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Uniqueness for an inverse problem in electromagnetism with partial data

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Abstract

A uniqueness result for the recovery of the electric and magnetic coefficients in the time-harmonic Maxwell equations from local boundary measurements is proven. No special geometrical condition is imposed on the *inaccessible* part of the boundary of the domain, apart from imposing that the boundary of the domain is $C^{1,1}$. The coefficients are assumed to coincide on a neighbourhood of the boundary, a natural property in applications.

Keywords: Inverse problem; Inverse boundary value problem; Maxwell equations; Partial data; Cauchy data set; Schrödinger equation

1. Introduction

Let μ, ε, σ be positive functions on a nonempty, bounded, open set Ω in $\mathbb{R}^3$, describing the permeability, permittivity and conductivity, respectively, of an inhomogeneous, isotropic medium Ω. Let $\partial\Omega$ denote the boundary of Ω and N the outward unit vector field normal to the boundary. Consider the electric and magnetic fields, E, H, satisfying the so-called time-harmonic Maxwell equations at a frequency $\omega > 0$, namely

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$$\begin{cases} \nabla \times H + i\omega\gamma E = 0, \\ \nabla \times E - i\omega\mu H = 0, \end{cases} \tag{1.1}$$

in Ω, where $\gamma = \varepsilon + i\sigma/\omega$, i denotes the imaginary unit, and $\nabla\times$ denotes the *curl* operator.

Let ε, σ, μ be non-negative coefficients and assume that ε, μ are bounded from below in Ω. Then there exist positive values of ω for which the equations (1.1), posed in proper spaces and domains, with the tangential boundary condition either $N \times H|_{\partial\Omega} = 0$ or $N \times E|_{\partial\Omega} = 0$, have non-trivial solutions (see [60,42]). Such values of ω are called *resonant frequencies*.

The boundary data corresponding to the inverse boundary value problem (IBVP) for the system (1.1) only can be given by a boundary mapping (the *impedance* or *admittance* map) if ω is not a resonant frequency. The fact that the position of resonant frequencies depends on the unknown coefficients (as it is stated in [52]) motivated Pedro Caro in [16] to consider a *Cauchy data set* instead of a boundary map as boundary data. Cauchy data sets have been used in [14,55,56,16,17,20].

This work is focused on the IBVP for the system (1.1) with local boundary measurements established by a Cauchy data set taken just on a part of $\partial\Omega$. More precisely, Definition 1.1 describes the conditions for the domain and the part of its boundary where the measurements are taken and Definition 1.2 (used in [17]) introduces the boundary data for the IBVP studied in this article.

Definition 1.1. Let $\Omega \subset \mathbb{R}^3$ be a non-empty, bounded domain in $\mathbb{R}^3$ with $C^{1,1}$ boundary $\partial\Omega$. Assume Γ is a smooth proper non-empty open subset of $\partial\Omega$. We call Γ the *accessible part* of the boundary $\partial\Omega$ and $\Gamma_c := \partial\Omega \setminus \overline{\Gamma}$ the *inaccessible part* of the boundary.

Definition 1.2. Let Ω and Γ be as in Definition 1.1. For a pair of smooth coefficients μ, γ on Ω according to Definition 1.4, define the Cauchy data set restricted to Γ, write $C(\mu, \gamma; \Gamma)$, at frequency $\omega > 0$ by the set of couples $(T, S) \in TH_0(\Gamma) \times TH(\Gamma)$ such that there exists a solution $(E, H) \in (H(\Omega; \mathrm{curl}))^2$ of (1.1) in Ω satisfying $N \times E|_{\partial\Omega} = T$ and $N \times H|_{\Gamma} = S$, where the spaces $H(\Omega; \mathrm{curl})$, $TH(\Gamma)$, $TH_0(\Gamma)$ are defined in Definition 1.3.

It is known that if the domain Ω is not convex and its boundary is not $C^{1,1}$, Maxwell equations may not admit solutions in $H^1(\Omega)$ even for boundary data in $H^{1/2}(\partial\Omega)$ (see [8,9,57,58,23]). Thus, for a less regular domain (e.g., Lipschitz), some non-standard Sobolev spaces are necessary. Some of them, which will be used in these notes, appear in the following

Definition 1.3. Let Ω and Γ be as in Definition 1.1. Define $H^{1/2}(\Gamma) = \{f|_\Gamma : f \in H^{1/2}(\partial\Omega)\}$, with norm $\|g\|_{H^{1/2}(\Gamma)} = \inf\{\|f\|_{H^{1/2}(\partial\Omega)} : f|_\Gamma = g\}$, and $H_0^{1/2}(\Gamma) = \{f \in H^{1/2}(\partial\Omega) : \operatorname{supp} f \subset \overline{\Gamma}\}$, with norm $\|f\|_{H_0^{1/2}(\Gamma)} = \|f\|_{H^{1/2}(\partial\Omega)}$. Write $H^{-1/2}(\partial\Omega)$ for the dual space of $H^{1/2}(\partial\Omega)$. Consider the space $H(\Omega; \mathrm{curl}) = \{u \in L^2(\Omega; \mathbb{C}^3) : \nabla \times u \in L^2(\Omega; \mathbb{C}^3)\}$ with the usual graph norm, and the following

$$TH(\partial\Omega) = \{u \in H^{-1/2}(\partial\Omega; \mathbb{C}^3) : N \times v|_{\partial\Omega} = u, \text{ for some } v \in H(\Omega; \mathrm{curl})\},$$

$$TH(\Gamma) = \{u|_\Gamma : u \in TH(\partial\Omega)\},$$

$$TH_0(\Gamma) = \{u \in TH(\partial\Omega) : \operatorname{supp} u \subset \overline{\Gamma}\},$$

with norms $\|u\|_{TH(\partial\Omega)} = \inf\{\|v\|_{H(\Omega;\mathrm{curl})} : v \in H(\Omega;\mathrm{curl}),\ N \times v|_{\partial\Omega} = u\}$, $\|u\|_{TH(\Gamma)} = \inf\{\|v\|_{TH(\partial\Omega)} : v|_\Gamma = u\}$, $\|u\|_{TH_0(\Gamma)} = \|u\|_{TH(\partial\Omega)}$. Some properties of these spaces can be found, e.g., in [47,16,17].

Next, the class of admissible coefficients for the uniqueness result proven in this article is set.

Definition 1.4. Let Ω be as in Definition 1.1. Let $M > 0$. The pair of coefficients μ, γ is *admissible* if $\mu, \gamma \in C^{1,1}(\overline{\Omega}) \cap W^{2,\infty}(\Omega)$, and the following conditions are satisfied:

- $\operatorname{Re}\gamma \geq M^{-1}$, $\mu \geq M^{-1}$ in Ω,
- $\|\gamma\|_{W^{2,\infty}(\Omega)} + \|\mu\|_{W^{2,\infty}(\Omega)} \leq M$.

The main result of this work reads as follows.

Theorem 1.1. *Let Ω and Γ be as in Definition 1.1 and $\omega > 0$ the time-harmonic frequency. Assume μ_j, γ_j (with $j = 1, 2$) are two pairs of admissible coefficients such that* $\operatorname{supp}(\mu_1 - \mu_2)$, $\operatorname{supp}(\gamma_1 - \gamma_2) \subset \Omega$. *Then, if $C(\mu_1, \gamma_1; \Gamma) = C(\mu_2, \gamma_2; \Gamma)$ then $\gamma_1 = \gamma_2$ and $\mu_1 = \mu_2$ in Ω.*

The IBVP for Maxwell equations can be seen as a vector generalisation of the inverse conductivity problem of Calderón. In his seminal paper [15], Calderón posed two questions as follows: Firstly, is it possible to uniquely determine the conductivity of an unknown object from boundary measurements? Secondly, in the affirmative case, can this conductivity be reconstructed? Here the boundary measurements are determined by the Dirichlet-to-Neumann map Λ_σ, which for a conductivity $\sigma \in L^\infty(\Omega)$ defined on a bounded domain Ω modelling the object, is defined by $\Lambda_\sigma f = \sigma|_{\partial\Omega}(\partial u/\partial N)|_{\partial\Omega}$, where $f \in H^{1/2}(\partial\Omega)$, $u \in H^1(\Omega)$ solves the Dirichlet problem

$$\nabla \cdot \sigma \nabla u = 0 \text{ in } \Omega, \qquad u = f \text{ on } \partial\Omega,$$

and $(\partial u/\partial N)|_{\partial\Omega}$ denotes the normal derivative of u on $\partial\Omega$.

Matti Lassas in [41] proved that the boundary measurements of the Calderón problem are a low-frequency limit of the boundary data (impedance map) of the IBVP for time-harmonic Maxwell equations under some restrictions.

Concerning the Calderón problem in the plane, there are three main global uniqueness proofs giving reconstruction D-bar methods based on complex geometrical optics (CGO) solutions: the Schrödinger equation approach for twice differentiable σ by Nachman [50], the first-order system approach for once differentiable σ by Brown and Uhlmann [12], and the Beltrami equation approach assuming no smoothness ($\sigma \in L^\infty(\Omega)$) by Astala and Päivärinta [4]. The assumption $\sigma \in L^\infty(\Omega)$ was the one originally used by Calderón in [15]. Several stability estimates have been proven: [6,7,21,27].

In dimension $n \geq 3$ the best known uniqueness result for the Calderón problem is due to Haberman and Tataru [30] for continuously differentiable conductivities. A novel argument of decay *in average* using Bourgain-type spaces is introduced there. We cite some previous uniqueness results: the foundational [63] for smooth conductivities by Sylvester and Uhlmann, [49] where Nachman presents a reconstruction algorithm, and [10,11,54]. Concerning conditional stability, the best result is by the third author et al. in [18] for $C^{1,\varepsilon}$ conductivities on Lipschitz domains using the method in [30]. A previous stability result was given by Heck in [31]. Roughly speaking, the method introduced by Alessandrini in [1] gives the main guidelines followed by most stability methods for both the scalar and vector problems.

To deal with inverse problems from partial data for scalar elliptic equations, two main approaches are found in the literature in dimension $n > 2$ (see [3] for $n = 2$): using Carleman estimates [12,38,32] and using reflection arguments [35,33]. This work applies to the vector case the density argument shown in [2] for the scalar Schrödinger equation by Gunther Uhlmann and Habib Ammari.

The IBVP for stationary Maxwell equations with global data was originally proposed by Somersalo et al. in [60], where the coefficients are supposed to deviate only slightly from constant values. The same year the unique recovery of the parameters from the scattering amplitude for μ constant was presented in [22], and a local uniqueness result for the IBVP from global data was proven in [62]. The first global determination result for the IBVP with general coefficients μ, γ from global boundary measurements was proven by Lassi Päivärinta et al. in [51], assuming C^3 smoothness on μ, γ on $C^{1,1}$ domains. The proof is constructive. Later on, the proof was simplified via a relation between Maxwell equations and a matrix Helmholtz equation with a potential in [53]. Boundary determination results appeared in [45] and [36] for smooth boundaries. Chiral media were studied in [46]. Stability from global boundary data was obtained in [16]. Other inverse problems in electromagnetism in settings different to the ones in this paper have been considered in [59,34,40,44,37,20].

