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Distressed Sales in OTC Markets

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ABSTRACT: We present a stylized model of the over-the-counter markets in the tradition of Duffie, Gârleanu, and Pedersen [14] with two distinctive features: (i) buyers have heterogeneous preferences and their willingness to pay is private information and (ii) sellers become financially distressed if they cannot sell for too long. A unique steady-state equilibrium exists and it is characterized by predatory buying. Specifically, during periods where sellers are more likely to become distressed (e.g. during economic crises, financial turmoils etc.) buyers become more selective and hold off purchasing despite the abundance of distressed sales and low prices. This reluctance triggers the number of distressed sellers to grow even further and forces them for additional price cuts.

Keywords: OTC markets, predation, liquidation sales

JEL: D8, G1

1 Introduction

Over-the-counter (OTC) markets, unlike exchanges, operate via search and matching. An investor who wants to sell or buy an asset must first search for a counterparty. Transactions are typically bilateral and private, and prices are determined strategically taking into account the outside option of each participant. In a seminal paper Duffie et al. [14] construct a search model of the OTC markets addressing these frictions.

Even though the model in [14] is based on search, it still portrays a rather standardized and transparent trading environment. In their model products are homogenous, investors are homogenous, and as the setup is based on complete information, every meeting automatically results in trade. OTC, however, is a blanket term covering a vast array of products with significantly different characteristics. Some products are indeed standardized and transparent and therefore fit to the portrayal above e.g. centrally cleared products such as interest rate derivatives traded on the inter-dealer clearing house SwapClear or equities traded over DCTCC. However, there exists a

range of other products that are not nearly as standardized and transparent, and therefore require a different modelling approach, e.g. mortgage-backed securities, emerging-market debt, equity derivatives, exotic derivatives including non-vanilla interest and currency derivatives. These products are highly differentiated and non-standard and they are traded by a diverse investor base with wide-ranging needs and objectives.

Consider, for instance, the over-the-counter equity derivatives (OTCED). With a total notional amount exceeding \$7 trillion, the OTCED market serves a wide variety of investors including large corporations, banks, insurance companies, hedge funds, public sector funds, sovereign wealth funds and so on. The flexibility in terms of product design and its private outlook helped the OTCED market flourish over the years. The market offers a significant number of products that are not available on exchanges or clearing houses with strict rules, where products are too standard to accommodate the particular requirements of an ever-expanding and diverse investor base.

Furthermore OTCED transactions are executed through bilateral meetings and, due to the private nature of these transactions, the market is characterized by a lack of pre-trade transparency. A recent report by the Financial Services Authority & HM Treasury states: "OTC derivative markets are not subject to formal pre and post trade transparency requirements. As a result some market participants have better access to better information."¹ Indeed, characterizations such as "opaque", "murky" or "anonymous" appear frequently in the financial press describing such products. The opaqueness of the OTCED market implies that for most products only a limited amount of pre-trade public data is available, and therefore, investors can find out detailed product features only after getting in contact with sellers.

The discussion thus far gives credit to construct a model where investors with heterogenous preferences operate in a non-transparent market that offers a wide range of heterogenous products. To capture the notion of preference and product heterogeneity, we assume that the dividend of an asset consists of two components: a market-wide deterministic and aggregate component x , plus an idiosyncratic component v , which is a random draw from a known cdf.² The realization of v determines how good a fit the asset is for a buyer's tastes and preferences. Furthermore, to capture the idea that the market is opaque and characterized by a lack of pre-trade transparency, we assume that the buyer realizes the quality of the fit v only after linking up with the seller. The realization of v is the buyer's private information and it cannot be observed by anyone else.

With these assumptions the probability of trade is endogenous; so, meetings are no longer guaranteed to result in trade. The search process, from buyers' perspective, amounts to finding a good fit and in doing so, they follow a threshold rule: if the quality of fit in a match is sufficiently high then the deal goes through, otherwise buyers walk away.

¹"Reforming OTC Derivative Markets: A UK perspective." Available at www.fsa.gov.uk

²This is a standard technique to accommodate product heterogeneity in the market and preference heterogeneity among buyers; see for instance Jovanovic [20] or Wolinsky [36].

There are two types of frictions in the model: the first is meeting a counterparty and the second is whether or not the transaction materializes. The literature, spurred by Duffie et al. [14] captures the former friction, but not the second. Those models are based on complete information, and therefore all meetings, by default, result in trade.³ In reality, however, it is not uncommon at all for parties to walk away without trading; disagreement, in fact, is the more likely outcome. In addition, with the advancing communication technology, getting in contact with potential traders is easier than ever; hence the key friction is the latter—that is, whether the buyer wants to purchase or not. This, in turn, depends on whether the asset is indeed what the buyer is looking for.

The endogeneity of the probability of trade implies that buyers can control, and in fact manipulate, the duration of sale, which brings us to the second component of the model; namely the fact that sellers can become financially distressed if they cannot sell for too long.⁴ Sellers in financial markets can become distressed for a variety of reasons including nearing margin calls, pressing debt obligations, hedging motives, being caught in a short squeeze and so on. To incorporate this notion, we assume that there is an adverse shock that pushes regular sellers into a state of permanent distress. It is sensible to think that such a shock is more likely to arrive during episodes of economic crises, recessions and financial turmoils. We show that during such periods customers become more selective and hold off purchasing despite the abundance of distressed sales and lower prices. By doing so they strategically slow down the speed of trade causing the percentage of distressed sellers to grow further. This, in turn, exerts more pressure on sellers forcing them for further price cuts. At the end, distressed sellers not only are forced to cut their already low prices, but also they find it more difficult to sell and exit thanks to buyers’ reluctance to trade. This cycle, which we label as *predatory buying*, dries up liquidity and increases the cost of liquidation sales for distressed sellers. Indeed, from their point of view liquidity disappears when it is mostly needed.

Though it lacks an agreed upon definition in the literature, predation is a prevalent feature of financial markets. A recent body of theoretical work explores various mechanisms through which predatory trading takes place e.g. Attari et al. [3], Brunnermeier and Pedersen [6], Carlin et al. [9]. In Section 4.2 we discuss these papers in more detail; but at this point we want to point out that the aforementioned papers are not based on search and matching. To the best of our knowledge, this is the first paper exploring predation in a search model of the OTC markets and

³See Duffie et al. [14], Lagos and Rocheteau [24], Rocheteau and Weill [30], Vayanos and Wang [34] among others.

⁴Albrecht et al. [2] and Selcuk [33] provide models on the housing market that have related notions of distressed sellers. The model in [2] produces various equilibrium matching patterns including “opportunistic matching” where regular searchers wait to meet with desperate searchers only. The setup in [33] is open loop in that trading agents leave the market and are replaced by clones, whereas ours, similar to the aforementioned papers in the OTC literature, has a closed loop setting where sellers become buyers, buyers become owners and owners become sellers again.

bridging the gap between the two strands of literature.

2 Model

We consider a continuous-time economy with a fixed supply $a > 0$ of indivisible assets that yield a flow of dividends q . Investors are risk neutral and divided into four categories; buyers, non-trading owners, regular sellers and distressed sellers. Similar to Duffie et al. [14] we have a ‘closed loop’ setting where no agent leaves the market and there is no entry from outside. The total measure of agents η is fixed and exceeds a . Each buyer wants to purchase one unit of the asset to consume its dividends. After trading, buyers become owners and remain so until they are hit by a liquidity shock that turns them into regular sellers. The shock arrives with a Poisson rate σ and reduces the flow value of dividends from q to zero, which is why sellers wish to trade and liquidate their holdings.⁵ Once the asset has been sold, the seller comes back to the market as a buyer (see the flowchart).

If regular sellers cannot trade for too long then they may become distressed. We model this notion by another idiosyncratic adverse shock, which, too, arrives at an exogenous Poisson rate μ . The shock is similar in nature to the liquidity shock above and may be associated with factors such as pressing debt obligations, margin calls from other positions and so on. Such difficulties are more likely to arise during financial crises or recessions, so it is sensible to think that μ rises during such periods. Buyers and regular sellers discount future utility at rate $e^{-\delta}$ whereas distressed sellers are more impatient with $\bar{\delta} > \delta$. A larger value of $\bar{\delta}$ implies a more severe shock.

As discussed in the Introduction, investors possess heterogeneous preferences. To implement this idea we assume that the dividend q of an asset consists of an aggregate component x plus an idiosyncratic component v , that is

$$q = x + v.$$

The aggregate component x is same across all assets, whereas the idiosyncratic component $v \in [0, 1]$ is a random draw from the unit interval via the cdf F . Buyers differ in terms of their tastes and preferences, so the realization of v determines how good a fit the asset is for a buyer’s preference. A high value of v indicates a good fit and a low value indicates a poor fit. We assume that v is independent across buyers, so the same asset may be liked by one buyer and disliked by another. From a buyer’s perspective the search process amounts to finding a high enough v . The value of v does not change over time; once an asset is purchased the buyer enjoys the same v forever. We impose the following assumption on F .

⁵The liquidity shock in the literature is typically associated with hedging needs arising from a position in another market; see, for instance, [14], [24] or [34].

Assumption 1. *The survival function $\Phi = 1 - F$ is log-concave, i.e.*

$$f^2(v) + f'(v)\Phi(v) > 0, \forall v.$$

The market is opaque and characterized by a lack of pre-trade transparency; hence we assume that the buyer realizes the value of v only after linking up with the seller and that this realization is private information. The seller cannot observe v (he only knows the cdf F that generates it), so he is unable to tailor the price individually, and therefore, he must quote the same price p for each customer. The probability of trade Φ_j is endogenous and depends on the seller's type, regular or distressed, denoted by $j = r, d$.

The market operates via search and matching and agents meet each according to a Poisson process. Specifically, a buyer meets a distressed seller at rate αm_d where $\alpha > 0$ denotes the search intensity and m_d denotes the steady state measure of distressed sellers. Similarly, the buyer meets a regular seller at rate αm_r , where m_r is the measure of regular sellers. Finally, a seller meets a buyer at rate αm_b , where m_b is the measure of buyers.⁶

Before proceeding to the analysis, a remark is in order to explain why the trading mechanism in our setup is price-posting and not bargaining. Indeed, existing papers in the literature consider Nash or Rubinstein bargaining procedures and it would be interesting to explore the implications of our model under these pricing mechanisms. However, modelling bargaining with private information is a non-trivial task as multiple or a continuum of equilibria are common in such models (see [21] for an extensive discussion.) With price posting however, equilibrium is unique and can be characterized analytically. In this paper our goal is to understand how the presence of distressed sellers affects prices, liquidity and buyers' search behavior and to that end the uniqueness of the equilibrium and the fact that equilibrium objects (prices, probabilities, measures of agents etc.) can be characterized analytically is indispensable. A second point in defense of using price posting is the result by Samuelson [31], who shows that in bargaining between informed and uninformed agents, where parties may bargain by any procedure they deem appropriate, the optimal mechanism is for the uninformed agent to make a take-it-or-leave offer. This result indicates that the optimal pricing mechanism in our model is indeed price posting where the seller (the uninformed party) advertises a take-it-or-leave-it price.

2.1 Discussion on Modelling Assumptions

The model operates via search-and-matching, and more significantly it exhibits product and preference heterogeneity. So, for the model to be relevant, the market in question ought to exhibit these traits, that is:-

⁶Duffie and Sun [16] present a formal proof of this argument. See also Vayanos and Wang [34].

- Investors should be somewhat in the dark about potential trading opportunities and it should take time to find and meet a suitable partner for a trade.
- Second, and more importantly, products ought to be heterogeneous and non-standard, making it difficult for an investor to assess whether or not a product is indeed suitable for his specific needs before meeting the seller and scrutinizing the underlying structure of the product.

Examples for such markets include markets for mortgage-backed securities, equity derivatives, collateralized debt obligations and other structured credit products, which are indeed heterogeneous and non-standard and usually exhibit complex contractual features. Investors often find out the specific details only after linking up with the seller and analysing the underlying structure of the asset. In [18]’s terminology, products in these markets are indeed *inspection goods*: buyers cannot resolve the pre-trade uncertainty pertaining the good before inspecting it (i.e. examining its contractual aspects). Furthermore, a wide variety of investors operate in these markets—insurance companies, hedge funds, public sector funds, sovereign wealth funds—all of which have vastly different priorities, constraints and requirements; thus an asset that is a good fit for a particular investor may well turn out to be improper fit for another.⁷

An OTC trade negotiation for such products is typically initiated when an investor finds and contacts a seller and asks for terms of trade. This process refers to the search and matching friction above and it is addressed in the paper by the random matching process. Communication could be by phone, by email, by electronic query systems or, in some markets, through a broker, though we ignore the role of brokers in this paper. At this step the investor obtains the necessary information pertaining the product, and after analysing its underlying structure and the terms of trade offered by the seller, he decides whether to carry on with the transaction or to walk away. This step refers to the second point mentioned above and this is where the parameter v comes into play.

