SPACE-TIME FRACTIONAL STOCHASTIC EQUATIONS ON REGULAR BOUNDED OPEN DOMAINS

V.V. Anh[1], **N.N. Leonenko**[2] **and M.D. Ruiz-Medina**[3]

Abstract

Fractional (in time and in space) evolution equations defined on Dirichlet regular bounded open domains, driven by fractional integrated in time Gaussian spatiotemporal white noise, are considered here. Sufficient conditions for the definition of a weak-sense Gaussian solution, in the mean-square sense, are derived. The temporal, spatial and spatiotemporal Hölder continuity, in the mean-square sense, of the derived solution is obtained, under suitable conditions, from the asymptotic properties of the Mittag-Leffler function, and the asymptotic order of the eigenvalues of a fractional polynomial of the Dirichlet negative Laplacian operator on such bounded open domains.

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1. Introduction

Space-time fractional diffusion equations are introduced when integer-order derivatives in space and in time are replaced by their fractional counterpart. In particular, they can model anomalous diffusion processes in physics (Meerschaert *et al.* [31]). Fractional diffusion equations are very popular in several fields of application (see Gorenflo and Mainardi [21]; Metzler and Klafter [34], among others).

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Since the pioneer papers by Bochner [11] and Feller [18] who proved the connection between the stable distribution and fractional calculus, the theory of α-stable distributions and processes has been extensively developed. Specifically, Bochner [11] formulated the Cauchy problem, whose solution is the symmetric α-stable distribution. Feller [18] extended these results to a more general situation by replacing the fractional Laplacian $-(-\Delta)^{\alpha/2}$ by a pseudodifferential operator with symbol

$$-|\lambda|^{\alpha}\exp\left(i\,\mathrm{sign}\,(\lambda)\,\theta\pi/2\right), \quad \lambda\in\mathbb{R}, \quad \alpha\in(0,2),$$

where α is the index of stability, and θ is the index of skewness (asymmetry). The corresponding solutions generate all stable distributions. Despite a large number of fractional operators (see Samko *et al.* [37]), there were few known specific examples of generating more general distributions. Actually, Anh, Leonenko and Sikorskii [4] analyze the Cauchy problem characterizing the main properties of Riesz-Bessel distribution.

The traditional model for spreading particles at the macroscopic level is the well-known heat equation

$$\partial_t u = \Delta u,$$

with Δ denoting the Laplacian operator, and ∂_t the partial derivative with respect to time. The relative particle concentration can be predicted in terms of the Gaussian probability density providing a point source solution of the heat equation. The paths of individual particles are described in terms of the realizations of Browninan motion. Dirichlet boundary value problems for the heat equation, as well as for more general equations, given in terms of elliptic diffusion operators can be seen in Bass [8] and Davies [15], among others. Particle sticking and trapping phenomena can be described when partial derivative in time ∂_t is replaced by fractional derivative ∂_t^{β} for $0<\beta<1$. While if negative Laplacian operator $(-\Delta)$ is replaced by fractional power $(-\Delta)^{\alpha/2}$, for $0<\alpha<2$, long particle jumps can be represented. The space-time fractional diffusion equation is defined in terms of both fractional derivative operators in time and in space:

$$\partial_t^{\beta} u = (-\Delta)^{\alpha/2} u, \tag{1.1}$$

whose solution displays self-similarity and heavy tails. The particle concentration profile provided by the corresponding probability density solution has sharper peak and heavy tails. A non-Markovian setting can then be introduced through an inverse stable subordinator time change (see also Barkai *et al.* [7]; Benson *et al.* [9]; Gorenflo and Mainardi [20]; [21]; Meerschaert *et al.* [31]; Schneider and Wyss [38], among others). The extension to the case of Riesz-Bessel subordinators is addressed in Anh and McVinish [5], considering, in the space-time fractional diffusion equation (**1**), the pseudodifferential operator $(\Delta)^{\alpha/2}(I-\Delta)^{\gamma/2}$. A different stochastic framework is analyzed in the papers by Anh and Leonenko [2], [3], where the spectral representation of the mean-square solution of the following space-time fractional diffusion equation with random initial conditions

$$\partial_t^{\beta} u = (-\Delta)^{\alpha/2} (I-\Delta)^{\gamma/2} u, \quad u_0(\mathbf{x}) = \eta(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n,$$

is derived. Here, η is a measurable random field defined on a complete probability space $(\Omega, \mathcal{A}, P)$. Gaussian and non-Gaussian limiting distributions of the renormalized solution are obtained as well. Also, in the context of stochastic evolution equations on an unbounded domain, a functional approach was adopted by Kelbert, Leonenko and Ruiz-Medina [25], where the spectral properties of the mean-square solution of fractional in time and in space evolution equations driven by random white noise are derived. These results are extended to the more general framework of stochastic partial differential equations driven by fractional Brownian motion in Leonenko, Ruiz-Medina, Taqqu [26], where, in particular, the correlation structure and spectral properties of the mean-square solution to fractional in time and in space evolution equations driven by fractional Brownian are analyzed.

In the context of fractional diffusion on bounded domains, we refer to the papers by Defterli, D'Elia, Du, Gunzburger, Lehoucq and Meerschaert [16]; Chen, Meerschaert and Nane [12], and Meerschaert, Nane and Vellaisamy [32], where strong solutions, and their probabilistic representation are obtained. On the other hand, Mijena and Nane [35] consider fractional heat equation on unbounded domains, with a non-linear random external force, involving space-time white noise. Sufficient conditions for the existence and uniqueness of mild solutions, as well as for their continuity are derived. We consider here a different framework. Specifically, we study the weak-sense solution of the following fractional in space and in time stochastic partial differential equation, with Dirichlet boundary conditions, and null initial condition:

$$\frac{\partial^{\beta}}{\partial t^{\beta}} c\,(t,\mathbf{x}) + (-\Delta_D)^{\alpha/2} (I-\Delta_D)^{\gamma/2} c\,(t,\mathbf{x}) = I_t^{1-\beta} \varepsilon\,(t,\mathbf{x})\,, \ \mathbf{x} \in D \tag{1.2}$$

$$c(t,\mathbf{x}) = 0, \quad \mathbf{x} \in \partial D, \quad \forall t, \quad c(0,\mathbf{x}) = 0, \quad \forall \mathbf{x} \in D \subset \mathbb{R}^n, \tag{1.3}$$

for $\beta \in (0,1)$, $\alpha + \gamma > n$, where equality is understood in the mean-square sense. Here, the driven process

$$I_t^{1-\beta}\varepsilon = \frac{1}{\Gamma(1-\beta)} \int_0^t (t-u)^{-\beta} \varepsilon(u) du \tag{1.4}$$

is constructed from space-time Gaussian white noise ε, defined on a basic probability space $(\Omega, \mathcal{A}, P)$, and satisfying

$$E[\varepsilon\,(t,\mathbf{x})\,\varepsilon\,(s,\mathbf{y})] = \delta(t-s)\delta(\mathbf{x}-\mathbf{y}),$$

for all $t, s \in \mathbb{R}_+$, and $\mathbf{x}, \mathbf{y} \in D$, with δ being the Dirac Delta distribution. Specifically, the driven process is the Riemannan–Liouville fractional integral of order $\beta - 1$, in time, of the space-time Gaussian white noise ε, where integration is understood in the mean-square sense (see, for example, Samko *et al.* [37]). It is well-known that the inverse of the Riemannan–Liouville fractional integral of order $\beta-1$ is the Riemann-Liouville fractional derivative of order $1-\beta$,

$$D_t^{1-\beta} f(t) = \frac{d}{dt} I_t^{\beta} f(t) = \frac{d}{dt} \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} f(\tau) d\tau,$$

that, in our case, coincides with the fractional-in-time derivative of order $1-\beta$, in the Caputo-Djrbashian sense (see equation (1.5) below), since from (1.3), we assume that $\varepsilon(0, \mathbf{x}) = 0$, for all $\mathbf{x} \in D$.

Although we refer here to the particular case where fractional derivatives in space are defined from the operator $(-\Delta_D)^{\alpha/2} (I - \Delta_D)^{\gamma/2}$, with $(-\Delta_D)$ representing the Dirichlet negative Laplacian operator on regular bounded open domain D, the results derived in this paper hold, in general, for a fractional polynomial of the Dirichlet negative Laplacian operator on D, with constant coefficients, as proved in Theorem **8.1** in Section **8**. In this paper, special attention has been paid to operator $(-\Delta_D)^{\alpha/2} (I - \Delta_D)^{\gamma/2}$, since, for suitable domains, e.g., for bounded open domain satisfying the exterior cone condition, the eigenvalues of such an operator provide two-sided estimates of the eigenvalues of the corresponding restriction of the inverse of the composition of Riesz and Bessel potentials, for certain range of parameter α (see, for example, Chen and Song [13]).

Our main goal is the study of the local regularity mean-square and sample-path properties (modulus of continuity) of the derived weak-sense Gaussian solution to equations (1.2)–(1.3). Sufficient conditions are formulated to obtain the mean-quadratic local asymptotic order of the temporal, spatial and spatiotemporal increment random fields, associated with the weak-sense Gaussian solution to equations (1.2)–(1.3) (see Theorems **4.1**, **5.1** and **6.1** below). Specifically, the results derived hold under the condition that the regular bounded open domain D is such that the eigenvectors of the Dirichlet negative Laplacian operator on D are uniformly bounded. Some examples of domains D, where this condition is satisfied, are provided in Section **7**. Furthermore, the mean-square Hölder continuity in time of the random field solution is obtained under some restrictions on the parameter space. While its mean-square Hölder continuity in space requires the Hölder continuity of the eigenvectors of the Dirichlet negative Laplacian operator on domain D. The mean-square Hölder continuity in space and time directly follows, under the above-referred conditions. Also, under such conditions, the sample-path local asymptotic orders are straightforwardly derived from Theorem 3.3.3, in p.57 of Adler [1] (see Theorem **6.2** below). Note that, although the formulated Gaussian solution c is not fractional differentiable in time in the strong-sense (its time fractional differentiation is understood in the weak-sense), under the conditions assumed in this paper, it is Hölder continuous in the mean-square sense.

Note that, we have not adopted the classical framework of diffusion processes and stochastic differential equations, characterized by the Kolmogorov forward or Fokker–Planck equation. In our case, the regularized fractional derivative in time, or fractional-in-time derivative in the Caputo-Djrbashian sense, and the FokkerPlanck operator with constant coefficients are applied, in the mean-square sense, to a spatiotemporal Gaussian random field for its *almost decorrelation in space and time*. Hence, the local self-similarity properties are observed in the correlation structure in space and in time of the weak-sense mean-square Gaussian solution c, as we will prove in this paper. The approach adopted is then clearly different from the previous one considered in Chen, Meerschaert and Nane [12], since, in the last case, the properties of the transition probability densities are investigated, while, in this paper, new classes of spatiotemporal Gaussian random fields, displaying local self-similarity, are introduced in the weak sense. In particular, their local exponents of self-similarity are computed in time, space and space-time, in the mean-square and sample-path sense.

Finally, we recall the interest of considering stochastic models, in particular, spatiotemporal Gaussian random fields defined on Dirichlet regular bounded open domains (see Fuglede [19]), including, as particular cases, bounded open C^{∞}- domains, domains with C^{1}-boundary, with Lipschitz continuous boundary, or with fractal boundary, among others. Special attention , in the current literature, has been paid to the unit ball and the unit sphere, motivated by the analysis of Cosmic Microwave Background (CMB) radiation (see, for example, Leonenko and Sakhno [27]; Malyarenko [29]; Marinucci and Peccati [30]). In this setting, tensor-valued random fields on the unit sphere are considered for the investigation of the combinations $Q \pm \ iU$, with Q and U respectively representing the linear and circular polarization Stokes parameters.

In the following, we consider the regularized fractional derivative in time or fractional-in-time derivative in the Caputo-Djrbashian sense: For $\beta \in (0,1]$,

$$\frac{\partial^{\beta} u}{\partial t^{\beta}} = \begin{cases} \frac{\partial u}{\partial t}\left(t,\mathbf{x}\right), & \text{if } \beta = 1 \\ \frac{1}{\Gamma(1-\beta)}\frac{\partial}{\partial t}\int_0^t \left(t-\tau\right)^{-\beta} u\left(\tau,\mathbf{x}\right) d\tau - \frac{u(0,\mathbf{x})}{t^{\beta}}, & \text{if } \beta \in (0,1),\ t \in (0,T] \end{cases} \tag{1.5}$$

(see Meerschaert and Sikorskii [33]; Podlubny [36]).

The outline of the paper is the following. Preliminary elements and results are presented in Section **2**. The derivation of a weak-sense mean-square Gaussian solution to equations (1.2)-(1.3) is established in Section **3**. The mean-quadratic local variation exponents in time of the derived solution are obtained in Section **4**. The mean-quadratic local variation exponents in space are given in Section **5**. Section **6** then provides the asymptotic local mean quadratic orders in space and time. The modulus of continuity of the sample paths of the weak-sense mean-square solution to equations (1.2)–(1.3) is also derived in this section . Some examples are provided in Section **7** for illustration purposes. The extended formulation of the results derived for fractional polynomials of the Dirichlet negative Laplacian operator are presented in Section **8**. Final comments and some open research lines are discussed in Section **9**.