The uniqueness and stability issue of the IBVP for Maxwell equations with local boundary data has been little studied. The only works in this direction the authors are aware of are [19, 17], where an extension of Isakov's method in [35] to Maxwell system is performed. Another extension of methods used in the scalar case for partial data to vector systems is in [56], where uniqueness for a Dirac-type system is proven following the ideas of [38].

Our proof uses the CGO solutions given in [16,17] to a matrix Schrödinger-type equation related to Maxwell system and a Dirac-type system not related to Maxwell equations. As in Lemma 3.2 of [37] we give an integral identity (Proposition 4.2) involving a solution Z_1 to the Schrödinger-type equation and a solution Y_2 to the Dirac system. In order that such an identity holds for boundary data restricted to Γ, the solutions have to satisfy certain local homogeneous boundary conditions on Γ_c. In [17] the solutions with such properties are constructed from the CGO solutions following the reflection principle in [35], arising this way the strong geometrical constraint on Γ_c of being plane or part of a sphere.

We manage to avoid this annoying restriction by the density argument given by Lemma 2 in [2] for the scalar Schrödinger equation adapted to a vector Helmholtz equation satisfied by the electric (and magnetic with different coefficients) field related to Z_1 and another matrix Schrödinger-type equation verified by Y_2. Here, unique continuation principles for the aforementioned vector equations and a stability estimate for the inverse of the Dirac-type operator are required.

The density argument makes the assumption that the boundary is $C^{1,1}$ and the coefficients coincide on a neighbourhood of the boundary, the latter being a natural property in applications. The rest of the proof is valid with just Lipschitz boundary.

This article is organised as follows. In Section 2 some key matrix equations are introduced with the novelty, compared to previous works, that we use a Schrödinger-type equation satisfied by the solutions to the Dirac-type equation $(P + W^*)\widehat{Y} = 0$ not related to the Maxwell system. The CGO solutions to some of these equations used in this article are recalled in Section 3. Section 4 is devoted to showing an orthogonality identity involving the potentials and solutions to matrix equations corresponding to two couples of admissible coefficients. Two density results which are essential in the proof are presented in Section 5. Finally, our proof of uniqueness is expounded in Section 6 which contains a demonstration of the bounded invertibility of a Dirac-type operator with certain boundary conditions.

Throughout this work the following notation is used.

Notation. Given a pair of coefficients (μ_j, γ_j) for $j = 1, 2$, denote $C^j_\Gamma := C(\mu_j, \gamma_j; \Gamma)$, where $C(\mu_j, \gamma_j; \Gamma)$ is defined in Definition 1.2. For an expression like $U \cdot V$, with U, V $\mathbb{C}^m$-valued vector fields and m a natural number, $\cdot$ denotes the analytic extension to $\mathbb{C}^m$ of the Euclidean real-inner product on $\mathbb{R}^m$. I_m stands for the $m \times m$ identity matrix. For a matrix of complex entries U, the expressions U^t and U^* stand for its transpose and conjugate transpose, $\overline{U}^{\,t}$, respectively. For the domain Ω and complex vector fields $U, V \in H^1(\Omega; \mathbb{C}^m)$, denote

$$(U|V)_\Omega := \int_\Omega V^* U \, dx, \qquad (U|V)_{\partial\Omega} := \int_{\partial\Omega} V^* U \, ds,$$

where ds denotes the restriction of the Lebesgue measure of $\mathbb{R}^3$ to $\partial\Omega$.

2. The equations

Here some differential systems related to Maxwell equations are presented.

Let us start with the classical Schrödinger-type equation approach by Petri Ola and Erkki Somersalo in [53]. Fix a frequency $\omega > 0$. Assume the coefficients μ, γ to be in $C^{1,1}(\overline{\Omega})$ on a bounded domain with boundary locally described by the graph of a Lipschitz function. Following the notation in [16], write

$$\alpha := \log \gamma, \qquad \beta := \log \mu, \qquad \kappa := \omega \mu^{1/2} \gamma^{1/2}.$$

The vector fields $E, H \in H(\Omega; \mathrm{curl})$ solve Maxwell equations (1.1) with coefficients μ, γ if and only if $X = (h \quad H^t \mid e \quad E^t)^t$ solves the so-called *augmented system* $(P + V)X = 0$ and the scalar fields e, h vanish, where

$$P := \left(\begin{array}{cc|cc} & & & D\cdot \\ & & D & -D\times \\ \hline & D\cdot & & \\ D & D\times & & \end{array}\right), \quad V := \left(\begin{array}{cc|cc} \omega\mu & & & D\alpha\cdot \\ & \omega\mu I_3 & D\alpha & \\ \hline & D\beta\cdot & \omega\gamma & \\ D\beta & & & \omega\gamma I_3 \end{array}\right), \tag{2.1}$$

with $D := 1/i\nabla$, ∇ denoting the *gradient* operator, and $\nabla\cdot$ the *divergence* operator. Further, X solves $(P + V)X = 0$ if and only if $Y = \mathrm{diag}(\mu^{1/2} I_4, \gamma^{1/2} I_4)\, X$ solves the *rescaled system* $(P + W)Y = 0$ with

$$W := \kappa I_8 + \frac{1}{2}\left(\begin{array}{cc|cc} & & & D\alpha\cdot \\ & & D\alpha & D\alpha\times \\ \hline & D\beta\cdot & & \\ D\beta & -D\beta\times & & \end{array}\right), \tag{2.2}$$

where $\times$ denotes the cross product. Define the terms

$$Q := WP - PW^t - WW^t, \qquad \widehat{Q} := W^*P - P\,\overline{W} - W^*\,\overline{W}. \tag{2.3}$$

It can be checked that the expressions $WP - PW^t$ and $W^*P - P\,\overline{W}$ are zeroth-order. Therefore, the second-order operators

$$-\Delta I_8 + Q = (P + W)(P - W^t), \qquad -\Delta I_8 + \widehat{Q} = (P + W^*)(P - \overline{W}),$$

which do not contain first-order terms, are Schrödinger-type. In [16,17] a further zeroth-order operator Q' is considered which it is not used in this work.

Note that if $(-\Delta I_8 + Q)Z = 0$ and $X := \mathrm{diag}(\mu^{-1/2} I_4, \gamma^{-1/2} I_4)(P - W^t)Z$ then $(P + V)X = 0$. If additionally the scalar fields in X are identically zero, the vector fields in X give the electromagnetic fields verifying Maxwell equations. Moreover, if $\widehat{Z}$ solves $(-\Delta I_8 + \widehat{Q})\widehat{Z} = 0$ then $\widehat{Y} := (P - \overline{W})\widehat{Z}$ is solution to $(P + W^*)\widehat{Y} = 0$.

Finally, a new matrix Schrödinger potential is introduced, namely $\widetilde{Q}$ in (2.4), satisfying Lemma 2.1 below.

Specify explicitly the (κ, α, β)-dependence of W, W^t and W^* by writing

$$W = W(\kappa, \alpha, \beta), \qquad W^t = W^t(\kappa, \alpha, \beta), \qquad W^* = W^*(\kappa, \alpha, \beta).$$

A straightforward computation gives $W^*(\kappa, \alpha, \beta) = -W^t(-\bar{\kappa}, \bar{\alpha}, \beta)$. Since the relation among κ, α and β is not involved in the proof of the fact that $-PW^t + WP$ is zeroth-order, we deduce that $-PW^t(-\bar{\kappa}, \bar{\alpha}, \beta) + W(-\bar{\kappa}, \bar{\alpha}, \beta)P$ is zeroth-order. Thus, the matrix operator

$$(P + W(-\bar{\kappa}, \bar{\alpha}, \beta))(P - W^t(-\bar{\kappa}, \bar{\alpha}, \beta)) = -\Delta I_8 + \widetilde{Q}$$

is Schrödinger-type, where

$$\widetilde{Q} := -PW^t(-\bar{\kappa}, \bar{\alpha}, \beta) + W(-\bar{\kappa}, \bar{\alpha}, \beta)P - W(-\bar{\kappa}, \bar{\alpha}, \beta)W^t(-\bar{\kappa}, \bar{\alpha}, \beta) \tag{2.4}$$

is zeroth-order. We deduce the following

Lemma 2.1. *Assume* $\widehat{Y} \in H^1(\Omega; \mathbb{C}^8)$. *If* $(P + W^*)\widehat{Y} = 0$ *then* $(-\Delta I_8 + \widetilde{Q})\widehat{Y} = 0$.

Notation. In the rest of the manuscript, for two pairs of coefficients μ_j, γ_j with $j = 1, 2$, we will write Q_j, $\widetilde{Q}_j$, W_j to refer to the zeroth-order matrix operators Q, $\widetilde{Q}$, W defined in (2.3), (2.4), (2.2), respectively, for the case $\mu = \mu_j$, $\gamma = \gamma_j$.

3. The special solutions

In this section we recall the almost exponentially growing solutions Z, Y constructed in [16] for the systems $(-\Delta I_8 + Q)Z = 0$, $(P + W^*)Y = 0$ based on ideas of the papers [63,10,53,37]. Here the coefficients μ_j, γ_j $(j = 1, 2)$ under Theorem 1.1's conditions have to be considered extended to the whole Euclidean space $\mathbb{R}^3$. We denote the extended coefficients in the same manner μ_j, γ_j. The extensions fulfil the properties as follows:

1. They are Whitney type (see [61] for their construction).
2. The extensions preserve the regularity and a priori conditions stated in Definition 1.4.

3. The extended functions μ_j, γ_j satisfy that $\mu_j = \mu_0, \gamma_j = \varepsilon_0$ outside a ball $B(\mathrm{O}, \rho)$ centred at origin O with radius $\rho > 0$ such that $\overline{\Omega} \subset B(\mathrm{O}, \rho)$, where $\mu_0, \varepsilon_0 > 0$ are constants.
4. Denoting likewise the matrices Q, $\widehat{Q}$ obtained by replacing the coefficients by their extensions, the matrix functions $\omega^2\varepsilon_0\mu_0 I_8 + Q$, $\omega^2\varepsilon_0\mu_0 I_8 + \widehat{Q}$ are compactly supported in $\overline{B(\mathrm{O}, \rho)}$.

These properties allow Caro to prove Proposition 9 and Proposition 11 in [16], which we present here for the reader's convenience.

Notation. $\|f\|^2_{L^2_\delta} = \int_{\mathbb{R}^3}(1+|x|^2)^\delta |f(x)|^2\,dx$, for $\delta \in \mathbb{R}$.

