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TWISTED K -THEORY AND OBSTRUCTIONS AGAINST POSITIVE SCALAR CURVATURE METRICS

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ABSTRACT. We decompose $\theta(M)$, the twisted index obstruction to a positive scalar curvature metric for closed oriented manifolds with spin universal cover, into a pairing of a twisted K -homology with a twisted K -theory class and prove that $\theta(M)$ does not vanish if M is a closed orientable enlargeable manifold with spin universal cover.

Key Words: enlargeable manifold, index theory, Dirac operator, positive scalar curvature, twisted index obstruction, twisted K -theory

Mathematics Subject Classification (2010): 19L50, 19K56

1. INTRODUCTION

In [22] Rosenberg constructed an index obstruction $\alpha(M) \in KO_n(C_{\mathbb{R}}^*(\pi_1(M)))$ for closed spin manifolds M of dimension n , which vanishes if M admits a metric of positive scalar curvature. It takes values in the K -theory of the (maximal or reduced) real group C^* -algebra associated to the fundamental group and relies on the existence of a spin structure. Gromov, Lawson and Rosenberg conjectured that $\alpha(M)$ is the only obstruction to the existence of a psc metric if $\dim(M) \geq 5$. This was proven to be true in the simply-connected case by Stolz [27], but is false in general as was shown by Schick [24].

Stolz generalized $\alpha(M)$ to the case where M itself may not be spin, but its universal cover \widetilde{M} still is [26], [23, Theorem 1.7]. The new invariant $\theta(M) \in KO(C^*\gamma)$ takes values in the K -theory of a real C^* -algebra associated to a *twisted* version of the fundamental group accounting for the missing spin structure on M . The latter is a $\mathbb{Z}/2\mathbb{Z}$ -extension $\widehat{\pi}$ of $\pi = \pi_1(M)$, therefore we will use the notation $C^*(\widehat{\pi} \rightarrow \pi)$ instead of $C^*\gamma$.

The element $\alpha(M)$ can be expressed as the pairing of the Dirac class $[D] \in KO_n(M)$ with the KO -theory class $[\mathcal{V}] \in KO_0(C(M, C^*\pi))$ of the Mishchenko-Fomenko bundle. We show that the same is true for $\theta(M)$ if one switches to the *twisted* versions of K -homology and K -theory [7, 2, 14]. For simplicity we treat the complex analogues of the real invariants and refer to [11, Section 1.4] for a nice exposition on how the different indices are related. We identify $\theta(M)$ as the pairing of the twisted fundamental K -homology class $[D^S] \in KK(C(M, \mathbb{C}\ell(M)), \mathbb{C})$ with a twisted version of the Mishchenko-Fomenko bundle representing an element $[\mathcal{V}^S] \in KK(\mathbb{C}, C(M, \mathbb{C}\ell(M)) \otimes C^*(\widehat{\pi} \rightarrow \pi))$:

$$\theta(M) = \text{ind}(D_+^{\mathcal{V}}) = [\mathcal{V}^S] \otimes_{C(M, \mathbb{C}\ell(M))} [D^S] .$$

The Dixmier-Douady class of the twist in this case is the element $W_3(M) := \beta(w_2(M)) \in H^3(M; \mathbb{Z})$, i.e. the Bockstein of the second Stiefel-Whitney class. The superscript S denotes the twisted spinor bundle over M . We will use the language of bundle gerbes developed by Murray [19] and the C^* -algebraic versions of their modules [4, 17] to obtain a geometric description of the twisted K -group with coefficients in a C^* -algebra. Most features of index theory are preserved in the twisted case: Murray and Singer proved the analogue of the Atiyah-Singer index theorem in this context [21] and Carey and Wang proved the Thom isomorphism [6] (see also [7]).

As an application of our techniques, we show that $\theta(M)$ does not vanish for closed orientable enlargeable manifolds (Definition 4.1) with spin universal cover and thereby enhance a result of Hanke and Schick [11, 12]:

Theorem 5.7. *Let M be a closed smooth orientable even-dimensional manifold with $\dim(M) \geq 3$ and \widetilde{M} spin that is enlargeable in the sense of Definition 4.1. Then we have $\theta^{\max}(M) \neq 0$.*

Extending the suspension argument from [11] it is easy to drop the assumption about even-dimensionality. Gromov and Lawson showed that closed enlargeable manifolds M do not allow a metric of positive scalar curvature. They worked with finite covers in the definition of enlargeability in [9], but later generalized to infinite ones [10]. We also allow the covers $\bar{M} \rightarrow M$ to be *non-compact* as in [12]. In this generality, transfer arguments fail, but twisted Hilbert A -module bundles provide a way to circumvent this. This result is independent of the injectivity of the twisted Baum-Connes map.

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Notation. A compact manifold without boundary will be called closed. Throughout the paper, M will denote a smooth closed orientable manifold and A will be a unital C^* -algebra if not stated otherwise. Moreover, \underline{F} will denote a trivial bundle with fiber F if the base space is clear. Whenever we have a surjective submersion $Y \rightarrow M$, the notation $Y^{[n]}$ will denote the n -fold fiber product over M .

2. TWISTED HILBERT A -MODULE BUNDLES

In this section we will discuss a way to represent classes in twisted K -theory with coefficients in a C^* -algebra A by twisted Hilbert A -module bundles. These are straightforward generalizations of bundle gerbe modules [4, 17]. For an introduction to Hilbert A -modules we refer the reader to [15]. We start by reviewing the geometric description of 2-cocycles via bundle gerbes [19, 20].

Definition 2.1. Let M be a smooth manifold and let $Y \rightarrow M$ be a surjective submersion. A *real* line bundle $L \rightarrow Y^{[2]}$ will be called a $\mathbb{Z}/2\mathbb{Z}$ -*bundle gerbe* (or simply *bundle gerbe* for short) if there exists a multiplication over $Y^{[3]}$, i.e. an isomorphism of line bundles $\mu: \pi_{12}^*L \otimes \pi_{23}^*L \rightarrow \pi_{13}^*L$, where $\pi_{ij}: Y^{[3]} \rightarrow Y^{[2]}$ denotes the canonical projections to the fiber product of the i th and j th factor, and such that over $Y^{[4]}$ the following diagram commutes:

$$\begin{array}{ccc} (\pi_{12}^*L \otimes \pi_{23}^*L) \otimes \pi_{34}^*L & \xlongequal{\quad} & \pi_{12}^*L \otimes (\pi_{23}^*L \otimes \pi_{34}^*L) \\ \mu \otimes \text{id} \downarrow & & \downarrow \text{id} \otimes \mu \\ \pi_{13}^*L \otimes \pi_{34}^*L & \xrightarrow{\quad \mu \quad} \pi_{14}^*L \xleftarrow{\quad \mu \quad} & \pi_{12}^*L \otimes \pi_{24}^*L \end{array}$$

The bundle gerbe $\delta Q = \pi_1^*Q \otimes \pi_2^*Q^* \rightarrow Y^{[2]}$ associated to a line bundle $Q \rightarrow Y$ will be called the *trivial bundle gerbe*. Let $L \rightarrow Y^{[2]}$ be a bundle gerbe. A choice of a line bundle $Q \rightarrow Y$ together with an isomorphism of bundle gerbes $L \rightarrow \delta Q$ will be called a *trivialization* of L .

Let $L_i \rightarrow Y_i^{[2]}$ for $i \in \{1, 2\}$ be two bundle gerbes and let $\pi_i: Y_1^{[2]} \times_M Y_2^{[2]} \rightarrow Y_i^{[2]}$ denote the projection. The exterior tensor product $L_1 \boxtimes L_2 = \pi_1^*L_1 \otimes \pi_2^*L_2$ is again a bundle gerbe. Each bundle gerbe L has a class $dd(L) \in H^2(M, \mathbb{Z}/2\mathbb{Z})$ canonically associated to it as explained in [17]. $dd(L)$ is called the *Dixmier-Douady class* of L and we summarize its properties in the following theorem proven in [17]:

Theorem 2.2. *Let $L \rightarrow Y^{[2]}$ and $L_i \rightarrow Y_i^{[2]}$ for $i \in \{1, 2\}$ be bundle gerbes. The Dixmier-Douady class has the following properties:*

- a) $dd(L) = 0$ if and only if L is isomorphic over $Y^{[2]}$ to a trivial bundle gerbe.
- b) $dd(L_1 \boxtimes L_2) = dd(L_1) + dd(L_2)$. □

Definition 2.3. Let L be a bundle gerbe. A covariant derivative $\nabla^L: \Omega^0(L) \rightarrow \Omega^1(L)$ on L is called a *bundle gerbe connection* if the multiplication isomorphism $\mu: \pi_{12}^*L \otimes \pi_{23}^*L \rightarrow \pi_{13}^*L$ pulls it back to the canonical connection on the tensor product, i.e.

$$(1) \quad \mu^* \pi_{13}^* \nabla^L = \pi_{12}^* \nabla^L \otimes 1 + 1 \otimes \pi_{23}^* \nabla^L .$$

Remark 2.4. The proof for the existence of such connections given in [19] works with the obvious changes for $\mathbb{Z}/2\mathbb{Z}$ -bundle gerbes as well. Since the structure group is discrete in our case, every $\mathbb{Z}/2\mathbb{Z}$ -bundle gerbe connection is automatically flat.

The main example of bundle gerbes will arise from the following construction.

Definition 2.5. Let Γ be a Lie group (possibly discrete) and let $q: \widehat{\Gamma} \rightarrow \Gamma$ be a central $\mathbb{Z}/2\mathbb{Z}$ -extension of Γ . Let $P \rightarrow M$ be a principal Γ -bundle over a manifold M . The line bundle L_P associated to the principal $\mathbb{Z}/2\mathbb{Z}$ -bundle $\widehat{L}_P \rightarrow P^{[2]}$ with $\widehat{L}_P = \{(p_1, p_2, \widehat{g}) \in P^{[2]} \times \widehat{\Gamma} \mid p_1 q(\widehat{g}) = p_2\}$ is a bundle gerbe, which will be called the *lifting bundle gerbe* associated to the extension.

For details about this construction see [20, section 6.1], [17, section 2.2]. Since the principal $\mathbb{Z}/2\mathbb{Z}$ -bundle associated to any trivialization Q of L_P is a lift of P to a principal $\widehat{\Gamma}$ -bundle, we have

Lemma 2.6 ([19]). *The class $dd(L_P)$ represents the obstruction lifting $P \rightarrow M$ to a principal $\widehat{\Gamma}$ -bundle. The isomorphism classes of trivialization of L_P are in 1 : 1-correspondence with the possible lifts of P to a principal $\widehat{\Gamma}$ -bundle.*

Definition 2.7. Let M be a smooth manifold, A a C^* -algebra and let $Y \rightarrow M$ be a surjective submersion. Let $L \rightarrow Y^{[2]}$ be a bundle gerbe. A (right) Hilbert A -module bundle $E \rightarrow Y$ together with an A -linear action $\gamma: L \otimes \pi_2^* E \rightarrow \pi_1^* E$ is called a *twisted Hilbert A -module bundle* for L if the following associativity diagram commutes:

$$\begin{array}{ccc} (\pi_{12}^* L \otimes \pi_{23}^* L) \otimes \pi_3^* E & \xlongequal{\quad} & \pi_{12}^* L \otimes (\pi_{23}^* L \otimes \pi_3^* E) \\ \mu \otimes \text{id} \downarrow & & \downarrow \text{id} \otimes \gamma \\ \pi_{13}^* L \otimes \pi_3^* E & \xrightarrow{\quad \gamma \quad} \pi_1^* E \xleftarrow{\quad \gamma \quad} & \pi_{12}^* L \otimes \pi_2^* E \end{array}$$

E will be called *finitely generated* and *projective*, if its fibers are finitely generated and projective as Hilbert A -modules. γ will be called the *twisting* of E .

