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On superalgebras of matrices with symmetry properties

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ABSTRACT

It is known that constant sum matrices form a $\mathbb{Z}_2$-graded algebra or superalgebra with the even and odd subspaces under centre-point reflection symmetry as the two components. We show that other symmetries which have been studied for square matrices give rise to similar superalgebra structures, pointing to novel symmetry types in their complementary parts. In particular, this provides a unifying framework for the composite 'most perfect square' symmetry and the related class of 'reversible squares'; moreover, the algebra of constant sum matrices is identified as part of a $\mathbb{Z}_2$-gradation of the general square matrix algebra. We derive explicit representation formulae for matrices of all symmetry types considered, which can be used to construct all such matrices.

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1. Introduction

In this paper we present a novel approach to the classification of particular families of $n \times n$ matrices, defined by their symmetry properties, in terms of $\mathbb{Z}_2$-graded algebras. The latter type of algebra (also known as *superalgebra*) has a decomposition into an 'even' subalgebra and an 'odd' complementary part which is a bimodule over the 'even' subalgebra and squares into it. The families of matrices considered here are derived from nine fundamental symmetry properties that generate corresponding matrix symmetry vector spaces. Using a block matrix representation introduced in [1], we find that these matrix spaces arrange into four $\mathbb{Z}_2$-graded algebras and a single algebra (the space R_n defined below).

These algebraic structures enable us to analyse more specialized algebras of matrices, defined by compositions of these symmetry properties. Such matrix families encompass some well-known symmetry types such as the sets of constant sum matrices [2,3], the associated constant sum matrices [4,5], most perfect square matrices, and the reversible square matrices of [6]. Accompanying these matrix families in their respective $\mathbb{Z}_2$-graded algebras, we find hitherto undocumented matrix symmetry types such as the symmetries (N), (Q) and (V) defined in this paper. The present findings build on recent work [1,4,5,7] to provide insight into the deeper algebraic structures underpinning an area of mathematics that has been of interest for many years. In the process, we derive matrix

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algebraic characterisations of the symmetries and find representation formulae for the matrices of each type, which can be used to construct them.

Our basic symmetries are distilled from the following known symmetry types.

Constant sum matrices are defined by the property that all rows and all columns add up to the same constant [3]. In an associated constant sum matrix, opposite entries with respect to the centre of the square also add up to the same number. A balanced constant sum matrix has the complementary property that the opposite entries are equal, so the matrix has a half-turn rotational symmetry.

Most perfect squares are constant sum matrices with the additional properties that all 2×2 blocks of numbers add up to the same constant and that the matrix has the strong pandiagonal property, so all pairs of entries half the size of the square apart along a general diagonal (i.e. any line parallel to either of the two main diagonals) add up to the same constant. Clearly this definition only makes sense for square matrices of even dimension.

Reversible squares are square matrices with the properties that all pairs of entries on a row or column which have the same distance from the centre of the row or column, resp., add up to the same constant, and that for any rectangular submatrix, the two pairs of diagonally opposite vertex entries add up to the same constant.

It was shown in [4] that, after the removal of a common two-sided ideal, the constant sum matrices, considered as square matrices with the usual matrix addition and multiplication operations, form an algebra which has the form of a $\mathbb{Z}_2$-graded algebra, with the balanced constant sum matrices as 'even' subalgebra and the associated constant sum matrices as 'odd' complementary direct summand. In Sections 3 and 4, we explore the algebraic properties of the other types of matrices mentioned above, and of more general symmetry types arising in their definitions. Along the way, we also establish representation formulae for matrices of different symmetry types, which make their algebraic behaviour more transparent and also provide a simple way of constructing matrices of a particular type. It turns out that the $\mathbb{Z}_2$-graded algebra structure recurs in various guises.

In Section 5 we study the separate algebra of matrices with symmetry (R).

The set of most perfect square matrices and the set of reversible square matrices do not themselves form subalgebras of the general algebra of square matrices. However, in Sections 6 and 7 we identify suitable complementing subalgebras which extend these sets to $\mathbb{Z}_2$-graded algebras. Again, we provide explicit construction formulae for all matrices of these symmetry types.

2. Matrix symmetry type spaces

We consider the following symmetries of a square matrix $M = \left(M_{i,j}\right)_{i,j\in\mathbb{Z}_n} \in \mathbb{R}^{n\times n}$. Note that the indices are considered to be elements of the cyclic ring $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$, and all calculations with indices are performed in this ring, i.e. modulo n. The top left corner of the matrix will have indices $(1,1) \in \mathbb{Z}_n^2$.

(S) Constant sum property of weight w:

$$\sum_{j\in\mathbb{Z}_n} M_{i,j} = \sum_{j\in\mathbb{Z}_n} M_{j,i} = nw \qquad (i \in \mathbb{Z}_n).$$

(A) Associated property of weight $w : M_{i,j} + M_{n+1-i,n+1-j} = 2w$ $(i,j \in \mathbb{Z}_n)$.

(B) Balanced property: $M_{i,j} - M_{n+1-i,n+1-j} = 0$ ($i, j \in \mathbb{Z}_n$).

(R) Row and column reverse property:

$$M_{i,j} + M_{i,n+1-j} = M_{i,k} + M_{i,n+1-k},$$
$$M_{i,j} + M_{n+1-i,j} = M_{k,j} + M_{n+1-k,j} \qquad (i, j, k \in \mathbb{Z}_n).$$

(V) Vertex cross sum property: $M_{i,j} + M_{k,l} = M_{i,l} + M_{k,j}$ ($i, j, k, l \in \mathbb{Z}_n$).

In the case where $n = 2\nu$ is even, we also consider the following symmetries.

(M) 2×2 array sum property of weight w: $M_{i,j} + M_{i,j+1} + M_{i+1,j} + M_{i+1,j+1} = 4w$ ($i, j \in \mathbb{Z}_n$)
and alternating sum property:

$$\sum_{i,j \in \mathbb{Z}_n} (-1)^{i+j} M_{i,j} = 0.$$

(N) Consecutive row and column alternating sum property:

$$\sum_{i \in \mathbb{Z}_n} (-1)^i (M_{i,j} + M_{i,j+1}) = \sum_{i \in \mathbb{Z}_n} (-1)^i (M_{j,i} + M_{j+1,i}) = 0 \qquad (j \in \mathbb{Z}_n).$$

(P) Strong pandiagonal property of weight w: $M_{i,j} + M_{i+\nu,j+\nu} = 2w$ ($i, j \in \mathbb{Z}_n$).

(Q) Quartered sum property: $M_{i,j} - M_{i+\nu,j+\nu} = 0$ ($i, j \in \mathbb{Z}_n$).

Remarks:

(1) The following two further symmetries are often considered. The first is the property that both main diagonals of a constant sum matrix of weight w add up to nw; this is then called a *diagonal constant sum matrix* [1,3]. This property evidently follows from (A) or (P). The second is the (weak) pandiagonal property, where all general (cyclically broken) diagonals of the matrix add up to nw; this clearly follows from (P). We do not consider these two symmetries in this paper, except where they naturally follow from stronger properties.

(2) Property (M) does not at face value presuppose even matrix dimension n. However, if n is odd, then only the null matrix

$$\mathcal{O}_n = (0)_{i,j=1}^n \qquad (1)$$

has this property; see Lemma 2.1 below. We note that there are odd-dimensional non-trivial matrices with property (N), e.g.

$$M = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ 0 & -2 & -2 \end{pmatrix}.$$

(3) A most perfect square matrix has properties (M), (P) and (S). Note that in the original definition by Ollerenshaw (see [6] page 12), the alternating sum property part of (M) was not stipulated, however it is already implied by (P) in the case of even dimension $n = 2\nu$; indeed, then

$$\begin{aligned}
&\sum_{i,j\in\mathbb{Z}_n} (-1)^{i+j} M_{i,j} \\
&\quad = \sum_{i,j=1}^{\nu} \big((-1)^{i+j} M_{i,j} + (-1)^{i+j+2\nu} \times M_{i+\nu,j+\nu} \\
&\qquad + (-1)^{i+j+\nu}(M_{i+\nu,j} + M_{i,j+\nu})\big) \\
&\quad = \sum_{i,j=1}^{\nu} (-1)^{i+j}(2w + (-1)^{\nu} 2w) = 0
\end{aligned}$$

both for even and odd ν. When property (M) is considered by itself, the additional alternating sum property is essential to give a clear separation from property (N), see Theorem 2.9.

(4) Reversible squares have properties (R) and (V). Moreover, as we shall see below in Corollary 2.10 and Theorem 4.7, property (V) somewhat surprisingly also plays a role as a complement to property (S). Reversible squares arose from Ollerenshaw's adaptation of a 1939 construction [8] of Rosser and Walker, which she used to enumerate the number of doubly-even order most perfect square matrices [6].

(5) Properties (N) and (Q) have not previously been studied; we identify them here as natural complements to properties (M) and (P), respectively. In the case of (Q) this is easy to understand; a matrix with symmetry (Q) is of the form

$$\begin{pmatrix} A & B \\ B & A \end{pmatrix} \tag{2}$$

with $A, B \in \mathbb{R}^{\nu\times\nu}$, resembling a quartered shield in heraldry, whereas matrices with symmetry (P) have the structure

$$\begin{pmatrix} A & B \\ -B & -A \end{pmatrix} \tag{3}$$

with $A, B \in \mathbb{R}^{\nu\times\nu}$, so obviously any $n \times n$ matrix can be written as a sum of type (P) and type (Q) matrices. Symmetry (N), which means that the alternating sum of each row is the negative of the alternating sum of a neighbouring row, and similarly for the columns, is not very intuitive; it arises as a complementary property to (M) shared by products of matrices which have property (M), see Corollary 2.10 and Theorem 4.8.

Lemma 2.1: *Let $n \in \mathbb{N}$ be odd and $M \in \mathbb{R}^{n\times n}$ a matrix with property (M). Then $M = \mathcal{O}_n$.*

***Proof*:** By the 2×2 array sum property, we have for each $i \in \mathbb{Z}_n$

$$\begin{aligned}
M_{i,1} + M_{i+1,1} &= 4w - M_{i,2} - M_{i+1,2} = M_{i,3} + M_{i+1,3} = 4w - M_{i,4} - M_{i+1,4} \\
&= \cdots = M_{i,n} + M_{i+1,n} = 4w - M_{i,1} - M_{i+1,1},
\end{aligned}$$

so $M_{i,j} + M_{i+1,j} = 2w$ for all $i, j \in \mathbb{Z}_n$. Hence, for each $j \in \mathbb{Z}_n$,

$$M_{1,j} = 2w - M_{2,j} = M_{3,j} = \cdots = M_{n,j} = 2w - M_{1,j},$$

which implies $M_{i,j} = w$ $(i,j \in \mathbb{Z}_n)$. But then the alternating sum property requires $w = 0$. $\square$

In [1,4] it was observed that the matrix

$$\mathcal{E}_n = (1)_{i,j=1}^n \tag{4}$$

generates a two-sided ideal in the algebra of matrices having property (S), and that this ideal is the intersection of the subspace of matrices with properties (A), (S) and the subalgebra of matrices with properties (B), (S). Also, subtracting $w\mathcal{E}_n$ from the matrices which have (A) with weight w gives weight 0 matrices with the same symmetry.

In fact, the very simple matrix $\mathcal{E}_n$ shares all of the above symmetries.

Lemma 2.2: *Let $n \in \mathbb{N}$. The matrix $\mathcal{E}_n$ has properties (S), (A), (B), (R), (V), and, if n is even, also (M), (N), (P) and (Q); where applicable, its weight is $w = 1$.*

In consequence, we can often restrict our attention to the weightless case $w = 0$ by subtracting a suitable multiple of $\mathcal{E}_n$ from the matrices under consideration; we shall do this regularly with properties (A), (M) and (P).

Furthermore, all of the above symmetry properties are linear (with weight either fixed to 0 or left variable) and hence give rise to vector spaces of matrices as follows. The additional requirement in the definition of V_n corresponds to the restriction to weight 0 in A_n, M_n and P_n.

Definition 2.3: Let $n \in \mathbb{N}$. We define the following matrix symmetry type spaces.

