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Limit theorems for Multifractal Products of Geometric Stationary Processes

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We investigate the properties of multifractal products of geometric Gaussian processes with possible long-range dependence and geometric Ornstein-Uhlenbeck processes driven by Lévy motion and their finite and infinite superpositions. We present the general conditions for the $\mathcal{L}_q$ convergence of cumulative processes to the limiting processes and investigate their q-th order moments and Rényi functions, which are nonlinear, hence displaying the multifractality of the processes as constructed. We also establish the corresponding scenarios for the limiting processes, such as log-normal, log-gamma, log-tempered stable or log-normal tempered stable scenarios.

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1. Introduction

Multifractal models have been used in many applications in hydrodynamic turbulence, finance, genomics, computer network traffic, etc. (see, for example, Kolmogorov (1941), Kolmogorov (1962), Kahane (1985), Kahane (1987), Novikov (1994), Frisch (1995), Mandelbrot (1997), Falconer (1997), Schertzer *et al* (1997), Harte (2001), Riedi (2003)). There are many ways to construct random multifractal models ranging from simple binomial cascades to measures generated by branching processes and the compound Poisson process (Kahane (1985), Kahane (1987), Falconer (1997), Schmitt (2003), Harte (2001), Barral Mandelbrot (2002), Barral and Mandelbrot (2010), Bacry and Muzy (2003), Riedi (2003), Mörters and Shieh (2004), Shieh and Taylor (2002), Schmitt (2003), Schertzer *et al* (1997), Barral *et al* (2009), Ludeña (2008), Jaffard *et al* (2010)) Jaffard (1999) showed that Lévy processes (except Brownian motion and Poisson processes) are multifractal; but since the increments of a Lévy process are independent, this class excludes the effects of dependence structures. Moreover, Lévy processes have a linear singularity spectrum while real data often exhibit a strictly concave spectrum.

Anh, Leonenko and Shieh (2008a,b, 2009a,b, 2010) considered multifractal products of stochastic processes as defined in Kahane (1985), Kahane (1987) and Mannersalo *et al* (2002). Especially Anh *et al* (2008a) constructed multifractal processes based on products of geometric Ornstein-Uhlenbeck (OU) processes driven by Lévy motion with inverse Gaussian or normal inverse Gaussian distribution. They also described the behaviour of the q-th order moments and Rényi functions, which are nonlinear, hence displaying the multifractality of the processes as constructed. In these papers a number of scenarios were obtained for $q \in Q \cap [1, 2]$, where Q is a set of parameters of marginal distribution of an OU processes driven by Lévy motion. The simulations show that for q outside this range, the scenarios still hold (see Anh *et al* (2010b)). In this paper we present a rigorous proof of these results and also construct new scenarios which generalize those corresponding to the inverse Gaussian and normal inverse Gaussian distributions obtained in Anh and Leonenko (2008), Anh *et al* (2008a). We use the theory of OU processes with tempered stable law and normal tempered stable law for their marginal distributions. Note that in their pioneering paper Calvet and Fisher (2002) proposed the simplified version of the construction of Mannersalo *et al* (2002).

The next section recaptures some basic results on multifractal products of stochastic processes as developed in Kahane (1985), Kahane (1987) and Mannersalo *et al* (2002). Section 3 contains the general $\mathcal{L}_q$ bounds for cumulative process of multifractal products of stationary processes. Section 4 establishes the general results on the scaling moments of multifractal products of geometric OU processes in terms of the marginal distributions of OU processes and their Lévy measures.

Our exposition extends results of Mannersalo *et al* (2002) on the basic properties of multifractal products of stochastic processes. We should also note some related results by Barndorff-Nielsen and Schmiegel (2004) who introduced some Lévy-based spatiotemporal models for parametric modelling of turbulence. Log-infinitely divisible scenarios related to independently scattered random measures were investigated in Schmitt (2003), Bacry and Muzy (2003), see also their references.

2. Multifractal products of stochastic processes

This section recaptures some basic results on multifractal products of stochastic processes as developed in Kahane (1985), Kahane (1987) and Mannersalo *et al* (2002). We provide an interpretation of their conditions based on the moment generating functions, which is useful for our exposition. Throughout the text the notation C, c is used for the generic constants which do not necessarily coincide.

We introduce the following conditions:

A′. Let $\Lambda(t),\ t \in \mathbb{R}_+ = [0, \infty)$, be a measurable, separable, strictly stationary, positive stochastic process with $\mathrm{E}\Lambda(t) = 1$.

We call this process the mother process and consider the following setting:

A″. Let $\Lambda(t) = \Lambda^{(i)},\ i = 0, 1, ...$ be independent copies of the mother process Λ, and

$\Lambda_b^{(i)}$ be the rescaled version of $\Lambda^{(i)}$:

$$\Lambda_b^{(i)}(t) \stackrel{d}{=} \Lambda^{(i)}(tb^i), \quad t \in \mathbb{R}_+, \quad i = 0, 1, 2, \ldots,$$

where the scaling parameter $b > 1$, and $\stackrel{d}{=}$ denotes equality in finite-dimensional distributions.

Moreover, in the examples, the stationary mother process satisfies the following conditions:

A‴. Let $\Lambda(t) = \exp\{X(t)\}$, $t \in \mathbb{R}_+$, where $X(t)$ is a strictly stationary process, such that there exist a marginal probability density function $\pi(x)$ and a bivariate probability density function $p(x_1, x_2; t_1 - t_2)$. Moreover, we assume that the moment generating function

$$M(\zeta) = \mathrm{E} \exp\{\zeta X(t)\} \tag{2.1}$$

and the bivariate moment generating function

$$M(\zeta_1, \zeta_2; t_1 - t_2) = \mathrm{E} \exp\{\zeta_1 X(t_1) + \zeta_2 X(t_2)\} \tag{2.2}$$

exist.

The conditions **A′**-**A‴** yield

$$\mathrm{E}\Lambda_b^{(i)}(t) = M(1) = 1; \mathrm{Var}\Lambda_b^{(i)}(t) = M(2) - 1 = \sigma_\Lambda^2 < \infty;$$

$$\mathrm{Cov}(\Lambda_b^{(i)}(t_1), \Lambda_b^{(i)}(t_2)) = M(1, 1; (t_1 - t_2)b^i) - 1, \ b > 1.$$

We define the finite product processes

$$\Lambda_n(t) = \prod_{i=0}^{n} \Lambda_b^{(i)}(t) = \exp\left\{\sum_{i=0}^{n} X^{(i)}(tb^i)\right\}, t \in [0, 1], \tag{2.3}$$

and the cumulative processes

$$A_n(t) = \int_0^t \Lambda_n(s)ds, \quad n = 0, 1, 2, \ldots, t \in [0, 1], \tag{2.4}$$

where $X^{(i)}(t), i = 0, ..., n, ...$, are independent copies of a stationary process $X(t), t \geq 0$.

We also consider the corresponding positive random measures defined on Borel sets B of $\mathbb{R}_+$:

$$\mu_n(B) = \int_B \Lambda_n(s)ds, \quad n = 0, 1, 2, \ldots \tag{2.5}$$

Kahane (1987) proved that the sequence of random measures μ_n converges weakly almost surely to a random measure μ. Moreover, given a finite or countable family of Borel sets B_j on $\mathbb{R}_+$, it holds that $\lim_{n\to\infty} \mu_n(B_j) = \mu(B_j)$ for all j with probability one. The almost sure convergence of $A_n(t)$ in countably many points of $\mathbb{R}_+$ can be extended to all points in $\mathbb{R}_+$ if the limit process $A(t)$ is almost surely continuous. In this case,

$\lim_{n\to\infty} A_n(t) = A(t)$ with probability one for all $t \in \mathbb{R}_+$. As noted in Kahane (1987), there are two extreme cases: (i) $A_n(t) \to A(t)$ in $\mathcal{L}_1$ for each given t, in which case $A(t)$ is not almost surely zero and and is said to be fully active (non-degenerate) on $\mathbb{R}_+$; (ii) $A_n(1)$ converges to 0 almost surely, in which case $A(t)$ is said to be degenerate on $\mathbb{R}_+$. Sufficient conditions for non-degeneracy and degeneracy in a general situation and relevant examples are provided in Kahane (1987) (Eqs. (18) and (19) respectively.) The condition for complete degeneracy is detailed in Theorem 3 of Kahane (1987). In our work we present general conditions for non-degeneracy in Theorem 3.

The Rényi function of a random measure μ, also known as the deterministic partition function, is defined for $t \in [0,1]$ as

$$T(q) = \liminf_{n\to\infty} \frac{\log \mathrm{E} \sum_{k=0}^{2^n-1} \mu^q\left(I_k^{(n)}\right)}{\log \left|I_k^{(n)}\right|} = \liminf_{n\to\infty} \left(-\frac{1}{n}\right) \log_2 \mathrm{E} \sum_{k=0}^{2^n-1} \mu^q\left(I_k^{(n)}\right),$$

where $I_k^{(n)} = [k2^{-n}, (k+1)2^{-n}]$, $\quad k = 0, 1, \ldots, 2^n - 1$, $\left|I_k^{(n)}\right|$ is its length, and $\log_b$ is log to the base b.

In the present paper we establish convergence

$$A_n(t) \stackrel{\mathcal{L}_q}{\to} A(t), \quad n \to \infty. \tag{2.6}$$

For the limiting process we show that for some constants $\overline{C}$ and $\underline{C}$,

$$\underline{C} t^{q - \log_b \mathrm{E}\Lambda^q(t)} \leqslant \mathrm{E}A^q(t) \leqslant \overline{C} t^{q - \log_b \mathrm{E}\Lambda^q(t)}, \tag{2.7}$$

which will be written as

$$\mathrm{E}A^q(t) \sim t^{q - \log_b \mathrm{E}\Lambda^q(t)}.$$

This allows us to find the scaling function

$$\varsigma(q) = q - \log_b \mathrm{E}\Lambda^q(t) = q - \log_b M(q). \tag{2.8}$$

As is shown in Leonenko and Shieh (2013) for the exponentially decreasing correlations and $q \in [1,2]$ there is a connection between Rényi function and the scaling function given by

$$T(q) = \varsigma(q) - 1. \tag{2.9}$$

The exact conditions are stated in Theorem 2 and Theorem 3.

An important contribution of our paper is that we proved (2.6) for general $q > 0$. In comparison, in Mannersalo *et al* (2002) convergence (2.6) was shown for $q \in [1,2]$ under an additional assumption $A(t) \in \mathcal{L}_q$. Additionally we simplified significantly the conditions under which equations (2.6) and (2.7) hold. Finally we provide a number of scenarios where scaling function can be written explicitly.

3. $\mathcal{L}_q$ convergence: general bound

This section contains a generalisation of the basic results on multifractal products of stochastic processes developed in Kahane (1985), Kahane (1987) and Mannersalo *et al* (2002).

Consider the cumulative process $A_n(t)$ defined in (2.4). For fixed t, the sequence $\{A_n(t), \mathcal{F}_n\}_{n=0}^{\infty}$ is a martingale. It is well known that for $q > 1$, $\mathcal{L}_q$ convergence is equivalent to the finiteness of

$$\sup_n \mathrm{E} A_n^q(t) < \infty.$$

3.1. $\mathcal{L}_2$ convergence

First we consider a simpler case $q = 2$, which was studied in Mannersalo *et al* (2002). The proof in the general case uses the same idea but is more complicated.

