A DIXMIER-DOUADY THEORY FOR STRONGLY SELF-ABSORBING
$C^\ast$-ALGEBRAS II: THE BRAUER GROUP

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Abstract. We have previously shown that the isomorphism classes of orientable locally trivial fields of $C^\ast$-algebras over a compact metrizable space $X$ with fiber $D \otimes K$, where $D$ is a strongly self-absorbing $C^\ast$-algebra, form an abelian group under the operation of tensor product. Moreover this group is isomorphic to the first group $\bar{E}^{1}_D(X)$ of the (reduced) generalized cohomology theory associated to the unit spectrum of topological K-theory with coefficients in $D$. Here we show that all the torsion elements of the group $\bar{E}^{1}_D(X)$ arise from locally trivial fields with fiber $D \otimes M_n(C)$, $n \geq 1$, for all known examples of strongly self-absorbing $C^\ast$-algebras $D$. Moreover the Brauer group generated by locally trivial fields with fiber $D \otimes M_n(C)$, $n \geq 1$ is isomorphic to $\text{Tor}(\bar{E}^{1}_D(X))$.

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1. Introduction

Let $X$ be a compact metrizable space. Let $K$ denote the $C^\ast$-algebra of compact operators on an infinite dimensional separable Hilbert space. It is well-known that $K \otimes K \cong K$ and $M_n(C) \otimes K \cong K$. Dixmier and Douady [7] showed that the isomorphism classes of locally trivial fields of $C^\ast$-algebras over $X$ with fiber $K$ form an abelian group under the operation of tensor product over $C(X)$ and this group is isomorphic to $H^3(X,\mathbb{Z})$. The torsion subgroup of $H^3(X,\mathbb{Z})$ admits the following description. Each element of $\text{Tor}(H^3(X,\mathbb{Z}))$ arises as the Dixmier-Douady class of a field $A$ which is isomorphic to the stabilization $B \otimes K$ of some locally trivial field of $C^\ast$-algebras $B$ over $X$ with all fibers isomorphic to $M_n(C)$ for some integer $n \geq 1$, see [8], [1].

In this paper we generalize this result to locally trivial fields with fiber $D \otimes K$ where $D$ is a strongly self-absorbing $C^\ast$-algebra [17]. For a $C^\ast$-algebra $B$, we denote by $\mathcal{C}_B(X)$ the isomorphism classes of locally trivial continuous fields of $C^\ast$-algebras over $X$ with fibers isomorphic to $B$. The isomorphism classes of orientable locally trivial continuous fields is denoted by $\mathcal{C}^0_B(X)$, see Definition 2.2. We have shown in [4] that $\mathcal{C}_{D \otimes K}(X)$ is an abelian group under the operation of tensor product over $C(X)$, and moreover, this group is isomorphic to the first group $E^1_D(X)$ of a generalized cohomology theory $E^*_D(X)$ which we have proven to be isomorphic to the theory associated to the unit spectrum of topological K-theory with coefficients in $D$, see [5]. Similarly $(\mathcal{C}^0_{D \otimes K}(X), \otimes) \cong \bar{E}^1_D(X)$ where $\bar{E}^*_D(X)$ is the reduced theory associated to $E^*_D(X)$. For $D = \mathbb{C}$, we have, of course, $\bar{E}^1_{\mathbb{C}}(X) \cong H^3(X,\mathbb{Z})$.

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We consider the stabilization map \( \sigma : \mathcal{C}_{D \otimes M_n(\mathbb{C})}(X) \to (\mathcal{C}_{D \otimes \mathbb{K}}(X), \otimes) \cong E_D^1(X) \) given by \([A] \mapsto [A \otimes \mathbb{K}]\) and show that its image consists entirely of torsion elements. Moreover, if \( D \) is any of the known strongly self-absorbing \( C^*\)-algebras, we show that the stabilization map

\[
\sigma : \bigcup_{n \geq 1} \mathcal{C}_{D \otimes M_n(\mathbb{C})}(X) \to \text{Tor}(E_D^1(X))
\]

is surjective, see Theorem 2.10. In this situation \( \mathcal{C}_{D \otimes M_n(\mathbb{C})}(X) \cong \mathcal{C}_{D \otimes M_n(\mathbb{C})}^0(X) \) by Lemma 2.2 and hence the image of the stabilization map is contained in the reduced group \( E_D^1(X) \). In analogy with the classic Brauer group generated by continuous fields of complex matrices \( M_n(\mathbb{C}) \) [8], we introduce a Brauer group \( Br_D(X) \) for locally trivial fields of \( C^*\)-algebras with fibers \( M_n(D) \) for \( D \) a strongly self-absorbing \( C^*\)-algebra and establish an isomorphism \( Br_D(X) \cong \text{Tor}(E_D^1(X)) \), see Theorem 2.15.

Our proof is new even in the classic case \( D = \mathbb{C} \) whose original proof relies on an argument of Serre, see [8, Thm.1.6], [1, Prop.2.1]. In the cases \( D = \mathbb{Z} \) or \( D = \mathcal{O}_\infty \) the group \( E_D^1(X) \) is isomorphic to \( H^1(X, BSU_\otimes) \), which appeared in [20], where its equivariant counterpart played a central role.

We introduced in [4] characteristic classes

\[
\delta_0 : E_D^1(X) \to H^1(X, K_0(D)^\mathbb{Z}_+) \quad \text{and} \quad \delta_k : E_D^1(X) \to H^{2k+1}(X, \mathbb{Q}), \quad k \geq 1.
\]

If \( X \) is connected, then \( E_D^1(X) = \ker(\delta_0) \). We show that an element \( a \) belongs to \( \text{Tor}(E_D^1(X)) \) if and only if \( \delta_0(a) \) is a torsion element and \( \delta_k(a) = 0 \) for all \( k \geq 1 \).

In the last part of the paper we show that if \( A^{op} \) is the opposite \( C^*\)-algebra of a locally trivial continuous field \( A \) with fiber \( D \otimes \mathbb{K} \), then \( \delta_k(A^{op}) = (-1)^k \delta_k(A) \) for all \( k \geq 0 \). This shows that in general \( A \otimes A^{op} \) is not isomorphic to a trivial field, unlike what happens in the case \( D = \mathbb{C} \).

Similar arguments show that in general \( [A^{op}]_{Br} \neq -[A]_{Br} \) in \( Br_D(X) \) for \( A \in \mathcal{C}_{D \otimes M_n(\mathbb{C})}(X) \), see Example 3.5.

We would like to thank Ilan Hirshberg for prompting us to seek a refinement of Theorem 2.10 in the form of Theorem 2.11.

2. Background and main result

The class of strongly self-absorbing \( C^*\)-algebras was introduced by Toms and Winter [17]. They are separable unital \( C^*\)-algebras \( D \) singled out by the property that there exists an isomorphism \( D \to D \otimes D \) which is unitarily homotopic to the map \( d \mapsto d \otimes 1_D \) [6], [19].

If \( n \geq 2 \) is a natural number we denote by \( M_{n,\infty} \) the UHF-algebra \( M_n(\mathbb{C})^\otimes_\infty \). If \( P \) is a nonempty set of primes, we denote by \( M_{P,\infty} \) the UHF-algebra of infinite type \( \bigotimes_{p \in P} M_{p,\infty} \). If \( P \) is the set of all primes, then \( M_{\infty} \) is the universal UHF-algebra, which we denote by \( M_\infty \).

The class \( D_{pi} \) of all purely infinite strongly self-absorbing \( C^*\)-algebras that satisfy the Universal Coefficient Theorem in KK-theory (UCT) was completely described in [17]. \( D_{pi} \) consists of the Cuntz algebras \( \mathcal{O}_2, \mathcal{O}_\infty \) and of all \( C^*\)-algebras \( M_{P,\infty} \otimes \mathcal{O}_\infty \) with \( P \) an arbitrary set of primes. Let \( D_{qd} \) denote the class of strongly self-absorbing \( C^*\)-algebras which satisfy the UCT and which are quasidiagonal. A complete description of \( D_{qd} \) has become possible due to the recent results of Matui and Sato [13, Cor. 6.2] that build on results of Winter [18], and Lin and Niu [12]. Thus \( D_{qd} \) consists of \( \mathbb{C} \), the Jiang-Su algebra \( \mathcal{Z} \) and all UHF-algebras \( M_{P,\infty} \) with \( P \) an arbitrary set of primes.
The class \( D = \mathcal{D}_{pt} \cup \mathcal{D}_{pt} \) contains all known examples of strongly self-absorbing \( C^* \)-algebras. It is closed under tensor products. If \( D \) is strongly self-absorbing, then \( K_0(D) \) is a unital commutative ring. The group of positive invertible elements of \( K_0(D) \) is denoted by \( K_0(D)_+ \).

