}, -\varphi_1 m_{14} \right\}$$

$$\sigma(\mathbf{e}_2) = \left\{ -\frac{\varphi_2}{4}, -\varphi_2 m_{12}, -\varphi_2 m_{23}, -\varphi_2 m_{24} \right\}$$

$$\sigma(\mathbf{e}_3) = \left\{ -\frac{\varphi_3}{4}, -\varphi_3 m_{13}, -\varphi_3 m_{23}, -\varphi_3 m_{24} \right\}$$

$$\sigma(\mathbf{e}_4) = \Big\{ -\frac{\varphi_4}{4}, -\varphi_4 m_{14}, -\varphi_4 m_{24}, -\varphi_4 m_{34} \Big\}.$$

They are all negative, therefore $\mathbf{e}_i$ are sinks, and the local stable manifold is four-dimensional. For the equilibria in the edges $\mathbf{e}^*_{ij}$ we have

$$\sigma(\mathbf{e}^*_{12}) = \left\{ \frac{\varphi_1\varphi_2}{\varphi_1+\varphi_2} m_{12}, \frac{\varphi_1\gamma_2}{\varphi_1+\varphi_2}\left(m_{12}-\frac{1}{2}\right), \frac{\varphi_1\varphi_2}{\varphi_1+\varphi_2} m_{12(3)}, \frac{\varphi_1\varphi_2}{\varphi_1+\varphi_2} m_{12(4)} \right\}$$

$$\sigma(\mathbf{e}^*_{13}) = \left\{ \frac{\varphi_1\varphi_3}{\varphi_1+\varphi_3} m_{13}, \frac{\varphi_1\varphi_3}{\varphi_1+\varphi_3}\left(m_{13}-\frac{1}{2}\right), \frac{\varphi_1\varphi_3}{\varphi_1+\varphi_3} m_{13(2)}, \frac{\varphi_1\varphi_3}{\varphi_1+\varphi_3} m_{13(4)} \right\}$$

$$\sigma(\mathbf{e}^*_{14}) = \left\{ \frac{\varphi_1\varphi_4}{\varphi_1+\varphi_4} m_{14}, \frac{\varphi_1\varphi_4}{\varphi_1+\varphi_4}\left(m_{14}-\frac{1}{2}\right), \frac{\varphi_1\varphi_4}{\varphi_1+\varphi_4} m_{14(2)}, \frac{\varphi_1\varphi_4}{\varphi_1+\varphi_4} m_{14(3)} \right\}$$

$$\sigma(\mathbf{e}^*_{23}) = \left\{ \frac{\varphi_2\varphi_3}{\varphi_2+\varphi_3} m_{23}, \frac{\varphi_2\varphi_3}{\varphi_2+\varphi_3}\left(m_{23}-\frac{1}{2}\right), \frac{\varphi_2\varphi_3}{\varphi_2+\varphi_3} m_{23(1)}, \frac{\varphi_2\varphi_3}{\varphi_2+\varphi_3} m_{23(4)} \right\}$$

$$\sigma(\mathbf{e}^*_{24}) = \left\{ \frac{\varphi_2\varphi_4}{\varphi_2+\varphi_4} m_{24}, \frac{\varphi_2\varphi_4}{\varphi_2+\varphi_4}\left(m_{24}-\frac{1}{2}\right), \frac{\varphi_2\varphi_4}{\varphi_2+\varphi_4} m_{24(1)}, \frac{\varphi_2\varphi_4}{\varphi_2+\varphi_4} m_{24(3)} \right\}$$

$$\sigma(\mathbf{e}^*_{34}) = \left\{ \frac{\varphi_3\varphi_4}{\varphi_3+\varphi_4} m_{34}, \frac{\varphi_3\varphi_4}{\varphi_3+\varphi_4}\left(m_{34}-\frac{1}{2}\right), \frac{\varphi_3\varphi_4}{\varphi_3+\varphi_4} m_{34(1)}, \frac{\varphi_3\varphi_4}{\varphi_3+\varphi_4} m_{34(2)} \right\}.$$

Then they are all generalized saddles. However the local stable manifolds dimensions may differ: $\mathbf{e}^*_{12}$, $\mathbf{e}^*_{23}$ and $\mathbf{e}^*_{34}$ are three-dimensional, $\mathbf{e}^*_{13}$ and $\mathbf{e}^*_{24}$ may be three-dimensional or two dimensional (if $m_{13(2)} > 0$, in the first case, or $m_{24(3)} > 0$) and $\mathbf{e}^*_{14}$ may be three-, two, o one dimensional. If $\phi \in \Phi^2_4$ the least-dimensional case and if $\phi \in \Phi^2_0$ the higher-dimensional case holds.

