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Citation for final published version:

Oertel, Timm , Wagner, Christian and Weismantel, Robert 2014. Integer convex minimization by mixed integer linear optimization. Operations Research Letters 42 (6-7) , pp. 424-428. 10.1016/j.orl.2014.07.005

Publishers page: http://dx.doi.org/10.1016/j.orl.2014.07.005

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Integer convex minimization by mixed integer linear optimization

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Abstract

Minimizing a convex function over the integral points of a bounded convex set is polynomial in fixed dimension [6]. We provide an alternative, short, and geometrically motivated proof of this result. In particular, we present an oracle-polynomial algorithm based on a mixed integer linear optimization oracle.

Keywords: convex minimization, integer optimization, polynomial algorithm

1. Introduction

In [13] it is shown that, given a system of linear inequalities with rational coefficients in fixed dimension, in polynomial time in the size of the encoding length of the input data one can either find an integral solution to the system, or show that all integral points of the solution set of the system lie on few parallel hyperplanes. This fact is used in [6, Theorem 6.7.10]) to show that, in fixed dimension, integer convex minimization is polynomial.

In this paper, we present two algorithmic ways to prove that integer convex minimization in fixed dimension can be polynomially reduced to mixed integer linear optimization. Let us first state our assumptions. Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be convex functions given by *first order evaluation oracles*, i.e. queried on a specific point such oracles return a function value and a subgradient of the subdifferential at this point. We further assume that (i) some $B \in \mathbb{N}$ is known satisfying $\{x \in \mathbb{Z}^n \mid g(x) \leq 0\} \subset [0, B]^n$, and (ii) the output of the evaluation oracles is of sufficient precision. More precisely, let ϵ and δ be given nonnegative constants. Then, if queried on $\bar{x}$, we assume that the evaluation oracle for f returns $\bar{f}$ and $\bar{h}$ such that

$$|f(\bar{x}) - \bar{f}| \leq \epsilon$$

and

$$\bar{h} = 0 \qquad \text{if} \qquad 0 \in \partial f(\bar{x})$$

$$\text{or} \quad \left\| \frac{h}{\|h\|_\infty} - \frac{\bar{h}}{\|\bar{h}\|_\infty} \right\|_\infty \leq \delta \qquad \text{for some } h \in \partial f(\bar{x}) \setminus \{0\},$$

where $\partial f(\bar{x})$ denotes the subdifferential of f at $\bar{x}$. Since we are only interested in the separating property of the subgradients, we assume that, whenever a subgradient is nonzero, then it is normalized, i.e. $\|\bar{h}\|_\infty = 1$. Analogously, the same precision applies to g. We aim at solving

$$\min\{f(x) \mid x \in \mathbb{Z}^n \text{ and } g(x) \leq 0\}. \tag{1}$$

Every bounded integer convex minimization problem (ICP) can be written in the above form. Besides the first order evaluation oracles, we assume to have at our disposal a *mixed integer linear optimization oracle*, that returns an optimal solution when fed with a mixed integer linear optimization problem (MILP) with a fixed number of integer variables. Our motivation for using such an oracle lies in the significant progress in developing efficient solution techniques for MILP's that has been achieved over the last decades. Today, one can solve MILP's that were considered out of reach twenty years ago. Moreover, if one intends to solve ICP's, it is natural to assume the existence of an oracle capable of solving easier optimization problems. Thus, it is plausible to postulate that the linear case can be solved.

The main result that is shown in this paper is stated below.

Theorem 1. *Let $n \in \mathbb{N}$ be fixed. Let $B \in \mathbb{N}$, and $\delta, \epsilon \geq 0$ with $\delta \in \mathcal{O}(B^{-n})$ be given and satisfying the assumptions (i) and (ii). Assume to have at hand first order evaluation oracles for f and g, and a mixed integer linear optimization oracle able to solve MILP's with at most n integer variables. Then problem (1) can be solved within a number of oracle calls bounded by a polynomial in the binary encoding of B. That is, we either find a point $\bar{x} \in \mathbb{Z}^n$ with $g(\bar{x}) \leq 2\epsilon$ and*

$$f(\bar{x}) \leq \min\{f(x) \mid x \in \mathbb{Z}^n \text{ and } g(x) \leq 0\} + 2\epsilon,$$

or show that $g(x) > 0$ for all $x \in \mathbb{Z}^n$.

