ROSENBLATT DISTRIBUTION SUBORDINATED TO GAUSSIAN RANDOM FIELDS WITH LONG-RANGE DEPENDENCE

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Abstract

The Karhunen-Loève expansion and the Fredholm determinant formula are used, to derive an asymptotic Rosenblatt-type distribution of a sequence of integrals of quadratic functions of Gaussian stationary random fields on $\mathbb{R}^d$ displaying long-range dependence. This distribution reduces to the usual Rosenblatt distribution when $d = 1$. Several properties of this new distribution are obtained. Specifically, its series representation, in terms of independent chi-squared random variables, is established. Its Lévy-Khintchine representation, and membership to the Thorin subclass of self-decomposable distributions are obtained as well. The existence and boundedness of its probability density then follow as a direct consequence.

Keywords: Asymptotics of eigenvalues, Fredholm determinant, Hermite polynomials, infinite divisible distributions, multiple Wiener-Itô stochastic integrals, non-central limit theorems, Rosenblatt-type distribution.

1 Introduction

The aim of this paper is to derive and study the properties of the limit distribution, as $T \longrightarrow \infty$, of the random integral

$$S_T = \frac{1}{d_T} \int_{D(T)} (Y^2(\mathbf{x}) - 1) d\mathbf{x}, \tag{1}$$

where the normalizing function d_T is given by

$$d_T = T^{d-\alpha} \mathcal{L}(T), \quad 0 < \alpha < d/2, \tag{2}$$

with $\mathcal{L}$ being a positive slowly varying function at infinity, that is

$$\lim_{T \to \infty} \mathcal{L}(T\|\mathbf{x}\|)/\mathcal{L}(T) = 1, \tag{3}$$

for every $\|\mathbf{x}\| > 0$, and $D(T) \subset \mathbb{R}^d$ denotes a homothetic transformation of a set $D \subset \mathbb{R}^d$, with center at the point $\mathbf{0} \in D$, and coefficient or scale factor $T > 0$. In the subsequent development, D is assumed to be a regular compact domain, whose interior has positive Lebesgue measure, and with boundary having null Lebesgue measure. Here, $\{Y(\mathbf{x}),\ \mathbf{x} \in \mathbb{R}^d\}$ is a zero-mean Gaussian homogeneous and isotropic random field with values in $\mathbb{R}$, displaying long-range dependence. That is, Y is assumed to satisfy the following condition:

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Condition A1. The random field $\{Y(\mathbf{x}),\ \mathbf{x} \in \mathbb{R}^d\}$ is a measurable zero-mean Gaussian homogeneous and isotropic mean-square continuous random field on a probability space $(\Omega, \mathcal{A}, P)$, with $\mathrm{E}Y^2(\mathbf{x}) = 1$, for all $\mathbf{x} \in \mathbb{R}^d$, and correlation function $\mathrm{E}[Y(\mathbf{x})Y(\mathbf{y})] = B(\|\mathbf{x} - \mathbf{y}\|)$ of the form:

$$B(\|\mathbf{z}\|) = \frac{\mathcal{L}(\|\mathbf{z}\|)}{\|\mathbf{z}\|^{\alpha}}, \quad \mathbf{z} \in \mathbb{R}^d, \quad 0 < \alpha < d/2. \tag{4}$$

From **Condition A1**, the correlation B of Y is a continuous function of $r = \|\mathbf{z}\|$. It then follows that $\mathcal{L}(r) = \mathcal{O}(r^{\alpha})$, $r \longrightarrow 0$. Note that the covariance function

$$B(\|\mathbf{z}\|) = \frac{1}{(1 + \|\mathbf{z}\|^{\beta})^{\gamma}}, \quad 0 < \beta \leq 2, \quad \gamma > 0, \tag{5}$$

is a particular case of the family of covariance functions (4) studied here with $\alpha = \beta\gamma$, and

$$\mathcal{L}(\|\mathbf{z}\|) = \|\mathbf{z}\|^{\beta\gamma}/(1 + \|\mathbf{z}\|^{\beta})^{\gamma}. \tag{6}$$

The limit random variable of (1) will be denoted as S_∞. The distribution of S_∞ will be referred to as the *Rosenblatt-type* distribution, or sometimes simply as the *Rosenblatt* distribution because this is how it is known in the case $d = 1$. In that case, a discretized version in time of the integral (1) first appears in the paper by Rosenblatt (1961), and the limit functional version is considered in Taqqu (1975) in the form of the Rosenblatt process. In this classical setting, the limit of (1) is represented by a double Wiener-Itô stochastic integral (see Dobrushin and Major, 1979; Taqqu, 1979). Other relevant references include, for example, Albin (1998), Anh, Leonenko and Olenko (2015), Fox and Taqqu (1985), Ivanov and Leonenko (1989), Leonenko and Taufer (2006), Rosenblatt (1979), to mention just a few. The general approach considered here for deriving the weak-convergence to the Rosenblatt distribution is inspired by the paper of Taqqu (1975), which is based on the convergence of characteristic functions. This approach has also been used, recently, in the paper by Leonenko and Taufer (2006), to study the characteristic functions of quadratic forms of strongly-correlated Gaussian random variables sequences.

We suppose here $d \geq 2$, and thus consider integrals of quadratic functions of long-range dependence zero-mean Gaussian stationary random fields. We pursue, however, a different methodology than in the case $d = 1$, which was based on the discretization of the parameter space. A direct extension of these techniques is not available when $d \geq 2$. Instead of discretizing the parameter space of the random field, we focus on the characteristic function for quadratic forms for Hilbert-valued Gaussian random variables (see, for example, Da Prato and Zabczyk, 2002), and take advantage of functional analytical tools, like the Karhunen-Loève expansion and the Fredholm determinant formula, to obtain the convergence in distribution to a limit random variable S_∞ with Rosenblatt-type distribution.

The double Wiener-Itô stochastic integral representation of S_∞ in the spectral domain leads to its series expansion in terms of independent chi-squared random variables, weighted by the eigenvalues of the integral operator introduced in equation (22) below. The asymptotics of these eigenvalues is given in Corollary 4.2. The infinitely divisible property of S_∞ is then obtained as a direct consequence of the previous results derived, in relation to the series expansion of S_∞, and the asymptotic properties of the eigenvalues. We also prove that the distribution of S_∞ is self-decomposable, and that it belongs, in particular, to the Thorin subclass. The existence and boundedness of the probability density of S_∞ then follows.

The outline of the paper is now described. In Section 2, we recall the Karhunen-Loève expansion, introduce the Fredholm determinant formula, and use the referred tools to obtain the characteristic function of (1). In Section 3, we prove the weak convergence of (1) to the random variable S_∞ with a Rosenblatt-type distribution. The double Wiener-Itô stochastic integral representation of S_∞, its series expansion in terms of independent chi-square random variables, and the asymptotics of the involved eigenvalues are established in Section 4. These results are applied in Section 5 to derive some properties of the Rosenblatt distribution, e.g., infinitely divisible property, self-decomposability, and, in particular, the membership to the Thorin subclass. Appendices A-C provide some auxiliary results and the proofs of some propositions and corollaries.

In this paper we consider the case of real-valued random fields. In what follows we use the symbols C, C_0, M_1, M_2, etc., to denote constants. The same symbol may be used for different constants appearing in the text.

2 Karhunen-Loéve expansion and related results

This section introduces some preliminary definitions, assumptions and lemmas hereafter used in the derivation of the main results of this paper. We start with the Karhunen-Loève Theorem for a zero-mean second-order random field $\{Y(\mathbf{x}),\ \mathbf{x} \in K \subset \mathbb{R}^d\}$, with continuous covariance function $B_0(\mathbf{x}, \mathbf{y}) = \mathrm{E}[Y(\mathbf{x})Y(\mathbf{y})],\ (\mathbf{x}, \mathbf{y}) \in K \times K \subset \mathbb{R}^d \times \mathbb{R}^d$, defined on a compact set K of $\mathbb{R}^d$ (see Adler and Taylor, 2007, Section 3.2). This theorem provides the following orthogonal expansion of the random field Y :

$$\begin{aligned} Y(\mathbf{x}) &= \sum_{j=1}^{\infty} \sqrt{\lambda_j}\phi_j(\mathbf{x})\eta_j, \quad \mathbf{x} \in K, \\ \lambda_k\phi_k(\mathbf{x}) &= \int_K B_0(\mathbf{x}, \mathbf{y})\phi_k(\mathbf{y})d\mathbf{y}, \quad k \in \mathbb{N}_*, \quad \langle \phi_i, \phi_j \rangle_{L^2(K)} = \delta_{i,j}, \quad i, j \in \mathbb{N}_*, \end{aligned} \tag{7}$$

where $\eta_k = \frac{1}{\sqrt{\lambda_k}} \int_K Y(\mathbf{x})\phi_k(\mathbf{x})d\mathbf{x}$, for each $k \geq 1$, and the convergence holds in the $L^2(\Omega, \mathcal{A}, P)$ sense. The eigenvalues of B_0 are considered to be arranged in decreasing order of magnitude, that is, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{k-1} \geq \lambda_k \geq \dots$. The orthonormality of the eigenfunctions $\phi_j,\ j \in \mathbb{N}_*$, leads to the uncorrelation of the random variables $\eta_j,\ j \in \mathbb{N}_*$, with variance one, since

$$E[\eta_j\eta_k] = \int_K \int_K B_0(\mathbf{x}, \mathbf{y})\phi_j(\mathbf{y})\phi_k(\mathbf{x})d\mathbf{y}d\mathbf{x} = \lambda_j \int_K \phi_j(\mathbf{x})\phi_k(\mathbf{x})d\mathbf{x} = \lambda_j\delta_{j,k},$$

with δ denoting the Kronecker delta function. In the Gaussian case, they are independent.

For each $T > 0$, let us fix some notation related to the Karhunen-Loève expansion of the restriction to the set $D(T)$ of Gaussian random field Y, with covariance function (4). By $R_{Y,D(T)}$ we denote the covariance operator of Y with covariance kernel $B_{0,T}(\mathbf{x}, \mathbf{y}) = \mathrm{E}[Y(\mathbf{x})Y(\mathbf{y})], \mathbf{x}, \mathbf{y} \in D(T)$, which, as an operator from $L^2(D(T))$ onto $L^2(D(T))$, satisfies

$$R_{Y,D(T)}(\phi_{l,T})(\mathbf{x}) = \int_{D(T)} B_{0,T}(\mathbf{x}, \mathbf{y})\phi_{l,T}(\mathbf{y})d\mathbf{y} = \lambda_{l,T}(R_{Y,D(T)})\phi_{l,T}(\mathbf{x}), \quad l \in \mathbb{N}_*,$$

where, in the following, by $\lambda_k(A)$ we will denote the kth eigenvalue of the operator A. In particular, $\{\lambda_{k,T}(R_{Y,D(T)})\}_{k=1}^{\infty}$ and $\{\phi_{k,T}\}_{k=1}^{\infty}$ respectively denote the eigenvalues and eigenfunctions of $R_{Y,D(T)}$, for each $T > 0$. Note that, as commented, $B_{0,T}$ refers to the covariance function of $\{Y(\mathbf{x}),\ \mathbf{x} \in D(T)\}$ as a function of $(\mathbf{x}, \mathbf{y}) \in D(T) \times D(T)$, which, under **Condition A1**, defines a non-negative, symmetric and continuous kernel on the compact set $D(T)$, satisfying the conditions assumed in Mercer's Theorem. Hence, the Karhunen-Loève expansion of random field Y holds on $D(T)$, and its covariance kernel $B_{0,T}$ also admits the series representation

$$B_{0,T}(\mathbf{x}, \mathbf{y}) = \sum_{j=1}^{\infty} \lambda_{j,T}(R_{Y,D(T)})\phi_{j,T}(\mathbf{x})\phi_{j,T}(\mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in D(T), \tag{8}$$

where the convergence is absolute and uniform (see, for example, Adler and Taylor, 2007, pp.70-74). The orthonormality of the eigenfunctions $\{\phi_{l,T}\}_{l=1}^{\infty}$ yields

$$\frac{1}{d_T}\int_{D(T)} Y^2(\mathbf{x})d\mathbf{x} = \frac{1}{d_T}\sum_{j=1}^{\infty}\lambda_{j,T}(R_{Y,D(T)})\eta_{j,T}^2. \tag{9}$$

In the derivation of the limit characteristic function of (1), we will use the Fredholm determinant formula of a trace operator. Recall first that a positive operator A on a separable Hilbert space H is a trace operator if

$$\|A\|_1 \equiv \mathrm{Tr}(A) \equiv \sum_k \left\langle (A^*A)^{1/2}\varphi_k, \varphi_k\right\rangle_H < \infty, \tag{10}$$

where A^* denotes the adjoint of A and $\{\varphi_k\}$ is an orthonormal basis of the Hilbert space H (see Reed and Simon, 1980, pp. 207-209). A sufficient condition for a compact and self-adjoint operator A to belong to the trace class is $\sum_{k=1}^{\infty}\lambda_k(A) < \infty$. For each finite $T > 0$, the operator $R_{Y,D(T)}$ is in the trace class, since from equation (8), applying the orthonormality of the eigenfunction system $\{\phi_{j,T},\ j \in \mathbb{N}_*\}$, and keeping in mind that $B_{0,T}(\mathbf{0}) = 1$, we have

$$Tr(R_{Y,D(T)}) = \sum_{j=1}^{\infty}\lambda_{j,T}(R_{Y,D(T)}) = \int_{D(T)} B_{0,T}(\mathbf{x},\mathbf{x})d\mathbf{x} = \int_{D(T)} d\mathbf{x} = T^d|D| < \infty, \tag{11}$$

where $|D|$ denotes the Lebesgue measure of the compact set D. Note that the class of compact and self-adjoint operators contains the class of trace and self-adjoint operators. Hence, under **Condition A1**, from equation (11), the restriction of Y to $D(T)$ admits a Karhunen-Loéve expansion, convergent in the mean-square sense (i.e., in the $L^2(\Omega,\mathcal{A},P)$-sense), for any $T > 0$, and for an arbitrary regular bounded domain D. Furthermore, for any $k \geq 1$,

$$R^k_{Y,D(T)}f(\mathbf{x}) = \int_{D(T)} B^{*(k)}_{0,T}(\mathbf{x},\mathbf{y})f(\mathbf{y})d\mathbf{y}, \quad f \in L^2(D(T)), \tag{12}$$

where $B^{*(k)}_{0,T}$ denotes

$$B^{*(1)}_{0,T}(\mathbf{x},\mathbf{y}) = B_{0,T}(\mathbf{x},\mathbf{y}), \quad k = 1,$$

$$B^{*(k)}_{0,T}(\mathbf{x},\mathbf{y}) = \int_{D(T)} B^{*(k-1)}_{0,T}(\mathbf{x},\mathbf{z})B_{0,T}(\mathbf{z},\mathbf{y})d\mathbf{z}, \quad k = 2,3,\ldots. \tag{13}$$

From equations (8) and (13), applying the orthonormality of $\phi_{j,T}$, $j \in \mathbb{N}_*$, one can obtain

$$Tr(R^k_{Y,D(T)}) = \sum_{j=1}^{\infty}\lambda^k_{j,T}(R_{Y,D(T)}) = \int_{D(T)} B^{*(k)}_{0,T}(\mathbf{x},\mathbf{x})d\mathbf{x} < \infty, \quad k \in \mathbb{N}_*, \tag{14}$$

since, for every $k \geq 1$, $|\lambda_k(R_{Y,D(T)})| \leq M|\lambda_k(R_{Y,D(T)})|^k = M|\lambda_k(R^k_{Y,D(T)})|$, for some positive constant M. In particular, in the homogeneous random field case,

$$\begin{aligned} Tr(R^k_{Y,D(T)}) &= \sum_{j=1}^{\infty}\lambda^k_{j,T}(R_{Y,D(T)}) = \int_{D(T)} B^{*(k)}_{0,T}(\mathbf{x}_k,\mathbf{x}_k)d\mathbf{x}_k \\ &= \int_{D(T)} \cdots \int_{D(T)} \left[\prod_{j=1}^{k-1} B_{0,T}(\mathbf{x}_{j+1}-\mathbf{x}_j)\right] B_{0,T}(\mathbf{x}_1-\mathbf{x}_k)d\mathbf{x}_1\ldots d\mathbf{x}_k, \end{aligned} \tag{15}$$

and, in the homogeneous and isotropic case, for $k = 2$,

$$Tr(R^2_{Y,D(T)}) = \sum_{j=1}^{\infty}\lambda^2_{j,T}(R_{Y,D(T)}) = \int_{D(T)}\int_{D(T)} \frac{\mathcal{L}^2(\|\mathbf{x}-\mathbf{y}\|)}{\|\mathbf{x}-\mathbf{y}\|^{2\alpha}}d\mathbf{y}d\mathbf{x}. \tag{16}$$

The following definition introduces the Fredholm determinant of an operator A, as a complex-valued function which generalizes the determinant of a matrix.