Let E, E' be two twisted Hilbert A -module bundles for the same bundle gerbe L and denote the twistings by γ and γ' . A right A -linear map $f: E \rightarrow E'$ will be called a *morphism of twisted Hilbert A -module bundles* or (*twisted morphism* for short) if the following diagram commutes:

$$\begin{array}{ccc} L \otimes \pi_2^* E & \xrightarrow{\quad \gamma \quad} & \pi_1^* E \\ \downarrow \text{id}_L \otimes \pi_2^* f & & \downarrow \pi_1^* f \\ L \otimes \pi_2^* E' & \xrightarrow{\quad \gamma' \quad} & \pi_1^* E' \end{array}$$

For $A = \mathbb{C}$ a twisted Hilbert A -module bundle is the same as a bundle gerbe module [4]. Let $L_i \rightarrow Y_i^{[2]}$ for $i \in \{1, 2\}$ be two bundle gerbes with $dd(L_1) = dd(L_2)$. A trivialization of $L_1^* \boxtimes L_2$ should be seen as a generalized morphism between L_1 and L_2 [28]. In particular, it induces a push-forward from twisted bundles for L_1 to twisted bundles for L_2 [17, Proposition 2 and 3]:

Lemma 2.8 ([17]). *Let $L_i \rightarrow Y_i^{[2]}$ for $i \in \{1, 2\}$ be two bundle gerbes with $dd(L_1) = dd(L_2)$. Let $Q \rightarrow Y_1 \times_M Y_2$ be a trivialization of $L_1^* \boxtimes L_2 \rightarrow (Y_1 \times_M Y_2)^{[2]}$. If $E \rightarrow Y_1$ is a twisted bundle for L_1 and $\rho: Y_1 \times_M Y_2 \rightarrow Y_1$ denotes the canonical projection, then $\rho^* E \otimes Q$ descends to a twisted bundle $Q(E) \rightarrow Y_2$ for L_2 . In particular if $L_2 = \underline{\mathbb{C}}$ over $M^{[2]} = M$, i.e. Q is a trivialization of L_1^* , we obtain an (untwisted) Hilbert A -module bundle $Q(E) \rightarrow M$.*

Definition 2.9. Let $L \rightarrow Y^{[2]}$ be a bundle gerbe over a surjective submersion $Y \rightarrow M$ and let A be a unital C^* -algebra. Denote by $K_{L,A}^0(M)$ the Grothendieck group of isomorphism classes of finitely generated projective twisted Hilbert A -module bundles for L with respect to \oplus .

Given a principal $PO(n)$ -bundle $P \rightarrow M$ there is a bundle of matrix algebras $\mathcal{K} \rightarrow M$ with fiber $M_n(\mathbb{C})$ associated to it via the inclusion $PO(n) \rightarrow PU(n)$ and the adjoint action of $PU(n)$. Let $L_P \rightarrow P^{[2]}$ be the corresponding lifting bundle gerbe for the extension $O(n) \rightarrow PO(n)$. We

denote a point in the fiber $(L_P)_{(p_1, p_2)}$ by $[\widehat{g}, \lambda]$, where $\widehat{g} \in O(n)$ is a lift of the element $g \in PO(n)$ with $p_1 g = p_2$, $\lambda \in \mathbb{R}$ and $(\widehat{g}, -\lambda) \sim (-\widehat{g}, \lambda)$ determines $[\widehat{g}, \lambda]$.

There is a *canonical twisted vector bundle* $S \rightarrow P$ for L_P associated to this construction as follows: Let $S = P \times \mathbb{C}^n$ (see Remark 2.17) with the twisting given by

$$(2) \quad \gamma: L_P \otimes \pi_2^* S \rightarrow \pi_1^* S \quad ; \quad \gamma([\widehat{g}, \lambda] \otimes (p_2, v)) = (p_1, \lambda \widehat{g} v)$$

Given a twisted Hilbert A -module bundle $E \rightarrow Y$ for some bundle gerbe L , note that $S^* \boxtimes E := \pi_P^* S^* \otimes \pi_Y^* E$ over $P \times_M Y$ is a twisted Hilbert A -module bundle for $L_P^* \boxtimes L$.

Theorem 2.10. *Let L be a bundle gerbe, let A be a unital C^* -algebra. Let P , L_P and \mathcal{K} be as in the above paragraph. Assume that $dd(L) = dd(L_P)$ and let Q be a trivialization of $L_P^* \boxtimes L$, then*

$$\kappa_Q: K_{L,A}^0(M) \rightarrow K_0(C(M, \mathcal{K}) \otimes A) \quad ; \quad [E] \mapsto [C(M, Q(S^* \boxtimes E))]$$

is a well-defined isomorphism.

Proof. Let V be the typical fiber of E . It is a finitely generated projective right Hilbert A -module, therefore $(\mathbb{C}^n)^* \otimes V$ is a finitely generated projective right Hilbert $M_n(\mathbb{C}) \otimes A$ -module in a canonical way. Thus, $S^* \boxtimes E$ is a bundle of finitely generated projective right Hilbert $M_n(\mathbb{C}) \otimes A$ -modules over $P \times_M Y$. When descending this bundle as in [4, Proposition 4.3], we first obtain bundles $(\mathbb{C}^n)^* \otimes E_i \rightarrow U_i$ over an open cover $U_i \subset M$. The transitions over double intersections act via the canonical action of $O(n) \subset U(n)$ on the first tensor factor and in an A -linear way on the second tensor factor. Therefore, $C(M, Q(S^* \boxtimes E))$ carries an action of $C(M, \mathcal{K} \otimes A)$ and the fiberwise $M_n(\mathbb{C}) \otimes A$ -valued inner product on $S^* \boxtimes E$ turns $C(M, Q(S^* \boxtimes E))$ into a right Hilbert $C(M, \mathcal{K} \otimes A)$ -module. Since V was finitely generated and projective, the same holds true for $C(M, Q(S^* \boxtimes E))$.

Let σ be a projection-valued section of the bundle $\mathcal{K} \otimes M_k(A) \rightarrow M$. Let $\pi: P \rightarrow M$ be the bundle projection, then $\sigma \circ \pi$ yields a section of $\pi^* \mathcal{K} \otimes M_k(A)$. This bundle has a canonical trivialization, such that $\sigma \circ \pi$ yields a continuous map $\widehat{\sigma}: P \rightarrow M_n(\mathbb{C}) \otimes M_k(A)$ with the property $\widehat{\sigma}(pg) = \widehat{g}^{-1} \sigma(p) \widehat{g}$. Let $E_\sigma = \{(p, v) \in P \times (\mathbb{C}^n \otimes A^k) \mid \widehat{\sigma}(p)v = v\}$. This is a finitely generated, projective twisted Hilbert A -module bundle for L_P . The map $[\sigma] \mapsto [E_\sigma]$ is a homomorphism $K_0(C(M, \mathcal{K}) \otimes A) \rightarrow K_{L_P, A}^0(M)$. The composition with the isomorphism $K_{L_P, A}^0(M) \rightarrow K_{L, A}^0(M)$ induced by Q^* yields the inverse of the above construction. \square

Lemma 2.11. *Consider principal $PO(n_i)$ -bundles P_i for $i \in \{1, 2\}$ and let \mathcal{K}_i be the associated bundles of complex matrix algebras. We have $dd(L_{P_1}) = dd(L_{P_2})$ if and only if $\mathcal{K}_1 \otimes \mathcal{K}_2 \cong \text{End}(\mathcal{V}) \otimes \mathbb{C}$ for a real vector bundle $\mathcal{V} \rightarrow M$. The isomorphism classes of trivializations of $L_{P_1}^* \boxtimes L_{P_2}$ are in 1 : 1-correspondence with the isomorphism classes of possible \mathcal{V} 's.*

Proof. Let $O(n_1) \otimes O(n_2)$ be the quotient of the product $O(n_1) \times O(n_2)$ by the diagonal $Z/2\mathbb{Z}$ -action. If $dd(L_{P_1}) = dd(L_{P_2})$, then $P_1 \times_M P_2$ lifts to a principal $O(n_1) \otimes O(n_2)$ -bundle P and $\mathcal{V} = P \times_\rho (\mathbb{R}^{n_1} \otimes \mathbb{R}^{n_2})$ satisfies the condition, where ρ is the standard representation. The isomorphism

class of \mathcal{V} only depends on the one of P and the choices of possible P 's are in 1 : 1-correspondence with the trivializations of $L_{P_1}^* \boxtimes L_{P_2}$.

On the other hand, if \mathcal{V} exists, let P be its principal $O(n_1 n_2)$ -bundle. $P/(\mathbb{Z}/2\mathbb{Z})$ reduces to the principal $PO(n_1) \times PO(n_2)$ -bundle $P_1 \times_M P_2$ and the pullback of $P \rightarrow P/(\mathbb{Z}/2\mathbb{Z})$ via $P_1 \times_M P_2 \rightarrow P/(\mathbb{Z}/2\mathbb{Z})$ is a lift of $P_1 \times_M P_2$ to a principal $O(n_1) \otimes O(n_2)$ -bundle, which shows that $dd(L_{P_1}) = dd(L_{P_2})$. \square

Remark 2.12. Note that $C(M, \mathcal{V} \otimes \mathbb{C}) \otimes A$ provides a Morita equivalence bimodule between $C(M, \mathcal{K}_1) \otimes A$ and $C(M, \mathcal{K}_2) \otimes A$ and $\psi_{\mathcal{V}}: K_0(C(M, \mathcal{K}_1) \otimes A) \rightarrow K_0(C(M, \mathcal{K}_2) \otimes A)$ with $\psi_{\mathcal{V}}([X]) = [X \otimes_{C(M, \mathcal{K}_1) \otimes A} C(M, \mathcal{V} \otimes \mathbb{C}) \otimes A]$ is the corresponding isomorphism on K -theory.