We assume that the dividend of a product consists of two components: a market-wide aggregate component x , plus an idiosyncratic component v , which is a random draw from a known cumulative density function. The value of v is uncertain until the buyer meets the seller and the realization of v can be interpreted as the inspection process mentioned above. Upon realizing the value of v the buyer understands how good a fit the asset is for his tastes and preferences. This is a standard

⁷Admittedly, there are other OTC markets, where products are rather standard, transactions are transparent, the volume of trade is high and the overall trading experience does not fit the description above, e.g. recently issued US government bonds and certain liquid OTC derivatives such as simple interest rate and currency swaps. These products are indeed natural candidates for exchange-based trade but they are somehow traded over the counter. From an academic point of view there is a lack convincing theories explaining why such simple products are traded over the counter and not on exchanges. In this regard, OTC markets where search and matching plays almost no role and where there is little scope for opaqueness and price and product uncertainty are beyond the consideration of this paper.

technique to accommodate the notion of product and preference heterogeneity in that no two assets are identical and an asset that turns out to be a good fit for one buyer may turn out to be a poor fit for another. An advantage of the technique above is the fact the by fixing the boundaries and the density function of v and varying x one can explore how equilibrium objects—prices, probabilities of trade—respond to the degree of product standardization in the market (see the discussion in Section 4).

3 Analysis

3.1 Steady State Measures

The asset is in fixed supply a , so the measures of agents in possession of the asset (owners + regular sellers + distressed sellers) add up to a , that is

$$m_o + m_r + m_d = a. \tag{1}$$

The total measure of agents η is also fixed and exceeds a . It follows that the steady state measure of buyers, too, is fixed and equals to

$$m_b = \eta - a > 0.$$

Without loss in generality fix $m_b = 1$ so that η equals to $1 + a$. Remaining measures m_o , m_r and m_d are endogenous and are determined by the fact that in steady state the inflow into a group of investors equals to the outflow from it. Similar to Duffie et al. [14], we have a ‘closed loop’ setup in the sense that no agent leaves the market and there is no inflow from outside (see Fig 1 below). Consider distressed sellers. The inflow μm_r consists of regular sellers hit by the adverse shock. The outflow $\alpha m_d \Phi_d$ comprises of sellers who trade and become buyers. Setting inflow equal to outflow yields

$$\alpha m_d \Phi_d = \mu m_r. \tag{2}$$

Now consider regular sellers. The inflow σm_o consists of owners hit by the liquidity shock. The outflow has two components: $\alpha m_r \Phi_r$ which are regular sellers who trade and become buyers plus μm_r which are regular sellers who become distressed. Therefore

$$\sigma m_o = \alpha m_r \Phi_r + \mu m_r. \tag{3}$$

Proposition 1 *Equations (1), (2) and (3) pin down the steady state measures m_o , m_d and m_r as*

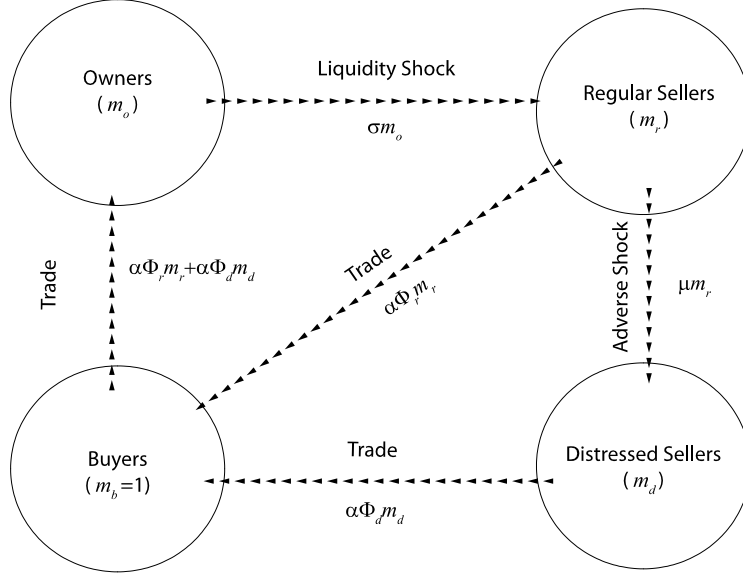


Figure 1: Flowchart

follows:

$$\begin{aligned}
 m_d &= a \left\{ \frac{\alpha}{\sigma} \Phi_d \left\{ 1 + \frac{\alpha}{\mu} \Phi_r \right\} + \frac{\alpha}{\mu} \Phi_d + 1 \right\}^{-1}, \\
 m_o &= m_d \times \frac{\alpha}{\sigma} \Phi_d \left\{ 1 + \frac{\alpha}{\mu} \Phi_r \right\}, \\
 m_r &= m_d \times \frac{\alpha}{\mu} \Phi_d.
 \end{aligned} \tag{4}$$

The measures depend on exogenous parameters α, a, μ and σ as well as the probabilities of trade Φ_r and Φ_d which are endogenous and controlled by buyers.⁸ The fraction of distressed sellers in the market is given by

$$\theta \equiv \frac{m_d}{m_d + m_r} = \frac{1}{1 + \frac{\alpha}{\mu} \Phi_d}. \tag{5}$$

Note that θ increases as Φ_d falls. Indeed if buyers squeeze Φ_d then distressed sellers cannot trade fast enough and their prolonged presence in the market causes θ to grow. The growing θ , in turn,

⁸The following table summarizes the signs of the partial derivatives of the measures with respect to the parameters of interest (the algebra is skipped):

	σ	μ	Φ_d	Φ_r
m_d	+	+	-	-
m_r	+	-	+	-
m_o	-	+	+	+

A rise in the arrival rate of the liquidity shock σ turns more owners into sellers, so m_d and m_r rise while m_o falls. Similarly a rise in the arrival rate of the adverse shock μ causes more relaxed sellers to become distressed; hence m_r falls while m_d goes up. The effect of μ on the measure of owners m_o is more subtle. The rising μ increases the fraction of distressed sellers, and distressed sellers trade faster than regular sellers; so, at the end, more buyers become owners, hence m_o goes up. Using similar arguments, and the flowchart, one can explain the signs wrt Φ_j .

intensifies competition among distressed sellers and forces them for further price cuts. This is the basic mechanism behind the predation result.

3.2 Value Functions of Owners, Buyers and Sellers

Letting Γ denote the value function of an owner, we have

$$\delta\Gamma = v + x + \sigma\{\Pi_r - \Gamma\}.$$

An owner keeps enjoying the idiosyncratic dividend v plus the aggregate dividend x until he is hit by the liquidity shock σ , which turns him into a regular seller, whose value function is denoted by Π_r . Rearranging yields

$$\Gamma = \frac{v + x + \sigma\Pi_r}{\sigma + \delta}. \quad (6)$$

Now turn to buyers. Letting Ω denote their value function we have

$$\delta\Omega = \alpha m_r I_r + \alpha m_d I_d,$$

where

$$I_j = \int_0^1 \max\{\Gamma(v) - p_j - \Omega, 0\} dF(v) \text{ for } j = r, d.$$

The expression I_j is the expected surplus to a buyer contingent on having met a type j seller. As long as the surplus $\Gamma(v) - p_j$ exceeds the opportunity cost Ω the buyer purchases, otherwise he walks away. For any given price p_j we conjecture an associated threshold (or ‘reservation value’) v_j leaving the buyer indifferent between buying and searching i.e. satisfying

$$p_j + \Omega = \Gamma(v_j).$$

After substituting for Γ , the indifference condition becomes

$$p_j + \Omega = \frac{v_j + x + \sigma\Pi_r}{\sigma + \delta}. \quad (7)$$

Buyers’ decision is simple: purchase if $v \geq v_j$ and keep searching otherwise. Clearly the probability of trade Φ_j is endogenous and equals to

$$\Phi_j = \Pr(v \geq v_j) = \Phi(v_j),$$

where $\Phi = 1 - F$ is the survival function. As mentioned earlier, not all meetings result in trade; for trade to occur the asset has to be a good match for the buyer. Substitute Γ from (6) into the

expression for I_j and use the indifference condition (7) to obtain

$$I_j = \int_{v_j}^1 \frac{v - v_j}{\sigma + \delta} dF(v) = \int_{v_j}^1 \frac{\Phi(v) dv}{\sigma + \delta}.$$

The second step follows from integration by parts. Substituting I_j we get a cleaner expression for buyers' value function:

$$\Omega = \alpha m_r \int_{v_r}^1 \frac{\Phi(v) dv}{\delta(\sigma + \delta)} + \alpha m_d \int_{v_d}^1 \frac{\Phi(v) dv}{\delta(\sigma + \delta)}. \quad (8)$$

Finally we turn to sellers. Desperate and regular sellers' value functions are given by

$$\bar{\delta}\Pi_d = X_d \quad \text{and} \quad \delta\Pi_r = X_r + \mu(\Pi_d - \Pi_r), \quad (9)$$

where

$$X_j = \alpha\Phi(v_j)(p_j + \Omega - \Pi_j).$$

Expression X_j is the expected net trade surplus to a type j seller. A seller encounters a buyer at rate α and the buyer purchases with probability $\Phi(v_j)$. If trade occurs the seller obtains price p_j plus Ω (he becomes a buyer now) minus Π_j (he is no longer a seller). With this information it is easy to interpret Π_d and Π_r . Note that a regular seller keeps track of the possibility of becoming distressed as well, whereas a distressed seller will remain distressed until he sells.

Note that

$$(\delta + \mu)\Pi_r = X_r + \frac{\mu}{\delta}X_d.$$

A type j seller solves

$$\max_{p_j \in \mathbb{R}_+} \Pi_j \quad \text{s.t.} \quad v_j = (\sigma + \delta)(p_j + \Omega) - x - \sigma\Pi_r$$

taking Ω as given.⁹ The function Π_j is a weighted average of X_j s; so, the optimal price p_j must, by the Bellman principle, maximize the net surplus X_j . The FOC, thus, is given by

$$p_j + \Omega - \Pi_j = \frac{\Phi(v_j)}{(\sigma + \delta)f(v_j)}. \quad (10)$$

Expression (10) is the net trade surplus for a seller and the fact that it is positive implies that, conditional on having met a buyer and that the buyer is willing to transact, the seller is willing to transact as well (instead of walking away). To see why, note that if the seller transacts then he

⁹Sellers are atomless and they fail to realize the effect of an individual price change on buyers' value of search; see [8] for a discussion.

obtains price p_j plus the value of becoming a buyer Ω , whereas if he waits then he continues to obtain Π_j . Since $p_j + \Omega > \Pi_j$, the former option outweighs the latter.¹⁰

It is easy to verify the second order condition; thus the solution above corresponds a maximum.¹¹ Inserting the FOC into X_j yields

$$X_j = \frac{\alpha \Phi^2(v_j)}{(\sigma + \delta) f(v_j)}.$$

Substituting this into (9) produces the closed form expressions for sellers' value functions:

$$\Pi_d(v_r, v_d) = \frac{\alpha \Phi^2(v_d)}{\bar{\delta}(\sigma + \delta) f(v_d)} \quad \text{and} \quad (11)$$

$$\Pi_r(v_r, v_d) = \frac{\alpha \Phi^2(v_r)}{(\sigma + \delta)(\delta + \mu) f(v_r)} + \frac{\mu \alpha \Phi^2(v_d)}{\bar{\delta}(\delta + \mu)(\sigma + \delta) f(v_d)}. \quad (12)$$

Now we can define the equilibrium.