Proposition 3.1. *Let $-1 < \delta < 0$ and $\zeta \in \mathbb{C}^3$ with $\zeta \cdot \zeta = \omega^2\varepsilon_0\mu_0$. Assume*

$$|\zeta| > C(\delta, \rho)\left(\sum_{j=1,2}\left\|\omega^2\varepsilon_0\mu_0 + q_j\right\|_{L^\infty(B_\rho)} + \left\|\omega^2\varepsilon_0\mu_0 I_8 + Q\right\|_{L^\infty(B_\rho;\mathcal{M}_{8\times 8})}\right),$$

where $\mathcal{M}_{8\times 8}$ denotes the space of 8×8 matrices with complex entries, and

$$q_1 = -\frac{1}{2}\Delta\beta - \kappa^2 - \frac{1}{4}(D\beta \cdot D\beta), \qquad q_2 = -\frac{1}{2}\Delta\alpha - \kappa^2 - \frac{1}{4}(D\alpha \cdot D\alpha).$$

Then there exists a solution

$$Z(x,\zeta) = e^{i\zeta\cdot x}(L(\zeta) + R(x,\zeta)),$$

to $(-\Delta I_8 + Q)Z = 0$ in $\mathbb{R}^3$ with $Z|_\Omega \in H^2(\Omega; \mathbb{C}^8)$, where

$$L(\zeta) = \frac{1}{|\zeta|}\begin{pmatrix} \zeta \cdot A_1 \\ \omega\varepsilon_0^{1/2}\mu_0^{1/2}B_1 \\ \hline \zeta \cdot B_1 \\ \omega\varepsilon_0^{1/2}\mu_0^{1/2}A_1 \end{pmatrix} \tag{3.1}$$

with A_1, B_1 constant complex vector fields, and

$$\|R\|_{L^2_\delta(\mathbb{R}^3;\mathbb{C}^8)} \le \frac{C(\delta,\rho)}{|\zeta|}|L|\left\|\omega^2\epsilon_0\mu_0 I_8 + Q\right\|_{L^\infty(B_\rho;\mathcal{M}_{8\times 8})}. \tag{3.2}$$

Furthermore, $Y := (P - W^t)Z$ solves $(P + W)Y = 0$ in $\mathbb{R}^3$ and has the form

$$Y = (0 \quad \mu^{1/2}H^t \mid 0 \quad \gamma^{1/2}E^t)^t,$$

with E, H solutions of (1.1) in $\mathbb{R}^3$.

Proposition 3.2. *Let $\zeta \in \mathbb{C}^3$ with $\zeta \cdot \zeta = \omega^2 \varepsilon_0 \mu_0$ and*

$$|\zeta| > C(\rho) \left\| \omega^2 \varepsilon_0 \mu_0 I_8 + \widehat{Q} \right\|_{L^\infty(B_\rho; \mathcal{M}_{8\times 8})}.$$

Then there exists a solution

$$\widehat{Y}(x,\zeta) = e^{i\zeta\cdot x}(M(\zeta) + S(x,\zeta))$$

to $(P + W^*)\widehat{Y} = 0$ in $\mathbb{R}^3$ with $\widehat{Y}|_\Omega \in H^1(\Omega; \mathbb{C}^8)$,

$$M(\zeta) = \frac{1}{|\zeta|} \left(\begin{array}{c} \zeta \cdot A_2 \\ -\zeta \times A_2 \\ \hline \zeta \cdot B_2 \\ \zeta \times B_2 \end{array} \right), \tag{3.3}$$

where A_2, B_2 are constant complex vector fields, and

$$\|S\|_{L^2(\Omega;\mathbb{C}^8)} \leq \frac{C(\rho,\Omega)}{|\zeta|} \left(\left\| \omega^2 \varepsilon_0 \mu_0 I_8 + \widehat{Q} \right\|_{L^\infty(B_\rho;\mathcal{M}_{8\times 8})} + \|W\|_{L^\infty(\Omega;\mathcal{M}_{8\times 8})} \right). \tag{3.4}$$

4. An orthogonality identity

This section is aimed at proving an orthogonality identity given by Proposition 4.2 involving solutions on the open set Ω to certain matrix partial differential equations whose traces contain information supported on $\overline{\Gamma}$.

Lemma 4.1. *Let μ_j, γ_j belong to $C^{0,1}(\overline{\Omega})$ for $j = 1, 2$. Let*

$$Y_1 = (0 \quad \mu_1^{1/2} H_1^t \,|\, 0 \quad \gamma_1^{1/2} E_1^t)^t,$$

with $E_1, H_1 \in H(\Omega; curl)$ solutions of

$$\nabla \times H_1 + i\omega\gamma_1 E_1 = 0, \tag{4.1}$$

$$\nabla \times E_1 - i\omega\mu_1 H_1 = 0, \tag{4.2}$$

in Ω such that $N \times E_1 = 0$ on Γ_c. In addition, suppose that

$$Y_2 = (f^1 \quad (u^1)^t \,|\, f^2 \quad (u^2)^t)^t$$

is a solution to $(P + W_2^*)Y_2 = 0$ in Ω with $f^j \in H^1(\Omega)$, $u^j \in H(\Omega; curl)$ and $f^1 = N \times u^2 = 0$ on Γ_c. Hence, for any pair E_2, H_2 in $H(\Omega; curl)$ of solutions to

$$\begin{cases} \nabla \times H_2 + i\omega\gamma_2 E_2 = 0, \\ \nabla \times E_2 - i\omega\mu_2 H_2 = 0, \end{cases} \tag{4.3}$$

in Ω such that $N \times E_2|_{\partial\Omega} = 0$ on Γ_c, the following estimate holds:

$$
\begin{aligned}
&|(Y_1|PY_2)_\Omega - (PY_1|Y_2)_\Omega| \\
&\quad \leq C\left(\|N \times (E_1 - E_2)|_{\partial\Omega}\|_{TH_0(\Gamma)} + \|N \times (H_1 - H_2)|_\Gamma\|_{TH(\Gamma)} \right) \\
&\qquad \times \left(\left\|\mu_2^{-1/2}\right\|_{C^{0,1}(\overline{\Gamma})} \left\|f^2\right\|_{H^{1/2}(\Gamma)} + \left\|\gamma_2^{1/2}\right\|_{C^{0,1}(\overline{\Gamma})} \left\|N \times u^1\right\|_{TH(\Gamma)} \right. \\
&\qquad + \left. \left\|\gamma_2^{-1/2}\right\|_{C^{0,1}(\overline{\Gamma})} \left\|f^1\right\|_{H_0^{1/2}(\Gamma)} + \left\|\mu_2^{1/2}\right\|_{C^{0,1}(\overline{\Gamma})} \left\|N \times u^2\right\|_{TH_0(\Gamma)} \right) \\
&\qquad + C\left(\|N \times E_1\|_{TH_0(\Gamma)} + \|N \times H_1\|_{TH(\Gamma)} \right) \\
&\qquad \times \left(\left\|\mu_1^{-1/2} - \mu_2^{-1/2}\right\|_{C^{0,1}(\overline{\Gamma})} \left\|f^2\right\|_{H^{1/2}(\Gamma)} + \left\|\gamma_1^{1/2} - \gamma_2^{1/2}\right\|_{C^{0,1}(\overline{\Gamma})} \left\|N \times u^1\right\|_{TH(\Gamma)} \right. \\
&\qquad + \left. \left\|\gamma_1^{-1/2} - \gamma_2^{-1/2}\right\|_{C^{0,1}(\overline{\Gamma})} \left\|f^1\right\|_{H_0^{1/2}(\Gamma)} + \left\|\mu_1^{1/2} - \mu_2^{1/2}\right\|_{C^{0,1}(\overline{\Gamma})} \left\|N \times u^2\right\|_{TH_0(\Gamma)} \right).
\end{aligned}
$$

Lemma 4.1 follows from the proof of Lemma 3.3 in [17] by making the solutions Y_1, Y_2 on Ω play the role of $\mathcal{Y}_1$, $\mathcal{Y}_2$ on U in Lemma 3.3 from [17]. This is achieved imposing directly to Y_1, Y_2 the appropriate boundary conditions on $\partial\Omega$, namely the tangential component of the electric field appearing in the structure of Y_1 vanishes on the inaccessible part of the boundary, and concerning Y_2, the trace of the first component and the tangential component of the second vector field also vanish on the inaccessible part of the boundary. In Lemma 3.3 from [17] such boundary conditions for $\mathcal{Y}_1$, $\mathcal{Y}_2$ come from a reflection argument using the special geometric conditions assumed to $\partial U \setminus \overline{\Gamma}$ there, which cannot be used here.

The proof of Lemma 4.1 uses Lemma 2.2, Lemma 2.4, Lemma 2.5 and Lemma 2.6 in [17]. Lemma 4.1's proof is omitted since, up to these comments, is identical to Lemma 3.3's proof in [17].

Proposition 4.2. *Let μ_j, γ_j be an admissible pair of coefficients ($j = 1, 2$) such that $C_\Gamma^1 = C_\Gamma^2$. Additionally, suppose $\mu_1 = \mu_2$, $\gamma_1 = \gamma_2$, $\partial_{x_l}\mu_1 = \partial_{x_l}\mu_2$ and $\partial_{x_l}\gamma_1 = \partial_{x_l}\gamma_2$ on $\overline{\Gamma}$ for $l = 1, 2, 3$. Then*

$$((Q_1 - Q_2)Z_1|Y_2)_\Omega = 0$$

holds for any $Z_1 \in H^1(\Omega; \mathbb{C}^8)$ solving $(P - W_1^t)Z_1 = Y_1$ in Ω with

$$Y_1 = (0 \quad \mu_1^{1/2} H_1^t \,|\, 0 \quad \gamma_1^{1/2} E_1^t)^t,$$

where $E_1, H_1 \in H(\Omega; curl)$ are solutions of

$$\nabla \times H_1 + i\omega\gamma_1 E_1 = 0, \tag{4.4}$$

$$\nabla \times E_1 - i\omega\mu_1 H_1 = 0, \tag{4.5}$$

in Ω and $N \times E_1|_{\partial\Omega} = 0$ on Γ_c, and for any $Y_2 \in H^1(\Omega; \mathbb{C}^8)$ verifying $(P + W_2^*)Y_2 = 0$ in Ω such that $Y_2|_{\partial\Omega} = 0$ on Γ_c.

Proof of Proposition 4.2. Following the proof of Proposition 1 in [17], we obtain the identity

$$((Q_1 - Q_2)Z_1|Y_2)_\Omega = ((W_1^t - W_2^t)Z_1|P_N Y_2)_{\partial\Omega} + (Y_1|PY_2)_\Omega - (PY_1|Y_2)_\Omega,$$

whose proof involves integration by parts and the relations $(P + W_2^*)Y_2 = 0$, $Y_1 = (P - W_1^t)Z_1$, $(P + W_1)Y_1 = 0$. Here, denote

$$P_N := \frac{1}{i}\left(\begin{array}{cc|cc} & & & N\cdot \\ & & N & -N\times \\ \hline & N\cdot & & \\ N & N\times & & \end{array}\right).$$

Since $\mu_1 = \mu_2$, $\gamma_1 = \gamma_2$, $\partial_{x_l}\mu_1 = \partial_{x_l}\mu_2$ and $\partial_{x_l}\gamma_1 = \partial_{x_l}\gamma_2$ on $\overline{\Gamma}$ (for $l = 1, 2, 3$), it follows that

$$W_1^t - W_2^t = 0 \qquad \text{on } \overline{\Gamma}. \tag{4.6}$$

From the fact that $Y_2|_{\partial\Omega}$ is supported on $\overline{\Gamma}$, we have

$$P_N Y_2|_{\Gamma_c} \equiv 0. \tag{4.7}$$

By (4.6), (4.7),

$$((W_1^t - W_2^t)Z_1|P_N Y_2)_{\partial\Omega} = 0. \tag{4.8}$$

Since $\mu_1 = \mu_2$ and $\gamma_1 = \gamma_2$ on $\overline{\Gamma}$,

$$\left\|\mu_1^{1/2} - \mu_2^{1/2}\right\|_{C^{0,1}(\overline{\Gamma})} = \left\|\mu_1^{-1/2} - \mu_2^{-1/2}\right\|_{C^{0,1}(\overline{\Gamma})} = \left\|\gamma_1^{1/2} - \gamma_2^{1/2}\right\|_{C^{0,1}(\overline{\Gamma})} \tag{4.9}$$

$$= \left\|\gamma_1^{-1/2} - \gamma_2^{-1/2}\right\|_{C^{0,1}(\overline{\Gamma})} = 0. \tag{4.10}$$