Let L be a bundle gerbe. Let \mathcal{K}_i, P_i for $i \in \{1, 2\}$ be as in Lemma 2.11. Suppose that $dd(L) = dd(L_{P_1}) = dd(L_{P_2})$. Trivializations Q_i of $L^* \boxtimes L_{P_i}$ induce a trivialization of $L_{P_1}^* \boxtimes L_{P_2}$, since $Q_1^* \boxtimes Q_2$ over $P_1 \times_M P \times_M P \times_M P_2$ descends to a line bundle Q_{12} over $P_1 \times_M P_2$ in a way compatible with the actions of L_{P_i} . The trivialization Q_{12} corresponds to a real vector bundle \mathcal{V}_{12} as described in Lemma 2.11. Let $\kappa_{Q_i}([E]) = [C(M, Q_i(S_i^* \boxtimes E))]$ be the homomorphism from Theorem 2.10 associated to the data Q_i and the canonical bundle S_i . A bundle gerbe L provides a canonical trivialization of $L^* \boxtimes L$ (in particular, if P is a principal $PO(n)$ -bundle, then the principal $PO(n) \times PO(n)$ -bundle $P^{[2]}$ has a canonical lift to a $O(n) \otimes O(n)$ -bundle). Let $\kappa_{L_{P_i}}$ be the associated isomorphism from Theorem 2.10. Then the the following diagram of isomorphisms commutes:

$$\begin{array}{ccccc}
& & K_0(C(M, \mathcal{K}_1) \otimes A) & \xleftarrow{\kappa_{L_{P_1}}} & K_{L_{P_1}, A}^0(M) \\
& \nearrow^{\kappa_{Q_1}} & \downarrow \psi_{\mathcal{V}} & & \downarrow Q_{12} \\
K_{L, A}^0(M) & & K_0(C(M, \mathcal{K}_2) \otimes A) & \xleftarrow{\kappa_{L_{P_2}}} & K_{L_{P_2}, A}^0(M) \\
& \searrow_{\kappa_{Q_2}} & & &
\end{array}$$

2.1. The twisted Mishchenko-Fomenko bundle. The obstruction $\alpha(M)$ is built out of the Dirac operator on M and a certain bundle of Hilbert $C^*(\pi_1(M))$ -modules called the Mishchenko-Fomenko bundle. In the twisted case the algebra as well as the bundle itself have to be replaced by twisted versions.

Definition 2.13. Let $1 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \hat{\pi} \rightarrow \pi \rightarrow 1$ be a central $\mathbb{Z}/2\mathbb{Z}$ -extension of a discrete group π . Let $C_r^*(\hat{\pi})$ be the reduced group C^* -algebra associated to $\hat{\pi}$, likewise let $C_{\max}^*(\hat{\pi})$ be its maximal group C^* -algebra. Let $e \in \mathbb{Z}/2\mathbb{Z}$ be the non-trivial element and observe that $q = \frac{1}{2}(1 - e) \in C_{r/\max}^*(\hat{\pi})$ is a central projection. Let $C_r^*(\hat{\pi} \rightarrow \pi) = q C_r^*(\hat{\pi}) = q C_r^*(\hat{\pi}) q$ and define $C_{\max}^*(\hat{\pi} \rightarrow \pi)$ analogously. It is straightforward to check that these are in fact C^* -algebras, which we will call the *reduced*, respectively *maximal twisted group C^* -algebra* [26, Definition 8.1].

Remark 2.14. An extension in the above definition can be classified by a group 2-cocycle $c_{\hat{\pi}} \in H_{\text{gr}}^2(\pi, \mathbb{Z}/2\mathbb{Z})$, which yields an alternative description of $C_r^*(\hat{\pi} \rightarrow \pi)$ using completions of the

twisted group ring $\mathbb{C}[\pi, c_{\widehat{\pi}}]$ [5, Definition 2.1 for $A = \mathbb{C}$]. The algebra $C_{\max}^*(\widehat{\pi} \rightarrow \pi)$ has the universal property that any projective representation $\rho: \pi \rightarrow U(H)$ for the cocycle $c_{\widehat{\pi}}$ extends to a $*$ -homomorphism $C_{\max}^*(\widehat{\pi} \rightarrow \pi) \rightarrow B(H)$.

Let M be a closed smooth oriented manifold with $\dim(M) \geq 3$, such that its universal cover \widetilde{M} carries a spin structure. Denote by P_{SO} the oriented frame bundle of M and let $\pi = \pi_1(M)$. Let $P_{SO(\widetilde{M})}$ be the frame bundle of the universal cover. Consider the exact sequence

$$(3) \quad \pi_2(\widetilde{M}) \rightarrow \pi_1(SO(n)) \rightarrow \pi_1(P_{SO(\widetilde{M})}) \rightarrow 1$$

The map $\pi_2(\widetilde{M}) \rightarrow \pi_1(SO(n))$ sends $f: S^2 \rightarrow \widetilde{M}$ to the transition map $\varphi_f: S^1 \rightarrow SO(n)$ obtained from the pullback $f^*P_{SO(\widetilde{M})}$. If \widetilde{M} carries a spin structure, then φ_f factors through $S^1 \rightarrow \text{Spin}(n)$ and is therefore nullhomotopic. Comparing (3) with the exact sequence

$$\pi_2(M) \rightarrow \pi_1(SO(n)) \rightarrow \pi_1(P_{SO}) \rightarrow \pi_1(M) \rightarrow 1$$

for the bundle $P_{SO} \rightarrow M$, a diagram chase shows that $\pi_2(M) \rightarrow \pi_1(SO(n))$ is also trivial in this case and we obtain a central $\mathbb{Z}/2\mathbb{Z}$ -extension [26]:

$$(4) \quad 1 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \pi_1(P_{SO}) \rightarrow \pi \rightarrow 1.$$

Definition 2.15. The lifting bundle gerbe $L_{\widehat{\pi}} \rightarrow \widetilde{M}^{[2]}$ associated to the above central extension (4) is called the *Mishchenko-Fomenko bundle gerbe*.

Note that a trivialization of $L_{\widehat{\pi}} \rightarrow \widetilde{M}^{[2]}$ as a line bundle is given by a split $\pi \rightarrow \pi_1(P_{SO}) = \widehat{\pi}$. After applying this trivialization, the bundle gerbe multiplication over points $\tilde{m}_1, \tilde{m}_2, \tilde{m}_3$ with $\tilde{m}_2 = \tilde{m}_1 g_1$, $\tilde{m}_3 = \tilde{m}_2 g_2$ for some $g_1, g_2 \in \pi$ is transformed to $\mu((\tilde{m}_1, \tilde{m}_2, z_1), (\tilde{m}_2, \tilde{m}_3, z_2)) = (\tilde{m}_1, \tilde{m}_3, z_1 z_2 c_{\widehat{\pi}}(g_1, g_2))$ for some cocycle $c_{\widehat{\pi}}: \pi \times \pi \rightarrow \mathbb{Z}/2\mathbb{Z}$ representing the extension in the group $H^2(\pi, \mathbb{Z}/2\mathbb{Z})$ and $z_i \in \mathbb{R}$.

Definition 2.16. Let $\widehat{\pi} = \pi_1(P_{SO})$ and let $1 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \widehat{\pi} \rightarrow \pi \rightarrow 1$ be the extension described above. The canonical twisted Hilbert A -module bundle for $A = C_{\max}^*(\widehat{\pi} \rightarrow \pi)$ given by $\mathcal{V}_{\max} = \widetilde{M} \times C_{\max}^*(\widehat{\pi} \rightarrow \pi)$ is called the *(maximal) twisted Mishchenko-Fomenko bundle*. Similarly, we define $\mathcal{V}_{\text{red}} = \widetilde{M} \times C_r^*(\widehat{\pi} \rightarrow \pi)$. In both cases the twisting is given as in (2).

Remark 2.17. Although \mathcal{V}_{red} is trivial as a bundle over \widetilde{M} , the twisting provides extra structure that replaces the associated bundle construction, which does not make sense in the twisted case. This is analogous to viewing a vector bundle E with fiber V and frame bundle P_{SO} as the equivariant vector bundle $P_{SO} \times V$ over P_{SO} and keeping track of the action.

Let $E \rightarrow \widetilde{M}$ be a twisted Hilbert A -module bundle for $L_{\widehat{\pi}}$. Trivializing the bundle gerbe $L_{\widehat{\pi}}$ (and thereby fixing a cocycle $c_{\widehat{\pi}}$) turns the twisting into a collection of bundle maps that are on the fibers given by $\gamma^g: E_{\tilde{m}} \rightarrow E_{\tilde{m}g^{-1}}$ with the property that $\gamma^g \circ \gamma^h = c_{\widehat{\pi}}(g, h)\gamma^{gh}$, i.e. it behaves like a projective representation.

Now suppose that E is smooth and comes equipped with a twisted connection ∇ . Fix $\tilde{m} \in \widetilde{M}$, let m be its image in M and let $c: I \rightarrow \widetilde{M}$ be a smooth curve in \widetilde{M} starting at \tilde{m} . Let $\mathcal{P}_c: I \times E_{\tilde{m}} \rightarrow E$ be the parallel transport with respect to ∇ . Let $\hat{g} \in \hat{\pi}$, let $g \in \pi$ be its image under the projection. Observe that \mathcal{P}_c is equivariant in the sense that $\mathcal{P}_{cg^{-1}}(t, \hat{g} \cdot v) = \hat{g} \cdot \mathcal{P}_c(t, v)$ for $t \in I$, $v \in E$. Let $\tau: S^1 \rightarrow M$ be a smooth loop in M at m representing an element $h \in \pi = \pi_1(M)$, let $\bar{\tau}: I \rightarrow \widetilde{M}$ be the lift of τ to \tilde{m} . Note that the endpoint of $\bar{\tau}$ is $\tilde{m}h^{-1}$. Define $\mathcal{P}(\tilde{m}, \tau)(v) := \mathcal{P}_{\bar{\tau}}(1, v)$ and let $\text{hol}(\tau, \tilde{m}): E_{\tilde{m}} \rightarrow E_{\tilde{m}}$ be given by $\text{hol}(\tau, \tilde{m}) = \gamma^{h^{-1}} \circ \mathcal{P}(\tilde{m}, \tau)$. It satisfies

$$\begin{aligned} c_{\hat{\pi}}(g, h)^{-1} \text{hol}(\bar{m}, \sigma * \tau) &= c_{\hat{\pi}}(g, h)^{-1} \gamma^{(gh)^{-1}} \mathcal{P}(\bar{m}, \sigma * \tau) = \gamma^{h^{-1}} \circ \gamma^{g^{-1}} \circ \mathcal{P}(\bar{m}g^{-1}, \tau) \circ \mathcal{P}(\bar{m}, \sigma) \\ &= \gamma^{h^{-1}} \circ \left(\gamma^{g^{-1}} \circ \mathcal{P}(\bar{m}g^{-1}, \tau) \circ \gamma^g \right) \circ \gamma^{g^{-1}} \circ \mathcal{P}(\bar{m}, \sigma) \\ &= \gamma^{h^{-1}} \circ \mathcal{P}(\bar{m}, \tau) \circ \gamma^{g^{-1}} \circ \mathcal{P}(\bar{m}, \sigma) = \text{hol}(\bar{m}, \tau) \circ \text{hol}(\bar{m}, \sigma), \end{aligned}$$

which proves that it is a projective representation of π on $E_{\tilde{m}}$ with respect to $c_{\hat{\pi}}$.

Definition 2.18. We will call $\text{hol}(\cdot, \tilde{m})$ the *projective holonomy of ∇ at \tilde{m}* .