We point out that the accuracy of 2ϵ in Theorem 1 comes from the fact that the evaluation oracles for f and g return the function value only with a precision of ϵ. When

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fed with a point $\bar{x} \in \mathbb{R}^n$, the evaluation oracle for g returns a value $\bar{g}$ such that $|g(\bar{x}) - \bar{g}| \leq \epsilon$. Hence, if $\bar{g} \leq \epsilon$, then $g(\bar{x}) \leq 2\epsilon$, and if $\bar{g} > \epsilon$ then $g(\bar{x}) > 0$. Moreover, the evaluation oracle for f is, in general, not able to distinguish the function values of two points $\bar{x}, \bar{y} \in \mathbb{R}^n$ with $|f(\bar{x}) - f(\bar{y})| \leq 2\epsilon$. Thus, the accuracy in Theorem 1 is best possible, assuming evaluation oracles as given. We note that even enumerating all points in $[0, B]^n \cap \mathbb{Z}^n$ would not lead to a better accuracy.

The standard approach to solve (1) is to set $F_\gamma := f - \gamma$ and to solve the feasibility problem for the level-set $\{x \in \mathbb{Z}^n : F_\gamma(x) \leq 0 \text{ and } g(x) \leq 0\}$ while applying binary search on γ. In particular, this procedure is used in the original proof that (1) is solvable in oracle-polynomial time in [6]. The drawback of this approach is that the underlying minimization problem (1) can only be solved up to a certain accuracy, even if the evaluation oracles provide exact output, i.e. $\epsilon = 0$ (and $\delta \in \mathcal{O}(B^{-n})$). Our methods solve (1) without binary search on the objective function value by only making use of the local information from oracle outputs. The virtue of our methods is that they solve (1) exactly when $\epsilon = 0$. Nevertheless, if $\epsilon > 0$, then all the approaches can approximate the optimal solution up to 2ϵ in polynomial time. Though, in the algorithm described in [6], ϵ enters the run-time while the run-time of our approach is independent of ϵ.

To the best of our knowledge it has never been stated directly that problems of type (1) are oracle-polynomially solvable in fixed dimension. However, this result is derivable from the work of Lenstra [13] and Grötschel et al. [6].

After the preliminaries we present in section 3 and 4 two constructive proofs for Theorem 1. Both use cutting plane methods. The first proof culminates in an algorithm that uses centroids – whose computation is time-consuming. The second proof results in an algorithm that avoids the computation of centroids at the expense of more iteration steps. The special feature of the second algorithm is that it only needs to solve MILP's as subproblems. This is of practical relevance as the computation of centroids is theoretically doable in fixed dimension, but intractable in practice. Moreover, to the best of our knowledge, it is not known how to accelerate the computation of centroids with an MILP oracle at hand.

Following the general cutting plane schemes presented in [14, Section 3.2.6], we see that there are parallels between the development of solution techniques for continuous convex optimization and the introduced integral techniques. The Ellipsoid Method (see [14, p. 154]) bears resemblance with the algorithms of [6, 12, 9, 10] to solve (1), using ellipsoidal approximations. Furthermore, the Method of Centers of Gravity (see [14, p. 152], or for a randomized version see [3]) exhibits many similarities to our centroid algorithm in Section 3. Finally, the Kelly Method [14, Section 3.3.2] and the Level Method [14, Section 3.3.3] can be seen as analogues to our MILP algorithm in Section 4 in the sense that linear techniques are applied to solve non-linear problems.

2. Preliminaries

In this section, we present auxiliary lemmata and observations that are needed for the proofs in Sections 3 and 4.

We first introduce some notation. Let $K \subset \mathbb{R}^n$ be a compact convex set and let $v \in \mathbb{Z}^n \setminus \{0\}$. We define the *width of K with respect to v* by

$$\omega(K, v) := \max\{v^\mathsf{T} x \mid x \in K\} - \min\{v^\mathsf{T} x \mid x \in K\}$$

and the *lattice width of K* by

$$\omega(K) := \min\{\omega(K, v) \mid v \in \mathbb{Z}^n \setminus \{0\}\}.$$

A vector $v \in \mathbb{Z}^n \setminus \{0\}$ with $\omega(K) = \omega(K, v)$ is called *flatness direction for K*.

Observation 2. *Let $n \in \mathbb{N}$ be fixed and $P = \{x \in \mathbb{R}^n : Ax \leq b\} \subset \mathbb{R}^n$ be a rational polytope. Given a mixed integer linear optimization oracle in n integer variables, we can compute a flatness direction for P in polynomial time.*

Proof. W.l.o.g. let $\operatorname{int}(P) \neq \emptyset$ and let $a_i^\top$ denote the i-th row vector of A. Further, we want to assume that $Ax \leq b$ has no redundant inequalities. By scaling the rows, we may assume that $b_i - \min_{x \in P} a_i x = 1$. Then $P_0 := P - P = \{x \in \mathbb{R}^n \mid -\mathbb{1} \leq Ax \leq \mathbb{1}\}$, where $\mathbb{1}$ denotes the all-one vector. Observe that $2\omega(P) = \omega(P_0)$. Let $P_0^\star := \{x \in \mathbb{R}^n \mid x^\top y \leq 1 \text{ for all } y \in P_0\}$ denote the *polar set* of P_0 and let $\|x\|_{P_0^\star} := \min\{\gamma \geq 0 \mid x \in \gamma P_0^\star\}$ denote the norm induced by $P_0^\star$. Then, we can reformulate the lattice width as follows

$$\omega(P_0) = 2 \min_{x \in \mathbb{Z}^n \setminus \{0\}} \max_{y \in P_0} x^\top y = 2 \min_{x \in \mathbb{Z}^n \setminus \{0\}} \|x\|_{P_0^\star}.$$