Since $(N \times E_1|_{\partial\Omega}, N \times H_1|_\Gamma) \in C_\Gamma^1 = C_\Gamma^2$, there exist solutions E_2, H_2 in $H(\Omega; \mathrm{curl})$ to the Maxwell system (4.3) such that $N \times E_2|_{\partial\Omega}$ is supported on $\overline{\Gamma}$ and

$$(N \times E_1|_{\partial\Omega}, N \times H_1|_\Gamma) = (N \times E_2|_{\partial\Omega}, N \times H_2|_\Gamma). \tag{4.11}$$

Applying Lemma 4.1 to Y_1, Y_2, E_2, H_2 and by (4.9)–(4.10), (4.11),

$$(Y_1|PY_2)_\Omega - (PY_1|Y_2)_\Omega = 0. \tag{4.12}$$

By identities (4.8), (4.12), we conclude $((Q_1 - Q_2)Z_1|Y_2)_\Omega = 0$. $\square$

5. Density and unique continuation results

In the remainder of the paper, let Ω and Γ be as in Definition 1.1. If $E, H \in H(\Omega; \text{curl})$ solve system (1.1) in Ω for certain coefficients μ, γ, then E, H are also solutions to the following second order system:

$$\nabla \times (\mu^{-1}\nabla \times E) - \omega^2 \gamma E = 0,$$
$$\nabla \times (\gamma^{-1}\nabla \times H) - \omega^2 \mu H = 0.$$

Notation. For known coefficients μ, γ and frequency $\omega > 0$, notation L will refer to the following Helmholtz-type vector second order differential operator defined in the sense of distributions for $U \in (C_0^\infty)'(\Omega; \mathbb{C}^3)$ by

$$L\,U = \nabla \times (\mu^{-1}\nabla \times U) - \omega^2 \gamma U. \tag{5.1}$$

For Lipschitz continuous functions μ, γ on Ω, assuming μ to be bounded from below, $L\,U$ is an L^2 vector field if $U \in H(\Omega; \text{curl})$ and $\nabla \times (\nabla \times U) \in L^2(\Omega; \mathbb{C}^3)$, since $\nabla \times (\mu^{-1}\nabla \times U) = (\nabla \mu^{-1}) \times (\nabla \times U) + \mu^{-1}\nabla \times (\nabla \times U)$.

Note the following integration by parts formula for any $E, F \in C^\infty(\Omega; \mathbb{C}^3)$:

$$\int_\Omega (L\,E) \cdot F\,dx = \int_\Omega E \cdot (L\,F)\,dx - \int_{\partial\Omega} \mu^{-1}\left(E \cdot (N \times (\nabla \times F)) + (\nabla \times E) \cdot (N \times F)\right) ds. \tag{5.2}$$

Next, the density result for the scalar Schrödinger equation given by Lemma 2 in [2] is adapted to Schrödinger-type matrix equations (Proposition 5.1) and the second order operator L (Proposition 5.2).

Proposition 5.1. *Let Ω' be an open subset of $\mathbb{R}^3$ with C^2 boundary. Assume $\Omega' \subset\subset \Omega$ and $\Omega \setminus \overline{\Omega'}$ is connected. Let $\widetilde{Q}$ be the zeroth-order 8×8 matrix operator defined in (2.4) for an admissible pair of coefficients μ, γ. Then the set*

$$\widetilde{K}(\Omega) := \{g \in H^2(\Omega; \mathbb{C}^8) : (-\Delta I_8 + \widetilde{Q})g = 0 \text{ in } \Omega,\ g|_{\partial\Omega} = 0 \text{ on } \Gamma_c\}$$

is dense in the space $K(\Omega) := \{v \in H^2(\Omega; \mathbb{C}^8) : (-\Delta I_8 + \widetilde{Q})v = 0 \text{ in } \Omega\}$ with respect to the topology in $L^2(\Omega'; \mathbb{C}^8)$. Here, g, v denote 8×1 vector fields.

Proposition 5.2. *Let Ω' be an open subset of $\mathbb{R}^3$ with C^2 boundary. Assume $\Omega' \subset\subset \Omega$ and $\Omega \setminus \overline{\Omega'}$ is connected. Let L be the differential operator defined in (5.1) for an admissible pair of coefficients μ, γ. Then the set $\widetilde{N}(\Omega)$ of vector functions $\widetilde{E} \in H(\Omega; \text{curl})$ such that $\nabla \times (\nabla \times \widetilde{E}) \in L^2(\Omega; \mathbb{C}^3)$, $L\,\widetilde{E} = 0$ in Ω, $N \times \widetilde{E}|_{\partial\Omega} = 0$ on Γ_c, is dense in the space*

$$N(\Omega) := \{E \in H(\Omega; \text{curl}) : \nabla \times (\nabla \times E) \in L^2(\Omega; \mathbb{C}^3),\ L\,E = 0 \text{ in } \Omega\}$$

with respect to the topology in $L^2(\Omega'; \mathbb{C}^3)$.

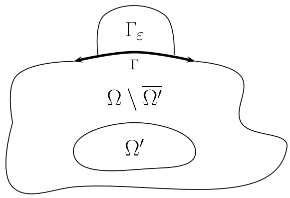

Fig. 1. Picture of possible choices of the sets Γ, Γ_ε, Ω, Ω' in the plane. This is for the sake of clarification only; remember that the open sets Γ_ε, Ω, Ω' are taken in the Euclidean topology of $\mathbb{R}^3$ and $\Gamma \subset \partial\Omega$.

The following unique continuation principles, Lemma 5.3 and Lemma 5.4, are used to prove Proposition 5.1 and Proposition 5.2.

Lemma 5.3 *(Unique continuation principle for matrix Schrödinger-type equations). Let $\widetilde{Q}$ be the zeroth-order 8×8 matrix operator defined in (2.4) for a pair μ, γ of admissible coefficients. Assume $\Omega' \subset\subset \Omega$ and $\Omega \setminus \overline{\Omega'}$ is connected with $\partial\Omega' \in C^2$. Hence,*

i) If $u \in H^2(\Omega; \mathbb{C}^8)$ satisfies $(-\Delta I_8 + \widetilde{Q})u = 0$ in Ω and $u = 0$ on B for some open ball B such that $\overline{B} \subset \Omega$ then $u = 0$ in Ω.

ii) Suppose $u \in H^2(\Omega \setminus \overline{\Omega'}; \mathbb{C}^8)$ verifies $(-\Delta I_8 + \widetilde{Q})u = 0$ in $\Omega \setminus \overline{\Omega'}$, $u = 0$ on $\partial\Omega$, $\left(\frac{\partial}{\partial\nu} I_8\right)u\big|_{\partial\Omega} = 0$ on Γ, where Γ is a smooth proper non-empty open subset of $\partial\Omega$. Then $u = 0$ on $\Omega \setminus \Omega'$.

Lemma 5.4 *(Unique continuation principle for L). Let $\mathcal{G}$ be a nonempty, open, bounded, connected subset of $\mathbb{R}^3$ with Lipschitz boundary $\partial\mathcal{G}$. Let L denote the operator (5.1) for scalar functions μ, $\gamma \in C^1(\overline{\mathcal{G}})$ with $\mu \geq C$, $Re\,\gamma \geq C$ in $\mathcal{G}$ for some constant $C > 0$.*

Further, assume $U \in H(\mathcal{G}; curl)$ and $\nabla \times (\nabla \times U) \in L^2(\mathcal{G}; \mathbb{C}^3)$. Therefore:

i) If $LU = 0$ in $\mathcal{G}$ and $U = 0$ in B for some open ball B such that $\overline{B} \subset \mathcal{G}$, then $U = 0$ in $\mathcal{G}$.

ii) Let Γ' denote a nonempty, smooth, open subset of $\partial\mathcal{G}$. If $LU = 0$ in $\mathcal{G}$, $N \times U|_{\partial\mathcal{G}} = 0$ on Γ' and $N \times (\nabla \times U)|_{\partial\mathcal{G}} = 0$ on Γ', then $U = 0$ in $\mathcal{G}$. Here N also denotes the outward unit vector field normal to $\partial\mathcal{G}$.

Proof of Lemma 5.3. Part i) of Lemma 5.3 can be proven by trivially rewriting Theorem 6.5.1's proof in [25] for the vector case taking $W_1 = \|\widetilde{Q}\|_{L^\infty(\Omega;\mathcal{M}_{8\times 8})}$, $W_2 = 0$, $C = \emptyset$. Part ii) follows from part i), remarking that the boundary conditions on Γ guarantee that the extension of the solution by zero on a neighbourhood Γ_ε in $\mathbb{R}^3$ such that $\Gamma_\varepsilon \cap \Omega = \emptyset$ and $\overline{\Gamma_\varepsilon} \cap \overline{\Omega}$ is an open subset of Γ with respect to the relative topology on $\partial\Omega$ induced by the Euclidean topology of $\mathbb{R}^3$, satisfies the same equation and maintains the H^2-regularity on $\mathrm{int}((\overline{\Omega} \setminus \Omega') \cup \overline{\Gamma_\varepsilon})$. Indeed, the kernel of the trace operator $(u|_{\partial\mathcal{G}}, (\partial u/\partial N)|_{\partial\mathcal{G}})$ defined for $u \in H^2(\mathcal{G})$, is the closure of $C_0^\infty(\mathcal{G})$ in $H^2(\mathcal{G})$ (usually denoted by $H_0^2(\mathcal{G})$), for any domain $\mathcal{G}$ with $C^{1,1}$ boundary $\partial\mathcal{G}$ (see [43] or e.g. [29, Theorem 1.5.1.5]). For clarity Fig. 1 illustrates the sets Γ_ε, Ω, Ω' in the plane (although they must be considered in $\mathbb{R}^3$). $\square$

Proof of Lemma 5.4. Under the conditions of part i), define $V := (i\omega\mu)^{-1}\nabla \times U$ and check that (U, V) solves in $\mathcal{G}$ the Maxwell equations $\nabla \times V + i\omega\gamma U = 0$, $\nabla \times U - i\omega\mu V = 0$, and $U, V \in H(\mathcal{G}; \mathrm{curl})$ with $\nabla \cdot U$, $\nabla \cdot V \in L^2(\mathcal{G})$. By [28, Chapter I, Corollary 2.10], $U, V \in H^1_{\mathrm{loc}}(\mathcal{G}; \mathbb{C}^3)$. Consider another open ball B' with $B \cap B' \neq \emptyset$ and $B' \subset \mathcal{G}$. Since the restrictions of U, V to B' are in $H^1(B'; \mathbb{C}^3)$, from the unique continuation result across C^2-surfaces by Eller and

Yamamoto [26, Corollary 1.2] for the Maxwell system with C^1 coefficients, we deduce that $U = V = 0$ on B'. Propagating this argument we conclude that U and V vanish on any neighbourhood in $\mathcal{G}$. This proves part i).