3. INDEX THEORY ON TWISTED BUNDLES

3.1. The Chern Character. Twisted versions of the Chern character have been developed in [4, Section 6], [21, Section 3.2]. Crucial in both cases is the idea that the endomorphism bundle $\text{End}(F) \rightarrow Y$ of a twisted vector bundle F for a bundle gerbe L descends to a vector bundle $\text{end}(F)$ over M . Let ∇^F be a twisted connection on F . Since we deal with the case of real bundle gerbes, where curving forms do not matter, the curvature descends to a form $\Omega_F \in \Omega^2(M, \text{end}(F))$.

Definition 3.1. The *twisted Chern character* $\text{ch}(F)$ is the class

$$(5) \quad \text{ch}(F) = \text{tr} \left(\exp \left(\frac{i\Omega_F}{2\pi} \right) \right) \in H^{\text{even}}(M, \mathbb{R}).$$

It induces a group homomorphism $\text{ch}: K_{L, \mathbb{C}}^0(M) \rightarrow H^{\text{even}}(M, \mathbb{R})$. If L_1, L_2 are two bundle gerbes with $dd(L_1) = dd(L_2)$ and Q is a trivialization of $L_1^* \boxtimes L_2$, then

$$\begin{array}{ccc} K_{L_1, \mathbb{C}}^0(M) & \xrightarrow{\text{ch}} & H^{\text{even}}(M, \mathbb{R}) \\ \downarrow Q & & \uparrow \\ K_{L_2, \mathbb{C}}^0(M) & \xrightarrow{\text{ch}} & H^{\text{even}}(M, \mathbb{R}) \end{array}$$

commutes, since Q is a flat line bundle.

Choose a lifting bundle gerbe L_P with $dd(L) = dd(L_P)$. The Künneth theorem and the canonical isomorphism $K_{L_P, \mathbb{C}}^0(M) \rightarrow K^0(C(M, \mathcal{K}))$ allows us to extend the above definition to a $K_0(A) \otimes \mathbb{R}$ -valued Chern character via

$$\text{ch}: K_{L, A}^0(M) \rightarrow K_{L_P, \mathbb{C}}^0(M) \otimes K_0(A) \otimes \mathbb{R} \rightarrow H^{\text{even}}(M, K_0(A) \otimes \mathbb{R}).$$

The first map is obtained from Theorem 2.10 using a trivialization Q of $L^* \boxtimes L_P$. As we have seen in Remark 2.12, the result of ch does not depend on the choice of P and L_P .

Lemma 3.2. *Let $L_i \rightarrow Y_i^{[2]}$ for $i \in \{1, 2\}$ be bundle gerbes. The following diagram commutes:*

$$\begin{array}{ccc} K_{L_1, A}^0(M) \times K_{L_2, \mathbb{C}}^0(M) & \xrightarrow{\boxtimes} & K_{L_1 \boxtimes L_2, A}^0(M) \\ \downarrow \text{ch} \times \text{ch} & & \downarrow \text{ch} \\ H^{\text{even}}(M, K_0(A) \otimes \mathbb{R}) \times H^{\text{even}}(M, \mathbb{R}) & \xrightarrow{\wedge} & H^{\text{even}}(M, K_0(A) \otimes \mathbb{R}) \end{array}$$

Proof. For $A = \mathbb{C}$ the proof is analogous to the untwisted case. For arbitrary A , choose a principal $PO(n)$ -bundle P and a trivialization Q of $L_1^* \boxtimes L_P$ as in Theorem 2.10. Note that Q and the canonical trivialization of $L_2^* \boxtimes L_2$ induce a trivialization of $(L_1 \boxtimes L_2)^* \boxtimes (L_P \boxtimes L_2)$. The statement is a consequence of the following commutative diagram

$$\begin{array}{ccc} K_{L_1, A}^0(M) \times K_{L_2, \mathbb{C}}^0(M) & \longrightarrow & K_{L_P, \mathbb{C}}^0(M) \otimes K_0(A) \otimes \mathbb{R} \times K_{L_2, \mathbb{C}}^0(M) \\ \downarrow & & \downarrow \\ K_{L_1 \boxtimes L_2, A}^0(M) & \longrightarrow & K_{L_P \boxtimes L_2, \mathbb{C}}^0(M) \otimes K_0(A) \otimes \mathbb{R} \end{array}$$

in which the vertical maps are given by tensor products and the horizontal ones are the decomposition maps from the Künneth theorem. \square

3.1.1. Chern character and traces. If A comes equipped with a trace τ , which we will assume for the rest of this section, there is a more direct approach to the Chern character, which was explored in the untwisted case in [25, Section 4]. If V is a finitely generated projective Hilbert A -module, then we have $\text{End}(V) = \mathcal{K}(V, V) \cong V \otimes_A \mathcal{K}(V, A)$, where $\mathcal{K}(V, W)$ denotes the compact adjointable A -linear operators. Setting $\tau_V(v \otimes T) = \tau(T(v))$ on elementary tensors, extends the trace to $\text{End}(V)$.

Let L be a bundle gerbe and E be a finitely generated projective twisted Hilbert A -module bundle for L . As was already noted above, $\text{End}(E)$ descends to a bundle of C^* -algebras $\text{end}(E)$ over M . Extending τ to the fibers yields a linear map $\Omega^n(M, \text{end}(E)) \rightarrow \Omega^n(M)$ on $\text{end}(E)$ -valued forms, which we will again denote by τ .

Definition 3.3. Let M be a closed smooth manifold, L be a bundle gerbe and E be a twisted Hilbert A -module bundle for L equipped with a twisted connection ∇^E that has curvature $\Omega_E \in \Omega^2(M, \text{end}(E))$. As in [25, Lemma 4.2] it follows that

$$\tau \left(\exp \left(\frac{i\Omega_E}{2\pi} \right) \right) \in \Omega^{\text{even}}(M)$$

is a closed form, which is independent of the choice of twisted connection. Its cohomology class $\text{ch}_\tau(E)$ is called the τ -Chern character of E .

To explain the connection between ch from Definition 3.1 and ch_τ we need the following concept.

Definition 3.4. Let V be a finitely generated projective right Hilbert A -module. Its *dimension* is defined to be $\dim_\tau(V) = \tau_V(\text{id}_V)$, where τ_V . If $V \cong tA^n$ for some projection $t \in M_n(A) = M_n(\mathbb{C}) \otimes A$, then $\dim_\tau(V) = (\text{tr} \otimes \tau)(t)$. \dim_τ yields a well-defined map $K_0(A) \rightarrow \mathbb{R}$.

Theorem 3.5. Let L be a bundle gerbe and let E be a twisted Hilbert A -module bundle for L . Then we have

$$(6) \quad \dim_\tau(\text{ch}(E)) = \text{ch}_\tau(E) \in H^{\text{even}}(M, \mathbb{R}) .$$

Proof. First observe that ch_τ is still multiplicative, in the sense that for a twisted Hilbert A -module bundle E and a twisted vector bundle F we have $\text{ch}_\tau(E \boxtimes F) = \text{ch}_\tau(E) \cup \text{ch}(F)$. In fact, this property rests only on the fact that the canonical tensor product connection $\nabla^{E \boxtimes F}$ has curvature $\Omega_{E \boxtimes F} = \Omega_E \otimes \text{id} + \text{id} \otimes \Omega_F \in \Omega^2(M, \text{end}(E) \otimes \text{end}(F))$ together with the calculation

$$\tau \left(\exp \left(\frac{i \Omega_{E \boxtimes F}}{2\pi} \right) \right) = \tau \left(\exp \left(\frac{i \Omega_E}{2\pi} \right) \right) \wedge \tau \left(\exp \left(\frac{i \Omega_F}{2\pi} \right) \right)$$

which is proven as in the untwisted case.

Choose P and a trivialization Q as in Theorem 2.10 and let F be a twisted vector bundle for L_P . Let V be a finitely generated projective Hilbert A -module. Observe that for the trivial bundle \underline{V} we have $\text{ch}_\tau(\underline{V}) = \dim_\tau(V)$. Thus,

$$\text{ch}_\tau(F \otimes V) = \text{ch}_\tau(F \boxtimes \underline{V}) = \text{ch}(F) \cup \text{ch}_\tau(\underline{V}) = \dim_\tau(V) \text{ch}(F) .$$

Likewise we have $\text{ch}(F \otimes V) = \text{ch}(F) \otimes [V] \in H^{\text{even}}(M, K_0(A) \otimes \mathbb{R})$. Therefore the following diagram commutes

$$\begin{array}{ccccc} K_{L_P, \mathbb{C}}^0(M) \otimes K_0(A) & \longrightarrow & K_0(C(M, \mathcal{K} \otimes A)) & \xrightarrow{\rho_Q^{-1}} & K_{L, A}^0(M) \\ & \searrow \text{ch} \otimes \dim_\tau & & \swarrow \text{ch}_\tau & \\ & & H^{\text{even}}(M, \mathbb{R}) & & \end{array}$$

where the first horizontal map is given by the canonical identification $K_{L_P, \mathbb{C}}^0(M) \cong K_0(C(M, \mathcal{K}))$ and the tensor product of modules. The statement follows if ch_τ vanishes on the second summand in the Künneth decomposition. We can identify $K_{L_P, \mathbb{C}}^1(M)$ with $K_{\pi^* L_P, \mathbb{C}}^0(\mathbb{R} \times M)$. Likewise we can represent classes in $K_1(A)$ as compactly supported virtual Hilbert A -module bundles over \mathbb{R} . If $F = (F_+, F_-, \varphi)$ represents an element in $K_{\pi^* L_P, \mathbb{C}}^0(\mathbb{R} \times M)$ and $W = (W_+, W_-, \psi)$ an element in $K_0(C_0(\mathbb{R}, A))$, then the graded tensor product $F \widehat{\boxtimes} W$ yields an element in $K_{\pi^* L_P, A}^0(\mathbb{R}^2 \times M)$ and Bott periodicity maps it to $b([F \widehat{\boxtimes} W]) \in K_{L_P, A}^0(M)$. It is a consequence of the multiplicativity that $\text{ch}_\tau(X) = \text{ch}_\tau(b(X)) \cup e$, where $e \in H_c^2(\mathbb{R}^2, \mathbb{R}) \cong \mathbb{R}$ denotes a generator and X is a triple

representing a class in $K_{\pi^*L_P,A}^0(\mathbb{R}^2 \times M)$. Thus,

$$\mathrm{ch}_\tau(b(F \boxtimes W)) \cup e = \mathrm{ch}_\tau(F \boxtimes W) = \mathrm{ch}(F) \cup \mathrm{ch}_\tau(W) = 0$$

since $\mathrm{ch}_\tau(W) \in H_c^{\mathrm{even}}(\mathbb{R}, \mathbb{R}) = 0$. \square

3.2. Local C^* -algebras and twisted bundles. A *local C^* -algebra* is a $*$ -algebra B equipped with a pre- C^* -norm such that $M_n(B)$ is closed under holomorphic functional calculus for all $n \in \mathbb{N}$. If $\mathcal{B} \rightarrow M$ is a locally trivial bundle with fiber B and structure group $\mathrm{Aut}(B)$ equipped with the point norm topology, then the section algebra $C(M, \mathcal{B})$ is again a local C^* -algebra as can be checked directly from the definitions using the naturality of holomorphic functional calculus.