Let \( B \) be a \( C^* \)-algebra. We denote by \( \text{Aut}_0(B) \) the path component of the identity of \( \text{Aut}(B) \) endowed with the point-norm topology. Recall that we denote by \( \mathcal{C}_B(X) \) the isomorphism classes of locally trivial continuous fields over \( X \) with fibers isomorphic to \( B \). The structure group of \( A \in \mathcal{C}_B(X) \) is \( \text{Aut}(B) \), and \( A \) is in fact given by a principal \( \text{Aut}(B) \)-bundle which is determined up to an isomorphism by an element of the homotopy classes of continuous maps from \( X \) to the classifying space of the topological group \( \text{Aut}(B) \), denoted by \([X, B\text{Aut}(B)]\).

**Definition 2.1.** A locally trivial continuous field \( A \) of \( C^* \)-algebras with fiber \( B \) is orientable if its structure group can be reduced to \( \text{Aut}_0(B) \), in other words if \( A \) is given by an element of \([X, B\text{Aut}_0(B)]\).

The corresponding isomorphism classes of orientable and locally trivial fields is denoted by \( \mathcal{C}^0_B(X) \).

**Lemma 2.2.** Let \( D \) be a strongly self-absorbing \( C^* \)-algebra satisfying the UCT. Then \( \text{Aut}(M_n(D)) = \text{Aut}_0(M_n(D)) \) for all \( n \geq 1 \) and hence \( \mathcal{C}_D \otimes M_n(\mathbb{C})(X) \cong \mathcal{C}^0_D \otimes M_n(\mathbb{C})(X) \).

**Proof.** First we show that for any \( \beta \in \text{Aut}(D \otimes M_n(\mathbb{C})) \) there exist \( \alpha \in \text{Aut}(D) \) and a unitary \( u \in D \otimes M_n(\mathbb{C}) \) such that \( \beta = u(\alpha \otimes \text{id}_{M_n(\mathbb{C})})u^* \). Let \( e_{11} \in M_n(\mathbb{C}) \) be the rank-one projection that appears in the canonical matrix units \((e_{ij})\) of \( M_n(\mathbb{C}) \) and let \( 1_n \) be the unit of \( M_n(\mathbb{C}) \). Then \( n[1_D \otimes e_{11}] = [1_D \otimes 1_n] \) in \( K_0(D) \) and hence \( n[\beta(1_D \otimes e_{11})] = n[1_D \otimes e_{11}] \) in \( K_0(D) \). Under the assumptions of the lemma, it is known that \( K_0(D) \) is torsion free (by [17]) and that \( D \) has cancellation of full projections by [19] and [15]. It follows that there is a partial isometry \( v \in D \otimes M_n(\mathbb{C}) \) such that \( v^*v = 1_D \otimes e_{11} \) and \( vv^* = \beta(1_D \otimes e_{11}) \). Then \( u = \sum_{i=1}^n \beta(1_D \otimes e_{11})v(1_D \otimes e_{11}) \in D \otimes M_n(\mathbb{C}) \) is a unitary such that the automorphism \( u^*\beta u \) acts identically on \( 1_D \otimes M_n(\mathbb{C}) \). It follows that \( u^*\beta u = \alpha \otimes \text{id}_{M_n(\mathbb{C})} \) for some \( \alpha \in \text{Aut}(D) \). Since both \( U(D \otimes M_n(\mathbb{C})) \) and \( \text{Aut}(D) \) are path connected by [17], [15] and respectively [6] we conclude that \( \text{Aut}(D \otimes M_n(\mathbb{C})) \) is path-connected as well. \( \square \)

Let us recall the following results contained in Cor. 3.7, Thm. 3.8 and Cor. 3.9 from [4]. Let \( D \) be a strongly self-absorbing \( C^* \)-algebra.

1. The classifying spaces \( BAut(D \otimes \mathbb{K}) \) and \( BAut_0(D \otimes \mathbb{K}) \) are infinite loop spaces giving rise to generalized cohomology theories \( E^2_D(X) \) and respectively \( \tilde{E}^2_D(X) \).
2. The monoid \((\mathcal{C}_{D \otimes \mathbb{K}}(X), \otimes)\) is an abelian group isomorphic to \( E^1_D(X) \). Similarly, the monoid \((\mathcal{C}^0_{D \otimes \mathbb{K}}(X), \otimes)\) is a group isomorphic to \( \tilde{E}^1_D(X) \). In both cases the tensor product is understood to be over \( C(X) \).
3. \( E^1_{M_0}(X) \cong H^1(X, \mathbb{Q}^\times) \oplus \bigoplus_{k \geq 1} H^{2k+1}(X, \mathbb{Q}) \), 
   \( E^1_{M_0 \otimes O_\infty}(X) \cong H^1(X, \mathbb{Q}^\times) \oplus \bigoplus_{k \geq 1} H^{2k+1}(X, \mathbb{Q}) \),
4. \( E^1_{M_0 \otimes O_\infty}(X) \cong E^1_{M_0 \otimes O_\infty}(X) \cong \bigoplus_{k \geq 1} H^{2k+1}(X, \mathbb{Q}) \).
5. If \( D \) satisfies the UCT then \( D \otimes M_0 \otimes O_\infty \cong M_0 \otimes O_\infty \), by [17]. Therefore the tensor product operation \( A \mapsto A \otimes M_0 \otimes O_\infty \) induces maps
   \( \mathcal{C}_{D \otimes \mathbb{K}}(X) \to \mathcal{C}_{M_0 \otimes O_\infty \otimes \mathbb{K}}(X), \quad \mathcal{C}^0_{D \otimes \mathbb{K}}(X) \to \mathcal{C}^0_{M_0 \otimes O_\infty \otimes \mathbb{K}}(X) \) and hence
\[E_D^1(X) \xrightarrow{\delta} E_D^1_{M_0 \otimes \mathcal{O}_X}(X) \cong H^1(X, \mathbb{Q}^\times) \oplus \bigoplus_{k \geq 1} H^{2k+1}(X, \mathbb{Q}),\]

\[\delta(A) = (\delta_0^A(A), \delta_1(A), \delta_2(A), \ldots), \quad \delta_k(A) \in H^{2k+1}(X, \mathbb{Q}),\]

\[\tilde{E}_D^1(X) \xrightarrow{\tilde{\delta}} \tilde{E}_D^1_{M_0 \otimes \mathcal{O}_X}(X) \cong \bigoplus_{k \geq 1} H^{2k+1}(X, \mathbb{Q}),\]

\[\tilde{\delta}(A) = (\tilde{\delta}_1(A), \tilde{\delta}_2(A), \ldots), \quad \tilde{\delta}_k(A) \in H^{2k+1}(X, \mathbb{Q}).\]

The invariants \(\delta_k(A)\) are called the rational characteristic classes of the continuous field \(A\), see [4, Def.4.6]. The first class \(\delta_0^A\) lifts to a map \(\delta_0^A : E_D^1(X) \to H^1(X, K_0(D)_+)\) induced by the morphism of groups \(\text{Aut}(D \otimes \mathbb{K}) \to \pi_0(\text{Aut}(D \otimes \mathbb{K})) \cong K_0(D)^+_+\). \(\delta_0^A\) represents the obstruction to reducing the structure group of \(A\) to \(\text{Aut}_0(D \otimes \mathbb{K})\).

**Proposition 2.3.** A continuous field \(A \in \mathcal{C}_D \otimes \mathbb{K}(X)\) is orientable if and only if \(\delta_0(A) = 0\). If \(X\) is connected, then \(\tilde{E}_D^1(X) \cong \ker(\delta_0)\).

