A DIXMIER-DOUADY THEORY FOR STRONGLY SELF-ABSORBING C^*-ALGEBRAS II: THE BRAUER GROUP

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Abstract. We have previously shown that the isomorphism classes of orientable locally trivial fields of C^*-algebras over a compact metrizable space X with fiber $D \otimes \mathbb{K}$, where D is a strongly self-absorbing C^*-algebra, form an abelian group under the operation of tensor product. Moreover this group is isomorphic to the first group $\bar{E}^1_D(X)$ of the (reduced) generalized cohomology theory associated to the unit spectrum of topological K-theory with coefficients in D. Here we show that all the torsion elements of the group $\bar{E}^1_D(X)$ arise from locally trivial fields with fiber $D \otimes M_n(\mathbb{C})$, $n \geq 1$, for all known examples of strongly self-absorbing C^*-algebras D. Moreover the Brauer group generated by locally trivial fields with fiber $D \otimes M_n(\mathbb{C})$, $n \geq 1$ is isomorphic to $Tor(\bar{E}^1_D(X))$.

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1. Introduction

Let X be a compact metrizable space. Let $\mathbb{K}$ denote the C^*-algebra of compact operators on an infinite dimensional separable Hilbert space. It is well-known that $\mathbb{K} \otimes \mathbb{K} \cong \mathbb{K}$ and $M_n(\mathbb{C}) \otimes \mathbb{K} \cong \mathbb{K}$. Dixmier and Douady [7] showed that the isomorphism classes of locally trivial fields of C^*-algebras over X with fiber $\mathbb{K}$ form an abelian group under the operation of tensor product over $C(X)$ and this group is isomorphic to $H^3(X,\mathbb{Z})$. The torsion subgroup of $H^3(X,\mathbb{Z})$ admits the following description. Each element of $Tor(H^3(X,\mathbb{Z}))$ arises as the Dixmier-Douady class of a field A which is isomorphic to the stabilization $B \otimes \mathbb{K}$ of some locally trivial field of C^*-algebras B over X with all fibers isomorphic to $M_n(\mathbb{C})$ for some integer $n \geq 1$, see [8], [1].

In this paper we generalize this result to locally trivial fields with fiber $D \otimes \mathbb{K}$ where D is a strongly self-absorbing C^*-algebra [17]. For a C^*-algebra B, we denote by $\mathscr{C}_B(X)$ the isomorphism classes of locally trivial continuous fields of C^*-algebras over X with fibers isomorphic to B. The isomorphism classes of orientable locally trivial continuous fields is denoted by $\mathscr{C}^0_B(X)$, see Definition 2.2. We have shown in [4] that $\mathscr{C}_{D\otimes\mathbb{K}}(X)$ is an abelian group under the operation of tensor product over $C(X)$, and moreover, this group is isomorphic to the first group $E^1_D(X)$ of a generalized cohomology theory $E^*_D(X)$ which we have proven to be isomorphic to the theory associated to the unit spectrum of topological K-theory with coefficients in D, see [5]. Similarly $(\mathscr{C}^0_{D\otimes\mathbb{K}}(X), \otimes) \cong \bar{E}^1_D(X)$ where $\bar{E}^*_D(X)$ is the reduced theory associated to $E^*_D(X)$. For $D = \mathbb{C}$, we have, of course, $E^1_{\mathbb{C}}(X) \cong H^3(X,\mathbb{Z})$.

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We consider the stabilization map $\sigma : \mathscr{C}_{D\otimes M_n(\mathbb{C})}(X) \to (\mathscr{C}_{D\otimes\mathbb{K}}(X), \otimes) \cong E^1_D(X)$ given by $[A] \mapsto [A \otimes \mathbb{K}]$ and show that its image consists entirely of torsion elements. Moreover, if D is any of the known strongly self-absorbing C^*-algebras, we show that the stabilization map

$$\sigma : \bigcup_{n\geq 1} \mathscr{C}_{D\otimes M_n(\mathbb{C})}(X) \to Tor(\bar{E}^1_D(X))$$

is surjective, see Theorem 2.10. In this situation $\mathscr{C}_{D\otimes M_n(\mathbb{C})}(X) \cong \mathscr{C}^0_{D\otimes M_n(\mathbb{C})}(X)$ by Lemma 2.2 and hence the image of the stabilization map is contained in the reduced group $\bar{E}^1_D(X)$. In analogy with the classic Brauer group generated by continuous fields of complex matrices $M_n(\mathbb{C})$ [8], we introduce a Brauer group $Br_D(X)$ for locally trivial fields of C*-algebras with fibers $M_n(D)$ for D a strongly self-absorbing C^*-algebra and establish an isomorphism $Br_D(X) \cong Tor(\bar{E}^1_D(X))$, see Theorem 2.15.

Our proof is new even in the classic case $D = \mathbb{C}$ whose original proof relies on an argument of Serre, see [8, Thm.1.6], [1, Prop.2.1]. In the cases $D = \mathcal{Z}$ or $D = \mathcal{O}_\infty$ the group $\bar{E}^1_D(X)$ is isomorphic to $H^1(X, BSU_\otimes)$, which appeared in [20], where its equivariant counterpart played a central role.

We introduced in [4] characteristic classes

$$\delta_0 : E^1_D(X) \to H^1(X, K_0(D)^\times_+) \quad \text{and} \quad \delta_k : E^1_D(X) \to H^{2k+1}(X, \mathbb{Q}), \quad k \geq 1.$$

If X is connected, then $\bar{E}^1_D(X) = \ker(\delta_0)$. We show that an element a belongs $Tor(E^1_D(X))$ if and only if $\delta_0(a)$ is a torsion element and $\delta_k(a) = 0$ for all $k \geq 1$.

In the last part of the paper we show that if A^{op} is the opposite C*-algebra of a locally trivial continuous field A with fiber $D \otimes \mathbb{K}$, then $\delta_k(A^{op}) = (-1)^k\delta_k(A)$ for all $k \geq 0$. This shows that in general $A \otimes A^{op}$ is not isomorphic to a trivial field, unlike what happens in the case $D = \mathbb{C}$. Similar arguments show that in general $[A^{op}]_{Br} \neq -[A]_{Br}$ in $Br_D(X)$ for $A \in \mathscr{C}_{D\otimes M_n(\mathbb{C})}(X)$, see Example 3.5.

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2. Background and main result

The class of strongly self-absorbing C^*-algebras was introduced by Toms and Winter [17]. They are separable unital C^*-algebras D singled out by the property that there exists an isomorphism $D \to D \otimes D$ which is unitarily homotopic to the map $d \mapsto d \otimes 1_D$ [6], [19].

If $n \geq 2$ is a natural number we denote by M_{n^∞} the UHF-algebra $M_n(\mathbb{C})^{\otimes\infty}$. If P is a nonempty set of primes, we denote by M_{P^∞} the UHF-algebra of infinite type $\bigotimes_{p\in P} M_{p^\infty}$. If P is the set of all primes, then M_{P^∞} is the universal UHF-algebra, which we denote by $M_\mathbb{Q}$.

The class $\mathcal{D}_{pi}$ of all purely infinite strongly self-absorbing C^*-algebras that satisfy the Universal Coefficient Theorem in KK-theory (UCT) was completely described in [17]. $\mathcal{D}_{pi}$ consists of the Cuntz algebras $\mathcal{O}_2$, $\mathcal{O}_\infty$ and of all C^*-algebras $M_{P^\infty} \otimes \mathcal{O}_\infty$ with P an arbitrary set of primes. Let $\mathcal{D}_{qd}$ denote the class of strongly self-absorbing C^*-algebras which satisfy the UCT and which are quasidiagonal. A complete description of $\mathcal{D}_{qd}$ has become possible due to the recent results of Matui and Sato [13, Cor. 6.2] that build on results of Winter [18], and Lin and Niu [12]. Thus $\mathcal{D}_{qd}$ consists of $\mathbb{C}$, the Jiang-Su algebra $\mathcal{Z}$ and all UHF-algebras M_{P^∞} with P an arbitrary set of primes.

The class $\mathcal{D} = \mathcal{D}_{qd} \cup \mathcal{D}_{pi}$ contains all known examples of strongly self-absorbing C^*-algebras. It is closed under tensor products. If D is strongly self-absorbing, then $K_0(D)$ is a unital commutative ring. The group of positive invertible elements of $K_0(D)$ is denoted by $K_0(D)_+^\times$.

Let B be a C^*-algebra. We denote by $\mathrm{Aut}_0(B)$ the path component of the identity of $\mathrm{Aut}(B)$ endowed with the point-norm topology. Recall that we denote by $\mathscr{C}_B(X)$ the isomorphism classes of locally trivial continuous fields over X with fibers isomorphic to B. The structure group of $A \in \mathscr{C}_B(X)$ is $\mathrm{Aut}(B)$, and A is in fact given by a principal $\mathrm{Aut}(B)$-bundle which is determined up to an isomorphism by an element of the homotopy classes of continuous maps from X to the classifying space of the topological group $\mathrm{Aut}(B)$, denoted by $[X, B\mathrm{Aut}(B)]$.