Definition 2.1. (see, for example, Simon, 2005, Chapter 5, pp.47-48, equation (5.12)) Let A be a trace operator on a separable Hilbert space H. The Fredholm determinant of A is

$$\mathcal{D}(\omega) = \det(I - \omega A) = \exp\left(-\sum_{k=1}^{\infty} \frac{\mathrm{Tr}A^k}{k}\omega^k\right) = \exp\left(-\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}[\lambda_l(A)]^k \frac{\omega^k}{k}\right), \quad (17)$$

for $\omega \in \mathbb{C}$, and $|\omega|\|A\|_1 < 1$. Note that $\|A^m\|_1 \leq \|A\|_1^m$, for A being a trace operator.

Lemma 2.1. *Let $\{Y(\mathbf{x}),\ \mathbf{x} \in D \subset \mathbb{R}^d\}$ be an integrable and continuous, in the mean-square sense, zero-mean, Gaussian random field, on a bounded regular domain $D \subseteq \mathbb{R}^d$ containing the point zero. Then, the following identity holds:*

$$\begin{aligned} \mathrm{E}\left[\exp\left(\mathrm{i}\xi \int_D Y^2(\mathbf{x})d\mathbf{x}\right)\right] &= \prod_{j=1}^{\infty}(1 - 2\lambda_j(R_{Y,D})\mathrm{i}\xi)^{-1/2} = (\mathcal{D}(2\mathrm{i}\xi))^{-1/2} \\ &= \exp\left(\frac{1}{2}\sum_{m=1}^{\infty}\frac{(2\mathrm{i}\xi)^m}{m} Tr(R_{Y,D}^m)\right), \end{aligned} \quad (18)$$

for $\|R_{Y,D}\|_1|2\mathrm{i}\xi| < 1$, as given in Definition 2.1.

Proof. The covariance operator $R_{Y,D}$ of Y, acting on the space $L^2(D)$, is in the trace class. From Definition 2.1, the following identities hold:

$$\begin{aligned} &E\left[\exp\left(\mathrm{i}\xi \int_D Y^2(\mathbf{x})d\mathbf{x}\right)\right] = E\left[\exp\left(\mathrm{i}\xi \sum_{j=1}^{\infty}\lambda_j(R_{Y,D})\eta_j^2\right)\right] \\ &= \prod_{j=1}^{\infty} E\left[\exp\left(\mathrm{i}\xi\lambda_j(R_{Y,D})\eta_j^2\right)\right] = \prod_{j=1}^{\infty}(1 - 2\lambda_j(R_{Y,D})\mathrm{i}\xi)^{-1/2} = (\mathcal{D}(2\mathrm{i}\xi))^{-1/2} \\ &= \left[\exp\left(-\sum_{m=1}^{\infty}\frac{(2\mathrm{i}\xi)^m}{m}\mathrm{Tr}(R_{Y,D}^m)\right)\right]^{-1/2} = \exp\left(\frac{1}{2}\sum_{m=1}^{\infty}\frac{(2\mathrm{i}\xi)^m}{m}\mathrm{Tr}(R_{Y,D}^m)\right), \end{aligned} \quad (19)$$

where the last two identities in equation (19) are finite for $|\xi| < \frac{1}{2|D|}$, from the Fredholm determinant formula (17). Note that

$$\mathrm{Tr}(R_{Y,\mathcal{D}}^m) = \sum_{j=1}^{\infty}\lambda_j^m(R_{Y,D}) \leq \lambda_1^{m-1}(R_{Y,D})\sum_{j=1}^{\infty}\lambda_j(R_{Y,D}) = \lambda_1^{m-1}(R_{Y,D})\|R_{Y,\mathcal{D}}\|_1 < \infty. \quad (20)$$

■

Remark 2.1. Similarly to equation (18), one can obtain the following identities, which will be used in the subsequent development: For a homothetic transformation $D(T)$ of $D \subset \mathbb{R}^d$, with center at the point $\mathbf{0} \in D$, and coefficient $T > 0$,

$$\begin{aligned} E\left[\exp\left(\mathrm{i}\xi \int_{D(T)} Y^2(\mathbf{x})d\mathbf{x}\right)\right] &= \prod_{j=1}^{\infty}(1 - 2\lambda_{j,T}(R_{Y,D(T)})\mathrm{i}\xi)^{-1/2} = (\mathcal{D}_T(2\mathrm{i}\xi))^{-1/2} \\ &= \exp\left(\frac{1}{2}\sum_{m=1}^{\infty}\frac{(2\mathrm{i}\xi)^m}{m}\mathrm{Tr}(R_{Y,D(T)}^m)\right), \end{aligned} \quad (21)$$

where $\lambda_{1,T}(R_{Y,D(T)}) \geq \lambda_{2,T}(R_{Y,D(T)}) \geq \cdots \geq \lambda_{j,T}(R_{Y,D(T)}) \geq \ldots$, with, as before, $\{\lambda_{j,T}(R_{Y,D(T)}),\ j \in \mathbb{N}_*\}$ denoting the system of eigenvalues of the covariance operator $R_{Y,D(T)}$ of Y, as an operator from $L^2(D(T))$ onto $L^2(D(T))$. The last identity in equation (21) holds for $\|R_{Y,D(T)}\|_1|2\mathrm{i}\xi| < 1$, i.e., for $\mathrm{Tr}(R_{Y,D(T)})|2\mathrm{i}\xi| = T^d|D||2\mathrm{i}\xi| < 1$, or equivalently for $|\xi| < \frac{1}{2T^d|D|}$.

3 Weak convergence of the random integral S_T

This section provides the weak convergence of the random integral (1) to a Rosenblatt-type distribution, in Theorem 3.2. This results is based on the asymptotic behavior of the eigenvalues of the integral operator $\mathcal{K}_\alpha$ (see Theorem 3.1 below)

$$\mathcal{K}_\alpha(f)(\mathbf{x}) = \int_D \frac{1}{\|\mathbf{x}-\mathbf{y}\|^\alpha} f(\mathbf{y})d\mathbf{y}, \quad \forall f \in \mathrm{Supp}(\mathcal{K}_\alpha), \quad 0<\alpha<d, \tag{22}$$

with Supp(A) denoting the support of operator A. Operator (22) can be related with the Riesz potential $(-\Delta)^{-\beta/2}$ of order β, $0<\beta<d$, on $\mathbb{R}^d$, formally defined as (see Stein, 1970, p.117)

$$(-\Delta)^{-\beta/2}(f)(\mathbf{x}) = \frac{1}{\gamma(\beta)} \int_{\mathbb{R}^d} \|\mathbf{x}-\mathbf{y}\|^{-d+\beta} f(\mathbf{y})d\mathbf{y}, \tag{23}$$

where $(-\Delta)$ denotes the negative Laplacian operator, and

$$\gamma(\beta) = \frac{\pi^{d/2}2^\beta\Gamma(\beta/2)}{\Gamma\left(\frac{d-\beta}{2}\right)} = \frac{1}{c(d,\beta)}, \quad 0<\beta<d. \tag{24}$$

Indeed, except a constant, the function $(1/\|\mathbf{x}-\mathbf{y}\|^\alpha)$ in equation (22) defines the kernel of the Riesz potential $(-\Delta)^{(\alpha-d)/2}$ of order $\beta=(d-\alpha)$, for $0<\alpha<d$. Similarly, $\left(1/\|\mathbf{x}-\mathbf{y}\|^{2\alpha}\right)$ is the kernel of the Riesz potential $(-\Delta)^{\alpha-d/2}$ of order $\beta=(d-2\alpha)$ on $\mathbb{R}^d$, for $0<\alpha<d/2$.

Recall that the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ is the space of of infinitely differentiable functions on $\mathbb{R}^d$, whose derivatives remain bounded when multiplied by polynomials, i.e., whose derivatives are rapidly decreasing. Particularly, $C_0^\infty(D) \subset \mathcal{S}(\mathbb{R}^d)$, with $C_0^\infty(D)$ denoting the infinitely differentiable functions with compact support contained in D.

The Fourier transform of the Riesz potential is understood in the weak sense, considering the space $\mathcal{S}(\mathbb{R}^d)$. The following lemma provides such a transform (see Lemma 1 of Stein, 1970, p.117):

Lemma 3.1. *Let us consider $0<\beta<d$.*

(i) The Fourier transform of the function $\|\mathbf{z}\|^{-d+\beta}$ is $\gamma(\beta)\|\mathbf{z}\|^{-\beta}$, in the sense that

$$\int_{\mathbb{R}^d} \|\mathbf{z}\|^{-d+\beta}\overline{\psi(\mathbf{z})}d\mathbf{z} = \int_{\mathbb{R}^d} \gamma(\beta)\|\mathbf{z}\|^{-\beta}\overline{\mathcal{F}(\psi)(\mathbf{z})}d\mathbf{z}, \quad \forall \psi \in \mathcal{S}(\mathbb{R}^d), \tag{25}$$

where

$$\mathcal{F}(\psi)(\mathbf{z}) = \int_{\mathbb{R}^d} \exp\left(-\mathrm{i}\left\langle \mathbf{x},\mathbf{z}\right\rangle\right)\psi(\mathbf{x})d\mathbf{x}$$

denotes the Fourier transform of ψ.

(ii) The identity $\mathcal{F}\left((-\Delta)^{-\beta/2}(f)\right)(\mathbf{z}) = \|\mathbf{z}\|^{-\beta}\mathcal{F}(f)(\mathbf{z})$ holds in the sense that

$$\int_{\mathbb{R}^d} (-\Delta)^{-\beta/2}(f)(\mathbf{x})\overline{g(\mathbf{x})}d\mathbf{x} = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \mathcal{F}(f)(\mathbf{x})\|\mathbf{x}\|^{-\beta}\overline{\mathcal{F}(g)(\mathbf{x})}d\mathbf{x}, \quad \forall f,g \in \mathcal{S}(\mathbb{R}^d). \tag{26}$$

In particular, the following convolution formula is obtained by iteration of (26) using (23):

$$\begin{aligned}
&\int_{\mathbb{R}^d}\left(\frac{1}{\gamma(\beta)}\int_{\mathbb{R}^d}\|\mathbf{x}-\mathbf{y}\|^{-d+\beta}\left[\frac{1}{\gamma(\beta)}\int_{\mathbb{R}^d}\|\mathbf{y}-\mathbf{z}\|^{-d+\beta}f(\mathbf{z})d\mathbf{z}\right]d\mathbf{y}\right)\overline{g(\mathbf{x})}d\mathbf{x} \\
&= \int_{\mathbb{R}^d}(-\Delta)^{-\beta/2}\left[(-\Delta)^{-\beta/2}(f)\right](\mathbf{x})\ \overline{g(\mathbf{x})}d\mathbf{x} \\
&= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\left[\mathcal{F}((-\Delta)^{-\beta/2}(f))(\mathbf{x})\right]\|\mathbf{x}\|^{-\beta}\overline{\mathcal{F}(g)(\mathbf{x})}d\mathbf{x}
\end{aligned}$$

$$= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathcal{F}(f)(\mathbf{x}) \|\mathbf{x}\|^{-\beta} \|\mathbf{x}\|^{-\beta} \overline{\mathcal{F}(g)(\mathbf{x})} d\mathbf{x}$$
$$= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathcal{F}(f)(\mathbf{x}) \|\mathbf{x}\|^{-2\beta} \overline{\mathcal{F}(g)(\mathbf{x})} d\mathbf{x}$$
$$= \int_{\mathbb{R}^d} (-\Delta)^{-\beta}(f)(\mathbf{x}) \overline{g(\mathbf{x})} d\mathbf{x}, \quad \forall f, g \in \mathcal{S}(\mathbb{R}^d), \quad 0 < \beta < d/2, \tag{27}$$

where we have used that if $f \in \mathcal{S}(\mathbb{R}^d)$, then $(-\Delta)^{-\beta/2}(f) \in \mathcal{S}(\mathbb{R}^d)$. From equation (27), and Lemma 3.1(i),

$$\int_{\mathbb{R}^d} \frac{1}{\gamma(2\beta)} \|\mathbf{z}\|^{-d+2\beta} \overline{f(\mathbf{z})} d\mathbf{z} = \int_{\mathbb{R}^d} \|\mathbf{z}\|^{-2\beta} \overline{\mathcal{F}(f)(\mathbf{z})} d\mathbf{z}$$
$$= \int_{\mathbb{R}^d} \frac{1}{[\gamma(\beta)]^2} \left[\int_{\mathbb{R}^d} \|\mathbf{z} - \mathbf{y}\|^{-d+\beta} \|\mathbf{y}\|^{-d+\beta} d\mathbf{y} \right] \overline{f(\mathbf{z})} d\mathbf{z}, \quad \forall f \in \mathcal{S}(\mathbb{R}^d), \quad 0 < \beta < d/2. \tag{28}$$

Let us now consider on the space of infinitely differentiable functions with compact support contained in D, $C_0^\infty(D) \subset \mathcal{S}(\mathbb{R}^d)$, the norm

$$\|f\|^2_{(-\Delta)^{\alpha-d/2}} = \left\langle (-\Delta)^{\alpha-d/2}(f), f \right\rangle_{L^2(\mathbb{R}^d)} = \left\langle (-\Delta)^{\alpha-d/2}(f), f \right\rangle_{L^2(D)}$$
$$= \int_{\mathbb{R}^d} (-\Delta)^{\alpha-d/2}(f)(\mathbf{x}) \overline{f(\mathbf{x})} d\mathbf{x} = \int_{\mathbb{R}^d} \frac{1}{\gamma(d-2\alpha)} \int_{\mathbb{R}^d} \frac{1}{\|\mathbf{x}-\mathbf{y}\|^{2\alpha}} f(\mathbf{y}) \overline{f(\mathbf{x})} d\mathbf{y} d\mathbf{x}$$
$$= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\mathcal{F}(f)(\boldsymbol{\lambda})|^2 \|\boldsymbol{\lambda}\|^{-(d-2\alpha)} d\boldsymbol{\lambda}, \quad \forall f \in C_0^\infty(D), \quad 0 < \alpha < d/2. \tag{29}$$

The associated inner product is given by

$$\begin{aligned} \langle f, g \rangle_{(-\Delta)^{\alpha-d/2}} &= \int_{\mathbb{R}^d} \frac{1}{\gamma(d-2\alpha)} \int_{\mathbb{R}^d} \frac{1}{\|\mathbf{x}-\mathbf{y}\|^{2\alpha}} f(\mathbf{y}) \overline{g(\mathbf{x})} d\mathbf{y} d\mathbf{x} \\ &= \int_{D} \frac{1}{\gamma(d-2\alpha)} \int_{D} \frac{1}{\|\mathbf{x}-\mathbf{y}\|^{2\alpha}} f(\mathbf{y}) \overline{g(\mathbf{x})} d\mathbf{y} d\mathbf{x}, \end{aligned} \tag{30}$$

for all $f, g \in C_0^\infty(D)$. The closure of $C_0^\infty(D)$ with the norm $\|\cdot\|_{(-\Delta)^{\alpha-d/2}}$, introduced in (29), defines a Hilbert space, which will be denoted as $\mathcal{H}_{2\alpha-d} = \overline{C_0^\infty(D)}^{\|\cdot\|_{(-\Delta)^{\alpha-d/2}}}$.