We have,

$$\mathrm{E}A_n^2(t) = \mathrm{E}\int_0^t \int_0^t \Lambda_n(s_1)\Lambda_n(s_2)ds_1ds_2 = \int_0^t \int_0^t \prod_{i=0}^{n} \mathrm{E}\Lambda^{(i)}(s_1)\Lambda^{(i)}(s_2)ds_1ds_2.$$

The process $\Lambda^{(i)}$ is stationary. Therefore,

$$\begin{aligned} \mathrm{E}A_n^2(t) &= 2\int_0^t \int_{s_1}^t \prod_{i=0}^{n} \mathrm{E}\Lambda^{(i)}(0)\Lambda^{(i)}(s_2 - s_1)ds_1ds_2 \\ &= 2\int_0^t \int_0^{t-s_1} \prod_{i=0}^{n} \rho(b^i(s_2 - s_1)ds_1ds_2 \leq 2t\int_0^t \prod_{i=0}^{n} \rho(b^i u)du, \end{aligned}$$

where

$$\rho(u) = \mathrm{E}\Lambda(0)\Lambda(u). \tag{3.1}$$

Hence, to show $\mathcal{L}_2$ convergence it is sufficient to show that

$$\sup_n \int_0^t \prod_{i=0}^{n} \rho(b^l u)du < \infty.$$

Theorem 1. *Assume that $\rho(u)$ as defined in (3.1) is monotone decreasing in u,*

$$b > \mathrm{E}\Lambda(0)^2 \tag{3.2}$$

and

$$\sum_{i=0}^{\infty}(\rho(b^i) - 1) < \infty. \tag{3.3}$$

Then $A_n(t)$ converges in $\mathcal{L}_2$ (and hence in $\mathcal{L}_q$ for $q \in [0, 2]$) for every fixed $t \in [0, 1]$.

Proof. First note that $\mathcal{L}_2$ convergence implies $\mathcal{L}_q$ convergence for all $q \in [0,2]$. This follows from the inequality $\mathrm{E}|A_n(t) - A(t)|^s \le (\mathrm{E}|A_n(t) - A(t)|^2)^{s/2}$ valid for any $s \le 2$. In turn the latter inequality follows from the Jensen inequality.

Without loss of generality let $t = 1$. Let $n(u) = [-\log_b u]$ be the integer part of $-\log_b u$. Then, using monotonicity of ρ we obtain

$$\prod_{i=0}^{n} \rho(b^i u) \le \rho(0)^{n(u)} \prod_{i=n(u)}^{n} \rho(b^i u).$$

Using monotonicity of ρ again,

$$\prod_{i=n(u)}^{n} \rho(b^i u) \le \prod_{i=0}^{n-n(u)} \rho(b^{i+n(u)} u) \le \Pi := \prod_{i=0}^{\infty} \rho(b^i).$$

Constant Π is finite due to the condition (3.3). For sufficiently small $\delta \in (0,1)$, by the condition (3.2), $b^{1-\delta} > \rho(0) = \mathrm{E}\Lambda(0)^2$. Therefore,

$$\sup_n \int_0^1 \prod_{i=0}^{n} \rho(b^i u) du \le \Pi \int_0^1 \rho(0)^{n(u)} du \le \Pi \int_0^1 b^{(1-\delta)n(u)} du \le \Pi \int_0^1 \frac{1}{u^{1-\delta}} du < \infty.$$

The proof of Theorem 1 is complete.

□

3.2. $\mathcal{L}_q$ convergence for $q > 2$

Now we are going to consider $q > 2$. Now we assume additionally that $A_n(t)$ is a cadlag process. Also, we strengthen condition (3.3). For that let

$$\rho(u_1, \dots, u_{q-1}) = \mathrm{E}\Lambda(0)\Lambda(u_1) \dots \Lambda(u_1 + \dots + u_{q-1}) \tag{3.4}$$

We require that the function $\rho(u_1, \dots, u_{q-1})$ satisfies certain mixing conditions. Namely, let $m < q - 1$ and $\mathcal{C} = \{i_1, \dots i_m\}$ be a subset of indices ordered in the increasing order $1 \le i_1 < \dots < i_m \le q - 1$. Consider the vector $(u_1, \dots, u_{q-1})$ such that $u_j = A$ if $j \in \mathcal{C}$ and $u_j = 0$ otherwise. Then we assume that for any set $\mathcal{C}$ the following mixing condition holds

$$\lim_{A \to \infty} \rho(u_1, \dots, u_{q-1}) = \mathrm{E}\Lambda(0)^{i_1} \mathrm{E}\Lambda(0)^{i_2 - i_1} \cdot \dots \cdot \mathrm{E}\Lambda(0)^{q - i_m}. \tag{3.5}$$

The starting point is the equality

$$\begin{aligned} \mathrm{E}A_n^q(t) &= \mathrm{E} \int_0^t \int_0^t \dots \int_0^t \Lambda_n(s_1)\Lambda_n(s_2) \dots \Lambda_n(s_q) ds_1 ds_2 \dots ds_q \\ &= q! \int_{0 < s_1 < \dots < s_q < t} \mathrm{E}\Lambda_n(s_1)\Lambda_n(s_2) \dots \Lambda_n(s_p) ds_1 ds_2 \dots ds_q. \end{aligned} \tag{3.6}$$

First we make change of variables

$$u_0 = s_1, u_1 = s_2 - s_1, \dots u_{q-1} = s_q - s_{q-1},$$

which transforms equality (3.6) into

$$\begin{aligned}
\mathrm{E}A_n^q(t) &= q! \int_{0<u_0,\dots,u_{q-1}}^{u_0+\dots+u_{q-1}\le t} \mathrm{E}\Lambda_n(u_0)\Lambda_n(u_0+u_1)\dots\Lambda_n(u_0+\dots+u_{q-1})du_0\dots du_{q-1} \\
&\le q! \int_{0<u_0,\dots,u_{q-1}<t} \mathrm{E}\Lambda_n(u_0)\Lambda_n(u_0+u_1)\dots\Lambda_n(u_0+\dots+u_{q-1})du_0 du_1\dots du_{q-1} \\
&= q! \int_{0<u_1,\dots,u_{q-1}<t} \mathrm{E}\Lambda_n(0)\Lambda_n(u_1)\dots\Lambda_n(u_1+\dots+u_{q-1})du_1\dots du_{q-1},
\end{aligned}$$

where we used stationarity of the process $\Lambda(t)$ to obtain the latter inequality. Thus it is sufficient to prove that

$$\sup_n \int_{0<u_1,\dots,u_{q-1}<t} \prod_{l=0}^{n} \rho(b^l u_1, \dots, b^l u_{q-1})du_1\dots du_{q-1} < \infty. \tag{3.7}$$

We are ready now to state the main result of this section.

Theorem 2. *Suppose that conditions $\boldsymbol{A}'$-$\boldsymbol{A}''$ hold. Assume that $\rho(u_1,\dots,u_{q-1})$ defined in (3.4) is monotone decreasing in all variables. Let*

$$b^{q-1} > \mathrm{E}\Lambda(0)^q \tag{3.8}$$

for some integer $q \ge 2$, and

$$\sum_{n=1}^{\infty} (\rho(b^n,\dots,b^n) - 1) < \infty. \tag{3.9}$$

Finally assume that the mixing condition (3.5) holds. Then,

$$\mathrm{E}A(t)^q < \infty, \tag{3.10}$$

and $A_n(t)$ converges to $A(t)$ in $\mathcal{L}_q$ (and hence in $\mathcal{L}_{\widetilde{q}}$ for $\widetilde{q} \in [0,q]$).

Proof of Theorem 2. As above $\mathcal{L}_q$ convergence implies $\mathcal{L}_{\widetilde{q}}$ convergence for all $\widetilde{q} \in [0,q]$. This follows from the inequality $\mathrm{E}|A_n(t) - A(t)|^{\widetilde{q}} \le (\mathrm{E}|A_n(t) - A(t)|^q)^{\widetilde{q}/q}$ valid for any $\widetilde{q} \le q$.

It is sufficient to prove that equation (3.7) holds. To simplify notation we put $t = 1$. First represent the integral in (3.7) as the sum of the integrals over different regions

$$\begin{aligned}
&\int_{0\le u_1,\dots,u_{q-1}\le 1} \prod_{l=0}^{n} \rho(b^l u_1,\dots,b^l u_{q-1})du_1\dots du_{q-1} \\
&\qquad = \sum_{i_1,\dots i_{q-1}} \int_{0\le u_{i_1}\le u_{i_2}\le\dots\le u_{i_{q-1}}\le 1} \prod_{l=0}^{n} \rho(b^l u_1,\dots,b^l u_{q-1})du_1\dots du_{q-1},
\end{aligned} \tag{3.11}$$

where the sum is taken over all possible permutations of numbers $(1, 2, \dots, q-1)$. Next we are going to bound the integrals on these separate regions. Put

$$u_{(1)} = u_{i_1}, u_{(2)} = u_{i_2}, \dots, u_{(q-1)} = u_{i_{q-1}}.$$

Fix a large number $A \ge 1$ which we define later and define an auxiliary function $n(u) = -[\log_b u/A]$. Note that this function is non-negative for $u \le 1$. Now let

$$l_1 = n(u_{(1)}), l_2 = n(u_{(2)}), \dots, l_{q-1} = n(u_{(q-1)}).$$

These numbers are decreasing

$$l_1 \ge l_2 \ge \dots \ge l_{q-1}. \tag{3.12}$$

Then we can split the product as

$$\prod_{l=0}^{n} \rho(b^l u_1, \dots, b^l u_{q-1}) = \prod_{l=0}^{l_{q-1}-1} \prod_{l=l_{q-1}}^{l_{q-2}-1} \dots \prod_{l=l_2}^{l_1-1} \prod_{l=l_1}^{n} \rho(b^l u_1, \dots, b^l u_{q-1}). \tag{3.13}$$

Further, using monotonicity of the function ρ we can estimate for $l < l_{q-1}$,

$$\rho(b^l u_1, \dots, b^l u_{q-1}) \le \rho(0, \dots, 0) = \mathrm{E}\Lambda(0)^q.$$

For $l \in [l_{q-1}, l_{q-2})$, we have

$$\rho(b^l u_1, \dots, b^l u_{q-1}) \le \rho(0, \dots, 0, A, 0 \dots, 0),$$

where i_{q-1}th argument of the function ρ is equal to A and all other arguments are equal to 0. Indeed this holds due to the fact that for $l > l_{q-1}$

$$b^l u_{(q-1)} \ge b^{l_{q-1}} u_{(q-1)} \ge \frac{A}{u_{(q-1)}} u_{(q-1)} = A$$

and the monotonicity of the function ρ. Here recall that $u_{(q-1)}$ corresponds to $u_{i_{q-1}}$. Fix a small number δ which we define later. Now we can note that mixing condition (3.5) implies that

$$\lim_{A\to\infty} \rho(0, \dots, 0, A, 0, \dots, 0) = \mathrm{E}\Lambda(0)^{i_{q-1}} \mathrm{E}\Lambda(0)^{q-i_{q-1}}$$