Definition 2.1. A locally trivial continuous field A of C^*-algebras with fiber B is *orientable* if its structure group can be reduced to $\mathrm{Aut}_0(B)$, in other words if A is given by an element of $[X, B\mathrm{Aut}_0(B)]$.

The corresponding isomorphism classes of orientable and locally trivial fields is denoted by $\mathscr{C}_B^0(X)$.

Lemma 2.2. *Let D be a strongly self-absorbing C^*-algebra satisfying the UCT. Then* $\mathrm{Aut}(M_n(D)) = \mathrm{Aut}_0(M_n(D))$ *for all $n \geq 1$ and hence $\mathscr{C}_{D\otimes M_n(\mathbb{C})}(X) \cong \mathscr{C}^0_{D\otimes M_n(\mathbb{C})}(X)$.*

Proof. First we show that for any $\beta \in \mathrm{Aut}(D \otimes M_n(\mathbb{C}))$ there exist $\alpha \in \mathrm{Aut}(D)$ and a unitary $u \in D \otimes M_n(\mathbb{C})$ such that $\beta = u(\alpha \otimes \mathrm{id}_{M_n(\mathbb{C})})u^*$. Let $e_{11} \in M_n(\mathbb{C})$ be the rank-one projection that appears in the canonical matrix units (e_{ij}) of $M_n(\mathbb{C})$ and let 1_n be the unit of $M_n(\mathbb{C})$. Then $n[1_D \otimes e_{11}] = [1_D \otimes 1_n]$ in $K_0(D)$ and hence $n[\beta(1_D \otimes e_{11})] = n[1_D \otimes e_{11}]$ in $K_0(D)$. Under the assumptions of the lemma, it is known that $K_0(D)$ is torsion free (by [17]) and that D has cancellation of full projections by [19] and [15]. It follows that there is a partial isometry $v \in D \otimes M_n(\mathbb{C})$ such that $v^*v = 1_D \otimes e_{11}$ and $vv^* = \beta(1_D \otimes e_{11})$. Then $u = \sum_{i=1}^n \beta(1_D \otimes e_{i1})v(1_D \otimes e_{1i}) \in D \otimes M_n(\mathbb{C})$ is a unitary such that the automorphism $u^*\beta\, u$ acts identically on $1_D \otimes M_n(\mathbb{C})$. It follows that $u^*\beta\, u = \alpha \otimes \mathrm{id}_{M_n(\mathbb{C})}$ for some $\alpha \in \mathrm{Aut}(D)$. Since both $U(D \otimes M_n(\mathbb{C}))$ and $\mathrm{Aut}(D)$ are path connected by [17], [15] and respectively [6] we conclude that $\mathrm{Aut}(D \otimes M_n(\mathbb{C}))$ is path-connected as well. □

Let us recall the following results contained in Cor. 3.7, Thm. 3.8 and Cor. 3.9 from [4]. Let D be a strongly self-absorbing C^*-algebra.

(1) The classifying spaces $B\mathrm{Aut}(D \otimes \mathbb{K})$ and $B\mathrm{Aut}_0(D \otimes \mathbb{K})$ are infinite loop spaces giving rise to generalized cohomology theories $E_D^*(X)$ and respectively $\bar{E}_D^*(X)$.
(2) The monoid $(\mathscr{C}_{D\otimes\mathbb{K}}(X), \otimes)$ is an abelian group isomorphic to $E_D^1(X)$. Similarly, the monoid $(\mathscr{C}^0_{D\otimes\mathbb{K}}(X), \otimes)$ is a group isomorphic to $\bar{E}_D^1(X)$. In both cases the tensor product is understood to be over $C(X)$.
(3) $E^1_{M_\mathbb{Q}}(X) \cong H^1(X, \mathbb{Q}_+^\times) \oplus \bigoplus_{k\geq 1} H^{2k+1}(X, \mathbb{Q})$,
$E^1_{M_\mathbb{Q}\otimes\mathcal{O}_\infty}(X) \cong H^1(X, \mathbb{Q}^\times) \oplus \bigoplus_{k\geq 1} H^{2k+1}(X, \mathbb{Q})$,
(4) $\bar{E}^1_{M_\mathbb{Q}\otimes\mathcal{O}_\infty}(X) \cong \bar{E}^1_{M_\mathbb{Q}\otimes\mathcal{O}_\infty}(X) \cong \bigoplus_{k\geq 1} H^{2k+1}(X, \mathbb{Q})$.
(5) If D satisfies the UCT then $D \otimes M_\mathbb{Q} \otimes \mathcal{O}_\infty \cong M_\mathbb{Q} \otimes \mathcal{O}_\infty$, by [17]. Therefore the tensor product operation $A \mapsto A \otimes M_\mathbb{Q} \otimes \mathcal{O}_\infty$ induces maps

$$\mathscr{C}_{D\otimes\mathbb{K}}(X) \to \mathscr{C}_{M_\mathbb{Q}\otimes\mathcal{O}_\infty\otimes\mathbb{K}}(X), \quad \mathscr{C}^0_{D\otimes\mathbb{K}}(X) \to \mathscr{C}^0_{M_\mathbb{Q}\otimes\mathcal{O}_\infty\otimes\mathbb{K}}(X) \quad \text{and hence maps}$$

$$E^1_D(X) \xrightarrow{\delta} E^1_{M_{\mathbb{Q}}\otimes\mathcal{O}_\infty}(X) \cong H^1(X,\mathbb{Q}^\times) \oplus \bigoplus_{k\geq 1} H^{2k+1}(X,\mathbb{Q}),$$
$$\delta(A) = (\delta_0^s(A), \delta_1(A), \delta_2(A), \dots), \quad \delta_k(A) \in H^{2k+1}(X,\mathbb{Q}),$$

$$\bar{E}^1_D(X) \xrightarrow{\bar{\delta}} \bar{E}^1_{M_{\mathbb{Q}}\otimes\mathcal{O}_\infty}(X) \cong \bigoplus_{k\geq 1} H^{2k+1}(X,\mathbb{Q}),$$
$$\bar{\delta}(A) = (\delta_1(A), \delta_2(A), \dots), \quad \delta_k(A) \in H^{2k+1}(X,\mathbb{Q}).$$

The invariants $\delta_k(A)$ are called the rational characteristic classes of the continuous field A, see [4, Def.4.6]. The first class δ_0^s lifts to a map $\delta_0 : E^1_D(X) \to H^1(X, K_0(D)^\times_+)$ induced by the morphism of groups $\mathrm{Aut}(D\otimes\mathbb{K}) \to \pi_0(\mathrm{Aut}(D\otimes\mathbb{K})) \cong K_0(D)^\times_+$. $\delta_0(A)$ represents the obstruction to reducing the structure group of A to $\mathrm{Aut}_0(D\otimes\mathbb{K})$.

Proposition 2.3. *A continuous field $A \in \mathscr{C}_{D\otimes\mathbb{K}}(X)$ is orientable if and only if $\delta_0(A) = 0$. If X is connected, then $\bar{E}^1_D(X) \cong \ker(\delta_0)$.*

Proof. Let us recall from [4, Cor. 2.19] that there is an exact sequence of topological groups

$$1 \to \mathrm{Aut}_0(D\otimes\mathbb{K}) \to \mathrm{Aut}(D\otimes\mathbb{K}) \xrightarrow{\pi} K_0(D)^\times_+ \to 1. \tag{1}$$

The map π takes an automorphism α to $[\alpha(1_D \otimes e)]$ where $e \in \mathbb{K}$ is a rank-one projection. If G is a topological group and H is a normal subgroup of G such that $H \to G \to G/H$ is a principal H-bundle, then there is a homotopy fibre sequence $G/H \to BH \to BG \to B(G/H)$ and hence an exact sequence of pointed sets $[X, G/H] \to [X, BH] \to [X, BG] \to [X, B(G/H)]$. In particular, in the case of the fibration (1) we obtain

$$[X, K_0(D)^\times_+] \to [X, B\mathrm{Aut}_0(D\otimes\mathbb{K})] \to [X, B\mathrm{Aut}(D\otimes\mathbb{K})] \xrightarrow{\delta_0} H^1(X, K_0(D)^\times_+). \tag{2}$$

A continuous field $A \in \mathscr{C}^0_{D\otimes\mathbb{K}}(X)$ is associated to a principal $\mathrm{Aut}(D\otimes\mathbb{K})$-bundle whose classifying map gives a unique element in $[X, B\mathrm{Aut}(D\otimes\mathbb{K})]$ whose image in $H^1(X, K_0(D)^\times_+)$ is denoted by $\delta_0(A)$. It is clear from (2) that the class $\delta_0(A) \in H^1(X, K_0(D)^\times_+)$ represents the obstruction for reducing this bundle to a principal $\mathrm{Aut}_0(D\otimes\mathbb{K})$-bundle. If X is connected, $[X, K_0(D)^\times_+] = \{*\}$ and hence $\bar{E}^1_D(X) \cong \ker(\delta_0)$. □

Remark 2.4. If $D = \mathbb{C}$ or $D = \mathcal{Z}$ then A is automatically orientable since in those cases $K_0(D)^\times_+$ is the trivial group.