The last minimization problem can be solved using MILP's. For that, note $\gamma P_0^\star = \gamma \operatorname{conv}\{\pm a_1, \ldots, \pm a_m\} = \{x \in \mathbb{R}^n \mid x = A^\top \lambda,\ \|\lambda\|_1 \leq \gamma\}$. For $i = 1, \ldots, n$ we solve the following MILP's that we call F^i.

$$\begin{array}{lll} \min & \gamma & \\ \text{s.t.} & x = A^\top \lambda, & \|\lambda\|_1 \leq \gamma, \\ & x \in \mathbb{Z}^n, & x_i \geq 1, \\ & \lambda \in \mathbb{R}^m, & \gamma \in \mathbb{R}. \end{array} \tag{F^i}$$

Let $(\bar{\gamma}^i, \bar{x}^i, \bar{\lambda}^i)$ be an optimal solution of F^i and let $\bar{\gamma}^j := \min_{i=1,\ldots,n} \bar{\gamma}^i$. Then $\omega(P) = \bar{\gamma}^j$ and $\bar{x}^j$ is a flatness direction for P. $\square$

The following lemma is similar to known results in literature, see [2] for instance. It states that a convex set is flat whenever its volume is sufficiently small. It thus defines the threshold at which to switch from adding cutting planes to enumerating lower-dimensional subproblems. As it is stated here we are not aware of a reference. This is why we outline a short proof.

Lemma 3. *Let $K \subset \mathbb{R}^n$ be a bounded convex set. If* $\operatorname{vol}(K) < 1$ *then*

$$\omega(K) \leq cn^{\frac{3}{2}}$$

for a universal constant c.

Proof. We show that K has a lattice-free translate, i.e. there exists a point $x \in \mathbb{R}^n$ such that $(x + K) \cap \mathbb{Z}^n = \emptyset$. Then $\omega(K) \leq cn^{\frac{3}{2}}$ for a universal constant c (see [2]).

Let χ_K denote the characteristic function of K and for a set $S \subset \mathbb{Z}^n$, $|S|$ denotes the cardinality of S. Assume that for all $x \in \mathbb{R}^n$ it holds that $|(x + K) \cap \mathbb{Z}^n| \geq 1$. Then

$$\begin{aligned}
\operatorname{vol}(K) &= \int_{\mathbb{R}^n} \chi_K(x)dx = \sum_{z\in\mathbb{Z}^n} \int_{[0,1)^n} \chi_K(x+z)dx \\
&= \int_{[0,1)^n} \sum_{z\in\mathbb{Z}^n} \chi_K(x+z)dx \\
&= \int_{[0,1)^n} \sum_{z\in\mathbb{Z}^n} \chi_{K-x}(z)dx \\
&= \int_{[0,1)^n} |(K-x) \cap \mathbb{Z}^n| dx \geq 1,
\end{aligned}$$

a contradiction. □

Given a point $\bar{x} \in [0, B]^n$ the evaluation oracle provides us with a vector $\bar{h} \in \mathbb{R}^n$. Either $\bar{h} = 0$ or $\|\bar{h}\|_\infty = 1$ and there exists a $h \in \partial f(\bar{x})$ (resp. $\in \partial g(\bar{x})$) such that $\|\frac{h}{\|h\|_\infty} - \bar{h}\|_\infty \leq \delta$. In the following we want to discuss the error of the oracle, i.e. the value δ, of the second possible outcome. Let us assume that $\bar{h} \neq 0$. For our algorithms in Sections 3 and 4 we need to investigate how the the error of the oracle affects the volume, i.e. the ratio between $\operatorname{vol}(\{x \in P \mid h^\mathsf{T} x \leq h^\mathsf{T} \bar{x}\})$ and $\operatorname{vol}(\{x \in P \mid \bar{h}^\mathsf{T} x \leq \bar{h}^\mathsf{T} \bar{x}\})$ for a polyhedron $P \subset [0, B]^n$. For that, we need the following observation.

Observation 4. $M_1 := \{x \in [0, B]^n \mid h^\mathsf{T} x \leq h^\mathsf{T} \bar{x}\} \subset \{x \in [0, B]^n \mid \bar{h}^\mathsf{T} x \leq \bar{h}^\mathsf{T} \bar{x} + nB\delta\} =: M_2$.