A right B -module is called *finitely generated* and *projective* if it is isomorphic to one of the form tB^n for a projection $t \in M_n(B)$. The groups $K_i(B)$, $i \in \{0, 1\}$ are now defined as in [3]. If A is the completion of B , then $K_i(B) \cong K_i(A)$. Twisted B -module bundles are defined just as in 2.7 and yield analogous groups $K_{L,B}^0(M)$ and $K_{L,B}^1(M) := K_{\pi_M^*L,B}^0(\mathbb{R}^2 \times M)$. Given a twisted B -module bundle \mathcal{E} , the fiberwise inner product $\mathcal{E} \otimes_B A$ is a twisted Hilbert A -module bundle. Thus we get a homomorphism $K_{L,B}^0(M) \rightarrow K_{L,A}^0(M)$

Let S be as in Theorem 2.10. The exterior tensor product $S^* \boxtimes \mathcal{E}$ involves no completion. Therefore the proof of that theorem still works and gives $K_{L,B}^0(M) \cong K_0(C(M, \mathcal{K} \otimes B))$, which also proves $K_{L,B}^0(M) \cong K_{L,A}^0(M)$. The Bott map $b: K_{L,B}^0(M) \rightarrow K_{\pi_M^*L,B}^0(\mathbb{R}^2 \times M)$ is also still well-defined and a comparison with the corresponding map for its completion shows that b still is an isomorphism. Likewise, there is a Künneth homomorphism

$$K_{L_P,\mathbb{C}}^0(M) \otimes K_0(B) \otimes \mathbb{R} \oplus K_{L_P,\mathbb{C}}^1(M) \otimes K_1(B) \otimes \mathbb{R} \rightarrow K_0(C(M, \mathcal{K} \otimes B)) \otimes \mathbb{R}$$

that just involves algebraic tensor products in the fibers. Comparison with the completion reveals this to be an isomorphism as well. This enables us to define $\mathrm{ch}: K_{L,B}^0(M) \rightarrow H^{\mathrm{even}}(M, K_0(B) \otimes \mathbb{R})$.

Let B be a unital local C^* -algebra carrying a trace τ . If \mathcal{V} is a finitely generated and projective B -module, then $\mathrm{End}(\mathcal{V})$ coincides with the finite rank operators and we have $\mathrm{End}(\mathcal{V}) \cong \mathcal{V} \otimes_B \mathrm{Hom}(\mathcal{V}, B)$, where the tensor product is a quotient of the algebraic tensor product. Repeating the constructions in Section 3.1.1 we can define $\mathrm{ch}_\tau: K_{L,B}^0(M) \rightarrow H^{\mathrm{even}}(M, \mathbb{R})$ and $\mathrm{dim}_\tau: K_0(B) \rightarrow \mathbb{R}$. The proof of Theorem 3.5 shows that $\mathrm{dim}_\tau(\mathrm{ch}(\mathcal{E})) = \mathrm{ch}_\tau(\mathcal{E})$.

3.3. The projective Dirac operator. Let (M, g) be a closed oriented Riemannian manifold of even dimension $n > 3$. The oriented frame bundle P_{SO} together with the defining central $\mathbb{Z}/2\mathbb{Z}$ -extension $\mathrm{Spin}(n) \rightarrow SO(n)$ gives a lifting bundle gerbe $L_{\mathrm{spin}} \rightarrow P_{SO}^{[2]}$. Since we assume M to be even-dimensional, the complex spinor module $\Delta_{\mathbb{C}}$ is $\mathbb{Z}/2\mathbb{Z}$ -graded, i.e. $\Delta_{\mathbb{C}} = \Delta_+ \oplus \Delta_-$. There is a $\mathbb{Z}/2\mathbb{Z}$ -graded twisted vector bundle $S = P_{SO} \times \Delta_{\mathbb{C}}$ with $S_{\pm} = P_{SO} \times \Delta_{\pm}$ for L_{spin} , where the action is given by $\gamma([\hat{g}, \lambda] \otimes v) = \lambda \hat{g} \cdot v$ for $\hat{g} \in \mathrm{Spin}(n)$, $\lambda \in \mathbb{C}$ and $v \in \Delta_{\mathbb{C}}$. If we fix a connection form $\eta \in \Omega^1(P_{SO}, \mathfrak{so}(n))$, then there is a canonical twisted connection ∇^S on S , which acts like $\nabla^S(\sigma) = d\sigma + \rho_*(\eta) \cdot \sigma$ for a section $\sigma \in C(P_{SO}, S)$, where ρ is induced by $O(n) \rightarrow U(n)$.

Let $E \rightarrow Y$ be a smooth twisted Hilbert A -module bundle for a bundle gerbe L with $dd(L) = -dd(L_{\text{spin}}) = dd(L_{\text{spin}})$ equipped with a twisted connection ∇^E . Choose a trivialization Q of $L_{\text{spin}} \boxtimes L$, then $Q(S \boxtimes E)$ is a Hilbert A -module bundle over M . Fixing a (flat) connection on Q , $\nabla^{S \boxtimes E} = \nabla^S \otimes \text{id} + \text{id} \otimes \nabla^E$ descends to a connection on $Q(S \boxtimes E)$. The latter bundle carries an action of the complex Clifford bundle $\mathbb{C}\ell(M)$. We have $\nabla_X^{S \boxtimes E}(c \cdot \sigma) = \nabla_X^{\mathbb{C}\ell(M)}(c) \cdot \sigma + c \cdot \nabla_X^{S \boxtimes E}(\sigma)$ for $X \in C^\infty(M, TM)$, $c \in C^\infty(M, \mathbb{C}\ell(M))$, $\sigma \in C^\infty(M, Q(S \boxtimes E))$.

Definition 3.6. Let S, E and Q be as above, then the elliptic first order differential operator $D^E: C^\infty(M, Q(S \boxtimes E)) \rightarrow C^\infty(M, TM \otimes Q(S \boxtimes E)) \rightarrow C^\infty(M, Q(S \boxtimes E))$ where the first map is induced by the connection $\nabla^{S \boxtimes E}$ and the metric and the second is Clifford multiplication will be called the *projective Dirac operator* twisted by E .

The bundle gerbe L_{spin} provides a canonical trivialization Q of $L_{\text{spin}} \boxtimes L_{\text{spin}}$ with the property that $Q(S^* \boxtimes S) \cong \mathbb{C}\ell(M)$ as a bundle of $\mathbb{C}\ell(M)$ - $\mathbb{C}\ell(M)$ -bimodules. This identifies the operator $D^S: C^\infty(M, \mathbb{C}\ell(M)) \rightarrow C^\infty(M, \mathbb{C}\ell(M))$ with the right Clifford linear Dirac operator. As described in [3, Section 24.5], D^S yields a class in the K -homology group $KK(C(M, \mathbb{C}\ell(M)), \mathbb{C})$.

Since the connection is even and Clifford multiplication is an odd operation, D^E has the decomposition $D_\pm^E: C^\infty(M, Q(S_\pm \boxtimes E)) \rightarrow C^\infty(M, Q(S_\pm \boxtimes E))$.

Theorem 3.7. Let S, E and Q be as in the last paragraph and denote by D_+^E the positive part of the projective Dirac operator twisted by E , then

$$\text{ind}(D_+^E) = \int_M \widehat{A}(M) \text{ch}(E) .$$

In particular, the index of D_+^E does not depend on the choice of Q .

Proof. Let $\pi_M: T^*M \rightarrow M$ be the bundle projection and denote by $D(T^*M)$ and $S(T^*M)$ the disc and sphere bundle of the cotangent bundle. The symbol $\sigma: \pi_M^*Q(S_+ \boxtimes E) \rightarrow \pi_M^*Q(S_- \boxtimes E)$ of D_+^E yields an element $[Q(S_+ \boxtimes E), Q(S_- \boxtimes E), \sigma] \in K_A^0(D(T^*M), S(T^*M)) \cong K_A^0(T^*M)$. Denote by

$$\phi: H^*(M, \mathbb{R}) \rightarrow H^{*+\dim(M)}(D(T^*M), S(T^*M); \mathbb{R})$$

the Thom isomorphism. From the Mishchenko-Fomenko index theorem [18] together with Lemma 3.2 we obtain

$$\begin{aligned} \text{ind}(D_+^E) &= \int_M \text{Td}(M) \phi^{-1}(\text{ch}([Q(S_+ \boxtimes E), Q(S_- \boxtimes E), \sigma])) \\ &= \int_M \text{Td}(M) \phi^{-1}(\text{ch}([S_+, S_-, \sigma_D])) \text{ch}(E) = \int_M \widehat{A}(M) \text{ch}(E) \end{aligned}$$

where the identification of $\text{Td}(M) \phi^{-1}(\text{ch}([S_+, S_-, \sigma_D]))$ with the \widehat{A} -genus described in [16] works in the twisted case as well. \square

Theorem 3.8. Let D be the projective Dirac operator. Let $E \rightarrow Y$ be a finitely generated projective twisted Hilbert A -module bundle for a bundle gerbe L with $dd(L) = dd(L_{\text{spin}})$. Let Q be a trivialization of $L^* \boxtimes L_{\text{spin}}$ and $[E^S] := \rho_Q([E]) = [C(M, Q(S^* \boxtimes E))] \in KK(\mathbb{C}, C(M, \mathbb{C}\ell(M) \otimes A))$. Then

$[E^S] \otimes_{C(M, \mathcal{C}\ell(M))} [D^S] \in KK(\mathbb{C}, A) \cong K_0(A)$ is the class representing the Mishchenko-Fomenko index of D^E .

Proof. It suffices to show that the intersection product coincides with the Fredholm module:

$$\left[L^2(Q(S^* \boxtimes E)), D^E \left(1 + (D^E)^2 \right)^{-\frac{1}{2}} \right] \in KK(\mathbb{C}, A) .$$

The proof of this is just a slight modification of [25, Theorem 5.22], therefore we omit it. \square

From this decomposition of the index in twisted K -theory, we obtain the following naturality result in analogy to the untwisted case [11, Lemma 3.1].

Corollary 3.9. *Let D be the projective Dirac operator and let $E \rightarrow Y$ be a twisted Hilbert A -module bundle for a bundle gerbe L with $dd(L) = dd(L_{spin})$. Given a C^* -algebra homomorphism $\varphi : A \rightarrow B$ define the twisted Hilbert B -module bundle F via $F = E \otimes_{\varphi} B$. Then: $\varphi_*([D^E]) = [D^F]$, where $\varphi_* : K_0(A) \rightarrow K_0(B)$ denotes the induced map on K -theory.*

Proof. Let $[E^S] := [C(M, Q(S^* \boxtimes E))] \in KK(\mathbb{C}, C(M, \mathcal{C}\ell(M) \otimes A))$ and choose a trivialization Q of $L^* \boxtimes L_{spin}$. The naturality of the Kasparov product yields

$$\begin{aligned} \varphi_*([D^E]) &= \varphi_*([E^S] \otimes_{C(M, \mathcal{C}\ell(M))} [D^S]) \\ &= ((\text{id}_{C(M, \mathcal{C}\ell(M))} \otimes \varphi_*)[C(M, Q(S^* \boxtimes E))]) \otimes_{C(M, \mathcal{C}\ell(M))} [D^S] \\ &= [C(M, Q(S^* \boxtimes F))] \otimes_{C(M, \mathcal{C}\ell(M))} [D^S] = [D^F] . \quad \square \end{aligned}$$

4. ENLARGEABLE MANIFOLDS WITH SPIN UNIVERSAL COVER

In this section we will extend the result of [11, 12] about the Rosenberg index obstruction for closed enlargeable spin manifolds to closed enlargeable orientable manifolds with spin structure on the universal cover. Since \widetilde{M} may be non-compact, we have to work in a twistedly equivariant setting.