Remark 2.5. Let Y be a compact metrizable space and let $X = \Sigma Y$ be the suspension of Y. Since the rational Künneth isomorphism and the Chern character on $K^0(X)$ are compatible with the ring structure on $K_0(C(Y)\otimes D)$, we obtain a ring homomorphism

$$\mathrm{ch}\colon K_0(C(Y)\otimes D) \to K^0(Y)\otimes K_0(D)\otimes\mathbb{Q} \to \prod_{k=0}^{\infty} H^{2k}(Y,\mathbb{Q}) =: H^{\mathrm{ev}}(Y,\mathbb{Q}) \ ,$$

which restricts to a group homomorphism $\mathrm{ch}\colon \bar{E}^0_D(Y) \to SL_1(H^{\mathrm{ev}}(Y,\mathbb{Q}))$, where the right hand side denotes the units, which project to $1 \in H^0(Y,\mathbb{Q})$. If A is an orientable locally trivial continuous field with fiber $D\otimes\mathbb{K}$ over X, then we have

$$\delta_k(A) = \log\mathrm{ch}(f_A) \in H^{2k}(Y,\mathbb{Q}) \cong H^{2k+1}(X,\mathbb{Q}) \ , \tag{3}$$

where $f_A\colon Y \to \Omega B\mathrm{Aut}_0(D\otimes\mathbb{K}) \simeq \mathrm{Aut}_0(D\otimes\mathbb{K})$ is induced by the transition map of A. The homomorphism $\log\colon SL_1(H^{\mathrm{ev}}(Y,\mathbb{Q})) \to H^{\mathrm{ev}}(Y,\mathbb{Q})$ is the rational logarithm from [14, Section 2.5]. For the proof of (3) it suffices to treat the case $D = M_{\mathbb{Q}}\otimes\mathcal{O}_\infty$, where it can be easily checked on the level of homotopy groups, but since $\bar{E}^0_D(Y)$ and $H^{\mathrm{ev}}(Y,\mathbb{Q})$ have rational vector spaces as coefficients this is enough.

Lemma 2.6. *Let D be a strongly self-absorbing C^*-algebra in the class $\mathcal{D}$. If $p \in D\otimes\mathbb{K}$ is a projection such that $[p]\neq 0$ in $K_0(D)$, then there is an integer $n\geq 1$ such that $[p]\in nK_0(D)^\times_+$. If $[p]\in nK_0(D)^\times_+$, then $p(D\otimes\mathbb{K})p\cong M_n(D)$. Moreover, if $n,m\geq 1$, then $M_n(D)\cong M_m(D)$ if and only if $nK_0(D)^\times_+ = mK_0(D)^\times_+$.*

Proof. Recall that $K_0(D)$ is an ordered unital ring with unit $[1_D]$ and with positive elements $K_0(D)_+$ corresponding to classes of projections in $D\otimes\mathbb{K}$. The group of invertible elements is denoted by $K_0(D)^\times$ and $K_0(D)^\times_+$ consists of classes $[p]$ of projections $p\in D\otimes\mathbb{K}$ such that $[p]\in K_0(D)^\times$. It was shown in [4, Lemma 2.14] that if $p\in D\otimes\mathbb{K}$ is a projection, then $[p]\in K_0(D)^\times_+$ if and only if $p(D\otimes\mathbb{K})p\cong D$. The ring $K_0(D)$ and the group $K_0(D)^\times_+$ are known for all $D\in\mathcal{D}$, [17]. In fact $K_0(D)$ is a unital subring of $\mathbb{Q}$, $K_0(D)_+ = \mathbb{Q}_+\cap K_0(D)$ if $D\in\mathcal{D}_{qd}$ and $K_0(D)_+ = K_0(D)$ if $D\in\mathcal{D}_{pi}$. Moreover:

$$K_0(\mathbb{C})\cong K_0(\mathcal{Z})\cong K_0(\mathcal{O}_\infty)\cong\mathbb{Z},\ K_0(\mathcal{O}_2)=\{0\},$$
$$K_0(M_{P^\infty})\cong K_0(M_{P^\infty}\otimes\mathcal{O}_\infty)\cong\mathbb{Z}[1/P]\cong\bigotimes_{p\in P}\mathbb{Z}[1/p]\cong\{np_1^{k_1}p_2^{k_2}\cdots p_r^{k_r}: p_i\in P, n,k_i\in\mathbb{Z}\},$$
$$K_0(\mathbb{C})^\times_+\cong K_0(\mathcal{Z})^\times_+=\{1\},\ K_0(\mathcal{O}_\infty)^\times_+=\{\pm1\},$$
$$K_0(M_{P^\infty})^\times_+\cong\{p_1^{k_1}p_2^{k_2}\cdots p_r^{k_r}: p_i\in P, k_i\in\mathbb{Z}\}.$$
$$K_0(M_{P^\infty}\otimes\mathcal{O}_\infty)^\times_+\cong\{\pm p_1^{k_1}p_2^{k_2}\cdots p_r^{k_r}: p_i\in P, k_i\in\mathbb{Z}\}.$$

In particular, we see that in all cases $K_0(D)_+ = \mathbb{N}\cdot K_0(D)^\times_+$, which proves the first statement. If $p\in D\otimes\mathbb{K}$ is a projection such that $[p]\in nK_0(D)^\times_+$, then there is a projection $q\in D\otimes\mathbb{K}$ such that $[q]\in K_0(D)^\times_+$ and $[p]=n[q]=[\mathrm{diag}(q,q,\dots,q)]$. Since D has cancellation of full projections, it follows then immediately that $p(D\otimes\mathbb{K})p\cong M_n(D)$ proving the second part.

To show the last part of the lemma, suppose now that $\alpha: D\otimes M_n(\mathbb{C})\to D\otimes M_m(\mathbb{C})$ is a $*$-isomorphism. Let $e\in M_n(\mathbb{C})$ be a rank one projection. Then $\alpha(1_D\otimes e)(D\otimes M_m(\mathbb{C}))\alpha(1_D\otimes e)\cong D$. By [4, Lemma 2.14] it follows that $\alpha_*[1_D]=[\alpha(1_D\otimes e)]\in K_0(D)^\times_+$. Since α is unital, $\alpha_*(n[1_D]) = m[1_D]$ and hence $m[1_D]\in nK_0(D)^\times_+$. This is equivalent to $nK_0(D)^\times_+ = mK_0(D)^\times_+$.

Conversely, suppose that $m[1_D]=nu$ for some $u\in K_0(D)^\times_+$. Let $\alpha\in\mathrm{Aut}(D\otimes\mathbb{K})$ be such that $[\alpha(1_D\otimes e)]=u$. Then $\alpha_*(n[1_D])=nu=m[1_D]$. This implies that α maps a corner of $D\otimes\mathbb{K}$ that is isomorphic to $M_n(D)$ to a corner that is isomorphic to $M_m(D)$. □

Corollary 2.7. *Let $D\in\mathcal{D}$ and let $\theta\colon D\otimes M_{n^r}(\mathbb{C})\to D\otimes M_{n^\infty}$ be a unital inclusion induced by some unital embedding $M_{n^r}(\mathbb{C})\to M_{n^\infty}$, where $n\geq 2, r\geq 0$. Let R be the set of prime factors of n. Then, under the canonical isomorphism $K_0(D\otimes M_{n^r}(\mathbb{C}))\cong K_0(D)$, we have*

$$\theta_*^{-1}(K_0(D\otimes M_{n^\infty})^\times_+) = \bigcup_r rK_0(D)^\times_+ \subset K_0(D)$$

where r runs through the set of all products of the form $\prod_{q\in R} q^{k_q}$, $k_q\in\mathbb{N}\cup\{0\}$.

