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A 3rd Order ENO-Like Multi-Moment Method for Solving Hyperbolic Conservation Laws

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ABSTRACT

We propose a conservative multi-moment method based on 3rd order polynomial interpolation functions and the essentially non-oscillatory (ENO) scheme. Based on CIP-CSL3, the proposed method employs two multi-moment based 3rd order polynomial interpolation functions; CIP-CSL3D where three constraints are used in the upwind cell with one constraint in the downwind cell-centre, and a new, complementary method CIP-CSL3U, which uses a constraint in the upwind cell-centre instead. Based on the ENO (Essentially Non-Oscillatory) scheme, either CSL3D or CSL3U is chosen using a local smoothness indicator.

Numerical results from common benchmark tests in one dimension (both linear and non-linear) show that this scheme maintains 4th order accuracy for smooth profiles while successfully reducing numerical oscillation around discontinuity without giving way to numerical diffusion. Its competitive results show that this is a promising high-order method for CFD applications given its algorithmic simplicity, and can be extended for multi-dimensions.

Key Words: *CIP-CSL; multi-moment; ENO; upwind scheme*

1. Introduction

Many numerical schemes have been developed for solving hyperbolic equations [1][2][3]. A way to improve order of accuracy is by increasing the stencil size or by employing the multi-moment method, where at least two types of moments are used and each is updated using a different formulation. For example, the CIP-CSL2 (constrained interpolation conservative semi-Lagrangian) method solves the conservation equation by using two boundary values updated using the finite difference formulation and a cell integrated average moment updated using the finite volume formulation.

Several variants of the CIP-CSL method have been proposed thus far (CSL3 [4], CSL3D [5]). We propose a variant of CIP-CSL3 method with an ENO-like formulation [6].

2. Numerical Method

The CIP-CSL method solves the conservation equation

$$\frac{\partial \phi}{\partial t} + \frac{\partial (u\phi)}{\partial x} = 0 \tag{1}$$

where ϕ is the property being transported at a constant velocity u along the x axis.

2.1. CIP-CSL3D method

The CIP-CSL3D uses 3 moments in the upwind cell (i.e. 1 cell average $\bar{\phi}_{i-1}$ and 2 cell boundary values $\phi_{i-\frac{1}{2}}$ and $\phi_{i-\frac{3}{2}}$) and 1 moment in the downwind cell centre ($\hat{\phi}_i$) which is itself interpolated from the downwind cell's boundary values and cell integrated average. Consequently it yields a cubic interpolation function as such, for $u > 0$

$$\Phi_{i-1}^{CSL3D}(x) = C_{3,i-1}^{CSL3D}(x - x_{i-\frac{1}{2}})^3 + C_{2,i-1}^{CSL3D}(x - x_{i-\frac{1}{2}})^2 + C_{1,i-1}^{CSL3D}(x - x_{i-\frac{1}{2}}) + \phi_{i-\frac{1}{2}} \tag{2}$$

to interpolate between $\phi_{i-\frac{1}{2}}$ and $\phi_{i-\frac{3}{2}}$.

Using this interpolation function, the boundary value $\phi_{i-\frac{1}{2}}$ is updated by the conservation equation in a finite difference form

$$\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} = -\phi\frac{\partial u}{\partial x} \tag{3}$$

and the cell average is updated by

$$\frac{\partial}{\partial t}\int_{x-\frac{1}{2}}^{x+\frac{1}{2}} \phi dx = -\frac{F_{i+1/2} - F_{i-1/2}}{\Delta x} \tag{4}$$

where $F_{i-1/2}$ is the flux at cell boundary $x_{i-1/2}$.

2.2. CIP-CSL3U

This is a proposed variant of CIP-CSL3 complementary to CIP-CSL3D, where an upwind moment in the cell centre ($\hat{\phi}_{i-2}$) is used as a constraint. The cubic interpolation function used to interpolate between $\phi_{i-\frac{1}{2}}$ and $\phi_{i-\frac{3}{2}}$ is similar as that used for CSL3D

$$\Phi_{i-1}^{CSL3U}(x) = C_{3,i-1}^{CSL3U}(x - x_{i-\frac{1}{2}})^3 + C_{2,i-1}^{CSL3U}(x - x_{i-\frac{1}{2}})^2 + C_{1,i-1}^{CSL3U}(x - x_{i-\frac{1}{2}}) + \phi_{i-\frac{1}{2}} \tag{5}$$

for $u > 0$. The cell boundary and cell integrated average are updated in the same manner as CSL3D.