Definition 2 *A steady-state symmetric equilibrium is characterized by value functions Γ , Ω , Π_d , Π_r given by (6), (8), (9) and the pair $\mathbf{v}^* = (v_r^*, v_d^*) \in [0, 1]^2$ and $\mathbf{p}^* = (p_r^*, p_d^*) \in \mathbb{R}_+^2$ satisfying indifference (7) and profit maximization (10). The steady state measures m_d^* , m_r^* and m_o^* , also implicitly part of the equilibrium, can be recovered from (4).*

¹⁰Recall that our setup is a closed loop setting with no entry or exit. Provided that the outside option associated with exiting the market is normalized to zero, the no-exit condition is non-binding. Indeed the fact that both Π_r and Π_d , given by (11) and (12), are positive implies that along the equilibrium path, all sellers, regular or distressed, would prefer to remain in the market even if they were allowed to exit. In the literature, the outside option is interpreted as the rate of return of a risk-free asset and it is typically normalized to zero e.g. Brunnermeier and Pedersen [6]. From a broader perspective, theoretical models studying OTC markets, including this one, are partial equilibrium settings: the focus is the OTC market and other alternatives that investors might turn to (e.g. the "risk-free" market) are treated exogenously. To meaningfully discuss exit and entry decisions one needs a general equilibrium setup where the rate of return in the alternative market, too, is endogenous and investors are free to self select themselves into whichever market they want. This, however, is beyond the scope of the current paper.

¹¹We have

$$X_j'' = -\alpha(\sigma + \delta) \times \{f'(v_j)(\sigma + \delta)\{p_j + \Omega - \Pi_j\} + 2f(v_j)\}.$$

Substitute the FOC (and omit the argument v_j) to obtain

$$X_j'' = -\alpha(\sigma + \delta) \times \{f'\Phi + 2f^2\} / f.$$

The expression is negative because of log concavity (Assumption 1).

4 Results

4.1 Existence of Equilibrium and Liquidation Sales

Combine indifference conditions in (7) with FOCs in (10) to obtain the following system of equations that pin down the equilibrium values of v_r^* and v_d^* :

$$\Delta_r(v_r, v_d) = \Phi(v_r)/f(v_r) + \delta\Pi_r - x - v_r = 0 \quad \text{and} \quad (13)$$

$$\Delta_d(v_r, v_d) = \Phi(v_d)/f(v_d) + (\sigma + \delta)\Pi_d - \sigma\Pi_r - x - v_d = 0. \quad (14)$$

Proposition 3 *The equilibrium exists and it is unique. In equilibrium distressed sellers pursue ‘liquidation sales’ as they accept to trade at lower prices and consequently sell faster, i.e. $p_d^* < p_r^*$ and $\Phi_d^* > \Phi_r^*$.*

In the proof we show that the locus of $\Delta_r = 0$ is downward sloping wrt v_r whereas the locus of $\Delta_d = 0$ is upward sloping; so, they intersect once in the $v_r - v_d$ space, which implies that there exists a unique \mathbf{v}^* satisfying (13) and (14) (the proof is in the appendix). Furthermore, the equilibrium is characterized by *liquidation sales*. After being hit by the adverse shock, a distressed seller grows impatient and quotes a lower price in an effort to quickly exit from his position. (In section 4.3 we provide numerical simulations exploring the cost of such sales.) The price-cut produces the desired outcome: the inequality $\Phi_d^* > \Phi_r^*$ implies that distressed trades materialize faster than regular trades.

Before moving on, we briefly comment on the link between the aggregate yield x and the probability of trade. As seen above, from a buyer’s point of view the search process amounts to finding a high enough v since all assets yield the same deterministic x . So, it may appear that the aggregate yield x plays no role in determining the probability of trade; however this is not true. As it turns out, buyers pay little or no attention to v if x is large enough.

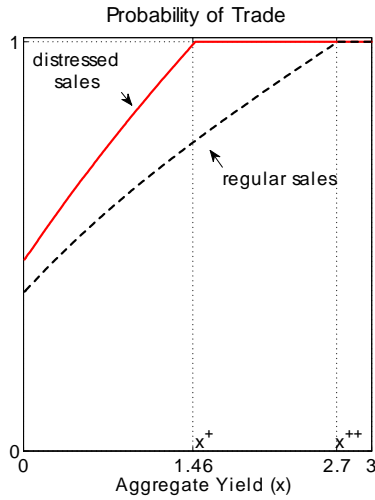
Remark 4 *Both Φ_r^* and Φ_d^* rise in the deterministic component x of a product. Specifically*

$$\begin{aligned} \Phi_r^* < \Phi_d^* < 1 & \quad \text{if } 0 < x < x^+ \\ \Phi_r^* < \Phi_d^* = 1 & \quad \text{if } x^+ \leq x < x^{++} \\ \Phi_r^* = \Phi_d^* = 1 & \quad \text{if } x^{++} \leq x, \end{aligned}$$

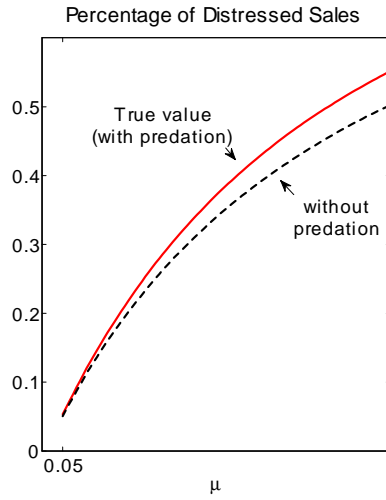
where x^+ and x^{++} are thresholds given by (29) and (34).

If x shrinks, then the idiosyncratic goodness of fit v becomes too important for buyers, and therefore no meeting is guaranteed to result in trade. Indeed if $x < x^+$, then even distressed sellers, who charge lower prices, face some uncertainty about whether or not they can sell. However, if x

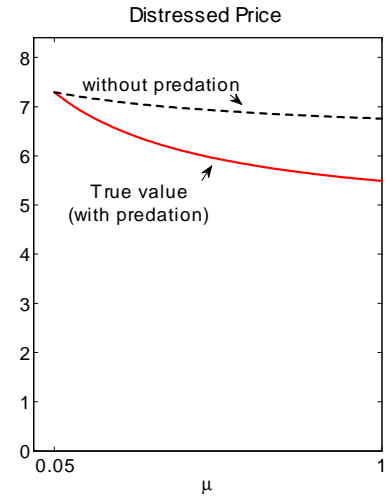
starts to grow, then buyers start paying less attention to v , and therefore Φ_d^* and Φ_r^* start to grow as well. For an illustration see Fig 2a.



2a



2b



2c

With some abuse in labelling, one can re-interpret this simulation by thinking of x as a proxy of product standardization in the market. To see why, note that $E[v]$ and $StDev[v]$ are fixed in the simulation whereas x ranges from 0 to 3. If a product has a high value of x , then the random fit v is relatively unimportant in that the product possesses little chance of not being compatible with buyers' preferences; hence it can be labelled as "fairly standard" (e.g. vanilla products traded via clearing houses). The opposite is true if x is small (e.g. exotic products traded through bilateral meetings). With this interpretation, panel 2a suggests that the probability of trade (and therefore the speed of trade) increases with product standardization: the more uniform the products the faster the trade. In addition, the probability of trade can be 1 if products are "standard enough" i.e. one does not need perfectly uniform products as in Duffie et al. [14] to ensure that all meetings result in trade.

4.2 Predation

Proposition 5 *If the adverse shock arrives more often, i.e. if μ rises, then the equilibrium price p_d^* falls, yet the probability of trade Φ_d^* decreases, i.e. buyers deliberately delay purchasing from distressed sellers despite the falling prices.*

It is sensible to think that the adverse shock μ is more likely to arrive during periods of turmoils and economic crises. The proposition says that during such times distressed sellers lower their prices, yet buyers become more reluctant to purchase. This behavior (labelled as ‘predation’) further increases the percentage of distressed sellers in the market and forces them for further price cuts.

The mechanism behind the result is this. An increase in μ causes sellers’ and buyers’ value functions to move in opposite directions making sellers worse off and buyers better off. Specifically, the fraction of distressed sellers θ rises with μ and intensifies the competition for distressed sellers. Realizing that many other sellers are in the same dire situation, distressed sellers are forced to cut their already low prices. The question is whether price cuts generate the desired outcome and the answer is no; indeed their probability of trade Φ_d^* falls instead of rising. To understand why note that distressed sales come with greater consumer surplus, which means that the rising θ boosts buyers’ value of search. Realizing that there are plenty of good deals in the market, buyers hold off purchasing and search longer, i.e. they lower Φ_d^* . This response has the following feedback effect. By lowering Φ_d^* buyers strategically slow down the speed of trade and cause θ to grow further. The growing θ , in turn, puts additional downward pressure on prices and so on. For an illustration of these arguments see Figure 2b and 2c. The solid lines in panels 2b and 2c are the true values of θ and p_d^* , whereas the dashed lines are what they would have been had the probabilities of trade remained intact. The difference between the two lines, therefore, is due to predation mechanism described above.¹²

Predation is a prevalent feature of financial markets, especially during financial crises where the adverse shock μ is indeed more likely to arrive. A recent body of theoretical work explores mechanisms through which different forms of predation may take place. For instance in Attari et al. [3] predators lend to financially fragile players in an effort to obtain higher profits by trading against them for a prolonged time. Carlin et al. [9] construct an equilibrium where cooperation among traders occasionally breaks down leading to predatory trading and episodic illiquidity. In Brunnermeier and Pedersen [6], which is arguably the most influential paper in this literature, if a distressed trader is forced to liquidate, other strategic traders initially sell in the same direction driving down the price even faster and then buy back at the low price.

The aforementioned models are not based on search and matching. They take either the aggregate demand function or market price equations parametrically without deriving them explicitly from the mechanics and frictions of a decentralized market (as in the literature spurred by Duffie et al [14]). To our knowledge this is the first paper studying predation in a search model of the OTC markets and bridging the gap between these two strands of literature.

¹²The dashed lines are obtained by fixing $\Phi_d^* = 0.89$ and $\Phi_r^* = 0.59$ which are the equilibrium values when $\mu = 0.05$. This is why in both pictures the solid and dashed lines intersect at $\mu = 0.05$.

In addition, the papers above are all based on settings with heterogenous investors in terms of their size and their ability to take multiple positions and these assumptions are vital in producing predatory mechanisms in those settings. For instance, in Brunnermeier and Pedersen [6] it is the presence of large strategic traders and their ability to impact market prices single-handedly that triggers prices to fall to artificially low levels. Without such large players the predation mechanism in their paper (described above) cannot function. On the contrary, in our setting players are atomless and can only have a single position.¹³ Our mechanism relies on the endogeneity of the probability of trade, which enables buyers to manipulate the speed of trade to their advantage. Such a mechanism does not require large investors; preference heterogeneity and private information are sufficient.

Finally, note that in Brunnermeier and Pedersen [6] during episodes of predation prices fall, but on the other hand, trade speeds up. In other words, distressed sellers are forced to sell at lower prices, but at least they are able sell and exit quicker than before. This is not the case in our model: during episodes of predation prices fall, yet distressed sellers find it more difficult to sell due to buyer's reluctance to trade. Anecdotal evidence suggests that our model's prediction is more in line with what happens in decentralized markets (financial or otherwise) during times of distress and turbulence. Indeed, during the last crisis the financial press was rife with news of buyers holding out despite the falling prices, slowness of trade and investors' apparent reluctance to make a move. Though such behavior may be attributed to uncertainty resolution or risk aversion, there is no doubt that some of that reluctance was indeed a strategic and deliberate effort to obtain better deals in the future.

4.3 A Numerical Example

In what follows we provide some sensitivity analysis via numerical simulations. We set the search intensity $\alpha = 125$ so that an agent expects to meet 125 other agents a year, which is equivalent to one counterparty per two business days. Following the calibration in Duffie et al. [15] the fraction of investors holding a position (sellers + owners) is assumed to be 0.8. To match this we set the supply of the asset $a = 4$.¹⁴ Recall that $\eta = a + 1$, so $\eta = 5$. The deterministic dividend is normalized to $x = 0$, whereas the idiosyncratic dividend v is assumed to be uniformly distributed in the unit interval. We set $\delta = 0.05$, which means that all agents, except distressed sellers, discount future utility at the annual rate $(1 + \delta)^{-1} \approx 95\%$. Again, following Duffie et al. [15], the arrival rate of the liquidity shock σ equals to 0.5 meaning that an owner remains so for an average of 2

¹³This assumption precludes regular sellers from preying on distressed sellers (as opposed to [6]).

¹⁴Recall that the measures of agents holding an asset (sellers and owners) must add up to a and that the measure of buyers is fixed at $m_b = 1$. It follows that the fraction of agents with a position is $a/(a + 1)$, which equals to 0.8 when $a = 4$.

years. The arrival rate of the adverse shock is set $\mu = 6$ i.e. on average a seller can last $12/\mu = 2$ months without trading before becoming distressed. Finally $\bar{\delta} = 9$, which means that distressed sellers discount future utility at the annual rate $(1 + \bar{\delta})^{-1} = 10\%$. The table below summarizes the baseline parameters.