2. Preliminaries

Some preliminary definitions and results needed in the development of this paper are now introduced. Specifically, some basic results on spectral calculus for self-adjoint operators on a Hilbert space are given in Section **2.1**. The Mittag-Leffler function is conisdered in Section **2.2**. Basic elements on fractional Sobolev spaces on a regular bounded open domain are presented in Section **2.3**.

2.1. Spectral theory of self-adjoint operators on a separable Hilbert space

Let us first consider some results on spectral calculus for self-adjoint operators on a Hilbert space.

Theorem **2.1**. (Dautray and Lions, 1990, pp. 119-120 [14]) Let H be a separable Hilbert space, then an injection mapping $\widehat{\sigma}$ exists from the set of spectral families in H into the set of self-adjoint operators on H. The following assertions hold:

Let $\mathbb{A}$ be the self-adjoint operator associated with the spectral family $\{E_\lambda\}_{\lambda\in\Lambda}$, where Λ denotes the spectrum of $\mathbb{A}$. The domain of $\mathbb{A}^k$ is defined by

$$D\left(\mathbb{A}^k\right) = \left\{ x \in H : \int_\Lambda \lambda^{2k} d\left(E_\lambda x, x\right) < \infty \right\}, \ k \geq 1. \qquad (2.1)$$

For all $x \in D\left(\mathbb{A}^k\right)$, and for all $y \in H$,

$$\langle \mathbb{A}^k x, y\rangle_H = \int_\Lambda \lambda^k d\left(E_\lambda x, y\right), \tag{2.2}$$

$$\|\mathbb{A}^k x\|_H^2 = \int_\Lambda \lambda^{2k} d\left(E_\lambda x, x\right). \tag{2.3}$$

If $P_k\left(\lambda\right)$ is a polynomial of degree k, then, for all $x \in D\left(\mathbb{A}^k\right)$, and for all $y \in H$, $P_k\left(\mathbb{A}\right)$ is given by

$$\langle P_k\left(\mathbb{A}\right) x, y\rangle_H = \int_\Lambda P_k\left(\lambda\right) d\left(E_\lambda x, y\right). \tag{2.4}$$

Finally, for a continuous function f on Λ, the following identities hold for every $x \in D\left(f\left(\mathbb{A}\right)\right)$, and $y \in H$,

$$\langle f\left(\mathbb{A}\right) x, y\rangle_H = \int_\Lambda f\left(\lambda\right) d\left(E_\lambda x, y\right). \tag{2.5}$$

Theorem **2.2**. (Dautray and Lions, 1990, pp. 140 [14]) Let $\mathbb{A}$ be a self-adjoint operator in a separable Hilbert space H. If we denote $\overline{f}$ the complex conjugate function for f, then $D\left(\overline{f}\left(\mathbb{A}\right)\right) = D\left(f\left(\mathbb{A}\right)\right)$. Moreover we have $\langle f\left(\mathbb{A}\right) x, y\rangle_H = \langle x^*, \overline{f}\left(\mathbb{A}\right) y^*\rangle$, for all $x,\ y \in D\left(f\left(\mathbb{A}\right)\right)$.

For $x \in D\left(f\left(\mathbb{A}\right)\right)$, and $y \in D\left(g\left(\mathbb{A}\right)\right)$, then

$$\langle f\left(\mathbb{A}\right) x, g\left(\mathbb{A}\right) y\rangle_H = \int_\Lambda f\left(\lambda\right) \overline{g}\left(\lambda\right) d\left(E_\lambda x, y\right). \tag{2.6}$$

Furthermore, $(f+g)\left(\mathbb{A}\right) x = f\left(\mathbb{A}\right) x + g\left(\mathbb{A}\right) x$, for all $x \in D\left(f\left(\mathbb{A}\right)\right) \cap D\left(g\left(\mathbb{A}\right)\right)$.
Finally, if $x \in D\left(f\left(\mathbb{A}\right)\right)$, with $\left(g \circ f\right)\left(\lambda\right) = g\left(\lambda\right) f\left(\lambda\right)$, then $\left[g\left(\mathbb{A}\right) f\left(\mathbb{A}\right)\right] x = \left(g \circ f\right)\left(\mathbb{A}\right) x$.

Theorem **2.1** is now applied to derive the asymptotic order of the eigenvalues of operator $(-\Delta)_D^{\alpha/2}(I-\Delta_D)^{\gamma/2}$, with, as before, $(-\Delta)_D$ representing the Dirichlet negative Laplacian operator on regular bounded open domain D.

Corollary **2.1**. The following asymptotic order holds for the eigenvalues of $(I-\Delta_D)^{\gamma/2}(-\Delta_D)^{\alpha/2}$:

$$\lim_{k\longrightarrow\infty} \frac{\lambda_k\left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)}{k^{\alpha+\gamma/n}} = \widetilde{c}(n, \alpha+\gamma)|D|^{-(\gamma+\alpha)/n}, \tag{2.7}$$

where $\widetilde{c}(n, \alpha+\gamma)$ is a positive constant depending on n, α and γ.

Futhermore, for $\{\phi_k\}_{k\geq 1}$ being the eigenvector system of the Dirichlet negative Laplacian operator $(-\Delta_D)$ on domain D, the following equality holds:

$$(-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\phi_k = \lambda_k\left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)\phi_k, \quad k\geq 1. \tag{2.8}$$

Proof. It is well-known that the eigenvalues $\{\gamma_k(-\Delta_D)\}_{k\geq 1}$ of the Dirichlet negative Laplacian operator on domain $D\subset\mathbb{R}^n$, arranged in decreasing order of their modulus magnitude satisfy (see, for example, Chen and Song [13]):

$$\gamma_k(-\Delta_D)\sim 4\pi\frac{\left(\Gamma\left(1+\frac{n}{2}\right)\right)^{2/n}}{|D|^{2/n}}k^{2/n}, \quad k\longrightarrow\infty, \tag{2.9}$$

where $f(k)\sim g(k)$ means that $\lim_{k\to\infty} f(k)/g(k)=C$, for certain positive constant C. In particular, $C=1$ in (2.9).

From equation (2.5) in Theorem **2.1**, considering $f(u)=u^{\alpha/2}(1+u)^{\gamma/2}$, we obtain

$$\lambda_k\left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right) = (\gamma_k(-\Delta_D))^{\alpha/2}\left(1+\gamma_k(-\Delta_D)\right)^{\gamma/2}. \tag{2.10}$$

Equation (2.7) then follows from equations (2.9) and (2.10).

Equation (2.8) is straightforwardly obtained from equation (2.5) in Theorem **2.1**, since in our case, i.e., for $f(u)=u^{\alpha/2}(1+u)^{\gamma/2}$, for all $x,y\in H=L^2(D)$, with $L^2(D)$ denoting the space of square integrable functions on D,

$$\begin{aligned}\int_\Lambda f(\lambda)d\left(E_\lambda x,y\right) &= \sum_{k=1}^{\infty}(\gamma_k(-\Delta_D))^{\alpha/2}\left(1+\gamma_k(-\Delta_D)\right)^{\gamma/2} \\ &\quad\times\int_{D\times D}\phi_k(\mathbf{u})\phi_k(\mathbf{v})x(\mathbf{u})y(\mathbf{v})d\mathbf{u}d\mathbf{v} \\ &= \sum_{k=1}^{\infty}(\gamma_k(-\Delta_D))^{\alpha/2}\left(1+\gamma_k(-\Delta_D)\right)^{\gamma/2}x_k y_k,\end{aligned} \tag{2.11}$$

with, as before, $\{\phi_k\}_{k\geq 1}$ being the eigenvector system of the Dirichlet negative Laplacian operator on D. Specifically, our spectral family is defined in terms of the spectral kernel $\sum_{k=1}^{\infty}\phi_k(\mathbf{u})\phi_k(\mathbf{v})$, and the spectral measure is given by a point or counting measure with atoms located at the eigenvalues. ■

2.2. Mittag-Leffler function

The weak-sense solution derived in the next section involves the Mittag-Leffler function. The definition of the Mittag-Leffler function, and a two-sided uniform inequality are now considered.

Definition **2.1**. The Mittag-Leffler function is given by

$$E_{\beta}(\mathbf{z}) = \sum_{j=0}^{\infty} \frac{(\mathbf{z})^{j}}{\Gamma(j\beta+1)}, \quad \mathbf{z} \in \mathbb{C}, \quad 0 < \beta \leq 1 \tag{2.12}$$

(see Erdély *et al.* [17]; Haubold, Mathai and Saxena [24], for a more detailed description of this function and its properties).

Lemma **2.1**. For every $\beta \in (0,1)$, the uniform estimate

$$\frac{1}{1+\Gamma(1-\beta)x} \leq E_{\beta}(-x) \leq \frac{1}{1+[\Gamma(1+\beta)]^{-1}x}$$

holds over $\mathbb{R}_{+}$ with optimal constants (see Simon [39], Theorem 4).

2.3. Fractional Sobolev spaces on regular bounded open domains

The scale of fractional Sobolev spaces is introduced within the spaces $\mathcal{S}(\mathbb{R}^{n})$, the space of C^{∞}-functions with rapid decay at infinity, and $\mathcal{D}(\mathbb{R}^{n})$, the space of C^{∞}-functions with compact support contained in $\mathbb{R}^{n}$. The dual of these spaces are respectively the space of tempered distributions, $\mathcal{S}'(\mathbb{R}^{n})$, and the space of distributions, $\mathcal{D}'(\mathbb{R}^{n})$.

For $s \in \mathbb{R}$, we denote by $H^{s}(\mathbb{R}^{n})$ the space of tempered distributions u such that $\left(1+\|\boldsymbol{\lambda}\|^{2}\right)^{s/2}\widehat{u} \in L_{2}(\mathbb{R}^{n})$, $\boldsymbol{\lambda} \in \mathbb{R}^{n}$. For a regular bounded open domain D in $\mathbb{R}^{n}$, we denote

$$\overline{H}^{s}(D) = \left\{u \in H^{s}(\mathbb{R}^{n}) : \operatorname{supp} u \subseteq \overline{D}\right\}, \tag{2.13}$$

$$H^{s}(D) = \left\{f \in \mathcal{D}'(D) : \exists F \in H^{s}(\mathbb{R}^{n}) \text{ such that } f = F_{D}\right\}, \tag{2.14}$$

where F_{D} denotes the restriction of F to D. With the quotient norm

$$\|f\|_{H^{s}(D)} = \inf_{\{F;F_{D}=f\}} \|F\|_{H^{s}(\mathbb{R}^{n})},$$

$H^{s}(D)$ is a Hilbert space (see Dautray and Lions, [14], p. 118).

3. The mean-square Gaussian solution in the weak sense

The preliminaries given in the previous section are now applied in the derivation of a zero-mean Gaussian solution to the stochastic boundary value problem (1.2)–(1.3), in the mean-square and weak senses. The following result first establishes the suitable range of parameter α and γ for the construction of a Green operator in the trace class, with kernel, the fundamental solution to the deterministic problem corresponding to (1.2)–(1.3). Namely, the following proposition states the ranges of parameters α and γ such that the sequence

$$\left\{E_\beta\left(-\lambda_k\left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)t^\beta\right),\ k\geq 1\right\} \tag{3.1}$$

is in the space l^1 of absolute summable sequences, for every $t>0$.

PROPOSITION **3.1**. For $n<\alpha+\gamma$,

$$\sum_{k=1}^{\infty} E_\beta\left(-\lambda_k\left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)t^\beta\right)<\infty, \tag{3.2}$$

for every $t>0$.