**Proof.** Let us recall from [4, Cor. 2.19] that there is an exact sequence of topological groups

\[(1) \quad 1 \to \text{Aut}_0(D \otimes \mathbb{K}) \to \text{Aut}(D \otimes \mathbb{K}) \xrightarrow{\pi} K_0(D)^+_+ \to 1.\]

The map \(\pi\) takes an automorphism \(\alpha\) to \([\alpha(1_D \otimes e)]\) where \(e \in \mathbb{K}\) is a rank-one projection. If \(G\) is a topological group and \(H\) is a normal subgroup of \(G\) such that \(H \to G \to G/H\) is a principal \(H\)-bundle, then there is a homotopy fibre sequence \(G/H \to BH \to BG \to B(G/H)\) and hence an exact sequence of pointed sets \([X, G/H] \to [X, BH] \to [X, BG] \to [X, B(G/H)]\). In particular, in the case of the fibration (1) we obtain

\[(2) \quad [X, K_0(D)^+_+] \to [X, B\text{Aut}_0(D \otimes \mathbb{K})] \to [X, B\text{Aut}(D \otimes \mathbb{K})] \xrightarrow{\delta_0} H^1(X, K_0(D)^+_+).\]

A continuous field \(A \in \mathcal{C}_D \otimes \mathbb{K}(X)\) is associated to a principal \(\text{Aut}(D \otimes \mathbb{K})\)-bundle whose classifying map gives a unique element in \([X, B\text{Aut}(D \otimes \mathbb{K})]\) whose image in \(H^1(X, K_0(D)^+_+)\) is denoted by \(\delta_0(A)\). It is clear from (2) that the class \(\delta_0(A) \in H^1(X, K_0(D)^+_+)\) represents the obstruction for reducing this bundle to a principal \(\text{Aut}_0(D \otimes \mathbb{K})\)-bundle. If \(X\) is connected, \([X, K_0(D)^+_+] = \{\ast\}\) and hence \(\tilde{E}_D^1(X) \cong \ker(\delta_0)\). \(\square\)

**Remark 2.4.** If \(D = \mathbb{C}\) or \(D = \mathbb{Z}\) then \(A\) is automatically orientable since in those cases \(K_0(D)^+_+\) is the trivial group.

**Remark 2.5.** Let \(Y\) be a compact metrizable space and let \(X = \Sigma Y\) be the suspension of \(Y\). Since the rational K"unneth isomorphism and the Chern character on \(K^0(X)\) are compatible with the ring structure on \(K_0(C(Y) \otimes D)\), we obtain a ring homomorphism

\[\text{ch} : K_0(C(Y) \otimes D) \to K^0(Y) \otimes K_0(D) \otimes \mathbb{Q} \to \prod_{k=0}^{\infty} H^{2k}(Y, \mathbb{Q}) =: H^\text{ev}(Y, \mathbb{Q}),\]

which restricts to a group homomorphism \(\text{ch} : \tilde{E}_D^0(Y) \to SL_1(H^\text{ev}(Y, \mathbb{Q}))\), where the right hand side denotes the units, which project to \(1 \in H^0(Y, \mathbb{Q})\). If \(A\) is an orientable locally trivial continuous field with fiber \(D \otimes \mathbb{K}\) over \(X\), then we have

\[(3) \quad \delta_k(A) = \log \text{ch}(f_A) \in H^{2k}(Y, \mathbb{Q}) \cong H^{2k+1}(X, \mathbb{Q}),\]
where \( f_A : Y \to \Omega B \text{Aut}_0(D \otimes \mathbb{K}) \simeq \text{Aut}_0(D \otimes \mathbb{K}) \) is induced by the transition map of \( A \). The homomorphism \( \log : SL_1(H^e(Y, \mathbb{Q})) \to H^e(Y, \mathbb{Q}) \) is the rational logarithm from [14, Section 2.5]. For the proof of (3) it suffices to treat the case \( D = M_\mathbb{Q} \otimes O_\infty \), where it can be easily checked on the level of homotopy groups, but since \( E^B_2(Y) \) and \( H^e(Y, \mathbb{Q}) \) have rational vector spaces as coefficients this is enough.

**Lemma 2.6.** Let \( D \) be a strongly self-absorbing \( C^* \)-algebra in the class \( D \). If \( p \in D \otimes \mathbb{K} \) is a projection such that \( [p] \neq 0 \) in \( K_0(D) \), then there is an integer \( n \geq 1 \) such that \( [p] \in nK_0(D)_+^\times \). If \( [p] \in nK_0(D)_+ \), then \( p(D \otimes \mathbb{K})p \cong M_n(D) \). Moreover, if \( n, m \geq 1 \), then \( M_n(D) \cong M_m(D) \) if and only if \( nK_0(D)_+ = mK_0(D)_+ \).

**Proof.** Recall that \( K_0(D) \) is an ordered unital ring with unit \([1_D]\) and with positive elements \( K_0(D)_+ \) corresponding to classes of projections in \( D \otimes \mathbb{K} \). The group of invertible elements is denoted by \( K_0(D)^\times \) and \( K_0(D)_+^\times \) consists of classes \([p]\) of projections \( p \in D \otimes \mathbb{K} \) such that \([p] \in K_0(D)^\times \). It was shown in [4, Lemma 2.14] that if \( p \in D \otimes \mathbb{K} \) is a projection, then \([p] \in K_0(D)_+^\times \) if and only if \( p(D \otimes \mathbb{K})p \cong D \). The ring \( K_0(D) \) and the group \( K_0(D)_+^\times \) are known for all \( D \in D \), [17]. In fact \( K_0(D) \) is a unital subring of \( \mathbb{Q} \), \( K_0(D)_+ = \mathbb{Q}_+ \cap K_0(D) \) if \( D \in D_{qd} \) and \( K_0(D)_+ = K_0(D) \) if \( D \in D_{pe} \). Moreover:

\[
K_0(\mathbb{C}) \cong K_0(\mathbb{Z}) \cong K_0(\mathbb{O}_\infty) \cong \mathbb{Z}, \quad K_0(\mathbb{O}_2) = \{0\},
\]

\[
K_0(M_\mathbb{P}) \cong K_0(M_\mathbb{P} \otimes \mathbb{O}_\infty) \cong \mathbb{Z}[1/P] \cong \bigotimes_{p \in P} \mathbb{Z}[1/p] \cong \{np_k^{k_1}p_2^{k_2} \cdots p_r^{k_r} : p_i \in P, n, k_i \in \mathbb{Z}\},
\]

\[
K_0(\mathbb{C})_+ \cong K_0(\mathbb{Z})_+ = \{1\}, \quad K_0(\mathbb{O}_\infty)_+ = \{\pm 1\},
\]

\[
K_0(M_\mathbb{P})_+ \cong \{p_1^{k_1}p_2^{k_2} \cdots p_r^{k_r} : p_i \in P, k_i \in \mathbb{Z}\},
\]

\[
K_0(M_\mathbb{P} \otimes \mathbb{O}_\infty)_+ \cong \{\pm p_1^{k_1}p_2^{k_2} \cdots p_r^{k_r} : p_i \in P, k_i \in \mathbb{Z}\}.
\]

In particular, we see that in all cases \( K_0(D)_+ = \mathbb{N} \cdot K_0(D)_+^\times \), which proves the first statement. If \( p \in D \otimes \mathbb{K} \) is a projection such that \([p] \in nK_0(D)_+^\times \), then there is a projection \( q \in D \otimes \mathbb{K} \) such that \([q] \in K_0(D)_+^\times \) and \([p] = n[q] = [\text{diag}(q, q, \ldots, q)] \). Since \( D \) has cancellation of full projections, it follows then immediately that \( p(D \otimes \mathbb{K})p \cong M_n(D) \) proving the second part.

To show the last part of the lemma, suppose now that \( \alpha : D \otimes M_n(\mathbb{C}) \to D \otimes M_m(\mathbb{C}) \) is a \(*\)-isomorphism. Let \( e \in M_n(\mathbb{C}) \) be a rank one projection. Then \( \alpha(1D \otimes e)(D \otimes M_m(\mathbb{C})) \alpha(1D \otimes e) \cong D \).

By [4, Lemma 2.14] it follows that \( \alpha_{1D} = [\alpha(1D \otimes e)] \in K_0(D)_+^\times \). Since \( \alpha \) is unital, \( \alpha_{n[1D]} = m[1D] \) and hence \( m[1D] \in nK_0(D)_+^\times \). This is equivalent to \( nK_0(D)_+^\times = mK_0(D)_+^\times \).

Conversely, suppose that \( m[1D] = nu \) for some \( u \in K_0(D)_+^\times \). Let \( \alpha \in \text{Aut}(D \otimes \mathbb{K}) \) be such that \([\alpha(1D \otimes e)] = u \). Then \( \alpha_{n[1D]} = nu = m[1D] \). This implies that \( \alpha \) maps a corner of \( D \otimes \mathbb{K} \) that is isomorphic to \( M_n(D) \) to a corner that is isomorphic to \( M_m(D) \). \( \square \)

**Corollary 2.7.** Let \( D \in D \) and let \( \theta : D \otimes M_{nr}(\mathbb{C}) \to D \otimes M_{n\infty} \) be a unital inclusion induced by some unital embedding \( M_{nr}(\mathbb{C}) \to M_{n\infty} \), where \( n \geq 2, r \geq 0 \). Let \( R \) be the set of prime factors of \( n \). Then, under the canonical isomorphism \( K_0(D \otimes M_{nr}(\mathbb{C})) \cong K_0(D) \), we have

\[
\theta^{-1}_r(K_0(D \otimes M_{n\infty})_+^\times) = \bigcup_{r} rK_0(D)_+^\times \subset K_0(D)
\]

where \( r \) runs through the set of all products of the form \( \prod_{q \in R} q^{k_q} \), \( k_q \in \mathbb{N} \cup \{0\} \).