$S_n = \{M \in \mathbb{R}^{n\times n} \mid M$ has property (S) with some weight $w\}$,
$A_n = \{M \in \mathbb{R}^{n\times n} \mid M$ has property (A) with weight $0\}$,
$B_n = \{M \in \mathbb{R}^{n\times n} \mid M$ has property (B)$\}$,
$R_n = \{M \in \mathbb{R}^{n\times n} \mid M$ has property (R)$\}$,
$V_n = \{M = (M_{i,j})_{i,j=1}^n \in \mathbb{R}^{n\times n} \mid M$ has property (V), and $\sum_{i,j=1}^n M_{i,j} = 0\}$.

For even n, we also define the symmetry type spaces

$M_n = \{M \in \mathbb{R}^{n\times n} \mid M$ has property (M) with weight $0\}$,
$N_n = \{M \in \mathbb{R}^{n\times n} \mid M$ has property (N)$\}$,
$P_n = \{M \in \mathbb{R}^{n\times n} \mid M$ has property (P) with weight $0\}$,
$Q_n = \{M \in \mathbb{R}^{n\times n} \mid M$ has property (Q)$\}$.

We shall extend the definitions of M_n and N_n to odd n below. Composite symmetry types are captured in the following intersections of the above spaces,

$RV_n = R_n \cap V_n$, $RS_n = R_n \cap S_n$, $AV_n = A_n \cap V_n$,
$AS_n = A_n \cap S_n$, $BS_n = B_n \cap S_n$

and, for even n,

$MPS_n = M_n \cap P_n \cap S_n$ and $NQS_n = N_n \cap Q_n \cap S_n$.

The composite symmetry spaces will be studied in Sections 6 and 7.

We use the convention of calling a direct sum $\Xi \oplus \mathrm{H}$, where Ξ, H are vector subspaces of $\mathbb{R}^{n\times n}$, a $\mathbb{Z}_2$-graded algebra if the first direct summand Ξ is the 'even' subalgebra and the second direct summand H is the 'odd' complement, i.e. if

$$\Xi\Xi \subset \Xi, \qquad \mathrm{H}\Xi \subset \mathrm{H}, \qquad \Xi\mathrm{H} \subset \mathrm{H}, \qquad \mathrm{H}\mathrm{H} \subset \Xi.$$

The following statements follow immediately from this definition.

Lemma 2.4: *Let $n \in \mathbb{N}$.*

(a) *If $\Xi \oplus \mathrm{H} \in \mathbb{R}^{n\times n}$ is a $\mathbb{Z}_2$-graded algebra and $\Gamma \subset \mathbb{R}^{n\times n}$ is a matrix algebra, then $(\Xi \cap \Gamma) \oplus (\mathrm{H} \cap \Gamma)$ is a $\mathbb{Z}_2$-graded algebra.*

(b) *If $\Xi \oplus \mathrm{H}, \Xi' \oplus \mathrm{H}' \subset \mathbb{R}^{n\times n}$ are $\mathbb{Z}_2$-graded algebras, then $(\Xi \cap \Xi') \oplus (\mathrm{H} \cap \mathrm{H}')$ is a $\mathbb{Z}_2$-graded algebra.*

As one of the central results of the present paper, we shall obtain the following symmetry superalgebras.

Theorem 2.5: *Let $n \in \mathbb{N}$.*

(a) *The following are $\mathbb{Z}_2$-graded algebras,*

$$\mathbb{R}^{n\times n} = B_n \oplus A_n = S_n \oplus V_n = N_n \oplus M_n;$$

if n is even, then also $\mathbb{R}^{n\times n} = Q_n \oplus P_n$.

(b) *For even n, the space of most perfect square matrices is the odd component of the $\mathbb{Z}_2$-graded algebra $NQS_n \oplus MPS_n$.*

(c) *The space of reversible square matrices is the odd component in the two $\mathbb{Z}_2$-graded algebras $R_n = RS_n \oplus RV_n$ and $BS_n \oplus RV_n$.*

The different symmetry properties were defined above by reference to the individual matrix entries. This is descriptive and helps visualize each particular matrix symmetry, but it is rather inconvenient for studying the algebraic properties of the symmetry type. We now give an equivalent characterisation of the symmetries in terms of matrix algebra.

We use the following notation. We write 0_n for the null vector in $\mathbb{R}^n$, and 1_n for the vector in this space which has all entries equal to 1. Moreover, we define the alternating vector $\S_n$ which has $(-1)^{j-1}$ for its jth entry, $j \in \{1, \dots, n\}$. These and other vectors in $\mathbb{R}^n$ are considered as column vectors; we denote row vectors by the transpose of column vectors, writing v^T for a row vector, where $v \in \mathbb{R}^n$. Thus for even $n \in \mathbb{N}$, we have

$$\S_n = (1, -1, 1, -1, \dots, 1, -1)^T \in \mathbb{R}^n,$$

and this vector is orthogonal on 1_n, but this is not the case if n is odd, since then $\S_n$ will have 1 as its last entry.

In addition to the matrices $\mathcal{E}_n$ and $\mathcal{O}_n$ already defined in (4) and (1), respectively, we use the special matrices $\mathcal{J}_n = (\delta_{i,n+1-j})_{i,j=1}^n \in \mathbb{R}^{n\times n}$, which has entries 1 on the antidiagonal and 0 otherwise, and the $n \times n$ unit matrix $\mathcal{I}_n = (\delta_{i,j})_{i,j=1}^n$, where $\delta_{i,j}$ is the Kronecker symbol. As usual, we denote by $X^\perp = \{u \in \mathbb{R}^n \mid u^T v = 0\ (v \in X)\}$ the orthogonal complement of a set $X \subset \mathbb{R}^n$.

Theorem 2.6: *Let $M \in \mathbb{R}^{n\times n}$, $n \in \mathbb{N}$. Then*

(a) *$M \in S_n$ if and only if $1_n^T M u = 0 = u^T M 1_n$ $(u \in \{1_n\}^\perp)$;*

(b) *$M \in A_n$ if and only if $M + \mathcal{J}_n M \mathcal{J}_n = \mathcal{O}_n$;*

(c) *$M \in B_n$ if and only if $M = \mathcal{J}_n M \mathcal{J}_n$;*

(d) $M \in R_n$ *if and only if* $(M + M\mathcal{J}_n)u = 0$ *and* $(M^T + M^T\mathcal{J}_n)u = 0$ $(u \in \{1_n\}^\perp)$*; these are equivalent to* $(M^T + \mathcal{J}_n M^T)\mathbb{R}^n \subset \mathbb{R}1_n$ *and* $(M + \mathcal{J}_n M)\mathbb{R}^n \subset \mathbb{R}1_n$*, respectively;*

(e) $M \in V_n$ *if and only if* $u^T M v = 0$ $(u, v \in \{1_n\}^\perp)$ *and* $1_n^T M 1_n = 0$*;*

(f) *if n is even, then* $M \in M_n$ *if and only if* $u^T M v = 0$ $(u, v \in \{\S_n\}^\perp)$ *and* $\S_n^T M \S_n = 0$*;*

(g) *if n is even, then* $M \in N_n$ *if and only if* $\S_n^T M u = 0 = u^T M \S_n$ $(u \in \{\S_n\}^\perp)$.

Remark: The symmetry types (Q) and (weightless) (P) are conveniently described by their block matrix structures (2) and (3). Note that S_n, V_n closely parallel N_n and M_n, respectively, with the vector $\S_n$ taking the role of 1_n for the latter pair.

For later use, we also define symmetry spaces M_n and N_n for *odd* n in terms of the properties in Theorem 2.6 (f), (g); this will be useful in Theorems 4.4 and 4.5. Note, however, that the elements of these spaces will *not* in general have the symmetries (N) or (M), respectively; in particular, M_n will contain non-trivial matrices notwithstanding Lemma 2.1.

Definition: Let $n \in \mathbb{N}$ be odd. Then we define the symmetry type spaces

$$M_n = \{M \in \mathbb{R}^{n\times n} \mid u^T M v = 0 \ (u, v \in \{\S_n\}^\perp), \S_n^T M \S_n = 0\},$$
$$N_n = \{M \in \mathbb{R}^{n\times n} \mid \S_n^T M u = 0 = u^T M \S_n \ (u \in \{\S_n\}^\perp)\}.$$

In the proof of Theorem 2.6 (a) and later on we shall use the following observation that the conditions in (a) and (g) are equivalent to an eigenvalue property of M and M^T.

Lemma 2.7: *Let $n \in \mathbb{N}$ and $y \in \mathbb{R}^n \setminus \{0_n\}$. Then $M \in \mathbb{R}^{n\times n}$ satisfies*

$$y^T M u = 0 = u^T M y \qquad (u \in \{y\}^\perp)$$

if and only if there is some $\lambda \in \mathbb{R}$ such that $My = \lambda y$, $M^T y = \lambda y$.

***Proof*:** Since $0 = u^T M y$ for all $u \in \{y\}^\perp$, we see that $My \in \{y\}^{\perp\perp} = \mathbb{R}y$, so there is some $\lambda \in \mathbb{R}$ such that $My = \lambda y$. Similarly, $0 = y^T M u = (u^T M^T y)^T$ shows that there is some $\lambda' \in \mathbb{R}$ such that $M^T y = \lambda' y$. Hence

$$\lambda y^T y = y^T M y = (M^T y)^T y = \lambda' y^T y,$$

and as $y^T y \neq 0$, it follows that $\lambda' = \lambda$. The converse statement is obvious. □

Proof of Theorem 2.6: (a) Property (S) can be rewritten in the form $M1_n = M^T 1_n = nw1_n$, so the equivalence follows by Lemma 2.7 with $y = 1_n$. (b) and (c) are straightforward, noting that conjugation with $\mathcal{J}_n$ rotates the matrix by a half-turn.

For (d), note that $(M + M\mathcal{J}_n)u = 0$ $(u \in \{1_n\}^\perp)$ means that $M + M\mathcal{J}_n = (M_{i,j} + M_{i,n+1-j})_{i,j\in\mathbb{Z}_n}$ has constant rows. Also, $(M + \mathcal{J}_n M)\mathbb{R}^n \subset \mathbb{R}1_n$ means that $M + \mathcal{J}_n M = (M_{i,j} + M_{n+1-i,j})_{i,j\in\mathbb{Z}_n}$ has constant columns. These statements are equivalent to (R). The other equivalent equations follow by considering M^T.

For (e), first note that $1_n^T M 1_n = \sum_{i,j=1}^n M_{i,j}$. Now consider the vectors v_j, $j \in \{1, n-1\}$, defined such that v_j has 1 in the jth and -1 in the $(j+1)$st positions, and zeros otherwise. These vectors form a basis of $\{1_n\}^\perp$; indeed, any vector $u = (u_1, u_2, \ldots, u_n)^T$ such that $\sum_{k=1}^n u_k = 0$ can be rewritten as

$$u = u_1 v_1 + (u_1 + u_2)v_2 + (u_1 + u_2 + u_3)v_3 + \cdots + (u_1 + u_2 + \cdots + u_{n-1})v_{n-1}.$$

Now, for any $j, k \in \{1, \dots, n-1\}$,

$$v_j^T M v_k = M_{j,k} + M_{j+1,k+1} - M_{j,k+1} - M_{j+1,k} = 0$$

by property (V), and (e) follows by bilinearity. Conversely, if (e) holds and $j, k, l, m \in \{1, \dots, n\}$, let u be the vector such that $u_j = 1$, $u_l = -1$ and all other entries vanish, and let v be the vector such that $v_k = 1$, $v_m = -1$ and all other entries vanish. Then $u, v \in \{1_n\}^{\perp}$, so

$$0 = u^T M v = M_{j,k} + M_{l,m} - M_{j,m} - M_{l,k},$$

and hence M has property (V).

For (f), note first that $\S_n^T M \S_n = \sum_{i,j=1}^{n} (-1)^{i+j} M_{i,j}$. Further, the 2×2 array sum property with weight 0 can be expressed as

$$v_i^T M v_j = 0 \qquad (i, j \in \{1, \dots, n\}),$$

where $v_k \in \mathbb{R}^n$ is the vector which has entries 1 in the kth and $k+1$st positions (in positions n and 1 if $k = n$) and 0 otherwise. Obviously, $\S_n^T v_k = 0$ ($k \in \{1, \dots, n\}$) and this holds for all linear combinations of the v_k, too. In fact, the vectors $\{v_1, v_2, \dots, v_{n-1}\}$ span the subspace $\{\S_n\}^{\perp}$: given $u \in \{\S_n\}^{\perp}$, we can take $\alpha_1 = u_1, \alpha_2 = u_2 - u_1, \alpha_3 = u_3 - u_2 + u_1$, etc. ending with $\alpha_n = u_n - u_{n-1} + u_{n-2} - \cdots + u_2 - u_1 = 0$; then $u = \sum_{j=1}^{n-1} \alpha_j v_j$. Therefore, by bilinearity a square matrix with property (M) also satisfies (f). The converse is straightforward.