Let Γ_ε be a nonempty, open, connected subset of $\mathbb{R}^3$ with Lipschitz boundary such that $\Gamma_\varepsilon \cap \mathcal{G} = \emptyset$, $\overline{\Gamma_\varepsilon} \cap \overline{\mathcal{G}}$ is an open subset of Γ' with respect to the relative topology on $\partial\mathcal{G}$ induced by the Euclidean topology of $\mathbb{R}^3$. Fig. 1 with $\Omega = \mathcal{G}$, $\Gamma = \Gamma'$ illustrates the choice of Γ_ε in the plane. The conditions of part ii) guarantee that the extension $\widetilde{U}$ of U by zero on Γ_ε verifies $\widetilde{U} \in H(\mathcal{G}'; \text{curl})$, $\nabla \times (\nabla \times \widetilde{U}) \in L^2(\mathcal{G}'; \mathbb{C}^3)$ and $L\widetilde{U} = 0$ in $\mathcal{G}'$, where $\mathcal{G}' := \text{int}(\overline{\mathcal{G}} \cup \overline{\Gamma_\varepsilon})$. This property follows from the fact that the $H(\text{curl})$-vector functions on a bounded, Lipschitz domain that can be approximated by smooth compactly supported functions in $H(\text{curl})$-norm are exactly those ones with zero tangential trace (see e.g. [48, Theorem 3.33] for details). By part i) of this Lemma 5.4, $\widetilde{U} = 0$ in $\mathcal{G}'$. In particular, $U = 0$ in $\mathcal{G}$. □

Regarding the aforementioned result in [26], note that a counterexample for the stationary Maxwell system with coefficients in the Hölder class C^α for every $\alpha < 1$ is provided in [24] by Demchenko.

Proof of Proposition 5.1. Following the lines of Lemma 2's proof in [2], suppose $v \in K(\Omega)$ satisfies $(g|v)_{\Omega'} = \int_{\Omega'} v^* g\, dx = 0$ for any $g \in \widetilde{K}(\Omega)$. We are going to prove that $v = 0$ in Ω.

Consider the Dirichlet Green's function G in Ω verifying for $x \in \Omega$,

$(-\Delta_y I_8)G(x, y) + G(x, y)\widetilde{Q}(y) = \delta(y - x)I_8$, for $y \in \Omega$, $G(x, y) = 0$, for $y \in \partial\Omega$, where δ denotes the Dirac delta function with pole at the origin and I_8 the 8×8 identity matrix. For $g \in \widetilde{K}(\Omega)$ and $x \in \Omega$ we have, by Green's formula,

$$\begin{aligned} g(x) &= \int_\Omega (\delta(y - x)I_8)g(y)dy = \int_\Omega (-(\Delta_y I_8)G(x, y) + G(x, y)\widetilde{Q}(y))g(y)dy \\ &= \int_\Omega G(x, y)(-\Delta_y I_8 + \widetilde{Q}(y))g(y)dy - \int_{\partial\Omega} \left(\left(\frac{\partial}{\partial\nu(y)} I_8\right)G(x, y)\right)g(y)\Big)ds(y) \\ &= -\int_\Gamma \left(\left(\frac{\partial}{\partial\nu(y)} I_8\right)G(x, y)\right)g(y)\, ds(y). \end{aligned}$$

In particular note that for $x \in \Omega$, $U(x) = \int_\Omega G(x, y)F(y)\, dy$ provided that $(-\Delta I_8 + \widetilde{Q})U = F$ in Ω and $U = 0$ on $\partial\Omega$.

By Fubini's theorem,

$$\begin{aligned} &\int_\Gamma \int_{\Omega'} v(x)^* \left(\left(\frac{\partial}{\partial\nu(y)} I_8\right)G(x, y)\right)dx\, g(y)\, ds(y) \\ &= \int_{\Omega'} v(x)^* \int_\Gamma \left(\left(\frac{\partial}{\partial\nu(y)} I_8\right)G(x, y)\right)g(y)\, ds(y)\, dx \\ &= -\int_{\Omega'} v(x)^* g(x)\, dx = 0. \end{aligned}$$

Thus, for $y \in \Gamma$,

$$\int_{\Omega'} v(x)^* \left(\left(\frac{\partial}{\partial \nu(y)} I_8 \right) G(x, y) \right) dx = 0. \tag{5.3}$$

Define the vector field u by

$$u(y)^* := \int_{\Omega'} v(x)^* G(x, y) \, dx.$$

Since $G(x, y) = 0$ for $y \in \partial\Omega$ and $x \in \Omega$, $u(y) = 0$ for $y \in \partial\Omega$. By (5.3),

$$\left(\frac{\partial}{\partial \nu(y)} I_8 \right) u(y) = 0 \qquad \text{for } y \in \Gamma. \tag{5.4}$$

Since $(-\Delta I_8 + (\widetilde{Q})^*)u = 0$ in $\Omega \setminus \overline{\Omega'}$, $u\big|_{\partial\Omega} = 0$ and by (5.4), it follows that $u = 0$ in $\Omega \setminus \Omega'$ by the unique continuation principle (Lemma 5.3). In particular,

$$u = \left(\frac{\partial}{\partial \nu} I_8 \right) u = 0 \qquad \text{on } \partial\Omega'.$$

Note that $(-\Delta I_8 + (\widetilde{Q})^*)u = v$ in Ω'. Now, we can write

$$\begin{aligned}
\int_{\Omega'} v(y)^* v(y) \, dy &= \int_{\Omega'} ((-\Delta_y I_8) u(y)^* + u(y)^* \widetilde{Q}(y)) v(y) \, dy \\
&= -\int_{\Omega'} u(y)^* (\Delta I_8) v(y) dy \\
&\quad + \int_{\partial\Omega'} \left(u(y)^* \left(\frac{\partial}{\partial \nu(y)} I_8 \right) v(y) - \left(\left(\frac{\partial}{\partial \nu(y)} I_8 \right) u(y)^* \right) v(y) \right) ds(y) \\
&\quad + \int_{\Omega'} u(y)^* \widetilde{Q}(y) v(y) dy \\
&= \int_{\Omega'} u(y)^* (-\Delta I_8 + \widetilde{Q}(y)) v(y) dy = 0,
\end{aligned} \tag{5.5}$$

where identity (5.5) follows from Green's formula. Hence $v = 0$ in Ω'. Since, $(-\Delta I_8 + \widetilde{Q})v = 0$ in Ω, by unique continuation (Lemma 5.3), $v = 0$ in Ω. □

Proof of Proposition 5.2. Fix $E \in N(\Omega)$ such that $\int_{\Omega'} \overline{E} \cdot \widetilde{E} \, dx = 0$ for any $\widetilde{E} \in \widetilde{N}(\Omega)$. Define E' as the solution to the equation $L E' = \chi_{\Omega'} \overline{E}$ in Ω satisfying $N \times E' = 0$ on $\partial\Omega$ in the trace sense, such that $E' \in H(\Omega; \mathrm{curl})$ and $\nabla \times \nabla \times E' \in L^2(\Omega; \mathbb{C}^3)$. Using the integration by parts formula (5.2), for any $\widetilde{E} \in \widetilde{N}(\Omega)$ we have

$$0 = \int_{\Omega'} \overline{E} \cdot \widetilde{E}\, dx = \int_{\Omega} L\, E' \cdot \widetilde{E}\, dx = \int_{\Omega} E' \cdot L\, \widetilde{E}\, dx \tag{5.6}$$

$$- \int_{\partial\Omega} \frac{1}{\mu} [(N \times E') \cdot (N \times (N \times \nabla \times \widetilde{E})) + (\nabla \times E') \cdot (N \times \widetilde{E})]\, ds \tag{5.7}$$

$$= - \int_{\Gamma} \frac{1}{\mu} (\nabla \times E') \cdot (N \times \widetilde{E})\, ds. \tag{5.8}$$

The trace of $\nabla \times E'$ on $\partial\Omega \in C^{1,1}$ can be decomposed into its tangential and normal components as follows:

$$\nabla \times E'|_{\partial\Omega} = -N \times (N \times (\nabla \times E'))|_{\partial\Omega} + (N \cdot (\nabla \times E'))\, N|_{\partial\Omega}. \tag{5.9}$$

On using the identity (5.9) in the integral (5.8), the second term in the right hand side of (5.9) gets cancelled. Therefore from (5.6)–(5.8) and (5.9) we deduce for each $\widetilde{E} \in \widetilde{N}(\Omega)$,

$$0 = \int_{\Gamma} \frac{1}{\mu} N \times (N \times (\nabla \times E'))|_{\partial\Omega} \cdot (N \times \widetilde{E})|_{\partial\Omega}\, ds.$$

So, $N \times (N \times (\nabla \times E'))|_{\partial\Omega}$ vanishes on Γ. As a result, $N \times (\nabla \times E')|_{\partial\Omega} = 0$ on Γ.

From the condition $N \times E'|_{\partial\Omega} = 0$ and the properties $N \times (\nabla \times E')|_{\partial\Omega} = 0$ on Γ and $L(E'|_{\Omega\setminus\overline{\Omega'}}) = 0$ in $\Omega \setminus \overline{\Omega'}$, it follows that $E' = 0$ in $\Omega \setminus \Omega'$ by the uniqueness result stated in part ii) of Lemma 5.4 for $\mathcal{G} = \Omega \setminus \overline{\Omega'}$ and $\Gamma' = \Gamma$. In particular,

$$E'|_{\partial\Omega'} = 0, \qquad \nabla \times E'|_{\partial\Omega'} = 0. \tag{5.10}$$

Now, we write

$$\int_{\Omega'} E^*\, E\, dx = \int_{\Omega'} (L\, E')^t\, E\, dx = \int_{\Omega'} (E')^t\, L\, E\, dx$$
$$- \int_{\partial\Omega'} \mu^{-1}((E')^t\, (N \times (\nabla \times E)) + (\nabla \times E')^t\, (N \times E))\, ds = 0,$$

where the last identity follows from formula (5.2). Hence, $E = 0$ in Ω'. Since $L\, E = 0$ in Ω, we deduce by the unique continuation principle stated in part i) of Lemma 5.4 with $\mathcal{G} = \Omega$, that $E = 0$ in Ω. $\square$

6. Proof of uniqueness

Here, the outline of [1], Section 3.2 in [16], Section 3.4 in [17] is adapted to prove Theorem 1.1.

Let $\omega > 0$ be the time-harmonic frequency. Assume μ_j, γ_j is an admissible pair of coefficients for each $j = 1, 2$, according to Definition 1.4, such that $\mathrm{supp}(\mu_1 - \mu_2), \mathrm{supp}(\gamma_1 - \gamma_2) \subset \Omega$. Let Ω' be an open subset of $\mathbb{R}^3$ with C^2 boundary such that $\Omega' \subset\subset \Omega$, $\Omega \setminus \overline{\Omega'}$ is connected and

$$\mu_1 = \mu_2 \qquad \text{and} \qquad \gamma_1 = \gamma_2 \qquad \text{in } \overline{\Omega} \setminus \Omega'. \tag{6.1}$$

Suppose $C_\Gamma^1 = C_\Gamma^2$. The extended coefficients to $\mathbb{R}^3$ according to the extensions described in Section 3 will be written likewise, μ_j, γ_j. Remember that $\mu_j = \mu_0$, $\gamma_j = \varepsilon_0$ outside the ball $B(\mathrm{O}, \rho)$, where $\overline{\Omega} \subset B(\mathrm{O}, \rho)$, and μ_0, ε_0 are constants.