Hence we can pick $A = A(\delta)$ sufficiently large to ensure that

$$\rho(0, \dots, 0, A, 0, \dots, 0) \le (1+\delta)\mathrm{E}\Lambda(0)^{i_{q-1}} \mathrm{E}\Lambda(0)^{q-i_{q-1}}.$$

Function $g(x) = \ln \mathrm{E}\Lambda(0)^x$ is convex. Hence we can apply Karamata majorization inequality Karamata (1932) to obtain that

$$g(i_{q-1}) + g(q - i_{q-1}) \le g(q-1) + g(1).$$

Therefore,

$$\mathrm{E}\Lambda(0)^{i_{q-1}}\mathrm{E}\Lambda(0)^{q-i_{q-1}} \le \mathrm{E}\Lambda(0)^{q-1}\mathrm{E}\Lambda(0) = \mathrm{E}\Lambda(0)^{q-1}$$

and

$$\rho(0,\dots,0,A,0,\dots,0) \le (1+\delta)\mathrm{E}\Lambda(0)^{q-1}.$$

Similarly, for $l \in [l_{q-2}, l_{q-3}]$, we have

$$\rho(b^l u_1,\dots,b^l u_{q-1}) \le \rho(0,\dots,0,A,\dots,0,A,0\dots,0),$$

where the arguments of the function ρ are equal to 0 except arguments i_{q-1} and i_{q-2} which are equal to A. Applying the mixing condition and increasing A if necessary we can ensure that for $l \in [l_{q-2}, l_{q-3}]$,

$$\rho(b^l u_1,\dots,b^l u_{q-1}) \le (1+\delta)\mathrm{E}\Lambda(0)^a\mathrm{E}\Lambda(0)^{b-a}\mathrm{E}\Lambda(0)^{q-b},$$

where $a = \min(i_{q-2}, i_{q-1})$, $b = \max(i_{q-2}, i_{q-1})$. We apply now Karamata's majorisation inequality twice. First application of the inequality gives

$$\mathrm{E}\Lambda(0)^a\mathrm{E}\Lambda(0)^{b-a} \le \mathrm{E}\Lambda(0)^{b-1}.$$

Second application of Karamata's inequality gives

$$\mathrm{E}\Lambda(0)^{b-1}\mathrm{E}\Lambda(0)^{q-b} \le \mathrm{E}\Lambda(0)^{q-2}.$$

Hence, for $l \in [l_{q-2}, l_{q-3})$ and sufficiently large A,

$$\rho(b^l u_1,\dots,b^l u^{q-1}) \le (1+\delta)\mathrm{E}\Lambda(0)^a\mathrm{E}\Lambda(0)^{b-a}\mathrm{E}\Lambda(0)^{q-b} \le (1+\delta)\mathrm{E}\Lambda(0)^{q-2}.$$

In exactly the same manner, using the mixing conditions and Karamata's majorisation inequality one can obtain for $l \in [l_j, l_{j-1})$ and $j = q-1, q-2, \dots, 2$

$$\rho(b^l u_1,\dots,b^l u^{q-1}) \le (1+\delta)\mathrm{E}\Lambda(0)^j.$$

Hence,

$$\begin{aligned}\prod_{l=0}^{l_1-1}\rho(b^l u_1,\dots,b^l u_{q-1}) &= \prod_{l=0}^{l_{q-1}-1}\prod_{l=l_{q-1}}^{l_{q-2}-1}\cdots\prod_{l=l_2}^{l_1-1}\rho(b^l u_1,\dots,b^l u_{q-1}) \\ &\le (1+\delta)^{l_1}\prod_{i=2}^{q}\prod_{l=l_i}^{l_{i-1}-1}\mathrm{E}\Lambda(0)^i = (1+\delta)^{l_1}\prod_{i=2}^{q}\left(\mathrm{E}\Lambda(0)^i\right)^{l_{i-1}-l_i},\end{aligned} \tag{3.14}$$

where $l_q = 0$. Rearranging the terms we can represent this product in a slightly different form

$$\prod_{i=2}^{q}\left(\mathrm{E}\Lambda(0)^i\right)^{l_{i-1}-l_i} = \prod_{i=1}^{q-1}\left(\frac{\mathrm{E}\Lambda(0)^{i+1}\mathrm{E}\Lambda(0)^{i-1}}{(\mathrm{E}\Lambda(0)^i)^2}\right)^{l_{q-1}+\cdots+l_i} \tag{3.15}$$

Now one can note that since l_i are decreasing, see (3.12),

$$l_{q-1}+\cdots+l_i \le \frac{q-i}{q-1}(l_1+\cdots+l_{q-1}),$$

for any $i=1,\ldots,q-1$. Indeed, the latter inequality is equivalent to

$$(i-1)(l_{q-1}+\cdots+l_i) \le (q-i)(l_{i-1}+\cdots+l_1),$$

which follows from

$$\frac{l_{q-1}+\cdots+l_i}{q-i} \le l_i \le l_{i-1} \le \frac{l_{i-1}+\cdots+l_1}{i-1}.$$

In addition, by the Karamata's majorization inequality,

$$\frac{\mathrm{E}\Lambda(0)^{i+1}\mathrm{E}\Lambda(0)^{i-1}}{(\mathrm{E}\Lambda(0)^i)^2} > 1.$$

Therefore,

$$\left(\frac{\mathrm{E}\Lambda(0)^{i+1}\mathrm{E}\Lambda(0)^{i-1}}{(\mathrm{E}\Lambda(0)^i)^2}\right)^{l_{q-1}+\cdots+l_i} \le \left(\frac{\mathrm{E}\Lambda(0)^{i+1}\mathrm{E}\Lambda(0)^{i-1}}{(\mathrm{E}\Lambda(0)^i)^2}\right)^{\frac{q-i}{q-1}(l_1+\ldots+l_{q-1})}$$

Hence we can continue (3.15) as follows

$$\begin{aligned}\prod_{i=2}^{q}\left(\mathrm{E}\Lambda(0)^i\right)^{l_{i-1}-l_i} &\le \prod_{i=2}^{q-1}\left(\frac{\mathrm{E}\Lambda(0)^{i+1}\mathrm{E}\Lambda(0)^{i-1}}{(\mathrm{E}\Lambda(0)^i)^2}\right)^{\frac{q-i}{q-1}(l_1+\ldots+l_{q-1})}\\ &= \left(\mathrm{E}\Lambda(0)^q\right)^{\frac{l_1+\cdots+l_{q-1}}{q-1}}\prod_{i=2}^{q-3}\left(\mathrm{E}\Lambda(0)^i\right)^{\frac{q-i+1-2(q-i)+q-i-1}{q-1}} = \left(\mathrm{E}\Lambda(0)^q\right)^{\frac{l_1+\cdots+l_{q-1}}{q-1}}. \quad (3.16)\end{aligned}$$

Plugging the latter estimate in (3.14) we arrive at

$$\prod_{l=0}^{l_1}\rho(b^l u_1,\ldots,b^l u_{q-1}) \le (1+\delta)^{l_1}\left(\mathrm{E}\Lambda(0)^q\right)^{\frac{l_1+\cdots+l_{q-1}}{q-1}}.$$

We can now make use of the condition (3.8) and by taking δ sufficiently small we can ensure that

$$\prod_{l=0}^{l_1}\rho(b^l u_1,\ldots,b^l u_{q-1}) \le b^{(1-\varepsilon)(l_1+\cdots+l_{q-1})} = (u_1u_2\ldots u_{q-1})^{-1+\varepsilon}A^{q(1-\varepsilon)} \quad (3.17)$$

for some small $\varepsilon>0$. We are left to estimate the product $\prod_{l=l_1}^{n}$ uniformly in n. For that we are going to use finiteness of the series in (3.9). First note that for $l\ge l_1$,

$b^l u_j \geq A b^{l-l_1}$. Then, by monotonicity of the function ρ, uniformly in n, for some $C > 0$

$$\prod_{l=l_1}^{n} \rho(b^l u_1, \ldots, b^l u_{q-1}) \leq \prod_{l=l_1}^{n} \rho(b^{l-l_1} A, \ldots, b^{l-l_1} A)$$
$$\leq \prod_{l=0}^{\infty} \rho(b^l, \ldots, \ldots, b^l) < C, \tag{3.18}$$

according to the finiteness of the series. Together (3.17) and (3.18) give us

$$\begin{aligned}
&\int_{0 \leq u_1, \ldots, u_{q-1} \leq 1} \prod_{l=0}^{n} \rho(b^l u_1, \ldots, b^l u_{q-1}) du_1 \ldots du_{q-1} \\
&= \sum_{i_1, \ldots i_{q-1}} \int_{0 < u_{i_1} \leq u_{i_2} \leq u_{i_{q-1}} \leq 1} \prod_{l=0}^{n} \rho(b^l u_1, \ldots, b^l u_{q-1}) du_1 \ldots du_{q-1} \\
&\leq C \sum_{i_1, \ldots i_{q-1}} \int_{0 < u_{i_1} \leq u_{i_2} \leq u_{i_{q-1}} \leq 1} (u_1 u_2 \ldots u_{q-1})^{-1+\varepsilon} du_1 \ldots du_{q-1} \\
&= C \int_{0 \leq u_1, \ldots, u_{q-1} \leq 1} (u_1 u_2 \ldots u_{q-1})^{-1+\varepsilon} du_1 \ldots du_{q-1}
\end{aligned} \tag{3.19}$$

which immediately gives a finite bound for $\mathrm{E} A_n(1)^q$ uniform in n.

□

Remark 1. *It is not difficult to show that (3.8) is sharp. Indeed suppose that*

$$b^{q-1} < \mathrm{E}\Lambda(0)^q$$

and that $\rho(u_1, \ldots, u_{q-1})$ is continuous at $(0, \ldots, 0)$. Then, for $\varepsilon > 0$,

$$\begin{aligned}
\mathrm{E} A_n^q(t) &= q! \int_{0 < u_0, \ldots, u_{q-1}}^{u_0 + \cdots + u_{q-1} \leq t} \mathrm{E}\Lambda_n(0)\Lambda_n(u_1) \ldots \Lambda_n(u_1 + \cdots + u_{q-1}) du_0 \ldots du_{q-1} \\
&= q! \int_{0 < u_0, \ldots, u_{q-1}}^{u_0 + \cdots + u_{q-1} \leq t} \prod_{l=0}^{n} \rho(b^l u_1, \ldots, b^l u_{q-1}) du_0 \ldots du_{q-1} \\
&\geq q! \int_{0 < u_0 < 1/2, 0 < u_1 \ldots, u_{q-1} \leq \varepsilon / b^n} \prod_{l=0}^{n} \rho(b^l u_1, \ldots, b^l u_{q-1}) du_0 \ldots du_{q-1} \\
&\geq \frac{q!}{2} \int_{0 < u_1 \ldots, u_{q-1} \leq \varepsilon / b^n} \prod_{l=0}^{n} \rho(\varepsilon, \ldots, \varepsilon) du_1 \ldots du_{q-1} = \frac{q!}{2} \varepsilon^{q-1} \left(\frac{\rho(\varepsilon, \ldots, \varepsilon)}{b^{q-1}} \right)^n
\end{aligned}$$

Since $\rho(\varepsilon, \ldots, \varepsilon)$ can be made arbitrarily close to $\rho(0, \ldots, 0) = \mathrm{E}\Lambda(0)^q$, then, for sufficiently small $\varepsilon > 0$, $\rho(\varepsilon, \ldots, \varepsilon) > b^{q-1}$, and

$$\mathrm{E} A_n(t) \geq \frac{q!}{2} \varepsilon^{q-1} \left(\frac{\rho(\varepsilon, \ldots, \varepsilon)}{b^{q-1}} \right)^n \to \infty, \quad n \to \infty.$$

4. Scaling of moments

The aim of this Section is to establish the scaling property (2.7). For $q > 1$ let

$$\rho_q(s) = \inf_{u\in[0,1]} \left(\frac{\mathrm{E}\Lambda(0)^{q-1}\Lambda(su)}{\mathrm{E}\Lambda(0)^q} - 1 \right). \tag{4.1}$$

Note that $\rho_q(s) \le 0$. For $q \in (0,1)$ let

$$\rho_q(s) = \sup_{u\in[0,1]} \left(\frac{\mathrm{E}\Lambda(0)^{q-1}\Lambda(su)}{\mathrm{E}\Lambda(0)^q} - 1 \right). \tag{4.2}$$

For $q \le 1$ it is easy to see that $\rho_q(s) \ge 0$.