To see that (g) is equivalent to the condition (N), consider the vectors v_k defined in the proof of part (f), which span the space $\{\S_n\}^{\perp}$. As $M\S_n$ and $\S_n^T M$ are the vectors of alternating row and column sums of M, respectively, (N) implies that

$$\S_n^T M v_k = 0 = v_k^T M \S_n \qquad (k \in \{1, \dots, n-1\}),$$

and hence (g) by linearity; the converse is obvious.

We now observe that the conditions in Theorem 2.6 (a) and (e), as well as those in (f) and (g), are essentially mutually exclusive.

Lemma 2.8: *Let $n \in \mathbb{N}$ and $y \in \mathbb{R}^n \setminus \{0_n\}$. If $M \in \mathbb{R}^{n \times n}$ satisfies*

(i) $y^T M u = 0 = u^T M y$ ($u \in \{y\}^{\perp}$),
(ii) $u^T M v = 0$ ($u, v \in \{y\}^{\perp}$), *and*
(iii) $y^T M y = 0$,

then $M = \mathcal{O}_n$.

***Proof*:** The matrix $P = (y^T y)^{-1} y y^T$ is symmetric and idempotent, $P^2 = P$; it follows that $\mathcal{I}_n - P$ also has these properties. If $u \in \mathbb{R}^n$, then Pu is a multiple of y and $y^T P u = y^T u$, so $(\mathcal{I}_n - P)u \in \{y\}^{\perp}$. Hence, for $u, v \in \mathbb{R}^n$, $u^T M v = (Pu)^T M P v + ((\mathcal{I}_n - P)u)^T M P v + (Pu)^T M (\mathcal{I}_n - P) v + ((\mathcal{I}_n - P)u)^T M (\mathcal{I}_n - P) v = 0$, where the first term vanishes by (iii), the second and third by (i) and the fourth by (ii). □

Theorem 2.9: *Let $n \in \mathbb{N}$. Then $A_n \cap B_n = \{\mathcal{O}_n\}$, $S_n \cap V_n = \{\mathcal{O}_n\}$ and $M_n \cap N_n = \{\mathcal{O}_n\}$. If n is even, then also $P_n \cap Q_n = \{\mathcal{O}_n\}$.*

Proof: The second and third identity follow from Lemma 2.8, taking $y = 1_n$ and $y = \$_n$, respectively. The first and fourth identity are immediate from combining (A), (B) and (P), (Q), respectively, with weight $w = 0$. □

Hence we obtain the following four ways of splitting the vector space of $n \times n$ square matrices into direct sums of symmetry subspaces.

Corollary 2.10: *Let $n \in \mathbb{N}$. Then $\mathbb{R}^{n\times n} = B_n \oplus A_n = S_n \oplus V_n = N_n \oplus M_n$.*
If n is even, then also $\mathbb{R}^{n\times n} = Q_n \oplus P_n$.

Proof: In view of Theorem 2.9, we only need to show that any $n \times n$ matrix can be written as a sum of matrices from each summand in all cases.

Let $M \in \mathbb{R}^{n\times n}$. Then

$$M = \frac{1}{2}(M + \mathcal{J}_n M \mathcal{J}_n) + \frac{1}{2}(M - \mathcal{J}_n M \mathcal{J}_n),$$

and using Theorem 2.6 (b), (c) and the fact that $\mathcal{J}_n^2 = \mathcal{I}_n$, we see that the first term is in B_n, the second in A_n.

Further, defining the projector P as in the proof of Lemma 2.8, we find

$$M = (PMP + (\mathcal{I}_n - P)M(\mathcal{I}_n - P)) + (PM(\mathcal{I}_n - P) + (\mathcal{I}_n - P)MP);$$

then for $y = 1_n$, the first bracket is in S_n, the second in V_n by Theorem 2.6 (a), (e); for $y = \$_n$, the first bracket is in N_n, the second in M_n by Theorem 2.6 (g), (f).

Finally, if $n = 2\nu$ is even, then we can split M into $\nu \times \nu$ blocks,

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \frac{1}{2}\begin{pmatrix} A+D & B+C \\ B+C & A+D \end{pmatrix} + \frac{1}{2}\begin{pmatrix} A-D & B-C \\ -(B-C) & -(A-D) \end{pmatrix},$$

with the first matrix on the right-hand side, of form (2), in Q_n, the second matrix, of form (3), in P_n. □

3. Representation formulae: the $B_n \oplus A_n$ and $Q_n \oplus P_n$ algebras

We now proceed to find representation formulae for the various symmetry types. These will give a way of constructing matrices of a particular symmetry type by an expression without or with much simpler constraints; in the cases of matrix spaces S_n, V_n, N_n and M_n with even n, there will be a recursive element in that the construction formula requires some lower-dimensional matrix of the same type (see Section 4 below). Furthermore, these representation formulae will make the relationship between symmetry types and their algebraic properties more transparent.

We start with the spaces A_n and B_n. As a template for this approach, consider the characterisation and construction of the combined symmetry matrices in $A_n \cap S_n$ and in $B_n \cap S_n$ considered in [1].

A crucial role is played by the matrix $\mathcal{X}_n$, which is used to transform square matrices to their *block representation* by conjugation; it takes the form

$$\mathcal{X}_n = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathcal{I}_\nu & \mathcal{J}_\nu \\ \mathcal{J}_\nu & -\mathcal{I}_\nu \end{pmatrix} \in \mathbb{R}^{n\times n}$$

if $n = 2\nu$ is even, and

$$\mathcal{X}_n = \begin{pmatrix} \frac{1}{\sqrt{2}}\mathcal{I}_\nu & 0_\nu & \frac{1}{\sqrt{2}}\mathcal{J}_\nu \\ 0_\nu^T & 1 & 0_\nu^T \\ \frac{1}{\sqrt{2}}\mathcal{J}_\nu & 0_\nu & -\frac{1}{\sqrt{2}}\mathcal{I}_\nu \end{pmatrix} \in \mathbb{R}^{n\times n}$$

if $n = 2\nu+1$ is odd [1]. The matrix $\mathcal{X}_n$ is an orthogonal symmetric involution, i.e. $\mathcal{X}_n^T = \mathcal{X}_n$ and $\mathcal{X}_n^2 = \mathcal{I}_n$. It follows that

$$(\mathcal{X}_n M \mathcal{X}_n)(\mathcal{X}_n M' \mathcal{X}_n) = \mathcal{X}_n(MM')\mathcal{X}_n \qquad (M, M' \in \mathbb{R}^{n\times n}), \tag{5}$$

so conjugation with $\mathcal{X}_n$ (which is also linear) is a matrix algebra homomorphism.

Specifically for the weight matrix $\mathcal{E}_n$, the block representation is

$$\mathcal{E}_n = \mathcal{X}_n \begin{pmatrix} 2\mathcal{E}_\nu & \mathcal{O}_\nu \\ \mathcal{O}_\nu & \mathcal{O}_\nu \end{pmatrix} \mathcal{X}_n$$

if $n = 2\nu$ is even, and

$$\mathcal{E}_n = \mathcal{X}_n \begin{pmatrix} 2\mathcal{E}_\nu & \sqrt{2}\,1_\nu & \mathcal{O}_\nu \\ \sqrt{2}\,1_\nu^T & 1 & 0_\nu^T \\ \mathcal{O}_\nu & 0_\nu & \mathcal{O}_\nu \end{pmatrix} \mathcal{X}_n, \tag{6}$$

if $n = 2\nu + 1$ is odd.

Lemma 3.1: *A matrix $M \in \mathbb{R}^{n\times n}$ is an element of A_n if and only if*

$$M = \mathcal{X}_n \begin{pmatrix} \mathcal{O} & \Psi \\ \Phi & \mathcal{O}_\nu \end{pmatrix} \mathcal{X}_n, \tag{7}$$

where $\Phi, \Psi \in \mathbb{R}^{\nu\times\nu}$ if $n = 2\nu$ is even, $\Phi \in \mathbb{R}^{\nu\times(\nu+1)}$, $\Psi \in \mathbb{R}^{(\nu+1)\times\nu}$ if $n = 2\nu + 1$ is odd, and the top left null matrix has matching size.

***Proof*:** In the case of even n, it follows from Theorem 2.6 (b) and

$$\mathcal{J}_n = \begin{pmatrix} \mathcal{O}_\nu & \mathcal{J}_\nu \\ \mathcal{J}_\nu & \mathcal{O}_\nu \end{pmatrix} \tag{8}$$

that we can write the weight zero generally associated matrix in the form

$$M = \begin{pmatrix} A & -\mathcal{J}_\nu B \mathcal{J}_\nu \\ B & -\mathcal{J}_\nu A \mathcal{J}_\nu \end{pmatrix}$$

with some $A, B \in \mathbb{R}^{\nu\times\nu}$; then its block representation is

$$\mathcal{X}_n \begin{pmatrix} A & -\mathcal{J}_\nu B \mathcal{J}_\nu \\ B & -\mathcal{J}_\nu A \mathcal{J}_\nu \end{pmatrix} \mathcal{X}_n = \begin{pmatrix} \mathcal{O}_\nu & A\mathcal{J}_\nu + \mathcal{J}_\nu B \mathcal{J}_\nu \\ \mathcal{J}_\nu A - B & \mathcal{O}_\nu \end{pmatrix}.$$

Conversely,

$$\mathcal{X}_n \begin{pmatrix} \mathcal{O}_\nu & \Psi \\ \Phi & \mathcal{O}_\nu \end{pmatrix} \mathcal{X}_n = \frac{1}{2} \begin{pmatrix} \Psi\mathcal{J}_\nu + \mathcal{J}_\nu\Phi & -(\Psi\mathcal{J}_\nu - \mathcal{J}_\nu\Phi)\mathcal{J}_\nu \\ \mathcal{J}_\nu(\Psi\mathcal{J}_\nu - \mathcal{J}_\nu\Phi) & -\mathcal{J}_\nu(\Psi\mathcal{J}_\nu + \mathcal{J}_\nu\Phi)\mathcal{J}_\nu \end{pmatrix},$$

which evidently satisfies Theorem 2.6 (b).

In the case of odd n, we have

$$\mathcal{J}_n = \begin{pmatrix} \mathcal{O}_\nu & 0_\nu & \mathcal{J}_\nu \\ 0_n^T & 1 & 0_n^T \\ \mathcal{J}_\nu & 0_\nu & \mathcal{O}_\nu \end{pmatrix}, \tag{9}$$

and the matrix is of the form

$$M = \begin{pmatrix} A & v & -\mathcal{J}_\nu B\mathcal{J}_\nu \\ w^T & 0 & -w^T\mathcal{J}_\nu \\ B & -\mathcal{J}_\nu v & -\mathcal{J}_\nu A\mathcal{J}_\nu \end{pmatrix},$$

where $A, B \in \mathbb{R}^{\nu\times\nu}$ and $v, w \in \mathbb{R}^\nu$. Then its block representation is

$$\mathcal{X}_n \begin{pmatrix} A & v & -\mathcal{J}_\nu B\mathcal{J}_\nu \\ w^T & 0 & -w^T\mathcal{J}_\nu \\ B & -\mathcal{J}_\nu v & -\mathcal{J}_\nu A\mathcal{J}_\nu \end{pmatrix} \mathcal{X}_n = \begin{pmatrix} \mathcal{O}_\nu & 0_\nu & A\mathcal{J}_\nu + \mathcal{J}_\nu B\mathcal{J}_n \\ 0_\nu^T & 0 & \sqrt{2}w^T\mathcal{J}_\nu \\ \mathcal{J}_\nu A - B & \sqrt{2}\mathcal{J}_\nu v_\nu & \mathcal{O}_\nu \end{pmatrix}.$$

For the converse, the relationship between A, B and the first ν columns of Φ and rows of Ψ is as in the even-dimensional case. □

Lemma 3.2: *A matrix $M \in \mathbb{R}^{n\times n}$ is an element of B_n if and only if*

$$M = \mathcal{X}_n \begin{pmatrix} \Upsilon & \mathcal{O} \\ \mathcal{O} & \Omega \end{pmatrix} \mathcal{X}_n \tag{10}$$

with matrices $\Upsilon, \Omega \in \mathbb{R}^{\nu\times\nu}$ if $n = 2\nu$ is even, $\Upsilon \in \mathbb{R}^{(\nu+1)\times(\nu+1)}$, $\Omega \in \mathbb{R}^{\nu\times\nu}$ if $n = 2\nu + 1$ is odd, and null matrices of matching size.