For the equilibria $\mathbf{e}^*_{ijk}$ the spectra are:

$$\sigma(\mathbf{e}_{123}) = \Big\{ -\frac{\varphi_1\varphi_2\varphi_3}{d_{123}}\left(m_{12}m_{12(3)} + m_{13}m_{13(2)} + m_{23}m_{23(1)} + 2m_{12}m_{13}m_{23}\right), -\frac{\varphi_1\varphi_2\varphi_3 m_{123}}{d_{123}},$$
$$-\frac{\varphi_1\varphi_2\varphi_3 m_{12}m_{13}m_{23}}{d_{123}}\left(1+(\Delta_{123})^{1/2}\right), -\frac{\varphi_1\varphi_2\varphi_3 m_{12}m_{13}m_{23}}{d_{123}}\left(1-(\Delta_{123})^{1/2}\right) \Big\}$$

where

$$\Delta_{123} \equiv 1 + \frac{m_{12(3)}m_{13(2)}m_{23(1)}}{m_{12}m_{13}m_{23}},$$

as $m_{13(2} \leq 0$ and $d_{123} < 0$, $\mathbf{e}^*_{123}$ it is a generalized saddle with a one-dimensional, if $m_{123} > 0$ or two dimensional saddle manifold, if $m_{123} < 0$;

$$\sigma(\mathbf{e}_{124}) = \left\{ -\frac{\varphi_1\varphi_2\varphi_4}{d_{124}} \left(m_{12}m_{12(4)} + m_{14}m_{14(2)} + m_{24}m_{24(1)} + 2m_{12}m_{14}m_{24}\right), -\frac{\varphi_1\varphi_2\varphi_4 m_{124}}{d_{124}}, \right.$$
$$\left. -\frac{\varphi_1\varphi_2\varphi_4 m_{12}m_{14}m_{24}}{d_{124}} \left(1 + (\Delta_{124})^{1/2}\right), -\frac{\varphi_1\varphi_2\varphi_4 m_{12}m_{14}m_{24}}{d_{124}} \left(1 - (\Delta_{124})^{1/2}\right) \right\}$$

where

$$\Delta_{124} \equiv 1 + \frac{m_{12(4)}m_{14(2)}m_{24(1)}}{m_{12}m_{14}m_{24}}$$

as $m_{14(2} \leq 0$ and $d_{124} < 0$, $\mathbf{e}^*_{124}$ it is a generalized saddle with a one-dimensional, if $m_{124} > 0$ or two dimensional saddle manifold, if $m_{124} < 0$;

$$\sigma(\mathbf{e}_{134}) = \left\{ -\frac{\varphi_1\varphi_3\varphi_4}{d_{134}} \left(m_{13}m_{13(4)} + m_{14}m_{14(3)} + m_{34}m_{34(1)} + 2m_{13}m_{14}m_{34}\right), -\frac{\varphi_1\varphi_3\varphi_4 m_{134}}{d_{134}}, \right.$$
$$\left. -\frac{\varphi_1\varphi_3\varphi_4 m_{13}m_{14}m_{34}}{d_{134}} \left(1 + (\Delta_{134})^{1/2}\right), -\frac{\varphi_1\varphi_3\varphi_4 m_{13}m_{14}m_{34}}{d_{134}} \left(1 - (\Delta_{134})^{1/2}\right), \right\}$$

where

$$\Delta_{134} \equiv 1 + \frac{m_{13(4)}m_{14(3)}m_{34(1)}}{m_{13}m_{14}m_{34}}$$

as $m_{14(3} \leq 0$ and $d_{134} < 0$, $\mathbf{e}^*_{134}$ it is a generalized saddle with a one-dimensional, if $m_{134} > 0$ or two dimensional saddle manifold, if $m_{134} < 0$;