**Proof.** From Lemma 2.6 we see that \( K_0(D) \cong \mathbb{Z}[1/P] \) for a (possibly empty) set of primes \( P \). The order structure is the one induced by \((\mathbb{Q}, \mathbb{Q}_+)\) if \( D \) is quasidiagonal or \( K_0(D)_+ = \mathbb{Z}[1/P] \) if \( D \) is
purely infinite. If $R \subseteq P$, then $\theta$ induces an isomorphism on $K_0$ and the statement is true, since $\theta_s$ is order preserving and $\mathbb{Z}[1/R]^\times \subseteq K_0(D)^\times$. Thus, we may assume that $R \nsubseteq P$. Let $S = P \cup R$ and thus $K_0(D \otimes M_n) \cong \mathbb{Z}[1/S]$. The map $\theta_s$ induces the canonical inclusion $\mathbb{Z}[1/P] \hookrightarrow \mathbb{Z}[1/S]$. We can write $x \in \mathbb{Z}[1/P]$ as

$$x = m \cdot \prod_{p \in P} p^{r_p} \cdot \prod_{q \in R \setminus P} q^{k_q}$$

with $m \in \mathbb{Z}$ relatively prime to all $p \in P$ and $q \in R$, only finitely many $r_p \in \mathbb{Z}$ non-zero and $k_q \in \mathbb{N} \cup \{0\}$. From this decomposition we see that $x$ is invertible in $\mathbb{Z}[1/S]$ if and only if $m = \pm 1$. This concludes the proof since $p^{r_p} \in K_0(D)_+^\times$.

**Remark 2.8.** Let $q \in D \otimes K$ be a projection and let $\alpha \in \text{Aut}(D \otimes K)$. As in [4, Lemma 2.14] we have that $[\alpha(q)] = [\alpha(1 \otimes e)] \cdot [q]$ with $[\alpha(1 \otimes e)] \in K_0(D)_+^\times$. Thus, the condition $[q] \in nK_0(D)_+^\times$ for $n \in \mathbb{N}$ is invariant under the action of $\text{Aut}(D \otimes K)$ on $K_0(D)$. Given $A \in \mathcal{E}_{D \otimes K}(X)$, a projection $p \in A$, $x_0 \in X$ and an isomorphism $\phi: A(x_0) \to D \otimes K$ the condition $[\phi(p(x_0))] \in nK_0(D)_+^\times$ is independent of $\phi$. Abusing the notation we will write this as $[p(x_0)] \in nK_0(D)_+^\times$.

**Corollary 2.9.** Let $D \in \mathcal{D}$ and let $A \in \mathcal{E}_{D \otimes K}(X)$ with $X$ a connected compact metrizable space. If $p \in A$ is a projection such that $[p(x_0)] \in nK_0(D)_+^\times$ for some point $x_0$, then $(pAp)(x) \cong M_n(D)$ for all $x \in X$ and hence $pAp \in \mathcal{E}_{D \otimes M_n(C)}(X)$. If $p \in A$ is a projection with $[p(x_0)] \in K_0(D)_+^\times$ for some $n \in \mathbb{N}$.

**Proof.** Let $V_1, \ldots, V_k$ be a finite cover of $X$ by compact sets such that there are bundle isomorphisms $\phi_i: A(V_i) \cong C(V_i) \otimes D \otimes K$. Let $p_i$ be the image of the restriction of $p$ to $V_i$ under $\phi_i$. After refining the cover $(V_i)$, if necessary, we may assume that $\|p_i(x) - p_i(y)\| < 1$ for all $x, y \in V_i$. This allows us to find a unitary $u_i$ in the multiplier algebra of $C(V_i) \otimes D \otimes K$ such that after replacing $\phi_i$ by $u_i \phi_i u_i^*$ and $p_i$ by $u_i p_i u_i^*$, we may assume that $p_i$ are constant projections. Since $X$ is connected and $[p(x_0)] \in nK_0(D)_+^\times$ by assumption, it follows from $[p_i(x_0)] \in nK_0(D)_+^\times$ for $x_0 \in V_i$ and the above remark that $[p_j(x)] \in nK_0(D)_+^\times$ for all $1 \leq j \leq k$ and all $x \in V_j$. Then Lemma 2.6 implies $(pAp)(V_j) \cong C(V_j) \otimes M_n(D)$. By Lemma 2.6 we also have that $[p(x_0)] \neq 0$ implies $[p(x_0)] \in nK_0(D)_+^\times$ for some $n \in \mathbb{N}$ proving the statement about the case $[p(x_0)] \in K_0(D)_+^\times$.

We study the image of the stabilization map

$$\mathcal{E}_{D \otimes M_n(C)}(X) \to \mathcal{E}_{D \otimes K}(X)$$

induced by the map $A \mapsto A \otimes K$, or equivalently by the map

$$\text{Aut}(D \otimes M_n(C)) \to \text{Aut}(D \otimes M_n(C) \otimes K) \cong \text{Aut}(D \otimes K).$$

Let us recall that $\mathcal{D}$ denotes the class of strongly self-absorbing $C^*$-algebras which satisfy the UCT and which are either quasidiagonal or purely infinite.

**Theorem 2.10.** Let $D$ be a strongly self-absorbing $C^*$-algebra in the class $\mathcal{D}$. Let $A$ be a locally trivial continuous field of $C^*$-algebras over a connected compact metrizable space $X$ such that $A(x) \cong D \otimes K$ for all $x \in X$. The following assertions are equivalent:

1. $\delta_k(A) = 0$ for all $k \geq 0$.
2. The field $A \otimes M_Q$ is trivial.
(3) There is an integer \( n \geq 1 \) and a unital locally trivial continuous field \( \mathcal{B} \) over \( X \) with all fibers isomorphic to \( M_n(D) \) such that \( A \cong B \otimes \mathbb{K} \).

(4) \( A \) is orientable and \( A^{\otimes m} \cong C(X) \otimes D \otimes \mathbb{K} \) for some \( m \in \mathbb{N} \).

**Proof.** The statement is immediately verified if \( D \cong \mathcal{O}_2 \). Indeed all locally trivial fields with fiber \( \mathcal{O}_2 \otimes \mathbb{K} \) are trivial since \( \text{Aut}(\mathcal{O}_2 \otimes \mathbb{K}) \) is contractible by [4, Cor. 17 & Thm. 2.17]. For the remainder of the proof we may therefore assume that \( D \not\cong \mathcal{O}_2 \).

(1) \( \iff \) (2) If \( D \in D_{pd} \), then it is known that \( D \otimes M_Q \cong M_Q \). Similarly, if \( D \in D_{pd} \) and \( D \not\cong \mathcal{O}_2 \) then \( D \otimes M_Q \cong \mathcal{O}_\infty \otimes M_Q \). If \( A \) is as in the statement, then \( A \otimes M_Q \) is a locally trivial field whose fibers are all isomorphic to either \( M_Q \otimes \mathbb{K} \) or to \( \mathcal{O}_\infty \otimes M_Q \otimes \mathbb{K} \). In either case, it was shown in [4, Cor. 4.5] that a field is trivial if and only if \( \delta_k(A) = 0 \) for all \( k \geq 0 \). As reviewed earlier in this section, this follows from the explicit computation of \( E^1_{M_Q}(X) \) and \( E^1_{M_Q \otimes \mathcal{O}_\infty}(X) \).