Remark 3.1. For a bounded open domain D, from Proposition 2.2. in Caetano (2000), with $D = n-1$, $p = q = 2$, and $s = 0$ (hence, $A^s_{pq}(D) = A^0_{22}(D) = L^2(D)$, where, as usual, $L^2(D)$ denotes the space of square integrable functions on D), we have

$$\overline{C_0^\infty(D)}^{\|\cdot\|_{L^2(\mathbb{R}^d)}} = L^2(D), \tag{31}$$

(see also Triebel, 1978, for the case of regular bounded open domains with $C^\infty-$boundaries). In addition, for all $f \in C_0^\infty(D)$, by definition of the norm (29),

$$\|f\|_{(-\Delta)^{\alpha-d/2}} \leq C\|f\|_{L^2(\mathbb{R}^d)},$$

that is, all convergent sequences of $C_0^\infty(D)$ in the $L^2(\mathbb{R}^d)$ norm are also convergent in the $\mathcal{H}_{2\alpha-d}$ norm. Hence, the closure of $C_0^\infty(D)$, with respect to the norm $\|\cdot\|_{L^2(\mathbb{R}^d)}$, is included in the closure of $C_0^\infty(D)$, with respect to the norm $\|\cdot\|_{(-\Delta)^{\alpha-d/2}}$. Therefore, from equation (31),

$$L^2(D) = \overline{C_0^\infty(D)}^{\|\cdot\|_{L^2(\mathbb{R}^d)}} \subseteq \overline{C_0^\infty(D)}^{\|\cdot\|_{(-\Delta)^{\alpha-d/2}}} = \mathcal{H}_{2\alpha-d}. \tag{32}$$

The asymptotic order of the eigenvalues of operator $\mathcal{K}_\alpha$, in the case $d \geq 2$, are given in the next result, for a suitable regular bounded open domain D (see, for example, Triebel and Yang, 2001, Widom, 1963, and Zhale, 2004, p.197). (See also Dostanic, 1998, and Veillette and Taqqu, 2013, for the case $d = 1$).

Theorem 3.1. *Let us consider the integral operator $\mathcal{K}_\alpha$ introduced in equation (22) as an operator on the space $L^2(D)$, with D denoting a bounded open domain. The following asymptotics is satisfied by the eigenvalues $\lambda_k(\mathcal{K}_\alpha)$, $k \geq 1$, of operator $\mathcal{K}_\alpha$:*

$$\lim_{k \longrightarrow \infty} \frac{\lambda_k(\mathcal{K}_\alpha)}{k^{-(d-\alpha)/d}} = \widetilde{c}(d,\alpha)|D|^{(d-\alpha)/d}, \tag{33}$$

where $|D|$ denotes, as before, the Lebesgue measure of domain D, and

$$\widetilde{c}(d,\alpha) = \pi^{\alpha/2}\left(\frac{2}{d}\right)^{(d-\alpha)/d} \frac{\Gamma\left(\frac{d-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)\left[\Gamma\left(\frac{d}{2}\right)\right]^{(d-\alpha)/d}}. \tag{34}$$

Proof.

We apply the results derived in Widom (1963), on the asymptotic behavior of the eigenvalues associated with certain class of integral equations. Specifically, the following integral equation is considered in that paper:

$$\int V^{1/2}(\mathbf{x})k(\mathbf{x}-\mathbf{y})V^{1/2}(\mathbf{y})f(\mathbf{y})d\mathbf{y} = \lambda f(\mathbf{x}), \tag{35}$$

where k is an integrable function over a Euclidean space E_d of dimension d, having positive Fourier transform, and where V is a bounded non-negative function with bounded support. In particular, Widom (1963) considers the case where $E_d = \mathbb{R}^d$, V is the indicator function of a bounded domain $D \subseteq \mathbb{R}^d$, and $k(\|\mathbf{x}-\mathbf{y}\|) = \|\mathbf{x}-\mathbf{y}\|^\alpha$, for $\alpha > -d$, and $\alpha \neq 0, 2, 4, \ldots$. Function k coincides in $\mathbb{R}^d \setminus D$ with a function whose Fourier transform $f(\boldsymbol{\xi})$ is asymptotically equal to

$$2^{d-\alpha}\pi^{d/2}\frac{\Gamma\left(\frac{d-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)}|\boldsymbol{\xi}|^{-d+\alpha}$$

(see also the right-hand side of equation (25) for $\beta = d - \alpha$, with $0 < \alpha < d$). For $\alpha > -d$, $\alpha \neq 0, 2, 4, \ldots$, equation (2) in Widom (1963) then leads to the following asymptotic of the eigenvalues for the associated integral operator on $L^2(D)$:

$$\lambda_k \sim \pi^{-\alpha/2}\left(\frac{2}{d}\right)^{\frac{d+\alpha}{d}} \frac{\Gamma\left(\frac{d+\alpha}{2}\right)}{\Gamma\left(\frac{-\alpha}{2}\right)\left[\Gamma\left(\frac{d}{2}\right)\right]^{(d+\alpha)/d}} \left[\int_{\mathbb{R}^d} [V(\mathbf{x})]^{d/(d+\alpha)}\, d\mathbf{x}\right]^{(d+\alpha)/d} k^{-(d+\alpha)/d}, \tag{36}$$

with

$$\int_{\mathbb{R}^d} [V(\mathbf{x})]^{d/(d-\alpha)}\, d\mathbf{x} = |D|.$$

The above-referred function k, studied in Widom (1963), coincides with the kernel of the integral operator $\mathcal{K}_\alpha$, appearing in equation (22), for $\alpha \in (-d, 0)$. Hence, from equation (36), the asymptotic of the eigenvalues of operator $\mathcal{K}_\alpha$ are given by

$$\lambda_k(\mathcal{K}_\alpha) \sim \pi^{\alpha/2}\left(\frac{2}{d}\right)^{\frac{d-\alpha}{d}} \frac{\Gamma\left(\frac{d-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)\left[\Gamma\left(\frac{d}{2}\right)\right]^{(d-\alpha)/d}} \left[\int_{\mathbb{R}^d} [V(\mathbf{x})]^{d/(d-\alpha)}\, d\mathbf{x}\right]^{(d-\alpha)/d} k^{-(d-\alpha)/d},$$

for $\alpha \in (0, d)$.

■

Remark 3.2. Similar results to those ones presented in Theorem 3.2 of Veillette and Taqqu (2013) can be derived for the spectral zeta function of the Dirichlet Laplacian on a bounded closed multidimensional interval of $\mathbb{R}^d$ (see also Dostanic, 1998, for the case of $d = 1$). For a continuous function of the negative Dirichlet Laplacian, the explicit computation of its trace cannot always be obtained in a general regular compact domain of $\mathbb{R}^d$. Specifically, the knowledge of the eigenvalues is guaranteed for highly symmetric regions like the the sphere, or regions bounded by parallel planes (see, for example, Müler, 1998; Park and Wojciechowski, 2002a; 2002b). In particular, for the torus $\mathbb{T}^2$ in $\mathbb{R}^2$, the Spectral Zeta Function can be explicitly computed (see, for example, Arendt and Schleich, 2009, Chapter 1, equation (1.49), pp. 28-29).

For the next result, Theorem 3.2, we suppose that the slowly varying function $\mathcal{L}$ satisfies the following condition.

Condition A2. For every $m \geq 2$ there exists a constant $C > 0$, such that

$$\int_D ..(m). \int_D \frac{\mathcal{L}(T\|\mathbf{x}_1 - \mathbf{x}_2\|)}{\mathcal{L}(T)\|\mathbf{x}_1 - \mathbf{x}_2\|^\alpha} \frac{\mathcal{L}(T\|\mathbf{x}_2 - \mathbf{x}_3\|)}{\mathcal{L}(T)\|\mathbf{x}_2 - \mathbf{x}_3\|^\alpha} \cdots \frac{\mathcal{L}(T\|\mathbf{x}_m - \mathbf{x}_1\|)}{\mathcal{L}(T)\|\mathbf{x}_m - \mathbf{x}_1\|^\alpha} d\mathbf{x}_1 d\mathbf{x}_2 \cdots d\mathbf{x}_m \leq$$

$$\leq C \int_D ...(m). \int_D \frac{d\mathbf{x}_1 d\mathbf{x}_2 \cdots d\mathbf{x}_m}{\|\mathbf{x}_1 - \mathbf{x}_2\|^\alpha \|\mathbf{x}_2 - \mathbf{x}_3\|^\alpha \cdots \|\mathbf{x}_m - \mathbf{x}_1\|^\alpha}. \tag{37}$$

Note that **Condition A2** is satisfied by slowly varying functions such that

$$\sup_{T, \mathbf{x}_1, \mathbf{x}_2 \in D} \frac{\mathcal{L}(T\|\mathbf{x}_1 - \mathbf{x}_2\|)}{\mathcal{L}(T)} \leq C_0, \tag{38}$$

for $0 < C_0 \leq 1$. This condition holds for bounded slowly varying functions as in (6), in the case where $D \subseteq \mathcal{B}_1(\mathbf{0})$, with $\mathcal{B}_1(\mathbf{0}) = \{\mathbf{x} \in \mathbb{R}^d, \ \|\mathbf{x}\| \leq 1\}$.

For the derivation of the limit distribution when $T \longrightarrow \infty$ of the functional (1), we first compute its variance, in terms of H_2, the Hermite polynomial of order 2. It is well-known that Hermite polynomials form a complete orthogonal system of the Hilbert space $L_2(\mathbb{R}, \varphi(u)du)$, the space of square integrable functions with respect to the standard normal density φ. They are defined as follows:

$$H_k(u) = (-1)^k e^{\frac{u^2}{2}} \frac{d^k}{du^k} e^{-\frac{u^2}{2}}, \quad k = 0, 1, \ldots.$$

In particular, for a zero-mean Gaussian random field Y, for $k \geq 1$,

$$\mathrm{E}\ H_k(Y(\mathbf{x})) = 0, \quad \mathrm{E}\ (H_k(Y(\mathbf{x}))\ H_m(Y(\mathbf{y}))) = \delta_{m,k}\ m!\ (\mathrm{E}[Y(\mathbf{x})Y(\mathbf{y})])^m \tag{39}$$

(see, for example, Peccati and Taqqu, 2011).

We use some ideas from the book by Ivanov and Leonenko (1989, Sections 1.4, 1.5 and 2.1). Consider the uniform distribution on $D(T)$ with the density:

$$P_{D(T)}(\mathbf{x}) = T^{-d}|D|^{-1}\mathbb{I}_{\mathbf{x} \in D(T)}, \quad \mathbf{x} \in \mathbb{R}^d, \tag{40}$$

where $\mathbb{I}_{\mathbf{x} \in D(T)}$ denotes the indicator function of set $D(T)$.

Let $\mathbf{U}$ and $\mathbf{V}$ be two independent and uniformly distributed inside the set $D(T)$ random vectors. We denote $\psi_{D(T)}(\rho)$, the density of the Euclidean distance $\|\mathbf{U} - \mathbf{V}\|$. Note that $\psi_{D(T)}(\rho) = 0$, if $\rho > \mathrm{diam}\,(D(T))$, and $\psi_{D(1)}(\rho)$ is bounded, where $\mathrm{diam}\,(D(T))$ is the diameter of the set $D(T)$.

Using the above notation, we obtain

$$\begin{aligned} \int_{D(T)} \int_{D(T)} G(\|\mathbf{x} - \mathbf{y}\|) d\mathbf{x} d\mathbf{y} &= |D(T)|^2 E\left[G\left(\|\mathbf{U} - \mathbf{V}\|\right)\right] \\ &= |D|^2 T^{2d} \int_0^{\mathrm{diam}(D(T))} G(\rho)\psi_{D(T)}(\rho) d\rho, \end{aligned} \tag{41}$$

for any Borel function G such that the Lebesgue integral (41) exists. In particular, under **Conditions A1–A2** for $0 < \alpha < d/2$, and $T \to \infty$, we obtain

$$\begin{aligned}\sigma^2(T) &= \mathrm{Var}\left[\int_{D(T)} H_2(Y(\mathbf{x}))d\mathbf{x}\right] = 2\int_{D(T)}\int_{D(T)} \frac{\mathcal{L}^2(\|\mathbf{x}-\mathbf{y}\|)}{\|\mathbf{x}-\mathbf{y}\|^{2\alpha}}d\mathbf{x}d\mathbf{y} \\ &= 2!|D|^2T^{2d}\int_0^{\mathrm{diam}(D(T))} \mathcal{L}^2(\rho)\rho^{-2\alpha}\psi_{D(T)}(\rho)d\rho. \end{aligned} \tag{42}$$

In equation (42), consider the change of variable $u = \rho/T$. Applying the consistency of the uniform distribution with a homothetic transformation, and the asymptotic properties of slowly varying functions (see Theorem 2.7 of Seneta, 1976) we get

$$\begin{aligned}\sigma^2(T) &= 2|D|^2T^{2d-2\alpha}\int_0^{\mathrm{diam}(D)} u^{-2\alpha}\mathcal{L}^2(uT)\psi_D(u)du \\ &= |D|^2T^{2d-2\alpha}\mathcal{L}^2(T)[a_d(D)]^2(1+o(1)),\ 0<\alpha<d/2, \quad T\to\infty, \end{aligned} \tag{43}$$

where, by (41),

$$a_d(D) = \left[2\int_0^{\mathrm{diam}(D)} u^{-2\alpha}\psi_D(u)du\right]^{1/2} = \left[2\int_D\int_D \frac{d\mathbf{x}d\mathbf{y}}{\|\mathbf{x}-\mathbf{y}\|^{2\alpha}}\right]^{1/2}. \tag{44}$$

More details, including properties of slowly varying functions, can be found in Anh, Leonenko and Olenko (2015).