Proof. From Lemma 2.6 we see that $K_0(D)\cong\mathbb{Z}[1/P]$ for a (possibly empty) set of primes P. The order structure is the one induced by $(\mathbb{Q},\mathbb{Q}_+)$ if D is quasidiagonal or $K_0(D)^+ = \mathbb{Z}[1/P]$ if D is purely infinite. If $R \subseteq P$, then θ induces an isomorphism on K_0 and the statement is true, since θ_* is order preserving and $\mathbb{Z}[1/R]^\times \subseteq K_0(D)^\times$. Thus, we may assume that $R \nsubseteq P$. Let $S = P \cup R$ and thus $K_0(D \otimes M_{n^\infty}) \cong \mathbb{Z}[1/S]$. The map θ_* induces the canonical inclusion $\mathbb{Z}[1/P] \hookrightarrow \mathbb{Z}[1/S]$. We can write $x \in \mathbb{Z}[1/P]$ as

$$x = m \cdot \prod_{p \in P} p^{r_p} \cdot \prod_{q \in R \setminus P} q^{k_q}$$

with $m \in \mathbb{Z}$ relatively prime to all $p \in P$ and $q \in R$, only finitely many $r_p \in \mathbb{Z}$ non-zero and $k_q \in \mathbb{N} \cup \{0\}$. From this decomposition we see that x is invertible in $\mathbb{Z}[1/S]$ if and only if $m = \pm 1$. This concludes the proof since $p^{r_p} \in K_0(D)_+^\times$. □

Remark 2.8. Let $q \in D \otimes \mathbb{K}$ be a projection and let $\alpha \in \mathrm{Aut}(D \otimes \mathbb{K})$. As in [4, Lemma 2.14] we have that $[\alpha(q)] = [\alpha(1 \otimes e)] \cdot [q]$ with $[\alpha(1 \otimes e)] \in K_0(D)_+^\times$. Thus, the condition $[q] \in nK_0(D)_+^\times$ for $n \in \mathbb{N}$ is invariant under the action of $\mathrm{Aut}(D \otimes \mathbb{K})$ on $K_0(D)$. Given $A \in \mathscr{C}_{D \otimes \mathbb{K}}(X)$, a projection $p \in A$, $x_0 \in X$ and an isomorphism $\phi\colon A(x_0) \to D \otimes \mathbb{K}$ the condition $[\phi(p(x_0))] \in nK_0(D)_+^\times$ is independent of ϕ. Abusing the notation we will write this as $[p(x_0)] \in nK_0(D)_+^\times$.

Corollary 2.9. *Let $D \in \mathcal{D}$ and let $A \in \mathscr{C}_{D \otimes \mathbb{K}}(X)$ with X a connected compact metrizable space. If $p \in A$ is a projection such that $[p(x_0)] \in nK_0(D)_+^\times$ for some point x_0, then $(pAp)(x) \cong M_n(D)$ for all $x \in X$ and hence $pAp \in \mathscr{C}_{D \otimes M_n(\mathbb{C})}(X)$. If $p \in A$ is a projection with $[p(x_0)] \in K_0(D) \setminus \{0\}$, then $[p(x_0)] \in nK_0(D)_+^\times$ for some $n \in \mathbb{N}$.*

Proof. Let $V_1, .., V_k$ be a finite cover of X by compact sets such that there are bundle isomorphisms $\phi_i : A(V_i) \cong C(V_i) \otimes D \otimes \mathbb{K}$. Let p_i be the image of the restriction of p to V_i under ϕ_i. After refining the cover (V_i), if necessary, we may assume that $\|p_i(x) - p_i(y)\| < 1$ for all $x, y \in V_i$. This allows us to find a unitary u_i in the multiplier algebra of $C(V_i) \otimes D \otimes \mathbb{K}$ such that after replacing ϕ_i by $u_i \phi_i u_i^*$ and p_i by $u_i p_i u_i^*$, we may assume that p_i are constant projections. Since X is connected and $[p(x_0)] \in nK_0(D)_+^\times$ by assumption, it follows from $[p_i(x_0)] \in nK_0(D)_+^\times$ for $x_0 \in V_i$ and the above remark that $[p_j(x)] \in nK_0(D)_+^\times$ for all $1 \leq j \leq k$ and all $x \in V_j$. Then Lemma 2.6 implies $(pAp)(V_j) \cong C(V_j) \otimes M_n(D)$. By Lemma 2.6 we also have that $[p(x_0)] \neq 0$ implies $[p(x_0)] \in nK_0(D)_+^\times$ for some $n \in \mathbb{N}$ proving the statement about the case $[p(x_0)] \in K_0(D) \setminus \{0\}$. □

We study the image of the stabilization map

$$\mathscr{C}_{D \otimes M_n(\mathbb{C})}(X) \to \mathscr{C}_{D \otimes \mathbb{K}}(X)$$

induced by the map $A \mapsto A \otimes \mathbb{K}$, or equivalently by the map

$$\mathrm{Aut}(D \otimes M_n(\mathbb{C})) \to \mathrm{Aut}(D \otimes M_n(\mathbb{C}) \otimes \mathbb{K}) \cong \mathrm{Aut}(D \otimes \mathbb{K}).$$

Let us recall that $\mathcal{D}$ denotes the class of strongly self-absorbing C^*-algebras which satisfy the UCT and which are either quasidiagonal or purely infinite.

Theorem 2.10. *Let D be a strongly self-absorbing C^*-algebra in the class $\mathcal{D}$. Let A be a locally trivial continuous field of C^*-algebras over a connected compact metrizable space X such that $A(x) \cong D \otimes \mathbb{K}$ for all $x \in X$. The following assertions are equivalent:*

(1) *$\delta_k(A) = 0$ for all $k \geq 0$.*
(2) *The field $A \otimes M_{\mathbb{Q}}$ is trivial.*

(3) *There is an integer $n \geq 1$ and a unital locally trivial continuous field $\mathcal{B}$ over X with all fibers isomorphic to $M_n(D)$ such that $A \cong \mathcal{B} \otimes \mathbb{K}$.*
(4) *A is orientable and $A^{\otimes m} \cong C(X) \otimes D \otimes \mathbb{K}$ for some $m \in \mathbb{N}$.*

Proof. The statement is immediately verified if $D \cong \mathcal{O}_2$. Indeed all locally trivial fields with fiber $\mathcal{O}_2 \otimes \mathbb{K}$ are trivial since $\mathrm{Aut}(\mathcal{O}_2 \otimes \mathbb{K})$ is contractible by [4, Cor. 17 & Thm. 2.17]. For the remainder of the proof we may therefore assume that $D \not\cong \mathcal{O}_2$.

(1) $\Leftrightarrow$ (2) If $D \in \mathcal{D}_{qd}$, then it is known that $D \otimes M_{\mathbb{Q}} \cong M_{\mathbb{Q}}$. Similarly, if $D \in \mathcal{D}_{pi}$ and $D \not\cong \mathcal{O}_2$ then $D \otimes M_{\mathbb{Q}} \cong \mathcal{O}_\infty \otimes M_{\mathbb{Q}}$. If A is as in the statement, then $A \otimes M_{\mathbb{Q}}$ is a locally trivial field whose fibers are all isomorphic to either $M_{\mathbb{Q}} \otimes \mathbb{K}$ or to $\mathcal{O}_\infty \otimes M_{\mathbb{Q}} \otimes \mathbb{K}$. In either case, it was shown in [4, Cor. 4.5] that such a field is trivial if and only if $\delta_k(A) = 0$ for all $k \geq 0$. As reviewed earlier in this section, this follows from the explicit computation of $E^1_{M_{\mathbb{Q}}}(X)$ and $E^1_{M_{\mathbb{Q}} \otimes \mathcal{O}_\infty}(X)$.

(2) $\Rightarrow$ (3) Assume now that $A \otimes M_{\mathbb{Q}}$ is trivial, i.e. $A \otimes M_{\mathbb{Q}} \cong C(X) \otimes D \otimes M_{\mathbb{Q}} \otimes \mathbb{K}$. Let $p \in A \otimes M_{\mathbb{Q}}$ be the projection that corresponds under this isomorphism to the projection $1 \otimes e \in C(X) \otimes D \otimes M_{\mathbb{Q}} \otimes \mathbb{K}$ where 1 is the unit of the C^*-algebra $C(X) \otimes D \otimes M_{\mathbb{Q}}$ and $e \in \mathbb{K}$ is a rank-one projection. Then $[p(x)] \neq 0$ in $K_0(A(x) \otimes M_{\mathbb{Q}})$ for all $x \in X$ (recall that $D \not\cong \mathcal{O}_2$). Let us write $M_{\mathbb{Q}}$ as the direct limit of an increasing sequence of its subalgebras $M_{k(i)}(\mathbb{C})$. Then $A \otimes M_{\mathbb{Q}}$ is the direct limit of the sequence $A_i = A \otimes M_{k(i)}(\mathbb{C})$. It follows that there exist $i \geq 1$ and a projection $p_i \in A_i$ such that $\|p - p_i\| < 1$. Then $\|p(x) - p_i(x)\| < 1$ and so $[p_i(x)] \neq 0$ in $K_0(A_i(x))$ for each $x \in X$, since its image in $K_0(A(x) \otimes M_{\mathbb{Q}})$ is equal to $[p(x)] \neq 0$. Let us consider the locally trivial unital field $\mathcal{B} := p_i(A \otimes M_{k(i)}(\mathbb{C}))p_i$. Since the fibers of $A \otimes M_{k(i)}(\mathbb{C})$ are isomorphic to $D \otimes \mathbb{K} \otimes M_{k(i)}(\mathbb{C}) \cong D \otimes \mathbb{K}$, it follows by Corollary 2.9 that there is $n \geq 1$ such that all fibers of $\mathcal{B}$ are isomorphic to $M_n(D)$. Since $\mathcal{B}$ is isomorphic to a full corner of $A \otimes \mathbb{K}$, it follows by [3] that $A \otimes \mathbb{K} \cong \mathcal{B} \otimes \mathbb{K}$. We conclude by noting that since A is locally trivial and each fiber is stable, then $A \cong A \otimes \mathbb{K}$ by [9] and so $A \cong \mathcal{B} \otimes \mathbb{K}$.