2.3. ENO-like Combination of CSL3D and CSL3U

The ENO scheme proposed a way to avoid interpolating across discontinuities to avoid oscillations. A hierarchy is developed beginning with 1 or 2 cells, and 1 cell is added at a time to the stencil from two candidates on the left and right based on its local smoothness determined by some smoothness indicator. Our proposed method departs from the ENO method by instead having 2 fixed stencils, which is selected depending on their respective smoothness.

The smoothness indicator used to select between the 2 stencils is a modified version of the indicator proposed by Zhang and Shu (2007) [7].

$$S_{CSL3U} = \int_{x-\frac{1}{8}}^{x+\frac{1}{8}} \Delta x\left(\frac{\delta\phi^{CSL3U}(x)}{\delta x}\right)^2 dx - \frac{7}{9}\int_{x-\frac{1}{8}}^{x+\frac{1}{8}} \Delta x^3\left(\frac{\delta^2\phi^{CSL3U}(x)}{\delta^2 x}\right)^2 dx \tag{6}$$

$$S_{CSL3D} = \int_{x-\frac{1}{8}}^{x+\frac{1}{8}} \Delta x\left(\frac{\delta\phi^{CSL3D}(x)}{\delta x}\right)^2 dx - \frac{7}{9}\int_{x-\frac{1}{8}}^{x+\frac{1}{8}} \Delta x^3\left(\frac{\delta^2\phi^{CSL3D}(x)}{\delta^2 x}\right)^2 dx \tag{7}$$

3. Results

We validate the proposed method using both linear and non-linear scalar transport problems. In this paper, the results by CSL3 and CSL3D are also included in this section for comparison.

3.1. *Complex wave propagation*

We test the proposed methodology using the complex wave propagation problem, with $u(x) = 1$, grid size $N = 200$, $\Delta t = 0.4\Delta x$, $\Delta x = 2/N$, and periodic boundary conditions.

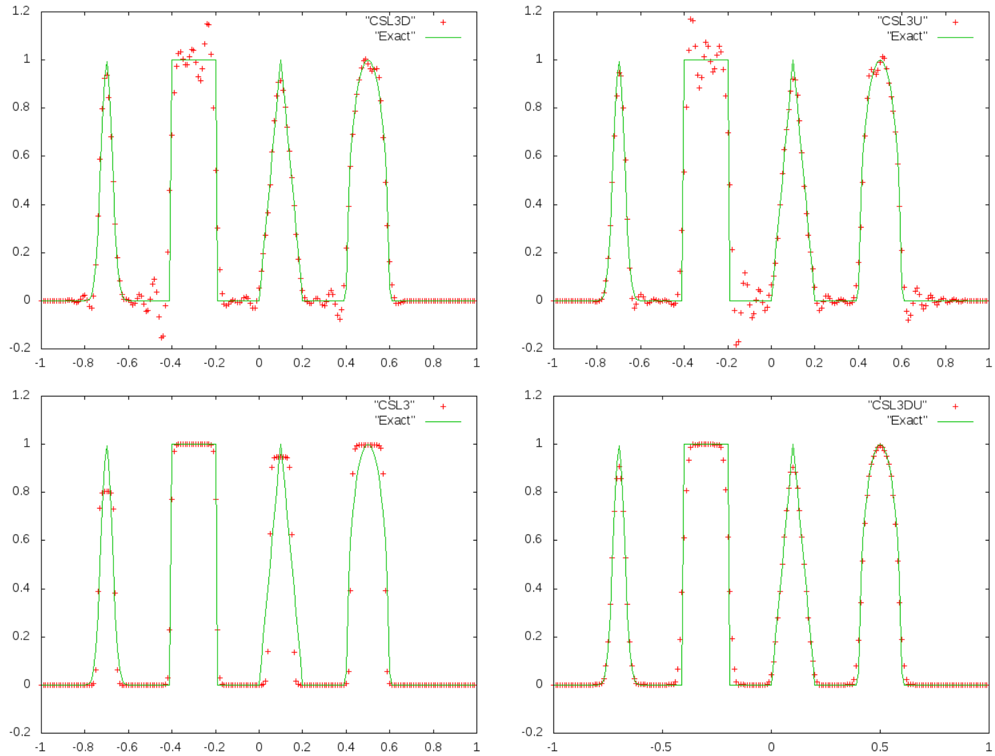

Figure 1: Numerical results of the complex wave propagation test at t=16 (4000 time steps) with N=200