Proof. From equation (2.7),

$$\lim_{k\longrightarrow\infty}\frac{\lambda_k\left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)}{k^{(\alpha+\gamma)/n}}=\widetilde{c}(n,\alpha+\gamma)|D|^{-(\alpha+\gamma)/n}. \tag{3.3}$$

Therefore, there exists k_0 such that for $k\geq k_0$,

$$L_1k^{(\alpha+\gamma)/n}\leq\lambda_k\left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)\leq L_2k^{(\alpha+\gamma)/n}, \tag{3.4}$$

for certain positive constants $0<L_1<L_2$, depending on k_0, and $\alpha+\gamma$ and n. In particular, for $k\geq k_0$,

$$\frac{1}{1+[\Gamma(1+\beta)]^{-1}\lambda_k\left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)t^\beta}\leq\frac{1}{1+[\Gamma(1+\beta)]^{-1}L_1k^{(\alpha+\gamma)/n}t^\beta}. \tag{3.5}$$

Now, applying Lemma **2.1**, for each fixed $t>0$,

$$
\begin{aligned}
&\sum_{k=1}^{\infty} E_{\beta}\left(-t^{\beta}\lambda_k\left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)\right)\\
&=\sum_{k=1}^{k_0} E_{\beta}\left(-t^{\beta}\lambda_k\left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)\right)\\
&+\sum_{k=k_0+1}^{\infty} E_{\beta}\left(-t^{\beta}\lambda_k\left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)\right)\\
&=M(\beta,\alpha,\gamma,n)+\sum_{k=k_0+1}^{\infty} E_{\beta}\left(-t^{\beta}\lambda_k\left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)\right)\\
&\leq M(\beta,\alpha,\gamma,n)+\sum_{k=k_0+1}^{\infty} E_{\beta}\left(-t^{\beta}L_1 k^{(\alpha+\gamma)/n}\right)\\
&\leq M(\beta,\alpha,\gamma,n)+\int_0^{\infty} E_{\beta}\left(-t^{\beta}L_1 x^{(\alpha+\gamma)/n}\right)dx\\
&=M(\beta,\alpha,\gamma,n)+\frac{t^{-\beta n/(\alpha+\gamma)}}{(\alpha+\gamma)/n}\int_0^{\infty} E_{\beta}\left(u\right)u^{\frac{n}{\alpha+\gamma}-1}du\\
&\leq M(\beta,\alpha,\gamma,n)+\frac{t^{-\beta n/(\alpha+\gamma)}}{(\alpha+\gamma)/n}\int_0^{\infty}\frac{u^{\frac{n}{\alpha+\gamma}-1}}{1+[\Gamma(1+\beta)]^{-1}u}du<\infty,
\end{aligned}
\tag{3.6}
$$

since

$$
M(\beta,\alpha,\gamma,n)=\sum_{k=1}^{k_0} E_{\beta}\left(-t^{\beta}\lambda_k\left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)\right)<\infty, \tag{3.7}
$$

and $\int_0^{\infty}\frac{u^{\frac{n}{\alpha+\gamma}-1}}{1+[\Gamma(1+\beta)]^{-1}u}du<\infty$, for $\alpha+\gamma>n$.

■

A mean-square Gaussian solution, in the weak sense, to equations (1.2)–(1.3) is formulated in Proposition **3.2**, considering D to be a Dirichlet-regular bounded open domain. Note that, in the classical theory of boundary value problems, given an open set D with compact closure $\overline{D}$ in $\mathbb{R}^n$, the classical Dirichlet problem consists of the extension of a given continuous function $\psi:\partial D\longrightarrow\mathbb{R}$ to a continuous function $\phi:\overline{D}\longrightarrow\mathbb{R}$ such that ϕ is harmonic, that is, satisfies the Laplace equation in D. The set D is termed regular if the Dirichlet problem has a (necessarily unique) solution for any continuous boundary function ψ. For example, every simply connected planar domain is regular, but may have a *bad* boundary, for instance, a fractal boundary (see Arendt and Schleich [6], pp. 54-55; Fuglede [19]). Dirichlet regularity implies that all the eigenfunctions of the Dirichlet Laplacian operator on D are bounded continuous functions on this domain that vanish continuously on the boundary. This fact will be exploited in the examples given in Section **7**, according to the conditions required on the eigenfunctions, in the derivation of the main results of this paper.

In a more general setting, we consider the following definition of Dirichlet-regular bounded open domain (see, for example, Brelot [10], p. 137 and Theorem 32, and Fuglede [19], p. 253).

DEFINITION **3.1**. For a connected bounded open domain D with boundary ∂D we say that $\mathbf{x}_0 \in \partial D$ is regular if and only if it has a Green kernel G^D such that, for each $\mathbf{x} \in D$,

$$\lim_{\mathbf{x}\to\mathbf{x}_0} G^D(\mathbf{x},\mathbf{y}) = 0, \quad \forall \mathbf{y} \in D. \tag{3.8}$$

The set D is regular if every point of ∂D is regular.

See also Chen *et al.* [12] for alternative characterizations of Dirichlet-regular bounded open domains in terms of the first exit time in the context of subordinate processes.

The following result provides a mean-square zero-mean Gaussian solution, in the weak sense, to the stochastic pseudodifferential boundary value problem (1.2)–(1.3) on a Dirichlet-regular bounded open domain D.

PROPOSITION **3.2**. Let c be defined as

$$c(t,\mathbf{x}) = \int_0^t \int_D G^D(t,\mathbf{x};s,\mathbf{y})\varepsilon(s,\mathbf{y})dsd\mathbf{y}, \tag{3.9}$$

where $\varepsilon(s,\mathbf{y})$ is space-time zero-mean Gaussian white noise as given in equation (1.2), and

$$\begin{aligned} & G^D(t,\mathbf{x};s,\mathbf{y}) = \\ & = \sum_{k\geq 1} E_\beta\left(-\lambda_k\left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)(t-s)^\beta\right)\phi_k(\mathbf{x})\phi_k(\mathbf{y}),\ t\geq s \\ & G^D(t,\mathbf{x};s,\mathbf{y}) = 0,\ s>t, \end{aligned} \tag{3.10}$$

with, as before, for each $k \geq 1$ (see Corollary **2.1**)

$$(-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\phi_k = \lambda_k\left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)\phi_k. \tag{3.11}$$

Assume that $\{\phi_k\{_{k\geq 1}$ are uniformly bounded by a constant $C(D)$, depending of the geometrical characteristics of the domain D, i.e.,

$$C(D) = \sup_{k\geq 1,\ \mathbf{x}\in D} \phi_k(\mathbf{x}).$$

Then, for $n < \alpha+\gamma$, c in (3.9) provides a mean-square zero-mean Gaussian solution to problem (1.2)–(1.3) on D, in the weak-sense in the space $\overline{H}^{\alpha+\gamma}(D)$. Equivalently,

$$\int_D \left[\frac{\partial^\beta}{\partial t^\beta} c\left(t,\mathbf{x}\right) + \left(-\Delta_D\right)^{\alpha/2}\left(I-\Delta_D\right)^{\gamma/2} c\left(t,\mathbf{x}\right)\right]\psi(\mathbf{x})d\mathbf{x}$$
$$\underset{\text{m.s.}}{=} \int_D I_t^{1-\beta}\varepsilon\left(t,\mathbf{x}\right)\psi(\mathbf{x})d\mathbf{x}, \quad \forall \mathbf{x}\in D, \quad \psi\in \overline{H}^{\alpha+\gamma}(D). \quad (3.12)$$

In addition, c has covariance kernel

$$R(t,\mathbf{x};s,\mathbf{y}) = E[c(t,\mathbf{x})c(s,\mathbf{y})] = \int_0^{t\wedge s}\int_D G^D(t,\mathbf{x};u,\mathbf{z})G^D(s,\mathbf{y};u,\mathbf{z})dud\mathbf{z}. \quad (3.13)$$

Proof. It is well-known that the solution to the eigenvalue equation

$$\frac{d^\beta}{dt^\beta}T(t) = -\mu T(t), \quad 0 < t \leq T, \quad (3.14)$$

is given by the Mittag-Leffler function $E_\beta(-\mu t^\beta)$, for any $\mu > 0$, with E_β being introduced in equation (2.12). For $\beta\in(0,1)$, from definition of G^D in equations (3.10)–(3.11), and the definition of the regularized fractional derivative in time (1.5),

$$\int_D \frac{\partial^\beta}{\partial t^\beta} G^D\left(t,\mathbf{x};0,\mathbf{y}\right)\psi(\mathbf{y})d\mathbf{y} = -\int_D \sum_{k=1}^{\infty}\lambda_k\left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)$$
$$\times E_\beta\left(-\lambda_k\left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)t^\beta\right)\phi_k(\mathbf{x})\phi_k(\mathbf{y})\psi(\mathbf{y})d\mathbf{y}$$
$$= -\sum_{k=1}^{\infty}\lambda_k\left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)E_\beta\left(-\lambda_k\left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)t^\beta\right)$$
$$\times\phi_k(\mathbf{x})\int_D \phi_k(\mathbf{y})\psi(\mathbf{y})d\mathbf{y}$$
$$= -(-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\sum_{k=1}^{\infty}E_\beta\left(-\lambda_k\left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)t^\beta\right)$$
$$\times\phi_k(\mathbf{x})\psi_k = -(-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\mathcal{G}_t^D\left(\psi\right), \quad (3.15)$$

where $\mathcal{G}_t^D$ denotes the integral operator on $L^2(D)$ with kernel $G^D\left(t,\mathbf{x};0,\mathbf{y}\right),$ for each $t>0$. Note that, since $\psi\in\overline{H}^{\alpha+\gamma}(D)$, with $\alpha+\gamma>n$, in a similar way to Proposition **3.1**, it can be proved that

$$\begin{aligned}
&\sum_{k=1}^{\infty}\frac{\partial^{\beta}}{\partial t^{\beta}}E_{\beta}\left(-\lambda_k\left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)t^{\beta}\right)\phi_k(\mathbf{x})\psi_k\\
&=\sum_{k=1}^{\infty}\lambda_k\left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)\\
&\qquad\times E_{\beta}\left(-\lambda_k\left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)t^{\beta}\right)\phi_k(\mathbf{x})\psi_k\\
&\leq C(D)\sum_{k=1}^{\infty}\lambda_k\left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)\\
&\qquad\times E_{\beta}\left(-\lambda_k\left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)t^{\beta}\right)\psi_k<\infty, \quad (3.16)
\end{aligned}$$

where, as before $\psi_k=\int_D\phi_k(\mathbf{y})\psi(\mathbf{y})d\mathbf{y}$.

Applying the regularized fractional derivative in time (1.5), from equation (3.15), we obtain

$$\begin{aligned}
&\int_D\frac{\partial^{\beta}}{\partial t^{\beta}}c\left(t,\mathbf{x}\right)\psi(\mathbf{x})d\mathbf{x}=\int_D\frac{d}{dt}\int_0^t(t-\tau)^{-\beta}\int_0^{\tau}\int_D G^D\left(\tau,\mathbf{x};s,\mathbf{y}\right)\\
&\qquad\qquad\times\varepsilon(s,\mathbf{y})\psi(\mathbf{x})d\mathbf{y}dsd\tau d\mathbf{x}\\
&=\int_D\frac{d}{dt}\int_0^t u^{-\beta}\int_0^{t-u}\int_D G^D\left(t-u,\mathbf{x};s,\mathbf{y}\right)\varepsilon(s,\mathbf{y})\psi(\mathbf{x})d\mathbf{y}dsdud\mathbf{x}\\
&=\int_D\left(\int_0^t u^{-\beta}\left[\int_D G^D\left(t-u,\mathbf{x};t-u,\mathbf{y}\right)\varepsilon(t-u,\mathbf{y})d\mathbf{y}\right]du\right)\psi(\mathbf{x})d\mathbf{x}\\
&+\int_D\left[\int_0^{t-u}\int_D\left[\frac{d}{dt}\int_0^t u^{-\beta}G^D\left(t-u,\mathbf{x};s,\mathbf{y}\right)du\right]\varepsilon(s,\mathbf{y})d\mathbf{y}ds\right]\psi(\mathbf{x})d\mathbf{x}\\
&=\int_D\left[\int_0^t u^{-\beta}\varepsilon(t-u,\mathbf{x})du\right]\psi(\mathbf{x})d\mathbf{x}\\
&+\int_D\left[\int_0^t\int_D\frac{\partial^{\beta}}{\partial t^{\beta}}G^D\left(t,\mathbf{x};s,\mathbf{y}\right)\varepsilon(s,\mathbf{y})d\mathbf{y}ds\right]\psi(\mathbf{x})d\mathbf{x}\\
&=\int_D I_t^{1-\beta}\varepsilon(t,\mathbf{x})\psi(\mathbf{x})d\mathbf{x}-\int_D\left[(-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\int_0^t\int_D G^D\left(t,\mathbf{x};s,\mathbf{y}\right)\right.\\
&\qquad\qquad\left.\times\varepsilon(s,\mathbf{y})d\mathbf{y}ds\right]\psi(\mathbf{x})d\mathbf{x}, \quad (3.17)
\end{aligned}$$

as we wanted to prove. Here, we have applied that

$$G^{D}(u,\mathbf{x};u,\mathbf{y}) = \sum_{k=1}^{\infty}\phi_k(\mathbf{x})\phi_k(\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y}), \quad \forall u>0,$$

with $\delta(\mathbf{x}-\mathbf{y})$ denoting the Dirac Delta distribution on $L^2(D)$ such that

$$\int_D \delta(\mathbf{x}-\mathbf{y})\varepsilon(t,\mathbf{y})d\mathbf{y} = \varepsilon(t,\mathbf{x}),$$

in the mean-square sense, and in the $L^2(D)$-weak sense.

Finally, equation (3.13) is obtained from straightforward computation of the covariance function of c in equation (3.9), since for $n/2 < \alpha+\gamma$, G^D defines a Hilbert-Schmidt operator. Consequently, its self-convolution defines a covariance operator $\mathcal{R}$ in the trace class. Thus, its covariance kernel R is continuous, and it can be defined pointwise from equation (3.13). ■

4. Mean-quadratic local variation in time

This section provides an upper bound for the mean-quadratic local variation of the temporal increments of the mean-square solution c defined in equation (3.9) of Proposition **3.2**. Note that although we have showed in Proposition **3.2** that c satisfies, in the mean-square sense, equation (1.2) over the test functions in $\overline{H}^{\alpha+\gamma}(D)$, for $\alpha+\gamma>n$, as we prove, in the following result, equation (3.9) defines a Hölder continuous, in time, spatiotemporal random field c, under a wider range of parameter $\alpha+\gamma$. Namely, Theorem **4.1** below holds for $\frac{n}{2}<\alpha+\gamma$, and $\beta<1/2$.