$v \sim U(0, 1)$	$x = 0$	$\alpha = 125$	$a = 4$	$\eta = 5$
$\sigma = 0.5$	$\mu = 6$	$\delta = 0.05$	$\bar{\delta} = 9$	

Table 1

Under the benchmark parameter values the model yields equilibrium prices $p_d^* = 0.98$, $p_r^* = 1.75$, probabilities of trade $\Phi_d^* = 0.63$, $\Phi_r^* = 0.21$ and measures of agents $m_o^* = 3.93$, $m_r^* = 0.065$, $m_d^* = 0.005$ (recall that $m_b^* = 1$). These numbers imply an annual turnover rate of

$$\frac{\alpha (m_r^* \Phi_r^* + m_d^* \Phi_d^*)}{m_r^* + m_d^* + m_o^*} = 49.2\%,$$

which is very close to the median annual turnover of 51.7% estimated by Edwards et al. [17]. Similarly about 78.5% of agents are owners, 1.5% are sellers and 20% are buyers. Again, these estimates are very close to their counterparts in the calibration exercise in Duffie et al. [15] (see Table 2 therein).

Before we get into simulations, note the analysis so far was based on a steady state setup. Therefore, the simulations below depict snapshots of the economy at various steady states; however they are silent about the transition process between those states. For the dynamic version of the model, where we explore this transition, see section 5.2.

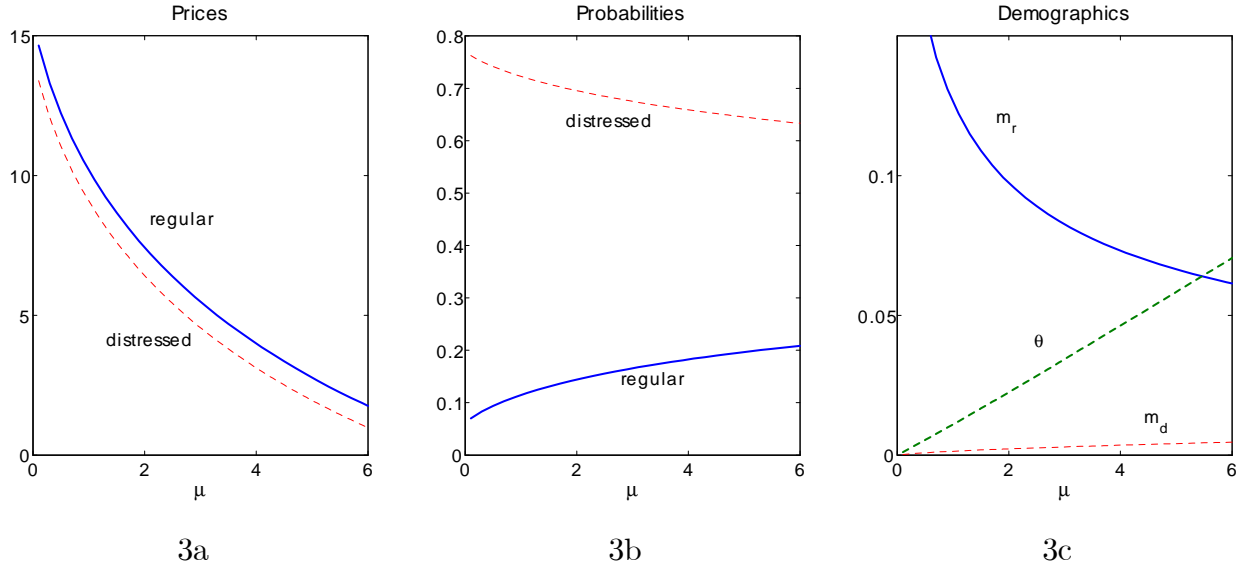


Figure 3 depicts prices, probabilities and measures of agents against the arrival rate of the adverse shock μ . An increase in μ has three consequences. First, more sellers become distressed and attempt liquidation sales; see the rising θ in panel 3c. Second, all sellers, regular and distressed, trade at lower prices (panel 3a). Third, customers become more reluctant to purchase from distressed sellers; the probability of trade for those sellers keeps falling (panel 3b). We have already discussed the mechanism behind this result, but there is a point to add here.

Regular sellers, too, are worse off because of the rising μ . Facing an increasing prospect of becoming distressed in the future, they significantly reduce their prices in an effort to quickly sell before being hit by the adverse shock (see the falling regular price in 3a). This reaction can be described as the *spillover* of distressed sales onto the regular sales and it can be indeed significant. To quantify this negative effect we start from a benchmark where $\mu = 2$ (the rest of the parameters are as in Table 1) and then we plot the percentage drop in the regular price against the increasing μ . When $\mu = 2$ a seller can go on for $12/\mu = 6$ months, on average, before becoming distressed and at that point the equilibrium value for the regular price p_r^* is about 7.42. In Figure 4a we plot the percentage drop $p_r^*(\mu)/7.42 - 1$ against μ and it is clear that the price drop can be substantial if μ increases significantly. For instance if μ rises from 2 to 6 then the regular price falls by about 75%.

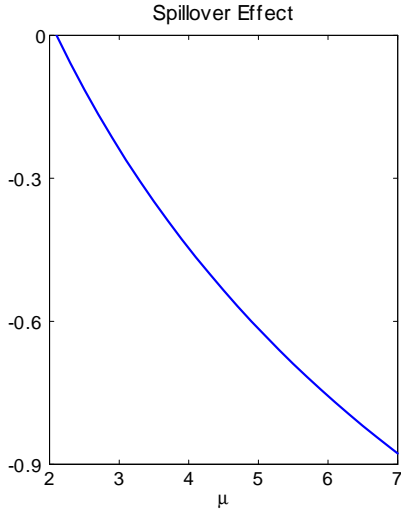


Figure 4a

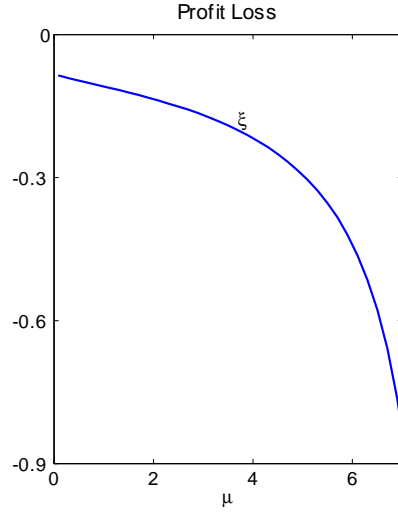


Figure 4b

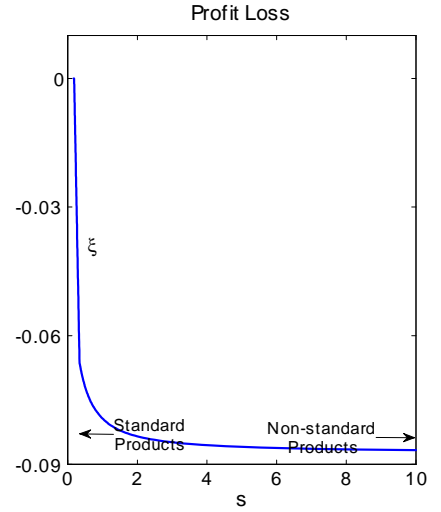


Figure 4c

4.4 Cost of Liquidation Sales

Distressed sellers accept substantially lower prices when they try to liquidate. For instance, under the benchmark parameters of Table 1 the equilibrium price in a distressed sale $p_d^* = 0.98$ is about 40% less than the price in a regular sale $p_r^* = 1.75$. The price cut produces the desired outcome: conditional on meeting a buyer, a distressed seller has a $\Phi_d^* = 63\%$ chance of trading as opposed to $\Phi_r^* = 21\%$ for a regular seller. Clearly, attempting a liquidation sale is costly. Had the seller not become distressed, he would have traded at p_r^* but the shock forces him to trade at the lower price p_d^* . We use the percentage-wise profit loss

$$\xi = p_d^*/p_r^* - 1$$

as a proxy for liquidity. The lower the value of ξ , the more costly the sale, the lower the liquidity. In panel 4b we plot ξ against the frequency of the adverse shock μ and it is clear that ξ falls exponentially in μ . Indeed when the shock is rather infrequent ($\mu \leq 3$), i.e. if a seller can go on for $12/\mu = 4$ months or more without being distressed, then the profit loss is less than 10%. However the loss grows rapidly as μ grows beyond 3.

The cost of liquidation is also related to the degree of product standardization; specifically the more standardized products a seller holds the less costly the liquidation sale. To establish this

relationship, start with the coefficient of variation

$$s \equiv \frac{StDev[v]}{x}$$

to measure how standard (or homogenous) the products are: the lower the value of s the more standardized the products (e.g. vanilla interest rate derivatives traded over clearing houses) and the higher the value of s the opposite (e.g. exotic swaps traded bilaterally). The simulation in 4c plots the profit loss ξ against s and it is clear that attempting a liquidation sale when holding standard (vanilla) products is significantly less costly than doing so when holding non-standard (exotic) products.¹⁵

These insights are in line with the empirical literature on forced asset sales, which provides several examples and anecdotes where transaction prices deviate from fundamental values due to forced sales. Pulvino [29] studies commercial aircraft transactions initiated by constrained versus unconstrained airlines and finds that commercial airplanes sold by distressed airlines bring 10 to 20 percent lower prices when compared to planes sold by regular airlines. Campbell et al. [7] consider forced selling in the real estate market due to events such as foreclosures and find large foreclosure discounts, about 27 percent on average. Coval and Stafford [11] examine institutional price pressure in equity markets and find that widespread selling by financially distressed mutual funds leads to transaction prices that are significantly below the fundamental value.

5 Extensions

5.1 Distressed Buyers

In this section we extend the benchmark model by considering the fact that buyers, too, may become distressed if they are unable to transact. Similar to sellers, buyers may have pressing reasons for acquiring a particular product due to, for instance, speculation, diversification or hedging purposes including spreading and shifting risk associated with a portfolio position. Consequently, they may become more eager to purchase if they cannot trade for too long.

Unlike the benchmark there are five groups of players now: owners (m_o), regular sellers ($m_{s,r}$), distressed sellers ($m_{s,d}$), regular buyers ($m_{b,r}$) and distressed buyers ($m_{b,d}$). The flowchart illustrates

¹⁵The random variable v is uniformly distributed over $[0, 1]$ hence $StDev[v]$ is fixed and equals to 0.5. We let $x \in [0, 5]$ in the simulation hence s ranges from 0.1 to 10, where 0.1 refers to a rather standard market with little product heterogeneity and 10 refers to a market with a high degree of product heterogeneity.

how players move across these groups.

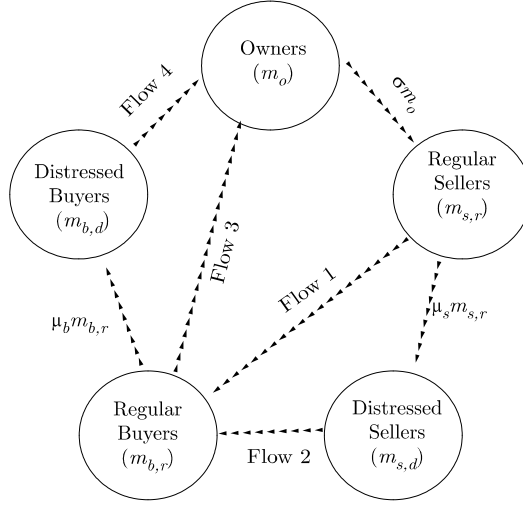


Figure 5 - Flowchart

A regular seller becomes distressed at rate μ_s . Given that there are $m_{s,r}$ such sellers in the market, the outflow is equal to $\mu_s m_{s,r}$. Similarly, a regular buyer becomes distressed at rate μ_b , hence the outflow is equal to $\mu_b m_{b,r}$. Owners become sellers at rate σ ; thus the outflow is σm_o . Remaining flows are given by

$$\text{Flow 1: } m_{s,r} [\alpha m_{b,r} \Phi(v_{r,r}) + \alpha m_{b,d} \Phi(v_{d,r})] \quad \text{Flow 2: } m_{s,d} [\alpha m_{b,r} \Phi(v_{r,d}) + \alpha m_{b,d} \Phi(v_{d,d})]$$

$$\text{Flow 3: } m_{b,r} [\alpha m_{s,r} \Phi(v_{r,r}) + \alpha m_{s,d} \Phi(v_{r,d})] \quad \text{Flow 4: } m_{b,d} [\alpha m_{s,r} \Phi(v_{d,r}) + \alpha m_{s,d} \Phi(v_{d,d})]$$