(2) \( \Rightarrow \) (3) Assume now that \( A \otimes M_Q \) is trivial, i.e. \( A \otimes M_Q \cong C(X) \otimes D \otimes M_Q \otimes \mathbb{K} \). Let \( p \in A \otimes M_Q \) be the projection that corresponds under this isomorphism to the projection \( 1 \otimes e \in C(X) \otimes D \otimes M_Q \otimes \mathbb{K} \) where \( 1 \) is the unit of the \( C^\ast \)-algebra \( C(X) \otimes D \otimes M_Q \) and \( e \in \mathbb{K} \) is a rank-one projection. Then \( [p(x)] \neq 0 \) in \( K_0(A(x) \otimes M_Q) \) for all \( x \in X \) (recall that \( D \not\cong \mathcal{O}_2 \)). Let us write \( M_Q \) as the direct limit of an increasing sequence of its subalgebras \( M_{k(i)}(\mathbb{C}) \). Then \( A \otimes M_Q \) is the direct limit of the sequence \( A_i = A \otimes M_{k(i)}(\mathbb{C}) \). It follows that there exist \( i \geq 1 \) and a projection \( p_i \in A_i \) such that \( \|p - p_i\| < 1 \). Then \( \|p(x) - p_i(x)\| < 1 \) and so \( [p_i(x)] \neq 0 \) in \( K_0(A_i(x)) \) for each \( x \in X \) that its image in \( K_0(A(x) \otimes M_Q) \) is equal to \( [p(x)] \neq 0 \). Let us consider the locally trivial unital field \( B := p_i(A \otimes M_{k(i)}(\mathbb{C}))p_i \). Since the fibers of \( A \otimes M_{k(i)}(\mathbb{C}) \) are isomorphic to \( D \otimes \mathbb{K} \otimes M_{k(i)}(\mathbb{C}) \cong D \otimes \mathbb{K} \), it follows by Corollary 2.9 that there is \( n \geq 1 \) such that all fibers of \( B \) are isomorphic to \( M_n(D) \). Since \( B \) is isomorphic to a full corner of \( A \otimes \mathbb{K} \), it follows by [3] that \( A \otimes \mathbb{K} \cong B \otimes \mathbb{K} \). We conclude by noting that since \( A \) is locally trivial and each fiber is stable, then \( A \cong A \otimes \mathbb{K} \) by [9] and so \( A \cong B \otimes \mathbb{K} \).

(3) \( \Rightarrow \) (2) This implication holds for any strongly self-absorbing \( C^\ast \)-algebra \( D \). Let \( A \) and \( B \) be as in (3). Let us note that \( B \otimes M_Q \) is a unital locally trivial field with all fibers isomorphic to the strongly self-absorbing \( C^\ast \)-algebra \( D \otimes M_Q \). Since \( \text{Aut}(D \otimes M_Q) \) is contractible by [4, Thm. 2.3], it follows that \( B \otimes M_Q \) is trivial. We conclude that \( A \otimes M_Q \cong (B \otimes M_Q) \otimes \mathbb{K} \cong C(X) \otimes D \otimes M_Q \otimes \mathbb{K} \).

(2) \( \iff \) (4) This equivalence holds for any strongly self-absorbing \( C^\ast \)-algebra \( D \) if \( A \) is orientable. In particular we do not need to assume that \( D \) satisfies the UCT. In the UCT case we note that since the map \( K_0(D) \to K_0(D \otimes M_Q) \) is injective, it follows that \( A \) is orientable if and only if \( A \otimes M_Q \) is orientable, i.e. \( \delta_0(A) = 0 \) if and only if \( \delta_0(A) = 0 \). Since \( \delta_0(A) = 0 \), \( A \) is determined up to isomorphism by its class \([A] \in \bar{E}_2^D(X)\). To complete the proof it suffices to show that the kernel of the map \( \tau : \bar{E}_2^D(X) \to \bar{E}_1^{D \otimes M_Q}(X) \), \( \tau[A] = [A \otimes M_Q] \), consists entirely of torsion elements. Consider the natural transformation of cohomology theories:

\[
\tau \otimes \text{id}_Q : \bar{E}_2^D(X) \otimes \mathbb{Q} \to \bar{E}_1^{D \otimes M_Q}(X) \otimes \mathbb{Q} \cong \bar{E}_1^{D \otimes M_Q}(X).
\]

If \( D \not\cong \mathbb{C} \), it induces an isomorphism on coefficients since \( \bar{E}_2^{-i}(pt) = \pi_i(\text{Aut}_0(D \otimes \mathbb{K})) \cong K_i(D) \) by [4, Thm. 2.18] and since the map \( K_i(D) \otimes \mathbb{Q} \to K_i(D \otimes M_Q) \) is bijective. We conclude that the kernel of \( \tau \) is a torsion group. The same property holds for \( D = \mathbb{C} \) since \( \bar{E}_2^C(X) \) is a direct summand of \( \bar{E}_2^C(X) \) by [4, Cor. 3.8].

\(\square\)
Theorem 2.11. Let $D$, $X$ and $A$ be as in Theorem 2.10 and let $n ≥ 2$ be an integer. The following assertions are equivalent:

1. The field $A ⊗ M_{n∞}$ is trivial.
2. There is a $k ∈ \mathbb{N}$ and a unital locally trivial continuous field $B$ over $X$ with all fibers isomorphic to $M_{nk}(D)$ such that $A ∼= B ⊗ K$.
3. $A$ is orientable and $A ⊗^{nk} ∼= C(X) ⊗ D ⊗ K$ for some $k ∈ \mathbb{N}$.

Proof. By reasoning as in the proof of Theorem 2.10, we may assume that $D ≠ O_2$.

1 $⇒$ 2: By assumption the continuous field $A ⊗ M_{n∞}$ is trivializable and hence it satisfies the global Fell condition of [4]. This means that there is a full projection $p_∞ ∈ A ⊗ M_{n∞}$ with the property that $p_∞(x) ∈ K_0(A(x) ⊗ M_{n∞})_+$ for all $x ∈ X$. Let $ν_t : M_{n^t}(C) → M_{n^t}$ be a unital inclusion map.

Since $A ⊗ M_{n∞}$ is the inductive limit of the sequence

$A → A ⊗ M_{n^t}(C) → \cdots → A ⊗ M_{n^t}(C) → A ⊗ M_{n^t+1}(C) → \cdots$

there is an $i ∈ \mathbb{N}$ and a full projection $p ∈ A ⊗ M_{n^t}(C)$ with $||(|id_A ⊗ ν_t|)(p) − p_∞|| < 1$. Fix a point $x_0 ∈ X$. Let $θ : A(x_0) ⊗ M_{n^t}(C) → A(x_0) ⊗ M_{n∞}$ be the unital inclusion induced by $ν_t$. Note that $θ_∗([p(x_0)]) = (id_A ⊗ ν_t)([p(x_0)]) = [p_∞(x_0)] ∈ K_0(A(x_0) ⊗ M_{n∞})_+$. By Corollary 2.7 this implies that $[p(x_0)] ∈ rK_0(A(x_0))_+$ for some $r ∈ \mathbb{N}$ that divides $nk$ for some $k ∈ \mathbb{N} ∪ \{0\}$. Then $B_0 := p(A ⊗ M_{n^t}(C))p ∈ C_{D⊗M_{n^t},i}(X)$ by Corollary 2.9. Write $nk = mr$ with $m ∈ \mathbb{N}$. It follows that $B := B_0 ⊗ M_{n^t}(C) ⊂ C_{D⊗M_{nk},i}(X)$. The fact that $B ⊗ K ∼= A$ follows just as in step (2) $⇒$ (3) in the proof of Theorem 2.10.

(2) $⇒$ (1): This is just the same argument as step (3) $⇒$ (2) in the proof of Theorem 2.10.

(1) $⇔$ (3): The orientability of $A$ follows from Theorem 2.10. Observe that the elements $[A] ∈ C_{D⊗K}^0(X) = E_D^1(X)$ such that $nk[A] = 0$ or equivalently $A ⊗^{nk}$ is trivializable for some $k ∈ \mathbb{N} ∪ \{0\}$ coincide precisely with the elements in the kernel of the group homomorphism $E_D^1(X) → E_D^1(X) ⊗ \mathbb{Z}^1/\mathbb{Z}^1$. Since $\mathbb{Z}^1/\mathbb{Z}^1$ is flat, it follows that $X → E_D^* (X) ⊗ \mathbb{Z}^1/\mathbb{Z}^1$ still satisfies all axioms of a generalized cohomology theory. In particular, we have the following commutative diagram of natural transformations of cohomology theories:

$$
\begin{array}{ccc}
\tilde{E}_D^1(X) & \longrightarrow & \tilde{E}_{D⊗M_{n∞}}^1(X) \\
\downarrow & & \downarrow \\
\tilde{E}_D^* (X) ⊗ \mathbb{Z}^1/\mathbb{Z}^1 & \longrightarrow & \tilde{E}_{D⊗M_{n∞}}^* (X) ⊗ \mathbb{Z}^1/\mathbb{Z}^1
\end{array}
$$

where the isomorphism on the right hand side can be checked on the coefficients. A similar argument shows that for $D ≠ C$ the bottom homomorphism is an isomorphism. Thus the kernel of the left vertical map agrees with the one of the upper horizontal map in this case. For $D = C$ we can use that $E_D^1(X)$ embeds as a direct summand into $E_Z^2(X)$ via the natural *-homomorphism $C → \mathbb{Z}$ [4, Cor. 4.8]. In particular, $\tilde{E}_C^* (X) ⊗ \mathbb{Z}^1/\mathbb{Z}^1 → \tilde{E}_D^* (X) ⊗ \mathbb{Z}^1/\mathbb{Z}^1$ is injective.