As before, we will consider the sequence of eigenvalues

$$\lambda_k\left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right), \quad k\geq 1,$$

arranged in increasing order of their modulus magnitude, with the associated eigenvectors ϕ_k, $k\geq 1$, in the same order.

Theorem **4.1**. *Let c be defined as in (3.9)–(3.11) of Proposition* **3.2**, *under the assumption that $C(D)=\sup_{k\geq 1,\ \mathbf{x}\in D}\phi_k(\mathbf{x})<\infty$. Then, for $\beta<1/2$, and $\frac{n}{2}<\alpha+\gamma$, the following inequality holds:*

$$E[c(t,\mathbf{x})-c(s,\mathbf{x})]^2 \leq [C(D)]^2 g(t-s), \tag{4.1}$$

where

$$g(t-s) = \mathcal{O}\left((t-s)^{\left(1-\frac{\beta n}{\alpha+\gamma}\right)\wedge(1-\beta)}\right), \quad s\to t,\ 0<s<t, \tag{4.2}$$

with $x \wedge y$ denoting the minimum of x and y, and $x \vee y$ denoting the maximum of x and y, for two real numbers x and y.

Proof. Since $E_\beta(-x)$ is a monotone decreasing functions with values in the interval $[0,1]$, for $x \in \mathbb{R}_+$, for $0 < s < t$, we obtain

$$
\begin{aligned}
E[c(t,\mathbf{x}) - c(s,\mathbf{x})]^2 &= \int_0^s \sum_{k=1}^{\infty} \phi_k^2(\mathbf{x}) \Big[E_\beta(-\lambda_k \left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)(t-u)^\beta) \\
&\quad - E_\beta\left(-\lambda_k \left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)(s-u)^\beta\right)\Big]^2 du \\
&+ \int_s^t \sum_{k=1}^{\infty} \phi_k^2(\mathbf{x}) E_\beta\left(-2\lambda_k \left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)(t-u)^\beta\right) du \\
&\leq [C(D)]^2 \left[\int_0^s \sum_{k=1}^{\infty} \Big[E_\beta\left(-\lambda_k \left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)(t-u)^\beta\right)\right. \\
&- E_\beta\left((-\lambda_k \left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)(s-u)^\beta\right)\Big]^2 du \\
&\quad \left. + \int_s^t \sum_{k=1}^{\infty} E_\beta\left(-2\lambda_k \left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)(t-u)^\beta\right) du \right] \\
&= [C(D)]^2 \left[\int_0^s \sum_{k=1}^{\infty} \Big[E_\beta\left(-\lambda_k \left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)(t-u)^\beta\right)\Big]^2 \right. \\
&\quad + \Big[E_\beta\left((-\lambda_k \left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)(s-u)^\beta\right)\Big]^2 \\
&- 2E_\beta\left(-\lambda_k \left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)(t-u)^\beta\right) \\
&\quad \times E_\beta\left((-\lambda_k \left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)(s-u)^\beta\right) du \\
&+ \left. \int_s^t \sum_{k=1}^{\infty} E_\beta\left(-2\lambda_k \left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)(t-u)^\beta\right) du \right] \\
&\leq [C(D)]^2 \left[\int_0^s \sum_{k=1}^{\infty} \Big[E_\beta\left(-\lambda_k \left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)(t-u)^\beta\right)\Big]^2 \right. \\
&+ \Big[E_\beta\left((-\lambda_k \left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)(s-u)^\beta\right)\Big]^2 \\
&- 2\Big[E_\beta\left(-\lambda_k \left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)(t-u)^\beta\right)\Big]^2 du
\end{aligned}
$$

$$
\begin{aligned}
&+\int_s^t \sum_{k=1}^{\infty} E_\beta\left(-2\lambda_k\left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)(t-u)^\beta\right)du\Bigg] \\
&\leq [C(D)]^2\left[\int_0^s \sum_{k=1}^{\infty}\left[E_\beta\left(-\lambda_k\left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)(s-u)^\beta\right)\right]^2 du\right. \\
&\qquad\left.+\int_s^t \sum_{k=1}^{\infty} E_\beta\left(-2\lambda_k\left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)(t-u)^\beta\right)du\right]
\end{aligned}
\tag{4.3}
$$

In a similar way to equation (3.6), from equation (4.3), we obtain

$$
\begin{aligned}
E[c(t,\mathbf{x})-c(s,\mathbf{x})]^2 &\leq [C(D)]^2\int_0^s \Bigg[\widetilde{M}(\beta,\alpha,\gamma,n,(s-u)^\beta) \\
&\qquad+\frac{(s-u)^{-\beta n/(\alpha+\gamma)}}{(\alpha+\gamma)/n}\int_0^\infty [E_\beta(x)]^2\, x^{\frac{n}{\alpha+\gamma}-1}dx\Bigg]du \\
&+[C(D)]^2\int_s^t \Bigg[M(\beta,\alpha,\gamma,n,(t-u)^\beta) \\
&\qquad+\frac{(t-u)^{-\beta n/(\alpha+\gamma)}}{(\alpha+\gamma)/n}\int_0^\infty E_\beta(2x)\, x^{\frac{n}{\alpha+\gamma}-1}dx\Bigg]du \\
&\leq [C(D)]^2\int_0^s \Bigg[\widetilde{M}(\beta,\alpha,\gamma,n,(s-u)^\beta) \\
&\qquad+\frac{(s-u)^{-\beta n/(\alpha+\gamma)}}{(\alpha+\gamma)/n}\int_0^\infty \frac{x^{\frac{n}{\alpha+\gamma}-1}}{(1+[\Gamma(1+\beta)]^{-1}x)^2}dx\Bigg]du \\
&+[C(D)]^2\int_s^t \Bigg[M(\beta,\alpha,\gamma,n,(t-u)^\beta) \\
&\qquad+\frac{(t-u)^{-\beta n/(\alpha+\gamma)}}{(\alpha+\gamma)/n}\int_0^\infty \frac{x^{\frac{n}{\alpha+\gamma}-1}}{1+[\Gamma(1+\beta)]^{-1}2x}dx\Bigg]du,
\end{aligned}
\tag{4.4}
$$

where

$$
\begin{aligned}
M(\beta,\alpha,\gamma,n,(s-u)^\beta) &= \sum_{k=1}^{k_0} E_\beta\left(-(s-u)^\beta\lambda_k\left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)\right) \\
\widetilde{M}(\beta,\alpha,\gamma,n,(s-u)^\beta) &= \sum_{k=1}^{k_0}\left[E_\beta\left(-(s-u)^\beta\lambda_k\left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)\right)\right]^2.
\end{aligned}
\tag{4.5}
$$

Hence,

$$E[c(t,\mathbf{x})-c(s,\mathbf{x})]^2 \leq [C(D)]^2 \Big[K_1(\beta,\alpha,\gamma,n)s^{1-2\beta} + K_2(\beta,\alpha,\gamma,n)s^{1-\beta n/(\alpha+\gamma)}$$

$$+K_3(\beta,\alpha,\gamma,n)(t-s)^{1-\beta} + K_4(\beta,\alpha,\gamma,n)(t-s)^{1-\beta n/(\alpha+\gamma)}\Big]. \tag{4.6}$$

Thus, when $s \to t$, $s < t$, we have

$$E[c(t,\mathbf{x})-c(s,\mathbf{x})]^2 = \mathcal{O}\left((t-s)^{1-(\beta n/(\alpha+\gamma))\wedge(1-\beta)}\right).$$

■

5. Mean-quadratic local variation in space

The fractional local exponent, in the mean-square sense, of the spatial increments of the solution c, derived in Proposition **3.2**, is obtained in the following result.

THEOREM **5.1**. *Let c be as given in equations (3.9)–(3.11), for $n < \alpha+\gamma$. Assume that for every $k \geq 1$,*

$$|\phi_k(\mathbf{x}+\mathbf{h}) - \phi_k(\mathbf{x})| = \mathcal{O}(\|\mathbf{h}\|^{\Upsilon}), \quad \|\mathbf{h}\| \to 0, \quad \Upsilon > 0.$$

In particular, for each $k \geq 1$, and for $\|\mathbf{h}\|$ small,

$$|\phi_k(\mathbf{x}+\mathbf{h}) - \phi_k(\mathbf{x})| \leq C_k\|\mathbf{h}\|^{\Upsilon},$$

for certain positive constant C_k. If $\sup_k C_k = C < \infty$, then, for each $t > 0$,

$$E[c(t,\mathbf{x})-c(t,\mathbf{y})]^2 = \mathcal{O}(\|\mathbf{x}-\mathbf{y}\|^{2\Upsilon}), \quad \|\mathbf{x}-\mathbf{y}\| \to 0, \quad \Upsilon > 0. \tag{5.1}$$

Thus, for $\|\mathbf{x}-\mathbf{y}\|$ sufficiently small,

$$E[c(t,\mathbf{x})-c(t,\mathbf{y})]^2 \leq Cg(t)\|\mathbf{x}-\mathbf{y}\|^{2\Upsilon}, \tag{5.2}$$

where

$$g(t) = t^{1-\beta}\sum_{k=1}^{\infty}\frac{\Gamma(1+\beta)}{\lambda_k\left((-\Delta)_D^{\alpha/2}(I-\Delta)_D^{\gamma/2}\right)}, \quad t > 0.$$

Proof.

Applying Hölder continuity of the eigenvectors, from Lemma **2.1**, we have, for every $t>0$,

$$
\begin{aligned}
& E[c(t,\mathbf{x})-c(t,\mathbf{y})]^2 \\
&= \int_0^t \sum_{k=1}^{\infty} \Big[E_\beta(-\lambda_k \left((-\Delta)_D^{\alpha/2}(I-\Delta)_D^{\gamma/2}\right)(t-u)^\beta)\phi_k(\mathbf{x}) \\
&- E_\beta(-\lambda_k \left((-\Delta)_D^{\alpha/2}(I-\Delta)_D^{\gamma/2}\right)(t-u)^\beta)\phi_k(\mathbf{y})\Big]^2 du \\
&= \int_0^t \sum_{k=1}^{\infty} [\phi_k(\mathbf{x})-\phi_k(\mathbf{y})]^2 \Big[E_\beta(-\lambda_k \left((-\Delta)_D^{\alpha/2}(I-\Delta)_D^{\gamma/2}\right)(t-u)^\beta)\Big]^2 du \\
&\leq C\|\mathbf{x}-\mathbf{y}\|^{2\Upsilon} \int_0^t \sum_{k=1}^{\infty} \Big[E_\beta(-\lambda_k \left((-\Delta)_D^{\alpha/2}(I-\Delta)_D^{\gamma/2}\right)(t-u)^\beta)\Big]^2 du \\
&\leq C\|\mathbf{x}-\mathbf{y}\|^{2\Upsilon} \int_0^t \sum_{k=1}^{\infty} \frac{\Gamma(1+\beta)}{\lambda_k \left((-\Delta)_D^{\alpha/2}(I-\Delta)_D^{\gamma/2}\right)\nu^\beta} d\nu \\
&= C\|\mathbf{x}-\mathbf{y}\|^{2\Upsilon} t^{1-\beta} \sum_{k=1}^{\infty} \frac{\Gamma(1+\beta)}{\lambda_k \left((-\Delta)_D^{\alpha/2}(I-\Delta)_D^{\gamma/2}\right)} = Cg(t)\|\mathbf{x}-\mathbf{y}\|^{2\Upsilon}, \qquad (5.3)
\end{aligned}
$$

as we wanted to prove. Note that, from Corollary **2.1**, for each fixed $t>0$,

$$
g(t) = t^{1-\beta} \sum_{k=1}^{\infty} \frac{\Gamma(1+\beta)}{\lambda_k \left((-\Delta)_D^{\alpha/2}(I-\Delta)_D^{\gamma/2}\right)} < \infty.
$$

6. Mean quadratic local variation in time and space

In this section we apply the results derived in Theorems **4.1** and **5.1** to obtain the mean-quadratic local variation properties of the spatiotemporal increments of the weak-sense solution c to equations (1.2)–(1.3).

THEOREM **6.1**. *Under conditions of Theorems* **4.1** *and* **5.1**, *let* c *be defined in equations (3.9)–(3.11). Then, as* $s\to t$, $s,t\in(0,T]$, *and* $\|\mathbf{x}-\mathbf{y}\|\to 0$,

$$
\begin{aligned}
& E[c(t,\mathbf{x})-c(s,\mathbf{y})]^2 \\
\leq\ & \widetilde{C}(D,T,\beta,\alpha,\gamma,\Upsilon)\|(t,\mathbf{x})-(s,\mathbf{y})\|^{\left(1-\frac{\beta n}{\alpha+\gamma}\right)\wedge(1-\beta)\wedge 2\Upsilon},
\end{aligned}
$$

where

$$\widetilde{C}(D,T,\beta,\alpha,\gamma,\Upsilon) = 8([C(D)]^2 \vee Cg(T)) \left(\frac{1}{2}\right)^{1/2\left[\left(1-\frac{\beta n}{\alpha+\gamma}\right)\wedge(1-\beta)\wedge 2\Upsilon\right]},$$

with, as before, $x \vee y$ representing the maximum of x and y, and $x \wedge y$ representing the minimum. Here, $C(D)$ is introduced in Theorem **4.1**, *and C and $g(T)$ are given in Theorem* **5.1**.