If D is the ball $\mathcal{B}_T(\mathbf{0}) = \{\mathbf{x}\in\mathbb{R}^d : \ \|\mathbf{x}\| \le T\}$, then (see Ivanov and Leonenko, 1989, Lemma 1.4.2)

$$\psi_{\mathcal{B}_T(\mathbf{0})}(\rho) = T^{-d}I_{1-(\frac{\rho}{2T})^2}\left(\frac{d+1}{2},\frac{1}{2}\right)d\rho, \quad 0\le\rho\le 2T, \tag{45}$$

where

$$I_\mu(p,q) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)}\int_0^\mu t^{p-1}(1-t)^{q-1}dt, \quad \mu\in[0,1], \quad p>0,\ q>0, \tag{46}$$

is the incomplete beta function. In this case, one can show (see Lemma 2.1.3 in Ivanov and Leonenko, 1989)

$$a_d(B_1(\mathbf{0})) = \frac{2^{d-2\alpha+2}\pi^{d-1/2}\Gamma\left(\frac{d-2\alpha+1}{2}\right)}{(d-2\alpha)\Gamma\left(\frac{d}{2}\right)\Gamma(d-\alpha+1)}. \tag{47}$$

For $d = 1$, $D = [0,1]$,

$$a_1([0,1]) = 2\int_0^1\int_0^1 \frac{dxdy}{|x-y|^{2\alpha}} = \frac{1}{(1-\alpha)(1-2\alpha)}, \quad 0<\alpha<1/2.$$

Theorem 3.2. *Let D be a regular bounded domain. Assume that* ***Conditions A1*** *and* ***A2*** *are satisfied. The following assertions then hold:*

(i) As $T \longrightarrow \infty$, the functional S_T in (1) converges in distribution sense to a zero-mean random variable S_∞. If $C = 1$ in ***A2****, it has characteristic function given by*

$$\psi(z) = E\left[\exp(izS_\infty)\right] = \exp\left(\frac{1}{2}\sum_{m=2}^{\infty}\frac{(2\mathrm{i}z)^m}{m}c_m\right), \quad z\in\mathbb{R}, \tag{48}$$

where

$$c_m = \int_D \overset{(m)}{\cdots} \int_D \frac{1}{\|\mathbf{x}_1-\mathbf{x}_2\|^\alpha}\frac{1}{\|\mathbf{x}_2-\mathbf{x}_3\|^\alpha}\cdots\frac{1}{\|\mathbf{x}_m-\mathbf{x}_1\|^\alpha}d\mathbf{x}_1\dots d\mathbf{x}_m. \tag{49}$$

(ii) The functional

$$S_T^H = \frac{1}{\mathcal{L}(T)T^{d-\alpha}}\left[\int_{D(T)} G(Y(\mathbf{x}))d\mathbf{x} - C_0^H T^d|D|\right]$$

converges in distribution sense, as $T \longrightarrow \infty$*, to the random variable* $\frac{1}{2}C_2^H S_\infty$*, with* S_∞ *having characteristic function (48), and with* $G \in L^2(\mathbb{R}, \varphi(x)dx)$ *having Hermite rank* $m = 2$*. Here,*

$$\begin{aligned} C_0^H &= \int_{\mathbb{R}} G(u)H_0(u)\varphi(u)du = \mathrm{E}[G(Y(\mathbf{x}))] \\ C_2^H &= \int_{\mathbb{R}} G(u)H_2(u)\varphi(u)du, \end{aligned}$$

respectively denote the 0*th and* 2*th Hermite coefficients of the function* G*.*

Remark 3.3. Note that **Condition A2** is satisfied by the slowly varying function (6) with $C = 1$, for $D = \mathcal{B}_1(\mathbf{0}) = \{\mathbf{x} : \|\mathbf{x}\| \leq 1\}$.

Proof. We first prove (i). Since $EY^2(\mathbf{x}) = 1$,

$$\int_{D(T)} d\mathbf{x} = \int_{D(T)} E\left[Y^2(\mathbf{x})\right] d\mathbf{x} = E\left[\int_{D(T)} Y^2(\mathbf{x})d\mathbf{x}\right] = \sum_{j=1}^{\infty} \lambda_{j,T}(R_{Y,D(T)})E\eta_j^2 = \sum_{j=1}^{\infty} \lambda_{j,T}(R_{Y,D(T)}).$$

From Definition 2.1, Lemma 2.1, and Remark 2.1, one has

$$\begin{aligned} \psi_T(z) &= E\left[\exp\left(\frac{\mathrm{i}z}{d_T}\int_{D(T)}(Y^2(\mathbf{x})-1)d\mathbf{x}\right)\right] \\ &= \exp\left(-\frac{\mathrm{i}z\sum_{j=1}^{\infty}\lambda_{j,T}(R_{Y,D(T)})}{d_T}\right)\prod_{j=1}^{\infty}\left(1-2\mathrm{i}z\frac{\lambda_{j,T}(R_{Y,D(T)})}{d_T}\right)^{-1/2} \\ &= \exp\left(-\frac{\mathrm{i}z\sum_{j=1}^{\infty}\lambda_{j,T}(R_{Y,D(T)})}{d_T}\right)\left[\mathcal{D}_T\left(\frac{2\mathrm{i}z}{d_T}\right)\right]^{-1/2} \\ &= \exp\left(-\frac{\mathrm{i}z\sum_{j=1}^{\infty}\lambda_{j,T}(R_{Y,D(T)})}{d_T}\right)\exp\left(\frac{1}{2}\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{2\mathrm{i}z}{d_T}\right)^m \mathrm{Tr}\left(R_{Y,D(T)}^m\right)\right) \\ &= \exp\left(-\frac{\mathrm{i}z\sum_{j=1}^{\infty}\lambda_{j,T}(R_{Y,D(T)})}{d_T}+\frac{\mathrm{i}z\sum_{j=1}^{\infty}\lambda_{j,T}(R_{Y,D(T)})}{d_T}\right. \\ &\qquad \left.+\frac{1}{2}\sum_{m=2}^{\infty}\frac{1}{m}\left(\frac{2\mathrm{i}z}{d_T}\right)^m \mathrm{Tr}\left(R_{Y,D(T)}^m\right)\right) \\ &= \exp\left(\frac{1}{2}\sum_{m=2}^{\infty}\frac{1}{m}\left(\frac{2\mathrm{i}z}{d_T}\right)^m \mathrm{Tr}\left(R_{Y,D(T)}^m\right)\right). \end{aligned} \tag{50}$$

Note that, from (3), for every $m \geq 2$,

$$\lim_{T\to\infty}\frac{\mathrm{Tr}\left(R_{Y,D(T)}^m\right)}{d_T^m} = \mathrm{Tr}\left(\mathcal{K}_\alpha^m\right), \tag{51}$$

which is finite from Theorem 3.1, that ensures the trace property of $\mathcal{K}_\alpha^2$. Hence, for every $m \geq 2$, and z,

$$\lim_{T\to\infty}\frac{1}{m}\left(\frac{2\mathrm{i}z}{d_T}\right)^m \mathrm{Tr}\left(R_{Y,D(T)}^m\right) = \frac{1}{m}(2\mathrm{i}z)^m\,\mathrm{Tr}\left(\mathcal{K}_\alpha^m\right) = \frac{1}{m}(2\mathrm{i}z)^m c_m, \tag{52}$$

with c_m being given in equation (49).

In addition, under **A2**, there exists a positive constant C such that

$$\begin{aligned}\frac{1}{d_T^2}\mathrm{Tr}\left(R^2_{Y,D(T)}\right) &= \int_D\int_D \frac{\mathcal{L}(T\|\mathbf{x}_1-\mathbf{x}_2\|)}{\mathcal{L}(T)}\frac{\mathcal{L}(T\|\mathbf{x}_2-\mathbf{x}_1\|)}{\mathcal{L}(T)}\frac{1}{\|\mathbf{x}_1-\mathbf{x}_2\|^{2\alpha}}d\mathbf{x}_1 d\mathbf{x}_2\\ &\leq C\int_D\int_D \frac{1}{\|\mathbf{x}_1-\mathbf{x}_2\|^{2\alpha}}d\mathbf{x}_1 d\mathbf{x}_2 = C\mathrm{Tr}\left(\mathcal{K}^2_\alpha\right)<\infty \end{aligned} \tag{53}$$

$$\begin{aligned}&\frac{1}{d_T^m}\mathrm{Tr}\left(R^m_{Y,D(T)}\right) =\\ &= \frac{1}{[\mathcal{L}(T)]^m}\int_D \overset{\dots}{\underset{(m)}{}} \int_D \frac{\mathcal{L}(T\|\mathbf{x}_1-\mathbf{x}_2\|)}{\|\mathbf{x}_1-\mathbf{x}_2\|^{\alpha}}\frac{\mathcal{L}(T\|\mathbf{x}_2-\mathbf{x}_3\|)}{\|\mathbf{x}_2-\mathbf{x}_3\|^{\alpha}}\cdots\frac{\mathcal{L}(T\|\mathbf{x}_m-\mathbf{x}_1\|)}{\|\mathbf{x}_m-\mathbf{x}_1\|^{\alpha}}d\mathbf{x}_1\dots d\mathbf{x}_m\\ &\leq C\int_D \overset{\dots}{\underset{(m)}{}} \int_D \frac{1}{\|\mathbf{x}_1-\mathbf{x}_2\|^{\alpha}}\frac{1}{\|\mathbf{x}_2-\mathbf{x}_3\|^{\alpha}}\cdots\frac{1}{\|\mathbf{x}_m-\mathbf{x}_1\|^{\alpha}}d\mathbf{x}_1\dots d\mathbf{x}_m\\ &= C\mathrm{Tr}\left(\mathcal{K}^m_\alpha\right)<\infty, \quad m>2, \end{aligned} \tag{54}$$

since $\|\mathcal{K}^m_\alpha\|_1 \leq \|\mathcal{K}^2_\alpha\|_1$, for $m>2$.

From equations (50)–(54), for every $T>0$,

$$\begin{aligned}|\psi_T(z)| &= \left|\exp\left(\frac{1}{2}\sum_{m=2}^{\infty}\frac{(-1)^{m-1}}{2m-2}\left(\frac{2z}{d_T}\right)^{2m-2}\mathrm{Tr}\left(R^{2m-2}_{Y,D(T)}\right)\right)\right|\\ &\times \left|\exp\left(\frac{i}{2}\sum_{m=3}^{\infty}\frac{(-1)^{m}}{2m-3}\left(\frac{2z}{d_T}\right)^{2m-3}\mathrm{Tr}\left(R^{2m-3}_{Y,D(T)}\right)\right)\right|\\ &= \left|\exp\left(\frac{1}{2}\sum_{m=2}^{\infty}\frac{(-1)^{m-1}}{2m-2}\left(\frac{2z}{d_T}\right)^{2m-2}\mathrm{Tr}\left(R^{2m-2}_{Y,D(T)}\right)\right)\right|\\ &\times \left[\cos^2\left(\frac{1}{2}\sum_{m=3}^{\infty}\frac{(-1)^{m}}{2m-3}\left(\frac{2z}{d_T}\right)^{2m-3}\mathrm{Tr}\left(R^{2m-3}_{Y,D(T)}\right)\right)\right.\\ &\left.+\sin^2\left(\frac{1}{2}\sum_{m=3}^{\infty}\frac{(-1)^{m}}{2m-3}\left(\frac{2z}{d_T}\right)^{2m-3}\mathrm{Tr}\left(R^{2m-3}_{Y,D(T)}\right)\right)\right]^{1/2} \end{aligned} \tag{55}$$

$$= \left|\exp\left(\frac{-1}{2}\sum_{m=2}^{\infty}\frac{(-1)^{m}}{2m-2}\left(\frac{2z}{d_T}\right)^{2m-2}\mathrm{Tr}\left(R^{2m-2}_{Y,D(T)}\right)\right)\right| \tag{56}$$

$$= \left|\exp\left(\frac{-1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{4n-2}\left(\frac{2z}{d_T}\right)^{4n-2}\mathrm{Tr}\left(R^{4n-2}_{Y,D(T)}\right)\right)\right|$$

$$\begin{aligned} &\times \left|\exp\left(\frac{-1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{2n+1}}{4n}\left(\frac{2z}{d_T}\right)^{4n}\mathrm{Tr}\left(R^{4n}_{Y,D(T)}\right)\right)\right| \\ &= \left|\exp\left(\frac{-1}{2}\sum_{n=1}^{\infty}\frac{1}{4n-2}\left(\frac{2z}{d_T}\right)^{4n-2}\mathrm{Tr}\left(R^{4n-2}_{Y,D(T)}\right)\right)\right|\\ &\times \left|\exp\left(\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{4n}\left(\frac{2z}{d_T}\right)^{4n}\mathrm{Tr}\left(R^{4n}_{Y,D(T)}\right)\right)\right| \end{aligned} \tag{57}$$

$$
\begin{aligned}
&\leq \left|\exp\left(\frac{-1}{2}\sum_{n=1}^{\infty}\frac{1}{4n-2}\left(\frac{2z}{d_T}\right)^{4n-2}\mathrm{Tr}\left(R_{Y,D(T)}^{4n-2}\right)\right)\right| \\
&\times \left|\exp\left(\frac{C}{2}\sum_{n=1}^{\infty}\frac{1}{4n}(2z)^{4n}\,\mathrm{Tr}\left(\mathcal{K}_{\alpha}^{4n}\right)\right)\right| \\
&\leq \left|\exp\left(\frac{C}{2}\sum_{n=1}^{\infty}\frac{1}{4n}(2z)^{4n}\,\mathrm{Tr}\left(\mathcal{K}_{\alpha}^{4n}\right)\right)\right| \\
&= \left|\exp\left(\frac{C}{8}\sum_{n=1}^{\infty}\frac{1}{n}\left(16z^4\right)^{n}\mathrm{Tr}\left((\mathcal{K}_{\alpha}^{4})^{n}\right)\right)\right| = \left[\mathcal{D}_{\mathcal{K}_{\alpha}^{4}}(16z^4)\right]^{-C/8} < \infty, \qquad (58)
\end{aligned}
$$

where we have applied, inside the argument of the exponential, the straightforward identities $i^{2m-2} = (i^2)^{m-1} = (-1)^{m-1}$, $|\exp(iu)| = \cos^2(u) + \sin^2(u) = 1$, and the fact that the sequence of natural numbers $m = 2, 3, 4, 5, 6 \dots = \mathbb{N} - \{0, 1\}$ can be obtained as the union of the sequences $\{2m-2\}_{m\geq 2} = 2, 4, 6, \dots$ and $\{2m-3\}_{m\geq 3} = 3, 5, 7, \dots$. Hence, in the above equation, the sum in $\mathbb{N} - \{0, 1\}$ can be splitted into the sums $\sum_{m=2}^{\infty}(-1)^{m-1} f(2m-2)$ and $\sum_{m=3}^{\infty}(-1)^{m} f(2m-3)$. Moreover, in (57), we consider the sequence $\{2m-2\}_{m\geq 2} = 2, 4, 6, 8, 10, 12, \dots$ as the union of the sequences $\{4n-2\}_{n\geq 1} = 2, 6, 10, \dots$, and $\{4n\}_{n\geq 1} = 4, 8, 12, \dots$, corresponding to the changes of variable $m = 2n$ and $m = 2n+1$. Thus, for $m = 2n$, $2m-2 = 4n-2$, and, for $m = 2n+1$, $2m-2 = 4n+2-2 = 4n$. The sum $\sum_{m=2}^{\infty}(-1)^{m} f(2m-2)$ can then be splitted into the two sums $\sum_{n=1}^{\infty}(-1)^{2n} f(4n-2)$ and $\sum_{n=1}^{\infty}(-1)^{2n+1} f(4n)$. Furthermore, the last identity in (58) is obtained from the Fredholm determinant formula

$$
\mathcal{D}_{\mathcal{K}_{\alpha}^{4}} = \det(I - \omega\mathcal{K}_{\alpha}^{4}) = \exp\left(-\sum_{k=1}^{\infty}\frac{\mathrm{Tr}[\mathcal{K}_{\alpha}^{4}]^{k}}{k}\omega^{k}\right) = \exp\left(-\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}[\lambda_l(\mathcal{K}_{\alpha}^{4})]^{k}\frac{\omega^{k}}{k}\right) \qquad (59)
$$

of $\mathcal{K}_{\alpha}^{4}$ at point $\omega = 16z^4$, which is finite for $|\omega| < \frac{1}{\|\mathcal{K}_{\alpha}^{4}\|_1}$, since $\mathcal{K}_{\alpha}^{4}$ is in the trace class in view of the trace property of $\mathcal{K}_{\alpha}^{2}$ (see Definition 2.1 and equation (17)). Note that, from Theorem 3.1, $\mathcal{K}_{\alpha}^{2}$ is in the trace class, i.e., considering equations (44) and (22),

$$
\mathrm{Tr}\left(\mathcal{K}_{\alpha}^{2}\right) = \int_D\int_D \frac{1}{\|\mathbf{x}-\mathbf{y}\|^{2\alpha}} d\mathbf{x}d\mathbf{y} = \frac{[a_d(D)]^2}{2} < \infty. \qquad (60)
$$