Corollary 2.12. Let $D$ and $X$ be as in Theorem 2.10. Then any element $x ∈ E_D^{11}(X)$ with $nx = 0$ is represented by the stabilization of a unital locally trivial field over $X$ with all fibers isomorphic to $M_{nk}(D)$ for some $k ≥ 1$. Moreover if $A ∈ E_{D⊗K}^{11}(X)$, then $A ⊗ M_{\mathbb{Q}}$ is trivial if and only if $A$ is orientable and $nk[A] = 0$ in $E_D^1(X)$ for some $k ∈ \mathbb{N}$ and some $n ∈ \mathbb{N}$. \[\Box\]
(An example from [1] for \( D = \mathbb{C} \) shows that in general one cannot always arrange that \( k = 1 \).)

**Proof.** The first part follows from Theorem 2.11. Indeed, condition (3) of that theorem is equivalent to requiring that \( A \) is orientable and \( n^k[A] = 0 \) in \( E_X^{1}(\mathbb{R}) \). The second part follows from Theorems 2.10 and 2.11. \( \square \)

**Definition 2.13.** Let \( D \) be a strongly self-absorbing \( C^* \)-algebra. If \( X \) is connected compact metrizable space we define the Brauer group \( Br_D(X) \) as equivalence classes of continuous fields \( A \in \bigcup_{n \geq 1} \mathcal{C}_{M_n(D)}(X) \). Two continuous fields \( A_i \in \mathcal{C}_{M_n(D)}(X) \), \( i = 1, 2 \) are equivalent, if

\[
A_1 \otimes p_1 C(X, M_{N_1}(D))p_1 \cong A_2 \otimes p_2 C(X, M_{N_2}(D))p_2,
\]

for some full projections \( p_i \in C(X, M_{N_i}(D)) \). We denote by \([A]_{Br} \) the class of \( A \) in \( Br_D(X) \). The multiplication on \( Br_D(X) \) is induced by the tensor product operation, after fixing an isomorphism \( D \otimes D \cong D \). We will show in a moment that the monoid \( Br_D(X) \) is a group.

**Remark 2.14.** It is worth noting the following two alternative descriptions of the Brauer group.

(a) If \( D \in \mathcal{D} \) is quasidiagonal, then two continuous fields \( A_i \in \mathcal{C}_{M_n(D)}(X) \), \( i = 1, 2 \) have equal classes in \( Br_D(X) \), if and only if \( A_1 \otimes p_1 C(X, M_{N_1}(\mathbb{C}))p_1 \cong A_2 \otimes p_2 C(X, M_{N_2}(\mathbb{C}))p_2 \), for some full projections \( p_i \in C(X, M_{N_i}(\mathbb{C})) \). (b) If \( D \in \mathcal{D} \) is purely infinite, then two continuous fields \( A_i \in \mathcal{C}_{M_n(D)}(X) \), \( i = 1, 2 \) have equal classes in \( Br_D(X) \), if and only if \( A_1 \otimes p_1 C(X, M_{N_1}(\mathbb{O}_\infty))p_1 \cong A_2 \otimes p_2 C(X, M_{N_2}(\mathbb{O}_\infty))p_2 \), for some full projections \( p_i \in C(X, M_{N_i}(\mathbb{O}_\infty)) \). In order to justify (a) we observe that if \( D \) is quasidiagonal, then every projection \( p \in C(X, M_{N}(D)) \) has a multiple \( p(m) : = p \otimes 1_{M_m}(\mathbb{C}) \) such that \( p(m) \) is Murray-Von Neumann equivalent to a projection in \( C(X, M_{N_m}(\mathbb{C})) \otimes 1_D \subset C(X, M_{N_m}(\mathbb{C})) \otimes D \) and that \( A_i \otimes D \cong A_i \) by [9]. For (b) we note that if \( D \) is purely infinite, then every projection \( p \in C(X, M_{N}(D)) \) has a multiple \( p \otimes 1_{M_m}(\mathbb{C}) \) that is Murray-Von Neumann equivalent to a projection in \( C(X, M_{N_m}(\mathbb{O}_\infty)) \otimes 1_D \).

One has the following generalization of a result of Serre, [8, Thm.1.6].

**Theorem 2.15.** Let \( D \) be a strongly self-absorbing \( C^* \)-algebra in \( D \).

(i) \( Tor(E_D(X)) = \ker \left( \overline{E_D^1(X)} \xrightarrow{\delta} \bigoplus_{k \geq 1} H^{2k+1}(X, \mathbb{Q}) \right) \)

(ii) The map \( \theta : Br_D(X) \rightarrow Tor(E_D^1(X)) \), \([A]_{Br} \mapsto [A \otimes \mathbb{K}] \) is an isomorphism of groups.

**Proof.** (i) was established in the last part of the proof of Theorem 2.10.

(ii) We denote by \( L_p \) the continuous field \( p C(X, M_{N}(D))p \). Since \( L_p \otimes \mathbb{K} \cong C(X, D \otimes \mathbb{K}) \) it follows that the map \( \theta \) is a well-defined morphism of monoids.

We use the following observation. Let \( \theta : S \rightarrow G \) be a unital surjective morphism of commutative monoids with units denoted by \( 1 \). Suppose that \( G \) is a group and that \( \{ s \in S : \theta(s) = 1 \} = \{ 1 \} \). Then \( S \) is a group and \( \theta \) is an isomorphism. Indeed if \( s \in S \), there is \( t \in S \) such that \( \theta(t) = \theta(s)^{-1} \) by surjectivity of \( \theta \). Then \( \theta(st) = \theta(s)\theta(t) = 1 \) and so \( st = 1 \). It follows that \( S \) is a group and that \( \theta \) is injective.

We are going to apply this observation to the map \( \theta : Br_D(X) \rightarrow Tor(E_D^1(X)) \). By condition (3) of Theorem 2.10 we see that \( \theta \) is surjective. Let us determine the set \( \theta^{-1}(\{ 0 \}) \). We are going to show that if \( B \in \mathcal{C}_{D \otimes M_n(C)}(X) \), then \( [B \otimes \mathbb{K}] = 0 \) in \( E_D^1(X) \) if and only if

\[
B \cong p(C(X) \otimes D \otimes M_n(C)p \cong L_{C(X,D)}(pC(X,D)^N)
\]
for some selfadjoint projection \( p \in C(X) \otimes D \otimes M_N(\mathbb{C}) \cong M_N(C(X, D)) \). Let \( B \in \mathcal{G}_{D \otimes M_n(\mathbb{C})}(X) \) be such that \([B \otimes \mathbb{K}] = 0\) in \( E^1_D(X)\). Then there is an isomorphism of continuous fields \( \phi : B \otimes \mathbb{K} \xrightarrow{\cong} C(X) \otimes D \otimes \mathbb{K} \). After conjugating \( \phi \) by a unitary we may assume that \( p := \phi(1_B \otimes e_{11}) \in C(X) \otimes D \otimes M_N(\mathbb{C}) \) for some integer \( N \geq 1 \). It follows immediately that the projection \( p \) has the desired properties. Conversely, if \( B \cong p(C(X) \otimes D \otimes M_N(\mathbb{C})) \) then there is an isomorphism of continuous fields \( B \otimes \mathbb{K} \cong C(X) \otimes D \otimes \mathbb{K} \) by [3]. We have thus shown that that \( \theta([B]_{Br}) = 0 \) if and only if \([B]_{Br} = 0\).

We are now able to conclude that \( Br_D(X) \) is a group and that \( \theta \) is injective by the general observation made earlier. \( \square \)

**Definition 2.16.** Let \( D \) be a strongly self-absorbing \( C^*\)-algebra. Let \( A \) be a locally trivial continuous field of \( C^*\)-algebras with fiber \( D \otimes \mathbb{K} \). We say that \( A \) is a torsion continuous field if \( A^\otimes k \) is isomorphic to a trivial field for some integer \( k \geq 1 \).

**Corollary 2.17.** Let \( A \) be as in Theorem 2.10. Then \( A \) is a torsion continuous field if and only if \( \delta_0(A) \in H^1(X, K_0(D)^\wedge) \) is a torsion element and \( \delta_k(A) = 0 \in H^{2k+1}(X, \mathbb{Q}) \) for all \( k \geq 1 \).