Theorem 3. *Assume that $A(t) \in \mathcal{L}_q, q \in \mathbf{R}_+$ and $\rho_q(s)$ defined in (4.1) and (4.2) is such that*

$$\sum_{n=1}^{\infty} |\rho_q(b^{-n})| < \infty. \tag{4.3}$$

Then,

$$\mathrm{E}A^q(t) \sim t^{q-\log_b \mathrm{E}\Lambda^q(t)}, \quad t \in [0,1]. \tag{4.4}$$

and process $A(t)$ is non-degenerate, that is $\mathbb{P}(A(t) > 0) > 0$.

Proof of Theorem 3

Our strategy in proving of (4.4) is to use martingale properties of the sequence $A_n(t)$. We concentrate mainly on $q > 1$, as the case $q < 1$ is symmetric. For the upper bound we obtain uniform in n bounds from above for $\mathrm{E}A_n(t)^q$. Then, since $A_n(t)$ converges to $A(t)$ in $\mathcal{L}_q$, the same estimates hold for $\mathrm{E}A(t)^q$. For the lower bound, we use the fact that as $A_n(t) \in \mathcal{L}_q$ for $q > 1$ the martingale $A_n(t)$ is closable. Hence it can be represented as $A_n(t) = \mathrm{E}(A(t)|A_1(t), \ldots, A_n(t))$. Therefore, for $q > 1$, by the conditional Jensen inequality,

$$\begin{aligned}
\mathrm{E}A_n(t)^q &= \mathrm{E}(\mathrm{E}(A(t)|A_1(t),\ldots,A_n(t)))^q \\
&\le \mathrm{E}(\mathrm{E}(A(t)^q|A_1(t),\ldots,A_n(t))) = \mathrm{E}A(t)^q.
\end{aligned}$$

Thus, we are going to obtain an estimate from below for $\mathrm{E}A_n(t)^q$ for a suitable choice of n. Clearly, by the latter inequality, this estimate will hold for $\mathrm{E}A(t)^q$ as well.

We start with a change of variable

$$A_n(t) = \int_0^t \Lambda_n(s)ds = t\int_0^1 \Lambda_n(ut)du \equiv t\widetilde{A}_n(t).$$

Clearly $\widetilde{A}_n(t)$ is a martingale for any fixed t.

We are going to treat the cases $q \ge 1$ and $q \le 1$ separately. This is due to the fact that for $q \ge 1$, the sequences $\widetilde{A}_n(t)^q$ and $A_n(t)^q$ are submartingales while for $q \in (0,1)$ the sequences are supermartingales with respect to the filtration $\mathcal{F}_n = \sigma(\Lambda^{(1)}, \ldots, \Lambda^{(n)})$.

We start with an upper bound for $q \geq 1$. Let $n_t = -[\log_b t]$ be the biggest integer such that $n_t \leq -\log_b t$. We use the Hölder inequality in the form,

$$\left(\int_0^1 |fg|\right)^q = \left(\int_0^1 |f||g|^{1/q}|g|^{1/p}\right)^q \leq \left(\int_0^1 |f|^q|g|\right)\left(\int_0^1 |g|\right)^{q/p},$$

where $1/q + 1/p = 1$. It follows from the latter inequality,

$$\left(\int_0^1 \prod_{k=0}^{n} \Lambda^{(k)}(ut)du\right)^q \leq \left(\int_0^1 \left(\prod_{k=0}^{n_t-1} \Lambda^{(k)}(ut)\right)^q \prod_{k=n_t}^{n} \Lambda^{(k)}(ut)du\right)\left(\int_0^1 \prod_{k=n_t}^{n} \Lambda^{(k)}(ut)du\right)^{q/p}.$$

Applying expectation to both sides we obtain, using independence of $\Lambda^{(k)}$ of each other,

$$\mathrm{E}\widetilde{A}_n(t)^q \leq \left(\int_0^1 \prod_{k=0}^{n_t-1} \mathrm{E}(\Lambda^{(k)})^q(ut) \prod_{k=n_t}^{n} \mathrm{E}\left(\Lambda^{(k)}(ut)\left(\int_0^1 \prod_{k=n_t}^{n} \Lambda^{(k)}(vt)dv\right)^{q/p}\right)du\right).$$

By the stationarity of the process $\Lambda(t)$ we have

$$\prod_{k=0}^{n_t-1} \mathrm{E}(\Lambda^{(k)})^q(ut) = (\mathrm{E}\Lambda(0)^q)^{n_t} \leq (\mathrm{E}\Lambda(0)^q)^{-\log_b t} = t^{-\log_b \mathrm{E}\Lambda(0)^q}.$$

Therefore,

$$\begin{aligned}
\mathrm{E}\widetilde{A}_n(t)^q &\leq t^{-\log_b \mathrm{E}\Lambda(0)^q}\mathrm{E}\int_0^1 \prod_{k=n_t}^{n}\left(\Lambda^{(k)}(ut)\left(\int_0^1 \prod_{k=n_t}^{n} \Lambda^{(k)}(vt)dv\right)^{q/p}\right)du \\
&= t^{-\log_b \mathrm{E}\Lambda(0)^q}\mathrm{E}\left(\int_0^1 \prod_{k=n_t}^{n} \Lambda^{(k)}(ut)du\right)^{1+q/p} = t^{-\log_b \mathrm{E}\Lambda(0)^q}\mathrm{E}\left(\int_0^1 \prod_{k=n_t}^{n} \Lambda^{(k)}(b^k ut)du\right)^q \\
&= t^{-\log_b \mathrm{E}\Lambda(0)^q}\mathrm{E}\left(\int_0^1 \prod_{k=0}^{n-n_t} \Lambda^{(k)}(b^k u b^{-[\log_b t]+\log_b t})du\right)^q = t^{-\log_b \mathrm{E}\Lambda(0)^q}\mathrm{E}\widetilde{A}_{n-n_t-1}(b^{-[\log_b t]+\log_b t})^q.
\end{aligned}$$

Now note that

$$\mathrm{E}\widetilde{A}_{n-n_t-1}(b^{-[\log_b t]+\log_b t})^q = b^{[log_b t]-\log_b t}\mathrm{E}A(b^{-[\log_b t]+\log_b t})^q \leq b \sup_{s\in[0,1]} \mathrm{E}A(s)^q.$$

This bound is uniform in n and therefore,

$$\mathrm{E}A(t)^q \leq b t^{q-\log_b \mathrm{E}\Lambda(0)^q} \sup_{s\in[0,1]} \mathrm{E}A(s)^q.$$

Now we turn to the lower bound for $q \geq 1$.

Since $\widetilde{A}_n(t)$ is a submartingale,

$$\mathrm{E}\widetilde{A}(t)^q \ge \mathrm{E}\widetilde{A}_{n_t}(t)^q,$$

where $n_t = [-\log_b t]$.

We are going to obtain a recursive estimate for $\mathrm{E}\widetilde{A}_n(t)$. First,

$$\begin{aligned}\mathrm{E}\widetilde{A}_{n+1}(t)^q &= \mathrm{E}\left(\int_0^1 \Lambda_n(ut)\Lambda^{(n+1)}(b^{n+1}ut)du\right)^q \\ &= \mathrm{E}\left(\int_0^1 \Lambda_n(ut)(\Lambda^{(n+1)}(b^{n+1}ut) - \Lambda^{(n+1)}(0))du + \widetilde{A}_n(t)\Lambda^{(n+1)}(0)\right)^q.\end{aligned}$$

Now we can use an elementary estimate of the form: if $a+b>0$ and $b>0$ then

$$(a+b)^q \ge qab^{q-1} + b^q \tag{4.5}$$

for $q \ge 1$. This estimate is easy to prove by analyzing the function $(1+t)^q - 1 - qt$ for $t \ge -1$. Applying (4.5) we obtain

$$\begin{aligned}\mathrm{E}\widetilde{A}_{n+1}(t)^q \ge q\mathrm{E}&\left[\left(\widetilde{A}_n(t)\Lambda^{(n+1)}(0)\right)^{q-1}\int_0^1 \Lambda_n(ut)(\Lambda^{(n+1)}(b^{n+1}ut) - \Lambda^{(n+1)}(0))du\right] \\ &+ \mathrm{E}\left(\widetilde{A}_n(t)\Lambda^{(n+1)}(0)\right)^q \equiv E_1 + E_2.\end{aligned} \tag{4.6}$$

The second expectation is straightforward,

$$E_2 = \mathrm{E}\left(\int_0^1 \Lambda_n(ut)\Lambda^{(n+1)}(0)du\right)^q = \mathrm{E}\Lambda(0)^q\mathrm{E}\widetilde{A}_n(t)^q, \tag{4.7}$$

where we use independence of Λ_n and $\Lambda^{(n+1)}$. For the first expectation, rearranging the terms, we have

$$\begin{aligned}E_1 &= q\mathrm{E}\left[\int_0^1 \widetilde{A}_n(t)^{q-1}\Lambda_n(ut)(\Lambda^{(n+1)}(0))^{q-1}(\Lambda^{(n+1)}(b^{n+1}ut) - \Lambda^{(n+1)}(0))du\right] \\ &= q\int_0^1 \mathrm{E}\widetilde{A}_n(t)^{q-1}\Lambda_n(ut)\mathrm{E}(\Lambda^{(n+1)}(0))^{q-1}(\Lambda^{(n+1)}(b^{n+1}ut) - \Lambda^{(n+1)}(0))du.\end{aligned}$$

By the definition of ρ_q, see (4.1), for all $u \in [0,1]$,

$$\mathrm{E}(\Lambda^{(n+1)}(0))^{q-1}(\Lambda^{(n+1)}(b^{n+1}ut) - \Lambda^{(n+1)}(0)) \ge \mathrm{E}\Lambda(0)^q\rho_q(b^{n+1}t).$$