(3) $\Rightarrow$ (2) This implication holds for any strongly self-absorbing C^*-algebra D. Let A and $\mathcal{B}$ be as in (3). Let us note that $\mathcal{B} \otimes M_{\mathbb{Q}}$ is a unital locally trivial field with all fibers isomorphic to the strongly self-absorbing C^*-algebra $D \otimes M_{\mathbb{Q}}$. Since $\mathrm{Aut}(D \otimes M_{\mathbb{Q}})$ is contractible by [4, Thm. 2.3], it follows that $\mathcal{B} \otimes M_{\mathbb{Q}}$ is trivial. We conclude that $A \otimes M_{\mathbb{Q}} \cong (\mathcal{B} \otimes M_{\mathbb{Q}}) \otimes \mathbb{K} \cong C(X) \otimes D \otimes M_{\mathbb{Q}} \otimes \mathbb{K}$.

(2) $\Leftrightarrow$ (4) This equivalence holds for any strongly self-absorbing C^*-algebra D if A is orientable. In particular we do not need to assume that D satisfies the UCT. In the UCT case we note that since the map $K_0(D) \to K_0(D \otimes M_{\mathbb{Q}})$ is injective, it follows that A is orientable if and only if $A \otimes M_{\mathbb{Q}}$ is orientable, i.e. $\delta_0(A) = 0$ if and only if $\delta_0^s(A) = 0$. Since $\delta_0(A) = 0$, A is determined up to isomorphism by its class $[A] \in \bar{E}^1_D(X)$. To complete the proof it suffices to show that the kernel of the map $\tau : \bar{E}^1_D(X) \to \bar{E}^1_{D \otimes M_{\mathbb{Q}}}(X)$, $\tau[A] = [A \otimes M_{\mathbb{Q}}]$, consists entirely of torsion elements. Consider the natural transformation of cohomology theories:

$$\tau \otimes \mathrm{id}_{\mathbb{Q}} : \bar{E}^*_D(X) \otimes \mathbb{Q} \to \bar{E}^*_{D \otimes M_{\mathbb{Q}}}(X) \otimes \mathbb{Q} \cong \bar{E}^*_{D \otimes M_{\mathbb{Q}}}(X).$$

If $D \neq \mathbb{C}$, it induces an isomorphism on coefficients since $\bar{E}^{-i}_D(pt) = \pi_i(\mathrm{Aut}_0(D \otimes \mathbb{K})) \cong K_i(D)$ by [4, Thm.2.18] and since the map $K_i(D) \otimes \mathbb{Q} \to K_i(D \otimes M_{\mathbb{Q}})$ is bijective. We conclude that the kernel of τ is a torsion group. The same property holds for $D = \mathbb{C}$ since $\bar{E}^*_{\mathbb{C}}(X)$ is a direct summand of $\bar{E}^*_{\mathcal{Z}}(X)$ by [4, Cor.3.8]. $\square$

Theorem 2.11. *Let D, X and A be as in Theorem 2.10 and let $n \geq 2$ be an integer. The following assertions are equivalent:*

1. *The field $A \otimes M_{n^\infty}$ is trivial.*
2. *There is a $k \in \mathbb{N}$ and a unital locally trivial continuous field $\mathcal{B}$ over X with all fibers isomorphic to $M_{n^k}(D)$ such that $A \cong \mathcal{B} \otimes \mathbb{K}$.*
3. *A is orientable and $A^{\otimes n^k} \cong C(X) \otimes D \otimes \mathbb{K}$ for some $k \in \mathbb{N}$.*

Proof. By reasoning as in the proof of Theorem 2.10, we may assume that $D \ncong \mathcal{O}_2$.

(1) $\Rightarrow$ (2): By assumption the continuous field $A \otimes M_{n^\infty}$ is trivializable and hence it satisfies the global Fell condition of [4]. This means that there is a full projection $p_\infty \in A \otimes M_{n^\infty}$ with the property that $p_\infty(x) \in K_0(A(x) \otimes M_{n^\infty})_+^\times$ for all $x \in X$. Let $\nu_i \colon M_{n^i}(\mathbb{C}) \to M_{n^\infty}$ be a unital inclusion map. Since $A \otimes M_{n^\infty}$ is the inductive limit of the sequence

$$A \to A \otimes M_n(\mathbb{C}) \to \cdots \to A \otimes M_{n^i}(\mathbb{C}) \to A \otimes M_{n^{i+1}}(\mathbb{C}) \to \dots$$

there is an $i \in \mathbb{N}$ and a full projection $p \in A \otimes M_{n^i}(\mathbb{C})$ with $\|(\mathrm{id}_A \otimes \nu_i)(p) - p_\infty\| < 1$. Fix a point $x_0 \in X$. Let $\theta \colon A(x_0) \otimes M_{n^i}(\mathbb{C}) \to A(x_0) \otimes M_{n^\infty}$ be the unital inclusion induced by ν_i. Note that $\theta_*([p(x_0)]) = (\mathrm{id}_{A(x_0)} \otimes \nu_i)_*([p(x_0)]) = [p_\infty(x_0)] \in K_0(A(x_0) \otimes M_{n^\infty})_+^\times$. By Corollary 2.7 this implies that $[p(x_0)] \in rK_0(A(x_0))_+^\times$ for some $r \in \mathbb{N}$ that divides n^k for some $k \in \mathbb{N} \cup \{0\}$. Then $\mathcal{B}_0 := p(A \otimes M_{n^i}(\mathbb{C}))p \in \mathscr{C}_{D \otimes M_r(\mathbb{C})}(X)$ by Corollary 2.9. Write $n^k = mr$ with $m \in \mathbb{N}$. It follows that $\mathcal{B} := \mathcal{B}_0 \otimes M_m(\mathbb{C}) \in \mathscr{C}_{D \otimes M_{n^k}(\mathbb{C})}(X)$. The fact that $\mathcal{B} \otimes \mathbb{K} \cong A$ follows just as in step (2) $\Rightarrow$ (3) in the proof of Theorem 2.10.

(2) $\Rightarrow$ (1): This is just the same argument as step (3) $\Rightarrow$ (2) in the proof of Theorem 2.10.

(1) $\Leftrightarrow$ (3): The orientability of A follows from Theorem 2.10. Observe that the elements $[A] \in \mathscr{C}^0_{D \otimes \mathbb{K}}(X) = \bar{E}^1_D(X)$ such that $n^k[A] = 0$ or equivalently $A^{\otimes n^k}$ is trivializable for some $k \in \mathbb{N} \cup \{0\}$ coincide precisely with the elements in the kernel of the group homomorphism $\bar{E}^1_D(X) \to \bar{E}^1_D(X) \otimes \mathbb{Z}[\frac{1}{n}]$. Since $\mathbb{Z}[\frac{1}{n}]$ is flat, it follows that $X \mapsto \bar{E}^*_D(X) \otimes \mathbb{Z}[\frac{1}{n}]$ still satisfies all axioms of a generalized cohomology theory. In particular, we have the following commutative diagram of natural transformations of cohomology theories:

$$\begin{array}{ccc} \bar{E}^*_D(X) & \longrightarrow & \bar{E}^*_{D \otimes M_{n^\infty}}(X) \\ \downarrow & & \downarrow \cong \\ \bar{E}^*_D(X) \otimes \mathbb{Z}[\frac{1}{n}] & \longrightarrow & \bar{E}^*_{D \otimes M_{n^\infty}}(X) \otimes \mathbb{Z}[\frac{1}{n}] \end{array}$$

where the isomorphism on the right hand side can be checked on the coefficients. A similar argument shows that for $D \neq \mathbb{C}$ the bottom homomorphism is an isomorphism. Thus the kernel of the left vertical map agrees with the one of the upper horizontal map in this case. For $D = \mathbb{C}$ we can use that $\bar{E}^*_{\mathbb{C}}(X)$ embeds as a direct summand into $\bar{E}^*_{\mathcal{Z}}(X)$ via the natural $*$-homomorphism $\mathbb{C} \to \mathcal{Z}$ [4, Cor. 4.8]. In particular, $\bar{E}^*_{\mathbb{C}}(X) \otimes \mathbb{Z}[\frac{1}{n}] \to \bar{E}^*_{\mathcal{Z}}(X) \otimes \mathbb{Z}[\frac{1}{n}]$ is injective. $\square$

Corollary 2.12. *Let D and X be as in Theorem 2.10. Then any element $x \in \bar{E}^1_D(X)$ with $nx = 0$ is represented by the stabilization of a unital locally trivial field over X with all fibers isomorphic to $M_{n^k}(D)$ for some $k \geq 1$. Moreover if $A \in \mathscr{C}_{D \otimes \mathbb{K}}(X)$, then $A \otimes M_{\mathbb{Q}}$ is trivial $\Leftrightarrow$ $A \otimes M_{n^\infty}$ is trivial for some $n \in \mathbb{N}$ $\Leftrightarrow$ A is orientable and $n^k[A] = 0$ in $\bar{E}^1_D(X)$ for some $k \in \mathbb{N}$ and some $n \in \mathbb{N}$.*

(An example from [1] for $D = \mathbb{C}$ shows that in general one cannot always arrange that $k = 1$.)