$$\sigma(\mathbf{e}_{234}) = \left\{ -\frac{\varphi_2\varphi_3\varphi_4}{d_{234}} \left(m_{23}m_{23(4)} + m_{24}m_{24(3)} + m_{34}m_{34(2)} + 2m_{23}m_{24}m_{34}\right), -\frac{\varphi_2\varphi_3\varphi_4 m_{234}}{d_{234}}, \right.$$
$$\left. -\frac{\varphi_2\varphi_3\varphi_4 m_{23}m_{24}m_{34}}{d_{234}} \left(1 + (\Delta_{234})^{1/2}\right), -\frac{\varphi_2\varphi_3\varphi_4 m_{23}m_{24}m_{34}}{d_{234}} \left(1 - (\Delta_{234})^{1/2}\right), \right\}$$

where

$$\Delta_{234} \equiv 1 + \frac{m_{23(4)}m_{24(3)}m_{34(2)}}{m_{23}m_{24}m_{34}}$$

as $m_{24(3} \leq 0$ and $d_{234} < 0$, $\mathbf{e}^*_{234}$ it is a generalized saddle with a one-dimensional, if $m_{234} > 0$ or two dimensional saddle manifold, if $m_{234} < 0$.

The expressions for the eigenvalues at the interior equilibria are too large to report., however if it exists the local stable manifold at $\hat{\mathbf{z}}$ should be one-dimensional. $\square$

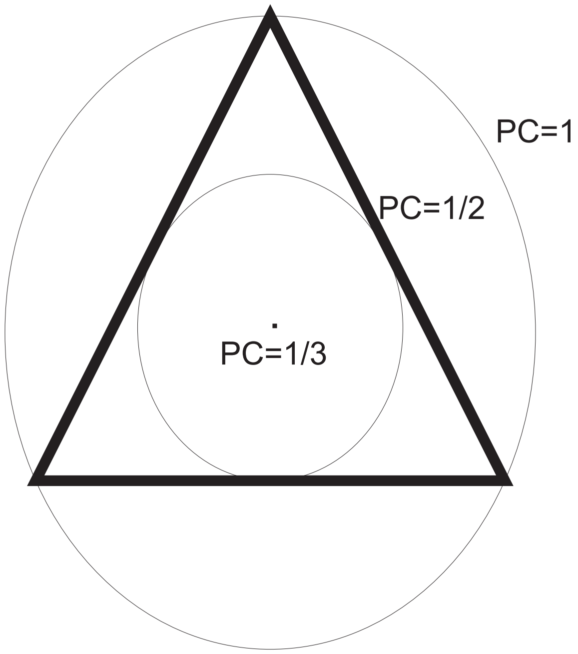

Figure 1: Consistency

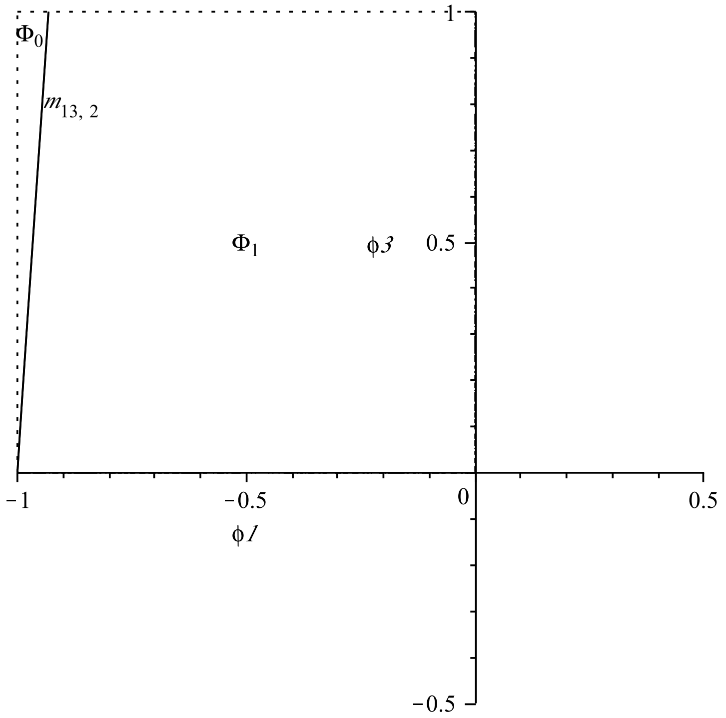

Figure 2: Bifurcation diagram in the space (φ_1, φ_3), for equally spaced conjectures $\varphi_2 = (\varphi_1 + \varphi_3)/2$.