Therefore,

$$\begin{aligned}E_1 &\ge q\int_0^1 \mathrm{E}\widetilde{A}_n(t)^{q-1}\Lambda_n(ut)du\mathrm{E}\Lambda(0)^q\rho_q(b^{n+1}t) \\ &\ge q\mathrm{E}\left[\widetilde{A}_n(t)^{q-1}\int_0^1 \Lambda_n(ut)du\right]\mathrm{E}\Lambda(0)^q\rho_q(b^{n+1}t) \\ &= q\mathrm{E}\widetilde{A}_n(t)^q\mathrm{E}\Lambda(0)^q\rho_q(b^{n+1}t)\end{aligned}$$

Therefore
$$E_1 \geq q\mathrm{E}\widetilde{A}_n(t)^q \mathrm{E}\Lambda(0)^q \rho_q(b^{n-n_t}).$$
The latter inequality together with (4.6) and (4.7) gives us
$$\mathrm{E}\widetilde{A}_{n+1}(t)^q \geq \mathrm{E}\widetilde{A}_n(t)^q \mathrm{E}\Lambda(0)^q \left(1 + q\rho_q(b^{n-n_t})\right) \tag{4.8}$$
Now we can iterate it. First fix N^* such that $|q\rho_q(b^{-n})| < 1$ for $n > N^*$. Then, iterating (4.8), we obtain
$$\mathrm{E}\widetilde{A}^q_{n_t-N^*} \geq (\mathrm{E}\Lambda(0)^q)^{n_t-N^*} \prod_{n=0}^{n_t-N*} \left(1 + q\rho_q(b^{n-n_t})\right) \geq (\mathrm{E}\Lambda(0)^q)^{n_t-N^*} \prod_{n=N^*}^{\infty} \left(1 + q\rho_q(b^{-n})\right). \tag{4.9}$$
It is sufficient to note that the latter product is strictly positive due to (4.3). As $\widetilde{A}_n(t)^q$ is a submartingale, we have $\mathrm{E}\widetilde{A}(t)^q \geq \mathrm{E}\widetilde{A}^q_{n_t-N^*}$ and the required lower bound for $q > 1$ follows. One can also see that $\widetilde{A}(t)$ is non-degenerate. Indeed, by our assumptions $\mathrm{E}\Lambda(0)^q > 0$ and the infinite product in (4.9) is strictly positive.

The proof for $q \in (0,1)$ is symmetric. For these values of q and a fixed t, the process $\widetilde{A}_n(t)^q$ is a supermartingale with respect to the natural filtration $\mathcal{F}_n = \sigma(\Lambda^{(1)}, \ldots, \Lambda^{(n)})$. The bound from below is proved using the reverse Hölder inequality for $q \in (0,1)$ and p such that $1/p + 1/q = 1$:
$$\left(\int_0^1 |fg|\right) \geq \left(\int_0^1 |f|^q\right)^{1/q} \left(\int_0^1 |g|^p\right)^{1/p}.$$
Note that p is negative. We are going to use this inequality in the form,
$$\left(\int_0^1 |fg|\right)^q = \left(\int_0^1 |f||g|^{1/q}|g|^{1/p}\right)^q \geq \left(\int_0^1 |f|^q|g|\right)\left(\int_0^1 |g|\right)^{q/p},$$
It follows from the latter inequality,
$$\left(\int_0^1 \prod_{k=0}^{n} \Lambda_k(ut)du\right)^q \geq \left(\int_0^1 \left(\prod_{k=0}^{n_t} \Lambda_k(ut)\right)^q \prod_{k=n_t+1}^{n} \Lambda_k(ut)du\right)\left(\int_0^1 \prod_{k=n_t+1}^{n} \Lambda_k(ut)du\right)^{q/p}.$$
The rest of the proof goes exactly as the proof of the upper bound for $q > 1$.

To prove the upper bound, we proceed similarly to the proof of the lower bound for $q > 1$. First we establish a recursive estimate. The elementary inequality (4.5) still holds (in the opposite direction), for $q \in (0,1)$,$(a+b)^q \leq qab^{q-1} + b^q$, for $a + b > 0, b > 0$. Repeating step by step the arguments for $q > 1$ we obtain an upper bound
$$\mathrm{E}\widetilde{A}_{n+1}(t)^q \leq \mathrm{E}\widetilde{A}_n(t)^q \mathrm{E}\Lambda(0)^q \left(1 + q\rho_q(b^{n-n_t})\right).$$
Applying this bound recursively
$$\mathrm{E}A^q_{n_t-N^*} \leq (\mathrm{E}\Lambda(0)^q)^{n_t-N^*} \prod_{n=0}^{n_t-N*} \left(1 + q\rho_q(b^{n-n_t})\right) \leq (\mathrm{E}\Lambda(0)^q)^{n_t-N^*} \prod_{n=N^*}^{\infty} \left(1 + q\rho_q(b^{-n})\right).$$

It is sufficient to note that the latter product converge due to (4.3). As $\widetilde{A}_n(t)^q$ is a supermartingale, we have $\mathrm{E}\widetilde{A}(t)^q \le \mathrm{E}\widetilde{A}^q_{n_t-N^*}$ and the required upper bound for $q<1$ follows.

□

5. Log-normal scenario with possible long-range dependence

The log-normal hypothesis of Kolmogorov Kolmogorov (1962) features prominently in turbulent cascades. In this section, we provide a related model, namely the log-normal scenario, for multifractal products of stochastic processes. In fact, this log-normal scenario has its origin in Kahane Kahane (1985, 1987). In this section we present a general result on log-normal scenario for a model with possible long-range dependence.

In this Section we consider a mother process of the form

$$\Lambda(t) = \exp\left\{X(t) - \frac{1}{2}\sigma_X^2\right\}, \tag{5.1}$$

where $X(t)$, $t\in[0,1]$ is a zero-mean Gaussian, measurable, separable stochastic process with covariance function

$$R_X(\tau) = \sigma_X^2 \mathrm{Corr}(X(t), X(t+\tau))) \tag{5.2}$$

We combine Theorems 2 and 3 for this special case in order to have a precise scaling law for the moments.

For the log-normal process we obtain the following specifications of the moment generating functions (2.1) and (2.2):

$$\begin{aligned}
M(\zeta) &= \mathrm{E}\exp\left\{\zeta\left(X(t) - \frac{1}{2}\sigma_X^2\right)\right\} = e^{\frac{1}{2}\sigma_X^2(\zeta^2-\zeta)}, \quad \zeta\in\mathbb{R}^1, \\
M(\zeta_1,\zeta_2;t_1-t_2) &= \mathrm{E}\exp\left\{\zeta_1\left(X(t_1) - \frac{1}{2}\sigma_X^2\right) + \zeta_2\left(X(t_2) - \frac{1}{2}\sigma_X^2\right)\right\} \\
&= \exp\left\{\frac{1}{2}\sigma_X^2\left[\zeta_1^2 - \zeta_1 + \zeta_2^2 - \zeta_2\right] + \zeta_1\zeta_2 R_X(t_1-t_2)\right\}, \quad \zeta_1,\zeta_2\in\mathbb{R}^1,
\end{aligned}$$

where $\sigma_X^2\in(0,\infty)$. It turns out that, in this case,

$$\begin{aligned}
M(1) &= 1; \quad M(2) = e^{\sigma_X^2}; \quad \sigma_\Lambda^2 = e^{\sigma_X^2} - 1; \\
\mathrm{Cov}(\Lambda(t_1),\Lambda(t_2)) &= M(1,1;t_1-t_2) - 1 = e^{R_X(t_1-t_2)} - 1
\end{aligned}$$

and

$$\log_b \mathrm{E}\Lambda(t)^q = \frac{(q^2-q)\sigma_X^2}{2\log b}, \quad q>0.$$

Note that

$$e^{R_X(t_1-t_2)} - 1 \geq R_X(t_1 - t_2).$$

Using Theorem 2 and Theorem 3 we obtain

Theorem 4. *Let $X(t)$ be a zero-mean Gaussian measurable separable stochastic process with the correlation function*

$$\mathrm{Corr}(X(t), X(t+\tau)) \leq C\tau^{-\alpha}, \quad \alpha > 0, \tag{5.3}$$

for sufficiently large τ, and for some $a > 0$,

$$1 - \mathrm{Corr}(X(t), X(t+\tau)) \leq C\left|\tau\right|^{a}, \tag{5.4}$$

for sufficiently small τ. Assume that

$$b > \exp\left\{q^{*}\sigma_X^2/2\right\}, \tag{5.5}$$

where $q^{*} \geq 2$ is a fixed integer. Then the stochastic processes

$$A_n(t) = \int_0^t \prod_{j=0}^{n} \Lambda^{(j)}\left(sb^j\right) ds, t \in [0,1]$$

converge in $\mathcal{L}_q, 0 < q \leq q^{*}$ to the stochastic process $A(t), t \in [0,1]$, as $n \to \infty$, such that

$$\mathrm{E}A(t)^q \sim t^{\varsigma(q)}, q \in [0, q^{*}], \tag{5.6}$$

and the scaling function is given by

$$\varsigma(q) = -aq^2 + (a+1)q, q \in [0, q^{*}],$$

where

$$a = \frac{\sigma_X^2}{2\log b}.$$

Moreover, if

$$\mathrm{Corr}(X(t), X(t+\tau)) = \frac{L(\tau)}{\left|\tau\right|^{\alpha}}, \alpha > 0,$$

where L is a slowly varying at infinity function, bounded on every bounded interval, then

$$VarA(t) \geqslant t^{2-\alpha}\sigma_X^2 \int_0^1 \int_0^1 \frac{L(t\left|u-v\right|)dudw}{L(t)\left|u-w\right|^{\alpha}}, 0 < \alpha < 1, \tag{5.7}$$

and

$$VarA(t) \geqslant 2t\sigma_X^2 \int_0^t (1 - \frac{u}{t})\frac{L(u)}{\left|u\right|^{\alpha}} du, \alpha \geq 1. \tag{5.8}$$

Remark 2. *We interpret the inequality (5.7) as a form of long-range dependence of the limiting process.*

Remark 3. *Note that the correlation function* $\mathrm{Corr}(X(t), X(t+\tau)) = (1+|\tau|^2)^{-\alpha/2}$, $\alpha > 0$, *satisfies all assumptions of the Theorem 2 (with* $L(\tau) = |\tau|^\alpha / (1+|\tau|^2)^{\alpha/2}$*), among the others.*

Proof. We will prove $\mathcal{L}_{q^*}$ convergence by applying Theorem 2, where $q^* \geq 2$ is an integer. Hence $\mathcal{L}_q$ convergence will hold forn any $q \geq q^*$. To simplify notation we will write q instead of q^* when proving $\mathcal{L}_{q^*}$ convergence.