Figure 1 shows that unlike CSL3, CSL3D and CSL3U capture sharp edges well. It is also evident that the oscillations generated by CSL3D and CSL3U are on opposite sides of each other (i.e. where CSL3D has oscillation on the downwind side, CSL3U has the same oscillation on the upwind side). CSL3DU successfully combines CSL3D and CSL3U by preserving sharp edges and suppressing oscillations.

3.2. *Square wave propagation in a non-uniform velocity field*

In this test, a square wave is transported in the following non-uniform velocity field

$$u(x, 0) = \frac{1}{1 + 0.4sin(2\pi x)} \tag{8}$$

Figure 2 shows that while individually, CSL3D and CSL3U generate large oscillations near discontinuities similar to Test 1, the combination of both eliminates oscillations to the same degree as CSL3 without giving way to diffusion.

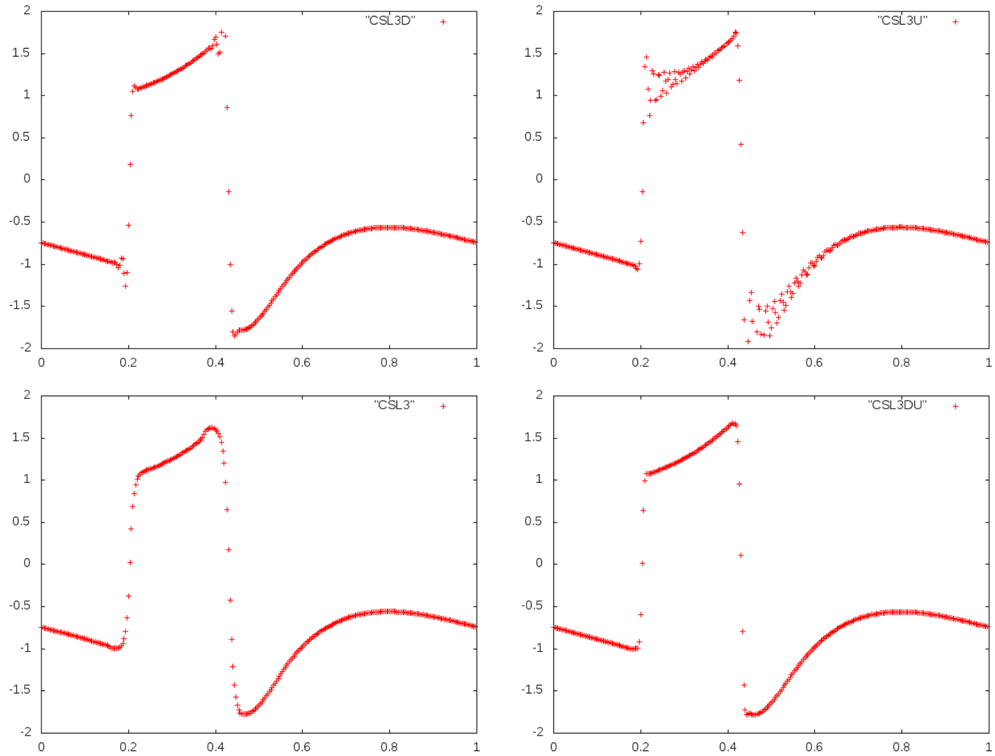

Figure 2: Numerical results of square wave propagation in a non-uniform velocity field at t = 1.8 and N = 300

4. Conclusions

We proposed the CIP-CSL3DU method which employs two 3rd order polynomial interpolation functions. One interpolation function is constructed based on 3 constraints in the upwind cell and 1 constraint in the downwind cell (CIP-CSL3D) and one interpolation is based on 3 constraints in the upwind cell and 1 constraint in the further upwind cell (CIP-CSL3U). By selecting the smoother of these two stencils, we produce CIP-CSL3DU which successfully suppresses Gibbs oscillation while maintaining numerical viscosity.

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