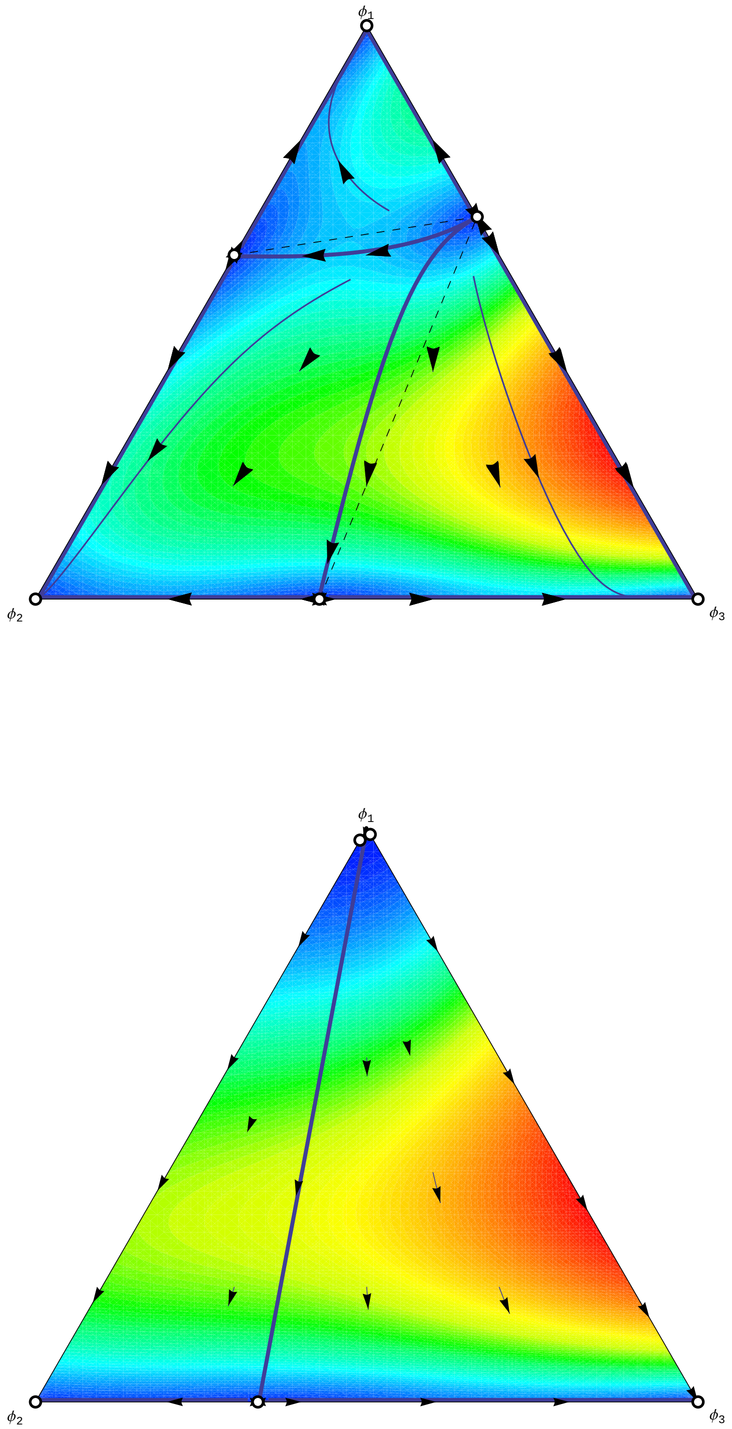

Figure 3: Phase diagrams over the simplex for equally spaced conjectures: the top panel is for case $\Gamma > 0$ where $(\varphi_1, \varphi_3) = (1, 2)$ and the bottom panel for $\Gamma < 0$ where $(\varphi_1, \varphi_3) = (0.01, 1.9)$.