***Proof*:** In the case of even n, it follows from Theorem 2.6 (c) and (8) that we can write the matrix in the form

$$M = \begin{pmatrix} A & \mathcal{J}_\nu B\mathcal{J}_\nu \\ B & \mathcal{J}_\nu A\mathcal{J}_\nu \end{pmatrix}$$

with $A, B \in \mathbb{R}^{\nu\times\nu}$; then its block representation is

$$\mathcal{X}_n \begin{pmatrix} A & \mathcal{J}_\nu B\mathcal{J}_\nu \\ B & \mathcal{J}_\nu A\mathcal{J}_\nu \end{pmatrix} \mathcal{X}_n = \begin{pmatrix} A + \mathcal{J}_\nu B & \mathcal{O}_\nu \\ \mathcal{O}_\nu & \mathcal{J}_\nu A\mathcal{J}_\nu - B\mathcal{J}_\nu \end{pmatrix}.$$

Conversely,

$$\mathcal{X}_n \begin{pmatrix} \Upsilon & \mathcal{O}_\nu \\ \mathcal{O}_\nu & \Omega \end{pmatrix} \mathcal{X}_n = \frac{1}{2} \begin{pmatrix} \Upsilon + \mathcal{J}_\nu\Omega\mathcal{J}_\nu & \mathcal{J}_\nu(\mathcal{J}_\nu\Upsilon - \Omega\mathcal{J}_\nu)\mathcal{J}_\nu \\ \mathcal{J}_\nu\Upsilon - \Omega\mathcal{J}_\nu & \mathcal{J}_\nu\Upsilon\mathcal{J}_\nu + \Omega \end{pmatrix}$$

clearly gives a balanced matrix.

In the case of odd n, the matrix is of the form

$$M = \begin{pmatrix} A & v & \mathcal{J}_\nu B \mathcal{J}_\nu \\ w^T & x & w^T \mathcal{J}_\nu \\ B & \mathcal{J}_\nu v & \mathcal{J}_\nu A \mathcal{J}_\nu \end{pmatrix},$$

with $A, B \in \mathbb{R}^{\nu\times\nu}$ and $v, w \in \mathbb{R}^\nu$, $x \in \mathbb{R}$. Then its block representation is

$$\mathcal{X}_n \begin{pmatrix} A & v & \mathcal{J}_\nu B \mathcal{J}_\nu \\ w^T & x & w^T \mathcal{J}_\nu \\ B & \mathcal{J}_\nu v & \mathcal{J}_\nu A \mathcal{J}_\nu \end{pmatrix} \mathcal{X}_n = \begin{pmatrix} A + \mathcal{J}_\nu B & \sqrt{2} v & \mathcal{O}_\nu \\ \sqrt{2} w^T & x & 0_n^T \\ \mathcal{O}_\nu & 0_n & \mathcal{J}_\nu A \mathcal{J}_\nu - B \mathcal{J}_\nu \end{pmatrix}.$$

For the converse, the relationship between A, B on the one hand and the top $\nu \times \nu$ submatrix of Υ and the matrix Ω is as in the even-dimensional case; v, w and x can be read off directly. □

The block representations of Lemma 3.1 and 3.2 make the splitting of a general matrix into its associated and balanced components very transparent. Moreover, it becomes obvious that this splitting gives $\mathbb{R}^{n\times n} = B_n \oplus A_n$ the structure of a $\mathbb{Z}_2$-graded algebra, with 'even' subalgebra B_n and 'odd' complement A_n.

Theorem 3.3: *Let $n \in \mathbb{N}$. Then*

$$B_n B_n \subset B_n, \qquad A_n A_n \subset B_n, \qquad A_n B_n \subset A_n, \qquad B_n A_n \subset A_n.$$

The same structure can be seen in the Q_n and P_n symmetry types; indeed, a straightforward calculation using directly the structures (2) and (3) shows that $\mathbb{R}^{n\times n} = Q_n \oplus P_n$ also is a $\mathbb{Z}_2$-graded algebra, with 'even' subalgebra Q_n.

Theorem 3.4: *Let $n \in \mathbb{N}$. Then*

$$Q_n Q_n \subset Q_n, \qquad P_n P_n \subset Q_n, \qquad P_n Q_n \subset P_n, \qquad Q_n P_n \subset P_n.$$

4. Representation formulae: the $S_n \oplus V_n$ and $N_m \oplus M_n$ algebras

Although the block representation by conjugation with the matrix $\mathcal{X}_n$ was originally devised to capture the structure of matrices with (A) or (B) symmetry, it also proves useful in the study of other symmetry types.

Theorem 4.1: *If $n = 2\nu$ is even, $M \in \mathbb{R}^{n\times n}$ is an element of S_n if and only if*

$$M = \mathcal{X}_n \begin{pmatrix} Y & V^T \\ W & Z \end{pmatrix} \mathcal{X}_n \tag{11}$$

with $Y \in S_\nu$, $V, W \in \mathbb{R}^{\nu\times\nu}$ with row sums 0, and $Z \in \mathbb{R}^{\nu\times\nu}$.

If $n = 2\nu + 1$ is odd, $M \in \mathbb{R}^{n\times n}$ is an element of S_n if and only if

$$M = \mathcal{X}_n \begin{pmatrix} Y + 2w\mathcal{E}_\nu & \sqrt{2}(w 1_\nu - Y 1_\nu) & V^T \\ \sqrt{2}(w 1_\nu - Y^T 1_\nu)^T & w + 2\, 1_\nu^T Y 1_\nu & -\sqrt{2}(V 1_\nu)^T \\ W & -\sqrt{2} W 1_\nu & Z \end{pmatrix} \mathcal{X}_n \tag{12}$$

with arbitrary $V, W, Y, Z \in \mathbb{R}^{\nu\times\nu}$; $w \in \mathbb{R}$ is the weight.

***Proof*:** First, consider the case of even n. Then

$$\mathcal{X}_n 1_n = \begin{pmatrix} \sqrt{2}\, 1_\nu \\ 0_\nu \end{pmatrix}, \tag{13}$$

and hence $u \in \{1_n\}^\perp$ if and only if $\mathcal{X}_n u = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$ with $\xi \in \{1_\nu\}^\perp$ and arbitrary $\eta \in \mathbb{R}^\nu$. Writing the block representation of M in the form (11), we find that the conditions of Theorem 2.6 (a) take the form

$$0 = \begin{pmatrix} \xi \\ \eta \end{pmatrix}^T \begin{pmatrix} Y & V^T \\ W & Z \end{pmatrix} \begin{pmatrix} 1_\nu \\ 0_\nu \end{pmatrix} = \xi^T Y 1_\nu + \eta^T W 1_\nu,$$
$$0 = \begin{pmatrix} 1_\nu \\ 0_\nu \end{pmatrix}^T \begin{pmatrix} Y & V^T \\ W & Z \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = 1^T Y \xi + (V 1_\nu)^T \eta$$

for all $\xi \in \{1_\nu\}$ and for any $\eta \in \mathbb{R}^\nu$. This is equivalent to $V1_\nu = 0_\nu$, $W1_\nu = 0_\nu$ and (again by Theorem 2.6 (a)) $Y \in S_\nu$.

The case of odd n is a bit more tricky. By Lemma 2.7, $M \in S_n$ is equivalent to 1_n being an eigenvector, for eigenvalue w, of both M and M^T. Hence, considering the matrix $M_0 := M - \frac{w}{n}\mathcal{E}_n$, we find $M_0 1_n = 0_n$, $M_0^T 1_n = 0_n$. Now observing that

$$\mathcal{X}_n 1_n = \begin{pmatrix} \sqrt{2}\, 1_\nu \\ 1 \\ 0_\nu \end{pmatrix} \tag{14}$$

and writing the block representation of M_0 in the form

$$M_0 = \mathcal{X}_n \begin{pmatrix} Y & v & V^T \\ y^T & \alpha & z^T \\ W & x & Z \end{pmatrix} \mathcal{X}_n \tag{15}$$

with $V, W, Y, Z \in \mathbb{R}^{\nu\times\nu}$, $x, y, v, z \in \mathbb{R}^\nu$ and $\alpha \in \mathbb{R}$, we see that these conditions on M_0 are equivalent to

$$\begin{pmatrix} 0_\nu \\ 0 \\ 0_\nu \end{pmatrix} = \begin{pmatrix} \sqrt{2}\, Y 1_\nu + v \\ \sqrt{2}\, y^T 1_\nu + \alpha \\ \sqrt{2}\, W 1_\nu + x \end{pmatrix}, \qquad \begin{pmatrix} 0_\nu \\ 0 \\ 0_\nu \end{pmatrix} = \begin{pmatrix} \sqrt{2}\, Y^T 1_\nu + y \\ \sqrt{2}\, v^T 1_\nu + \alpha \\ \sqrt{2}\, V 1_\nu + z \end{pmatrix}.$$

Thus $x = -\sqrt{2}\, W 1_\nu$, $y = -\sqrt{2}\, Y^T 1_\nu$, $v = -\sqrt{2}\, Y 1_\nu$, $z = -\sqrt{2}\, V 1_\nu$ and $\alpha = -\sqrt{2}\, 1_\nu^T v = 2\, 1_\nu^T Y 1_\nu = -\sqrt{2}\, y^T 1_\nu$, giving (12) in view of the block representation of $\mathcal{E}_n$, Equation (6). □

Theorem 4.2: *If $n = 2\nu$ is even, then $M \in \mathbb{R}^{n\times n}$ is an element of V_n if and only if*

$$M = \mathcal{X}_n \begin{pmatrix} Y & 1_\nu a^T \\ b 1_\nu^T & \mathcal{O}_\nu \end{pmatrix} \mathcal{X}_n \tag{16}$$

with $Y \in V_\nu$ and $a, b \in \mathbb{R}^\nu$.

If $n = 2\nu + 1$ is odd, then $M \in \mathbb{R}^{n\times n}$ is an element of V_n if and only if

$$M = \mathcal{X}_n \begin{pmatrix} \sqrt{2}(v1_\nu^T + 1_\nu y^T) - \frac{2\sqrt{2}}{2\nu-1}(1_\nu^T(v+y))\mathcal{E}_\nu & v & \sqrt{2}\,1_\nu z^T \\ y^T & \frac{\sqrt{2}}{2\nu-1}1_\nu^T(v+y) & z^T \\ \sqrt{2}\,x1_\nu^T & x & \mathcal{O}_\nu \end{pmatrix} \mathcal{X}_n \quad (17)$$

with arbitrary $v, x, y, z \in \mathbb{R}^\nu$.

We note that although the formula for matrices in S_n or V_n with even n looks simpler, it has a recursive condition on Y, whereas the formula for odd n has no restrictions. We also note that if n is odd, then the dimension of V_n is $4\nu = 2n - 2$. If n is even, then the dimension of V_n is the dimension of $V_{n/2}$ plus n; as V_2 has dimension 2, this also works out as $2n-2$, which correctly implies that the dimension of S_n is given by $n^2 - 2n + 2$ (see also [4]).

Proof: For even n, the first condition in Theorem 2.6 (e) translates, in analogy to the beginning of the proof of Theorem 4.1, into

$$0 = \begin{pmatrix} \xi \\ \eta \end{pmatrix}^T \begin{pmatrix} Y & V^T \\ W & Z \end{pmatrix} \begin{pmatrix} \xi' \\ \eta' \end{pmatrix} = \xi^T Y \xi' + \xi^T V^T \eta' + \eta^T W \xi' + \eta^T Z \eta',$$

for any $\xi, \xi' \in \{1_\nu\}^\perp$ and $\eta, \eta' \in \mathbb{R}^\nu$, in the block representation. When we take $\xi = \xi' = 0_\nu$, this implies $Z = \mathcal{O}_\nu$. Taking one or both of η, η' to be 0_ν, we find $V\xi = 0$, $W\xi = 0$ ($\xi \in \{1_\nu\}^\perp$) and $\xi^T Y \xi' = 0$ ($\xi, \xi' \in \{1_\nu\}^\perp$). Similarly, the second condition in Theorem 2.6 (e) gives $1_\nu^T Y 1_\nu = 0$. Hence $Y \in V_\nu$, and as the rows of V, W must be elements of $\{1_\nu\}^{\perp\perp} = \{1_\nu\}$, these matrices are of the stated form.