From (58), there exists $\widetilde{\psi}(z) = \lim_{T\to\infty}|\psi_T(z)| < \infty$, for $0 < z < \left[1/16\|\mathcal{K}_{\alpha}^{4}\|_1\right]^{1/4}$. An analytic continuation argument (see Lukacs, 1970, Th. 7.1.1) guarantees that $\widetilde{\psi}$ defines the unique limit characteristic function for all real values of z.

From (58), we now prove that $\widetilde{\psi}(z) = \psi(z)$, with $\psi(z)$ given in (48)–(49), for $C = 1$, in **Condition A2**.

From equations (51)–(52), the sequence of functions $\{g_{T,z}(m)\}_{T\geq 0}$, with

$$
g_{T,z}(m) = \frac{1}{m}\left(\frac{2\mathrm{i}z}{d_T}\right)^{m}\mathrm{Tr}\left(R_{Y,D(T)}^{m}\right),
$$

converges pointwise to the function $g_z(m) = \frac{1}{m}(2\mathrm{i}z)^m\,\mathrm{Tr}\left(\mathcal{K}_{\alpha}^{m}\right) = \frac{1}{m}(2\mathrm{i}z)^m c_m$, for each fixed z, as $T \to \infty$. Moreover, from equation (58),

$$
\begin{aligned}
\lim_{T\to\infty}|\psi_T(z)| &= \lim_{T\to\infty}\left|\exp\left(\frac{1}{2}\sum_{m=2}^{\infty}g_{T,z}(m)\right)\right| \\
&\leq \lim_{T\to\infty}\exp\left(\frac{C}{8}\sum_{m=1}^{\infty}h_z(m)\right) = \exp\left(\frac{C}{8}\sum_{m=1}^{\infty}h_z(m)\right) < \infty, \qquad (61)
\end{aligned}
$$

for $0 < z < \left[1/16\|\mathcal{K}_\alpha^4\|_1\right]^{1/4}$, where $h_z(n) = \frac{1}{n}\left(16z^4\right)^n \operatorname{Tr}\left((\mathcal{K}_\alpha^4)^n\right)$, for each $n \geq 1$. Thus, as proved, $\{\sum_{m=2}^{\infty} g_{T,z}(m)\}_{T\geq 0}$ is a convergent sequence in T. From (55),

$$\frac{\operatorname{Re}(\psi_T(z))}{\operatorname{Re}(\psi(z))} = \frac{\exp\left(\frac{-1}{2}\sum_{n=1}^{\infty}\frac{1}{4n-2}\left(\frac{2z}{d_T}\right)^{4n-2}\operatorname{Tr}\left(R_{Y,D(T)}^{4n-2}\right)\right)}{\exp\left(\frac{-1}{2}\sum_{n=1}^{\infty}\frac{1}{4n-2}(2z)^{4n-2}\operatorname{Tr}\left(\mathcal{K}_\alpha^{4n-2}\right)\right)} \times \frac{\exp\left(\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{4n}\left(\frac{2z}{d_T}\right)^{4n}\operatorname{Tr}\left(R_{Y,D(T)}^{4n}\right)\right)}{\exp\left(\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{4n}\left(\frac{2z}{d_T}\right)^{4n}\operatorname{Tr}\left(\mathcal{K}_\alpha^{4n}\right)\right)}. \quad (62)$$

From Fatou's Lemma (considering integration with respect to a counting or point measure), we obtain

$$\sum_{n=1}^{\infty}\frac{1}{4n-2}(2z)^{4n-2}\operatorname{Tr}\left(\mathcal{K}_\alpha^{4n-2}\right) \leq \liminf_{T\to\infty}\sum_{n=1}^{\infty}\frac{1}{4n-2}\left(\frac{2z}{d_T}\right)^{4n-2}\operatorname{Tr}\left(R_{Y,D(T)}^{4n-2}\right).$$

In particular,

$$\exp\left(-\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{4n-2}(2z)^{4n-2}\operatorname{Tr}\left(\mathcal{K}_\alpha^{4n-2}\right)\right) \geq \liminf_{T\to\infty}\exp\left(-\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{4n-2}\left(\frac{2z}{d_T}\right)^{4n-2}\operatorname{Tr}\left(R_{Y,D(T)}^{4n-2}\right)\right). \quad (63)$$

Since, as commented, from equation (61), $\{\sum_{m=2}^{\infty} g_{T,z}(m)\}_{T\geq 0}$ is a convergent sequence in T, in particular, from equation (62),

$$\liminf_{T\to\infty}\exp\left(-\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{4n-2}\left(\frac{2z}{d_T}\right)^{4n-2}\operatorname{Tr}\left(R_{Y,D(T)}^{4n-2}\right)\right) = \lim_{T\to\infty}\exp\left(-\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{4n-2}\left(\frac{2z}{d_T}\right)^{4n-2}\operatorname{Tr}\left(R_{Y,D(T)}^{4n-2}\right)\right). \quad (64)$$

Thus, from equations (62), (63) and (64),

$$\lim_{T\to\infty}\frac{\exp\left(\frac{-1}{2}\sum_{n=1}^{\infty}\frac{1}{4n-2}\left(\frac{2z}{d_T}\right)^{4n-2}\operatorname{Tr}\left(R_{Y,D(T)}^{4n-2}\right)\right)}{\exp\left(\frac{-1}{2}\sum_{n=1}^{\infty}\frac{1}{4n-2}(2z)^{4n-2}\operatorname{Tr}\left(\mathcal{K}_\alpha^{4n-2}\right)\right)} \leq \frac{\exp\left(\frac{-1}{2}\sum_{n=1}^{\infty}\frac{1}{4n-2}(2z)^{4n-2}\operatorname{Tr}\left(\mathcal{K}_\alpha^{4n-2}\right)\right)}{\exp\left(\frac{-1}{2}\sum_{n=1}^{\infty}\frac{1}{4n-2}(2z)^{4n-2}\operatorname{Tr}\left(\mathcal{K}_\alpha^{4n-2}\right)\right)} = 1. \quad (65)$$

In addition, under **A2**,

$$\lim_{T\to\infty}\frac{\exp\left(\frac{-1}{2}\sum_{n=1}^{\infty}\frac{1}{4n-2}\left(\frac{2z}{d_T}\right)^{4n-2}\operatorname{Tr}\left(R_{Y,D(T)}^{4n-2}\right)\right)}{\exp\left(\frac{-1}{2}\sum_{n=1}^{\infty}\frac{1}{4n-2}(2z)^{4n-2}\operatorname{Tr}\left(\mathcal{K}_\alpha^{4n-2}\right)\right)}$$

$$\geq \frac{\exp\left(\frac{-1}{2}\sum_{n=1}^{\infty}\frac{1}{4n-2}\left(2z\right)^{4n-2}C\mathrm{Tr}\left(\mathcal{K}_{\alpha}^{4n-2}\right)\right)}{\exp\left(\frac{-1}{2}\sum_{n=1}^{\infty}\frac{1}{4n-2}\left(2z\right)^{4n-2}\mathrm{Tr}\left(\mathcal{K}_{\alpha}^{4n-2}\right)\right)}$$

$$= \exp\left(\frac{1-C}{2}\sum_{n=1}^{\infty}\frac{1}{4n-2}\left(2z\right)^{4n-2}\mathrm{Tr}\left(\mathcal{K}_{\alpha}^{4n-2}\right)\right), \tag{66}$$

where C is given in **Condition A2** (see equation (37)).

Applying again Fatou's Lemma,

$$\lim_{T\to\infty}\frac{\exp\left(\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{4n}\left(\frac{2z}{d_T}\right)^{4n}\mathrm{Tr}\left(R_{Y,D(T)}^{4n}\right)\right)}{\exp\left(\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{4n}\left(\frac{2z}{d_T}\right)^{4n}\mathrm{Tr}\left(\mathcal{K}_{\alpha}^{4n}\right)\right)}$$

$$= \frac{\liminf_{T\to\infty}\exp\left(\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{4n}\left(\frac{2z}{d_T}\right)^{4n}\mathrm{Tr}\left(R_{Y,D(T)}^{4n}\right)\right)}{\exp\left(\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{4n}\left(\frac{2z}{d_T}\right)^{4n}\mathrm{Tr}\left(\mathcal{K}_{\alpha}^{4n}\right)\right)}$$

$$= \frac{\exp\left(\frac{1}{2}\liminf_{T\to\infty}\sum_{n=1}^{\infty}\frac{1}{4n}\left(\frac{2z}{d_T}\right)^{4n}\mathrm{Tr}\left(R_{Y,D(T)}^{4n}\right)\right)}{\exp\left(\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{4n}\left(\frac{2z}{d_T}\right)^{4n}\mathrm{Tr}\left(\mathcal{K}_{\alpha}^{4n}\right)\right)}$$

$$\geq \frac{\exp\left(\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{4n}\left(\frac{2z}{d_T}\right)^{4n}\mathrm{Tr}\left(\mathcal{K}_{\alpha}^{4n}\right)\right)}{\exp\left(\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{4n}\left(\frac{2z}{d_T}\right)^{4n}\mathrm{Tr}\left(\mathcal{K}_{\alpha}^{4n}\right)\right)} = 1. \tag{67}$$

Moreover, under **A2**,

$$\lim_{T\to\infty}\frac{\exp\left(\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{4n}\left(\frac{2z}{d_T}\right)^{4n}\mathrm{Tr}\left(R_{Y,D(T)}^{4n}\right)\right)}{\exp\left(\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{4n}\left(\frac{2z}{d_T}\right)^{4n}\mathrm{Tr}\left(\mathcal{K}_{\alpha}^{4n}\right)\right)}$$

$$\leq \exp\left(\frac{C-1}{2}\sum_{n=1}^{\infty}\frac{1}{4n}\left(\frac{2z}{d_T}\right)^{4n}\mathrm{Tr}\left(\mathcal{K}_{\alpha}^{4n}\right)\right). \tag{68}$$

From equations (62)–(68),

$$[\mathrm{Re}(\psi(z))]^{2-C} \leq \lim_{T\to\infty}\mathrm{Re}(\psi_T(z)) \leq [\mathrm{Re}(\psi(z))]^{C}. \tag{69}$$

From equation (55),

$$\frac{\mathrm{Im}(\psi_T(z))}{\mathrm{Im}(\psi(z))} = \frac{\exp\left(\frac{-1}{2}\sum_{n=2}^{\infty}\frac{1}{4n-5}\left(\frac{2z}{d_T}\right)^{4n-5}\mathrm{Tr}\left(R_{Y,D(T)}^{4n-5}\right)\right)}{\exp\left(\frac{-1}{2}\sum_{n=2}^{\infty}\frac{1}{4n-5}\left(2z\right)^{4n-5}\mathrm{Tr}\left(\mathcal{K}_{\alpha}^{4n-5}\right)\right)} \times \frac{\exp\left(\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{4n+1}\left(\frac{2z}{d_T}\right)^{4n+1}\mathrm{Tr}\left(R_{Y,D(T)}^{4n+1}\right)\right)}{\exp\left(\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{4n+1}\left(\frac{2z}{d_T}\right)^{4n+1}\mathrm{Tr}\left(\mathcal{K}_{\alpha}^{4n+1}\right)\right)}. \tag{70}$$

From equation (70), applying again Fatou's Lemma, under **Condition A2**, proceeding in a similar way to the computations made for the real part, we obtain

$$[\mathrm{Im}(\psi(z))]^{2-C} \leq \lim_{T\to\infty} \mathrm{Im}(\psi_T(z)) \leq [\mathrm{Im}(\psi(z))]^{C}, \tag{71}$$

where C is given in **Condition A2** (see equation (37)).

Thus, $\widetilde{\psi}(z) = \lim_{T\to\infty} \psi_T(z) = \psi(z)$, if $C = 1$ in **Condition A2**.