Proof. The proof follows from Theorems **4.1** and **5.1**. Specifically, from equations (4.1)–(4.2), and (5.1)–(5.2), as $s \to t$, and $\|\mathbf{x}-\mathbf{y}\| \to 0$,

$$\begin{aligned}
&E[c(t,\mathbf{x}) - c(s,\mathbf{y})]^2 = E[c(t,\mathbf{x}) - c(s,\mathbf{x}) + c(s,\mathbf{x}) - c(s,\mathbf{y})]^2 \\
&= E[c(t,\mathbf{x}) - c(s,\mathbf{x})]^2 + E[c(s,\mathbf{x}) - c(s,\mathbf{y})]^2 \\
&\qquad +2E\left[(c(t,\mathbf{x}) - c(s,\mathbf{x}))(c(s,\mathbf{x}) - c(s,\mathbf{y}))\right] \\
&\leq E[c(t,\mathbf{x}) - c(s,\mathbf{x})]^2 + E[c(s,\mathbf{x}) - c(s,\mathbf{y})]^2 \\
&\qquad +2\left|E\left[(c(t,\mathbf{x}) - c(s,\mathbf{x}))(c(s,\mathbf{x}) - c(s,\mathbf{y}))\right]\right| \\
&\leq E[c(t,\mathbf{x}) - c(s,\mathbf{x})]^2 + E[c(s,\mathbf{x}) - c(s,\mathbf{y})]^2 \\
&\qquad +2\sqrt{E\left[(c(t,\mathbf{x}) - c(s,\mathbf{x}))^2\right]}\sqrt{E\left[(c(s,\mathbf{x}) - c(s,\mathbf{y}))^2\right]} \\
&\leq [C(D)]^2|t-s|^{\left(1-\frac{\beta n}{\alpha+\gamma}\right)\wedge(1-\beta)} + Cg(s)\|\mathbf{x}-\mathbf{y}\|^{2\Upsilon} \\
&\qquad +2\sqrt{[C(D)]^2|t-s|^{\left(1-\frac{\beta n}{\alpha+\gamma}\right)\wedge(1-\beta)}Cg(s)\|\mathbf{x}-\mathbf{y}\|^{2\Upsilon}} \\
&\leq [C(D)]^2|t-s|^{\left(1-\frac{\beta n}{\alpha+\gamma}\right)\wedge(1-\beta)} + Cg(s)\|\mathbf{x}-\mathbf{y}\|^{2\Upsilon} \\
&\qquad +2\sqrt{\left([C(D)]^2|t-s|^{\left(1-\frac{\beta n}{\alpha+\gamma}\right)\wedge(1-\beta)}\right)^2 + (Cg(s)\|\mathbf{x}-\mathbf{y}\|^{2\Upsilon})^2} \\
&\leq 2([C(D)]^2 \vee Cg(T))\left[|t-s|^{\left(1-\frac{\beta n}{\alpha+\gamma}\right)\wedge(1-\beta)}\right. \\
&\qquad \left. +\|\mathbf{x}-\mathbf{y}\|^{2\Upsilon} + \sqrt{\left(|t-s|^{\left(1-\frac{\beta n}{\alpha+\gamma}\right)\wedge(1-\beta)}\right)^2 + (\|\mathbf{x}-\mathbf{y}\|^{2\Upsilon})^2}\right] \\
&\leq 4([C(D)]^2 \vee Cg(T))\left[\frac{1}{2}\left(|t-s|^{\left(1-\frac{\beta n}{\alpha+\gamma}\right)\wedge(1-\beta)\wedge 2\Upsilon}\right.\right. \\
&\qquad \left.\left. +\|\mathbf{x}-\mathbf{y}\|^{\left(1-\frac{\beta n}{\alpha+\gamma}\right)\wedge(1-\beta)\wedge 2\Upsilon}\right)\right.
\end{aligned}$$

$$+\sqrt{\frac{1}{4}\left(|t-s|^{\left(1-\frac{\beta n}{\alpha+\gamma}\right)\wedge(1-\beta)\wedge 2\Upsilon}\right)^{2}+\left(\|\mathbf{x}-\mathbf{y}\|^{\left(1-\frac{\beta n}{\alpha+\gamma}\right)\wedge(1-\beta)\wedge 2\Upsilon}\right)^{2}}\Bigg]. \tag{6.1}$$

Under the conditions assumed, $0<\left(1-\frac{\beta n}{\alpha+\gamma}\right)\wedge(1-\beta)\wedge 2\Upsilon<1$, hence, we can apply Jensen's inequality for concave function x^{ξ}, $0<\xi<1$, obtaining from the last inequality in (6.1),

$$\begin{aligned}
&E[c(t,\mathbf{x})-c(s,\mathbf{y})]^2 \leq 4([C(D)]^2\vee Cg(T))\\
&\times\left[\left(\frac{1}{2}\right)^{1/2\left[\left(1-\frac{\beta n}{\alpha+\gamma}\right)\wedge(1-\beta)\wedge 2\Upsilon\right]}\left(|t-s|^2\right.\right.\\
&\qquad\qquad\left.+\|\mathbf{x}-\mathbf{y}\|^2\right)^{1/2\left[\left(1-\frac{\beta n}{\alpha+\gamma}\right)\wedge(1-\beta)\wedge 2\Upsilon\right]}\\
&\left.+\sqrt{\left(\frac{1}{4}\right)^{\left(1-\frac{\beta n}{\alpha+\gamma}\right)\wedge(1-\beta)\wedge 2\Upsilon}\left(|t-s|^2+\|\mathbf{x}-\mathbf{y}\|^2\right)^{\left(1-\frac{\beta n}{\alpha+\gamma}\right)\wedge(1-\beta)\wedge 2\Upsilon}}\right]\\
&\leq 4([C(D)]^2\vee Cg(T))\left(\frac{1}{2}\right)^{1/2\left[\left(1-\frac{\beta n}{\alpha+\gamma}\right)\wedge(1-\beta)\wedge 2\Upsilon\right]}\\
&\qquad\qquad\times\left[\left(|t-s|^2+\|\mathbf{x}-\mathbf{y}\|^2\right)^{1/2\left[\left(1-\frac{\beta n}{\alpha+\gamma}\right)\wedge(1-\beta)\wedge 2\Upsilon\right]}\right.\\
&\left.+\sqrt{\left(|t-s|^2+\|\mathbf{x}-\mathbf{y}\|^2\right)^{\left(1-\frac{\beta n}{\alpha+\gamma}\right)\wedge(1-\beta)\wedge 2\Upsilon}}\right]\\
&=2C(D,T,\beta,\alpha,\gamma,\Upsilon)\|(t,\mathbf{x})-(s,\mathbf{y})\|^{\left(1-\frac{\beta n}{\alpha+\gamma}\right)\wedge(1-\beta)\wedge 2\Upsilon},
\end{aligned}\tag{6.2}$$

as we wanted to prove, with $\widetilde{C}(D,T,\beta,\alpha,\gamma,\Upsilon)=2C(D,T,\beta,\alpha,\gamma,\Upsilon)$, and

$$C(D,T,\beta,\alpha,\gamma,\Upsilon)=4([C(D)]^2\vee Cg(T))\left(\frac{1}{2}\right)^{1/2\left[\left(1-\frac{\beta n}{\alpha+\gamma}\right)\wedge(1-\beta)\wedge 2\Upsilon\right]}.$$

■

6.1. Sample-path properties

The following result provides the sample path local regularity properties of the mean-square weak-sense Gaussian solution c to equations (1.2)–(1.3). Note that the derived Gaussian solution c is not fractional differentiable in time in the strong-sense (see Proposition **3.2**). However, it is Hölder continuous, in the mean square sense, under the conditions assumed in the previous sections.

Theorem **6.2**. *Let c be defined in equations (3.9)–(3.11). Under the conditions of Theorems* **4.1** *and* **5.1**, *with probability one, the following inequalities hold, as $s \to t$, and $\|\mathbf{x}-\mathbf{y}\| \to 0$,*

$$\begin{aligned}
& \sup_{|t-s|<\delta} |c(t,\mathbf{x})-c(s,\mathbf{x})|^2 \\
\leq \ & Z\delta^{\left(1-\frac{\beta n}{\alpha+\gamma}\right)\wedge(1-\beta)} + H_1 \delta^{\left(1-\frac{\beta n}{\alpha+\gamma}\right)\wedge(1-\beta)} \left[\log\left(\frac{1}{\delta}\right)\right]^{1/2} \\
& \sup_{\|\mathbf{x}-\mathbf{y}\|<\delta} |c(t,\mathbf{x})-c(t,\mathbf{y})|^2 \\
\leq \ & Y\delta^{2\Upsilon} + H_2\delta^{2\Upsilon}\left[\log\left(\frac{1}{\delta}\right)\right]^{1/2} \\
& \sup_{\|(t,\mathbf{x})-(s,\mathbf{y})\|<\delta} |c(t,\mathbf{x})-c(s,\mathbf{y})|^2 \\
\leq \ & X\delta^{\left(1-\frac{\beta n}{\alpha+\gamma}\right)\wedge(1-\beta)\wedge 2\Upsilon} \\
& + H_3\delta^{\left(1-\frac{\beta n}{\alpha+\gamma}\right)\wedge(1-\beta)\wedge 2\Upsilon}\left[\log\left(\frac{1}{\delta}\right)\right]^{1/2},
\end{aligned} \tag{6.3}$$

where Z, Y and X are positive random variables, and H_i, $i = 1, 2, 3$, are positive constants that could depend on the geometrical characteristics of the domain D considered, like the boundary.

The proof directly follows from Theorems **4.1**–**6.1**, and Theorem 3.3.3, p.57, by Adler [1].

7. Examples

In the following subsections we consider some special cases of domain D, where the derived results can be applied. Specifically, in the examples introduced below, the eigenfunctions of the Dirichlet negative Laplacian operator can be explicitly computed, and Theorem **2.1** allows us to define the weak-sense mean-square solution to (1.2)–(1.3), in terms of such eigenfunctions as given in equations (3.9)–(3.10) in Proposition **3.2** (see, for example, Grebenkov and Nguyen [22]). The conditions required in Theorems **4.1**, **5.1**, **6.1** and **6.2**, for the continuity of c in the mean-square sense and in the sample path sense, are also verified.

7.1. Intervals, rectangles, parallelepipeds

Let us consider the case where $D = (0, L_1), \dots, (0, L_n) \subset \mathbb{R}^n$, where $L_i > 0$, for $i = 1, \dots, n$, the method of separation of variables yields the following eigenvectors for the Dirichlet negative Laplacian operator:

$$\phi_{k_1,\dots,k_n}(x_1, \dots, x_n) = \prod_{i=1}^{n} \sin\left(\frac{\pi(k_i + 1)x_i}{L_i}\right), \quad (k_1, \dots, k_n) \in \mathbb{N}_*^n, \quad (7.1)$$

with $\lambda_{k_1,\dots,k_n} = \lambda_{k_1} + \dots, +\lambda_{k_n}$, and $\lambda_{k_i} = \frac{\pi^2(k_i+1)^2}{L_i^2}$, for $i = 1, \dots, n$. In this case, the fundamental solution to the fractional space-time pseudodifferential equation

$$\frac{\partial^\beta c}{\partial t^\beta}(t, \mathbf{x}) = -(-\Delta_D)^{\alpha/2}(I - \Delta_D)^{\gamma/2} c(t, \mathbf{x}) \quad (7.2)$$
$$t \in \mathbb{R}_+, \quad \mathbf{x} \in D = (0, L_1), \dots, (0, L_n)$$

is given, for $\alpha + \gamma > n$, by

$$\begin{aligned} G_{t-s}(x_1, \dots, x_n; y_1, \dots, y_n) = &\sum_{(k_1,\dots k_n)\in\mathbb{N}_*^n} E_\beta\left(-\left(\sum_{i=1}^{n}\frac{\pi^2(k_i+1)^2}{L_i^2}\right)^{\alpha/2}\right. \\ &\left.\times\left(1 + \left(\sum_{i=1}^{n}\frac{\pi^2(k_i+1)^2}{L_i^2}\right)\right)^{\gamma/2}(t-s)^\beta\right) \\ &\times\left[\prod_{i=1}^{n}\sin\left(\frac{\pi(k_i+1)x_i}{L_i}\right)\right]\left[\prod_{i=1}^{n}\sin\left(\frac{\pi(k_i+1)y_i}{L_i}\right)\right], \quad t \geq s \\ &\text{and} \quad G(t, \mathbf{x}; s, \mathbf{y}) = 0, \quad s > t. \end{aligned}$$

From Hölder's inequality (see, for example, equation (6.4) in Grebenkov and Nguyen [22]),

$$|\phi_{k_1,\dots,k_n}(x_1, \dots, x_n)| \leq \sqrt{\prod_{i=1}^{n} L_i}\frac{\prod_{i=1}^{n} L_i}{2^{n/2+n}\pi^{n/2}}.$$

Therefore, in the previous computations in Theorems **4.1**, for $\beta < 1/2$, we consider

$$C(D) = \frac{\left[\prod_{i=1}^{n} L_i\right]^{3/2}}{2^{n/2+n}\pi^{n/2}}.$$

Theorem **5.1** also holds, since $\prod_{i=1}^{n}\sin\left(\frac{\pi(k_i+1)x_i}{L_i}\right)$, $(k_1, \dots, k_n) \in \mathbb{N}_*^n$, are continuously differentiable, and hence, Hölder continuous. Theorems **6.1** and **6.2** then follow for $\beta < 1/2$, and, as before, for $\alpha + \gamma > n$.