For odd n, the first condition of Theorem 2.6 (e) takes the block form

$$\begin{aligned} 0 &= \begin{pmatrix} \xi \\ -\sqrt{2}\,1_\nu^T \xi \\ \eta \end{pmatrix} \begin{pmatrix} Y & v & V^T \\ y^T & \alpha & z^T \\ W & x & Z \end{pmatrix} \begin{pmatrix} \xi' \\ -\sqrt{2}\,1_\nu^T \xi' \\ \eta' \end{pmatrix} \\ &= \xi^T Y \xi' - \sqrt{2}\left(\xi^T v\right)\left(1_\nu^T \xi'\right) + \xi V^T \eta' \\ &\quad - \sqrt{2}\left(\xi^T 1_\nu\right)\left(y^T \xi'\right) + 2\left(\xi^T 1_\nu\right)\left(1\_\nu^T \xi'\right)\alpha \\ &\quad - \sqrt{2}\left(\xi^T 1_\nu\right)\left(z^T \eta'\right) + \eta^T W \xi' - \sqrt{2}\left(\eta^T x\right)\left(1_\nu^T \xi'\right) + \eta^T Z \eta', \end{aligned} \quad (18)$$

for all $\xi, \xi', \eta, \eta' \in \mathbb{R}^\nu$.

Taking $\xi = \xi' = 0_\nu$, we conclude that $Z = \mathcal{O}_\nu$. Then, taking $\xi = 0_\nu$, we see that $\eta^T W \xi' = \sqrt{2}\,\eta^T x 1_\nu^T \xi'$ for all $\eta, \xi' \in \mathbb{R}^\nu$, which implies $W = \sqrt{2}\,x1^T$. Similarly, taking $\xi' = 0_\nu$ gives $V = \sqrt{2}\,z1^T$. This leaves (18) in the form

$$0 = \xi^T Y \xi' - \xi^T \left(\sqrt{2}\,v1_\nu^T + \sqrt{2}\,1_\nu y^T - 2\alpha 1_\nu 1_\nu^T\right)\xi'$$

for all $\xi, \xi' \in \mathbb{R}^\nu$, so $Y = \sqrt{2}\left(v1_\nu^T + 1_\nu y^T\right) - 2\alpha\mathcal{E}_\nu$.

Furthermore, the second condition in Theorem 2.6 (e) takes the block representation form

$$0 = \begin{pmatrix} \sqrt{2}\, 1_\nu \\ 1 \\ 0_\nu \end{pmatrix} \begin{pmatrix} Y & v & V^T \\ y^T & \alpha & z^T \\ W & x & Z \end{pmatrix} \begin{pmatrix} \sqrt{2}\, 1_\nu \\ 1 \\ 0_\nu \end{pmatrix} = 2\, 1_\nu^T Y 1_\nu + \sqrt{2}\, 1_\nu^T v + \sqrt{2}\, y^T 1_\nu + \alpha,$$

which together with the previous identity for Y gives $\alpha = \frac{\sqrt{2}}{2\nu - 1}\left(1_\nu^T v + y^T 1\right)$, and hence (17).

It is a straightforward calculation to check that, conversely, (17) satisfies Theorem 2.6 (e). □

Corollary 4.3: *If $M \in V_n$, $n = 2^k\,(2l+1)$ with $k, l \in \mathbb{N}_0$, then* rank $M \le 4 + k$.

***Proof*:** If $k = 0$, so n is odd, then in the block representation (17) the range of the top ν rows is spanned by the vectors 1_ν and v, the range of the bottom ν rows is spanned by the vector x, so taking into account the middle row, we see that the rank of this matrix does not exceed 4.

If $k > 0$, then it follows from (16) that the rank of M is no greater than the rank of $Y \in V_\nu$ plus 1, where $\nu = n/2$, and the estimate rank $M \le 4 + k$ follows by induction. □

Theorem 4.4: *If $n = 2\nu$ is even, then $M \in \mathbb{R}^{n\times n}$ is an element of N_n if and only if*

$$M = \mathcal{X}_n \begin{pmatrix} Y & V^T \\ W & Z \end{pmatrix} \mathcal{X}_n \tag{19}$$

with $Y \in \mathbb{R}^{\nu\times\nu}$, $V, W \in \mathbb{R}^{\nu\times\nu}$ such that $V^T \S_\nu = W^T \S_\nu = 0_\nu$, and $Z \in N_\nu$.

If $n = 2\nu + 1$ is odd, then $M \in \mathbb{R}^{n\times n}$ is an element of N_n if and only if

$$M = \mathcal{X}_n \begin{pmatrix} Y + 2\lambda \S_\nu \S_\nu^T & \pm\sqrt{2}\,(\lambda \S_\nu - Y\S_\nu) & V^T \\ \pm\sqrt{2}\,(\lambda\S_\nu - Y^T\S_\nu)^T & \lambda + 2\,\S_\nu^T Y \S_\nu & \mp\sqrt{2}\,(V\S_\nu)^T \\ W & \mp\sqrt{2}\, W\S_\nu & Z \end{pmatrix} \mathcal{X}_n \tag{20}$$

with arbitrary $V, W, Y, Z \in \mathbb{R}^{\nu\times\nu}$ and $\lambda \in \mathbb{R}$; here the upper signs apply if ν is even, the lower signs if ν is odd.

***Proof*:** The proof of Theorem 4.4 is largely analogous to the proof of Theorem 4.1, with the vector $\S_n$ taking the role of the vector 1_n, so we just detail the differences.

In the case of even n, we note that $J_\nu \S_\nu = \mp\S_\nu$ (where the upper sign applies if ν is even, the lower sign if ν is odd), so

$$\mathcal{X}_n \S_n = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathcal{I}_\nu & \mathcal{J}_\nu \\ \mathcal{J}_\nu & -\mathcal{I}_\nu \end{pmatrix} \begin{pmatrix} \S_\nu \\ \pm\S_\nu \end{pmatrix} = \mp\sqrt{2} \begin{pmatrix} 0_\nu \\ \S_\nu \end{pmatrix}. \tag{21}$$

Thus $u \in \{\S_n\}^\perp$ if and only if $\mathcal{X}_n u = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$ with arbitrary $\xi \in \mathbb{R}^\nu$ and $\eta \in \{\S_\nu\}^\perp$. The conditions of Theorem 2.6 (g) take the form

$$0 = \xi^T V^T \S_\nu + \eta^T Z \S_\nu, \qquad 0 = \S_\nu^T W \xi + \S_\nu^T Z \eta,$$

and give the conditions on V, W, Z stated in the theorem.

In the case of odd n, we apply Lemma 2.7 with $y = \S_n$ to find that $\S_n$ is an eigenvector, for eigenvalue $\lambda \in \mathbb{R}$, of both M and M^T, and consider the matrix $M_0 := M - \frac{\lambda}{n}\S_n\S_n^T$. Then $M_0\S_n = 0_n$, $M_0\S_n = 0_n$. Now

$$\mathcal{X}_n\S_n = \mathcal{X}_n\begin{pmatrix}\S_\nu\\ \pm 1\\ \mp\S_\nu\end{pmatrix} = \begin{pmatrix}\sqrt{2}\,\S_\nu\\ \pm 1\\ 0_n\end{pmatrix}, \tag{22}$$

and writing the block representation of M_0 as in (15), we see that the conditions on M_0 are equivalent to

$$\begin{pmatrix}0_\nu\\ 0\\ 0_\nu\end{pmatrix} = \begin{pmatrix}\sqrt{2}\,Y\S_\nu \pm v\\ \sqrt{2}\,y^T\S_\nu \pm \alpha\\ \sqrt{2}\,W\S_\nu \pm x\end{pmatrix}, \qquad \begin{pmatrix}0_\nu\\ 0\\ 0_\nu\end{pmatrix} = \begin{pmatrix}\sqrt{2}\,Y^T\S_\nu \pm y\\ \sqrt{2}\,v^T\S_\nu \pm \alpha\\ \sqrt{2}\,V\S_\nu \pm z\end{pmatrix},$$

from which v, x, y, z and α can be expressed in terms of V, W and Y. Equation (20) follows by observing that

$$\mathcal{X}_n\S_n\S_n^T\mathcal{X}_n = \mathcal{X}_n\S_n(\mathcal{X}_n\S_n)^T = \begin{pmatrix}2\,\S_\nu\S_\nu^T & \pm\sqrt{2}\,\S_\nu & 0_\nu\\ \pm\sqrt{2}\,\S_\nu^T & 1 & 0_\nu^T\\ \mathcal{O}_\nu & 0_\nu & \mathcal{O}_\nu\end{pmatrix}.$$

□

Theorem 4.5: *If $n = 2\nu$ is even, then $M \in \mathbb{R}^{n\times n}$ is an element of M_n if and only if*

$$M = \mathcal{X}_n\begin{pmatrix}\mathcal{O}_\nu & a\S_\nu^T\\ \S_\nu b^T & Z\end{pmatrix}\mathcal{X}_n \tag{23}$$

with $Z \in M_\nu$ and $a, b \in \mathbb{R}^\nu$.

If $n = 2\nu + 1$ is odd, then $M \in \mathbb{R}^{n\times n}$ is an element of M_n if and only if

$$M = \mathcal{X}_n\begin{pmatrix}\pm\sqrt{2}\,(v\S_\nu^T + \S_\nu y^T) \mp \frac{2\sqrt{2}}{2\nu-1}\,(\S_\nu^T(v+y))\S_\nu\S_\nu^T & v & \pm\sqrt{2}\,\S_\nu z^T\\ y^T & \frac{\pm\sqrt{2}}{2\nu-1}\,\S_\nu^T(v+y) & z^T\\ \pm\sqrt{2}\,x\S_\nu^T & x & \mathcal{O}_\nu\end{pmatrix}\mathcal{X}_n \tag{24}$$

with arbitrary $v, x, y, z \in \mathbb{R}^\nu$; the upper sign applies if ν is even, the lower sign if ν is odd.

***Proof*:** In the case of even n, we use the formal block representation (19), and by (21), the conditions of Theorem 2.6 (f) become

$$0 = \xi^T Y\xi' + \xi^T V^T\eta' + \eta^T W\xi' + \eta^T Z\eta',$$
$$0 = \S_\nu^T Z\S_\nu,$$

for all $\xi, \xi' \in \mathbb{R}^\nu$ and $\eta, \eta' \in \{\S_\nu\}^\perp$, so the stated properties of V, W, Y, Z follow as in the proof of Theorem 4.2.

For odd n, the reasoning is very similar to the proof of Theorem 4.2. However, in view of (22) we now have $u \in \{\S_n\}^{\perp}$ if and only if

$$\mathcal{X}_n u = \begin{pmatrix} \xi \\ \mp\sqrt{2}\,\S_\nu^T \xi \\ \eta \end{pmatrix}, \quad (\xi, \eta \in \mathbb{R}^\nu),$$

so the analogue of (18) takes the form

$$\begin{aligned} 0 = \xi^T Y \xi' &\pm \sqrt{2}\left(\xi^T v\right)\left(\S_\nu^T \xi'\right) + \xi V^T \eta' \\ &\pm \sqrt{2}\left(\xi^T \S_\nu\right)\left(y^T \xi'\right) + 2\left(\xi^T \S_\nu\right)\left(\S\_\nu^T \xi'\right)\alpha \\ &\pm \sqrt{2}\left(\xi^T \S_\nu\right)\left(z^T \eta'\right) + \eta^T W \xi' \pm \sqrt{2}\left(\eta^T x\right)\left(\S_\nu^T \xi'\right) + \eta^T Z \eta' \end{aligned}$$

for all $\xi, \eta \in \mathbb{R}^\nu$; the remaining calculations are as before. □

The splittings $\mathbb{R}^{n\times n} = S_n \oplus V_n$ and $\mathbb{R}^{n\times n} = N_n \oplus M_n$ again have the structure of a $\mathbb{Z}_2$-graded algebra, with subalgebra S_n and N_n, respectively. This follows from the next, more general, observation, which uses the symmetry properties in their matrix algebra form (Theorem 2.6) directly, rather than the block representations of Theorems 4.1 and 4.2.