Proof. The first part follows from Theorem 2.11. Indeed, condition (3) of that theorem is equivalent to requiring that A is orientable and $n^k[A] = 0$ in $\bar{E}^1_D(X)$. The second part follows from Theorems 2.10 and 2.11. □

Definition 2.13. Let D be a strongly self-absorbing C^*-algebra. If X is connected compact metrizable space we define the Brauer group $Br_D(X)$ as equivalence classes of continuous fields $A \in \bigcup_{n\geq 1} \mathscr{C}_{M_n(D)}(X)$. Two continuous fields $A_i \in \mathscr{C}_{M_{n_i}(D)}(X)$, $i = 1, 2$ are equivalent, if

$$A_1 \otimes p_1 C(X, M_{N_1}(D))p_1 \cong A_2 \otimes p_2 C(X, M_{N_2}(D))p_2,$$

for some full projections $p_i \in C(X, M_{N_i}(D))$. We denote by $[A]_{Br}$ the class of A in $Br_D(X)$. The multiplication on $Br_D(X)$ is induced by the tensor product operation, after fixing an isomorphism $D \otimes D \cong D$. We will show in a moment that the monoid $Br_D(X)$ is a group.

Remark 2.14. It is worth noting the following two alternative descriptions of the Brauer group. (a) If $D \in \mathcal{D}$ is quasidiagonal, then two continuous fields $A_i \in \mathscr{C}_{M_{n_i}(D)}(X)$, $i = 1, 2$ have equal classes in $Br_D(X)$, if and only if $A_1 \otimes p_1 C(X, M_{N_1}(\mathbb{C}))p_1 \cong A_2 \otimes p_2 C(X, M_{N_2}(\mathbb{C}))p_2$, for some full projections $p_i \in C(X, M_{N_i}(\mathbb{C}))$. (b) If $D \in \mathcal{D}$ is purely infinite, then two continuous fields $A_i \in \mathscr{C}_{M_{n_i}(D)}(X)$, $i = 1, 2$ have equal classes in $Br_D(X)$, if and only if $A_1 \otimes p_1 C(X, M_{N_1}(\mathcal{O}_\infty))p_1 \cong A_2 \otimes p_2 C(X, M_{N_2}(\mathcal{O}_\infty))p_2$, for some full projections $p_i \in C(X, M_{N_i}(\mathcal{O}_\infty))$. In order to justify (a) we observe that if D is quasidiagonal, then every projection $p \in C(X, M_N(D))$ has a multiple $p(m) := p \otimes 1_{M_m}(\mathbb{C})$ such that $p(m)$ is Murray-Von Neumann equivalent to a projection in $C(X, M_{Nm}(\mathbb{C})) \otimes 1_D \subset C(X, M_{Nm}(\mathbb{C})) \otimes D$ and that $A_i \otimes D \cong A_i$ by [9]. For (b) we note that if D is purely infinite, then then every projection $p \in C(X, M_N(D))$ has a multiple $p \otimes 1_{M_m}(\mathbb{C})$ that is Murray-Von Neumann equivalent to a projection in $C(X, M_{Nm}(\mathcal{O}_\infty)) \otimes 1_D$.

One has the following generalization of a result of Serre, [8, Thm.1.6].

Theorem 2.15. *Let D be a strongly self-absorbing C^*-algebra in $\mathcal{D}$.*

(i) $Tor(\bar{E}^1_D(X)) = ker\left(\bar{E}^1_D(X) \xrightarrow{\bar{\delta}} \bigoplus_{k\geq 1} H^{2k+1}(X, \mathbb{Q})\right)$

(ii) *The map $\theta : Br_D(X) \to Tor(\bar{E}^1_D(X))$, $[A]_{Br} \mapsto [A \otimes \mathbb{K}]$ is an isomorphism of groups.*

Proof. (i) was established in the last part of the proof of Theorem 2.10.

(ii) We denote by L_p the continuous field $p\,C(X, M_N(D))p$. Since $L_p \otimes \mathbb{K} \cong C(X, D \otimes \mathbb{K})$ it follows that the map θ is a well-defined morphism of monoids.

We use the following observation. Let $\theta : S \to G$ be a unital surjective morphism of commutative monoids with units denoted by 1. Suppose that G is a group and that $\{s \in S : \theta(s) = 1\} = \{1\}$. Then S is a group and θ is an isomorphism. Indeed if $s \in S$, there is $t \in S$ such that $\theta(t) = \theta(s)^{-1}$ by surjectivity of θ. Then $\theta(st) = \theta(s)\theta(t) = 1$ and so $st = 1$. It follows that S is a group and that θ is injective.

We are going to apply this observation to the map $\theta : Br_D(X) \to Tor(\bar{E}^1_D(X))$. By condition (3) of Theorem 2.10 we see that θ is surjective. Let us determine the set $\theta^{-1}(\{0\})$. We are going to show that if $B \in \mathscr{C}_{D\otimes M_n(\mathbb{C})}(X)$, then $[B \otimes \mathbb{K}] = 0$ in $\bar{E}^1_D(X)$ if and only if

$$B \cong p\,(C(X) \otimes D \otimes M_N(\mathbb{C}))\,p \cong \mathcal{L}_{C(X,D)}(p\,C(X, D)^N)$$

for some selfadjoint projection $p \in C(X) \otimes D \otimes M_N(\mathbb{C}) \cong M_N(C(X,D))$. Let $B \in \mathscr{C}_{D\otimes M_n(\mathbb{C})}(X)$ be such that $[B \otimes \mathbb{K}] = 0$ in $\bar{E}^1_D(X)$. Then there is an isomorphism of continuous fields $\phi : B\otimes\mathbb{K} \xrightarrow{\cong} C(X)\otimes D\otimes\mathbb{K}$. After conjugating ϕ by a unitary we may assume that $p := \phi(1_B\otimes e_{11}) \in C(X) \otimes D \otimes M_N(\mathbb{C})$ for some integer $N \geq 1$. It follows immediately that the projection p has the desired properties. Conversely, if $B \cong p\,(C(X) \otimes D \otimes M_N(\mathbb{C}))\,p$ then there is an isomorphism of continuous fields $B \otimes \mathbb{K} \cong C(X) \otimes D \otimes \mathbb{K}$ by [3]. We have thus shown that that $\theta([B]_{Br}) = 0$ iff and only if $[B]_{Br} = 0$.

We are now able to conclude that $Br_D(X)$ is a group and that θ is injective by the general observation made earlier. □

Definition 2.16. Let D be a strongly self-absorbing C^*-algebra. Let A be a locally trivial continuous field of C^*-algebras with fiber $D \otimes \mathbb{K}$. We say that A is a *torsion continuous field* if $A^{\otimes k}$ is isomorphic to a trivial field for some integer $k \geq 1$.

Corollary 2.17. *Let A be as in Theorem 2.10. Then A is a torsion continuous field if and only if $\delta_0(A) \in H^1(X, K_0(D)^\times_+)$ is a torsion element and $\delta_k(A) = 0 \in H^{2k+1}(X,\mathbb{Q})$ for all $k \geq 1$.*

Proof. Let $m \geq 1$ be an integer such that $m\delta_0(A) = 0$. Then $\delta_0(A^{\otimes m}) = 0$. We conclude by applying Theorem 2.10 to the orientable continuous field $A^{\otimes m}$. □

3. Characteristic classes of the opposite continuous field

Given a C^*-algebra B denote by B^{op} the *opposite C^*-algebra* with the same underlying Banach space and norm, but with multiplication given by $b^{\mathrm{op}}\cdot a^{\mathrm{op}} = (a\cdot b)^{\mathrm{op}}$. The *conjugate C^*-algebra* $\overline{B}$ has the conjugate Banach space as its underlying vector space, but the same multiplicative structure. The map $a \mapsto a^*$ provides an isomorphism $B^{\mathrm{op}} \to \overline{B}$. Any automorphism $\alpha \in \mathrm{Aut}(B)$ yields in a canonical way automorphisms $\bar{\alpha}\colon \overline{B} \to \overline{B}$ and $\alpha^{\mathrm{op}}\colon B^{\mathrm{op}} \to B^{\mathrm{op}}$ compatible with $*\colon B^{\mathrm{op}} \to \overline{B}$. Therefore we have group isomorphisms $\theta\colon \mathrm{Aut}(B) \to \mathrm{Aut}(\overline{B})$ and $\mathrm{Aut}(B) \to \mathrm{Aut}(B^{\mathrm{op}})$. Note that $\alpha \in \mathrm{Aut}(B)$ is equal to $\theta(\alpha)$ when regarded as set-theoretic maps $B \to B$. Given a locally trivial continuous field A with fiber B, we can apply these operations fiberwise to obtain the locally trivial fields A^{op} and $\overline{A}$, which we will call the *opposite* and the *conjugate field*. They are isomorphic to each other and isomorphic to the conjugate and the opposite C^*-algebras of A.