**Proof.** Let \( m \geq 1 \) be an integer such that \( m\delta_0(A) = 0 \). Then \( \delta_0(A^\otimes m) = 0 \). We conclude by applying Theorem 2.10 to the orientable continuous field \( A^\otimes m \). \( \square \)

3. **Characteristic classes of the opposite continuous field**

Given a \( C^*\)-algebra \( B \) denote by \( B^{op} \) the *opposite \( C^*\)-algebra* with the same underlying Banach space and norm, but with multiplication given by \( b^{op}a^{op} = (a-b)^{op} \). The *conjugate \( C^*\)-algebra* \( \overline{B} \) has the conjugate Banach space as its underlying vector space, but the same multiplicative structure. The map \( a \mapsto a^* \) provides an isomorphism \( B^{op} \to \overline{B} \). Any automorphism \( \alpha \in \text{Aut}(B) \) yields in a canonical way automorphisms \( \overline{\alpha} : \overline{B} \to \overline{B} \) and \( \alpha^{op} : B^{op} \to B^{op} \) compatible with \( * : B^{op} \to \overline{B} \). Therefore we have group isomorphisms \( \theta : \text{Aut}(B) \to \text{Aut}(\overline{B}) \) and \( \text{Aut}(B) \to \text{Aut}(B^{op}) \). Note that \( \alpha \in \text{Aut}(B) \) is equal to \( \theta(\alpha) \) when regarded as set-theoretic maps \( B \to B \). Given a locally trivial continuous field \( A \) with fiber \( B \), we can apply these operations fiberwise to obtain the locally trivial fields \( A^{op} \) and \( \overline{A} \), which we will call the *opposite* and the *conjugate field*. They are isomorphic to each other and isomorphic to the conjugate and the opposite \( C^*\)-algebras of \( A \).

A *real form* of a complex \( C^*\)-algebra \( A \) is a real \( C^*\)-algebra \( A^\mathbb{R} \) such that \( A \cong A^\mathbb{R} \otimes \mathbb{C} \). A real form is not necessarily unique [2] and not all \( C^*\)-algebras admit real forms [16]. If two \( C^*\)-algebras \( A \) and \( B \) admit real forms \( A^\mathbb{R} \) and \( B^\mathbb{R} \), then \( A^\mathbb{R} \otimes_{\mathbb{R}} B^\mathbb{R} \) is a real form of \( A \otimes B \).

**Example 3.1.** All known strongly self-absorbing \( C^*\)-algebras \( D \in D \) admit a real form.

Indeed, the real Cuntz algebras \( O^\mathbb{R}_2 \) and \( O^\mathbb{R}_\infty \), are defined by the same generators and relations as their complex versions. Alternatively \( O^\mathbb{R}_\infty \) can be realized as follows. Let \( H^\mathbb{R} \) be a separable infinite dimensional real Hilbert space and let \( \mathcal{F}^\mathbb{R}(H^\mathbb{R}) = \bigoplus_{n=0}^{\infty} H^\mathbb{R}_n^\mathbb{R} \) be the real Fock space associated to it. Every \( \xi \in H^\mathbb{R} \) defines a shift operator \( s_\xi(\eta) = \xi \otimes \eta \) and we denote the algebra spanned by the \( s_\xi \) and their adjoints \( s_\xi^* \) by \( O^\mathbb{R}_\infty \). If \( \mathcal{F}(H^\mathbb{R} \otimes \mathbb{C}) \) denotes the Fock space associated to the complex Hilbert space \( H = H^\mathbb{R} \otimes \mathbb{C} \), then we have \( \mathcal{F}^\mathbb{R} \otimes \mathbb{C} \cong \mathcal{F}(H) \). If we represent \( O^\mathbb{R}_\infty \) on \( \mathcal{F}(H) \) using the above construction, then the map \( s_\xi + is_\xi' \mapsto s_{\xi+i\xi'} \) induces an isomorphism \( O^\mathbb{R}_\infty \otimes \mathbb{C} \to \mathcal{O}_\infty \). Likewise define \( M^\mathbb{R}_2 \) to be the infinite tensor product \( M_2(\mathbb{R}) \otimes M_3(\mathbb{R}) \otimes M_4(\mathbb{R}) \otimes \ldots \).
Since $M_n(\mathbb{C}) \cong M_n(\mathbb{R}) \otimes \mathbb{C}$, we obtain an isomorphism $M^n_0 \otimes \mathbb{C} \cong M^n_0$ on the inductive limit. Let $K^n_0$ be the compact operators on $H^n_\mathbb{R}$ and $K$ those on $H$, then we have $K^n_0 \otimes \mathbb{C} \cong K$. Thus, $M^n_0 \otimes O_\infty \otimes \mathbb{K}$ is the complexification of the real $C^*$-algebra $M^n_0 \otimes O_\infty \otimes \mathbb{K}$. 

The Jiang-Su algebra $\mathcal{Z}$ admits a real form $\mathcal{Z}^\mathbb{K}$ which can be constructed in the same way as $\mathcal{Z}$. Indeed, one constructs $\mathcal{Z}^\mathbb{K}$ as the inductive limit of a system 

$$\cdots \rightarrow C([0, 1], M_{p_n q_n}(\mathbb{R})) \xrightarrow{\phi_n} C([0, 1], M_{p_{n+1} q_{n+1}}(\mathbb{R})) \rightarrow \cdots$$

where the connecting maps $\phi_n$ are defined just as in the proof of [11, Prop. 2.5] with only one modification. Specifically, one can choose the matrices $u_0$ and $u_1$ to be in the special orthogonal group $SO(p_n q_n)$ and this will ensure the existence of a continuous path $u_t$ in $O(p_n q_n)$ from $u_0$ to $u_1$ as required.

If $B$ is the complexification of a real $C^*$-algebra $B^\mathbb{R}$, then a choice of isomorphism $B \cong B^\mathbb{R} \otimes \mathbb{C}$ provides an isomorphism $c: B \rightarrow \overline{B}$ via complex conjugation on $\mathbb{C}$. On automorphisms we have $\text{Ad}_{c^{-1}}: \text{Aut}(\overline{B}) \rightarrow \text{Aut}(B)$. Let $\eta = \text{Ad}_{c^{-1}} \circ \theta: \text{Aut}(B) \rightarrow \text{Aut}(B)$. Now we specialize to the case $B = D \otimes K$ with $D \in D$ and study the effect of $\eta$ on homotopy groups, i.e., $\eta_*: \pi_{2k}(\text{Aut}(B)) \rightarrow \pi_{2k}(\text{Aut}(B))$. By [4, Theorem 2.18] the groups $\pi_{2k+1}(\text{Aut}(B))$ vanish.

Let $R$ be a commutative ring and denote by $[K^0(S^{2k} \otimes R)]^\times_1$ the group of units of the ring $K^0(S^{2k} \otimes R)$. Let $[K^0(S^{2k} \otimes R)]^\times_1$ be the kernel of the morphism of multiplicative groups $[K^0(S^{2k} \otimes R)]^\times \rightarrow R^\times$. This is the group of virtual rank 1 vector bundles with coefficients in $R$ over $S^{2k}$. Let $c_S: K^0(S^{2k}) \rightarrow K^0(S^{2k})$ and $c_R: K_0(D) \rightarrow K_0(D)$ be the ring automorphisms induced by complex conjugation.

**Lemma 3.2.** Let $D$ be a strongly self-absorbing $C^*$-algebra in the class $D$, let $R = K_0(D)$ and let $k > 0$. There is an isomorphism $\pi_{2k}(\text{Aut}(D \otimes K)) \rightarrow [K^0(S^{2k} \otimes R)]^\times_1 (k > 0)$ such that the following diagram commutes

$$\begin{array}{ccc}
\pi_{2k}(\text{Aut}(D \otimes K)) & \xrightarrow{\eta_*} & \pi_{2k}(\text{Aut}(D \otimes K)) \\
\downarrow & & \downarrow \\
[K^0(S^{2k} \otimes R)]^\times_1 & \xrightarrow{c_S \otimes c_R} & [K^0(S^{2k} \otimes R)]^\times_1
\end{array}$$

**Proof.** Observe that $\pi_{2k}(\text{Aut}(D \otimes K)) = \pi_{2k}(\text{Aut}_0(D \otimes K))$ (for $k > 0$) and $\text{Aut}_0(D \otimes K)$ is a path connected group, therefore $\pi_{2k}(\text{Aut}(D \otimes K)) = [S^{2k}, \text{Aut}_0(D \otimes K)]$. Let $e \in K$ be a rank 1 projection such that $e(1D \otimes e) = 1D \otimes e$. It follows from the proof of [4, Theorem 2.22] that the map $\alpha \mapsto \alpha(1 \otimes e)$ induces an isomorphism $[S^{2k}, \text{Aut}_0(D \otimes K)] \rightarrow K_0(C(S^{2k}) \otimes D)^\times_1 = 1 + K_0(C_0(S^{2k} \setminus x_0) \otimes D)$. We have $\eta(\alpha)(1 \otimes e) = c^{-1}(\alpha(c(1 \otimes e))) = c^{-1}(\alpha(1 \otimes e))$, i.e., the isomorphism intertwines $\eta$ and $c^{-1}$. Consider the following diagram of rings:

$$\begin{array}{ccc}
K^0(S^{2k} \otimes R) & \xrightarrow{c_S \otimes c_R} & K^0(S^{2k} \otimes R) \\
\downarrow & & \downarrow \\
K_0(C(S^{2k}) \otimes D) & \xrightarrow{p \rightarrow c^{-1}(p)} & K_0(C(S^{2k}) \otimes D)
\end{array}$$
The vertical maps arise from the Künneth theorem. Since \( K_1(D) = 0 \), these are isomorphisms. Since \( c_S \) corresponds to the operation induced on \( K_0(C(S^{2k})) \) by complex conjugation on \( \mathbb{K} \), the above diagram commutes. \( \square \)