The moment generating function of the multidimensional normal distribution is given by the following expression

$$M(\zeta_1, \zeta_2, \ldots, \zeta_q) = \mathrm{E} e^{\zeta_1 X(s_1) + \cdots + \zeta_p X(s_q)} = \exp\left\{ \frac{1}{2} \sum_{i=1}^q \sum_{j=1}^q \zeta_i \zeta_j R_X(|s_i - s_j|) \right\}.$$

One can immediately see that

$$\mathrm{E}\left(\Lambda(s_1)\Lambda(s_2)\ldots\Lambda(s_q)\right) = \mathrm{E} e^{X(s_1) - \frac{1}{2}\sigma_X^2} \ldots e^{X(s_q) - \frac{1}{2}\sigma_X^2}$$

$$= M(1, 1, \ldots, 1) e^{-\frac{q}{2}\sigma_X^2} = e^{\frac{1}{2}\sum_{i=1}^q \sum_{j=1}^q R_X(|s_i - s_j|)} = e^{-\frac{q}{2}\sigma_X^2 = e^{\sum_{1 \leq i < j \leq q} R_X(s_j - s_i)}}.$$

We can now substitute this into (3.4) and obtain

$$\rho(u_1, u_2, \ldots, u_{q-1}) = \exp\left\{ \sum_{1 \leq i < j \leq q-1} R_X(u_i + \cdots + u_j) \right\}.$$

Since the function $R_X(u)$ is monotone decreasing in u, function $\rho(u_1, \ldots, u_{q-1})$ is monotone decreasing in all arguments. Next we need to check the mixing condition (3.5). Let $1 \leq i_1 < i_2 < \ldots \leq i_m$ and $u_i = A$ if $i \in \{i_1, \ldots, i_m\}$ and 0 otherwise. Then, as $A \to \infty$, and $i_0 = 0, i_{m+1} = q$

$$\lim_{A \to \infty} \rho(u_1, \ldots, u_{q-1}) = \exp\left\{ \sum_{1 \leq k \leq m+1} \sum_{i_{k-1} < i < j < i_k} R_X(u_i + \cdots + u_j) \right\}$$
$$= \mathrm{E}\Lambda(0)^{i_1} \mathrm{E}\Lambda(0)^{i_2 - i_1} \cdot \ldots \mathrm{E}\Lambda(0)^{q - i_m}, \quad (5.9)$$

where we used that $\mathrm{E}\Lambda(0)^l = e^{\frac{l(l-1)}{2}\sigma_X^2}$. Finally, we should check the convergence of the series (3.9). We have,

$$\exp\{R_X(qb^n\} \leq \rho(b^n, \ldots, b^n) \leq \exp\left\{ \frac{q(q-1)}{2} R_X(b^n \right\}.$$

As $n \to \infty$, $R_X(b^n) \to 0$. Hence

$$(1 + o(1))R_X(qb^n) \le \rho(b^n, \dots, b^n) - 1 \le (1 + o(1))\frac{q(q-1)}{2}R_X(b^n).$$

As both sums

$$\sum_{n=1}^{\infty} R_X(qb^n) < \infty, \quad \sum_{n=1}^{\infty} R_X(b^n) < \infty,$$

the convergence of the series (3.9) follows. Condition (3.8) becomes

$$b^{q-1} > \mathrm{E}\Lambda(0)^q = \exp\left\{\frac{q(q-1)}{2}\sigma_X^2\right\},$$

which is equivalent to (5.5).

Next we are going to prove scaling (5.6). For that we apply the results of Section 4. We now do not assume that q is an integer. We need to show that (4.3) holds for ρ_q, where $q \in (0, q^\star)$ and ρ_q is defined in (4.1) and (4.2). For $q > 1$ we have, for sufficiently small s,

$$\begin{aligned}|\rho_q(s)| &= -\inf_{u\le 1}\left(\frac{\mathrm{E}\Lambda(0)^{q-1}\Lambda(su)}{\mathrm{E}\Lambda(0)^q} - 1\right) = -\inf_{u\le 1}\left(e^{\sigma_X^2((q-1)\rho_X(su)+1-q)} - 1\right)\\ &\le \sup_{u\le 1}\left(1 - e^{(1-q)\sigma_X^2(su)^a}\right) \le 1 - e^{(1-q)\sigma_X^2(s)^a} \le (q-1)\sigma_X^2 s^a.\end{aligned}$$

Thus using condition (5.4) one can immediately see that the series (4.3) converges. For $q < 1$, the same arguments give the bound

$$\rho_q(s) \le (1-q)\sigma_X^2 s^a.$$

Using condition (5.4) one can immediately see that the series (4.3) converges. Therefore, by the results of Section 4 scaling (5.6) holds. □

6. Geometric Ornstein-Uhlenbeck processes

This section reviews a number of known results on Lévy processes (see Bertoin (1996), Kyprianou (2006)) and OU type processes (see Barndorff-Nielsen (1998), Barndorff-Nielsen and Shephard (2001)) The geometric OU type processes have been studied also by Matsui and Shieh (2009).

As standard notation we will write

$$\kappa(z) = C\{z; X\} = \log \mathrm{E}\exp\{izX\}, \quad z \in \mathbb{R}$$

for the cumulant function of a random variable X, and

$$K\{\zeta; X\} = \log \mathrm{E}\exp\{\zeta X\}, \quad \zeta \in D \subseteq \mathbb{C}$$

for the Lévy exponent or Laplace transform or cumulant generating function of the random variable X. Its domain D includes the imaginary axis and frequently larger areas.

A random variable X is infinitely divisible if its cumulant function has the Lévy-Khintchine form

$$C\{z;X\} = iaz - \frac{d}{2}z^2 + \int_{\mathbb{R}} \left(e^{izu} - 1 - izu\mathbf{1}_{[-1,1]}(u)\right)\nu(du), \tag{6.1}$$

where $a \in \mathbb{R}$, $d \geq 0$ and ν is the Lévy measure, that is, a non-negative measure on $\mathbb{R}$ such that

$$\nu(\{0\}) = 0, \qquad \int_{\mathbb{R}} \min\left(1, u^2\right)\nu(du) < \infty. \tag{6.2}$$

The triplet (a, d, ν) uniquely determines the random variable X. For a Gaussian random variable $X \sim N(a, d)$, the Lévy triplet takes the form $(a, d, 0)$.

A random variable X is self-decomposable if, for all $c \in (0,1)$, the characteristic function $f(z)$ of X can be factorized as $f(z) = f(cz)\, f_c(z)$ for some characteristic function $f_c(z)$, $z \in \mathbb{R}$. A homogeneous Lévy process $Z = \{Z(t), t \geq 0\}$ is a continuous (in probability), càdlàg process with independent and stationary increments and $Z(0) = 0$ (recalling that a càdlàg process has right-continuous sample paths with existing left limits.) For such processes we have $C\{z; Z(t)\} = tC\{z; Z(1)\}$ and $Z(1)$ has the Lévy-Khintchine representation (6.1).

If X is self-decomposable, then there exists a stationary stochastic process $\{X(t),$ $t \geq 0\}$, such that $X(t) \stackrel{d}{=} X$ and

$$X(t) = e^{-\lambda t}X(0) + \int_{(0,t]} e^{-\lambda(t-s)}dZ(\lambda s), \tag{6.3}$$

for all $\lambda > 0$ (see Barndorff-Nielsen (1998)). Conversely, if $\{X(t), t \geq 0\}$ is a stationary process and $\{Z(t), t \geq 0\}$ is a Lévy process, independent of $X(0)$, such that $X(t)$ and $Z(t)$ satisfy the Itô stochastic differential equation

$$dX(t) = -\lambda X(t)\, dt + dZ(\lambda t), \tag{6.4}$$

for all $\lambda > 0$, then $X(t)$ is self-decomposable. A stationary process $X(t)$ of this kind is said to be an OU type process. The process $Z(t)$ is termed the background driving Lévy process (BDLP) corresponding to the process $X(t)$. In fact (6.3) is the unique (up to indistinguishability) strong solution to Eq. (6.4).

Let $X(t)$ be a square integrable OU process. Then $X(t)$ has the correlation function

$$\mathrm{Corr}(X(0), X(t)) = r_X(t) = \exp\{-\lambda|t|\}. \tag{6.5}$$

The cumulant transforms of $X = X(t)$ and $Z(1)$ are related by

$$C\{z;X\} = \int_0^\infty C\left\{e^{-s}z; Z(1)\right\}ds = \int_0^z C\{\xi; Z(1)\}\frac{d\xi}{\xi}, C\{z;Z(1)\} = z\frac{\partial C\{z;X\}}{\partial z}.$$

Suppose that the Lévy measure ν of X has a density function $p(u), u \in \mathbb{R}$, which is differentiable. Then the Lévy measure $\tilde{\nu}$ of $Z(1)$ has a density function $q(u), u \in \mathbb{R}$, and p and q are related by

$$q(u) = -p(u) - up'(u) \tag{6.6}$$

(see Barndorff-Nielsen (1998)).

The logarithm of the characteristic function of a random vector $(X(t_1), ..., X(t_m))$ is of the form

$$\log \mathrm{E} \exp \{i(z_1 X(t_1) + ... + z_m X(t_m)\} = \int_{\mathbb{R}} \kappa(\sum_{j=1}^{m} z_j e^{-\lambda(t_j - s)} \mathbf{1}_{[0,\infty)}(t_j - s))ds, \tag{6.7}$$

where

$$\kappa(z) = \log \mathrm{E} \exp \{izZ(1)\} = C\{z; Z(1)\},$$

and the function (6.7) has the form (6.1) with Lévy triplet $(\tilde{a}, \tilde{d}, \tilde{\nu})$ of $Z(1)$.

The logarithms of the moment generation functions (if they exist) take the forms

$$\log \mathrm{E} \exp \{\zeta X(t)\} = \zeta a + \frac{d}{2}\zeta^2 + \int_{\mathbb{R}} (e^{\zeta u} - 1 - \zeta u \mathbf{1}_{[-1,1]}(u))\nu(du),$$

where (a, d, ν) is the Lévy triplet of $X(0)$, or in terms of the Lévy triplet $(\tilde{a}, \tilde{d}, \tilde{\nu})$ of $Z(1)$

$$\begin{aligned} \log \mathrm{E} \exp \{\zeta X(t)\} &= \tilde{a} \int_{\mathbb{R}} (\zeta e^{-\lambda(t-s)} \mathbf{1}_{[0,\infty)}(t-s))ds + \frac{\tilde{d}}{2}\zeta^2 \int_{\mathbb{R}} (\zeta e^{-\lambda(t-s)} \mathbf{1}_{[0,\infty)}(t-s))^2 ds \\ &+ \int_{\mathbb{R}} \int_{\mathbb{R}} [\exp\left\{u\zeta e^{-\lambda(t-s)} \mathbf{1}_{[0,\infty)}(t-s)\right\} - 1 - u\left(\zeta e^{-\lambda(t-s)} \mathbf{1}_{[0,\infty)}(t-s)\right) \mathbf{1}_{[-1,1]}(u)]\tilde{\nu}(du)\, ds, \end{aligned} \tag{6.8}$$

and

$$\begin{aligned} &\log \mathrm{E} \exp\{\zeta_1 X(t_1) + \zeta_2 X(t_2)\} \\ &= \tilde{a} \int_{\mathbb{R}} \left(\sum_{j=1}^{2} \zeta_j e^{-\lambda(t_j - s)} \mathbf{1}_{[0,\infty)}(t_j - s))ds + \frac{\tilde{d}}{2}\zeta^2 \int_{\mathbb{R}} (\sum_{j=1}^{2} \zeta_j e^{-\lambda(t_j - s)} \mathbf{1}_{[0,\infty)}(t_j - s))^2 \right) ds \\ &\quad + \int_{\mathbb{R}} \int_{\mathbb{R}} [\exp\left\{ u \sum_{j=1}^{2} \zeta_j e^{-\lambda(t_j - s)} \mathbf{1}_{[0,\infty)}(t_j - s) \right\} - 1 \\ &\quad - u \left(\sum_{j=1}^{2} \zeta_j e^{-\lambda(t_j - s)} \mathbf{1}_{[0,\infty)}(t_j - s) \right) \mathbf{1}_{[-1,1]}(u)]\tilde{\nu}(du)\, ds. \end{aligned} \tag{6.9}$$