Flow 1 represents the number of regular sellers who trade and become buyers. A regular seller meets a regular buyer at rate $\alpha m_{b,r}$, and the buyer accepts to purchase with probability $\Phi(v_{r,r})$. Similarly the seller meets a distressed buyer at rate $\alpha m_{b,d}$, and the buyer accepts to purchase with probability $\Phi(v_{d,r})$. Since there are $m_{s,r}$ regular sellers in the market, the total number of such sellers who manage to sell and become buyers is given by $m_{s,r} [\alpha m_{b,r} \Phi(v_{r,r}) + \alpha m_{b,d} \Phi(v_{d,r})]$. Flows 2, 3 and 4 can be interpreted similarly. In the steady state the inflow into a pool must be equal to the outflow from it; so we have

$$\sigma m_o = \mu_s m_{s,r} + \alpha m_{b,r} m_{s,r} \Phi(v_{r,r}) + \alpha m_{b,d} m_{s,r} \Phi(v_{d,r}) \quad (15)$$

$$\mu_s m_{s,r} = \alpha m_{b,r} m_{s,d} \Phi(v_{r,d}) + \alpha m_{b,d} m_{s,d} \Phi(v_{d,d}) \quad (16)$$

$$\mu_b m_{b,r} = \alpha m_{b,d} m_{s,r} \Phi(v_{d,r}) + \alpha m_{b,d} m_{s,d} \Phi(v_{d,d}) \quad (17)$$

The first line focuses on the pool of regular sellers. The inflow σm_o consists of owners who are hit by the liquidity shock. The outflow has two components. The first one is Flow 1, which is discussed above. The second one is $\mu_s m_{s,r}$ which is the number of regular sellers who become distressed. The second line deals with the pool of distressed sellers. The inflow consists of regular sellers who become distressed ($\mu_s m_{s,r}$), whereas the outflow consists of distressed sellers who trade and become buyers (Flow 2). Finally the third line deals with the pool of distressed buyers. The inflow consists of regular buyers who become distressed ($\mu_s m_{s,r}$), and the outflow consists of distressed buyers who trade and become owners (Flow 4).¹⁶ The asset is in fixed supply, so we have

$$m_o + m_{s,r} + m_{s,d} = a. \quad (18)$$

The total measure of agents η is also fixed and exceeds a . It follows that the steady state measure of buyers, too, is fixed and equals to

$$m_{b,r} + m_{b,d} = \eta - a > 0. \quad (19)$$

Next we turn to the value functions. Unlike the benchmark, there are now two value functions for buyers: one for distressed buyers, denoted by Ω_d , and the other for regular buyers, denoted by Ω_r . We have

$$\begin{aligned} \bar{\delta}\Omega_d &= \alpha m_{s,r} I_{d,r} + \alpha m_{s,d} I_{d,d} \quad \text{and} \\ \delta\Omega_r &= \alpha m_{s,r} I_{r,r} + \alpha m_{s,d} I_{r,d} + \mu_b (\Omega_d - \Omega_s) \end{aligned}$$

where

$$I_{i,j} = \int_0^1 \max \{ \Gamma(v) - \Omega_i - p_j, 0 \} dF(v), \quad i, j = r, d.$$

The expression $I_{i,j}$ is the conditional expected utility of a type i buyer who meets a type j seller. As long as the net surplus of becoming an owner, given by $\Gamma(v) - p_j$, exceeds the opportunity cost of remaining as a buyer, given by Ω_i , the buyer purchases; otherwise he walks away. Note that, unlike the benchmark model, the value function of a regular buyer now has a component, given by $\mu_b (\Omega_d - \Omega_s)$, that deals with the possibility of the buyer becoming distressed. For a given pair Ω_i and p_j there is an associated threshold (or ‘reservation value’) $v_{i,j}$ leaving the buyer indifferent between buying and searching i.e.

$$\Gamma(v_{i,j}) = \Omega_i + p_j \Leftrightarrow \frac{v_{i,j} + x + \sigma \Pi_r}{\sigma + \delta} = \Omega_i + p_j, \quad \text{where } i, j = r, d. \quad (20)$$

¹⁶One can consider the inflows and outflows from the pools of owners and regular buyers as well; however, it is easy to verify that those equations are already implied by the system (15), (16) and (17).

Note that there are four different thresholds valuations: $v_{r,r}$, $v_{r,d}$, $v_{d,r}$ and $v_{d,d}$ (recall that in the benchmark there were only two, v_r and v_d). So, the probability that a meeting between a type i buyer and type j seller results in trade is equal to $\Phi(v_{i,j})$. Going through the algebra steps in Section 3.2 one can show that value functions Ω_d and Ω_r can be re-written as follows

$$\begin{aligned}\Omega_d &= \alpha m_{s,r} \int_{v_{d,r}}^1 \frac{\Phi(v)}{\bar{\delta}(\sigma + \delta)} dv + \alpha m_{s,d} \int_{v_{d,d}}^1 \frac{\Phi(v)}{\bar{\delta}(\sigma + \delta)} dv \quad \text{and} \\ \Omega_r &= \alpha m_{s,r} \int_{v_{r,r}}^1 \frac{\Phi(v)}{(\sigma + \delta)(\delta + \mu_b)} dv + \alpha m_{s,d} \int_{v_{r,d}}^1 \frac{\Phi(v)}{(\sigma + \delta)(\delta + \mu_b)} dv + \frac{\mu_b}{\delta + \mu_b} \Omega_d.\end{aligned}$$

Now turn to sellers. The value functions for distressed and regular sellers are given by

$$\begin{aligned}\bar{\delta}\Pi_d &= [\alpha m_{b,r}\Phi(v_{r,d}) + \alpha m_{b,d}\Phi(v_{d,d})] (p_d + \Omega_r - \Pi_d) \quad \text{and} \\ \delta\Pi_r &= [\alpha m_{b,r}\Phi(v_{r,r}) + \alpha m_{b,d}\Phi(v_{d,r})] (p_r + \Omega_r - \Pi_r) + \mu_s (\Pi_d - \Pi_r)\end{aligned}$$

Consider the first line. A distressed seller encounters a regular buyer at rate $\alpha m_{b,r}$ and a distressed buyer at rate $\alpha m_{b,d}$. In the first scenario the buyer purchases with probability $\Phi(v_{r,d})$ and in the second scenario with $\Phi(v_{d,d})$. If trade occurs the seller obtains price p_d plus Ω_r (he becomes a regular buyer now) minus Π_d (he is no longer a seller). The second line is the same, except for the part $\mu_s (\Pi_d - \Pi_r)$, which addresses the possibility of the regular seller becoming distressed in the future. The first order condition for a type $j = r, d$ seller is given by

$$p_j + \Omega_r - \Pi_j = \frac{m_{b,r}\Phi(v_{r,j}) + m_{b,d}\Phi(v_{d,j})}{(\sigma + \delta) [m_{b,r}f(v_{r,j}) + m_{b,d}f(v_{d,j})]}. \quad (21)$$

Substituting the FOC into the value functions yields

$$\begin{aligned}\Pi_d &= \frac{\alpha [m_{b,r}\Phi(v_{r,d}) + m_{b,d}\Phi(v_{d,d})]^2}{\bar{\delta}(\sigma + \delta) [m_{b,r}f(v_{r,d}) + m_{b,d}f(v_{d,d})]} \quad \text{and} \\ \Pi_r &= \frac{\alpha [m_{b,r}\Phi(v_{r,r}) + m_{b,d}\Phi(v_{d,r})]^2}{(\delta + \mu_s)(\sigma + \delta) [m_{b,r}f(v_{r,r}) + m_{b,d}f(v_{d,r})]} + \frac{\mu_s}{\delta + \mu_s} \Pi_d.\end{aligned}$$

Observe that Ω_r , Ω_d , Π_r and Π_d are all functions of thresholds $v_{r,r}$, $v_{r,d}$, $v_{d,r}$ and $v_{d,d}$. One can pin down $v_{i,j}$ via the indifference equations in (20) and then calculate the prices p_r and p_d via (21) and the measures of players m_o , $m_{s,r}$, $m_{s,d}$, $m_{b,r}$, $m_{b,d}$ via (15)-(19). However, even though the methodology is straightforward the algebra does not lend itself for an analytical solution, so we proceed via numerical simulations. The parameters of interest are μ_s and μ_b , which are the arrival rates of the adverse shocks that push sellers and buyers into the state of distress. It is sensible to think that μ_s and μ_b rise and fall together, so we have $\mu_b = c\mu_s$ for some positive c . Below we

simulate the equilibrium objects (prices and probabilities of trade) for $c = 1$ i.e. $\mu_b = \mu_s = \mu$ (simulations with different values of c produce similar results). Remaining parameters are as in Table 1.

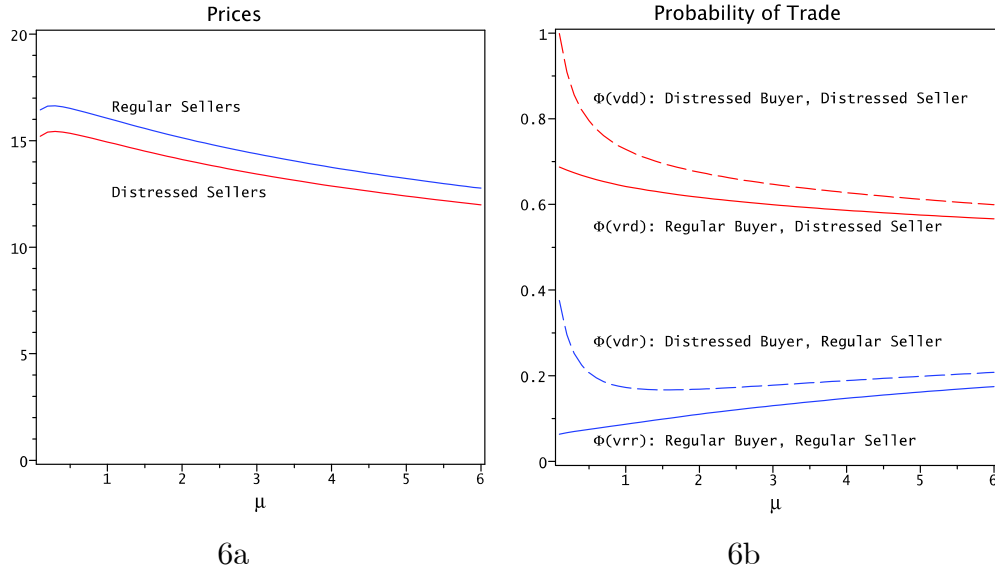


Figure 6a and 6b depict, respectively, the prices and the probabilities of trade against the arrival rate of the adverse shock μ . Distressed sellers are impatient to transact, so they post lower prices in order to sell quickly (6a). Distressed buyers are also impatient, so they become less selective and start paying little attention to the idiosyncratic component of the asset, which means that they are more likely to buy when compared to a regular buyer under the same circumstances. Indeed, note that $\Phi(v_{d,r}) > \Phi(v_{r,r})$ and $\Phi(v_{d,d}) > \Phi(v_{r,d})$ in 6b. It follows that the meeting that is most likely to result in trade is the one where a distressed seller meets a distressed buyer as both parties are most eager to trade. Similarly, the meeting that is least likely to result in trade is the opposite case where a regular seller meets a regular buyer as neither party is in a hurry. The other two scenarios, where a regular seller meets a distressed buyer and a distressed seller meets a regular buyer, lie between these two extremes.¹⁷

The simulation in 6a further reveals that prices are hump-shaped in μ . To understand why note that from a seller's perspective a rise in μ has two contrasting effects. On the positive side it increases the number of distressed buyers, who are ready to pay more, which induces sellers

¹⁷Note that when $\mu \approx 0$ a distressed seller is highly likely to make a sale: if he meets a distressed buyer the probability of trade is almost 1, and if he meets a regular buyer the probability of trade is close to 40%, which is significantly higher than a regular seller's chance of making a sale under the same circumstances (less 10%). The reason is that when μ is so small there are very few distressed players in the market. A buyer who meets a distressed seller knows that he is very unlikely to encounter another seller with such a low price, so the buyer becomes significantly less selective about how good a fit the asset is.

to raise their prices. On the negative side the rising μ causes more sellers to become distressed, which in turn induces them to lower their prices. The simulations suggest that the positive effect is dominant if μ is small and the negative effect is dominant if μ is large: if $\mu \approx 0$ (if shocks are highly unlikely) then sellers increase their prices in order to take advantage of the few distressed buyers present in the market; however as μ grows large the second effect starts to kick in, bringing down the prices.