We now turn to the proof of (ii). Under **Condition A1**, since $B(\|\mathbf{x}\|) \leq 1$, and $B(0) = 1$, we have

$$B^j(\|\mathbf{x}\|) \leq B^3(\|\mathbf{x}\|), \quad j \geq 3.$$

Hence,

$$\begin{aligned}
K_T &= \left[\frac{1}{\mathcal{L}^2(T)T^{2d-2\alpha}}\right] E\left[\left(\int_{D(T)} G(Y(\mathbf{x}))\, d\mathbf{x} - C_0^H T^d\,|D| - \frac{C_2^H}{2}\int_{D(T)} H_2(Y(\mathbf{x}))\, d\mathbf{x}\right)\right]^2 \\
&= \left[\frac{1}{\mathcal{L}^2(T)T^{2d-2\alpha}}\right] \sum_{j=3}^{\infty} \frac{(C_j^H)^2}{j!} \int_{D(T)}\int_{D(T)} B^j(\|\mathbf{x}-\mathbf{y}\|) d\mathbf{x} d\mathbf{y} \leq \\
&\leq \left[\frac{1}{\mathcal{L}^2(T)T^{2d-2\alpha}}\right] \int_{D(T)}\int_{D(T)} B^3(\|\mathbf{x}-\mathbf{y}\|) d\mathbf{x} d\mathbf{y} \left[\sum_{j=3}^{\infty} \frac{(C_j^H)^2}{j!}\right].
\end{aligned} \tag{72}$$

By **Condition A1**, for any $\varepsilon > 0$, there exists $A_0 > 0$, such that for$\|\mathbf{x}-\mathbf{y}\| > A_0$, $B(\|\mathbf{x}-\mathbf{y}\|) < \varepsilon$. Let $\mathcal{D}_1 = \{(\mathbf{x},\mathbf{y}) \in D(T)\times D(T) : \|\mathbf{x}-\mathbf{y}\| \leq A_0)\}$, $\mathcal{D}_2 = \{(\mathbf{x},\mathbf{y}) \in D(T)\times D(T) : \|\mathbf{x}-\mathbf{y}\| > A_0)\}$,

$$\int_{D(T)}\int_{D(T)} B^3(\|\mathbf{x}-\mathbf{y}\|) d\mathbf{x} d\mathbf{y} = \left\{\int\int_{\mathcal{D}_1} + \int\int_{\mathcal{D}_2}\right\} B^3(\|\mathbf{x}-\mathbf{y}\|) d\mathbf{x} d\mathbf{y} = S_T^{(1)} + S_T^{(2)}. \tag{73}$$

Using the bound $B^3(\|\mathbf{x}-\mathbf{y}\|) \leq 1$ on $\mathcal{D}_1$, and the bound $B^3(\|\mathbf{x}-\mathbf{y}\|) < \epsilon B^2(\|\mathbf{x}-\mathbf{y}\|)$ on $\mathcal{D}_2$, we obtain,

$$\left|S_T^{(1)}\right| \leq \int\int_{\mathcal{D}_1} \left|B^3(\|\mathbf{x}-\mathbf{y}\|)\right| d\mathbf{x} d\mathbf{y} \leq M_1 T^d$$

for a suitable constant $M_1 > 0$, and for T sufficiently large, under **A2**,

$$\begin{aligned}
\left|S_T^{(2)}\right| &\leq \int\int_{\mathcal{D}_2} \left|B^3(\|\mathbf{x}-\mathbf{y}\|)\right| d\mathbf{x} d\mathbf{y} \leq \epsilon \int\int_{\mathcal{D}_2} B^2(\|\mathbf{x}-\mathbf{y}\|) d\mathbf{x} d\mathbf{y} \\
&\leq \epsilon \int_{D(T)}\int_{D(T)} B^2(\|\mathbf{x}-\mathbf{y}\|) d\mathbf{x} d\mathbf{y} = \epsilon \int_{D(T)}\int_{D(T)} \frac{\mathcal{L}^2(\|\mathbf{x}-\mathbf{y}\|)}{\|\mathbf{x}-\mathbf{y}\|^{2\alpha}} d\mathbf{x} d\mathbf{y} \\
&= \epsilon \mathcal{L}^2(T) T^{2d-2\alpha} \int_{D(1)}\int_{D(1)} \frac{\mathcal{L}^2(T\|\mathbf{x}-\mathbf{y}\|)}{\|\mathbf{x}-\mathbf{y}\|^{2\alpha}\mathcal{L}^2(T)} d\mathbf{x} d\mathbf{y} \\
&\leq \epsilon C \mathcal{L}^2(T) T^{2d-2\alpha} \int_{D(1)}\int_{D(1)} \frac{d\mathbf{x} d\mathbf{y}}{\|\mathbf{x}-\mathbf{y}\|^{2\alpha}} < \infty, \quad 0 < \alpha < d/2.
\end{aligned} \tag{74}$$

Thus, for

$$M_2 = C \int_{D(1)}\int_{D(1)} \frac{d\mathbf{x} d\mathbf{y}}{\|\mathbf{x}-\mathbf{y}\|^{2\alpha}},$$

from (72)–(74), we have

$$\begin{aligned} K_T &\leq \left[\frac{1}{\mathcal{L}(T)T^{d-\alpha}}\right]^2 \left[\sum_{j=3}^{\infty}\frac{(C_j^H)^2}{j!}\right]\int_{D(T)}\int_{D(T)} B^3(\|\mathbf{x}-\mathbf{y}\|)d\mathbf{x}d\mathbf{y} \\ &\leq (M_1 \vee M_2)\left[\frac{T^d}{\mathcal{L}^2(T)T^{2d-2\alpha}} + \epsilon\frac{T^{2d-2\alpha}\mathcal{L}^2(T)}{\mathcal{L}^2(T)T^{2d-2\alpha}}\right] \end{aligned} \tag{75}$$

is arbitrarily small together with $\epsilon > 0$ as $T \to \infty$. The desired result on weak-convergence then follows. ■

Remark 3.4. Consider the case of $d = 1$ and discrete time. That is, let $\{Y(t),\ t \in \mathbb{Z}\}$ be a stationary zero-mean Gaussian sequence with unit variance and covariance function of the form

$$B(t) = \frac{\mathcal{L}(t)}{|t|^{\alpha}},$$

for $0 < \alpha < 1/2$. The proof of the weak convergence result in Rosenblatt (1961) and Taqqu (1975) is based on the following formula for the characteristic function of a quadratic form of strong-correlated Gaussian random variables:

$$\begin{aligned} &E\left[\exp\left\{\mathrm{i}z\frac{1}{d_T}\sum_{t=0}^{T-1}(Y^2(t)-1)\right\}\right] = \exp\left\{-\mathrm{i}zTd_T^{-1}\right\}\left[\det\left(I_T - 2\mathrm{i}zd_T^{-1}R_T\right)\right]^{-1/2} \\ &= \exp\left\{\sum_{k=2}^{\infty}(2\mathrm{i}zd_T^{-1})^k\frac{\mathrm{Sp}R_T^k}{k}\right\}, \end{aligned} \tag{76}$$

where

$$\frac{1}{d_T^k}\mathrm{Sp}R_T^k = \frac{1}{d_T^k}\sum_{i_1=0}^{T-1}\cdots\sum_{i_k=0}^{T-1}B(|i_1-i_2|)B(|i_2-i_3|)\dots B(|i_k-i_1|), \tag{77}$$

with $d_T = T^{1-\alpha}\mathcal{L}(T)$, $R_T = E[Y\bar{Y}']$, $Y = (Y(0),\dots,Y(T-1))'$, $\mathrm{Sp}R_T$ denoting the trace of the matrix R_T, and I_T representing the identity matrix of size T (see p.39 of the book by Mathai and Provost, 1992). One can get a direct extension of formulae (76) and (77) to the stationary zero-mean Gaussian random process case in continuous time $\{Y(t),\ t \in \mathbb{R}\}$ (see Leonenko and Taufer, 2006), but for $d \geq 2$ direct extensions of (76) and (77) are not available. The present paper addresses this problem by applying alternative functional tools, like the Karhunen-Loève expansion and Fredholm determinant formula, to overcome this difficulty of discretization of the multidimensional parameter space. Note that the Fredholm determinant formula appears in the definition of the characteristic functional of quadratic forms defined in terms of Hilbert-valued zero-mean Gaussian random variables (see, for example, Proposition 1.2.8 of Da Prato and Zabczyk, 2002).

Remark 3.5. Expanding around zero the characteristic function (48), we obtain the cumulants of random variable S_∞, that is, $\kappa_1 = 0$, and

$$\kappa_k = 2^{k-1}(k-1)!c_k, \quad k \geq 2, \tag{78}$$

where c_k are defined as in equation (49). The derivation of explicit expressions for c_k would lead to the computation of the moments or cumulants of the limit distribution. This aspect will constitute the subject of a subsequent paper.

4 Infinite series representation and eigenvalues

The representation of the Rosenblatt-type distribution as the sum of an infinite series of weighted independent chi-squared random variables is derived in this section. As in the classical case (see Proposition 2 of Dobrushin and Major, 1979), this series expansion is obtained from the double Wiener-Itô stochastic integral representation of S_∞ in the spectral domain (see Theorem 4.1). Proposition 4.1 and Corollary 4.2 below establish the connection between the eigenvalues of operator $\mathcal{K}_\alpha$ in (22) and the weights appearing in the series representation derived.
The following condition will be required for the derivation of Theorem 4.1(ii) below.

Condition A3. Suppose that **Condition A1** holds, and there exists a spectral density $f_0(\|\boldsymbol{\lambda}\|)$, $\boldsymbol{\lambda} \in \mathbb{R}^d$, being decreasing function for $\|\boldsymbol{\lambda}\| \in (0, \varepsilon]$, with $\varepsilon > 0$.

If **Condition A3** holds, from equation (4), applying a Tauberian Theorem (see Doukhan, León and Soulier, 1996, and Theorems 4 and 11 in Leonenko and Olenko, 2014),

$$f_0(\|\boldsymbol{\lambda}\|) \sim c(d,\alpha)\mathcal{L}\left(\frac{1}{\|\boldsymbol{\lambda}\|}\right)\|\boldsymbol{\lambda}\|^{\alpha-d}, \quad 0<\alpha<d, \quad \|\boldsymbol{\lambda}\| \to 0. \tag{79}$$

Here, $c(d,\alpha) = \frac{\Gamma\left(\frac{d-\alpha}{2}\right)}{2^\alpha \pi^{d/2}\Gamma\left(\frac{\alpha}{2}\right)}$ is defined in (24).

Condition A3 holds, in particular, for the correlation function (5), with the isotropic spectral density

$$f_0(\|\boldsymbol{\lambda}\|) = \frac{\|\boldsymbol{\lambda}\|^{1-\frac{d}{2}}}{2^{\frac{d}{2}-1}\pi^{\frac{d}{2}+1}} \int_0^\infty K_{\frac{d}{2}-1}(\|\boldsymbol{\lambda}\|u) \frac{\sin\left(\gamma \arg\left(1+u^\beta \exp\left(\frac{i\pi\beta}{2}\right)\right)\right)}{\left|1+u^\beta \exp\left(\frac{i\pi\beta}{2}\right)\right|^\gamma} u^{\frac{d}{2}}du, \tag{80}$$

where $K_\nu(z)$ is the modified Bessel function of the second kind. By Corollary 3.10 in Lim and Teo (2010), the spectral density (80) satisfies (79), with $\alpha = \beta\gamma < d$.

The zero-mean Gaussian random field Y with an absolutely continuous spectrum has the isonormal representation

$$Y(\mathbf{x}) = \int_{\mathbb{R}^d} \exp\left(i\left\langle \boldsymbol{\lambda}, \mathbf{x}\right\rangle\right)\sqrt{f_0(\|\boldsymbol{\lambda}\|)}Z(d\boldsymbol{\lambda}), \tag{81}$$

where Z is a complex white noise Gaussian random measure with Lebesgue control measure.

Theorem 4.1. *Let D be a regular bounded domain.*

(i) For $0 < \alpha < d/2$, the following identities hold:

$$\int_{\mathbb{R}^{2d}} |K(\boldsymbol{\lambda}_1+\boldsymbol{\lambda}_2, D)|^2 \frac{d\boldsymbol{\lambda}_1 d\boldsymbol{\lambda}_2}{(\|\boldsymbol{\lambda}_1\|\,\|\boldsymbol{\lambda}_2\|)^{d-\alpha}} = \left[\frac{a_d\gamma(\alpha)}{\sqrt{2}|D|}\right]^2 = \frac{[\gamma(\alpha)]^2 Tr(\mathcal{K}_\alpha^2)}{|D|^2} < \infty, \tag{82}$$

where a_d is defined in (44), $\gamma(\alpha)$ is introduced in equation (24), and K is the characteristic function of the uniform distribution over set D, given by

$$K(\boldsymbol{\lambda}, D) = \int_D e^{\mathrm{i}\langle \boldsymbol{\lambda}, \mathbf{x}\rangle} p_D(\mathbf{x})\, d\mathbf{x} = \frac{1}{|D|}\int_D e^{\mathrm{i}\langle \boldsymbol{\lambda}, \mathbf{x}\rangle} d\mathbf{x} = \frac{\vartheta(\boldsymbol{\lambda})}{|D|}, \tag{83}$$

with associated probability density function $p_D(\mathbf{x}) = 1/|D|$ if $\mathbf{x} \in D$, and 0 otherwise.

(ii) *Assume that **Conditions A1, A2, A3** hold. Then, the random variable S_∞ admits the following double Wiener-Itô stochastic integral representation:*

$$S_\infty \quad = \quad |D| c\,(d,\alpha) \int_{\mathbb{R}^{2d}}^{\prime\prime} H(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) \frac{Z\,(d\boldsymbol{\lambda}_1)\, Z\,(d\boldsymbol{\lambda}_2)}{\|\boldsymbol{\lambda}_1\|^{\frac{d-\alpha}{2}}\, \|\boldsymbol{\lambda}_2\|^{\frac{d-\alpha}{2}}}, \tag{84}$$

where Z is a complex white noise Gaussian measure with Lebesgue control measure, and the notation $\int_{\mathbb{R}^{2d}}^{\prime\prime}$ means that one does not integrate on the hyperdiagonals $\boldsymbol{\lambda}_1 = \pm\boldsymbol{\lambda}_2$. Here, the kernel H is given by:

$$H\,(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) = K\,(\boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2, D)\,, \tag{85}$$

and $c\,(d,\alpha) = \frac{\Gamma\left(\frac{d-\alpha}{2}\right)}{\pi^{d/2} 2^\alpha \Gamma(\alpha/2)} = \frac{1}{\gamma(\alpha)}$.

Remark 4.1. Our goal in this paper is to focus of the case of Hermite rank $m = 2$, which has very special properties not shared by the higher orders, such as the existence of eigenvalues. We are aware of the extension to all Hermite ranks, as described, for example, in the more general and different approach presented in the monograph by Major (1981).