Definition 4.1. A connected closed oriented manifold M with fixed metric g is called *enlargeable* if the following holds: For every $\varepsilon > 0$, there is a connected cover $\bar{M} \rightarrow M$ carrying a spin structure and an ε -contracting map

$$\bar{\psi}_{\varepsilon} : (\bar{M}, \bar{g}) \rightarrow (S^n, g_0)$$

which is constant outside a compact subset of \bar{M} and of nonzero degree. Here, \bar{g} denotes the induced metric on \bar{M} and g_0 is the standard metric on S^n .

Definition 4.2. Let M be a closed even-dimensional oriented Riemannian manifold and let \mathcal{V}_{\max} be the maximal twisted Mishchenko-Fomenko bundle (Def. 2.16). The (*maximal*) *twisted Stolz-Rosenberg obstruction* is given by $\theta^{\max}(M) = \text{ind}(D^{\mathcal{V}_{\max}}) \in K_0(C_{\max}^*(\widehat{\pi} \rightarrow \pi))$.

4.1. Almost flat twisted bundles. In our proof of the non-vanishing of $\theta^{\max}(M)$ for closed enlargeable M with spin universal cover we will extend the argument given in [11, 12] to the twisted setting. Thus, it will rely on almost flat *twisted* bundles.

Definition 4.3. Let $L_{\widehat{\pi}}$ be the Mishchenko-Fomenko bundle gerbe. A sequence $E_i \rightarrow \widetilde{M}$, $i \in \mathbb{N}$ of smooth twisted vector bundles for $L_{\widehat{\pi}}$ equipped with connections ∇^i will be called a *sequence of almost flat twisted bundles*, if $\lim_{i \rightarrow \infty} \|\Omega_i\| = 0$, where Ω_i is the curvature of the connection ∇^i . The norm is induced by the pointwise norm on $\text{end}(E_i) \rightarrow M$ and the maximum norm on the unit sphere bundle in $\Lambda^2(M)$. Moreover, the twistings $\gamma_i^g: E_i \rightarrow g^*E_i$ considered as sections $C(\widetilde{M}, \text{Hom}(E_i, g^*E_i))$ should be locally Lipschitz continuous with a global Lipschitz constant C independent of i , i.e. each point $\widetilde{m} \in \widetilde{M}$ has an open neighborhood U of \widetilde{m} , such that E_i and g^*E_i are trivial over U and such that $\gamma_i^g|_U$ viewed as an element in $C(U, U(V))$ (V being the typical fiber of E_i) satisfies

$$\|\gamma_i^g(\widetilde{m}_1) - \gamma_i^g(\widetilde{m}_2)\| \leq C d(\widetilde{m}_1, \widetilde{m}_2)$$

where the metric is induced by the Riemannian structure pulled back to \widetilde{M} .

Let M be a closed orientable smooth manifold and let \bar{M} be a cover of M equipped with a spin structure, let $\pi = \pi_1(M)$ and $\bar{\pi} = \pi_1(\bar{M}) \subset \pi$. The bundle projection $P_{\text{Spin}(\bar{M})} \rightarrow \bar{M}$ of the principal $\text{Spin}(n)$ -bundle induces an isomorphism $\pi_1(P_{\text{Spin}(\bar{M})}) \rightarrow \bar{\pi}$, which in turn yields an injective group homomorphism $\sigma: \bar{\pi} \rightarrow \pi_1(P_{\text{Spin}(\bar{M})}) \rightarrow \pi_1(P_{\text{SO}(\bar{M})}) \rightarrow \widehat{\pi}$, where the first map is the inverse of the above isomorphism and the other two homomorphism arise from the respective coverings. We call a (set-theoretic) split $s: \pi \rightarrow \widehat{\pi}$ *compatible*, if $s(gh) = s(g)\sigma(h)$ for $g \in \pi$ and $h \in \bar{\pi}$.

Lemma 4.4. *Let M , \bar{M} , π and $\bar{\pi}$ be as in the last paragraph. The choice of a compatible set-theoretic split $s: \pi \rightarrow \widehat{\pi}$ turns the Hilbert space $H = \ell^2(\pi/\bar{\pi})$ into a projective representation of π , which corresponds to an honest representation of $\widehat{\pi}$.*

Proof. Let $e \in \widehat{\pi}$ be the image of the non-trivial element of $\mathbb{Z}/2\mathbb{Z}$ and let $q = \frac{1}{2}(1 - e) \in \mathbb{C}[\widehat{\pi}]$. Since q is central in the group ring, it yields a well-defined projection $q: \ell^2(\widehat{\pi}/\sigma(\bar{\pi})) \rightarrow \ell^2(\widehat{\pi}/\sigma(\bar{\pi}))$. The space $q\ell^2(\widehat{\pi}/\sigma(\bar{\pi}))$ is a representation of $\widehat{\pi}$, on which e acts by multiplication with -1 , i.e. a projective representation of π .

A compatible split now yields an injective map $\pi/\bar{\pi} \rightarrow \widehat{\pi}/\sigma(\bar{\pi})$ via $[g] \mapsto [s(g)]$. It induces an isometric isomorphism $\ell^2(\pi/\bar{\pi}) \rightarrow q\ell^2(\widehat{\pi}/\sigma(\bar{\pi}))$ \square

Theorem 4.5. *Let M be an even-dimensional closed orientable manifold that is enlargeable in the sense of Definition 4.1. Let $i \in \mathbb{N}$ be a positive natural number. Then there is a C^* -algebra C_i (which will be constructed in the proof) and a twisted Hilbert C_i -module bundle $E_i \rightarrow \widetilde{M}$ for the Mishchenko-Fomenko bundle gerbe $L_{\widehat{\pi}}$ together with a twisted connection ∇_i that has the following properties: The curvature Ω_i of E_i satisfies*

$$\|\Omega_i\| \leq \frac{1}{i} C$$

where C is a constant depending only on $\dim(M)$. Moreover, there is a split extension $0 \rightarrow \mathbb{K} \rightarrow C_i \rightarrow X_i \rightarrow 0$ with a certain C^* -algebra X_i . In particular, each $K_0(C_i)$ splits off a $\mathbb{Z} = K_0(\mathbb{K})$ summand and the $K_0(\mathbb{K})$ -part of the index of the projective Dirac operator $D_+^{E_i}$ is different from 0.

Proof. Let $2n = \dim(M)$ and $\pi = \pi_1(M)$. Since the Chern character is rationally an isomorphism, there is a vector bundle $F \rightarrow S^{2n}$ with non-vanishing top Chern class $c_n(F) \neq 0$. Choose a connection η_F on F and fix $i \in \mathbb{N}$. Since M is enlargeable, there exists a spin covering $\bar{M} \rightarrow M$ together with a $\frac{1}{i}$ -contracting map $\psi: \bar{M} \rightarrow S^{2n}$, which is constant outside a compact subset K of \bar{M} . Let P_F be the principal $U(n)$ -bundle of frames in F . Since ψ is constant on $\bar{M} \setminus K$ we can choose a trivialization for the principal $U(n)$ -bundle ψ^*P_F over this set:

$$(\psi^*P_F)|_{\bar{M} \setminus K} \cong (\bar{M} \setminus K) \times U(n)$$

such that the pullback connection $\psi^*\eta_F|_{\bar{M} \setminus K}$ is flat. Let $\rho: \bar{M} \rightarrow M$ be the covering map, $\tilde{\rho}: \tilde{M} \rightarrow M$ the universal cover and $\tilde{\pi} = \pi_1(\tilde{M})$. As described in the proof of [12, Proposition 1.5] we can cover M by open sets $U_j, j \in I$, such that each component $V_{\lambda,j} \subset \bar{M}$ of $\rho^{-1}(U_j)$ maps diffeomorphically onto U_j , intersects only one component $V_{\mu,k}$ of $\rho^{-1}(U_k)$ for any k and such that $\psi^*P_F|_{V_{\lambda,j}}$ trivializes. Let $J_j = \pi_0(\rho^{-1}(U_j))$ be the index set labeling the components, likewise set $\tilde{J}_j = \pi_0(\tilde{\rho}^{-1}(U_j))$. Let $\tilde{\varphi}_{\alpha,j}: \tilde{J}_j \rightarrow \pi/\tilde{\pi}$ be the map that sends αg to $[g] \in \pi/\tilde{\pi}$ for $g \in \pi$, where π acts on \tilde{J}_j by deck transformations. Since $\bar{M} = \tilde{M}/\tilde{\pi}$, this induces bijections $\varphi_{\lambda,j}: J_j \rightarrow \pi/\tilde{\pi}$ for each $\lambda \in J_j$. Note that

$$(7) \quad \varphi_{[\alpha g],j} = g^{-1} \cdot \varphi_{[\alpha],j} .$$

Moreover, if $\lambda \in J_j$ and $\mu \in J_k$ belong to components with non-empty intersection, then $\varphi_{\lambda,j}(\kappa) = \varphi_{\mu,k}(\tau)$ if τ and κ intersect.

Consider the Hilbert space $H = \ell^2(\pi/\tilde{\pi}) \otimes \mathbb{C}^n$. Let $C_S \subset \mathcal{B}(H)$ be the C^* -algebra generated by the group of all permutations of $\pi/\tilde{\pi}$ and all multiplications by functions $f: \pi/\tilde{\pi} \rightarrow S^1$. So we have permutation operators with S^1 -entries as a generating set of C_S . Let $C_T \subset \mathcal{B}(H)$ be the C^* -algebra generated by linear transformations, which are of the form

$$T: H \rightarrow H \quad ; \quad T([g] \otimes v) = [h] \otimes T'v \quad \text{and} \quad T|_{([g] \otimes \mathbb{C}^n)^\perp} = 0 .$$

for some matrix $T' \in M_n(\mathbb{C})$ and $[g], [h] \in \pi/\tilde{\pi}$. Let $C_{S,T}$ be the C^* -algebra generated by C_S and C_T inside of $\mathcal{B}(H)$ and note that C_T is a 2-sided ideal in $C_{S,T}$. Moreover, C_T is isomorphic either to the compact operators or to a matrix algebra. Applying the stabilization trick of [12] we can without loss of generality assume that the former is the case. Let $C_i = \{(c_1, c_2) \in C_{S,T} \times C_{S,T} \mid c_1 - c_2 \in C_T\}$. This algebra fits into a split exact sequence $0 \rightarrow C_T \rightarrow C_i \rightarrow C_{S,T} \rightarrow 0$ with the splitting induced by the diagonal map, $C_T \rightarrow C_i$ via $a \mapsto (a, 0)$ and $C_i \rightarrow C_{S,T}$ via $(a, b) \mapsto b$.