Notice that even though prices may eventually fall, the drop is not as sharp as it was in the benchmark model. For instance, if μ rises from 2 to 6 then in the benchmark model prices fall by about 75% (see 3a), whereas in here they fall by about 15% (both simulations are based on the parameter values in Table 1, so they are comparable). The reason is that in the benchmark model μ did not affect buyers whereas in here it does, which, from a seller's perspective, is a welcome outcome. Indeed the rising μ pushes more buyers into a state of distress, making them willing to transact at higher prices. This effect prevents prices from falling as sharply as they did in the benchmark.

A final observation is this. The trajectories of the probabilities in 6b have generally the opposite pattern of the trajectories of prices in 6a: if prices rise then the probabilities fall and if prices fall then the probabilities rise. The exception is distressed sellers: even though they decrease their prices, their probabilities of trade $\Phi(v_{r,d})$ and $\Phi(v_{d,d})$ still keep falling. This is the predation result discussed in the benchmark, and the simulation suggests that it is present in this version of the model as well. Notice, however, the extent of predation is significantly less pronounced in here than it was in the benchmark. Indeed, a comparison between Figure 3 and Figure 6 reveals that the percentage-wise drops in prices as well as in probabilities are larger in the benchmark than they are in here.¹⁸ This is not surprising, because, as mentioned above, in addition to its impact on sellers, μ now has an adverse impact on buyers as well, so if μ rises then everyone in the market, not just sellers, face a higher likelihood of becoming distressed. This consideration filters into sellers' and buyers' value functions and thereby prevents prices and the probabilities of trade from decreasing as sharply as they did before.

From a risk management point of view, these observations suggest that in periods of high volatility (e.g. when μ rises) firms must not rush into lowering prices. Instead they ought to assess whether and to what extent potential customers may become distressed and they should make their pricing decisions accordingly.

¹⁸Note that in 6b the initial drops in $\Phi(v_{r,d})$ and $\Phi(v_{d,d})$ are due to the rising prices. Predation can be identified only after prices start to fall.

5.2 Dynamics

We now construct a dynamic version of the benchmark model. This extension will allow us to explore the transition process through which the economy responds to an exogenous shock and how it approaches to the new steady state. Specifically we are interested in the shock's immediate effect on equilibrium objects and the time-pattern of the recovery. To start, note that the measures of agents evolve according to

$$\begin{aligned}\dot{m}_d &= -\alpha m_d \Phi_d + \mu m_r \\ \dot{m}_r &= -\alpha m_r \Phi_r - \mu m_r + \sigma m_o \\ \dot{m}_o &= \alpha m_r \Phi_r + \alpha m_d \Phi_d - \sigma m_o.\end{aligned}$$

These expressions are similar to their counterparts in the benchmark except now we have the time differentials \dot{m}_d , \dot{m}_r and \dot{m}_o . Buyers' value function is given by

$$\delta \Omega = \alpha m_r \int_{v_r}^1 \frac{\Phi(v) dv}{\sigma + \delta} + \alpha m_d \int_{v_d}^1 \frac{\Phi(v) dv}{\sigma + \delta} + \dot{\Omega},$$

where at each point in time thresholds $v_d(t)$ and $v_r(t)$ satisfy the indifference condition

$$p_j + \Omega = \frac{v_j + x + \sigma \Pi_r}{\sigma + \delta} \text{ for } j = r, d.$$

Sellers' value functions are given by

$$\begin{aligned}\bar{\delta} \Pi_d &= \alpha \Phi(v_d) (p_d + \Omega - \Pi_d) + \dot{\Pi}_d \quad \text{and} \\ \delta \Pi_r &= \alpha \Phi(v_r) (p_r + \Omega - \Pi_r) + \mu (\Pi_d - \Pi_r) + \dot{\Pi}_r.\end{aligned}$$

A type j seller solves

$$\max_{p_j \in \mathbb{R}_+} \Pi_j \quad \text{s.t.} \quad p_j + \Omega = \frac{v_j + x + \sigma \Pi_r}{\sigma + \delta}.$$

Recall that $\Omega' = 0$ i.e. an individual seller fails to realize how his pricing decision affects buyers' value function Ω . Here we further assume that $\dot{\Pi}'_r = \dot{\Pi}'_d = 0$ i.e. sellers fails to internalize the effects of their pricing decisions on time differentials $\dot{\Pi}_r$ and $\dot{\Pi}_d$. With this simplification, the FOC of a type j seller is given by

$$p_j + \Omega - \Pi_j = \frac{\Phi(v_j)}{(\sigma + \delta) f(v_j)},$$

and therefore

$$\Pi_d = \Pi_d^{ss} + \frac{\dot{\Pi}_d}{\bar{\delta}} \quad \text{and} \quad \Pi_r = \Pi_r^{ss} + \frac{\mu \dot{\Pi}_d}{(\delta + \mu) \bar{\delta}} + \frac{\dot{\Pi}_r}{\delta + \mu},$$

where Π_d^{ss} and Π_r^{ss} are the steady state value functions in the benchmark, which were given by (11) and (12). Combining the indifference conditions with the FOCs yields

$$0 = \frac{\Phi(v_r)}{f(v_r)} + \delta\Pi_r - v_r - x \quad \text{and} \quad (22)$$

$$0 = \frac{\Phi(v_d)}{f(v_d)} + (\sigma + \delta)\Pi_d - \sigma\Pi_r - v_d - x. \quad (23)$$

These expressions are identical to their counterparts in the benchmark (compare with 13 and 14), except of course Π_d and Π_r have dynamic components $\dot{\Pi}_d$ and $\dot{\Pi}_r$. In what follows we express these components in terms of \dot{v}_d and \dot{v}_r . To do so, first substitute the FOC into the indifference condition to obtain

$$\Pi_j + \frac{\Phi(v_j)}{(\sigma + \delta)f(v_j)} = \frac{v_j + x + \sigma\Pi_r}{\sigma + \delta}.$$

Totally differentiating this equation wrt to time yields (recall that $\dot{\Pi}'_r = \dot{\Pi}'_d = 0$)

$$(\sigma + \delta) \left\{ \frac{\partial\Pi_j}{\partial v_r} \dot{v}_r + \frac{\partial\Pi_j}{\partial v_d} \dot{v}_d + \dot{\Pi}_j \right\} - \left\{ 1 + \frac{f'(v_j)\Phi(v_j)}{f^2(v_j)} \right\} \dot{v}_j = \dot{v}_j + \sigma \left\{ \frac{\partial\Pi_r}{\partial v_r} \dot{v}_r + \frac{\partial\Pi_r}{\partial v_d} \dot{v}_d + \dot{\Pi}_r \right\},$$

where $\partial\Pi_j/\partial v_r$ and $\partial\Pi_j/\partial v_d$ are given by (25) and (26) in the appendix. These relationships pin down $\dot{\Pi}_r$ and $\dot{\Pi}_d$, and therefore Π_r and Π_d , as functions of \dot{v}_r and \dot{v}_d . Substituting the resulting expressions into (22) and (23) yields the system of ODE that pin down the equilibrium values of v_r and v_d at any given time t . Specifically,

$$\begin{bmatrix} \lambda_{1,1} & \lambda_{1,2} \\ \lambda_{2,1} & \lambda_{2,2} \end{bmatrix} \begin{bmatrix} \dot{v}_r \\ \dot{v}_d \end{bmatrix} = \begin{bmatrix} \Phi(v_r)/f(v_r) + \delta\Pi_r^{ss} - v_r - x \\ \Phi(v_d)/f(v_d) + (\sigma + \delta)\Pi_d^{ss} - \sigma\Pi_r^{ss} - v_d - x \end{bmatrix},$$

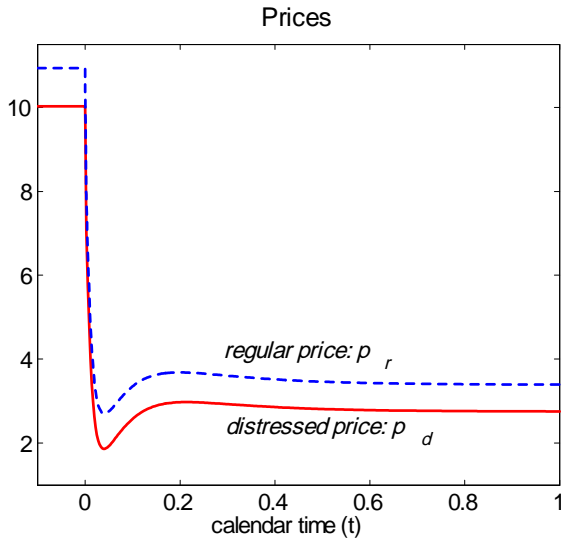
where

$$\begin{aligned} \lambda_{1,1} &= \left\{ \frac{1}{\delta + \mu} + \frac{\mu\sigma}{(\delta + \mu)(\sigma + \delta)\delta} \right\} \left\{ 2 + \frac{f'(v_r)\Phi(v_r)}{f^2(v_r)} \right\} - \frac{\delta}{\delta + \mu} \frac{\partial\Pi_r}{\partial v_r} \\ \lambda_{1,2} &= \frac{\delta\mu}{\delta(\delta + \mu)(\sigma + \delta)} \left\{ 2 + \frac{f'(v_d)\Phi(v_d)}{f^2(v_d)} - (\sigma + \delta) \frac{\partial\Pi_d}{\partial v_d} - \frac{\delta(\sigma + \delta)}{\mu} \frac{\partial\Pi_r}{\partial v_d} \right\} \\ \lambda_{2,1} &= \frac{\sigma(\sigma + \delta + \mu)}{\delta(\delta + \mu)(\sigma + \delta)} - \frac{\sigma}{(\delta + \mu)\delta} \left\{ 2 + \frac{f'(v_r)\Phi(v_r)}{f^2(v_r)} \right\} + \frac{\sigma}{\delta + \mu} \frac{\partial\Pi_r}{\partial v_r} \\ \lambda_{2,2} &= \frac{\delta(\sigma + \delta + \mu)}{\delta(\delta + \mu)} \left\{ \frac{2}{(\sigma + \delta)} + \frac{f'(v_d)\Phi(v_d)}{(\sigma + \delta)f^2(v_d)} - \frac{\partial\Pi_d}{\partial v_d} \right\} + \frac{\sigma}{\delta + \mu} \frac{\partial\Pi_r}{\partial v_d}. \end{aligned}$$

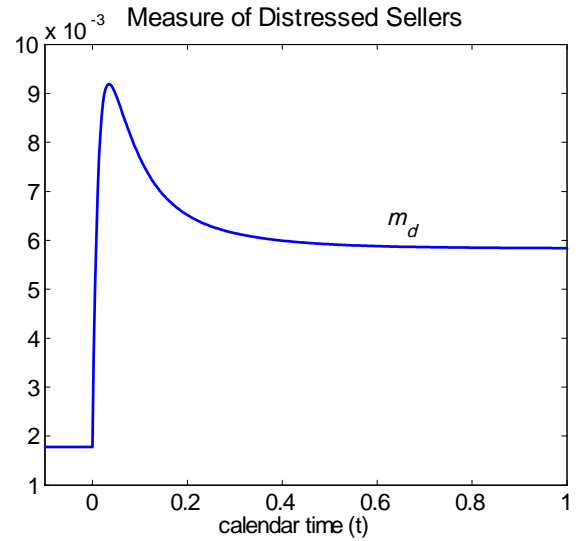
Observe that substituting $\dot{v}_r = \dot{v}_d = 0$ yields the steady state equilibrium conditions in the benchmark. However, analytically characterizing the solution of this system is a non-trivial task. To proceed we simulate the system using the parameters in Table 1 and the initial conditions $v_r(0)$ and $v_d(0)$. We assume that at time $t = 0$ the economy is at steady state; hence $v_r(0)$ and $v_d(0)$

correspond to the equilibrium values of v_r^* and v_d^* in the benchmark. Given $v_r(0)$ and $v_d(0)$ one can, then, pin down the starting values of measures of agents and the steady state value functions Π_d^{ss} and Π_r^{ss} .

In what follows we simulate how the economy responds to a sudden and permanent rise in μ . We set the initial value of μ to 1 indicating that regular seller, on average, lasts $12/\mu = 12$ months without becoming distressed. At date $t = 0$ the value of μ suddenly jumps to 6. Figures 7a and 7b depict trajectories of prices as well as the measure of distressed sellers in response to this shock.