Let us consider a geometric OU-type process as the mother process:

$$\Lambda(t) = e^{X(t) - c_X}, c_X = \log \mathrm{E} e^{X(0)}, M(\zeta) = \mathrm{E} e^{\zeta(X(t) - c_X)}, M_0(\zeta) = \mathrm{E} e^{\zeta X(t)}$$

where $X(t), t \in \mathbb{R}_+$, is the OU-type stationary process (6.3). Note that

$$\frac{M_0(q)}{M_0(1)^q} = \frac{M(q)}{M(1)^q}.$$

Then the correlation function of the mother process is of the form.

$$\operatorname{Corr}(\Lambda(t), \Lambda(t+\tau)) = \frac{M(1,1;\tau) - 1}{M(2) - 1}, \tag{6.10}$$

where now

$$\begin{aligned} M(\zeta_1, \zeta_2; \tau) &= \mathrm{E}\exp\{\zeta_1(X(t_1) - c_X) + \zeta_2(X(t_2) - c_X)\} \\ &= \exp\{-(\zeta_1 + \zeta_2)c_X\}\,\mathrm{E}\exp\{\zeta_1 X(t_1) + \zeta_2 X(t_2)\}, \end{aligned} \tag{6.11}$$

and $\mathrm{E}\exp\{\zeta_1 X(t_1) + \zeta_2 X(t_2)\}$ is defined by (6.9).

To prove that a geometric OU process satisfies the covariance decay condition (4.3) in Theorem *3*, the expression given by (6.9) is not ready to yield the decay as $t_2 - t_1 \to \infty$.

The following result plays a key role in multifractal analysis of geometric OU processes.

Theorem 5. *Let $X(t), t \in \mathbb{R}_+$ be an OU-type stationary process (6.3) such that the Lévy measure ν in (6.1) of the random variable $X(0)$ satisfies the condition: for an integer $q^* \geq 2$,*

$$\int_{|x|\geq 1} x e^{q^* x} \nu(dx) < \infty. \tag{6.12}$$

Then, for any fixed b such that

$$b > \left\{ \frac{M_0(q^*)}{M_0(1)^{q^*}} \right\}^{\frac{1}{q^*-1}}, \tag{6.13}$$

the sequence of stochastic processes

$$A_n(t) = \int_0^t \prod_{j=0}^{n} \Lambda^{(j)}\left(sb^j\right) ds, t \in [0,1]$$

converges in $\mathcal{L}_q$ to the stochastic process $A(t) \in \mathcal{L}_q$, as $n \to \infty$, for every fixed $t \in [0,1]$. The limiting process $A(t), t \in [0,1]$ satisfies

$$\mathrm{E}A^q(t) \sim t^{q - \log_b \mathrm{E}\Lambda^q(t)}, \quad q \in [0, q^*].$$

The scaling function is given by

$$\varsigma(q) = q - \log_b \mathrm{E}\Lambda^q(t) = q\left(1 + \frac{c_X}{\log b}\right) - \log_b M_0(q), \quad q \in [0, q^*]. \tag{6.14}$$

In addition,

$$\operatorname{Var}A(t) \geqslant 2t \int_0^t \left(1 - \frac{s}{t}\right)(M(1,1;s) - 1) ds, \tag{6.15}$$

where the bivariate moment generating function $M(\zeta_1, \zeta_2; t_1 - t_2)$ is given by (6.11)

Proof of Theorem 5 We are starting with $\mathcal{L}_q$ convergence. To show the convergence we apply Theorem 2. It is sufficient to show the convergence for $q = q^*$ since the convergence for $q < q^*$ immediately follows from the convergence for $q = q^*$. First we will derive a suitable explicit expression for $\rho(u_1, \dots, u_{q-1})$. Put $s_1 = 0 \le s_2 = u_1 \le s_2 = u_1 + u_2, \dots, s_q = u_1 + \cdots + u_{q-1}$. Then,

$$\rho(u_1, \dots, u_{q-1}) = \mathrm{E}\Lambda(s_1) \dots \Lambda(s_q) = \mathrm{E}\exp\left\{X(s_1) + \dots + X(s_q) - qc_X\right\}.$$

Using representation (6.3) one can obtain

$$X(s_q) = e^{-\lambda(s_q - s_{q-1})} X(s_{q-1}) + \int_{(s_{q-1}, s_q]} e^{-\lambda(s_q - s)} dZ(\lambda s).$$

Then,using independence of $X(s_{q-1})$ and the integral $\int_{(s_{q-1}, s_q]} e^{-\lambda(s_q - s)} dZ(\lambda s)$ we obtain

$$\begin{aligned}
&\mathrm{E}\exp\left\{X(s_1) + \dots + X(s_q)\right\} \\
&\quad = \mathrm{E}\exp\left\{X(s_1) + \dots + (1 + e^{-\lambda(s_q - s_{q-1})}) X(s_{q-1})\right\} \mathrm{E}e^{\int_{(s_{q-1}, s_q]} e^{-\lambda(s_q - s)} dZ(\lambda s)} \\
&\quad = \mathrm{E}\exp\left\{X(s_1) + \dots + (1 + e^{-\lambda(s_q - s_{q-1})}) X(s_{q-1})\right\} \frac{\mathrm{E}e^{X(s_q)}}{\mathrm{E}e^{e^{-\lambda(s_q - s_{q-1})} X(0)}} \\
&\quad = \mathrm{E}\exp\left\{X(s_1) + \dots + (1 + e^{-\lambda(s_q - s_{q-1})}) X(s_{q-1})\right\} \frac{M_0(1)}{M_0(e^{-\lambda(s_q - s_{q-1})})}.
\end{aligned}$$

Proceeding further by induction we obtain

$$\begin{aligned}
&\mathrm{E}\exp\left\{X(s_1) + \dots + X(s_q)\right\} \\
&= M_0(1) \frac{M_0(1 + e^{-\lambda u_{q-1}}) M_0(1 + e^{-\lambda u_{q-2}} + e^{-\lambda(u_{q-1} + u_{q-2})}) \dots M_0(1 + e^{-\lambda u_1} + \dots + e^{-\lambda(u_1 + \dots + u_{q-1})})}{M_0(e^{-\lambda u_{q-1}}) M_0(e^{-\lambda u_{q-2}} + e^{-\lambda(u_{q-1} + u_{q-2})}) \dots M_0(e^{-\lambda u_1} + \dots + e^{-\lambda(u_1 + \dots + u_{q-1})})}.
\end{aligned}$$

Hence

$$\begin{aligned}
\rho(u_1, \dots, u_{q-1}) &= \frac{M_0(1 + e^{-\lambda u_{q-1}})}{M_0(1) M_0(e^{-\lambda u_{q-1}})} \frac{M_0(1 + e^{-\lambda u_{q-2}} + e^{-\lambda(u_{q-1} + u_{q-2})})}{M_0(1) M_0(e^{-\lambda u_{q-2}} + e^{-\lambda(u_{q-1} + u_{q-2})})} \dots \times \\
&\qquad \times \frac{M_0(1 + e^{-\lambda u_1 + \dots + e^{-\lambda(u_1 + \dots + u_{q-1})}})}{M_0(1) M_0(e^{-\lambda u_1 + \dots + e^{-\lambda(u_1 + \dots + u_{q-1})}})}. \qquad (6.16)
\end{aligned}$$

This representation allows us to show monotonicity of $\rho(u_1, \dots, u_{q-1})$. For that we use the following inequality

$$\frac{M_0(1 + s)}{M_0(s)} \le \frac{M_0(1 + t)}{M_0(t)} \qquad (6.17)$$

for $s \le t$. This inequality follows from the fact that $\ln M_0(t)$ is a convex function and the Karamata majorisation inequality. Hence

$$\frac{M_0(1 + s)}{M_0(1) M_0(s)}$$

is monotone increasing in s. Since $e^{-\lambda u}$ is monotone decreasing in u the representation (6.16) implies that $\rho(u_1,\ldots,\ldots,u_{q-1})$ is monotone decreasing in all variables.

Condition (3.8) becomes

$$b^{q-1} > \mathrm{E}\Lambda(0)^q = \frac{M_0(q)}{M_0(1)^q},$$

which is equivalent to (6.13).

To show the finiteness of the series (3.9) we are going to use the following statement.

Lemma 1. *For $s\in[0,1]$, the following estimate holds*

$$\frac{M_0(1+s)}{M_0(1)M(s)} \le \left(\frac{M_0(2)}{M_0(1)e^{\mathrm{E}X(1)}}\right)^s. \tag{6.18}$$

Proof of Lemma 1 Function $\ln M_0(t)$ is convex. Therefore,

$$\ln M_0(1+s) = \ln M_0((1-s)+2s) \le (1-s)\ln M_0(1) + s\ln M_0(2).$$

In addition, by the Jensen inequality,

$$M_0(s) = \mathrm{E}e^{sX(1)} \ge e^{s\mathrm{E}X(1)}.$$

Together these inequalities imply,

$$\frac{M_0(1+s)}{M_0(1)M_0(s)} \le \frac{M_0(1)^{1-s}M_0(2)^s}{M_0(1)e^{s\mathrm{E}X(1)}} = \left(\frac{M_0(2)}{M_0(1)e^{\mathrm{E}X(1)}}\right)^s.$$

□

Now, using (6.16) and monotone decrease of $M_0(1+s)/M_0(s)$

$$\begin{aligned} 1 \le \rho(b^n,\ldots,b^n) &\le \left(\frac{M_0(1+e^{-\lambda b^n})}{M_0(1)M_0(e^{-\lambda b^n})}\right)^q \\ &\le C^{qe^{-\lambda b^n}} \le 1 + o(1)\ln Cqe^{-\lambda b^n}, \end{aligned} \tag{6.19}$$

where the former inequality follows from Lemma 1 with $C = M_0(2)/(M_0(1)e^{\mathrm{E}X(1)})$. Then, convergence of the series (6.16) follows from the finiteness of the series $\sum_{n=1}^\infty e^{-\lambda b^n}$.