7.2. Balls

Let us consider equations (1.2)–(1.3) on the ball. That is,

$$\frac{\partial^{\beta} c}{\partial t^{\beta}}(t,\mathbf{x}) = -(-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}c(t,\mathbf{x}), \tag{7.3}$$

$$t\in\mathbb{R}_+,\quad \mathbf{x}\in D=\{\mathbf{x}\in\mathbb{R}^n,\quad \|\mathbf{x}\|<R\}=\mathcal{B}_R(\mathbf{0}),\ R>0.$$

For $\alpha+\gamma>n$, the Green function is defined as

$$G_{t-s}(\rho,\theta;\rho',\theta') = \sum_{l=0}^{\infty}\sum_{r=1}^{\infty}\sum_{m=1}^{h(n,l)} E_{\beta}\left(-(\mu_{l,r})^{\alpha/2}(1+\mu_{l,r})^{\gamma/2}(t-s)^{\beta}\right)\right)$$
$$\times\phi_{l,r,m}(\rho,\theta)\phi_{l,r,m}(\rho',\theta'),\quad t\geq s$$
$$\text{and}\quad G(t,\mathbf{x};s,\mathbf{y})=0,\quad s>t,$$

where we have considered the spherical coordinates $\mathbf{x}=(\rho,\theta)$, $\mathbf{y}=(\rho',\theta')$, and, as before, E_β is the Mittag-Leffler function. Moreover,

$$(-\Delta)_D\phi_{l,r,m}(\rho,\theta) = \mu_{l,r}\phi_{l,r,m}(\rho,\theta),$$

for $l\in\mathbb{N}$, with $m=1,2,\dots,h(n,l)$, $r\in\mathbb{N}_*$ and $h(n,l)=\frac{(2l+n-2)(l+n-3)!}{(n-2)!l!}$, the number of spherical harmonics. Here, the eigenvalues are given by $\mu_{l,r}=\left(\frac{\xi_{l,r}^2}{R^2}\right)$, and the eigenvectors $\phi_{l,r,m}(\rho,\theta)=c_{l,r,m}\mathcal{J}_{l+\frac{n-2}{2}}\left(\xi_{l+\frac{n-2}{2},r}\frac{\rho}{R}\right)\mathcal{S}_{l,m}(\theta)$ are defined in terms of the Bessel function of the first kind of order ν, $\mathcal{J}_\nu$, and the orthonormal spherical harmonics on the sphere of radius one, $\mathcal{S}_{l,m}(\theta)$. Note that $c_{l,r,m}$ is the normalizing constant, and $\xi_{\nu,r}$ is the rth positive root of $\mathcal{J}_\nu$.

From Hölder inequality, since $\{\phi_{l,r,m}\}$ are normalized in the space of square integrable functions over the ball of radius R, i.e.., $\|\phi_{l,r,m}\|_2=1$, we obtain

$$|\phi_{l,r,m}(\rho,\theta)|\leq\|\phi_{l,r,m}\|_1\leq\|\phi_{l,r,m}\|_2\sqrt{\int_{\mathcal{B}_R(\mathbf{0})}d\mathbf{x}}=\sqrt{\frac{\pi^{n/2}R^n}{\Gamma\left(\frac{n}{2}+1\right)}}$$

(see, for example, equation (6.4) in Grebenkov and Nguyen [22]). Thus, in the previous computations in Theorem **4.1**, we can consider $C(D)=|\mathcal{B}_R(\mathbf{0})|^{1/2}$. Theorem **5.1** also holds, since Bessel functions of the first kind and order ν, on a closed interval, and the orthonormal spherical harmonics on the sphere of radius one, are Hölder continuous. Theorems **6.1** and **6.2** then follow for $\beta<1/2$, and, as before, for $\alpha+\gamma>n$.

Circular annulus. Let us now consider, for $\alpha+\gamma>n$, and $\beta<1/2$,

$$\frac{\partial^\beta c}{\partial t^\beta}(t,\mathbf{x}) = -(-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}c(t,\mathbf{x}), \tag{7.4}$$

$$t\in\mathbb{R}_+,\quad \mathbf{x}\in D=\{(x_1,x_2)\in\mathbb{R}^2;\quad R_0<|x|<R\}.$$

In polar coordinates, $x_1=r\cos\varphi$, $x_2=r\sin\varphi$, the Laplace operator Δ admits the expression

$$\Delta = \frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}.$$

The fundamental solution (in polar coordinates) of

$$\frac{\partial^\beta c}{\partial t^\beta}(t,\mathbf{x}) = -(-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}c(t,\mathbf{x}),$$

with

$$-\Delta_D = -\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}\right),$$

is then given by

$$G_{t-s}(r,\varphi,r',\varphi') = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\sum_{l=1}^{2} E_\beta\left(-\left(\alpha_{n,k}^2/R^2\right)^{\alpha/2}\left(1+\left(\alpha_{n,k}^2/R^2\right)\right)^{\gamma/2}(t-s)^\beta\right)$$
$$\times u_{n,k,l}(r,\varphi)u_{n,k,l}(r',\varphi'),$$

where

$$u_{n,k,l}(r,\varphi) = J_n(r\alpha_{n,k}/R)+c_{n,k}Y_n(r\alpha_{n,k}/R)\times\begin{cases}\cos(n\varphi), & l=1\\ \sin(n\varphi), & l=2\ (n\neq 0),\end{cases}$$

with J_n and Y_n being the Bessel functions of the first and second kind, and the coefficients $\alpha_{n,k}$ and $c_{n,k}$ being set by the boundary conditions at $r=R_0$, and $r=R$.

$$\begin{aligned}
0 &= \frac{\alpha_{n,k}}{R}\left[J_n'(\alpha_{n,k})+c_{n,k}Y_n'(\alpha_{n,k})\right]+h\left[J_n(\alpha_{n,k})+c_{n,k}Y_n(\alpha_{n,k})\right]\\
0 &= -\frac{\alpha_{n,k}}{R}\left[J_n'\left(\alpha_{n,k}\frac{R_0}{R}\right)+c_{n,k}Y_n'\left(\alpha_{n,k}\frac{R_0}{R}\right)\right]\\
&+ h\left[J_n\left(\alpha_{n,k}\frac{R_0}{R}\right)+c_{n,k}Y_n\left(\alpha_{n,k}\frac{R_0}{R}\right)\right].
\end{aligned}$$

Using Hölder's inequality,

$$|u_{n,k,l}(r,\varphi)| \le \sqrt{\frac{\pi}{\Gamma(2)}R^2-\frac{\pi}{\Gamma(2)}R_0^2}\,\|u_{n,k,l}\|_2,$$

where

$$\begin{aligned}\|u_{n,k,l}\|_2^2 &= \frac{\pi(2-\delta_{n,0})R^2}{2\alpha_{n,k}^2}\left[\left(\alpha_{n,k}^2+h^2R^2-n^2\right)v_{n,k}^2(R)\right.\\ &\quad \left.-\left((\alpha_{n,k}^2+h^2R^2)\frac{R_0^2}{R^2}-n^2\right)v_{n,k}^2(R_0)\right], \qquad (7.5)\end{aligned}$$

with

$$v_{n,k}(r)=J_n(\alpha_{n,k}r/R)+c_{n,k}Y_n(\alpha_{n,k}r/R). \qquad (7.6)$$

From equations (7.5) and (7.6), the uniform upper bound for the eigenvectors of Dirichlet negative Laplacian operator on circular annulus is then given by

$$C(D)=\sqrt{\frac{\pi}{\Gamma(2)}R^2-\frac{\pi}{\Gamma(2)}R_0^2}\pi R^2(1+M_1)M_2,$$

where

$$\begin{aligned}\frac{h^2R^2}{\alpha_{n,k}^2} &\leq M_1\\ v_{n,k}^2(R) &\leq M_2,\end{aligned}$$

since Bessel functions of the first and second kind are uniformly bounded for $0<R_0<r<R$. Furthermore, Bessel functions of the first and second kind on a closed bounded interval, as well as sine and cosine are also Hölder continuous. Thus, Theorem **5.1** holds. Summarizing, Theorems **4.1**, **5.1**, **6.1** and **6.2** hold for $\alpha+\gamma>n$, and $\beta<1/2$.

Elliptical annulus. In elliptic coordinates, $x_1=a\cosh r\cos\theta$, $x_2=a\sinh r\sin\theta$, the Laplace operator adopts the form:

$$\Delta=\frac{1}{a^2(\sinh^2 r+\sin^2\theta)}\left(\frac{\partial^2}{\partial r^2}+\frac{\partial^2}{\partial\theta^2}\right),$$

where $r\geq 0$, and $0\leq\theta<2\pi$, are the radial and angular coordinates, and $a>0$, is the prescribed distance between the origin and the foci. An ellipse is a curve of constant $r=R$ so that its points (x_1,x_2) satisfy $x_1^2/A^2+x^2/B^2=1$, where $A=a\cosh R$, and $B=a\sinh R$ are the major and minor semi-axes, and R denotes the radius of the ellipse. The eccentricity $e=a/A=1/\cosh R$ is strictly positive. The interior of an ellipse is characterized by $0\leq\theta<2\pi$ and $0\leq r<R$. An elliptical annulus, the interior between two ellipses with the same foci, can be characterized in elliptic coordinates (r,θ) with $R_0<r<R$ and $0\leq\theta<2\pi$.

In elliptic coordinates, the variable separation method, $u(r,\theta)=g(\theta)f(r)$, leads to the following equations, after considering that the two differential equations in θ and r are equal to a constant κ,

$$g''(\theta) + (\kappa - 2q\cos 2\theta)g(\theta) = 0 \tag{7.7}$$
$$f''(r) - (\kappa - 2q\cosh 2r\theta)f(r) = 0. \tag{7.8}$$

These equations are respectively known as the Mathieu equation and the modified Mathieu equation, where $q = \lambda a^2/4$, and the parameter κ is called the characteristic value of Mathieu functions, whose values lead to a real integer value of the characteristic exponent ν of the solution defined according to Floquet's theorem. The two linearly independent periodic solutions of equation (7.7) are known as the angular Mathieu functions, and they are respectively denoted as $\kappa e_n(\theta, q)$ and $se_{n+1}(\theta, q)$, $n = 0, 1, 2, \ldots$. That is, we consider κ such that the characteristic exponent ν satisfies $\nu(\kappa, q) \in \mathbb{Z}$, leading to the referred angular periodic Mathieu functions. There are two linearly independent oscillatory radial Mathieu functions of the first kind, solution to equation (7.8), respectively denoted as $M\kappa_n^{(1)}(r, q)$ and $M\kappa_n^{(2)}(r, q)$, corresponding to the same κ as $\kappa e_n(\theta, q)$. In addition, there is two linearly independent oscillatory radial Mathieu functions of the second kind $Ms_{n+1}^{(1)}(r, q)$ and $Ms_{n+1}^{(2)}(r, q)$ corresponding to the same s as $se_{n+1}(\theta, q)$ (see, for example, Gutiérrez-Vega *et. al.* [23]). Thus we have four families $l = 1, 2, 3, 4$, of eigenfunctions of Laplacian operator in an elliptical domain:

$$\begin{aligned}
u_{nk1}(r, \theta) &= \kappa e_n(\theta, q_{nk1}) M\kappa_n^{(1)}(r, q_{nk1}) \\
u_{nk1}(r, \theta) &= \kappa e_n(\theta, q_{nk2}) M\kappa_n^{(2)}(r, q_{nk2}) \\
u_{nk1}(r, \theta) &= se_{n+1}(\theta, q_{nk3}) Ms_{n+1}^{(1)}(r, q_{nk3}) \\
u_{nk1}(r, \theta) &= se_{n+1}(\theta, q_{nk4}) Ms_{n+1}^{(2)}(r, q_{nk4}).
\end{aligned} \tag{7.9}$$

For the elliptical annulus with Dirichlet boundary conditions having radius $0 < R_0 < R$, there are eight individual equations defining the parameter q for each $n = 0, 1, 2, \ldots$,

$$M\kappa_n^{(1)}(R, q_{nk1}) = 0;\ M\kappa_n^{(2)}(R, q_{nk2});\ Ms_{n+1}^{(1)}(R, q_{nk3}) = 0;\ Ms_{n+1}^{(2)}(R, q_{nk4}) = 0.$$

$$M\kappa_n^{(1)}(R_0, q_{nk1}) = 0;\ M\kappa_n^{(2)}(R_0, q_{nk2});\ Ms_{n+1}^{(1)}(R_0, q_{nk3}) = 0;\ Ms_{n+1}^{(2)}(R_0, q_{nk4}) = 0.$$

The fundamental solution (in elliptic coordinates) of

$$\frac{\partial^\beta c}{\partial t^\beta}(t, \mathbf{x}) = -(-\Delta_D)^{\alpha/2}(I - \Delta_D)^{\gamma/2} c(t, \mathbf{x}),$$

for $\alpha+\gamma>n$, with $-\Delta_D = -\frac{1}{a^2(\sinh^2 r+\sin^2\theta)}\left(\frac{\partial^2}{\partial r^2}+\frac{\partial^2}{\partial\theta^2}\right)$, is then given by

$$
\begin{aligned}
G_{t-s}(r,\theta,r',\theta') = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\sum_{l=1}^{4} E_\beta \Big(&- \left([4q_{nkl}/a^2]\right)^{\alpha/2} \\
&\times \left(1+\left([4q_{nkl}/a^2]\right)\right)^{\gamma/2}(t-s)^\beta\Big) \\
&\times u_{n,k,l}(r,\theta)u_{n,k,l}(r',\theta'),
\end{aligned}
$$

where $\{u_{n,k,l}\}$ are given in equation (7.9).