Proof. Suppose that $x \in M_1 \setminus M_2$. Then $0 \leq -h^\mathsf{T}(x - \bar{x})$ and $nB\delta < \bar{h}^\mathsf{T}(x - \bar{x})$. Adding these inequalities yields $nB\delta < (x - \bar{x})^\mathsf{T}(\bar{h} - h) \leq \sum_{i=1}^n |x_i - \bar{x}_i| \cdot |\bar{h}_i - h_i| \leq nB\delta$, a contradiction. □

The following lemma states a lower bound for the ratio between the volumes of $K \cap H'$ and K, provided that a lower bound for the ratio between the volumes of $K \cap H$ and K is known, where $K \subset [0, B]^n$ is a convex set and H and H' are half-spaces whose boundary hyperplanes are translates.

Lemma 5. *Let $K \subset [0, B]^n$ be a bounded convex set with* $\operatorname{vol}(K) \geq 1$. *Let $H := \{x \in \mathbb{R}^n \mid \alpha^\mathsf{T} x \leq \beta\}$ and $H' := \{x \in \mathbb{R}^n \mid \alpha^\mathsf{T} x \leq \beta - \kappa\}$ with $\|\alpha\|_\infty = 1$, $\beta \in \mathbb{R}$ and $\kappa \geq 0$. Moreover, let* $\operatorname{vol}(K \cap H) \geq C \operatorname{vol}(K)$ *for a constant $C > 0$. Let σ denote the volume of the five-dimensional unit ball. If $\kappa \leq \frac{C}{2\sigma}(\frac{2}{\sqrt{n}B)})^{n-1}$, then*

$$\operatorname{vol}(K \cap H') \geq \frac{C}{2} \operatorname{vol}(K).$$

Proof. It holds $\operatorname{vol}(K \cap H') = \operatorname{vol}(K \cap H) - (\operatorname{vol}(K \cap H) - \operatorname{vol}(K \cap H')) \geq C \operatorname{vol}(K) - (\operatorname{vol}(K \cap H) - \operatorname{vol}(K \cap H'))$. Let $S := \{x \in \mathbb{R}^n \mid \|x - \frac{B}{2}\mathbb{1}\|_2 \leq \frac{\sqrt{n}B}{2}\} \supset [0, B]^n$. Then

$$\begin{aligned}
&\operatorname{vol}(K \cap H') - C \operatorname{vol}(K) \\
&\geq - \operatorname{vol}(\{x \in K \mid \beta - \kappa \leq \alpha^\mathsf{T} x \leq \beta\}) \\
&\geq - \operatorname{vol}(\{x \in [0, B]^n \mid \beta - \kappa \leq \alpha^\mathsf{T} x \leq \beta\}) \\
&\geq - \operatorname{vol}\left(\left\{x \in S \;\middle|\; \frac{\beta - \kappa}{\|\alpha\|_2} \leq \frac{\alpha^\mathsf{T} x}{\|\alpha\|_2} \leq \frac{\beta}{\|\alpha\|_2}\right\}\right) \\
&= - \int_{-\frac{\kappa}{\|\alpha\|_2}}^{0} \operatorname{vol}_{n-1}\left(\left\{x \in S \;\middle|\; \frac{\alpha^\mathsf{T} x}{\|\alpha\|_2} = \frac{\beta}{\|\alpha\|_2} - y\right\}\right) dy \\
&\geq - \frac{\kappa}{\|\alpha\|_2} \operatorname{vol}_{n-1}\left(\left\{x \in S \;\middle|\; \alpha^\mathsf{T} x = \alpha^\mathsf{T} \frac{B}{2}\mathbb{1}\right\}\right) \\
&\geq - \kappa \operatorname{vol}_{n-1}\left(\left\{x \in \mathbb{R}^{n-1} \;\middle|\; \|x\|_2 \leq \frac{\sqrt{n}B}{2}\right\}\right) \\
&= - \kappa \left(\frac{\sqrt{n}B}{2}\right)^{n-1} \operatorname{vol}_{n-1}\left(\{x \in \mathbb{R}^{n-1} \mid \|x\|_2 \leq 1\}\right) \\
&\geq - \kappa \left(\frac{\sqrt{n}B}{2}\right)^{n-1} \sigma \geq -\frac{C}{2}.
\end{aligned}$$

Note that the second and third inequality follow from the fact that $K \subset [0, B]^n \subset S$. For the first equation we apply Cavalieri's principle. Then, in the fourth inequality we use that $\operatorname{vol}_{n-1}\left(\{x \in S \mid \alpha^\mathsf{T} x = y\}\right)$ is maximal for $y = \alpha^\mathsf{T} \frac{B}{2}\mathbb{1}$. In the fifth inequality we exploit that $\|\alpha\|_2 \geq 1$ and that the $(n-1)$-dimensional ball

$$\left\{x \in S \;\middle|\; \alpha^\mathsf{T} x = \alpha^\mathsf{T} \frac{B}{2}\mathbb{1}\right\}$$

is equivalent, up-to rotation and translation, to

$$\left\{x \in \mathbb{R}^{n-1} \mid \|x\|_2 \leq \frac{\sqrt{n}B}{2}\right\}.$$

Finally, in the last inequality we use that the n-dimensional volume of an n-dimensional unit ball (i.e. $\pi^{n/2}/\Gamma(n/2+1)$) is maximal for $n = 5$.