7a



7b

The sudden rise in the arrival rate of the adverse shock triggers an immediate and sharp drop in prices, which is then followed by an extended reversal phase. Note that prices initially over-react to the shock and fall below their new steady state level, only to recover afterwards. The recovery phase occurs within the first month whereas the convergence to the new steady state appears to take about six months. Duffie [13], in his presidential address to the AFA, explores various mechanisms causing similar overreactions in price dynamics. These include the relatively small subset of risk-bearing capacity that is immediately available to absorb a shock on short notice, institutional impediments to capital movement and investors' occasional lack of attention to trade. In our case the underlying reason behind the overreaction is the temporary glut of distressed sellers in the market. Indeed, the simulation in panel 7b reveals that within a few weeks after the sudden rise in μ there are four times as many distressed sellers in the market as they were before. Consequently, prices overreact to this glut and fall below their steady state level. As the glut resolves prices recover and converge to the new steady state.

Notice that the pattern of the prices in response to the shock and the prolonged amount of recovery time lends support to the search model in the context of OTC markets. There is significant empirical evidence that supply and demand shocks in asset markets, in addition to triggering an instant price reaction, lead to corrections that take a relatively prolonged amount of time. For instance, after major downgrades or defaults in OTC corporate bond markets one typically observes large price drops which are followed by delayed recovery phases e.g. see [19] and [10]. A similar scenario is reported in [25], who found that after large capital redemptions in 2005, convertible bond prices dropped immediately and rebounded only after several months.

In these examples the time pattern of the prices after the external shock reveals that the friction at work is not a transaction cost for trade. Indeed, if this were the case then investors would instantaneously modify their portfolios and the new price would be established very soon after the shock and it would remain there until the arrival of the next shock. In these examples, however, the price initially over-reacts to the shock and the correction takes a prolonged amount of time. The speed of adjustment, at least in part, is a reflection of search frictions in the market—i.e. the fact that it takes considerable time and effort (especially during times of volatility e.g. after an external shock) to find new investors and to negotiate the new terms with them.

What is remarkable, our simulations show that, just as observed empirically, prices initially overreact to the shock and it takes a significant amount of time until they recover and approach to their new steady state levels. This is indeed in line with the empirical papers referenced above, and therefore it gives further credit to the search-and-matching model to be used in the context of OTC markets.

5.3 Risk Management

The arrival rate of the adverse shock is exogenous. As a result there is not much a regular seller can do for not being hit by the shock except for trying to sell as quick as possible. The exogeneity of the adverse shock, admittedly, hinders our model’s ability to talk about strategies on how to prevent the shock or perhaps how to delay it; but, nevertheless, our results still offer some valuable insights into risk management.

First, the under shooting of the price is indeed significant. The simulation in panel 7a suggests that both the regular price and the distressed price initially fall about 25% below the new steady state level and the recovery phase takes about a month. These observations indicate that risk management should take into account the time frame in which assets are marked-to-market, especially if the position is secured via collateralized financing. Risk management should be mindful of the fact that during times of financial distress (e.g. when μ suddenly goes up) it takes significantly longer to sell (because of predation) and that the market price undershoots significantly before it recovers. These details ought to be built into the contract when obtaining the loan to secure the

position. Otherwise the loan provider, e.g. the broker, might mark-to-market too aggressively in an effort to trigger a margin call and liquidate the collateral.¹⁹

Furthermore, in the light of the spill-over result discussed earlier, one should take into account the inter-dependant nature of financial markets and the indirect effects and repercussions of a potential shock hitting trading partners or related investors. Said differently, a prudent risk management strategy should depend on financial standing of related traders and should have scenarios drawn against the possibility that they may fall into distress. An example for such a measure is JP Morgan's dealer exit stress test, which assesses the risk that a rival is forced to withdraw from the market.²⁰

6 Conclusion

This paper contributes to a recent literature spurred by Duffie et al. [14] studying the OTC markets via search and matching and complements this literature by assuming that (i) buyers' preferences are heterogenous and their willingness to pay is private information and that (ii) sellers are heterogeneous in terms of their urgency to sell. A search equilibrium exists and it is unique. In equilibrium distressed sellers pursue liquidation sales—that is, they significantly undercut their competitors in an effort to quickly trade and exit from their positions. Liquidation sales are associated with considerable profit losses, but more importantly they open the door for predation. Indeed we demonstrate that during periods where an increasing number of sellers become distressed, buyers deliberately hold off purchasing from such sellers, which in turn exerts more pressure on them and forces them for further price cuts—an outcome which we call predatory buying.

¹⁹Brunnermeier and Pedersen [6] provide an anecdote for such an outcome involving Granite Partners (Askin Capital Management), who held very illiquid fixed income securities: "[Granite's] main brokers—Merrill Lynch, DLJ, and others—gave the fund less than 24 hours to meet a margin call. Merrill Lynch and DLJ then allegedly sold off collateral assets at below market prices at an insider-only auction in which bids were solicited from a restricted number of other brokers excluding retail institutional investors."

²⁰David Remstein, JP Morgan's Global Head of Investment Performance, in his Investment Analytics & Consulting newsletter (Second Quarter 2012) states that "Another popular approach to building an extreme event is to consider the interrelationship of financial markets and the effect a liquidity shock may have on it. [...] An example of this technique might be an event causing a sharp increase in market risk and dealers exiting positions to avoid breaching trading limits. This contributes to further volatility and triggers action to be taken by other market participants [...]. The behaviour spreads to other markets through the deterioration in liquidity and the inability to implement hedging strategies, thus causing further increases in volatility."

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Appendix

Proof of Proposition 3. The proof involves three steps.

Step 0. Preliminaries. We have

$$\Pi_d = \frac{\alpha\Phi^2(v_d)}{\delta(\sigma+\delta)f(v_d)} \quad \text{and} \quad \Pi_r = \frac{\alpha}{(\sigma+\delta)(\mu+\delta)} \left[\frac{\Phi^2(v_r)}{f(v_r)} + \frac{\mu\Phi^2(v_d)}{\delta f(v_d)} \right]. \quad (24)$$

The following partial derivatives will be useful

$$\frac{\partial \Pi_d}{\partial v_d} = -\frac{\alpha\Phi(v_d)}{\delta(\sigma+\delta)} \times \frac{2f^2(v_d)+f'(v_d)\Phi(v_d)}{f^2(v_d)} \quad \text{and} \quad \frac{\partial \Pi_r}{\partial v_d} = \frac{\mu}{\mu+\delta} \times \frac{\partial \Pi_d}{\partial v_d}, \quad (25)$$

$$\frac{\partial \Pi_r}{\partial v_r} = -\frac{\alpha\Phi(v_r)}{(\sigma+\delta)(\mu+\delta)} \times \frac{2f^2(v_r)+f'(v_r)\Phi(v_r)}{f^2(v_r)} \quad \text{and} \quad \frac{\partial \Pi_d}{\partial v_r} = 0. \quad (26)$$

All partial derivatives (except for $\frac{\partial \Pi_d}{\partial v_r}$) are negative because Assumption 1 (log concavity).

Step 1. Existence- We will show that the locus of $\Delta_r = 0$ and that of $\Delta_d = 0$ intersect once in $v_r - v_d$ space, where Δ_r and Δ_d are given by (13) and (14). To start, let

$$\kappa_r(v_r) \doteq \{v_d \in [0, 1] \mid \Delta_r(v_r, v_d) = 0\}$$

be the locus of $\Delta_r(v_r, v_d)$. Similarly let $\kappa_d(v_r)$ be the locus of Δ_d . We will establish that κ_r is downward sloping whereas κ_d is upward sloping wrt v_r . Differentiating (13) and (14) wrt v_r and v_d we have:

$$\begin{aligned} \frac{\partial \Delta_r}{\partial v_r} &= -\frac{f^2(v_r)+f'(v_r)\Phi(v_r)}{f^2(v_r)} + \delta \frac{\partial \Pi_r}{\partial v_r} - 1 < 0 & \frac{\partial \Delta_r}{\partial v_d} &= \delta \frac{\partial \Pi_r}{\partial v_d} < 0 \\ \frac{\partial \Delta_d}{\partial v_d} &= -\frac{f^2(v_d)+f'(v_d)\Phi(v_d)}{f^2(v_d)} + (\sigma + \delta) \frac{\partial \Pi_d}{\partial v_d} - \sigma \frac{\partial \Pi_r}{\partial v_d} - 1 < 0 & \frac{\partial \Delta_d}{\partial v_r} &= -\sigma \frac{\partial \Pi_r}{\partial v_r} > 0 \end{aligned} \quad (27)$$

Focus on $\frac{\partial \Delta_r}{\partial v_r}$. The first term and $\frac{\partial \Pi_r}{\partial v_r}$ are both negative because of log concavity; hence $\frac{\partial \Delta_r}{\partial v_r} < 0$. Similarly $\frac{\partial \Delta_r}{\partial v_d} < 0$ since $\frac{\partial \Pi_r}{\partial v_d}$ is negative. Therefore $\Delta_r(v_r, v_d) = 0$ defines $v_d = \kappa_r(v_r)$ as an implicit function of v_r (Implicit Function Theorem) with

$$\frac{d\kappa_r}{dv_r} = -\frac{\partial \Delta_r / \partial v_r}{\partial \Delta_r / \partial v_d} < 0,$$

i.e. the locus of $\Delta_r = 0$ is downward sloping wrt v_r . Similarly one can verify that $\frac{\partial \Delta_d}{\partial v_r} < 0$ and $\frac{\partial \Delta_d}{\partial v_d} > 0$; therefore

$$\frac{d\kappa_d}{dv_r} = -\frac{\partial \Delta_d / \partial v_r}{\partial \Delta_d / \partial v_d} > 0,$$

which means that the locus of $\Delta_d = 0$ is upward sloping.

Now we prove that $\kappa_r(0) > \kappa_d(0)$ and $\kappa_r(1) < \kappa_d(1)$. Start by substituting $(v_r, v_d) = (0, 0)$ into

Δ_r and Δ_d and observe that $\Delta_r(0, 0) > \Delta_d(0, 0)$ because $\bar{\delta} > \delta$. In addition note that $\frac{\partial \Delta_d}{\partial v_d} < \frac{\partial \Delta_r}{\partial v_d} < 0$ (this follows from log-concavity and that $\frac{\partial \Pi_d}{\partial v_d} < \frac{\partial \Pi_r}{\partial v_d}$). It follows that $\Delta_r(0, v_d) > \Delta_d(0, v_d)$ for all $v_d > 0$. This, in turn, implies that $\kappa_r(0) > \kappa_d(0)$. Similarly $(v_r, v_d) = (0, 0)$ into Δ_r and Δ_d and observe that $\Delta_r(0, 0) = \Delta_d(0, 0) = -(1+x)$. Since $\frac{\partial \Delta_d}{\partial v_d} < \frac{\partial \Delta_r}{\partial v_d} < 0$ we have $\Delta_r(1, v_d) < \Delta_d(1, v_d)$ for all $v_d < 1$. This inequality implies that $\kappa_r(1) < \kappa_d(1)$.

Since (i) $\frac{d\kappa_r}{dv_r} < 0$ and $\frac{d\kappa_d}{dv_r} > 0$, (ii) $\kappa_r(0) > \kappa_d(0)$ and (iii) $\kappa_r(1) < \kappa_d(1)$, the Intermediate Value Theorem guarantees existence of a unique $v_r^* \in (0, 1)$ such that $\kappa_r(v_r^*) = \kappa_d(v_r^*) = v_d^*$.

Step 2. Liquidation Sales. First we will show that $v_d^* < v_r^*$, which, in turn, implies that $\Phi(v_d^*) > \Phi(v_r^*)$. Recall that $\frac{\partial \Delta_r}{\partial v_r} < 0$ and $\frac{\partial \Delta_d}{\partial v_r} > 0$; hence the difference $\Delta_r - \Delta_d$ decreases in v_r . Now, by contradiction suppose that $v_r^* = v_d^* = v$ and notice that

$$\Delta_r(v, v) - \Delta_d(v, v) = \Pi_r(v, v) - \Pi_d(v, v) = \frac{\alpha \Phi^2(v)(\bar{\delta} - \delta)}{f(v)\bar{\delta}(\mu + \delta)} > 0.$$

The expression is positive because $\bar{\delta} > \delta$. The fact that $\Delta_r(v, v) > \Delta_d(v, v)$ implies that $v_r^* \neq v_d^*$ because in equilibrium we must have $\Delta_r(v_r^*, v_d^*) = \Delta_d(v_r^*, v_d^*)$. The inequality gets worse if $v_d^* > v_r^*$ because $\Delta_r - \Delta_d$ decreases in v_r . The equilibrium condition can be satisfied only if $v_d^* < v_r^*$.