A *real form* of a complex C*-algebra A is a real C*-algebra $A^{\mathbb{R}}$ such that $A \cong A^{\mathbb{R}} \otimes \mathbb{C}$. A real form is not necessarily unique [2] and not all C*-algebras admit real forms [16]. If two C*-algebras A and B admit real forms $A^{\mathbb{R}}$ and $B^{\mathbb{R}}$, then $A^{\mathbb{R}} \otimes_{\mathbb{R}} B^{\mathbb{R}}$ is a real form of $A \otimes B$.

Example 3.1. All known strongly self-absorbing C*-algebras $D \in \mathcal{D}$ admit a real form.

Indeed, the real Cuntz algebras $\mathcal{O}_2^{\mathbb{R}}$ and $\mathcal{O}_\infty^{\mathbb{R}}$ are defined by the same generators and relations as their complex versions. Alternatively $\mathcal{O}_\infty^{\mathbb{R}}$ can be realized as follows. Let $H_{\mathbb{R}}$ be a separable infinite dimensional real Hilbert space and let $\mathcal{F}^{\mathbb{R}}(H_{\mathbb{R}}) = \bigoplus_{n=0}^\infty H_{\mathbb{R}}^{\otimes n}$ be the real Fock space associated to it. Every $\xi \in H_{\mathbb{R}}$ defines a shift operator $s_\xi(\eta) = \xi \otimes \eta$ and we denote the algebra spanned by the s_ξ and their adjoints s_ξ^* by $\mathcal{O}_\infty^{\mathbb{R}}$. If $\mathcal{F}(H_{\mathbb{R}} \otimes \mathbb{C})$ denotes the Fock space associated to the complex Hilbert space $H = H_{\mathbb{R}} \otimes \mathbb{C}$, then we have $\mathcal{F}^{\mathbb{R}} \otimes \mathbb{C} \cong \mathcal{F}(H)$. If we represent $\mathcal{O}_\infty$ on $\mathcal{F}(H)$ using the above construction, then the map $s_\xi + i\, s_{\xi'} \mapsto s_{\xi + i\,\xi'}$ induces an isomorphism $\mathcal{O}_\infty^{\mathbb{R}} \otimes \mathbb{C} \to \mathcal{O}_\infty$. Likewise define $M_{\mathbb{Q}}^{\mathbb{R}}$ to be the infinite tensor product $M_2(\mathbb{R}) \otimes M_3(\mathbb{R}) \otimes M_4(\mathbb{R}) \otimes \ldots$.

Since $M_n(\mathbb{C}) \cong M_n(\mathbb{R})\otimes\mathbb{C}$, we obtain an isomorphism $M_{\mathbb{Q}}^{\mathbb{R}}\otimes\mathbb{C} \cong M_{\mathbb{Q}}$ on the inductive limit. Let $\mathbb{K}^{\mathbb{R}}$ be the compact operators on $H_{\mathbb{R}}$ and $\mathbb{K}$ those on H, then we have $\mathbb{K}^{\mathbb{R}}\otimes\mathbb{C} \cong \mathbb{K}$. Thus, $M_{\mathbb{Q}}\otimes\mathcal{O}_\infty\otimes\mathbb{K}$ is the complexification of the real C^*-algebra $M_{\mathbb{Q}}^{\mathbb{R}} \otimes \mathcal{O}_\infty^{\mathbb{R}} \otimes \mathbb{K}^{\mathbb{R}}$.

The Jiang-Su algebra $\mathcal{Z}$ admits a real form $\mathcal{Z}^{\mathbb{R}}$ which can be constructed in the same way as $\mathcal{Z}$. Indeed, one constructs $\mathcal{Z}^{\mathbb{R}}$ as the inductive limit of a system

$$\cdots \to C([0,1], M_{p_nq_n}(\mathbb{R})) \xrightarrow{\phi_n} C([0,1], M_{p_{n+1}q_{n+1}}(\mathbb{R})) \to \cdots$$

where the connecting maps ϕ_n are defined just as in the proof of [11, Prop. 2.5] with only one modification. Specifically, one can choose the matrices u_0 and u_1 to be in the special orthogonal group $SO(p_nq_n)$ and this will ensure the existence of a continuous path u_t in $O(p_nq_n)$ from u_0 to u_1 as required.

If B is the complexification of a real C^*-algebra $B^{\mathbb{R}}$, then a choice of isomorphism $B \cong B^{\mathbb{R}}\otimes\mathbb{C}$ provides an isomorphism $c\colon B \to \overline{B}$ via complex conjugation on $\mathbb{C}$. On automorphisms we have $\mathrm{Ad}_{c^{-1}}\colon \mathrm{Aut}(\overline{B}) \to \mathrm{Aut}(B)$. Let $\eta = \mathrm{Ad}_{c^{-1}} \circ \theta\colon \mathrm{Aut}(B) \to \mathrm{Aut}(B)$. Now we specialize to the case $B = D \otimes \mathbb{K}$ with $D \in \mathcal{D}$ and study the effect of η on homotopy groups, i.e. $\eta_*\colon \pi_{2k}(\mathrm{Aut}(B)) \to \pi_{2k}(\mathrm{Aut}(B))$. By [4, Theorem 2.18] the groups $\pi_{2k+1}(\mathrm{Aut}(B))$ vanish.

Let R be a commutative ring and denote by $\left[K^0(S^{2k}) \otimes R\right]^\times$ the group of units of the ring $K^0(S^{2k}) \otimes R$. Let $\left[K^0(S^{2k}) \otimes R\right]_1^\times$ be the kernel of the morphism of multiplicative groups $\left[K^0(S^{2k}) \otimes R\right]^\times \to R^\times$. This is the group of virtual rank 1 vector bundles with coefficients in R over S^{2k}. Let $c_S\colon K^0(S^{2k}) \to K^0(S^{2k})$ and $c_R\colon K_0(D) \to K_0(D)$ be the ring automorphisms induced by complex conjugation.

Lemma 3.2. *Let D be a strongly self-absorbing C^*-algebra in the class $\mathcal{D}$, let $R = K_0(D)$ and let $k > 0$. There is an isomorphism $\pi_{2k}(\mathrm{Aut}(D \otimes \mathbb{K})) \to \left[K^0(S^{2k}) \otimes R\right]_1^\times$ ($k > 0$) such that the following diagram commutes*

$$\begin{array}{ccc}
\pi_{2k}(\mathrm{Aut}(D \otimes \mathbb{K})) & \xrightarrow{\eta_*} & \pi_{2k}(\mathrm{Aut}(D \otimes \mathbb{K})) \\
\downarrow & & \downarrow \\
\left[K^0(S^{2k}) \otimes R\right]_1^\times & \xrightarrow{c_S\otimes c_R} & \left[K^0(S^{2k}) \otimes R\right]_1^\times
\end{array}$$

Proof. Observe that $\pi_{2k}(\mathrm{Aut}(D \otimes \mathbb{K})) = \pi_{2k}(\mathrm{Aut}_0(D \otimes \mathbb{K}))$ (for $k > 0$) and $\mathrm{Aut}_0(D \otimes \mathbb{K})$ is a path connected group, therefore $\pi_{2k}(\mathrm{Aut}(D\otimes\mathbb{K})) = [S^{2k}, \mathrm{Aut}_0(D\otimes\mathbb{K})]$. Let $e \in \mathbb{K}$ be a rank 1 projection such that $c(1_D\otimes e) = 1_D\otimes e$. It follows from the proof of [4, Theorem 2.22] that the map $\alpha \mapsto \alpha(1\otimes e)$ induces an isomorphism $[S^{2k}, \mathrm{Aut}_0(D \otimes \mathbb{K})] \to K_0(C(S^{2k}) \otimes D)_1^\times = 1 + K_0(C_0(S^{2k} \setminus x_0) \otimes D)$. We have $\eta(\alpha)(1 \otimes e) = c^{-1}(\alpha(c(1 \otimes e))) = c^{-1}(\alpha(1 \otimes e))$, i.e. the isomorphism intertwines η and c^{-1}. Consider the following diagram of rings:

$$\begin{array}{ccc}
K^0(S^{2k}) \otimes R & \xrightarrow{c_S\otimes c_R} & K^0(S^{2k}) \otimes R \\
\downarrow & & \downarrow \\
K_0(C(S^{2k}) \otimes D) & \xrightarrow{p\mapsto c^{-1}(p)} & K_0(C(S^{2k}) \otimes D)
\end{array}$$