From Hölder inequality

$$|u_{n,k,l}(r,\theta)| \leq M\left[\pi(a\cosh R)(a\sinh R)-\pi(a\cosh R_0)(a\sinh R_0)\right],$$

where $M=\sup_{n,k,l}\max_{(r,\theta)}u_{nkl}(r,\theta)$, since angular Mathieu functions, for $0\leq\theta<2\pi$, and radial Mathieu functions of first and second kind, for $0<R_0<r<R$, are uniformly bounded (see, for example, Gutiérrez-Vega *et. al.* [23]). The uniform upper bound, in Theorems **4.1**, for the eigenvectors of Dirichlet negative Laplacian operator on D is then given by

$$C(D) = M\left[\pi(a\cosh R)(a\sinh R)-\pi(a\cosh R_0)(a\sinh R_0)\right].$$

Finally, since angular Mathieu functions, and radial Mathieu functions of the first and second kind on a closed bounded interval are Hölder continuous functions, Theorem **5.1** also holds. As before, Theorems **4.1**, **5.1**, **6.1** and **6.2** follow for $\alpha+\gamma>n$, and $\beta<1/2$.

8. Fractional polynomials of the Dirichlet negative Laplacian operator on D

The results formulated in this paper hold under a more general scenario. Specifically, in equations (1.2)–(1.3), we can replace $(-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}$ by a fractional elliptic polynomial of the form

$$P\left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right) = \sum_{l=0}^{p} c_l\left[(-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right]^l, \qquad (8.1)$$

of degree p, and with constant coefficients $c_l\geq 0$, $l=0,\dots,p-1$, and $c_p>0$. Thus, the following reformulation of equation (1.2) is considered

$$\frac{\partial^\beta}{\partial t^\beta}c(t,\mathbf{x}) + P\left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)c(t,\mathbf{x}) = I_t^{1-\beta}\varepsilon(t,\mathbf{x}), \quad \mathbf{x}\in D, \qquad (8.2)$$

with the boundary and initial conditions given in (1.3), and with $P\left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)$ being defined in (8.1). Here, as before, ε represents Gaussian space-time white noise. The next result provides the extension of the previously established statements, for equations (1.2)–(1.3), to equation (8.2).

THEOREM **8.1**. *The following assertions hold:*

(i) *For $n < p(\alpha+\gamma)$,*

$$\sum_{k=1}^{\infty} E_\beta\left(-\lambda_k\left(P\left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)\right)t^\beta\right) < \infty, \tag{8.3}$$

for every $t > 0$, where, for each $k \geq 1$

$$P\left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)\phi_k = \lambda_k\left(P\left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)\right)\phi_k. \tag{8.4}$$

For $n < p(\alpha+\gamma)$, the weak-sense solution on $\overline{H}^{p(\alpha+\gamma)}(D)$ to (8.2), in the mean-square sense, with boundary and initial conditions (1.3) is then given by

$$c(t,\mathbf{x}) = \int_0^t\int_D G^D(t,\mathbf{x};s,\mathbf{y})\varepsilon(s,\mathbf{y})ds d\mathbf{y}, \tag{8.5}$$

where the integral is understood in the mean-square sense, $\varepsilon(s,\mathbf{y})$ is space-time zero-mean Gaussian white noise as given in equation (8.2), and, for $t \geq s$,

$$\begin{aligned} &G^D(t,\mathbf{x};s,\mathbf{y}) = \\ &= \sum_{k\geq 1} E_\beta\left(-\lambda_k\left(P\left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)\right)(t-s)^\beta\right)\phi_k(\mathbf{x})\phi_k(\mathbf{y}), \\ &G^D(t,\mathbf{x};s,\mathbf{y}) = 0, \ s > t, \end{aligned} \tag{8.6}$$

with $\{\phi_k\}_{k\geq 1}$ and $\left\{\lambda_k\left(P\left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)\right)\right\}_{k\geq 1}$ satisfying (8.4).

(ii) *For $\beta < 1/2$, and $\frac{n}{2} < p(\alpha+\gamma)$, the following inequality holds:*

$$E[c(t,\mathbf{x}) - c(s,\mathbf{x})]^2 \leq [C(D)]^2 g(t-s), \tag{8.7}$$

where $C(D)$ is defined as in Theorem **4.1**, *and*

$$g(t-s) = \mathcal{O}\left((t-s)^{\left(1-\frac{\beta n}{p(\alpha+\gamma)}\right)\wedge(1-\beta)}\right), \quad s \to t, \ 0 < s < t. \tag{8.8}$$

(iii) *For $n < p(\alpha+\gamma)$, assume that the uniform Hölder continuity of the Dirichlet negative Laplacian operator eigenvectors holds, as given in Theorem* **5.1**, *considering $\|\mathbf{x}-\mathbf{y}\|$ sufficiently small,*

$$E[c(t,\mathbf{x})-c(t,\mathbf{y})]^2 \leq Cg(t)\|\mathbf{x}-\mathbf{y}\|^{2\Upsilon}, \tag{8.9}$$

where C is given in Theorem **5.1**, *and*

$$g(t) \quad = \quad t^{1-\beta}\sum_{k=1}^{\infty}\frac{\Gamma(1+\beta)}{\lambda_k\left(P\left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)\right)}, \quad t>0.$$

(iv) *For $\beta < 1/2$, and $n < p(\alpha+\gamma)$, assume that the uniform Hölder continuity of the Dirichlet negative Laplacian operator eigenvectors holds, as $s \to t$, $s,t \in (0,T]$, and $\|\mathbf{x}-\mathbf{y}\| \to 0$,*

$$\begin{aligned} & E[c(t,\mathbf{x})-c(s,\mathbf{y})]^2 \\ \leq \; & \widetilde{C}(D,T,\beta,\alpha,\gamma,p,\Upsilon)\|(t,\mathbf{x})-(s,\mathbf{y})\|^{\left(1-\frac{\beta n}{p(\alpha+\gamma})\right)\wedge(1-\beta)\wedge 2\Upsilon}, \end{aligned}$$

where

$$\widetilde{C}(D,T,\beta,\alpha,\gamma,p,\Upsilon) = 8([C(D)]^2 \vee Cg(T))\left(\frac{1}{2}\right)^{1/2\left[\left(1-\frac{\beta n}{(\alpha+\gamma)p}\right)\wedge(1-\beta)\wedge 2\Upsilon\right]}.$$

(v) *For $\beta < 1/2$, and $n < p(\alpha+\gamma)$, under the uniform Hölder continuity of the Dirichlet negative Laplacian operator eigenvectors, as $s \to t$, $s,t \in (0,T]$, and $\|\mathbf{x}-\mathbf{y}\| \to 0$,*

$$\begin{aligned} & \sup_{|t-s|<\delta}|c(t,\mathbf{x})-c(s,\mathbf{x})|^2 \\ \leq \; & \widetilde{Z}\delta^{\left(1-\frac{\beta n}{(\alpha+\gamma)p}\right)\wedge(1-\beta)} + \widetilde{H}_1\delta^{\left(1-\frac{\beta n}{(\alpha+\gamma)p}\right)\wedge(1-\beta)}\left[\log\left(\frac{1}{\delta}\right)\right]^{1/2} \\ & \sup_{\|\mathbf{x}-\mathbf{y}\|<\delta}|c(t,\mathbf{x})-c(t,\mathbf{y})|^2 \\ \leq \; & \widetilde{Y}\delta^{2\Upsilon} + \widetilde{H}_2\delta^{2\Upsilon}\left[\log\left(\frac{1}{\delta}\right)\right]^{1/2} \\ & \sup_{\|(t,\mathbf{x})-(s,\mathbf{y})\|<\delta}|c(t,\mathbf{x})-c(s,\mathbf{y})|^2 \\ \leq \; & \widetilde{X}\delta^{\left(1-\frac{\beta n}{(\alpha+\gamma)p}\right)\wedge(1-\beta)\wedge 2\Upsilon} \\ & +\widetilde{H}_3\delta^{\left(1-\frac{\beta n}{(\alpha+\gamma)p}\right)\wedge(1-\beta)\wedge 2\Upsilon}\left[\log\left(\frac{1}{\delta}\right)\right]^{1/2}, \end{aligned} \tag{8.10}$$

where $\widetilde{Z}, \widetilde{Y}$ and $\widetilde{X}$ are positive random variables, and $\widetilde{H}_i$, $i = 1, 2, 3$, are positive constants that could depend on the geometrical characteristics of the domain D considered, like the boundary.

Proof. (i) As, in Corollary **2.1**, we apply equation (2.5) in Theorem **2.1**, considering $f(u) = P(u^{\alpha/2}(1+u)^{\gamma/2})$, with $P(u) = \sum_{l=0}^{p} c_l u^l$ given in equation (8.1), to obtaining

$$\lambda_k\left(P\left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)\right) = \sum_{l=0}^{p} c_l \left[(\gamma_k(-\Delta_D))^{\alpha/2}\,(1+\gamma_k(-\Delta_D))^{\gamma/2}\right]^l, \tag{8.11}$$

where, as before, $\{\gamma_k(-\Delta_D)\}_{k\geq 1}$ denotes the eigenvalues of the Dirichlet negative Laplacian operator arranged in decreasing order of their magnitude. Since

$$\gamma_k(-\Delta_D) \sim 4\pi \frac{\left(\Gamma\left(1+\frac{n}{2}\right)\right)^{2/n}}{|D|^{2/n}} k^{2/n}, \quad k \longrightarrow \infty, \tag{8.12}$$

(see, for example, Chen and Song [13]), we obtain

$$\lim_{k\longrightarrow\infty} \frac{\lambda_k\left(P\left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)\right)}{k^{(\alpha+\gamma)p/n}} = \widetilde{c}(n, \alpha+\gamma, p)|D|^{-p(\gamma+\alpha)/n}, \tag{8.13}$$

where $\widetilde{c}(n, \alpha+\gamma)$ is a positive constant depending on n, α, γ and p.

Futhermore, equation (2.5) in Theorem **2.1** also implies the following equality: For $k \geq 1$,

$$P\left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)\phi_k = \lambda_k\left(P\left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)\right)\phi_k, \tag{8.14}$$

for the eigenvector system $\{\phi_k\}_{k\geq 1}$ of the Dirichlet negative Laplacian operator $(-\Delta_D)$ on domain D.

From equation (8.13), there exists k_0 such that for $k \geq k_0$,

$$\widetilde{L}_1 k^{p(\alpha+\gamma)/n} \leq \lambda_k\left(P\left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)\right) \leq \widetilde{L}_2 k^{p(\alpha+\gamma)/n}, \tag{8.15}$$

for certain positive constants $0 < \widetilde{L}_1 < \widetilde{L}_2$, depending on k_0, and $p(\alpha+\gamma)$ and n. In particular, for $k \geq k_0$,

$$\begin{aligned} &\frac{1}{1+[\Gamma(1+\beta)]^{-1}\lambda_k\left(P\left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)\right)t^{\beta}} \\ &\qquad \leq \frac{1}{1+[\Gamma(1+\beta)]^{-1}\widetilde{L}_1 k^{p(\alpha+\gamma)/n} t^{\beta}}. \end{aligned} \tag{8.16}$$

From Lemma **2.1**, in a similar way to Proposition **3.1**, for each fixed $t > 0$,

$$\sum_{k=1}^{\infty} E_{\beta}\left(-t^{\beta}\lambda_k\left(P\left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)\right)\right)$$
$$\leq M(\beta,\alpha,\gamma,p,n)+\frac{t^{-\beta n/p(\alpha+\gamma)}}{(\alpha+\gamma)/n}\int_0^{\infty}\frac{u^{\frac{n}{p(\alpha+\gamma)}-1}}{1+[\Gamma(1+\beta)]^{-1}u}du<\infty, \tag{8.17}$$

since

$$M(\beta,\alpha,\gamma,p,n)=\sum_{k=1}^{k_0} E_{\beta}\left(-t^{\beta}\lambda_k\left(P\left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)\right)\right)<\infty, \tag{8.18}$$

and $\int_0^{\infty}\frac{u^{\frac{n}{p(\alpha+\gamma)}-1}}{1+[\Gamma(1+\beta)]^{-1}u}du<\infty$, for $p(\alpha+\gamma)>n$.