Let $j \in \{1, 2\}$. Define

$$\alpha_j := \log \gamma_j, \qquad \beta_j := \log \mu_j, \qquad \kappa_j := \omega \mu_j^{1/2} \gamma_j^{1/2},$$
$$f := \chi_\Omega \cdot \left(\frac{1}{2} \Delta(\alpha_1 - \alpha_2) + \frac{1}{4} (\nabla\alpha_1 \cdot \nabla\alpha_1 - \nabla\alpha_2 \cdot \nabla\alpha_2) + (\kappa_2^2 - \kappa_1^2) \right),$$
$$g := \chi_\Omega \cdot \left(\frac{1}{2} \Delta(\beta_1 - \beta_2) + \frac{1}{4} (\nabla\beta_1 \cdot \nabla\beta_1 - \nabla\beta_2 \cdot \nabla\beta_2) + (\kappa_2^2 - \kappa_1^2) \right),$$

where χ_Ω denotes the characteristic function of Ω.

Fix $\xi \in \mathbb{R}^3 \setminus 0$. Let $\mathbb{S}^2$ denote the unit sphere in $\mathbb{R}^3$. Assume $\tau \geq 1$ and take $\eta_1, \eta_2 \in \mathbb{S}^2$ with $\eta_2 \cdot \eta_1 = \eta_1 \cdot \xi = \eta_2 \cdot \xi = 0$, and

$$\zeta_1 = -\frac{1}{2} \xi + i \left(\tau^2 + \frac{|\xi|^2}{4} \right)^{1/2} \eta_1 + (\tau^2 + \omega^2 \varepsilon_0 \mu_0)^{1/2} \eta_2,$$
$$\zeta_2 = \frac{1}{2} \xi - i \left(\tau^2 + \frac{|\xi|^2}{4} \right)^{1/2} \eta_1 + (\tau^2 + \omega^2 \varepsilon_0 \mu_0)^{1/2} \eta_2.$$

Note that $\zeta_j \in \mathbb{C}^3$ satisfies $\zeta_j \cdot \zeta_j = \omega^2 \varepsilon_0 \mu_0$, and

$$|\zeta_j| = (|\mathrm{Re}(\zeta_j)|^2 + |\mathrm{Im}(\zeta_j)|^2)^{1/2} = (|\xi|^2/2 + 2\tau^2 + \omega^2 \varepsilon_0 \mu_0)^{1/2}.$$

Further, $\zeta_1 - \overline{\zeta}_2 = -\xi$, and as $\tau \to \infty$,

$$\frac{\zeta_1}{|\zeta_1|} = \frac{1}{\sqrt{2}} (i\, \eta_1 + \eta_2) + \mathcal{O}(\tau^{-1}), \qquad \frac{\zeta_2}{|\zeta_2|} = \frac{1}{\sqrt{2}} (-i\, \eta_1 + \eta_2) + \mathcal{O}(\tau^{-1}),$$

where the implicit constants depend on $|\xi|$ (and ω, ε_0, μ_0).

Consider the special solutions

$$Z_1(x, \zeta_1) = e^{i \zeta_1 \cdot x} (L_1(\zeta_1) + R_1(x, \zeta_1)), \qquad Y_2(x, \zeta_2) = e^{i \zeta_2 \cdot x} (M_2(\zeta_2) + S_2(x, \zeta_2))$$

from Proposition 3.1 and Proposition 3.2 applied to the case $\mu = \mu_1$, $\gamma = \gamma_1$, $\zeta = \zeta_1$ and $\mu = \mu_2$, $\gamma = \gamma_2$, $\zeta = \zeta_2$, respectively, so that Z_1, Y_2 solve $(-\Delta I_8 + Q_1) Z_1 = 0$, $(P + W_2^*) Y_2 = 0$ in $\mathbb{R}^3$. Choosing such solutions with $B_j = 0$ and A_j such that

$$\left(i \frac{\eta_1}{\sqrt{2}} + \frac{\eta_2}{\sqrt{2}} \right) \cdot A_1 = \left(i \frac{\eta_1}{\sqrt{2}} + \frac{\eta_2}{\sqrt{2}} \right) \cdot \overline{A_2} = 1,$$

one obtains

$$((Q_1 - Q_2)Z_1|Y_2)_\Omega = \hat{f}(\xi) + \mathcal{O}(\tau^{-1}), \tag{6.2}$$

as $\tau \to \infty$. Analogously, choosing Z_1, Y_2 with $A_j = 0$ and B_j such that

$$\left(i\frac{\eta_1}{\sqrt{2}} + \frac{\eta_2}{\sqrt{2}}\right) \cdot B_1 = \left(i\frac{\eta_1}{\sqrt{2}} + \frac{\eta_2}{\sqrt{2}}\right) \cdot \overline{B_2} = 1,$$

one can prove

$$((Q_1 - Q_2)Z_1|Y_2)_\Omega = \hat{g}(\xi) + \mathcal{O}(\tau^{-1}), \tag{6.3}$$

as $\tau \to \infty$. In (6.2), (6.3) the implicit constant depends on M, ξ, $|\Omega|$, ρ, ω, ε_0, μ_0.

Fix $\epsilon > 0$. For each choice of Z_1, define $Y_1 := (P - W_1^t)Z_1$. Hence, $(P + W_1)Y_1 = 0$ in $\mathbb{R}^3$ and by Proposition 3.1, Y_1 reads $Y_1 = (0 \;\; \mu_1^{1/2} H_1^t \,|\, 0 \;\; \gamma_1^{1/2} E_1^t)^t$, with E_1, H_1 solutions of

$$\begin{aligned} \nabla \times H_1 + i\omega\gamma_1 E_1 &= 0, \\ \nabla \times E_1 - i\omega\mu_1 H_1 &= 0, \end{aligned} \tag{6.4}$$

in $\mathbb{R}^3$. In particular, $L_1 E_1 = 0$ in Ω, where L_1 denotes the second order operator L defined in (5.1) for μ_1, γ_1. By Proposition 3.1, $Z_1|_\Omega \in H^2(\Omega; \mathbb{C}^8)$. Thus, by the Lipschitz regularity and the a priori bounds from below for μ_1, γ_1, and from equation (6.4) we deduce that $E_1|_\Omega \in H(\Omega; \mathrm{curl})$ and $\nabla \times (\nabla \times E_1)|_\Omega \in L^2(\Omega; \mathbb{C}^3)$.

By Proposition 5.2, there exists $\widetilde{E}_1 \in H(\Omega; \mathrm{curl})$ such that $\nabla \times (\nabla \times \widetilde{E}_1) \in L^2(\Omega; \mathbb{C}^3)$, $L_1 \widetilde{E}_1 = 0$ in Ω, $N \times \widetilde{E}_1|_{\partial\Omega} = 0$ on Γ_c, and $\left\| E_1 - \widetilde{E}_1 \right\|_{L^2(\Omega'; \mathbb{C}^3)} < \epsilon$.

Due to the a priori condition $\mu_1 \geq M^{-1}$, μ_1 does not vanish. Define $\widetilde{H}_1 := (1/i\omega\mu_1)\nabla \times \widetilde{E}_1$. Therefore $\widetilde{E}_1$, $\widetilde{H}_1 \in H(\Omega; \mathrm{curl})$ solve

$$\begin{aligned} \nabla \times \widetilde{H}_1 + i\omega\gamma_1 \widetilde{E}_1 &= 0, \\ \nabla \times \widetilde{E}_1 - i\omega\mu_1 \widetilde{H}_1 &= 0, \end{aligned}$$

in Ω. Define $\widetilde{Y}_1 := (0 \;\; \mu_1^{1/2} \widetilde{H}_1^t \,|\, 0 \;\; \gamma_1^{1/2} \widetilde{E}_1^t)^t$.

6.1. *Invertibility of the Dirac-type operator P*

Along this subsection the letters M, N, E refer to mathematical entities which are different from their meanings in the rest of the paper.

Jochen Brüning and Matthias Lesch in [13] generalise the analysis of Dirac-type operators considered in the well-known paper by Atiyah, Patodi and Singer [5]. Concerning the general Dirac-type operators studied there on compact manifolds with boundary, in [13, Section 1.B] an operator D is introduced acting on sections of a hermitian vector bundle E over an open subset M of a compact oriented Riemannian manifold $\widetilde{M}$ such that its boundary $N = \partial M$ is a compact hypersurface in $\widetilde{M}$. The authors call $\widetilde{E}$ the vector bundle over $\widetilde{M}$, and $E_N := \widetilde{E} \upharpoonright N$. The differential operator D is said to be of Dirac type if it is first order, symmetric and elliptic in $L^2(E)$ with domain $C_0^\infty(E)$ verifying that D^2 has scalar principal symbol given by the metric tensor.

Taking $M = \Omega'$ and $E = M \times \mathbb{C}^8$ the trivial bundle over M, each fiber equipped with the standard hermitian inner product of $\mathbb{C}^8$, the operator P on Ω' defined in (2.1) falls into the category of these Dirac type operators, since $P^2 = -\Delta I_8$, $\langle PU, V\rangle_{\Omega'} = \langle U, PV\rangle_{\Omega'}$ for any U, $V \in C_0^\infty(\Omega'; \mathbb{C}^8)$ and the characteristic form of P, namely $Q(\lambda) = \det(\Lambda(\lambda)) = -i|\lambda|^8$, does not vanish for any $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3 \setminus 0$. Here, $\Lambda(\lambda)$ denotes the symbol of P given by the matrix form

$$\Lambda(\lambda) = \frac{1}{i}\left(\begin{array}{c|c} 0 & \mathcal{A}(\lambda) \\ \hline \mathcal{B}(\lambda) & 0 \end{array}\right),$$

with

$$\mathcal{A}(\lambda) = \begin{pmatrix} 0 & \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1 & 0 & \lambda_3 & -\lambda_2 \\ \lambda_2 & -\lambda_3 & 0 & \lambda_1 \\ \lambda_3 & \lambda_2 & -\lambda_1 & 0 \end{pmatrix}, \qquad \mathcal{B}(\lambda) = \begin{pmatrix} 0 & \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1 & 0 & -\lambda_3 & \lambda_2 \\ \lambda_2 & \lambda_3 & 0 & -\lambda_1 \\ \lambda_3 & -\lambda_2 & \lambda_1 & 0 \end{pmatrix}.$$

In [13] it is proved that D admits self-adjoint extensions by imposing non-local boundary conditions given by an orthogonal projection π in $L^2(E_N)$, which is a classical pseudodifferential operator on E_N satisfying a certain symmetry property (condition (1.13) in [13]) related to the structure of the operator (see [13, Lemma 1.1] for details). For such π and by [13, Theorem 1.5] and the interpretation by Y. Kurylev and M. Lassas [39, Theorem 2.1], it turns out that $P \restriction \mathcal{D}$ is self-adjoint with empty essential spectrum and finite-dimensional eigenspaces, where $\mathcal{D} := \{U \in H^1(\overline{\Omega'}; \mathbb{C}^8) : \pi(U|_{\partial\Omega'}) = 0\}$.