Finally we need to check the mixing condition (3.5). Let $1\le i_1 < i_2 < \ldots \le i_m$ and $u_i = A$ if $i\in\{i_1,\ldots,i_m\}$ and 0 otherwise. In this context it is convenient to use (6.16) in the form

$$\rho(u_1,\ldots,u_{q-1}) = \prod_{j=1}^{q-1} \frac{M_0(1+\sum_{k=j}^{q-1} e^{-\lambda\sum_{l=j}^k u_l})}{M_0(1)M_0(\sum_{k=j}^{q-1} e^{-\lambda\sum_{l=j}^k u_l})}$$

Then, as $A \to \infty$, and $i_0 = 0, i_{m+1} = q$

$$\lim_{A\to\infty} \rho(u_1, \ldots, u_{q-1}) = \prod_{\alpha=1}^{m} \prod_{j=i_\alpha+1}^{i_{\alpha+1}-1} \frac{M_0(1+\sum_{k=j}^{i_{\alpha+1}-1} 1)}{M_0(1)M_0(\sum_{k=j}^{i_{\alpha+1}-1} 1)}$$
$$= \prod_{\alpha=1}^{m} \frac{M_0(i_{\alpha+1}-i_\alpha)}{M_0(1)^{i_{\alpha+1}-i_\alpha}} = \prod_{\alpha=1}^{m} \mathrm{E}\Lambda(0)^{i_{\alpha+1}-i_\alpha}. \quad (6.20)$$

This proves (3.5). Therefore Theorem 2 gives $\mathcal{L}_q$ convergence of $A_n(t)$.

To prove the scaling property we are going to use the results of Theorem 3. First using representation (6.3), we have for any q,

$$\begin{aligned}
\mathrm{E}\Lambda(t)\Lambda(0)^{q-1} &= \mathrm{E}\exp\{(q-1)X(0) + X(t) - qc_X\} \\
&= \mathrm{E}\exp\{(q-1+e^{-\lambda t})X(0) + \int_{(0,t)} e^{-\lambda(t-s)} dZ(\lambda s) - qc_X\} \\
&= \mathrm{E}\exp\left\{(q-1+e^{-\lambda t})X(0) - qc_X\right\} \frac{\mathrm{E}\exp\{e^{-\lambda t}X(0) + \int_{(0,t)} e^{-\lambda(t-s)} dZ(\lambda s)\}}{\mathrm{E}e^{e^{-\lambda t}X(0)}} \\
&= \mathrm{E}\exp\{(q-1+e^{-\lambda t})X(0) - qc_X\} \frac{\mathrm{E}\exp\{X(t)\}}{\mathrm{E}e^{e^{-\lambda t}X(0)}} \\
&= \frac{M_0(q-1+e^{-\lambda t})}{M_0(1)^{q-1}M(e^{-\lambda t})}.
\end{aligned}$$

Then,

$$\frac{\mathrm{E}\Lambda(t)^{q-1}\Lambda(0)^{q-1}}{\mathrm{E}\Lambda(0)^q} = \frac{M_0(q-1+e^{-\lambda t})M_0(1)}{M_0(q)M_0(e^{-\lambda t})}.$$

For $q > 1$ the latter function is monotone decreasing in t, as follows from the Karamata motorization inequality. Hence,

$$|\rho_q(s)| = \sup_{u\in[0,1]} \left(1 - \frac{\mathrm{E}\Lambda(su)\Lambda(0)^{q-1}}{\mathrm{E}\Lambda(0)^q}\right) = 1 - \frac{M_0(q-1+e^{-\lambda s})M_0(1)}{M_0(q)M_0(e^{-\lambda s})}. \quad (6.21)$$

Function $f(x) = \ln M(x)$ is convex. Condition (6.12) ensures that the derivative $f'(q)$ exists for $q \le q^*$. Then, for any $x \le q$,

$$f(x) - f(q) \ge (x-q)f'(q).$$

In particular for $x = q - 1 + e^{-\lambda s}$,

$$f(q-1+e^{-\lambda s}) - f(q) \ge (-1+e^{-\lambda s})f'(q).$$

In addition, by the Jensen inequality,

$$M_0(e^{-\lambda s}) = \mathrm{E}e^{e^{-\lambda s}X(0)} \le (\mathrm{E}e^{X(0)})^{e^{-\lambda s}} = M_0(1)^{e^{-\lambda s}}.$$

The latter two inequalities give

$$\begin{aligned} |\rho_q(s)| &\leq 1 - e^{(-1+e^{-\lambda s})(f'(q)-f(1))} \\ &\leq (1-e^{-\lambda s})(f'(q)-f(1)) \leq \lambda s(f'(q)-f(1)). \end{aligned} \tag{6.22}$$

Then

$$0 \leq \sum_{n=1}^{\infty} |\rho_q(b^{-n})| \leq \lambda(f'(q)-f(1)) \sum_{n=1}^{\infty} b^{-n} < \infty.$$

Since we have already shown that $A(t) \in \mathcal{L}_q$ for $q < q^*$, we can apply Theorem 3.

□

As an example consider a stationary OU-type process $X(t)$, defined in (6.4), with marginal normal inverse Gaussian distribution $NIG(\alpha, \beta, \delta, \mu)$, which is self-decomposable, and hence infinitely divisible, see Barndorff-Nielsen (1998). The moment generating function of $NIG(\alpha, \beta, \delta, \mu)$ is given by the formula:

$$\log M_0(\zeta) = \mu\zeta + \delta\left[\sqrt{\alpha^2-\beta^2} - \sqrt{\alpha^2-(\beta+\zeta)^2}\right],\ |\beta+\zeta| < \alpha,$$

and the set of parameters satisfies the following constraints

$$\delta > 0, 0 \leq |\beta| \leq \alpha, \mu \in \mathbb{R}, \gamma^2 = \alpha^2 - \beta^2.$$

The Lévy triplet pf the process $X(t)$ is of the form $(a, 0, \nu)$, where

$$a = \mu + 2\pi^{-1}\delta\alpha \int_0^1 \sinh(\beta x) K_1(\alpha x)\, dx, \nu(du) = \pi^{-1}\delta\alpha\, |u|^{-1} K_1(\alpha\, |u|)\, e^{\beta u} du,$$

where the modified Bessel function of the third kind of index λ :

$$K_\lambda(z) = \int_0^\infty \exp\{-z\cosh(x)\}\cosh(\lambda x)\, dx, \quad \mathbf{Re}\ \lambda > 0.$$

Consider a mother process of the form

$$\Lambda(t) = \exp\{X(t) - c_X\}, c_X = \mu + \delta\sqrt{\alpha^2-\beta^2} - \sqrt{\alpha^2-(\beta+1)^2}, |\beta+1| < \alpha,$$

Let $q^* \leq \alpha - |\beta|$ be an integer and put

$$Q = \{q : 0 < q < q^*, |\beta+1| < \alpha, \mu \in \mathbb{R},\ \delta > 0\}$$

If

$$b > \exp\left\{-\delta\sqrt{\alpha^2-\beta^2} + \frac{\delta\sqrt{\alpha^2-(\beta+q^*)^2} - q^*\delta\sqrt{\alpha^2-(\beta+1)^2}}{1-q^*}\right\},$$

then the statement of Theorem 5 holds for $q \in Q$ with the scaling function

$$\varsigma(q) = \left(1 - \frac{\delta\left[\sqrt{\beta^2+\gamma^2-(\beta+1)^2} - \gamma\right]}{\log b}\right) q + \frac{\delta}{\log b}\sqrt{\beta^2+\gamma^2-(\beta+q)^2} - \frac{\delta\gamma}{\log b} - 1,$$

that is the log-normal inverse Gaussian scenario holds. This is an extension to Theorem 5 of Anh, Leonenko and Shieh (2008a).

Some other scenarios can be found in an extended version of this paper available on ArxivDenisov and Leonenko (2011).

7. Connections and prospects

Both papers Muzy and Bacry (2002) and Barral and Jin (2012) (see also their references) introduce multifractal random measures μ as a limit of positive martingales μ_j defined in a framework of log-infinitely divisible cascades constructed as independently scattered random measures on some cones on the plane. In particular, Barral and Jin (2012) extended some classical results valid for canonical multiplicative cascades to exact scaling of log-infinitely divisible cascades.

If $\psi(z)$ is the characteristic Lévy exponent with Lévy triplet (a, d, ν), see (6.2), and using notation of the paper, let $\varphi(q) = \log_2 \mathrm{E}(W^q) - (q-1) = \psi(-\mathrm{i}q) - (q-1)$, for some infinitely divisible random variable W, which generates cascade, then

(i) the necessary and sufficient condition for non-generacy of μ, is of the form: $\varphi'(1^-) < 0$, and (ii) the necessary and sufficient condition for E $(\|\mu\|^q) < \infty$, is of the form: $\varphi(q) < 0, q > 1$. Also, if $\psi(-\mathrm{i}2) < \infty$, the increments of limiting multifractal measure is stationary process with long-range dependence, see again Barral and Jin (2012).

Bacry and Muzy's construction uses other shapes for the cone, but in the notation above the condition of non-generacity is of the form i), while the condition of L_2 convergence and E $(\|\mu\|^2) < \infty$, is of the form: $\bar{\psi}(2) < 1$, where $\bar{\psi}(q)$ is the Laplace exponent of Lévy-Khintchin representation of some infinitely divisible random variable. In this case no long-range dependence between the increments of the multifractal measure, and in order to have long-range dependence they used the so-called multifractal random walk, that is superposition of fractional Brownian motion and limiting multifractal process, assuming that they are independent.

Our construction has connection to both papers. Firstly, we present general results on $\mathcal{L}_q$ convergence (Theorems 1 and 2) without any assumptions about log-infinitely divisibility of mother process. These results are more general then results of the above papers. To see this, one can apply these results (for $q = 2$) for the geometric stationary diffusion mother process, in which cases several scenarios are possible, including log-beta scenario, which is not log-infinitely divisible, see [6] for more details. Both short-range dependence and long-range dependence potentially covered by Theorems 1 and 2. Then we consider the geometric OU processes, which have log-self-decomposable marginal distributions, this is a subclass of log-infinitely divisible distributions, and inclusion is strict. In this case our results are less general in terms of possible scenarios, as well as our conditions, see Theorem 4. In particular our condition for log-gamma scenario required $\alpha > 2$ (see Theorem 9), while in the framework of the paper Muzy and Bacry (2002) for log-gamma scenario one needs only $\alpha > 1$. Also, for a α-stable OU process, the results Musy and Barcy (2002) and Barral and Jin (2012) hold for $\alpha \in (0, 2)$, while our condition (6.12) does not hold. Next, the results of the papers Muzy and Bacry (2002) and Barral and Jin

(2012) can be applied for discrete infinitely divisible distributions, i.e., to get log-poisson scenario, while our results of section 6 can not be applied for discrete distributions, since they are not self-decomposable. However by using results of sections 2 and 3, one can obtain log-poisson scenarios (among the others) by using the multiplicative products of ergodic birth-death processes, see [5] for details. As far as dependence is concern our approach allows to model both short- and long-range dependence, this question will be considered in a subsequent paper .

In the same spirit one can obatin the log Meixner or more generally log-z multifractal scenario (see Anh *et al* (2008a)) or log-Euler's gamma multifractal scenario (see Anh *et al* (2008b)). In principle, it is possible to obtain the log-hyperbolic scenarios for which there exist exact forms of Lévy measures of the OU process and the BDLP Lévy process; however some analytical work is still to be carried out. This will be done elsewhere.

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