For $\psi\in\overline{H}^{p(\alpha+\gamma)}(D)$, with $p(\alpha+\gamma)>n$, in a similar way to (8.17), it can be proved that

$$\sum_{k=1}^{\infty}\frac{\partial^{\beta}}{\partial t^{\beta}}E_{\beta}\left(-\lambda_k\left(P\left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)\right)t^{\beta}\right)\phi_k(\mathbf{x})\psi_k$$
$$=\sum_{k=1}^{\infty}\lambda_k\left(P\left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)\right)$$
$$\times E_{\beta}\left(-\lambda_k\left(P\left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)\right)t^{\beta}\right)\phi_k(\mathbf{x})\psi_k$$
$$\leq C(D)\sum_{k=1}^{\infty}\lambda_k\left(P\left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)\right)$$
$$\times E_{\beta}\left(-\lambda_k\left(P\left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)\right)t^{\beta}\right)\psi_k<\infty, \tag{8.19}$$

where, as before, $\psi_k=\int_D\phi_k(\mathbf{y})\psi(\mathbf{y})d\mathbf{y}$.

Applying the regularized fractional derivative in time (1.5), we then obtain, in a similar way to Proposition **3.2**,

$$\int_D\frac{\partial^{\beta}}{\partial t^{\beta}}c\,(t,\mathbf{x})\,\psi(\mathbf{x})d\mathbf{x}=\int_D I_t^{1-\beta}\varepsilon(t,\mathbf{x})\psi(\mathbf{x})d\mathbf{x}$$
$$-\int_D\left[P\left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)\int_0^t\int_D G^D\left(t,\mathbf{x};s,\mathbf{y}\right)\varepsilon(s,\mathbf{y})d\mathbf{y}ds\right]\psi(\mathbf{x})d\mathbf{x}, \tag{8.20}$$

for every $\psi \in \overline{H}^{p(\alpha+\gamma)}(D)$, as we wanted to prove.

(ii) In a similar way to Theorem **4.1**,

$$
\begin{aligned}
E[c(t,\mathbf{x})-c(s,\mathbf{x})]^2 \leq [C(D)]^2 \int_0^s \Big[\widetilde{M}(\beta,\alpha,\gamma,p,n,(s-u)^{\beta}) \\
+\frac{(s-u)^{-\beta n/p(\alpha+\gamma)}}{p(\alpha+\gamma)/n} \int_0^{\infty} \frac{x^{\frac{n}{p(\alpha+\gamma)}-1}}{(1+[\Gamma(1+\beta)]^{-1}x)^2} dx\Big]\, du \\
+[C(D)]^2 \int_s^t \Big[M(\beta,\alpha,\gamma,p,n,(t-u)^{\beta}) \\
+\frac{(t-u)^{-\beta n/p(\alpha+\gamma)}}{p(\alpha+\gamma)/n} \int_0^{\infty} \frac{x^{\frac{n}{p(\alpha+\gamma)}-1}}{1+[\Gamma(1+\beta)]^{-1}2x} dx\Big]\, du,
\end{aligned}
\tag{8.21}
$$

where

$$
\begin{aligned}
M(\beta,\alpha,\gamma,p,n,(s-u)^{\beta}) &= \sum_{k=1}^{k_0} E_{\beta}\left(-(s-u)^{\beta}\lambda_k\left(P\left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)\right)\right) \\
\widetilde{M}(\beta,\alpha,\gamma,p,n,(s-u)^{\beta}) &= \sum_{k=1}^{k_0} \left[E_{\beta}\left(-(s-u)^{\beta}\lambda_k\left(P\left((-\Delta_D)^{\alpha/2}(I-\Delta_D)^{\gamma/2}\right)\right)\right)\right]^2 .
\end{aligned}
\tag{8.22}
$$

Hence,

$$
\begin{aligned}
E[c(t,\mathbf{x})-c(s,\mathbf{x})]^2 \leq [C(D)]^2 \Big[K_1(\beta,\alpha,\gamma,p,n)s^{1-2\beta} \\
+K_2(\beta,\alpha,\gamma,p,n)s^{1-\beta n/(p(\alpha+\gamma))} + K_3(\beta,\alpha,\gamma,p,n)(t-s)^{1-\beta} \\
+K_4(\beta,\alpha,\gamma,p,n)(t-s)^{1-\beta n/((\alpha+\gamma)p)}\Big] .
\end{aligned}
\tag{8.23}
$$

Thus, when $s \to t$, $s < t$, we have

$$
E[c(t,\mathbf{x})-c(s,\mathbf{x})]^2 = \mathcal{O}\left((t-s)^{1-(\beta n/p(\alpha+\gamma))\wedge(1-\beta)}\right).
$$

(iii) Applying Hölder continuity of the eigenvectors, from Lemma **2.1**, in a similar way to Theorem **5.1**, for every $t > 0$,

$$\begin{aligned}
E[c(t,\mathbf{x}) - c(t,\mathbf{y})]^2 &\leq C\|\mathbf{x}-\mathbf{y}\|^{2\Upsilon} \\
&\qquad \times \int_0^t \sum_{k=1}^{\infty} \frac{\Gamma(1+\beta)}{\lambda_k\left(P\left((-\Delta)_D^{\alpha/2}(I-\Delta)_D^{\gamma/2}\right)\right)\nu^{\beta}} d\nu \\
&= C\|\mathbf{x}-\mathbf{y}\|^{2\Upsilon}\left[t^{1-\beta}\sum_{k=1}^{\infty}\frac{\Gamma(1+\beta)}{\lambda_k\left(P\left((-\Delta)_D^{\alpha/2}(I-\Delta)_D^{\gamma/2}\right)\right)}\right] \\
&= Cg(t)\|\mathbf{x}-\mathbf{y}\|^{2\Upsilon}, \qquad (8.24)
\end{aligned}$$

as we wanted to prove. Here, for each fixed $t > 0$,

$$g(t) = t^{1-\beta}\sum_{k=1}^{\infty}\frac{\Gamma(1+\beta)}{\lambda_k\left(P\left((-\Delta)_D^{\alpha/2}(I-\Delta)_D^{\gamma/2}\right)\right)} < \infty, \qquad (8.25)$$

for $p(\alpha+\gamma) > n$.

(iv) In a similar way to Theorem **6.1**, since under the conditions assumed, $0 < \left(1 - \frac{\beta n}{p(\alpha+\gamma)}\right) \wedge (1-\beta) \wedge 2\Upsilon < 1$, applying Jensen's inequality we obtain,

$$\begin{aligned}
E[c(t,\mathbf{x}) - c(s,\mathbf{y})]^2 &\leq 4([C(D)]^2 \vee Cg(T))\left(\frac{1}{2}\right)^{1/2\left[\left(1-\frac{\beta n}{p(\alpha+\gamma)}\right)\wedge(1-\beta)\wedge 2\Upsilon\right]} \\
&\qquad \times \left[\left(|t-s|^2 + \|\mathbf{x}-\mathbf{y}\|^2\right)^{1/2\left[\left(1-\frac{\beta n}{p(\alpha+\gamma)}\right)\wedge(1-\beta)\wedge 2\Upsilon\right]}\right. \\
&\qquad \left. + \sqrt{\left(|t-s|^2 + \|\mathbf{x}-\mathbf{y}\|^2\right)^{\left(1-\frac{\beta n}{p(\alpha+\gamma)}\right)\wedge(1-\beta)\wedge 2\Upsilon}}\right] \\
&= 2C(D,T,\beta,\alpha,\gamma,p,\Upsilon)\|(t,\mathbf{x}) - (s,\mathbf{y})\|^{\left(1-\frac{\beta n}{p(\alpha+\gamma)}\right)\wedge(1-\beta)\wedge 2\Upsilon}, \qquad (8.26)
\end{aligned}$$

where $\widetilde{C}(D,T,\beta,\alpha,\gamma,p,\Upsilon) = 2C(D,T,\beta,\alpha,\gamma,p,\Upsilon)$, and

$$C(D,T,\beta,\alpha,\gamma,p,\Upsilon) = 4([C(D)]^2 \vee Cg(T))\left(\frac{1}{2}\right)^{1/2\left[\left(1-\frac{\beta n}{p(\alpha+\gamma)}\right)\wedge(1-\beta)\wedge 2\Upsilon\right]}.$$

(v) The sample-path regularity properties follow straightforward from (ii)–(iv), by applying Theorem 3.3.3, in p.57 of Adler [1].

■

9. Final comments

Under the conditions assumed in Proposition **3.1**, a mean-square solution to equations (1.2)-(1.3) is derived in Proposition **3.2**, in the weak-sense on the space $\overline{H}^{\alpha+\gamma}(D)$ (respectively on the space $\overline{H}^{p(\alpha+\gamma)}(D)$, in the general case considered in Theorem **8.1**). In particular, from the results derived, we can define a $H^{-(\alpha+\gamma)}(D)-$valued stochastic process $\{\mathcal{C}_t,\ t\in\mathbb{R}^+\}$, on the basic probability space $(\Omega,\mathcal{A},P)$, satisfying equation (1.2) a.s., i.e.,

$$\left\langle \mathcal{C}_t(\cdot,\omega), \frac{\partial^\beta}{\partial t^\beta} + (-\Delta_D)^{\alpha/2}\,(I-\Delta_D)^{\gamma/2}\,\psi(\cdot)\right\rangle_{L^2(D)}$$
$$= \left\langle I_t^{1-\beta}\varepsilon(\cdot,\omega),\psi(\cdot)\right\rangle_{L^2(D)}, \quad \text{a.s,} \quad \forall\psi\in\overline{H}^{\alpha+\gamma}(D),\ t\in\mathbb{R}_+. \tag{9.1}$$

(Note that similar assertions hold for the derived solution to equation (8.2) in Theorem **8.1**). In this derivation, the orthogonality in $\mathcal{L}^2(\Omega,\mathcal{A},P)$ of the random components of ε is applied, i.e., we have applied that $E[\varepsilon(t,\mathbf{x})]=0$, and $E[\varepsilon(t,\mathbf{x})\varepsilon(s,\mathbf{y})]=\delta(t-s)\delta(\mathbf{x}-\mathbf{y})$, for $t,s\in\mathbb{R}_+$, and $\mathbf{x},\mathbf{y}\in D\subset\mathbb{R}^n$. It is well-known that this property holds for any white noise measure on $L^2(\mathbb{R}_+\times D)$, beyond the Gaussian case. Furthermore, the fractional integration in the definition of the driven process $I_t^{1-\beta}\varepsilon$ in equation (1.2) is understood in the mean-square sense on a suitable space of test functions, as given in Proposition **3.2**. Thus, we have only considered the properties of the second-order moments of the distribution of the driven process in equation (1.2). Hence, Proposition **3.2** also holds when the Gaussian space-time white noise on $L^2(\mathbb{R}_+\times D)$ is replaced by an arbitrary white noise random measure $\widetilde{\varepsilon}$ on $L^2(\mathbb{R}_+\times D)$. In particular, Lévy noise can be considered. In that case, Theorems **4.1**, **5.1** and **6.1** respectively provide the Hölder continuity, in the mean-square sense (i.e., the continuity of the second-order moments), in time, space, and space and time of the weak-sense solution, defined by integration with respect to Lévy noise $d\eta$, as

$$c(t,\mathbf{x}) = \int_0^t\int_D G^D(t,\mathbf{x};s,\mathbf{y})d\eta(s,\mathbf{y})dsd\mathbf{y}, \tag{9.2}$$

with, as before, G^D being defined in (3.10), for the case of Proposition **3.2**, and in (8.6), for the case of Theorem **8.1**. In both cases, we can interpret the integral (9.2) as a multiparameter Itô integral with respect to $n+1$-parameter Lévy process η, since, for $\alpha+\gamma>n$, and for each $t\in\mathbb{R}_+$, G_t^D defines a trace operator $\mathcal{G}_t$ on $L^2(D)$, as proved in Proposition **3.1** (see, for example, Løkka, Øksendal and Proske [28], for an alternative interpretation and derivation of solutions in that Lévy noise case, in terms of functions with values in the Kondratiev space of stochastic distributions). Summarizing, the derived results provide the characterization of the second-order regularity properties of the weak-sense solution to equations (1.2)–(1.3) (respectively, to equation (8.2) in Theorem **8.1**). For the non-Gaussian case, further research should be developed in order to obtain the distributional characteristics of (9.2), beyond the second-order moments. This subject will be considered in a subsequent paper. Note also that Theorem **6.2**, on the characterization of the sample-path regularity properties of the weak-sense solution to equations (1.2)-(1.3), in the mean-square sense (respectively, (v) of Theorem **8.1**) only holds for the Gaussian case.

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