**Remark 3.3.** (i) If \( D \in \mathcal{D} \) then \( R = K_0(D) \subset \mathbb{Q} \) with \( [1_D] = [1_{D^\mathbb{R}}] = 1 \). Thus \( c^{-1}(1_D) = 1_D \) and this shows that the above automorphism \( c_R \) is trivial. The \( K^0 \)-ring of the sphere is given by \( K^0(S^{2k}) \cong \mathbb{Z}[X_k]/(X_k^2) \). The element \( X_k \) is the \( k \)-fold reduced exterior tensor power of \( H-1 \), where \( H \) is the tautological line bundle over \( S^2 \). Since \( c_S \) maps \( H-1 \) to \( 1-H \), it follows that \( X_k \) is mapped to \(-X_k \) if \( k \) is odd and to \( X_k \) if \( k \) is even. We have \( [K^0(S^2) \otimes R] \cong \{1 + t X_k \mid t \in \mathbb{R} \} \subset R[X_k]/(X_k^2) \). Thus, \( c_S \) maps \( 1 + t X_k \) to its inverse \( 1 - t X_k \) if \( k \) is odd and acts trivially if \( k \) is even.

(ii) By [4, Theorem 2.18] there is an isomorphism \( \pi_0(\text{Aut}(D \otimes \mathbb{K})) \cong K_0(D)^{\times} \) given by \( [\alpha] \mapsto [\alpha(1 \otimes e)] \). Arguing as in Lemma 3.2 we see that the action of \( \eta \) on this groups is given by \( c_R = \text{id} \).

**Theorem 3.4.** Let \( X \) be a compact metrizable space and let \( A \) be a locally trivial continuous field with fiber \( D \otimes \mathbb{K} \) for a strongly self-absorbing \( C^* \)-algebra \( D \in \mathcal{D} \). Then we have for \( k \geq 0 \):

\[
\delta_k(A^\mathbb{Q}) = \delta_k(A) = (-1)^k \delta_k(A) \in H^{2k+1}(X, \mathbb{Q}) .
\]

**Proof.** Let \( D^\mathbb{R} \) be a real form of \( D \). The group isomorphism \( \eta : \text{Aut}(D \otimes \mathbb{K}) \rightarrow \text{Aut}(D \otimes \mathbb{K}) \) induces an infinite loop map \( B\eta : B\text{Aut}(D \otimes \mathbb{K}) \rightarrow B\text{Aut}(D \otimes \mathbb{K}) \), where the infinite loop space structure is the one described in [4, Section 3]. If \( f : X \rightarrow B\text{Aut}(D \otimes \mathbb{K}) \) is the classifying map of a locally trivial field \( A \), then \( B\eta \circ f \) classifies \( \tilde{A} \). Thus the induced map \( \eta_* : E^1_D(X) \rightarrow E^1_D(X) \) has the property that \( \eta_*[A] = [\tilde{A}] \).

The unital inclusion \( D^\mathbb{R} \rightarrow B^\mathbb{R} := D^\mathbb{R} \otimes \mathcal{O}_{\mathbb{R}} \otimes M^\mathbb{R}_\mathbb{Q} \) induces a commutative diagram

\[
\text{Aut}(D \otimes \mathbb{K}) \xrightarrow{\eta} \text{Aut}(D \otimes \mathbb{K}) \quad \quad \quad \quad \text{Aut}(B \otimes \mathbb{K}) \xrightarrow{\eta} \text{Aut}(B \otimes \mathbb{K})
\]

with \( B := B^\mathbb{R} \otimes \mathbb{C} \). From this we obtain a commutative diagram

\[
E^1_D(X) \xrightarrow{\eta_*} E^1_D(X) \xrightarrow{\delta} E^1_B(X) \xrightarrow{\eta_*} E^1_B(X)
\]

As explained earlier, \( B \cong M^\mathbb{R} \otimes \mathcal{O}_{\mathbb{R}} \). Recall that \( E^1_{M^\mathbb{R} \otimes \mathcal{O}_{\mathbb{R}}}(X) \cong H^1(X, \mathbb{Q}^\mathbb{R}) \oplus \bigoplus_{k \geq 1} H^{2k+1}(X, \mathbb{Q}) \).

By Lemma 3.2 and Remark 3.3(i) the effect of \( \eta \) on \( H^{2k+1}(X, \pi_{2k}(\text{Aut}(B))) \cong H^{2k+1}(X, \mathbb{Q}) \) is given by multiplication with \((-1)^k \) for \( k > 0 \). By Remark 3.3(ii) \( \eta \) acts trivially on \( H^1(X, \pi_0(\text{Aut}(B))) = H^1(X, \mathbb{Q}^\mathbb{R}) \).

**Example 3.5.** Let \( \mathcal{Z} \) be the Jiang-Su algebra. We will show that in general the inverse of an element in the Brauer group \( Br_{\mathcal{Z}}(X) \) is not represented by the class of the opposite algebra. Let \( Y \) be the space obtained by attaching a disk to a circle by a degree three map and let \( X_n = S^n \wedge Y \) be \( n \)th reduced suspension of \( Y \). Then \( E^1_{\mathcal{Z}}(X_3) \cong K^0(X_2)^{\times} = 1 + \tilde{K}^0(X_2) \) by [4, Thm.2.22].
Since this is a torsion group, $Br_Z(X_3) \cong E^2_{2,0}(X_3)$ by Theorem 2.15. Using the Künneth formula, $Br_Z(X_3) \cong 1 + K^0(S^2) \otimes \mathbb{Z}/3$. Reasoning as in Lemma 3.2 with $X_2$ in place of $S^{2k}$, we identify the map $\eta : E^2_{2,0}(X_3) \to E^2_{2,0}(X_2)$ with the map $K^0(X_3) \to K^0(X_2)$ that sends the class $x = [V_1] - [V_2]$ to $\overline{x} = [V_1] - [V_2]$, where $V_i$ is the complex conjugate bundle of $V_i$. If $V$ is complex vector bundle, and $c_1$ is the first Chern class, $c_1(V) = -c_1(V)$ by [10, p.206]. Since conjugation is compatible with the Künneth formula, we deduce that $x = \overline{x}$ for $x \in K^0(X_2)$. Indeed, if $\beta \in \mathbb{K}^0(S^2)$, $y \in \mathbb{K}^0(Y)$ and $x = 1 + \beta y$, then $\overline{x} = 1 + (-\beta)(-y) = x$. Let $A$ be a continuous field over $X_3$ with fibers $M_N(\mathbb{Z})$ such that $[A]_{Br} = 1 + \beta y$ in $Br(X_3) \cong 1 + K^0(S^2) \otimes \mathbb{K}(Y) \cong 1 + \mathbb{Z}/3$, where $\beta$ a generator of $\mathbb{K}^0(S^2)$ and $y$ is a generator of $\mathbb{K}^0(Y)$. Then $[A]_{Br} = 1 + (-\beta)(-y) = [A]_{Br}$ and hence

$$[A \otimes_{C(X_3)} A]_{Br} = (1 + \beta y)^2 = 1 + 2\beta y \neq 1.$$ 

**Corollary 3.6.** Let $X$ be a compact metrizable space and let $A$ be a locally trivial continuous field with fiber $D \otimes \mathbb{K}$ with $D$ in the class $D$. If $H^{4k+1}(X, \mathbb{Q}) = 0$ for all $k \geq 0$, then there is an $N \in \mathbb{N}$ such that

$$(A \otimes_{C(X)} A_{op}) \otimes N \cong C(X, D \otimes \mathbb{K}).$$

**Proof.** If $H^{4k+1}(X, \mathbb{Q}) = 0$, then $\delta_{2k}(A \otimes_{C(X)} A_{op}) = 0$ for all $k \geq 0$. Moreover, $\delta_{2k+1}(A \otimes_{C(X)} A_{op}) = \delta_{2k+1}(A) - \delta_{2k+1}(A) = 0$. The statement follows from Corollary 2.17. $\square$

**References**


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