The domain $\mathcal{D}$ in $H^1(\Omega'; \mathbb{C}^8)$ with the graph norm associated with P is continuously embedded into $H^1(\Omega'; \mathbb{C}^8)$. The space $H^1(\Omega'; \mathbb{C}^8)$ is compactly embedded into $L^2(\Omega'; \mathbb{C}^8)$. Since $W \in L^\infty(\Omega'; \mathcal{M}_{8\times 8})$ for admissible μ, γ, the operator of multiplication by W^t, which we write $\mathcal{M}_{W^t}$, is bounded and linear in $L^2(\Omega'; \mathbb{C}^8)$. Therefore, $\mathcal{M}_{W^t}$ is $(P \restriction \mathcal{D})$-compact.

Thus, $P - W^t$ has also empty essential spectrum and finite-dimensional eigenspaces. If 0 is in the spectrum of $P - W^t$, then 0 must be an eigenvalue with finitely many linearly independent eigen- and associated functions. We can make 0 no longer be an eigenvalue by choosing a new set of boundary conditions which are not satisfied by any of the finitely many linearly independent eigen- and associated functions in the root spaces associated with 0. Let us keep denoting the resultant boundary operator by π so that the condition $\pi(Z|_{\partial\Omega'}) = 0$ guarantees the existence of a constant C_{stblty} independent of Z such that

$$\|Z\|_{L^2(\Omega';\mathbb{C}^8)} \leq C_{\text{stblty}} \left\|(P - W^t)Z\right\|_{L^2(\Omega';\mathbb{C}^8)}, \tag{6.5}$$

provided that $Z, (P - W^t)Z \in L^2(\Omega'; \mathbb{C}^8)$.

The argument presented in Subsection 6.1, together with a trick based on an auxiliary system which improves the regularity of $(P - W^t)Z$ when Maxwell equations are satisfied, leads to the following

Lemma 6.1. *For admissible coefficients μ, γ, assume that $(P - W^t)Z = Y$ in Ω', where Y reads $Y = (0 \;\; \mu^{1/2}H^t \,|\, 0 \;\; \gamma^{1/2}E^t)^t$, with E, H verifying (1.1) in Ω'. Additionally, suppose $\pi(Z|_{\partial\Omega'}) = 0$ for the boundary operator π introduced in Subsection 6.1. Then there exists a constant C only depending on C_{stblty}, M, ω, such that $\|Z\|_{L^2(\Omega';\mathbb{C}^8)} \leq C \, \|E\|_{L^2(\Omega';\mathbb{C}^3)}$.*

Proof of Lemma 6.1. Writing $Z = (Z_1 \;\; Z_H^t \,|\, Z_2 \;\; Z_E^t)^t$, where Z_j $(j = 1, 2)$ are scalar fields and Z_H, Z_E vector fields, and defining $Z_{\rm aux} := (Z_1 \;\; Z_H^t \,|\, Z_2 \;\; (Z_E')^t)^t$ with $Z_E' := Z_E + \mu^{-1/2}\omega^{-1}E$, the dependence on the electric field E of the vector function $(P - W^t)Z_{\rm aux}$ is zeroth-order. Indeed, it is straightforward to check that

$$(P - W^t)Z_{\rm aux} = \left(\begin{array}{c} (-i/\omega)(-2\nabla\mu^{-1/2} + \mu^{-1/2}\nabla\alpha)\cdot E \\ 0 \\ \hline 0 \\ (\gamma^{1/2} - \kappa\mu^{-1/2}\omega^{-1})E \end{array}\right).$$

By (6.5),

$$\begin{aligned} \|Z\|_{L^2(\Omega';\mathbb{C}^8)} &\le \|Z_{\rm aux}\|_{L^2(\Omega';\mathbb{C}^8)} + M^{1/2}\omega^{-1}\,\|E\|_{L^2(\Omega';\mathbb{C}^3)} \\ &\le C_{\rm stblty}\left\|(P - W^t)Z_{\rm aux}\right\|_{L^2(\Omega';\mathbb{C}^8)} + M^{1/2}\omega^{-1}\,\|E\|_{L^2(\Omega';\mathbb{C}^3)} \\ &\le (C_{\rm stblty}C(M,\omega) + M^{1/2}\omega^{-1})\,\|E\|_{L^2(\Omega';\mathbb{C}^3)}. \qquad \square \end{aligned}$$

For $\widetilde{Y}_1$ defined above, let $\widetilde{Z}_1$ be a solution to the system $(P - W_1^t)\widetilde{Z}_1 = \widetilde{Y}_1$ in Ω' such that $\pi((Z_1 - \widetilde{Z}_1)|_{\partial\Omega'}) = 0$. By Lemma 6.1,

$$\left\|Z_1 - \widetilde{Z}_1\right\|_{L^2(\Omega';\mathbb{C}^8)} \le C\,\left\|E_1 - \widetilde{E}_1\right\|_{L^2(\Omega';\mathbb{C}^3)} \le \epsilon C. \tag{6.6}$$

For each choice of Y_2 (with $Y_2|_\Omega \in H^1(\Omega;\mathbb{C}^8)$), since $(-\Delta I_8 + \widetilde{Q}_2)Y_2 = 0$ in Ω by Lemma 2.1, where $\widetilde{Q}_2$ denotes the zeroth order matrix operator $\widetilde{Q}$ defined in (2.4) for μ_2, γ_2, by elliptic regularity and Proposition 5.1 there exists $\widetilde{Y}_2 \in H^2(\Omega;\mathbb{C}^8)$ verifying $(-\Delta I_8 + \widetilde{Q}_2)\widetilde{Y}_2 = 0$ in Ω with $\widetilde{Y}_2|_{\partial\Omega} = 0$ on Γ_c, and

$$\left\|Y_2 - \widetilde{Y}_2\right\|_{L^2(\Omega';\mathbb{C}^8)} < \epsilon. \tag{6.7}$$

Condition (6.1) implies that $\mu_1 = \mu_2$, $\gamma_1 = \gamma_2$, $\partial_{x_l}\mu_1 = \partial_{x_l}\mu_2$ and $\partial_{x_l}\gamma_1 = \partial_{x_l}\gamma_2$ on $\overline{\Gamma}$ (for $l = 1, 2, 3$). By Proposition 4.2,

$$((Q_1 - Q_2)\widetilde{Z}_1|\widetilde{Y}_2)_\Omega = 0. \tag{6.8}$$

Applying (6.6) and (6.7) write

$$\begin{aligned} &|((Q_1 - Q_2)Z_1|Y_2)_{\Omega'} - ((Q_1 - Q_2)\widetilde{Z}_1|\widetilde{Y}_2)_{\Omega'}| \\ &\quad = \left|\int_{\Omega'}(Q_1 - Q_2)Z_1\cdot\left(\overline{Y_2 - \widetilde{Y}_2}\right)dx - \int_{\Omega'}(Q_1 - Q_2)(\widetilde{Z}_1 - Z_1)\cdot\overline{\widetilde{Y}_2}\,dx\right| \\ &\quad \le \|Q_1 - Q_2\|_{L^\infty(\Omega';\mathcal{M}_{8\times 8})}\left(\|Z_1\|_{L^2(\Omega';\mathbb{C}^8)}\left\|Y_2 - \widetilde{Y}_2\right\|_{L^2(\Omega';\mathbb{C}^8)}\right. \\ &\qquad \left. + \left\|\widetilde{Y}_2\right\|_{L^2(\Omega';\mathbb{C}^8)}\left\|Z_1 - \widetilde{Z}_1\right\|_{L^2(\Omega';\mathbb{C}^8)}\right) \end{aligned}$$

$$\leq C(C_{\text{stblty}}, M, \omega)\epsilon \left(\|Z_1\|_{L^2(\Omega;\mathbb{C}^8)} + \|\widetilde{Y}_2\|_{L^2(\Omega';\mathbb{C}^8)} \right)$$
$$\leq \epsilon C(M, \Omega, \rho, \xi, \omega, \varepsilon_0, \mu_0, C_{\text{stblty}})\, e^{c(\tau+|\xi|)},$$

where $c = c(\Omega)$ and last inequality follows from the fact

$$\|\widetilde{Y}_2\|_{L^2(\Omega';\mathbb{C}^8)} \leq \|Y_2 - \widetilde{Y}_2\|_{L^2(\Omega';\mathbb{C}^8)} + \|Y_2\|_{L^2(\Omega';\mathbb{C}^8)} \leq 1 + \|Y_2\|_{L^2(\Omega;\mathbb{C}^8)},$$

the exponential behaviour of Z_1, Y_2 and estimates (3.2), (3.4). Therefore, denoting

$$\epsilon'(\epsilon) := \epsilon C(\Omega, \rho, \xi, \omega, \varepsilon_0, \mu_0, C_{\text{stblty}}),$$

since $Q_1 = Q_2$ in $\Omega \setminus \Omega'$ and by (6.8), we have

$$|((Q_1 - Q_2)Z_1|Y_2)_\Omega| = |((Q_1 - Q_2)Z_1|Y_2)_{\Omega'}| \tag{6.9}$$
$$\leq |((Q_1 - Q_2)\widetilde{Z}_1|\widetilde{Y}_2)_{\Omega'}| + \epsilon'(\epsilon)\, e^{c(\tau+|\xi|)} = \epsilon'(\epsilon)\, e^{c(\tau+|\xi|)}. \tag{6.10}$$

Thus, for fixed τ and ξ, letting $\epsilon \to 0$ in (6.9)–(6.10), we get

$$((Q_1 - Q_2)Z_1|Y_2)_\Omega = 0, \tag{6.11}$$

for both choices of Z_1, Y_2. By (6.2), (6.3) and (6.11), we have for large enough τ,

$$|\hat{f}(\xi)| + |\hat{g}(\xi)| \leq \frac{C}{\tau},$$

where $C = C(M, \xi, |\Omega|, \rho, \omega, \mu_0, \varepsilon_0)$. For any fixed $\xi \in \mathbb{R}^3$, by letting $\tau \to \infty$ deduce that $\hat{f}(\xi) = \hat{g}(\xi) = 0$. Hence, $f = g = 0$.

Using a Carleman estimate, Pedro Caro in [17] proves the following inequality

$$e^{d_1/h} \sum_{j=1,2} (h\,\|\phi_j\|^2_{L^2(\Omega)} + h^3\,\|\nabla\phi_j\|^2_{L^2(\Omega)}) \leq C\, e^{d_2/h} \tag{6.12}$$
$$\times \left(h^4 \big(\|f\|^2_{L^2(\Omega)} + \|g\|^2_{L^2(\Omega)} \big) + \sum_{j=1,2} \big(h\,\|\phi_j\|^2_{L^2(\partial\Omega)} + h^3\,\|\nabla\phi_j\|^2_{L^2(\partial\Omega)} \big) \right), \tag{6.13}$$

where $\phi_1 := \gamma_1^{1/2} - \gamma_2^{1/2}$, $\phi_2 := \mu_1^{1/2} - \mu_2^{1/2}$, $C = C(\Omega, M)$, $0 < h < C^{-1/3} \leq 1$, and

$$d_1 := \inf\{|x - x_0|^2 : x \in \Omega\}, \qquad d_2 := \sup\{|x - x_0|^2 : x \in \Omega\},$$

for certain point $x_0 \notin \overline{\Omega}$. Under Theorem 1.1's conditions the summation term of norms on $\partial\Omega$ in (6.13) vanishes. From this fact together with $f = g = 0$ in Ω, we conclude $\mu_1 = \mu_2$ and $\gamma_1 = \gamma_2$ in Ω.