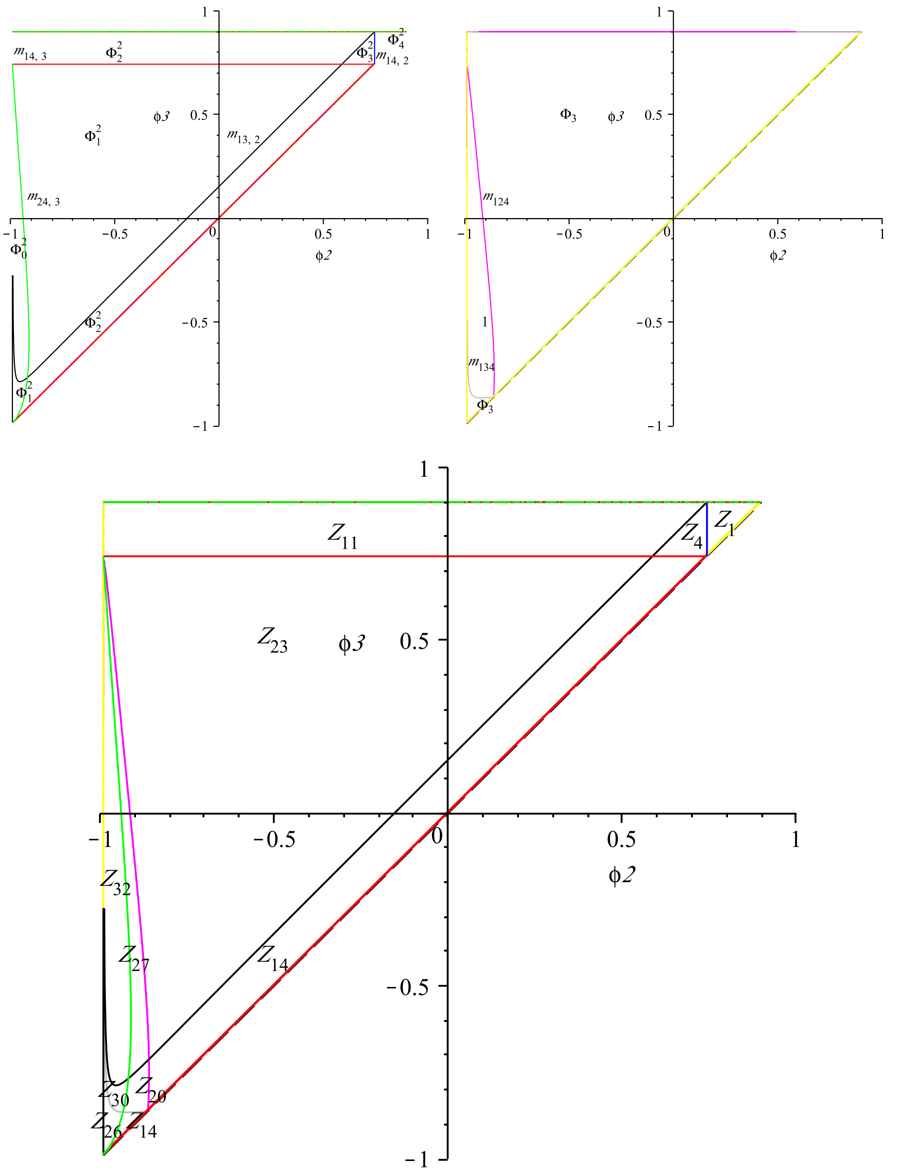

Figure 4: Bifurcation diagram in the space $(\phi_2, \phi_3)$for $\phi 1 = -0.99$ and $\phi_4 = 0.9$. The top left graph refers to set $\Phi^2_{s_2}$ and the top right graph refers to Φ^3. In the bottom graph the areas correspond to eleven multiple steady states distributions: $\mathcal{Z}_4$, $\mathcal{Z}_1$, $\mathcal{Z}_{11}$ (in this case $\mathbf{e}^*_{13}$ is Nash), $\mathcal{Z}_{14}$ ($\mathbf{e}^*_{14}$ is Nash), $\mathcal{Z}_{20}$ ($\mathbf{e}^*_{14}$ is Nash), $\mathcal{Z}_{23}$ ($\mathbf{e}^*_{13}$ and $\mathbf{e}^*_{14}$ are Nash), $\mathcal{Z}_{26}$ ($\mathbf{e}^*_{23}$ and $\mathbf{e}^*_{14}$ are Nash), $\mathcal{Z}_{27}$ ($\mathbf{e}^*_{13}$ and $\mathbf{e}^*_{14}$ are Nash), $\mathcal{Z}_{30}$ ($\mathbf{e}^*_{23}$ and $\mathbf{e}^*_{14}$ are Nash), $\mathcal{Z}_{32}$ (all Nash).