Hence $\operatorname{vol}(K \cap H') \geq C(\operatorname{vol}(K) - \frac{1}{2}) \geq \frac{C}{2} \operatorname{vol}(K)$. □

3. Cutting plane scheme based on centroids

In this section, we present our first algorithm to solve (1).

Let $K \subset \mathbb{R}^n$ be a compact convex set. The centroid of K is defined to be the point $c_K := \operatorname{vol}(K)^{-1} \int_K x dx$. In the case where K is a polytope, one possible way of computing the centroid is to triangulate K into simplices $S_1, \ldots, S_r$ and to compute the centroids $c_{S_1}, \ldots, c_{S_r}$ of the simplices. In turn, the centroid of a simplex S with vertices $v_0, \ldots, v_n$ is $c_S = \frac{1}{n+1} \sum_{i=0}^n v_i$. Finally, $c_K = \operatorname{vol}(K)^{-1} \sum_{i=1}^r c_{S_i} \cdot \operatorname{vol}(S_i)$. We note that the computation of a triangulation of the polytope K can be done in polynomial time in the number of vertices of K (see, for instance, [5] or [4, Chapter 8]).

For a given compact convex body K and a $0 \leq \lambda \leq 1$ we define $K_\lambda := \lambda(K - c_K) + c_K$, i.e. the scaling of K by the factor λ with respect to its centroid. Note that $K_1 = K$ and $K_0 = \{c_K\}$. Again, in the case where K is a polytope and is given by linear inequalities, i.e. $K = \{x \in \mathbb{R}^n \mid Ax \leq b\}$, then $K_\lambda = \{x \in \mathbb{R}^n \mid Ax \leq \lambda b + (1-\lambda)Ac_K\}$.

In the following lemma we give a straightforward generalization of a theorem of Grünbaum [8, Theorem 2].

Lemma 6. *Let $0 \leq \lambda \leq 1$. Let $K \subset \mathbb{R}^n$ be a closed convex set, and let $H \subset \mathbb{R}^n$ be a half-space. If $K_\lambda \cap H \neq \emptyset$, then*

$$\operatorname{vol}(K \cap H) \geq (1-\lambda)^n \cdot \left(\frac{n}{n+1}\right)^n \operatorname{vol}(K).$$

Proof. In [8], Grünbaum defined the set

$$S := \{x \in \mathbb{R}^n \mid \text{for all half-spaces } G \text{ with } x \in G \text{ holds } \operatorname{vol}(K \cap G) \geq \left(\frac{n}{n+1}\right)^n \operatorname{vol}(K)\}.$$

In the proof of [8, Theorem 2] it is shown that $c_K \in S$. This implies that if $c_K \in H$, then $\operatorname{vol}(K \cap H) \geq (\frac{n}{n+1})^n \operatorname{vol}(K)$. Note that $K = K_\lambda + K_{1-\lambda} - c_K$. Let $x \in K_\lambda \cap H$ and let $K^x := x + K_{1-\lambda} - c_K$. Then $K^x \subset K$ and $\operatorname{vol}(K \cap H) \geq \operatorname{vol}(K^x \cap H)$. Since $c_{K^x} = x$ we have $c_{K^x} \in H$. Hence $\operatorname{vol}(K^x \cap H) \geq (\frac{n}{n+1})^n \operatorname{vol}(K^x)$. We can rewrite $\operatorname{vol}(K^x)$ in terms of $\operatorname{vol}(K)$, namely $\operatorname{vol}(K^x) = (1-\lambda)^n \operatorname{vol}(K)$. Then the lemma follows. □

Observation 7. *If $\operatorname{int}(K_\lambda) \cap \mathbb{Z}^n = \emptyset$, then*

$$\omega(K) = \frac{1}{\lambda}\omega(K_\lambda) \leq \frac{1}{\lambda} cn\frac{3}{2}$$

for a universal constant c (see [2]).

We are now ready to give a first algorithmic proof of Theorem 1.