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References

[1] G. Alessandrini, Stable determination of conductivity by boundary measurements, Appl. Anal. 27 (1988) 153–172.

[2] H. Ammari, G. Uhlmann, Reconstruction of the potential from partial Cauchy data for the Schrödinger equation, Indiana Univ. Math. J. 53 (1) (2004) 169–184.

[3] K. Astala, M. Lassas, L. Päivärinta, Calderón's inverse problem for anisotropic conductivity in the plane, Comm. Partial Differential Equations 30 (2005) 207–224.

[4] K. Astala, L. Päivärinta, Calderón's inverse conductivity problem in the plane, Ann. of Math. 163 (2006) 265–299.

[5] M.F. Atiyah, V.K. Patodi, I.M. Singer, Spectral asymmetry and Riemannian geometry, I, Math. Proc. Cambridge Philos. Soc. 77 (1975) 43–69;
M.F. Atiyah, V.K. Patodi, I.M. Singer, Spectral asymmetry and Riemannian geometry, II, Math. Proc. Cambridge Philos. Soc. 78 (1975) 405–432;
M.F. Atiyah, V.K. Patodi, I.M. Singer, Spectral asymmetry and Riemannian geometry, III, Math. Proc. Cambridge Philos. Soc. 79 (1976) 71–99.

[6] J.A. Barceló, T. Barceló, A. Ruiz, Stability of the inverse conductivity problem in the plane for less regular conductivities, J. Differential Equations 173 (2001) 231–270.

[7] T. Barceló, D. Faraco, A. Ruiz, Stability of Calderón inverse conductivity problem in the plane, J. Math. Pures Appl. 88 (2007) 522–556.

[8] M. Birman, M. Solomyak, L^2-theory of the Maxwell operator in arbitrary domains, Russian Math. Surveys 42 (1987) 75–96.

[9] M. Birman, M. Solomyak, On the main singularities of the electric component of the electro-magnetic field in regions with screen, St. Petersburg Math. J. 5 (1993) 125–139.

[10] R. Brown, Global uniqueness in the impedance-imaging problem for less regular conductivities, SIAM J. Math. Anal. 27 (1996) 1049–1056.

[11] R.M. Brown, R.H. Torres, Uniqueness in the inverse conductivity problem for conductivities with $3/2$ derivatives in L^p, $p > 2n$, J. Fourier Anal. Appl. 9 (2003) 563–574.

[12] R. Brown, G. Uhlmann, Uniqueness in the inverse conductivity problem for nonsmooth conductivities in two dimensions, Comm. Partial Differential Equations 22 (1997) 1009–1027.

[13] J. Brüning, M. Lesch, On boundary value problems for Dirac type operators. I. Regularity and self-adjointness, J. Funct. Anal. 185 (2001) 1–62.

[14] A. Bukhgeim, G. Uhlmann, Recovering a potential from partial Cauchy data, Comm. Partial Differential Equations 27 (2002) 653–668.

[15] A.P. Calderón, On an inverse boundary value problem, in: Seminar on Numerical Analysis and Its Applications to Continuum Physics, Sociedade Brasileira de Matematica, Rio de Janeiro, 1980, pp. 65–73.

[16] P. Caro, Stable determination of the electromagnetic coefficients by boundary measurements, Inverse Probl. 26 (2010) 105014, 25 pp.

[17] P. Caro, On an inverse problem in electromagnetism with local data: stability and uniqueness, Inverse Probl. Imaging 5 (2011) 297–322.

[18] P. Caro, A. García, J.M. Reyes, Stability of the Calderón problem for less regular conductivities, J. Differential Equations 254 (2013) 469–492.

[19] P. Caro, P. Ola, M. Salo, Inverse boundary value problem for Maxwell equations with local data, Comm. Partial Differential Equations 34 (2009) 1425–1464.

[20] P. Caro, T. Zhou, Global uniqueness for an IBVP for the time-harmonic Maxwell equations, Anal. PDE 7 (2) (2014) 375–405.

[21] A. Clop, D. Faraco, A. Ruiz, Stability of Calderón's inverse conductivity problem in the plane for discontinuous conductivities, Inverse Probl. Imaging 4 (2010) 49–91.

[22] D. Colton, L. Päivärinta, The uniqueness of a solution to an inverse scattering problem for electromagnetic waves, Arch. Ration. Mech. Anal. 119 (1992) 59–70.

[23] M. Costabel, M. Dauge, Singularities of electromagnetic fields in polyhedral domains, Arch. Ration. Mech. Anal. 151 (2000) 221–276.

[24] M.N. Demchenko, Nonunique continuation for the Maxwell system, J. Math. Sci. 185 (4) (2012) 554–566.

[25] M.S.P. Eastham, H. Kalf, Schrödinger-Type Operators with Continuous Spectra, Research Notes in Mathematics, vol. 65, Pitman (Advanced Publishing Program), 1982.

[26] M.M. Eller, M. Yamamoto, A Carleman inequality for the stationary anisotropic Maxwell system, J. Math. Pures Appl. 86 (2006) 449–462.

[27] D. Faraco, K. Rogers, The Sobolev norm of characteristic functions with applications to the Calderón inverse problem, Quart. J. Math. 64 (2013) 133–147.

[28] V. Girault, P.-A. Raviart, Finite Element Methods for Navier–Stokes Equations. Theory and Algorithms, Springer-Verlag, 1986.

[29] P. Grisvard, Elliptic Problems in Nonsmooth Domains, Pitman Advanced Publishing Program, 1985.

[30] B. Haberman, D. Tataru, Uniqueness in Calderón's problem with Lipschitz conductivities, Duke Math. J. 162 (3) (2013) 497–516.

[31] H. Heck, Stability estimates for the inverse conductivity problem for less regular conductivities, Comm. Partial Differential Equations 34 (2009) 107–118.

[32] H. Heck, J.-N. Wang, Stability estimates for the inverse boundary value problem by partial Cauchy data, Inverse Probl. 22 (2006) 1787–1796.

[33] H. Heck, J.-N. Wang, Optimal stability estimate for the inverse boundary value problem by partial measurements, preprint, arXiv:0708.3289v1, 2007.

[34] P. Hähner, Stability of the inverse electromagnetic inhomogeneous medium problem, Inverse Probl. 16 (2000) 155–174.

[35] V. Isakov, On uniqueness in the inverse conductivity problem with local data, Inverse Probl. Imaging 1 (2007) 95–105.

[36] M. Joshi, S.R. McDowall, Total determination of material parameters from electromagnetic boundary information, Pac. J. Math. 193 (2000) 107–129.

[37] C.E. Kenig, M. Salo, G. Uhlmann, Inverse problems for the anisotropic Maxwell equations, Duke Math. J. 157 (2011) 369–419.

[38] C.E. Kenig, J. Sjöstrand, G. Uhlmann, The Calderón problem with partial data, Ann. of Math. 165 (2007) 567–591.

[39] Y. Kurylev, M. Lassas, Inverse problems and index formulae for Dirac operators, Adv. Math. 221 (2009) 170–216.

[40] Y. Kurylev, M. Lassas, E. Somersalo, Maxwell's equations with a polarization independent wave velocity: direct and inverse problems, J. Math. Pures Appl. 86 (2006) 237–270.

[41] M. Lassas, The impedance imaging problem as a low-frequency limit, Inverse Probl. 13 (6) (1997) 1503–1518.

[42] R. Leis, Initial Boundary Value Problems in Mathematical Physics, Wiley, New York, 1986.

[43] J.-L. Lions, E. Magenes, Problèmes aux limites non homogènes et applications I, II, Travaux et Recherches Mathématiques, vol. 17, Dunod, Paris, 1968 (engl. transl. Springer-Verlag, 1972).

[44] H. Liu, M. Yamamoto, J. Zou, Reflection principle for the Maxwell equations and its application to inverse electromagnetic scattering, Inverse Probl. 23 (2007) 2357–2366.

[45] S.R. McDowall, Boundary determination of material parameters from electromagnetic boundary information, Inverse Probl. 13 (1997) 153–163.

[46] S.R. McDowall, An electromagnetic inverse problem in chiral media, Trans. Amer. Math. Soc. 352 (2000) 2993–3013.

[47] M. Mitrea, Sharp Hodge decomposition, Maxwell's equations and vector Poisson problems on non-smooth, three-dimensional Riemannian manifolds, Duke Math. J. 125 (2004) 467–547.

[48] P. Monk, Finite Element Methods for Maxwell's Equations, Oxford Science Publications, Clarendon Press, Oxford, 2003.

[49] A. Nachman, Reconstruction from boundary measurements, Ann. of Math. 128 (1988) 531–576.

[50] A. Nachman, Global uniqueness for a two-dimensional inverse boundary value problem, Ann. of Math. 143 (1995) 71–96.

[51] P. Ola, L. Päivärinta, E. Somersalo, An inverse boundary value problem in electrodynamics, Duke Math. J. 70 (1993) 617–653.

[52] P. Ola, L. Päivärinta, E. Somersalo, Inverse Problems for Time Harmonic Electrodynamics, Inside Out: Inverse Problems and Applications, Math. Sci. Res. Inst. Publ., vol. 47, Cambridge University Press, Cambridge, 2003, pp. 169–191.
[53] P. Ola, E. Somersalo, Electromagnetic inverse problems and generalized Sommerfeld potentials, SIAM J. Appl. Math. 56 (1996) 1129–1145.
[54] L. Päivärinta, A. Panchenko, G. Uhlmann, Complex geometrical optics solutions for Lipschitz conductivities, Rev. Mat. Iberoam. 19 (2003) 57–72.
[55] M. Salo, L. Tzou, Carleman estimates and inverse problems for Dirac operators, Math. Ann. 344 (2009) 161–184.
[56] M. Salo, L. Tzou, Inverse problems with partial data for a Dirac system: a Carleman estimate approach, Adv. Math. 225 (2010) 487–513.
[57] J. Saranen, Über das Verhalten der Lösungen der Maxwellschen Randwertaufgabe in Gebieten mit Kegelspitzen, Math. Methods Appl. Sci. 2 (1980) 235–250.
[58] J. Saranen, Über das Verhalten der Lösungen der Maxwellschen Randwertaufgabe in einigen nichtglatten Gebieten, Ann. Acad. Sci. Fenn., Ser. A 1 Math. 6 (1981) 15–28.
[59] E. Sarkola, A unified approach to direct and inverse scattering for acoustic and electromagnetic waves, Ann. Acad. Sci. Fenn. Math. Diss. 101 (1995).
[60] E. Somersalo, D. Isaacson, M. Cheney, A linearized inverse boundary value problem for Maxwell's equations, J. Comput. Appl. Math. 42 (1992) 123–136.
[61] E. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton University Press, 1970.
[62] Z.Q. Sun, G. Uhlmann, An inverse boundary value problem for Maxwell's equations, Arch. Ration. Mech. Anal. 119 (1) (1992) 71–93.
[63] J. Sylvester, G. Uhlmann, A global uniqueness theorem for an inverse boundary value problem, Ann. of Math. 125 (1987) 153–169.