Lemma 4.6: *Let $n \in \mathbb{N}$ and $y \in \mathbb{R}^N \setminus \{0_n\}$. Let* (i), (ii), (iii) *denote the conditions listed in Lemma 2.8.*

(a) *If $M, M' \in \mathbb{R}^{n\times n}$ either both satisfy condition* (i) *or both satisfy conditions* (ii) *and* (iii), *then MM' satisfies condition* (i).
(b) *If $M \in \mathbb{R}^{n\times n}$ satisfies condition* (i) *and $M' \in \mathbb{R}^{n\times n}$ satisfies conditions* (ii) *and* (iii), *then MM' and $M'M$ satisfy conditions* (ii) *and* (iii).

Proof: Let P be the projector defined in the proof of Lemma 2.8, and $u, v \in \{y\}^{\perp}$. Then, for (a) we observe

$$y^T MM'u = y^T MPM'u + y^T M(\mathcal{I}_n - P)M'u = 0,$$

as in each of the two situations one half of each term vanishes; and similarly $uMM'y^T = 0$. For (b), we note that

$$u^T MM'v = u^T MPM'v + u^T M(\mathcal{I}_n - P)M'v = 0,$$

since $u^T MP = 0_n$ and $(\mathcal{I}_n - P)M'v = 0_n$; and

$$y^T MM'y = y^T MPM'y + y^T M(\mathcal{I}_n - P)M'y = 0,$$

since $PM'y = 0_n$ and $y^T M(\mathcal{I}_n - P) = 0_n$. □

In view of Theorem 2.6 (a) and (e), the choice $y = 1_n$ immediately gives the following result.

Theorem 4.7: *Let $n \in \mathbb{N}$. Then*

$$S_nS_n \subset S_n, \qquad V_nV_n \subset S_n, \qquad V_nS_n \subset V_n, \qquad S_nV_n \subset V_n.$$

Similarly, by Theorem 2.6 (f) and (g), the choice $y = \S_n$ gives the following statement.

Theorem 4.8: *Let $n \in \mathbb{N}$. Then*

$$N_nN_n \subset N_n, \qquad M_nM_n \subset N_n, \qquad M_nN_n \subset M_n, \qquad N_nM_n \subset M_n.$$

5. Representation formulae: the algebra R_n

We now turn to symmetry type (R). This does not directly fit into the scheme of pairings we observed in the other symmetry types; nevertheless, the block representation turns out to be a valuable tool for constructing the matrices in R_n and for understanding their properties.

Theorem 5.1: *If $n = 2\nu$ is even, then $M \in \mathbb{R}^{n\times n}$ is an element of R_n if and only if*

$$M = \mathcal{X}_n \begin{pmatrix} \gamma\mathcal{E}_\nu & 1_\nu z^T \\ x1_\nu^T & Z \end{pmatrix} \mathcal{X}_n \tag{25}$$

with $Z \in \mathbb{R}^{\nu\times\nu}$, $x, z \in \mathbb{R}^\nu$ and $\gamma \in \mathbb{R}$.

If $n = 2\nu + 1$ is odd, then $M \in \mathbb{R}^{n\times n}$ is an element of R_n if and only if

$$M = \mathcal{X}_n \begin{pmatrix} \sqrt{2}\,\gamma\mathcal{E}_\nu & \gamma 1_\nu & \sqrt{2}\,1_\nu z^T \\ \gamma 1_\nu^T & \frac{\gamma}{\sqrt{2}} & z^T \\ \sqrt{2}\,x1_\nu^T & x & Z \end{pmatrix} \mathcal{X}_n \tag{26}$$

with $Z \in \mathbb{R}^{\nu\times\nu}$, $x, z \in \mathbb{R}^\nu$ and $\gamma \in \mathbb{R}$.

***Proof*:** By Theorem 2.6 (d), $M \in R_n$ if and only if $(M + \mathcal{J}_nM)\mathbb{R}^n \subset \mathbb{R}1_n$ and $(M^T + \mathcal{J}_nM^T)\mathbb{R}^n \subset \mathbb{R}1_n$.

In the case of even n, this means, using the formal block representation (11), Equation (13) and

$$\mathcal{X}_n\mathcal{J}_n\mathcal{X}_n = \begin{pmatrix} \mathcal{I}_\nu & \mathcal{O}_\nu \\ \mathcal{O}_\nu & -\mathcal{I}_\nu \end{pmatrix}, \tag{27}$$

that

$$2\begin{pmatrix} Y & V^T \\ \mathcal{O}_\nu & \mathcal{O}_\nu \end{pmatrix}\mathbb{R}^n = \left(\begin{pmatrix} Y & V^T \\ W & Z \end{pmatrix} + \begin{pmatrix} \mathcal{I}_\nu & \mathcal{O}_\nu \\ \mathcal{O}_\nu & -\mathcal{I}_\nu \end{pmatrix}\begin{pmatrix} Y & V^T \\ W & Z \end{pmatrix}\right)\mathbb{R}^n \subset \mathbb{R}\begin{pmatrix} 1_\nu \\ 0_\nu \end{pmatrix},$$
$$2\begin{pmatrix} Y^T & W^T \\ \mathcal{O}_\nu & \mathcal{O}_\nu \end{pmatrix}\mathbb{R}^n = \left(\begin{pmatrix} Y^T & W^T \\ V & Z^T \end{pmatrix} + \begin{pmatrix} \mathcal{I}_\nu & \mathcal{O}_\nu \\ \mathcal{O}_\nu & -\mathcal{I}_\nu \end{pmatrix}\begin{pmatrix} Y^T & W^T \\ V & Z^T \end{pmatrix}\right)\mathbb{R}^n \subset \mathbb{R}\begin{pmatrix} 1_\nu \\ 0_\nu \end{pmatrix},$$

equivalent to all columns of Y, Y^T, V^T and W^T being multiples of 1_ν.

Similarly, in the case of odd n, we use (14) and the formal block representation (15) along with

$$\mathcal{X}_n\mathcal{J}_n\mathcal{X}_j = \begin{pmatrix} \mathcal{I}_\nu & 0_\nu & \mathcal{O}_\nu \\ 0_\nu^T & 1 & 0_\nu^T \\ \mathcal{O}_\nu & 0_\nu & -\mathcal{I}_\nu \end{pmatrix}$$

to rewrite the above conditions as

$$2\begin{pmatrix} Y & v & V^T \\ y^T & \alpha & z^T \\ \mathcal{O}_\nu & 0_\nu & \mathcal{O}_\nu \end{pmatrix} \mathbb{R} \subset \mathbb{R}\begin{pmatrix} \sqrt{2}\,1_\nu \\ 1 \\ 0_\nu \end{pmatrix}, \qquad 2\begin{pmatrix} Y^T & y & W^T \\ v^T & \alpha & x^T \\ \mathcal{O}_\nu & 0_\nu & \mathcal{O}_\nu \end{pmatrix} \mathbb{R} \subset \mathbb{R}\begin{pmatrix} \sqrt{2}\,1_\nu \\ 1 \\ 0_\nu \end{pmatrix}.$$

Hence we conclude that $Y = \sqrt{2}\,\gamma 1_\nu 1_\nu^T$ for some $\gamma \in \mathbb{R}$, $v = y = \gamma 1_\nu$ and $\alpha = \frac{\gamma}{\sqrt{2}}$. Moreover, $V^T = \sqrt{2}\,1_\nu z^T$ and $W^T = \sqrt{2}\,1_\nu x^T$. □

From the block representations of Theorem 5.1, it is apparent that the space R_n is a subalgebra of the matrix algebra $\mathbb{R}^{n\times n}$.

Theorem 5.2: *Let $n \in \mathbb{N}$. Then $R_nR_n \subset R_n$.*

***Proof*:** It is sufficient to show that the product of block representations of type R matrices is the block representation of a type R matrix.

Let $\gamma, \gamma' \in \mathbb{R}$, $x, z, x', z' \in \mathbb{R}^\nu$ and $Z, Z' \in \mathbb{R}^{\nu\times\nu}$. Then, for even $n = 2\nu$,

$$\begin{pmatrix} \gamma\mathcal{E}_\nu & 1_\nu z^T \\ b1_\nu^T & Z \end{pmatrix}\begin{pmatrix} \gamma'\mathcal{E}_\nu & 1_\nu z'^T \\ x'1_\nu^T & Z' \end{pmatrix} = \begin{pmatrix} (\gamma\gamma'\nu + z^Tx')\mathcal{E}_\nu & 1_\nu(\gamma\nu z'^T + z^TZ') \\ (\gamma'\nu x + Zx')1_\nu^T & \nu x z'^T + ZZ' \end{pmatrix},$$

which is of the form (25). For odd $n = 2\nu + 1$,

$$\begin{pmatrix} \sqrt{2}\,\gamma\mathcal{E}_\nu & \gamma 1_\nu & \sqrt{2}\,1_\nu z^T \\ \gamma 1_\nu^T & \frac{\gamma}{\sqrt{2}} & z^T \\ \sqrt{2}\,x1_\nu^T & x & Z \end{pmatrix}\begin{pmatrix} \sqrt{2}\,\gamma'\mathcal{E}_\nu & \gamma' 1_\nu & \sqrt{2}\,1_\nu z'^T \\ \gamma' 1_\nu^T & \frac{\gamma'}{\sqrt{2}} & z'^T \\ \sqrt{2}\,x'1_\nu^T & x' & Z' \end{pmatrix}$$

$$= \begin{pmatrix} (n\gamma\gamma' + 2z^Tx')\mathcal{E}_\nu & \frac{n\gamma\gamma'+2z^Tx'}{\sqrt{2}}1_\nu & 1_\nu(n\gamma z'^T + \sqrt{2}\,zZ') \\ \frac{n\gamma\gamma'+2z^Tx'}{\sqrt{2}}1_\nu^T & \frac{n\gamma\gamma'+2z^Tx'}{2} & \frac{n\gamma z'^T+\sqrt{2}\,zZ'}{\sqrt{2}} \\ (n\gamma' x + \sqrt{2}\,Zx')1_\nu^T & \frac{n\gamma' x+\sqrt{2}\,Zx'}{\sqrt{2}} & nxz'^T + ZZ' \end{pmatrix},$$

which is of the form (26). □

Remark: In view of the $\mathbb{Z}_2$-graded algebras $B_n \oplus A_n$, $N_n \oplus M_n$, $S_n \oplus V_n$ and $Q_n \oplus P_n$, it seems a natural question to look for a complementary space to R_n, hoping to establish another $\mathbb{Z}_2$-graded algebra. The complement to the (even-dimension) block representation (25) would be

$$\begin{pmatrix} Y & V^T \\ W & \mathcal{O}_\nu \end{pmatrix}$$

with $V1_\nu = W1_\nu = 0_\nu$, $1_\nu^TY1_\nu = 0$. As $\mathcal{X}_n\begin{pmatrix} 1_\nu \\ u \end{pmatrix} = 1_n + \begin{pmatrix} \mathcal{J}_\nu u \\ -u \end{pmatrix}$, this is equivalent to the matrix M having the property that $(1_n + u)^TM(1_n + v) = 0$ for all $u, v \in \mathbb{R}^n$ such that $\mathcal{J}_nu = -u$, $\mathcal{J}_nv = -v$. This defines another space which directly sums with R_n to the whole matrix space; however, on first appearance it does not seem to form a $\mathbb{Z}_2$-graded algebra with R_n, although their relationship may well merit further examination.

6. Composite symmetry: most perfect squares

After studying the basic symmetry types defined in Section 2, we now proceed to the more complicated symmetries of most perfect square matrices and, in the next section, reversible square matrices. As these matrix spaces arise as intersections of some basic symmetry spaces, their algebraic properties as well as construction formulae can be readily deduced from the results in the preceding sections by means of Lemma 2.4.