Proof of Theorem 1. We follow the idea of the Method of Centers of Gravity in [14, p. 152]. Our proof uses induction on n. We fix $\Lambda \in (0,1)$. Further we assume that

$$\delta \leq \frac{1}{4\sigma\sqrt{n}} \left(\frac{2n(1-\Lambda)}{(n+1)\sqrt{n}B}\right)^n,$$

where σ denotes the volume of the five-dimensional unit ball. We define P^0 to be the box $[0,B]^n$, to which we add cutting planes until we can reduce the original problem to a small number of lower-dimensional subproblems. Among all points visited in the course of the algorithm, we keep record of the feasible point with smallest objective function value.

In the following, we construct a sequence of polytopes $P^0 \supset P^1 \supset P^2 \ldots$, such that P^{i+1} arises from P^i by intersecting P^i with a half-space $H = \{x \in \mathbb{R}^n \mid \bar{h}^\intercal x \leq \bar{h}^\intercal \bar{x} + nB\delta\}$. Here, $\bar{x}$ is an integral point of P^i and $\bar{h}$ is a vector provided by the evaluation oracles. Note that, in order to avoid cutting off an optimal integral point – if any feasible integral point exists – we correct the oracle error δ by increasing the right hand side from $\bar{h}^\intercal \bar{x}$ to $\bar{h}^\intercal \bar{x} + nB\delta$ (see Observation 4)). Also, note that $P^i \cap \mathbb{Z}^n \neq \emptyset$ for all i.

The construction works as follows. Let $P^i = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ be given, where $A \in \mathbb{R}^{m \times n}$ with rows $a_i^\intercal \in \mathbb{R}^n$ and $\|a_i\|_\infty = 1$ for all $i = 1, \ldots, m$.

We solve the mixed integer linear minimization problem

$$\begin{array}{lr} \min & \lambda \\ \text{s.t.} & Ax + (Ac_{P^i} - b)\lambda \leq Ac_{P^i} \\ & \lambda \in \mathbb{R}_+ \\ & x \in \mathbb{Z}^n. \end{array} \qquad \text{(MILP-1)}$$

Let (λ^*, x^*) be an optimal solution. Note that (MILP-1) always has a solution. Further, observe that $x^* \in P^i_{\lambda^*} = \{x \in \mathbb{R}^n \mid Ax + (Ac_{P^i} - b)\lambda^* \leq Ac_{P^i}\}$ and that $P^i_{\lambda^*}$ is lattice-free.

We distinguish two cases.

Case 1 If $\lambda^* > \Lambda$, then we compute a flatness direction $v \in \mathbb{Z}^n \setminus \{0\}$ for P^i (see Observation 2). Further, we compute $s := \lceil \min_{x \in P_i} v^\intercal x \rceil$ and $k := \lfloor \max_{x \in P_i} v^\intercal x \rfloor - s$. Let $H_j = \{x \in \mathbb{R}^n \mid v^\intercal x = s + j\}$, for $j = 1, \ldots, k$. It holds that $P^i \cap \mathbb{Z}^n \subset H_1 \cup \ldots \cup H_k$ and, by Observation 7,

$$k \leq \left\lfloor \frac{cn^{\frac{3}{2}}}{\lambda^*} \right\rfloor + 1 \leq \left\lfloor \frac{cn^{\frac{3}{2}}}{\Lambda} \right\rfloor + 1 =: \psi.$$

So we need to solve at most ψ subproblems of dimension $n-1$. For all $j = 1, \ldots, k$, we solve the lower-dimensional problems $\min\{f(x) \mid x \in H_j \cap \mathbb{Z}^n \text{ and } g(x) \leq 0\}$. Among all feasible points, if they exist, we return the point with smallest objective function value. We have reduced our initial problem to a polynomial number of $(n-1)$-dimensional subproblems. By the induction hypothesis, all these subproblems can be solved in polynomial time.

Case 2 Let $\lambda^* \leq \Lambda$. Let $\bar{g}$ be the output of the evaluation oracle of g at x^*. Depending on the value of $\bar{g}$, we use either the evaluation oracle for f or the one for g to get $\bar{h}$: if $\bar{g} \leq \epsilon$ we use the evaluation oracle for f or otherwise, if $\bar{g} > \epsilon$, we use the evaluation oracle for g. Let $\bar{h}$ be the output of the evaluation oracle at x^*. If $\bar{h} = 0$, then either x^* is the optimum solution or Problem 1 is infeasible. In both cases are be done; we either return x^* or state that $g(x) > 0$ for all $x \in \mathbb{Z}^n$. Hence let us assume that $\bar{h} \neq 0$. Then we define $P^{i+1} := P^i \cap \{x \in \mathbb{R}^n \mid \bar{h}^\intercal x \leq \bar{h}^\intercal x^* + nB\delta\}$. With $C = (1-\Lambda)^n(\frac{n}{n+1})^n$ as in Lemma 6 and with $\kappa = nB\delta$, it follows from Lemma 5 that

$$\operatorname{vol}(P^{i+1}) \leq \left(1 - \frac{1}{2}(1-\Lambda)^n \left(\frac{n}{n+1}\right)^n\right) \operatorname{vol}(P^i). \qquad (2)$$