Proof. (i) From equation (29) and the proof of Theorem 3.1,

$$\begin{aligned} \|1_D\|^2_{\mathcal{H}_{2\alpha-d}} &= \int_D \frac{1}{\gamma(d-2\alpha)} \int_D \frac{1}{\|\mathbf{x}-\mathbf{y}\|^{2\alpha}} d\mathbf{y} d\mathbf{x} \\ &= \frac{a_d^2}{2\gamma(d-2\alpha)} = \frac{1}{\gamma(d-2\alpha)} \sum_{j=1}^{\infty} \lambda_j^2(\mathcal{K}_\alpha^2) = \frac{\mathrm{Tr}(\mathcal{K}_\alpha^2)}{\gamma(d-2\alpha)} < \infty, \end{aligned} \tag{86}$$

since $\mathcal{K}_\alpha^2$ is in the trace class. Therefore, 1_D belongs to the Hilbert space $\mathcal{H}_{2\alpha-d}$ with the inner product introduced in equation (30). From equation (29), we then obtain

$$\frac{a_d^2}{2\gamma(d-2\alpha)} = \|1_D\|^2_{\mathcal{H}_{2\alpha-d}} = \frac{|D|^2}{(2\pi)^d} \int_{\mathbb{R}^d} |K(\boldsymbol{\omega}_1, D)|^2 \|\boldsymbol{\omega}_1\|^{-(d-2\alpha)} d\boldsymbol{\omega}_1.$$

It is well-known that the Fourier transform defines an automorphism on the Schwartz space, which, in particular, contains $C_0^\infty(D)$. Thus, the Fourier transform of any function in the space $\mathcal{H}_{2\alpha-d}$ can be defined as the limit in the space $\mathcal{H}_{2\alpha-d}$ of the Fourier transforms of functions in $C_0^\infty(D)$. Therefore, from equation (28) with $f(\mathbf{z}) = |D|^2 |K(\mathbf{z}, D)|^2$,

$$\begin{aligned} \frac{a_d^2}{2\gamma(d-2\alpha)} &= \|1_D\|^2_{\mathcal{H}_{2\alpha-d}} = \frac{|D|^2}{(2\pi)^d} \int_{\mathbb{R}^d} |K(\boldsymbol{\omega}_1, D)|^2 \|\boldsymbol{\omega}_1\|^{-d+2\alpha} d\boldsymbol{\omega}_1 \\ &= \frac{|D|^2}{(2\pi)^d} \frac{\gamma(2\alpha)}{[\gamma(\alpha)]^2} \int_{\mathbb{R}^d} |K(\boldsymbol{\omega}_1, D)|^2 \left[\int_{\mathbb{R}^d} \|\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2\|^{-d+\alpha} \|\boldsymbol{\omega}_2\|^{-d+\alpha} d\boldsymbol{\omega}_2 \right] d\boldsymbol{\omega}_1 \\ &= \frac{|D|^2 \gamma(2\alpha)}{(2\pi)^d [\gamma(\alpha)]^2} \int_{\mathbb{R}^{2d}} |K(\boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2, D)|^2 \frac{d\boldsymbol{\lambda}_1 d\boldsymbol{\lambda}_2}{(\|\boldsymbol{\lambda}_1\| \|\boldsymbol{\lambda}_2\|)^{d-\alpha}}. \end{aligned}$$

Hence,

$$\frac{a_d^2}{2} = \left[\frac{|D|}{\gamma(\alpha)} \right]^2 \int_{\mathbb{R}^{2d}} |K(\boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2, D)|^2 \frac{d\boldsymbol{\lambda}_1 d\boldsymbol{\lambda}_2}{(\|\boldsymbol{\lambda}_1\| \|\boldsymbol{\lambda}_2\|)^{d-\alpha}}, \tag{87}$$

since $\frac{\gamma(2\alpha)\gamma(d-2\alpha)}{(2\pi)^d} = 1$. Note that, we also have applied the fact that, from Remark 3.1,

$$1_D \star 1_D(\mathbf{x}) = \int_{\mathbb{R}^d} 1_D(\mathbf{y}) 1_D(\mathbf{x}+\mathbf{y}) d\mathbf{y} = \int_D 1_D(\mathbf{x}+\mathbf{y}) d\mathbf{y} \in L^2(D) \subseteq \mathcal{H}_{2\alpha-d},$$

since

$$\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d} 1_D(\mathbf{y})1_D(\mathbf{x}+\mathbf{y})d\mathbf{y}\right|^2 d\mathbf{x} \leq \left|\mathcal{B}_{R(D)}(\mathbf{0})\right|^3,$$

where, as before, $|\mathcal{B}_{R(D)}|$ denotes the Lebesgue measure of the ball of center $\mathbf{0}$ and radius $R(D)$, with $R(D)$ being equal to two times the diameter of the regular compact set D containing the point $\mathbf{0}$. Hence, $\mathcal{F}(1_D \star 1_D)(\boldsymbol{\lambda}) = |D|^2|K(\boldsymbol{\lambda}, D)|^2$ belongs to the space of Fourier transforms of functions in $\mathcal{H}_{2\alpha-d}$. Summarizing, equation (87) provides the finiteness of (82), i.e., assertion (i) holds due to the trace property of $\mathcal{K}_\alpha^2$ for the regular bounded domain D considered (see Theorem 3.1).

(ii) The proof of this part of Theorem 4.1 can be obtained as a particular case of Theorem 5 in Leonenko and Olenko (2014) (see also Remark 6 in that paper). Note that convexity is not used in the proof of Theorem 5 of Leonenko and Olenko (2014). An outline of the proof of Theorem 5 in Leonenko and Olenko (2014) for the case of Hermite rank equal to two is now given.

Under **Conditions A1, A3** (see also (81)),

$$Y(\mathbf{x}) = \frac{|D(T)|}{(2\pi)^d}\int_{\mathbb{R}^d} \exp\left(\mathrm{i}\left\langle \mathbf{x}, \boldsymbol{\lambda}\right\rangle\right) K\left(\boldsymbol{\lambda}, D(T)\right) f_0^{1/2}(\boldsymbol{\lambda})Z(d\boldsymbol{\lambda}), \quad \mathbf{x}\in D(T). \tag{88}$$

Using the self-similarity of Gaussian white noise, and the Itô formula (see, for example, Dobrushin and Major, 1979; Major, 1981), we obtain from equation (88)

$$\begin{aligned} S_T &= = \frac{1}{T^{d-\alpha}\mathcal{L}(T)}\int_{D(T)} H_2(Y(\mathbf{x}))d\mathbf{x} \\ &= \frac{c\,(d,\alpha)\,|D(T)|}{T^{d-\alpha}\mathcal{L}(T)}\int_{\mathbb{R}^{2d}}^{\prime\prime} K\left(\boldsymbol{\lambda}_1+\boldsymbol{\lambda}_2, D(T)\right)\left(\frac{1}{c\,(d,\alpha)}\prod_{j=1}^{2} f_0^{1/2}(\boldsymbol{\lambda}_j)\right) Z\left(d\boldsymbol{\lambda}_1\right) Z\left(d\boldsymbol{\lambda}_2\right) \\ &\underset{d}{=} \frac{c\,(d,\alpha)\,|D|}{T^{d-\alpha}\mathcal{L}(T)}\int_{\mathbb{R}^{2d}}^{\prime\prime} K\left(\boldsymbol{\lambda}_1+\boldsymbol{\lambda}_2, D\right)\left(\frac{1}{c\,(d,\alpha)}\prod_{j=1}^{2} f_0^{1/2}(\boldsymbol{\lambda}_j/T)\right) Z\left(d\boldsymbol{\lambda}_1\right) Z\left(d\boldsymbol{\lambda}_2\right). \end{aligned} \tag{89}$$

By the isometry property of multiple stochastic integrals

$$\begin{aligned} &\mathrm{E}\left[S_T - c\,(d,\alpha)\,|D|\int_{\mathbb{R}^{2d}}^{\prime\prime} H(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2)\frac{Z\left(d\boldsymbol{\lambda}_1\right) Z\left(d\boldsymbol{\lambda}_2\right)}{\|\boldsymbol{\lambda}_1\|^{\frac{d-\alpha}{2}}\,\|\boldsymbol{\lambda}_2\|^{\frac{d-\alpha}{2}}}\right]^2 = \\ &= \int_{\mathbb{R}^{2d}} \left|K\left(\boldsymbol{\lambda}_1+\boldsymbol{\lambda}_2, D\right)\right|^2 [c\,(d,\alpha)\,|D|]^2 Q_T(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2)\frac{d\boldsymbol{\lambda}_1 d\boldsymbol{\lambda}_2}{\|\boldsymbol{\lambda}_1\|^{d-\alpha}\,\|\boldsymbol{\lambda}_2\|^{d-\alpha}}, \end{aligned} \tag{90}$$

where

$$Q_T(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2) = \left(\left[\frac{\|\boldsymbol{\lambda}_1\|^{(d-\alpha)/2}\,\|\boldsymbol{\lambda}_2\|^{(d-\alpha)/2}}{T^{d-\alpha}\mathcal{L}(T)c\,(d,\alpha)}\prod_{j=1}^{2} f_0^{1/2}(\boldsymbol{\lambda}_j/T)\right] - 1\right)^2. \tag{91}$$

From equation (79), under **Condition A3**, we obtain the pointwise convergence of $Q_T(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2)$ to 0, as $T\to\infty$. By Lebesgue's Dominated Convergence Theorem, the integral converges to zero if there is some integrable function which dominates integrands for all T. This fact can be proved as in pp. 21–22 of Leonenko and Olenko (2014), applying previous assertion (i) derived in this theorem. ■

Alternatively, in the proof of Theorem 4.1(ii), the class $\widetilde{\mathcal{LC}}$ of slowly varying functions, introduced in Definition 9 in Leonenko and Olenko (2013), can also be considered. Note that an infinitely differentiable function $\mathcal{L}(\cdot)$ belongs to the class $\widetilde{\mathcal{LC}}$ if

1. for any $\delta > 0$, there exists $\lambda_0(\delta) > 0$ such that $\lambda^{-\delta}\mathcal{L}(\lambda)$ is decreasing and $\lambda^{\delta}\mathcal{L}(\lambda)$ is increasing if $\lambda > \lambda_0(\delta)$;
2. $\mathcal{L}_j \in \mathcal{SL}$, for all $j \geq 0$, where $\mathcal{L}_0(\lambda) := \mathcal{L}$, $\mathcal{L}_{j+1}(\lambda) := \lambda\mathcal{L}_j'(\lambda)$, with $\mathcal{SL}$ being the class of functions that are slowly varying at infinity and bounded on each finite interval.

In that case, the following lemma should be applied for the proof of Theorem 4.1(ii).

Lemma 4.1. *Let $\alpha \in (0, d)$, $S \in C^{\infty}(s_{n-1}(1))$, and $\mathcal{L} \in \widetilde{\mathcal{L}}\mathcal{C}$. Let $\{\xi(\mathbf{x}),\ \mathbf{x} \in \mathbb{R}^d\}$ be a mean-square continuous homogeneous random field with zero mean. Let the field $\xi(\mathbf{x})$ has the spectral density $f_0(\mathbf{u})$, $\mathbf{u} \in \mathbb{R}^d$, which is infinitely differentiable for all $\mathbf{u} \neq 0$. If the covariance function $B(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^d$, of the field has the following behavior*

(a) $\|\mathbf{x}\|^{\alpha} B(\mathbf{x}) \sim S\left(\frac{\mathbf{x}}{\|\mathbf{x}\|}\right)\mathcal{L}(\|\mathbf{x}\|), \quad \mathbf{x} \longrightarrow \infty,$

the spectral density satisfies the condition

(b) $\|\mathbf{u}\|^{d-\alpha} f_0(\mathbf{u}) \sim \widetilde{S}_{\alpha,d}\left(\frac{\mathbf{u}}{\|\mathbf{u}\|}\right)\mathcal{L}\left(\frac{1}{\|\mathbf{u}\|}\right), \quad \|\mathbf{u}\| \longrightarrow 0.$

On the other hand, from Theorems 3.1 and 4.1(i), the spectral asymptotics of $\mathcal{K}_{\alpha}$ and the Dirichlet Laplacian operator on $L^2(D)$ can be applied to verifying the finiteness of (82) for a wide class of compact sets. Drum and fractal drum are two families of well-known regular compact sets where Weyl's classical theorem on the asymptotic behavior of the eigenvalues has been extended (see, for example, Gordon, Webb and Wolpert, 1992; Lapidus, 1991; Triebel, 1997). In particular, as illustration of Theorem 4.1(i), we now refer to the case of regular compact domains constructed from the finite union of convex compact sets like balls, or by their difference which is the case, for instance, of circular rings.

Examples

Let $D = \mathcal{B}_1(\mathbf{0}) \cup \mathcal{B}_1((2,0)) \subset \mathbb{R}^2$, with $\mathcal{B}_1(\mathbf{0}) = \{(x_1, x_2) \in \mathbb{R}^2 :\ \sqrt{x_1^2 + x_2^2} \leq 1\}$, and $\mathcal{B}_1((2,0)) = \{(x_1, x_2) \in \mathbb{R}^2 :\ \sqrt{(x_1 - 2)^2 + x_2^2} \leq 1\}$. It is well-known (see Ivanov and Leonenko, 1989, p. 57, Lemma 2.1.3) that, for $\mathcal{B}_1(\mathbf{0}) \subset \mathbb{R}^2$ and $0 < \alpha < 1$,

$$Tr([\mathcal{K}_{\alpha}^{\mathcal{B}_1(\mathbf{0})}]^2) = \int_{\mathcal{B}_1(\mathbf{0})}\int_{\mathcal{B}_1(\mathbf{0})} \frac{1}{\|\mathbf{x}-\mathbf{y}\|^{2\alpha}} d\mathbf{y}d\mathbf{x} = \frac{2^{2-2\alpha+1}\pi^{2-\frac{1}{2}}\Gamma(\frac{2-2\alpha+1}{2})}{(2-2\alpha)\Gamma(2-\alpha+1)},$$

where, to avoid confusion, for a subset S, we have used the notation $\mathcal{K}_{\alpha}^{S}$ to represent operator $\mathcal{K}_{\alpha}$ acting on the space $L^2(S)$, and $[\mathcal{K}_{\alpha}^{S}]^2 = \mathcal{K}_{\alpha}^{S}\mathcal{K}_{\alpha}^{S}$.

Hence,

$$\begin{aligned} & \int_{\mathcal{B}_1(\mathbf{0})\cup\mathcal{B}_1((2,0))}\int_{\mathcal{B}_1(\mathbf{0})\cup\mathcal{B}_1((2,0))} \frac{1}{\|\mathbf{x}-\mathbf{y}\|^{2\alpha}} d\mathbf{y}d\mathbf{x} \\ \leq & \int_{\mathcal{B}_3(\mathbf{0})}\int_{\mathcal{B}_3(\mathbf{0})} \frac{1}{\|\mathbf{x}-\mathbf{y}\|^{2\alpha}} d\mathbf{y}d\mathbf{x} \\ = & \ \mathrm{Tr}\left([\mathcal{K}_{\alpha}^{\mathcal{B}_3(\mathbf{0})}]^2\right) = 3^{4-2\alpha}\mathrm{Tr}\left([\mathcal{K}_{\alpha}^{\mathcal{B}_1(\mathbf{0})}]^2\right) = \frac{3^{4-2\alpha}2^{2-2\alpha+1}\pi^{2-\frac{1}{2}}\Gamma(\frac{2-2\alpha+1}{2})}{(2-2\alpha)\Gamma(2-\alpha+1)} < \infty. \end{aligned} \tag{92}$$

From Theorem 4.1(i), equation (92) provides the finiteness of (82) for non-convex compact set $D = \mathcal{B}_1(\mathbf{0}) \cup \mathcal{B}_1((2,0))$.

These computations can be easily extended to the finite union of balls with the same or with different radius, and to the case $d > 2$, considering the value of the integral

$$\int_{\mathcal{B}_R(\mathbf{0})}\int_{\mathcal{B}_R(\mathbf{0})} \frac{1}{\|\mathbf{x}-\mathbf{y}\|^{2\alpha}} d\mathbf{y}d\mathbf{x} = R^{2d-2\alpha} a_d(\mathcal{B}_1(\mathbf{0}))\frac{1}{2},$$

where the constant $a_d(\mathcal{B}_1(\mathbf{0}))$ is defined in (47), for $0 < \alpha < d/2$ (see Ivanov and Leonenko, 1989, p. 57, Lemma 2.1.3).