We recall that $MPS_n = M_n \cap P_n \cap S_n$ is indeed the space of all weightless most perfect square matrices, although the second part of property (M), corresponding to the last condition in Theorem 2.6 (f), was not stipulated in the original definition of most perfect squares; indeed it is implied by condition (P) as follows. Using the fact that $\S_n = \begin{pmatrix} \S_\nu \\ \pm\S_\nu \end{pmatrix}$, where the upper sign always refers to the case of even ν, the lower to the case of odd ν, Equation (3) gives

$$\S_n^T M \S_n = \begin{pmatrix} \S_\nu \\ \pm\S_\nu \end{pmatrix}^T \begin{pmatrix} A & B \\ -B & -A \end{pmatrix} \begin{pmatrix} \S_\nu \\ \pm\S_\nu \end{pmatrix} = 0.$$

Since $\mathcal{E}_n$ is a most perfect square matrix, the general most perfect square matrices form the space $MSP_n \oplus \mathbb{R}\mathcal{E}_n$.

The elements of MSP_n have the following block representation.

Theorem 6.1: *Let $M \in \mathbb{R}^{n\times n}$, $n = 2\nu$ even. Then $M \in MPS_n$ if and only if there are vectors $a, b \in \{1_\nu\}^\perp$ with $\mathcal{J}_\nu a = \mp a$, $\mathcal{J}_\nu b = \mp b$, where the upper sign applies if ν is even, the lower sign if ν is odd, and a matrix $Z \in A_\nu \cap M_\nu$, such that*

$$M = \mathcal{X}_n \begin{pmatrix} \mathcal{O}_\nu & a\S_\nu^T \\ \S_\nu b^T & Z \end{pmatrix} \mathcal{X}_n.$$

Proof: Let $M \in MPS_n$. Combining the block representations of Theorems 4.1 and 4.5, we find that

$$M = \mathcal{X}_n \begin{pmatrix} \mathcal{O}_\nu & a\S_\nu^T \\ \S_\nu b^T & Z \end{pmatrix} \mathcal{X}_n,$$

where $a^T 1_\nu = 0 = b^T 1_\nu$ and $Z \in M_\nu$. From (3), we see that

$$\mathcal{X}_n M \mathcal{X}_n = \frac{1}{2} \begin{pmatrix} A + B\mathcal{J}_\nu - \mathcal{J}_\nu B - \mathcal{J}_\nu A \mathcal{J}_\nu & A\mathcal{J}_\nu - B - \mathcal{J}_\nu B \mathcal{J}_\nu + \mathcal{J}_\nu A \\ \mathcal{J}_\nu A + \mathcal{J}_\nu B \mathcal{J}_\nu + B + A\mathcal{J}_\nu & \mathcal{J}_\nu A \mathcal{J}_\nu - \mathcal{J}_\nu B + B\mathcal{J}_\nu - A \end{pmatrix},$$

and the calculation

$$\mathcal{J}_\nu(\mathcal{J}_\nu A \pm B \pm \mathcal{J}_\nu B \mathcal{J}_\nu + A\mathcal{J}_\nu)\mathcal{J}_\nu = A\mathcal{J}_\nu \pm \mathcal{J}_\nu B \mathcal{J}_\nu \pm B + \mathcal{J}_\nu A$$

shows that $\S_\nu a^T, \S_\nu b^T \in B_\nu$. Thus, by Theorem 2.6 (c), $\S_\nu a^T = \mathcal{J}_\nu \S_\nu a^T \mathcal{J}_\nu = \mp \S_\nu a^T \mathcal{J}_\nu$, and hence $a = \mp \mathcal{J}_\nu a$; and similarly for b. Also,

$$\mathcal{J}_\nu Z \mathcal{J}_\nu = \frac{1}{2} \mathcal{J}_\nu(\mathcal{J}_\nu A \mathcal{J}_\nu - \mathcal{J}_\nu B + B\mathcal{J}_\nu - A)\mathcal{J}_\nu = \frac{1}{2} A - B\mathcal{J}_\nu + \mathcal{J}_\nu B - \mathcal{J}_\nu A \mathcal{J}_\nu = -Z,$$

so $Z \in A_\nu$ by Theorem 2.6 (b).

Conversely, let

$$M = \mathcal{X}_n \begin{pmatrix} \mathcal{O}_\nu & a\S_\nu^T \\ \S_\nu b^T & Z \end{pmatrix} \mathcal{X}_n,$$

where a, b, Z have the properties stated in the theorem. Then $M \in M_n \cap S_n$ by Theorems 4.1 and 4.5, and

$$M = \frac{1}{2} \begin{pmatrix} a\S_\nu^T \mathcal{J}_\nu + \mathcal{J}_\nu \S_\nu b^T + \mathcal{J}_\nu Z \mathcal{J}_\nu & -a\S_\nu^T + \mathcal{J}_\nu \S_\nu b^T \mathcal{J}_\nu - \mathcal{J}_\nu Z \\ \mathcal{J}_\nu a \S_\nu^T \mathcal{J}_\nu - \S_\nu b^T - Z\mathcal{J}_\nu & -\mathcal{J}_\nu a \S_\nu^T - \S_\nu b^T \mathcal{J}_\nu + Z \end{pmatrix}$$

is of the form (3), as can be checked by a straightforward calculation. □

The block representation of Theorem 6.1 can be used as a simple method of constructing most perfect squares, as illustrated in the example

$$2\mathcal{X}_6 \begin{pmatrix} 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -2 & 2 & -2 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ -2 & 4 & -2 & 1 & 0 & -1 \\ 2 & -4 & 2 & -1 & 0 & 1 \\ -2 & 4 & -2 & 1 & 0 & -1 \end{pmatrix} \mathcal{X}_6 = \begin{pmatrix} -2 & 3 & 0 & -4 & 5 & -2 \\ 1 & -2 & -1 & 5 & -6 & 3 \\ -2 & 3 & 0 & -4 & 5 & -2 \\ 4 & -5 & 2 & 2 & -3 & 0 \\ -5 & 6 & -3 & -1 & 2 & 1 \\ 4 & -5 & 2 & 2 & -3 & 0 \end{pmatrix}.$$

However, it turns out that the structure and construction of most perfect square matrices is even more simple. Indeed, they can be conveniently characterized, even without the use of the block representation, in the following way.

Theorem 6.2: *A matrix $M \in \mathbb{R}^{n\times n}$, $n = 2\nu$ even, is an element of MPS_n if and only if*

$$M = \gamma \S_n^T + \S_n \delta^T, \tag{28}$$

where

(a) *in case ν is even,* $\gamma = \begin{pmatrix} \tilde{\gamma} \\ -\tilde{\gamma} \end{pmatrix}$, $\delta = \begin{pmatrix} \tilde{\delta} \\ -\tilde{\delta} \end{pmatrix}$, $\tilde{\gamma}, \tilde{\delta} \in \mathbb{R}^\nu$,

(b) *in case ν is odd,* $\gamma = \begin{pmatrix} \tilde{\gamma} \\ \tilde{\gamma} \end{pmatrix}$, $\delta = \begin{pmatrix} \tilde{\delta} \\ \tilde{\delta} \end{pmatrix}$, $\tilde{\gamma}, \tilde{\delta} \in \{1_\nu\}^T \subset \mathbb{R}^\nu$.

The vectors γ, δ can be obtained from M as $\gamma = \frac{1}{n} M \S_n$, $\delta = \frac{1}{n} M^T \S_n$.

***Proof*:** If $M \in MPS_n$, then it follows from the second equation in Theorem 2.6 (f) that $\gamma := \frac{1}{n} M \S_n \in \{\S_n\}^\perp$.

Let $v \in \{\S_n\}^\perp$; then, by the first equation in Theorem 2.6 (f), $Mv \in \{\S_n\}^{\perp\perp} = \mathbb{R}\S_n$. Hence $Mv = f(v)\S_n$ ($v \in \{\S_n\}^\perp$), where f is a linear form on $\{\S_n\}^\perp$. By the Riesz representation theorem, there is a vector $\delta \in \{\S_n\}^\perp$ such that $f(v) = \delta^T v$, so $Mv = \S_n \delta^T v$ ($v \in \{\S_n\}^\perp$).

Now any $x \in \mathbb{R}^n$ can be written in the form $x = \alpha \S_n + v$, with $\alpha \in \mathbb{R}$ and $v \in \{\S_n\}^\perp$; then

$$\begin{aligned} Mx &= \alpha M \S_n + Mv = \alpha n \gamma + \S_n \delta^T v = (\gamma \S_n^T + \S_n \delta^T)(\alpha \S_n + v) \\ &= (\gamma \S_n^T + \S_n \delta^T)x, \end{aligned}$$

showing that M is of the form (28).

Writing $\gamma = \begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}$, $\delta = \begin{pmatrix}\delta_1\\ \delta_2\end{pmatrix}$, with $\gamma_1, \gamma_2, \delta_1, \delta_2 \in \mathbb{R}^\nu$, we find

$$M = \begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}\begin{pmatrix}\S_\nu\\ \pm\S_\nu\end{pmatrix}^T + \begin{pmatrix}\S_\nu\\ \pm\S_\nu\end{pmatrix}\begin{pmatrix}\delta_1\\ \delta_2\end{pmatrix}^T = \begin{pmatrix}\gamma_1\S_\nu^T + \S_\nu\delta_1^T & \pm\gamma_1\S_\nu^T + \S_\nu\delta_2^T\\ \gamma_2\S_\nu^T \pm \S_\nu\delta_1^T & \pm\gamma_2\S_\nu^T \pm \S_\nu\delta_2^T\end{pmatrix}.$$

Viewed in conjunction with (3), this implies $(\gamma_1 \pm \gamma_2)\S_\nu^T + \S_\nu(\delta_1 \pm \delta_2)^T = \mathcal{O}_\nu$. As

$$0 = \S_n^T\gamma = \begin{pmatrix}\S_\nu\\ \pm\S_\nu\end{pmatrix}^T\begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix} = \S_\nu^T(\gamma_1 \pm \gamma_2),$$

it follows that

$$0_\nu = \S_\nu^T(\gamma_1 \pm \gamma_2)\S_\nu^T + \S_\nu^T\S_\nu(\delta_1 \pm \delta_2)^T = 0_\nu + \nu(\delta_1 \pm \delta_2)^T,$$

so $\delta_2 = \mp\delta_1$. An analogous calculation gives $\gamma_2 = \mp\gamma_1$.

By Lemma 2.7 and Theorem 2.6 (a), (f) (with $u = v = 1_n$), 1_n is an eigenvector with eigenvalue 0 for both M and M^T. Since $\S_n^T 1_n = 0$, it follows that $\gamma, \delta \in \{1_n\}^\perp$. If ν is even, this will be satisfied for any γ_1, δ_1 in view of the above structure; if ν is odd, it gives the further condition that γ_1, δ_1 are orthogonal to 1_ν.

For the converse, a straightforward calculation shows that any M of the form (28), with γ, δ satisfying the hypotheses, has the properties listed in Theorem 2.6 (a), (f) and (3). □

The two terms in the representation (28) are obviously rank 1 matrices (if non-null), so we can immediately draw the following conclusion.

Corollary 6.3: *A most perfect square matrix has at most rank 2 if its weight is 0, at most rank 3 in general.*

As a further consequence, we find an equivalent criterion for *parasymmetry* of a most perfect (and in particular magic) square matrix M, defined in [1] as symmetry of its square M^2. Indeed, as $\S_n^T\gamma = \delta^T\S_n = 0$ and therefore $M^2 = 2n\gamma\delta^T + (\delta^T\gamma)\S_n\S_n^T$, the following is evident.

Corollary 6.4: *A weight* 0 *most perfect square*

$$M = \gamma\S_n^T + \S_n\delta^T \in MPS_n$$

is parasymmetric if and only if γ, δ are linearly dependent.

By Lemma 2.4, MPS_n is the complement to a product symmetry type $NQS_n := N_n \cap Q_n \cap S_n$, so that $NQS_n \oplus MPS_n$ is again a $\mathbb{Z}_2$-graded algebra.

NQS-type matrices have the following block representation.

Theorem 6.5: *Let $M \in \mathbb{R}^{n\times n}$, $n = 2\nu$ even. Then $M \in NQS_n$ if and only if*

$$M = \mathcal{X}_n\begin{pmatrix}Y & V^T\\ W & Z\end{pmatrix}\mathcal{X}_n, \qquad (29)$$

where $Y \in B_\nu \cap S_\nu$, $Z \in B_\nu \cap N_\nu$, *and* $V, W \in A_\nu$ *have the properties*

$$V1_\nu = W1_\nu = 0_\nu, \qquad V^T \S_\nu = W^T \S_\nu = 0_\nu. \tag{30}$$

The conditions on V and W in Theorem 6.5 mean that these matrices have symmetry (A) with weight 0, and all their row sums and all alternating column sums vanish.