In particular, (2) guarantees that after at most

$$\left\lfloor \frac{-\log(B^n)}{\log\left(1 - \frac{1}{2}(1-\Lambda)^n \left(\frac{n}{n+1}\right)^n\right)} \right\rfloor + 1$$

iterations we obtain a polytope P^l with $\mathrm{vol}(P^l) < 1$. We compute a flatness direction $v \in \mathbb{Z}^n \setminus \{0\}$ for P^l (see Observation 2). Next, we can construct as in Case 1 the value $s \in \mathbb{Z}$ and, by Lemma 3, a

$$k \leq \left\lfloor cn^{\frac{3}{2}} \right\rfloor + 1 =: \phi$$

and parallel hyperplanes $H_j = \{x \in \mathbb{R}^n \mid v^\top x = s + j\}$, $j = 1, \ldots, k$, such that $P^l \cap \mathbb{Z}^n \subset H_1 \cup \ldots \cup H_k$. So we need to solve at most ϕ subproblems of dimension $n-1$. For all $j = 1, \ldots, k$, we solve the lower-dimensional problems $\min\{f(x) \mid x \in H_j \cap \mathbb{Z}^n \text{ and } g(x) \leq 0\}$. Among all feasible points, if they exist, we return the point with smallest objective function value. We have reduced our initial problem to a polynomial number of $(n-1)$-dimensional subproblems. By the induction hypothesis, all these subproblems can be solved in polynomial time.

Each iteration needs a constant number of oracle calls and the number of iterations is polynomial in $\log(B)$. □

4. Cutting plane scheme based on mixed integer linear programs

In this section, we propose an alternative algorithm that avoids the computation of centroids. For that, we sacrifice on the fraction of volume decrease of our polytopes in every iteration. Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a full-dimensional polytope with $P \cap \mathbb{Z}^n \neq \emptyset$. We assume that $b \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m \times n}$ is a matrix with rows $a_i^\top$ and $\|a_i\|_\infty = 1$ for $i = 1, \ldots, m$. Further, we want to assume that $Ax \leq b$ has no redundant inequalities.

For $i = 1, \ldots, m$ let $l_i := \min_{x \in P} a_i^\top x$, and let $l := (l_1, \ldots, l_m)^\top$. Since P is bounded and $\mathrm{int}(P) \neq \emptyset$, l_i exists and $l_i < b_i$ for all i. Consider the following problem in variables $x = (x_1, \ldots, x_n)^\top$ and λ.

$$\begin{array}{lc} \max & \lambda \\ \text{s.t.} & Ax + (b-l)\lambda \leq b \\ & \lambda \in \mathbb{R}_+ \\ & x \in \mathbb{Z}^n. \end{array} \tag{MILP-2}$$

Since $P \cap \mathbb{Z}^n \neq \emptyset$, (MILP-2) has an optimal solution. We will use (MILP-2) to replace (MILP-1).

The following observation relates feasible points of (MILP-2) with the difference body of P.

Observation 8. $\{(\lambda, x) \in \mathbb{R}_+ \times \mathbb{R}^n \mid Ax + (b-l)\lambda \leq b\} = \{(\lambda, x) \in \mathbb{R}_+ \times \mathbb{R}^n \mid x + \lambda(P - P) \subset P\}$.

Proof. Let $(\lambda, x) \in \mathbb{R}_+ \times \mathbb{R}^n$.

Assume that $Ax + (b-l)\lambda \leq b$. For any $z \in P - P$ there exist $x_1, x_2 \in P$ such that $z = x_1 - x_2$ and $l \leq Ax_1 \leq b$, $l \leq Ax_2 \leq b$. It follows that $A(x + \lambda z) = Ax + \lambda A(x_1 - x_2) \leq Ax + \lambda(b - l) \leq b$.

Assume that $x + \lambda(P - P) \subset P$. Then for each $i = 1, \ldots, m$ there exists a pair $x_1, x_2 \in P$ such that $a_i^\top x_1 = b_i$ and $a_i^\top x_2 = l_i$ (we assumed that there are no redundant inequalities). Hence, $a_i^\top x + (b_i - l_i)\lambda = a_i^\top x + (a_i^\top x_1 - a_i^\top x_2)\lambda = a_i^\top(x + \lambda(x_1 - x_2)) \leq b_i$. □

The next lemma is an analogue to Lemma 6 for the new algorithm in this section.