We choose trivializations of ψ^*P_F over the sets $V_{\lambda,j} \subset \bar{M}$, where we take the trivialization fixed above if $V_{\lambda,j}$ is a subset of $\bar{M} \setminus K$. This way we get a cocycle on the double intersections $V_{(\lambda,\mu),(j,k)} = V_{\lambda,j} \cap V_{\mu,k}$:

$$T'_{(\lambda,\mu),(j,k)}: V_{(\lambda,\mu),(j,k)} \rightarrow U(n) .$$

We can extend $T'_{(\lambda,\mu),(j,k)}$ to a cocycle with values in the unitary group $U(C_{S,T})$ as follows:

$$T_{(\lambda,\mu),(j,k)}^1(x)(\varphi_{\lambda,j}(\kappa) \otimes v) = \varphi_{\mu,k}(\tau) \otimes T'_{(\kappa,\tau),(j,k)}(\bar{x})(v) ,$$

where $\tau \in \pi/\bar{\pi}$ is the index of the component of $\rho^{-1}(U_k)$ that intersects $V_{\kappa,j}$ and \bar{x} denotes the lift of $\rho(x)$ to the component $V_{\kappa,j}$. This map actually does nothing to the first tensor factor by our previous considerations. Let $T_{(\lambda,\mu),(j,k)}^2$ be the constant map with value $1 \in U(C_{S,T})$. $T'_{(\kappa,\tau),(j,k)}$ is different from the identity only for finitely many pairs (κ, τ) . Thus,

$$T_{(\lambda,\mu),(j,k)}: V_{(\lambda,\mu),(j,k)} \rightarrow U(C_i) \quad ; \quad T_{(\lambda,\mu),(j,k)} = (T_{(\lambda,\mu),(j,k)}^1, T_{(\lambda,\mu),(j,k)}^2) .$$

is a well-defined cocycle with values in $U(C_i)$. We therefore get a smooth Hilbert C_i -module bundle $\bar{E}_i \rightarrow \bar{M}$, whose pullback to \widetilde{M} will be $E_i = \widetilde{M} \times_M \bar{E}_i$. By Lemma 4.4, the space $\ell^2(\pi/\bar{\pi})$ carries a projective unitary representation of π , which induces a projective representation $r: \pi \rightarrow U(C_i)$. For $\alpha, \beta \in \widetilde{J}_j$ denote the corresponding components of $\widetilde{\rho}^{-1}(U_j)$ by $W_{\alpha,j}$ and $W_{\beta,j}$ respectively. Let $\lambda = [\alpha]$, $\mu = [\beta]$, $\lambda' = [\alpha g^{-1}]$ and $\mu' = [\beta g^{-1}] \in J_j$. We define

$$\gamma^g: W_{\alpha,j} \times C_i \rightarrow W_{\alpha g^{-1},j} \times C_i$$

by left multiplication with $r(g)$. Due to equation (7) and with $\varphi_{\lambda,j}(\kappa) = \varphi_{\mu,k}(\tau) = [h] \in \pi/\bar{\pi}$ we have

$$\begin{aligned} (T_{(\lambda',\mu'),(j,k)}(x) \cdot r(g)) (\varphi_{\lambda,j}(\kappa) \otimes v) &= c_{\bar{\pi}}(g, h) \varphi_{\mu',j}(\tau) \otimes T'_{(\kappa,\tau),(j,k)}(v) \\ &= (r(g) \cdot T_{(\lambda,\mu),(j,k)}(x)) (\varphi_{\lambda,j}(\kappa) \otimes v) . \end{aligned}$$

Thus, γ^g intertwines the transition functions of E_i and $g^*(E_i)$. Therefore it yields a well-defined twisting map $\gamma^g: E_i \rightarrow g^*(E_i)$. This clearly satisfies the Lipschitz condition, since it even is locally constant.

Let $\eta_{\kappa,j} \in \Omega^1(V_{\kappa,j}, \mathfrak{u}(n))$ be the pullback of η_F via the trivialization. These induce forms in $\Omega^1(V_{\lambda,j}, C_{S,T}^a)$, where $C_{S,T}^a$ denotes the anti-selfadjoint operators in $C_{S,T}$, via

$$\left(\eta_{\lambda,j}^{E_i} \right)_x (\xi) (\varphi_{\lambda,j}(\kappa) \otimes v) = \varphi_{\lambda,j}(\kappa) \otimes (\eta_{\kappa,j})_x (\xi) \cdot v .$$

Since $\eta_{\kappa,j}$ is non-zero only for finitely many κ , we can extend $\eta_{\lambda,j}^{E_i}$ to a well-defined 1-form with values in the anti-selfadjoint operators of C_i by setting it to zero in the second component. These 1-forms inherit their transformation behaviour from the forms $\eta_{\kappa,j}$. Thus, they yield a C_i -linear connection ∇^i on sections of E_i . Just like above it follows from (7) that ∇^i is a twisted connection. Since the norm of the curvature Ω_i of ∇^i coincides with that of $\psi^*\Omega_F$, we have

$$\|\Omega_i\| = \|\psi^*\Omega_F\| \leq \frac{1}{i} C .$$

It remains to be shown that the $K_0(\mathbb{K})$ -part of $\text{ind}(D_+^{E_i})$ does not vanish. Here we proceed exactly as in [12]: Let $\mathcal{T} \subset C_T \cong \mathbb{K}$ be the trace class ideal and let D_i be the algebra given by

$$D_i = \{(c_1, c_2) \in C_{S,T} \times C_{S,T} \mid c_1 - c_2 \in \mathcal{T}\} .$$

Since the proof of Lemma [12, lemma 2.4] applies to D_i with the changed $C_{S,T}$ as well, D_i is a unital local C^* -algebra with a trace $\tau(c_1, c_2) = \text{tr}(c_1 - c_2)$, which coincides with the trace of the element after projecting it from C_i to \mathcal{T} . Its C^* -completion is C_i . Since $K_0(C_i) \cong K_0(D_i)$, we can extend \dim_τ from Definition 3.4 to a functional on $K_0(C_i)$ and it suffices to prove that $\dim_\tau(\text{ind}(D_+^{E_i})) \neq 0$. The transition functions in the definition of E_i actually take values in $U(D_i)$ and thus lead to a twisted D_i -module bundle \mathcal{E}_i in the sense of Section 3.2. By Theorem 3.7 we have

$$\begin{aligned} \dim_\tau(\text{ind}(D_+^{E_i})) &= \int_M \widehat{A}(M) \dim_\tau(\text{ch}(E_i)) \\ &= \int_M \widehat{A}(M) \dim_\tau(\text{ch}(\mathcal{E}_i)) = \int_M \widehat{A}(M) \text{ch}_\tau(\mathcal{E}_i) . \end{aligned}$$

We can identify $\Omega_{\mathcal{E}_i} \in \Omega^2(M, \text{end}(\mathcal{E}_i))$ with an equivariant form in $\Omega^2(\widetilde{M}, \text{End}(\mathcal{E}_i))$. If we carry out the integration over a single subset $U_j \subset M$, we could integrate instead over the subset $W_{\alpha,j} \subset \widetilde{M}$ for some $\alpha \in \widetilde{J}_j$. This is independent of the choice of α by equivariance. But over $W_{\alpha,j}$ the form $\tau(\Omega_{\mathcal{E}_i} \wedge \cdots \wedge \Omega_{\mathcal{E}_i})|_{W_{\alpha,j}} \in \Omega^{\text{even}}(W_{\alpha,j}, \mathbb{R})$ coincides with the sum of all $\Omega_{\psi^*F} \wedge \cdots \wedge \Omega_{\psi^*F}|_{V_{\kappa,j}} \in \Omega^{\text{even}}(V_{\kappa,j}, \mathbb{R})$ over $\kappa \in J_j$ by the definition of the trace. Using a partition of unity on M we see that

$$\int_M \widehat{A}(M) \text{ch}_\tau(\mathcal{E}_i) = \int_{\widetilde{M}} \widehat{A}(\widetilde{M}) \text{ch}(\psi^*F - \underline{\mathbb{C}}^n) ,$$

Since the class of $\text{ch}(\psi^*F - \underline{\mathbb{C}}^n)$ is concentrated in degree n we see that the above term is non-vanishing. \square

Remark 4.6. Due to the stabilization trick mentioned in the proof the fibers of E_i are isomorphic to $t_i C_i$ for some projection $t_i \in C_i$, where $t_i = 1$ if \widetilde{M} is non-compact.

Having the sequence E_i of almost flat twisted bundles at hand, we can form the C^* -algebra

$$A = \prod_{i \in \mathbb{N}} C_i$$

of bounded sequences with i th entry in C_i , in which the norm closure of the sequences with only finitely many non-zero entries

$$A' = \overline{\bigoplus_{i \in \mathbb{N}} C_i}^{\|\cdot\|}$$

is a two-sided ideal and we set $Q = A/A'$. Let A_i be the ideal in A consisting of sequences that are 0 everywhere, but in the i th entry.

Theorem 4.7. *There is a smooth twisted Hilbert A -module bundle $E \rightarrow \widetilde{M}$ together with a twisted connection*

$$\nabla^E : C^\infty(\widetilde{M}, E) \rightarrow C^\infty(\widetilde{M}, T^*\widetilde{M} \otimes E)$$

such that the following holds:

- $E \cdot A_i$ is isomorphic to E_i as a twisted Hilbert C_i -module bundle.
- The connection preserves the subbundles $E \cdot A_i$.

- The sequence of curvatures $\Omega_i \in \Omega^2(M, \text{end}(E \cdot A_i))$ of the connection induced on $E \cdot A_i$ by ∇^E satisfies $\lim_{i \rightarrow \infty} \|\Omega_i\| = 0$.