Proof: If $M \in NQS_n$, then by combining the block representations of Theorems 4.1 and 4.4, we find that (29) holds with $Y \in S_n$, $Z \in N_n$ and with V, W satisfying (30). As M also has property (Q), we find from (2) that

$$\mathcal{X}_n M \mathcal{X}_n = \begin{pmatrix} A + B\mathcal{J}_\nu + \mathcal{J}_\nu B + \mathcal{J}_\nu A \mathcal{J}_\nu & A\mathcal{J}_\nu - B + \mathcal{J}_\nu B \mathcal{J}_\nu - \mathcal{J}_\nu A \\ \mathcal{J}_\nu A + \mathcal{J}_\nu B \mathcal{J}_\nu - B - A\mathcal{J}_\nu & \mathcal{J}_\nu A \mathcal{J}_\nu - \mathcal{J}_\nu B - B\mathcal{J}_\nu + A \end{pmatrix}$$

and can read off, upon multiplication with $\mathcal{J}_\nu$ on both sides, that $V^T, W \in A_\nu$ and $Y, Z \in B_\nu$.

Conversely, assume M is given by (29), where V, W, Y, Z have the stated properties. Then M is semimagic by Theorems 4.1 and 4.4, and

$$M = \frac{1}{2}\begin{pmatrix} Y + V^T\mathcal{J}_\nu + \mathcal{J}_\nu W + \mathcal{J}_\nu Z \mathcal{J}_\nu & Y\mathcal{J}_\nu - V^T + \mathcal{J}_\nu W \mathcal{J}_\nu - \mathcal{J}_\nu Z \\ \mathcal{J}_\nu Y + \mathcal{J}_\nu V^T \mathcal{J}_\nu - W - Z\mathcal{J}_\nu & \mathcal{J}_\nu Y \mathcal{J}_\nu - \mathcal{J}_\nu V^T - W\mathcal{J}_\nu + Z \end{pmatrix}$$

has property (Q). □

Note that the $\mathbb{Z}_2$-graded algebra $NQS_n \oplus MPS_n$ is a subalgebra of S_n, but not all of S_n.

As the 'odd' part of a $\mathbb{Z}_2$-graded algebra, MPS_n is not itself a subalgebra of the full matrix algebra; however, it has the property that the product of any *three* elements of MPS_n is again an element of MPS_n. If, taking the formula (28) as a motivation, we introduce the notation $(\gamma; \delta) := \gamma \S_n^T + \S_n \delta^T$ for most perfect square matrices, then the triple product can be expressed as

$$(\gamma_1; \delta_1)(\gamma_2; \delta_2)(\gamma_3; \delta_3) = n\,((\delta_2^T \gamma_3)\gamma_1; (\delta_1^T \gamma_2)\delta_3).$$

7. Composite symmetry: reversible squares

We now turn to reversible square matrices, defined as those which have symmetry properties (R) and (V). Although the definition of these properties does not refer to a weight w, it turns out that reversible squares always have the associated symmetry property (A) and hence a hidden weight.

Lemma 7.1: *Any reversible square matrix has property (A) with some weight* $w \in \mathbb{R}$.

Proof: Let $M \in \mathbb{R}^{n \times n}$ be a reversible matrix, so $M \in R_n$ and M has property (V), which, following the proof of Theorem 2.6 (e), can be seen to be equivalent to

$$u^T M v^T = 0 \qquad (u, v \in \{1_n\}^\perp). \tag{31}$$

By Theorem 2.6 (b), we only need to show that there is $w \in \mathbb{R}$ such that $M + \mathcal{J}_n M \mathcal{J}_n = 2w\mathcal{E}_n$.

Consider the two orthogonal projectors (symmetric idempotent matrices) $P = \frac{1}{2}(\mathcal{I}_n + \mathcal{J}_n)$, $Q = \frac{1}{2}(\mathcal{I}_n - \mathcal{J}_n)$; clearly $P^T = P, Q^T = Q$ and $P + Q = \mathcal{I}_n$. Moreover, $\mathcal{J}_n P = P = P\mathcal{J}_n$ and $\mathcal{J}_n Q = -Q = Q\mathcal{J}_n$. Also, $P1_n = 1_n$ and $Q1_n = 0_n$. Using these properties, we deduce

$$\begin{aligned} M + \mathcal{J}_n M \mathcal{J}_n &= (P+Q)^T (M + \mathcal{J}_n M \mathcal{J}_n)(P+Q) \\ &= PMP + P\mathcal{J}_n M \mathcal{J}_n P + PMQ + P\mathcal{J}_n M \mathcal{J}_n Q \\ &\quad + QMP + Q\mathcal{J}_n M \mathcal{J}_n P + QMQ + Q\mathcal{J}_n M \mathcal{J}_n Q \\ &= 2PMP + 2QMQ = 2PMP, \end{aligned}$$

observing in the last step that, by (31), $uQMQv = 0$ for all $u, v \in \mathbb{R}^n$, since $1_n^T Qu = (Q1_n)^T u = 0$ and similarly for v. By analogous reasoning, if $u \in \{1_n\}^\perp$, then also $Pu \in \{1_n\}^\perp$, so

$$(M + \mathcal{J}_n M \mathcal{J}_n)u = 2PMPu = PMPu + PM(\mathcal{J}_n P)u = P(M + M\mathcal{J}_n)Pu = \mathcal{O}_n$$

by Theorem 2.6 (d). Thus the dimension of the kernel of $M + \mathcal{J}M\mathcal{J}$ is at least $n-1$, so this matrix has at most rank 1. If it has rank 0, then $M \in A_n$ by Theorem 2.6 (b).

Assuming rank 1 in the following, we can rewrite $M + \mathcal{J}_n M \mathcal{J}_n = P(M + \mathcal{J}_n M)P$ as above; by Theorem 2.6 (d), the range of $M + \mathcal{J}_n M$ is $\mathbb{R}1_n$, which is invariant under the action of P. Thus, the range of the rank 1 matrix $M + \mathcal{J}_n M \mathcal{J}_n$ is $\mathbb{R}1_n$.

In summary, both the kernel and the range of the rank 1 matrix $M + \mathcal{J}_n M \mathcal{J}_n$ are equal to the kernel and range of $\mathcal{E}_n$, respectively. As a rank 1 matrix is determined up to a multiplicative constant by its kernel and range, it follows that $M + \mathcal{J}_n M \mathcal{J}_n = 2w\mathcal{E}_n$ for some $w \in \mathbb{R}$. □

The set V_n has, in addition to (V), the defining requirement that the sum of all matrix entries vanish. For a matrix with property (A) this is the case if and only if the weight w vanishes. Hence we can see that any reversible square matrix is a sum of an element of $R_n \cap V_n$ and a multiple of $\mathcal{E}_n$ (this decomposition being unique since $\mathcal{E}_n \notin V_n$).

Therefore it makes sense to focus on the space of *weightless reversible squares* $RV_n := R_n \cap V_n$. Its elements have the following block representation.

Theorem 7.2: *Let $M \in \mathbb{R}^{n\times n}$. Then M is an element of RV_n if and only if there are vectors $a, b \in \mathbb{R}^\nu$ such that*

$$M = \mathcal{X}_n \begin{pmatrix} \mathcal{O}_\nu & 1_\nu a^T \\ b1_\nu^T & \mathcal{O}_\nu \end{pmatrix} \mathcal{X}_n \tag{32}$$

if $n = 2\nu$ is even,

$$M = \mathcal{X}_n \begin{pmatrix} \mathcal{O}_\nu & 0_\nu & \sqrt{2}\,1_\nu a^T \\ 0_\nu^T & 0 & a^T \\ \sqrt{2}\,b1_\nu^T & b & \mathcal{O}_\nu \end{pmatrix} \mathcal{X}_n$$

if $n = 2\nu + 1$ is odd.

Proof: In the even-dimensional case, the result follows from comparison of the block representations in Theorems 5.1 and 4.2, noting that $\mathcal{E}_\nu \notin V_\nu$ and therefore $\gamma = 0$ in (25). Similarly, in the odd-dimensional case, comparison of (26) with (17) shows that

$v = y = \gamma 1_\nu$, and hence, in the central matrix entry, $\frac{4\nu}{2\nu-1}\gamma = \gamma$, which forces $\gamma = 0$ (as does the identity of the top left $\nu \times \nu$ blocks). □

Remark: From the formulae in Theorem 7.2, the block representation of a general reversible square can be obtained by adding a suitable multiple of the block representation of $\mathcal{E}_n$; this gives

$$M = \mathcal{X}_n \begin{pmatrix} 2w\mathcal{E}_\nu & 1_\nu a^T \\ b1_\nu^T & \mathcal{O}_\nu \end{pmatrix} \mathcal{X}_n$$

if $n = 2\nu$ is even,

$$M = \mathcal{X}_n \begin{pmatrix} 2w\mathcal{E}_\nu & \sqrt{2}w\,1_\nu & \sqrt{2}\,1_\nu a^T \\ \sqrt{2}w\,1_\nu^T & w & a^T \\ \sqrt{2}\,b1_\nu^T & b & \mathcal{O}_\nu \end{pmatrix} \mathcal{X}_n$$

if $n = 2\nu + 1$ is odd.

Comparison of the block structures in Theorem 7.2 and Lemma 3.1 shows that matrices in RV_n are elements of A_n with rank 1 blocks with constant rows and columns, respectively. This gives the following rank estimate.

Corollary 7.3: *If $M \in RV_n$, $n \in \mathbb{N}$, then* rank $M \le 2$.

The block representation of Theorem 7.2 reveals a further connection of the space RV_n with the space $AS_n := A_n \cap S_n$ of (weightless) associated constant sum matrices.

Corollary 7.4: *Let $n \in \mathbb{N}$. Then $RV_n = AV_n$. Consequently, $A_n = RV_n \oplus AS_n$.*

***Proof*:** From the block representations in Lemma 3.1 and Theorem 4.2, we see that matrices in AV_n have the same block representation as those in RV_n. The second statement follows from $S_n \oplus V_n = \mathbb{R}^{n\times n}$. □

Corollary 7.4 shows that reversible square matrices can equivalently be defined as square matrices with the properties (A) and (V); as $\mathcal{E}_n$ has both of these properties, this includes general weighted reversible square matrices.

The space RV_n does not form a matrix algebra by itself; however, as we know that $S_n \oplus V_n$ is a $\mathbb{Z}_2$-graded algebra by Theorem 4.7, and R_n is an algebra by Theorem 5.2, it follows by Lemma 2.4 (a) that $R_n = RS_n \oplus RV_n$ (where $RS_n := R_n \cap S_n$) is a $\mathbb{Z}_2$-graded algebra.

From Theorems 4.1 and 5.1 it is apparent that the elements of RS_n have the block representation

$$M = \mathcal{X}_n \begin{pmatrix} \gamma\mathcal{E}_\nu & \mathcal{O}_\nu \\ \mathcal{O}_\nu & Z \end{pmatrix} \mathcal{X}_n$$

in the even-dimensional case and

$$M = \mathcal{X}_n \begin{pmatrix} 2\gamma\mathcal{E}_\nu & \sqrt{2}\,\gamma 1_\nu & \mathcal{O}_\nu \\ \sqrt{2}\,\gamma 1_\nu^T & \gamma & 0_\nu^T \\ \mathcal{O}_\nu & 0_\nu & Z \end{pmatrix} \mathcal{X}_n$$

in the odd-dimensional case; equivalently, they are the sum of a multiple of $\mathcal{E}_n$ and of a matrix of form

$$\mathcal{X}_n \begin{pmatrix} \mathcal{O}_{n-\nu} & \mathcal{O} \\ \mathcal{O} & Z \end{pmatrix} \mathcal{X}_n \in B_n \cap S_n$$

with arbitrary $Z \in \mathbb{R}^{\nu\times\nu}$.

While this shows that RS_n is only a proper subspace of $BS_n := B_n \cap S_n$, we note, applying Lemma 2.4 (b) to $B_n \oplus A_n$ and $S_n \oplus V_n$, that $BS_n \oplus RV_n$ is another $\mathbb{Z}_2$-graded algebra; in particular, the product of a weightless reversible square and a balanced semimagic square matrix is a weightless reversible square matrix. The latter statement clearly extends to general, weighted reversible square matrices.

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