Lemma 9. *Let $(\lambda, x) \in \mathbb{R}_+ \times \mathbb{Z}^n$ be a feasible point of* (MILP-2)*, and let $H \subset \mathbb{R}^n$ be a half-space. If $x \in H$, then $\mathrm{vol}(P \cap H) \geq 2^{n-1}\lambda^n \mathrm{vol}(P)$.*

Proof. By Observation 8, we have $x + \lambda(P - P) \subset P$. Furthermore, due to the central symmetry of the difference body $P - P$, we have

$$\begin{aligned} \mathrm{vol}(P \cap H) &\geq \mathrm{vol}\Big(\big(x + \lambda(P - P)\big) \cap H\Big) \\ &\geq \frac{1}{2}\,\mathrm{vol}\big(\lambda(P - P)\big) \\ &= \frac{\lambda^n}{2}\,\mathrm{vol}(P - P) \geq 2^{n-1}\lambda^n\,\mathrm{vol}(P). \end{aligned}$$

The last inequality follows from the Brunn-Minkowski inequality (see, for instance, Gruber [7, Theorem 8.5]), stating that $2^n\,\mathrm{vol}(P) \leq \mathrm{vol}(P - P)$. □

The following lemma is an analogue to Observation 7.

Lemma 10. *Let $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ and let $P' = \{x \in \mathbb{R}^n \mid Ax + (b-l)\lambda \leq b\}$ for some $\lambda \in [0, \frac{1}{2n})$. If $\mathrm{int}(P') \cap \mathbb{Z}^n = \emptyset$ then*

$$\omega(P) \leq \frac{cn^{\frac{5}{2}}}{1 - 2\lambda n}$$

for a universal constant c.

Proof. Since $\mathrm{int}(P') \cap \mathbb{Z}^n = \emptyset$ it holds that $\omega(P') \leq cn^{\frac{3}{2}}$ for a universal constant c (see [2]).

By Observation 8, $P' = \{x \in \mathbb{R}^n \mid x + \lambda(P - P) \subset P\}$. By John's characterization of inscribed ellipsoids of maximal volume (see John [11] and Ball [1]), there exists an ellipsoid E centered at the origin, and a point q such that $q + E \subset P \subset q + nE$. By the definition of the difference body $P - P$, it follows that $2E = E - E \subset P - P \subset nE - nE = 2nE$. This implies $\lambda(P - P) \subset 2\lambda nE$ and thus $(1 - 2\lambda n)E + \lambda(P - P) \subset E \subset P - q$. Hence, $q + (1 - 2\lambda n)E + \lambda(P - P) \subset P$. This implies that $q + (1 - 2\lambda n)E \subset P'$. Thus, we obtain $P \subset q + nE \subset q + \frac{n}{1 - 2\lambda n}(P' - q)$. Hence $\omega(P) \leq \frac{n}{1 - 2\lambda n}\omega(P')$. □

We now give an alternative proof of Theorem 1.

Proof of Theorem 1. The main structure remains equivalent to the proof in Section 3. This time we set the threshold value $\Lambda \in (0, \frac{1}{2n})$. Further we assume that

$$\delta \leq \frac{1}{8\sigma\sqrt{n}}\left(\frac{4\Lambda}{\sqrt{n}B}\right)^n,$$

where σ denotes the volume of the five-dimensional unit ball. We replace (MILP-1) by (MILP-2). Let (λ^*, x^*) be an optimal solution of (MILP-2). We define $P^i_{\lambda^*} := \{x \in \mathbb{R}^n \mid Ax + (b - l)\lambda^* \leq b\}$. Again, we distinguish two cases.

In Case 1, if $\lambda^* \leq \Lambda$, we apply Lemma 10. It follows that we have to solve at most ψ subproblems, where

$$\psi := \left\lfloor \frac{cn^{\frac{5}{2}}}{1 - 2\lambda^* n} \right\rfloor + 1 \leq \left\lfloor \frac{cn^{\frac{5}{2}}}{1 - 2\Lambda n} \right\rfloor + 1.$$

In Case 2, let $\lambda^* > \Lambda$. We set $C = 2^{n-1}\Lambda^n$ as in Lemma 9 and $\kappa = nB\delta$. Then we apply Lemma 5. Thus, we ensure to reduce the volume of P^i by a constant factor of $1 - 2^{n-2}\Lambda^n$. This guarantees that after at most

$$\left\lfloor \frac{-\log(B^n)}{\log(1 - (1 - 2^{n-2}\Lambda^n)} \right\rfloor + 1$$

iterations we obtain a polytope P^k with $\mathrm{vol}(P^k) < 1$. Again, each iteration needs a constant number of oracle calls and the number of iterations is polynomial in $\log(B)$. □

It is straightforward to extend the algorithms to the mixed integer setting, provided that we can solve continuous minimization problems with a sufficient precision. Further, the approach can also be extended to quasi-convex functions, provided that instead of first order oracles delivering subgradients we have access to separation oracles.

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