The vertical maps arise from the Künneth theorem. Since $K_1(D) = 0$, these are isomorphisms. Since c_S corresponds to the operation induced on $K_0(C(S^{2k}))$ by complex conjugation on $\mathbb{K}$, the above diagram commutes. $\square$

Remark 3.3. (i) If $D \in \mathcal{D}$ then $R = K_0(D) \subset \mathbb{Q}$ with $[1_D] = [1_{D^{\mathbb{R}}}] = 1$. Thus $c^{-1}(1_D) = 1_D$ and this shows that the above automorphism c_R is trivial. The K^0-ring of the sphere is given by $K^0(S^{2k}) \cong \mathbb{Z}[X_k]/(X_k^2)$. The element X_k is the k-fold reduced exterior tensor power of $H-1$, where H is the tautological line bundle over $S^2 \cong \mathbb{C}P^1$. Since c_S maps $H-1$ to $1-H$, it follows that X_k is mapped to $-X_k$ if k is odd and to X_k if k is even. We have $\left[K^0(S^2) \otimes R\right]_1^{\times} = \{1 + t\,X_k \mid t \in R\} \subset R[X_k]/(X_k^2)$. Thus, c_S maps $1 + t\,X_k$ to its inverse $1 - t\,X_k$ if k is odd and acts trivially if k is even.

(ii) By [4, Theorem 2.18] there is an isomorphism $\pi_0(\mathrm{Aut}(D \otimes \mathbb{K})) \cong K_0(D)_+^{\times}$ given by $[\alpha] \mapsto [\alpha(1 \otimes e)]$. Arguing as in Lemma 3.2 we see that the action of η on this groups is given by $c_R = \mathrm{id}$.

Theorem 3.4. *Let X be a compact metrizable space and let A be a locally trivial continuous field with fiber $D \otimes \mathbb{K}$ for a strongly self-absorbing C^*-algebra $D \in \mathcal{D}$. Then we have for $k \geq 0$:*

$$\delta_k(A^{\mathrm{op}}) = \delta_k(\overline{A}) = (-1)^k\,\delta_k(A) \in H^{2k+1}(X, \mathbb{Q}) \ .$$

Proof. Let $D^{\mathbb{R}}$ be a real form of D. The group isomorphism $\eta\colon \mathrm{Aut}(D \otimes \mathbb{K}) \to \mathrm{Aut}(D \otimes \mathbb{K})$ induces an infinite loop map $B\eta\colon B\mathrm{Aut}(D \otimes \mathbb{K}) \to B\mathrm{Aut}(D \otimes \mathbb{K})$, where the infinite loop space structure is the one described in [4, Section 3]. If $f\colon X \to B\mathrm{Aut}(D \otimes \mathbb{K})$ is the classifying map of a locally trivial field A, then $B\eta \circ f$ classifies $\overline{A}$. Thus the induced map $\eta_* : E^1_D(X) \to E^1_D(X)$ has the property that $\eta_*[A] = [\overline{A}]$.

The unital inclusion $D^{\mathbb{R}} \to B^{\mathbb{R}} := D^{\mathbb{R}} \otimes \mathcal{O}_\infty^{\mathbb{R}} \otimes M_{\mathbb{Q}}^{\mathbb{R}}$ induces a commutative diagram

$$\begin{array}{ccc} \mathrm{Aut}(D \otimes \mathbb{K}) & \xrightarrow{\ \eta\ } & \mathrm{Aut}(D \otimes \mathbb{K}) \\ \downarrow & & \downarrow \\ \mathrm{Aut}(B \otimes \mathbb{K}) & \xrightarrow{\ \eta\ } & \mathrm{Aut}(B \otimes \mathbb{K}) \end{array}$$

with $B := B^{\mathbb{R}} \otimes \mathbb{C}$. From this we obtain a commutative diagram

$$\begin{array}{ccc} E^1_D(X) & \xrightarrow{\ \eta_*\ } & E^1_D(X) \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\delta} \\ E^1_B(X) & \xrightarrow{\ \eta_*\ } & E^1_B(X) \end{array}$$

As explained earlier, $B \cong M_{\mathbb{Q}} \otimes \mathcal{O}_\infty$. Recall that $E^1_{M_{\mathbb{Q}} \otimes \mathcal{O}_\infty}(X) \cong H^1(X, \mathbb{Q}^{\times}) \oplus \bigoplus_{k \geq 1} H^{2k+1}(X, \mathbb{Q})$. By Lemma 3.2 and Remark 3.3(i) the effect of η on $H^{2k+1}(X, \pi_{2k}(\mathrm{Aut}(B))) \cong H^{2k+1}(X, \mathbb{Q})$ is given by multiplication with $(-1)^k$ for $k > 0$. By Remark 3.3(ii) η acts trivially on $H^1(X, \pi_0(\mathrm{Aut}(B))) = H^1(X, \mathbb{Q}^{\times})$. $\square$

Example 3.5. Let $\mathcal{Z}$ be the Jiang-Su algebra. We will show that in general the inverse of an element in the Brauer group $Br_{\mathcal{Z}}(X)$ is not represented by the class of the opposite algebra. Let Y be the space obtained by attaching a disk to a circle by a degree three map and let $X_n = S^n \wedge Y$ be n^{th} reduced suspension of Y. Then $E^1_{\mathcal{Z}}(X_3) \cong K^0(X_2)_+^{\times} \cong 1 + \widetilde{K}^0(X_2)$ by [4, Thm.2.22].

Since this is a torsion group, $Br_{\mathcal{Z}}(X_3) \cong E^1_{\mathcal{Z}}(X_3)$ by Theorem 2.15. Using the Künneth formula, $Br_{\mathcal{Z}}(X_3) \cong 1+\widetilde{K}^0(S^2)\otimes\widetilde{K}^0(Y) \cong 1+\mathbb{Z}/3$. Reasoning as in Lemma 3.2 with X_2 in place of S^{2k}, we identify the map $\eta_* : E^1_{\mathcal{Z}}(X_3) \to E^1_{\mathcal{Z}}(X_3)$ with the map $K^0(X_2)^\times_+ \to K^0(X_2)^\times_+$ that sends the class $x = [V_1]-[V_2]$ to $\overline{x} = [\overline{V}_1]-[\overline{V}_2]$, where $\overline{V}_i$ is the complex conjugate bundle of V_i. If V is complex vector bundle, and c_1 is the first Chern class, $c_1(\overline{V}) = -c_1(V)$ by [10, p.206]. Since conjugation is compatible with the Künneth formula, we deduce that $x = \overline{x}$ for $x \in K^0(X_2)^\times_+$. Indeed, if $\beta \in \widetilde{K}^0(S^2)$, $y \in \widetilde{K}^0(Y)$ and $x = 1+\beta y$, then $\overline{x} = 1+(-\beta)(-y) = x$. Let A be a continuous field over X_3 with fibers $M_N(\mathcal{Z})$ such that $[A]_{Br} = 1+\beta y$ in $Br_{\mathcal{Z}}(X_3) \cong 1+\widetilde{K}^0(S^2)\otimes\widetilde{K}^0(Y) \cong 1+\mathbb{Z}/3$, where β a generator of $\widetilde{K}^0(S^2)$ and y is a generator of $\widetilde{K}^0(Y)$. Then $[\overline{A}]_{Br} = 1+(-\beta)(-y) = [A]_{Br}$ and hence

$$[\overline{A} \otimes_{C(X_3)} A]_{Br} = (1+\beta y)^2 = 1+2\beta y \neq 1.$$

Corollary 3.6. *Let X be a compact metrizable space and let A be a locally trivial continuous field with fiber $D \otimes \mathbb{K}$ with D in the class $\mathcal{D}$. If $H^{4k+1}(X,\mathbb{Q}) = 0$ for all $k \geq 0$, then there is an $N \in \mathbb{N}$ such that*

$$(A \otimes_{C(X)} A^{\mathrm{op}})^{\otimes N} \cong C(X, D \otimes \mathbb{K}) \ .$$

Proof. If $H^{4k+1}(X,\mathbb{Q}) = 0$, then $\delta_{2k}(A \otimes_{C(X)} A^{\mathrm{op}}) = 0$ for all $k \geq 0$. Moreover, $\delta_{2k+1}(A \otimes_{C(X)} A^{\mathrm{op}}) = \delta_{2k+1}(A) - \delta_{2k+1}(A) = 0$. The statement follows from Corollary 2.17. □

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