The inequality $p_r^* > p_d^*$ is follows from the indifference conditions (7) implying

$$p_r^* - p_d^* = (v_r^* - v_d^*) / (\sigma + \delta) > 0,$$

which is positive because $v_d^* < v_r^*$. ■

Proof of Remark 4. The first part of the remark deals with the signs of $\Phi(v_r^*)$ and $\Phi(v_d^*)$ wrt x . Recall that

$$\text{sign}(dv_j^*/du) = \text{sign}(\det B_j(u)) \text{ for } j = r, d$$

where $\det(B_r(u))$ and $\det(B_d(u))$ are given by (35). Below we show that $\det B_r(x)$ and $\det B_d(x)$ are both negative. Note that

$$\frac{\partial \Delta_d}{\partial x} = \frac{\partial \Delta_r}{\partial x} = -\frac{1}{\sigma + \delta}.$$

It follows that

$$\begin{aligned} \det B_r(x) &= \frac{1}{\sigma + \delta} \left[\frac{\partial \Delta_d}{\partial v_d} - \frac{\partial \Delta_r}{\partial v_d} \right] < 0 \\ \det B_d(x) &= \frac{1}{\sigma + \delta} \left[\frac{\partial \Delta_r}{\partial v_r} - \frac{\partial \Delta_d}{\partial v_r} \right] < 0 \end{aligned}$$

In the first line, the expression in square brackets is negative because $\frac{\partial \Delta_d}{\partial v_d} < \frac{\partial \Delta_r}{\partial v_d} < 0$; see the proof of Proposition 3. The expression in the second line is negative because $\frac{\partial \Delta_r}{\partial v_r} < 0$ and $\frac{\partial \Delta_d}{\partial v_r} > 0$. The signs of the determinants imply that both v_r^* and v_d^* fall and therefore $\Phi(v_r^*)$ and $\Phi(v_d^*)$ rise in x .

Characterization of Corner Solutions. Let v_r^+ be the specific value of v_r satisfying

$$\frac{\alpha\Phi^2(v_r^+)}{(\mu+\delta)f(v_r^+)} + \frac{\Phi(v_r^+)}{f(v_r^+)} - v_r^+ - \frac{1}{f(0)} \left[1 + \frac{\alpha\delta}{\bar{\delta}(\mu+\delta)} \right] = 0. \quad (28)$$

Basic algebra reveals that if $x = x^+$, where

$$x^+ \doteq \frac{\delta(1+\alpha/\bar{\delta})}{(\sigma+\delta)f(0)} - \frac{\sigma}{\sigma+\delta} \left[v_r^+ - \frac{\Phi(v_r^+)}{f(v_r^+)} \right], \quad (29)$$

then $\Delta_r(v_r^+, 0) = \Delta_d(v_r^+, 0) = 0$; hence the the pair $\mathbf{v}^* = (v_r^+, 0)$ correspond to an equilibrium.

Recall that v_r^* and v_d^* both fall in x . So, if $x > x^+$ then v_d^* falls below 0, implying that the probability of trade $\Phi(v_d^*)$ exceeds 1, which, of course, is impossible. In this parameter region, distressed sellers' FOC no longer holds with equality. The concavity of sellers' objective function implies that distressed sellers pick the price p_d satisfying $v_d = 0$ (not the FOC) and the indifference condition (7). More specifically, p_d satisfies

$$\frac{x+\sigma\Pi_r}{\sigma+\delta} = p_d + \Omega, \quad (30)$$

where the equation is obtained by substituting $v_d = 0$ into (7). Substitute (30) and $v_d = 0$ into distressed sellers value function Π_d to obtain

$$\Pi_d = \frac{\alpha(x+\sigma\Pi_r)}{(\alpha+\bar{\delta})(\sigma+\delta)}.$$

Relaxed sellers' problem is still the same. We conjecture that (to be verified below) their FOC

$$p_r + \Omega - \Pi_r = \frac{\Phi(v_r)}{(\sigma+\delta)f(v_r)} \quad (31)$$

holds with equality. Substitute (31) along with Π_d from above and $v_d = 0$ into Π_r to obtain

$$\Pi_r = c_3 \frac{\alpha\Phi^2(v_r)(\alpha+\bar{\delta})}{f(v_r)} + c_3\mu\alpha x \quad (32)$$

where

$$c_3 = [(\delta + \mu)(\alpha + \bar{\delta})(\sigma + \delta) - \alpha\mu\sigma]^{-1} \in (0, 1).$$

Relaxed sellers face the indifference condition

$$\frac{v_r+x+\sigma\Pi_r}{\sigma+\delta} = p_r + \Omega. \quad (33)$$

Combine the indifference condition with their FOC (31) above to obtain

$$\Delta_r(v_r) = \frac{\Phi(v_r)}{f(v_r)} + \delta\Pi_r - v_r - x = 0.$$

This function looks similar to the equilibrium condition in (13), but unlike the former, this one does not depend on v_d anymore (now Π_r is a function of v_r only). Substitute Π_r from (32) into Δ_r to obtain

$$\Delta_r(v_r) = \frac{\Phi(v_r)}{f(v_r)} + c_3 \frac{\alpha\delta\Phi^2(v_r)(\alpha+\bar{\delta})}{f(v_r)} - v_r - c_3x(\sigma+\delta) [\delta(\alpha+\bar{\delta}) + \mu\bar{\delta}] = 0.$$

It is easy to verify that Δ_r falls in v_r (assuming log-concavity). In addition $\Delta_r(1) < 0$. So if $\Delta_r(0) > 0$ then there exists an interior $v_r^* \in (0, 1)$ satisfying $\Delta_r(v_r) = 0$. Note that

$$\Delta_r(0) > 0 \Leftrightarrow x < x^{++},$$

where

$$x^{++} \doteq \frac{(\alpha+\bar{\delta})\{(\delta+\mu)(\sigma+\delta)+\alpha\delta\}-\alpha\mu\sigma}{f(0)(\sigma+\delta)\{\delta(\alpha+\bar{\delta})+\mu\bar{\delta}\}}. \quad (34)$$

Therefore if $x < x^{++}$ relaxed sellers' FOC holds with equality and the optimal v_r^* is interior; hence

$\Phi(v_r^*) < 1$. If, however, $x \geq x^{++}$ then, relaxed sellers, too, set p_r satisfying $v_r = 0$ and their indifference condition (33). In this parameter region both $\Phi(v_d^*)$ and $\Phi(v_r^*)$ are equal to 1. The value functions and other equilibrium objects can be obtained using the steps above. ■

Proof of Proposition 5. Recall that v_r^* and v_d^* simultaneously satisfy

$$\Delta_r(v_r^*, v_d^*) = 0 \text{ and } \Delta_d(v_r^*, v_d^*) = 0.$$

Omit the superscript * when understood and note that (General Implicit Function Theorem)

$$\frac{dv_j}{du} = \frac{\det B_j(u)}{\det A}, \text{ for } u = \mu, x, \bar{\delta}, \sigma \text{ and } j = r, d,$$

where

$$B_r(u) = \begin{bmatrix} -\frac{\partial\Delta_r}{\partial u} & \frac{\partial\Delta_r}{\partial v_d} \\ -\frac{\partial\Delta_d}{\partial u} & \frac{\partial\Delta_d}{\partial v_d} \end{bmatrix}, \quad B_d(u) = \begin{bmatrix} \frac{\partial\Delta_r}{\partial v_r} & -\frac{\partial\Delta_r}{\partial u} \\ \frac{\partial\Delta_d}{\partial v_r} & -\frac{\partial\Delta_d}{\partial u} \end{bmatrix}, \quad A = \begin{bmatrix} \frac{\partial\Delta_r}{\partial v_r} & \frac{\partial\Delta_r}{\partial v_d} \\ \frac{\partial\Delta_d}{\partial v_r} & \frac{\partial\Delta_d}{\partial v_d} \end{bmatrix}.$$

Note that

$$\det A = \frac{\partial\Delta_r}{\partial v_r} \frac{\partial\Delta_d}{\partial v_d} - \frac{\partial\Delta_d}{\partial v_r} \frac{\partial\Delta_r}{\partial v_d} > 0.$$

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The signs of the partial derivatives follow from (27). It follows that

$$\text{sign}(dv_j/du) = \text{sign}(\det B_j(u)),$$

where

$$\det B_r(u) = \frac{\partial \Delta_r}{\partial v_d} \frac{\partial \Delta_d}{\partial u} - \frac{\partial \Delta_d}{\partial v_d} \frac{\partial \Delta_r}{\partial u} \quad \text{and} \quad \det B_d(u) = \frac{\partial \Delta_d}{\partial v_r} \frac{\partial \Delta_r}{\partial u} - \frac{\partial \Delta_r}{\partial v_r} \frac{\partial \Delta_d}{\partial u}. \quad (35)$$

The setup is general and it can be used to analyze the signs of the partial derivatives of v_r^* and v_d^* wrt any one of the parameters $\mu, x, \bar{\delta}, \sigma$; but this proposition is about the sign of v_d^* wrt μ , so below we focus on $\det B_d(\mu)$. To start, note that

$$\frac{\partial \Delta_d}{\partial \mu} = -\sigma \frac{\partial \Pi_r}{\partial \mu} \quad \text{and} \quad \frac{\partial \Delta_r}{\partial \mu} = \delta \frac{\partial \Pi_r}{\partial \mu},$$

where

$$\frac{\partial \Pi_r}{\partial \mu} = -\frac{\alpha}{(\sigma+\delta)(\mu+\delta)^2} \left[\frac{\Phi^2(v_r^*)}{f(v_r^*)} - \frac{\delta}{\bar{\delta}} \frac{\Phi^2(v_d^*)}{f(v_d^*)} \right]. \quad (+)$$

Note that $\frac{\partial \Pi_r}{\partial \mu}$ is negative, because the expression in the square brackets (call it T_1) is positive.²¹

Now, substitute $\frac{\partial \Delta_d}{\partial v_r}$ and $\frac{\partial \Delta_d}{\partial v_d}$, which are given in (27), into $\det B_d(\mu)$ to obtain

$$\det B_d(\mu) = -\frac{\sigma}{\sigma+\delta} \times \frac{\partial \Pi_r}{\partial \mu} \times \frac{2f^2(v_r)+f'(v_r)\Phi(v_r)}{f^2(v_r)} > 0. \quad \begin{matrix} (-) & (+) & (+) \end{matrix}$$

The last expression is positive because of log concavity. We have already established that $\partial \Pi_r / \partial \mu$ is negative; hence $\det B_d(\mu)$ is positive, which implies that $dv_d^*/d\mu$ is positive, which in turn implies that the equilibrium probability of sale $\Phi(v_d^*)$ falls in μ .

Now we will show that p_d^* , too, falls in μ . Use the FOC (10) and the expression for Π_d , given by (24), to obtain

$$p_d^* + \Omega = \frac{\Phi(v_d^*)}{(\sigma+\delta)f(v_d^*)} \left[1 + \frac{\alpha}{\bar{\delta}} \Phi(v_d^*) \right].$$

Call the expression on the right hand side T_2 and notice that

$$\frac{dp_d^*}{d\mu} = \frac{\partial T_2}{v_d^*} \frac{dv_d^*}{d\mu} - \frac{d\Omega}{d\mu}. \quad \begin{matrix} (-) & (+) & (+) \end{matrix}$$

²¹To see why combine the FOCs, given by (10), with the value functions Π_d and Π_r , given by (24), to obtain

$$p_r^* - p_d^* = \frac{\Phi(v_r^*)}{f(v_r^*)(\sigma+\delta)} - \frac{\Phi(v_d^*)}{f(v_d^*)(\sigma+\delta)} + \frac{\alpha}{(\mu+\delta)(\sigma+\delta)} \times T_1 > 0.$$

This expression is positive since we have established that in equilibrium $p_r^* > p_d^*$. Now focus on the first two terms on the right hand side. The expression $\Phi(v)/f(v)$ falls in v because of log concavity. Since $v_r^* > v_d^*$ in equilibrium, it follows that the summation of the first two terms is negative. This means that, for $p_r^* > p_d^*$ to hold T_1 must be positive. Hence $\partial \Pi_r / \partial \mu$ is negative.

It is easy to verify that $\frac{\partial T_2}{v_d^*}$ is negative because of log-concavity; $\frac{dv_d^*}{d\mu}$ is positive from above. In addition $\frac{d\Omega}{d\mu} > 0$. Hence $\frac{dp_d^*}{d\mu}$ is negative. ■