Proof. The idea is to see that the product bundle $E_L = \Delta_{\widetilde{M}}^* (\prod_{i \in \mathbb{N}} E_i)$, where

$$\Delta_{\widetilde{M}}: \widetilde{M} \rightarrow \prod_{i \in \mathbb{N}} \widetilde{M}$$

is the diagonal map, has locally Lipschitz continuous transition functions. This parallels the construction given in the proof of [11, Lemma 2.1] with the only difference that we have to work equivariantly over \widetilde{M} , so we just sketch the differences and refer to [11] for the details: We cover M by subsets U_j , each of them diffeomorphic to I^n , where $I = [0, 1]$, such that $\widetilde{M} \rightarrow M$ is trivial over U_j via

$$\phi_j: U_j \times \pi \rightarrow \widetilde{M} \Big|_{U_j}.$$

We can find trivializations

$$\psi_{i,j}^1: \phi_j^* E_i \Big|_{U_j \times \{1\}} \rightarrow I^n \times t_i C_i$$

of $\phi_j^* E_i \Big|_{U_j \times \{1\}}$, such that constant sections of $\phi_j^* E_i$ over $I^k \times \{0, \dots, 0\}$ are parallel with respect to ∇_{∂_l} for $1 \leq l \leq k$, where ∇ denotes the connection induced by ∇^{E_i} . Using the twisting we can extend $\psi_{i,j}^1$ to a trivialization $\psi_{i,j}$ of $\phi_j^* E_i \Big|_{U_j \times \pi}$ with components $\psi_{i,j}^g$ with $g \in \pi$. Let $\eta_{i,j}^g \in \Omega^1(I^n, t_i C_i t_i)$ be the pullback of the connection 1-form of ∇^{E_i} . The way the trivializations are constructed is crucial to prove the estimate given in [11, Lemma 2.3], which now still holds and we have $\|\eta_{i,j}^g\| \leq n \cdot \|\Omega_{i,j}^g\|$, where $\Omega_{i,j}^g$ denotes the curvature of $\eta_{i,j}^g$. The right hand side of the above inequality is independent of $g \in \pi$. Thus, our control of the curvatures carries over to an upper bound on the local connection 1-forms. The trivializations $\psi_{i,j}$ induce transition maps

$$\psi_{i,(j,k)}: (U_j \cap U_k) \times \pi \rightarrow U(t_i C_i t_i)$$

and the upper bound on the local connection 1-forms yields an upper bound on the norm of the derivative $D_{(x,g)} \psi_{i,(j,k)}$ just as described in [11, Lemma 2.5, Proposition 2.6] proving Lipschitz continuity of the transition functions. The Lipschitz condition on the twisting maps ensures that the product of the γ_i^g is continuous, when considered as an element in $C(\widetilde{M}, \text{Hom}(E_L, g^* E_L))$. Thus, E_L is a continuous twisted Hilbert A -module bundle. Note that $\pi = \pi_1(M)$ acts via the adjoint action unitarily on C_i and we set $\mathcal{A} = \widetilde{M} \times_{\text{Ad}} C_i$. E_L corresponds to a projection $t_L \in C(M, \mathcal{A})$, which we can approximate by a projection in $C^\infty(M, \mathcal{A})$ to obtain a smooth twisted Hilbert A -module bundle E by the construction given in Theorem 2.10. We have $E \cdot A_i \cong E_L \cdot A_i = E_i$. The isomorphism may only be continuous, but it can be smoothed.

To construct the connection ∇^E we only need to give local connection 1-forms over the sets $\phi_j(U_j \times \{1\}) \subset \widetilde{M}$ and extend them equivariantly via γ^g to get connection forms over the images of $U_j \times \pi$, which can be patched together with a partition of unity on M . The construction takes

the local forms of the E_i and uses a convolution argument to get a smooth form on the product. This is exactly the same as in [11]. \square

The twisting γ^g maps the subbundle $E \cdot A'$ into itself, therefore the quotient $\mathcal{W} = E/(E \cdot A')$ is a smooth twisted Hilbert Q -module bundle equipped with a *flat* connection ∇^Q and typical fiber tQ for some projection $t \in Q$. If we now fix a basepoint $\tilde{m} \in \tilde{M}$, we get a projective holonomy representation in the sense of Definition 2.18: $(\pi, c_{\hat{\pi}}) \rightarrow \text{End}(\mathcal{W}_{\tilde{m}}) = tQt$. By the universal property of the maximal twisted group C^* -algebra, this extends to a $*$ -homomorphism $\phi: C_{\max}^*(\hat{\pi} \rightarrow \pi) \rightarrow Q$.

As a consequence of Corollary 3.9, the induced map $\phi_*: K_0(C_{\max}^*(\hat{\pi} \rightarrow \pi)) \rightarrow K_0(Q)$ maps $\theta^{\max}(M)$ to $\text{ind}(D_+^{\mathcal{W}'})$, where

$$\mathcal{W}' = \tilde{M} \times tQ = \tilde{M} \times \mathcal{W}_{\tilde{m}} .$$

Using parallel transport, its equivariance as described before Definition 2.18 with respect to γ^g and flatness of Q we see that \mathcal{W}' is isomorphic to \mathcal{W} as a twisted Hilbert Q -module bundle.

Theorem 4.8. *Let M be a closed smooth orientable even-dimensional manifold with $\dim(M) \geq 3$ and \tilde{M} spin that is enlargeable in the sense of Definition 4.1. Then we have*

$$\theta^{\max}(M) \neq 0 \in K_0(C_{\max}^*(\hat{\pi} \rightarrow \pi)) .$$

Proof. As we saw above, we have $\phi_*(\theta^{\max}(M)) = \text{ind}(D_+^{\mathcal{W}'}) \in K_0(Q)$. By [12], the group $K_0(Q)$ splits off a summand

$$\prod_{i \in \mathbb{N}} K_0(\mathbb{K}) / \bigoplus_{i \in \mathbb{N}} K_0(\mathbb{K}) \cong \prod_{i \in \mathbb{N}} \mathbb{Z} / \bigoplus_{i \in \mathbb{N}} \mathbb{Z}$$

and the image of $\text{ind}(D_+^{\mathcal{W}'})$ in the latter group corresponds to the sequence

$$z_i = \left[p_* \left(\text{ind}(D_+^{E \cdot A_i}) \right) \right] = \left[p_* \left(\text{ind}(D_+^{E_i}) \right) \right] ,$$

where $p: C_i \rightarrow \mathbb{K}$ is the projection. By Theorem 4.5 it has only non-vanishing entries. \square

Remark 4.9. Relaxing the condition about the orientability of M requires incorporating orientation twists of K -theory into the setup, which can be seen as a special case of twisted $\mathbb{Z}/2\mathbb{Z}$ -equivariant K -theory as has been observed by Karoubi [14, Remark 6.16], [13]. These can also be described by gerbes (see the Jandl gerbes in [8] and the functor $K_{\pm}(X)$ in [1], which is naturally equivalent with Karoubi's definition), therefore the above argument generalizes to closed non-orientable manifolds as well. Nevertheless, it seems to be impossible to drop the spin condition for the covers \tilde{M} in the definition of enlargeability, since our construction of a projective representation with the right cocycle relies on it.

REFERENCES

- [1] Michael Atiyah and Michael Hopkins. A variant of K -theory: K_{\pm} . In *Topology, geometry and quantum field theory*, volume 308 of *London Math. Soc. Lecture Note Ser.*, pages 5–17. Cambridge Univ. Press, Cambridge, 2004. 19
- [2] Michael Atiyah and Graeme Segal. Twisted K -theory. *Ukr. Mat. Visn.*, 1(3):287–330, 2004. 1

- [3] Bruce Blackadar. *K-theory for operator algebras*, volume 5 of *Mathematical Sciences Research Institute Publications*. Cambridge University Press, Cambridge, second edition, 1998. [11](#), [12](#)
- [4] Peter Bouwknegt, Alan L. Carey, Varghese Mathai, Michael K. Murray, and Danny Stevenson. Twisted K -theory and K -theory of bundle gerbes. *Comm. Math. Phys.*, 228(1):17–45, 2002. [2](#), [4](#), [5](#), [8](#)
- [5] Robert C. Busby and Harvey A. Smith. Representations of twisted group algebras. *Trans. Amer. Math. Soc.*, 149:503–537, 1970. [7](#)
- [6] Alan L. Carey and Bai-Ling Wang. Thom isomorphism and push-forward map in twisted K -theory. *J. K-Theory*, 1(2):357–393, 2008. [2](#)
- [7] P. Donovan and M. Karoubi. Graded Brauer groups and K -theory with local coefficients. *Inst. Hautes Études Sci. Publ. Math.*, (38):5–25, 1970. [1](#), [2](#)
- [8] Jürgen Fuchs, Thomas Nikolaus, Christoph Schweigert, and Konrad Waldorf. Bundle gerbes and surface holonomy. In *European Congress of Mathematics*, pages 167–195. Eur. Math. Soc., Zürich, 2010. [19](#)
- [9] Mikhael Gromov and H. Blaine Lawson, Jr. Spin and scalar curvature in the presence of a fundamental group. I. *Ann. of Math. (2)*, 111(2):209–230, 1980. [2](#)
- [10] Mikhael Gromov and H. Blaine Lawson, Jr. Positive scalar curvature and the Dirac operator on complete Riemannian manifolds. *Inst. Hautes Études Sci. Publ. Math.*, (58):83–196 (1984), 1983. [2](#)
- [11] Bernhard Hanke and Thomas Schick. Enlargeability and index theory. *J. Differential Geom.*, 74(2):293–320, 2006. [1](#), [2](#), [13](#), [14](#), [18](#), [19](#)
- [12] Bernhard Hanke and Thomas Schick. Enlargeability and index theory: infinite covers. *K-Theory*, 38(1):23–33, 2007. [2](#), [13](#), [14](#), [15](#), [16](#), [17](#), [19](#)
- [13] Max Karoubi. Algèbres de Clifford et K -théorie. *Ann. Sci. École Norm. Sup. (4)*, 1:161–270, 1968. [19](#)
- [14] Max Karoubi. Twisted K -theory—old and new. In *K-theory and noncommutative geometry*, EMS Ser. Congr. Rep., pages 117–149. Eur. Math. Soc., Zürich, 2008. [1](#), [19](#)
- [15] E. C. Lance. *Hilbert C^* -modules*, volume 210 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1995. A toolkit for operator algebraists. [2](#)
- [16] H. Blaine Lawson, Jr. and Marie-Louise Michelsohn. *Spin geometry*, volume 38 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1989. [12](#)
- [17] Varghese Mathai, Michael K. Murray, and Danny Stevenson. Type-I D-branes in an H -flux and twisted KO -theory. *J. High Energy Phys.*, (11):053, 23 pp. (electronic), 2003. [2](#), [3](#), [4](#)
- [18] A. S. Miščenko and A. T. Fomenko. The index of elliptic operators over C^* -algebras. *Izv. Akad. Nauk SSSR Ser. Mat.*, 43(4):831–859, 1967, 1979. [12](#)
- [19] M. K. Murray. Bundle gerbes. *J. London Math. Soc. (2)*, 54(2):403–416, 1996. [2](#), [3](#), [4](#)
- [20] Michael K. Murray. An introduction to bundle gerbes. In *The many facets of geometry*, pages 237–260. Oxford Univ. Press, Oxford, 2010. [2](#), [3](#)

- [21] Michael K. Murray and Michael A. Singer. Gerbes, Clifford modules and the index theorem. *Ann. Global Anal. Geom.*, 26(4):355–367, 2004. [2](#), [8](#)
- [22] Jonathan Rosenberg. C^* -algebras, positive scalar curvature, and the Novikov conjecture. III. *Topology*, 25(3):319–336, 1986. [1](#)
- [23] Jonathan Rosenberg. Manifolds of positive scalar curvature: a progress report. In *Surveys in differential geometry. Vol. XI*, volume 11 of *Surv. Differ. Geom.*, pages 259–294. Int. Press, Somerville, MA, 2007. [1](#)
- [24] Thomas Schick. A counterexample to the (unstable) Gromov-Lawson-Rosenberg conjecture. *Topology*, 37(6):1165–1168, 1998. [1](#)
- [25] Thomas Schick. L^2 -index theorems, KK -theory, and connections. *New York J. Math.*, 11:387–443 (electronic), 2005. [9](#), [13](#)
- [26] Stephan Stolz. Concordance classes of positive scalar curvature metrics. *preprint*. [1](#), [6](#), [7](#)
- [27] Stephan Stolz. Simply connected manifolds of positive scalar curvature. *Ann. of Math. (2)*, 136(3):511–540, 1992. [1](#)
- [28] Konrad Waldorf. More morphisms between bundle gerbes. *Theory Appl. Categ.*, 18:No. 9, 240–273, 2007. [4](#)

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