For the case of a circular ring, that is, for

$$D = \mathcal{B}_{R_1}(\mathbf{0})\backslash\mathcal{B}_{R_2}(\mathbf{0}) = \{\mathbf{x} \in \mathbb{R}^2 : \ R_2 < \|\mathbf{x}\| < R_1\}, \quad R_1 > R_2 > 0,$$

we can proceed in a similar way to the above-considered example. Specifically,

$$\begin{aligned} & \int_{\mathcal{B}_{R_1}(\mathbf{0})\backslash\mathcal{B}_{R_2}(\mathbf{0})} \int_{\mathcal{B}_{R_1}(\mathbf{0})\backslash\mathcal{B}_{R_2}(\mathbf{0})} \frac{1}{\|\mathbf{x}-\mathbf{y}\|^{2\alpha}} d\mathbf{y} d\mathbf{x} \\ \leq & \int_{\mathcal{B}_{R_1}(\mathbf{0})} \int_{\mathcal{B}_{R_1}(\mathbf{0})} \frac{1}{\|\mathbf{x}-\mathbf{y}\|^{2\alpha}} d\mathbf{y} d\mathbf{x} \end{aligned}$$

$$= \quad \mathrm{Tr}\left([\mathcal{K}_{\alpha}^{\mathcal{B}_{R_1}(\mathbf{0})}]^2\right) = R_1^{4-2\alpha}\mathrm{Tr}\left([\mathcal{K}_{\alpha}^{\mathcal{B}(\mathbf{0})}]^2\right) = \frac{R_1^{4-2\alpha} 2^{2-2\alpha+1} \pi^{2-\frac{1}{2}} \Gamma(\frac{2-2\alpha+1}{2})}{(2-2\alpha)\Gamma(2-\alpha+1)} < \infty.$$

From Theorem 4.1(i), equation (82) is finite for $D = \mathcal{B}_{R_1}(\mathbf{0})\backslash\mathcal{B}_{R_2}(\mathbf{0})$. Similarly, these computations can be extended to the finite union of circular rings.

Remark 4.2. Note that for a ball $D = \mathcal{B}_1(\mathbf{0}) = \mathcal{B}(\mathbf{0}) = \{\mathbf{x} \in \mathbb{R}^d; \ \|\mathbf{x}\| \leq 1\}$, the function $\vartheta(\boldsymbol{\lambda})$ in (83) is of the form $\int_{\mathcal{B}_1(\mathbf{0})} \exp\left(\mathrm{i}\left\langle \mathbf{x}, \boldsymbol{\lambda}\right\rangle\right) d\mathbf{x} = (2\pi)^{d/2} \frac{\mathcal{J}_{d/2}(\|\boldsymbol{\lambda}\|)}{\|\boldsymbol{\lambda}\|^{d/2}}$, for $d \geq 2$, where $\mathcal{J}_{\nu}(\mathbf{z})$ is the Bessel function of the first kind and order $\nu > -1/2$. For a rectangle, $D = \prod = \{a_i \leq x_i \leq b_i, \ i = 1, \dots, d\}$, with $\mathbf{0} \in \prod$, we have $\vartheta(\boldsymbol{\lambda}) = \prod_{j=1}^{d} \left(\exp\left(\mathrm{i}\lambda_j b_j\right) - \exp\left(\mathrm{i}\lambda_j a_j\right)\right)/\mathrm{i}\lambda_j$, for $d \geq 1$.Moreover for $d = 2$, considering the non-convex set $D = \mathcal{B}_1(\mathbf{0}) \cup \mathcal{B}_1((2,0)) \subset \mathbb{R}^2$,

$$\vartheta(\boldsymbol{\lambda}) = \vartheta(\lambda_1, \lambda_2) = \int_{\mathcal{B}_1(\mathbf{0})\cup\mathcal{B}_1((2,0))} \exp\left(\mathrm{i}\left\langle \mathbf{x}, \boldsymbol{\lambda}\right\rangle\right) d\mathbf{x} = \frac{2\pi\mathcal{J}_1(\|\boldsymbol{\lambda}\|)}{\|\boldsymbol{\lambda}\|}\left(1 + \exp\left(2\mathrm{i}\lambda_1\right)\right),$$

and for $D = \mathcal{B}_{R_1}(\mathbf{0})\backslash\mathcal{B}_{R_2}(\mathbf{0})$, $\vartheta(\boldsymbol{\lambda}) = (2\pi R_1)\mathcal{J}_1\left(\|\boldsymbol{\lambda}\|R_1\right)/\|\boldsymbol{\lambda}\| - (2\pi R_2)\mathcal{J}_1\left(\|\boldsymbol{\lambda}\|R_2\right)/\|\boldsymbol{\lambda}\|$.

The following corollary is an extension of Proposition 2 in Dobrushin and Major (1979).

Corollary 4.1. *Assume that the conditions of Theorem 4.1 hold. Then, the limit random variable S_{∞} admits the following series representation:*

$$S_{\infty} \underset{d}{=} c(d,\alpha)|D| \sum_{\mathbf{n}\in\mathbb{N}_*^d} \mu_{\mathbf{n}}(\mathcal{H})(\varepsilon_{\mathbf{n}}^2 - 1) = \sum_{\mathbf{n}\in\mathbb{N}_*^d} \lambda_{\mathbf{n}}(S_{\infty})(\varepsilon_{\mathbf{n}}^2 - 1), \tag{93}$$

where

$$c(d,\alpha) = \frac{\Gamma\left(\frac{d-\alpha}{2}\right)}{\pi^{d/2} 2^{\alpha} \Gamma(\alpha/2)} = \frac{1}{\gamma(\alpha)}$$

was already introduced in (24), $\varepsilon_{\mathbf{n}}$ are independent and identically distributed standard Gaussian random variables, and $\mu_{\mathbf{n}}(\mathcal{H})$, $\mathbf{n} \in \mathbb{N}_*^d$, is a sequence of non-negative real numbers, which are the eigenvalues of the self-adjoint Hilbert-Schmidt operator

$$\mathcal{H}(h)(\boldsymbol{\lambda}_1) = \int_{\mathbb{R}^d} H_1\left(\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2\right) h\left(\boldsymbol{\lambda}_2\right) G_{\alpha}(d\boldsymbol{\lambda}_2) : L_E^2(\mathbb{R}^d, G_{\alpha}) \longrightarrow L_E^2(\mathbb{R}^d, G_{\alpha}), \tag{94}$$

with

$$G_{\alpha}(d\mathbf{x}) = \frac{1}{\|\mathbf{x}\|^{d-\alpha}} d\mathbf{x}, \tag{95}$$

and $L^2_E(\mathbb{R}^d, G_\alpha)$ denotes the collection of linear combinations with real-valued coefficients of complex-valued and Hermitian functions that are square integrable with respect to $G_\alpha(d\mathbf{x})$. Note that $L^2_E(\mathbb{R}^d, G_\alpha)$ is a real Hilbert space, endowed with the scalar product

$$\langle \psi_1, \psi_2 \rangle_{G_\alpha} = \int_{\mathbb{R}^d} \psi_1(\mathbf{x})\overline{\psi_2(\mathbf{x})} G_\alpha(d\mathbf{x})$$

(see Peccati and Taqqu, 2011, pp. 159-161, for the case of $L^2_E(\mathbb{R}, d\beta)$ spaces). The symmetric kernel $H_1(\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2) = H(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2)$, is defined from H introduced in equation (85), in terms of the characteristic function K given in equation (83).

The proof can be derived as in Proposition 2 of Dobrushin and Major (1979), replacing the cube in $\mathbb{R}^d$ by a regular compact domain D, since Theorem 4.1(i) provides the equality between the traces of operators $\frac{\mathcal{K}^2_\alpha}{[|D|c(d,\alpha)]^2}$ and $\mathcal{H}^2$, with, as before, $\mathcal{H}$ having kernel $H(\cdot, \cdot)$ given in equation (85) (see also Appendix A).

In the following proposition the explicit relationship between the eigenvalues of $\mathcal{K}_\alpha$ and $\mathcal{H}$ is derived.

Proposition 4.1. *The operators $\mathcal{A}_\alpha : L^2_E(\mathbb{R}^d, G_\alpha) \longrightarrow L^2_E(\mathbb{R}^d, G_\alpha)$*

$$\mathcal{A}_\alpha(f)(\boldsymbol{\lambda}_1) = c(d,\alpha) \int_{\mathbb{R}^d} H_1(\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2) f(\boldsymbol{\lambda}_2) G_\alpha(d\boldsymbol{\lambda}_2),$$

and $|D|^{-1}\mathcal{K}_\alpha : L^2(D) \longrightarrow L^2(D)$ have the same eigenvalues. Here, $c(d,\alpha)$ was already introduced in (24), $H_1(\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2) = H(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2)$ with kernel H being given in equation (85), G_α is introduced in (95), *and $\mathcal{K}_\alpha$ is defined in (22).*

The proof of this result is given in Appendix A. (See Veillette and Taqqu, 2013, for $d = 1$).

Corollary 4.2. *Let $\{\lambda_k(S_\infty),\ k \geq 1\}$ be the eigenvalues appearing in representation (93), arranged into a decreasing order of their modulus magnitudes. Then, Theorem 3.1 holds for this system of eigenvalues.*

The proof directly follows from Corollary 4.1, Proposition 4.1 and Theorem 3.1.

5 Properties of Rosenblatt-type distribution

This section provides the Lévy-Khintchine representation of the limit random variable S_∞ (see Veillette and Taqqu, 2013, for $d = 1$, in the discrete time case), as well as its membership to a subclass of selfdecomposable distributions, given by the Thorin class. The absolute continuity of the law of S_∞, and the boundedness of its probability density is then obtained.

It is well-known that the distribution of a random variable X is infinitely divisible if for any integer $n \geq 1$, there exist $X_j^{(n)}$, $j = 1, 2, \ldots, n$, independent and identically distributed (i.i.d.) random variables such that $X \underset{d}{=} X_1^{(n)} + \cdots + X_n^{(n)}$. Let $\mathcal{ID}(\mathbb{R})$ be the class of infinitely divisible distributions or random variables. Recall that the cumulant function of an infinitely divisible random variable X admits the Lévy-Khintchine representation

$$\log \mathrm{E}\left[\exp\left(\mathrm{i}\theta X\right)\right] = \mathrm{i}a\theta - \frac{b}{2}\theta^2 + \int_{-\infty}^{\infty} (\exp(\mathrm{i}\theta u) - 1 - \mathrm{i}\tau(u)\theta)\mu(du), \quad \theta \in \mathbb{R}, \tag{96}$$

where $a \in \mathbb{R}$, $b \geq 0$, and

$$\tau(u) = \begin{cases} u & |u| \leq 1 \\ \frac{u}{|u|} & |u| > 1, \end{cases} \tag{97}$$

and where the Lévy measure μ is a Radon measure on $\mathbb{R} \setminus \{0\}$ such that $\mu(\{0\}) = 0$ and

$$\int \min(u^2, 1)\mu(du) < \infty.$$

An infinitely divisible random variable X (or its law) is selfdecomposable if its characteristic function $\phi(\theta) = E[\mathrm{i}\theta X]$, $\theta \in \mathbb{R}$, has the property that for every $c \in (0,1)$ there exists a characteristic function ϕ_c such that $\phi(\theta) = \phi(c\theta)\phi_c(\theta)$, $\theta \in \mathbb{R}$. It is known (see Sato, 1999, p.95, Corollary 15.11) that an infinitely divisible law is selfdecomposable if its Lévy measure has a density q satisfying

$$q(u) = \frac{h(u)}{|u|}, \quad u \in \mathbb{R},$$

with $h(u)$ being increasing on $(-\infty, 0)$ and decreasing on $(0,\infty)$. Let $\mathcal{SD}(\mathbb{R})$ be the class of selfdecomposable distributions or random variables. If $Y \in \mathcal{SD}(\mathbb{R})$ then (see Jurek and Vervaat, 1983)

$$Y \underset{d}{=} \int_0^\infty \exp(-s)dZ(s) \underset{d}{=} \int_0^\infty \exp(-s\lambda)dZ(s\lambda), \quad \lambda > 0, \tag{98}$$

where $\{Z(t),\ t \geq 0\}$ is a Lévy process whose law is determined by that of Y.

We next define the Thorin class on $\mathbb{R}$ (see Thorin, 1978; Bandorff-Nielsen *et al.*, 2006; James *et al.*, 2008) as follows: We refer to γx as an *elementary gamma random variable* if x is nonrandom non-zero vector in $\mathbb{R}$, and γ is a gamma random variable on $\mathbb{R}_+$. Then, the Thorin class on $\mathbb{R}$ (or the class of extended generalized gamma convolutions), denoted by $T(\mathbb{R})$, is defined as the smallest class of distributions that contains all elementary gamma distributions on $\mathbb{R}$, and is closed under convolution and weak convergence. It is known that $T(\mathbb{R}) \subset \mathcal{SD}(\mathbb{R}) \subset \mathcal{ID}(\mathbb{R})$, and inclusions are strict. Since any selfdecomposable distribution on $\mathbb{R}$ is absolutely continuous (see, for instance, Example 27.8 of Sato, 1999) and is unimodal (by Yamazato, 1978; see also Theorem 53.1 of Sato, 1999), then, any selfdecomposable distribution has a bounded density function.

If a probability distribution function F belongs to $T(\mathbb{R})$, then, its characteristic function has the form (see Thorin, 1978, Barndorff-Nielsen *et al*, 2006)

$$\phi(\theta) = \exp\left(\mathrm{i}\theta a - \frac{b\theta^2}{2} - \int_{\mathbb{R}} \left[\log\left(1 - \frac{\mathrm{i}\theta}{u}\right) + \frac{\mathrm{i}u\theta}{1+u^2}\right] U(du)\right), \tag{99}$$

where $a \in \mathbb{R}$, $b \geq 0$, and $U(du)$ is a non-decreasing measure on $\mathbb{R}\backslash\{0\}$, called Thorin measure, such that

$$U(0) = 0, \quad \int_{-1}^{1} |\log|u||\, U(du) < \infty, \quad \int_{-\infty}^{-1} \frac{1}{u^2} U(du) + \int_1^\infty \frac{1}{u^2} U(du) < \infty.$$

The Lévy density of a distribution from Thorin class is such that

$$|u|q(u) = \begin{cases} \int_0^\infty \exp(-yu)U(dy), & u > 0 \\ \int_0^\infty \exp(yu)U(dy), & u < 0, \end{cases} \tag{100}$$

where $U(du)$ is the Thorin measure. In other words, the Lévy density is of the form $h(|u|)/|u|$, where $h(|u|) = h_0(r)$, $r \geq 0$, is a completely monotone function over $(0,\infty)$.

The following result establishes the Lévy-Khintchine representation of S_∞, as well as the asymptotic orders at zero and at infinity of its associated Lévy density. The membership to the Thorin self-decomposable subclass is then obtained. As a direct consequence, we then have the existence and boundedness of the probability density of S_∞ (see, for instance, Example 27.8 of Sato, 1999).

Theorem 5.1. *Let S_∞ be given as in Theorem 3.2 with $0 < \alpha < d/2$. Let us consider $\lambda_k(S_\infty)$, $k \geq 1$, the sequence of eigenvalues introduced in Corollary 4.1 satisfying the properties stated in Theorem 3.1 (see Corollary 4.2). Then,*

(i) $S_\infty \in \mathcal{ID}(\mathbb{R})$ with the following Lévy-Khintchine representation:

$$\phi(\theta) = E[\mathrm{i}\theta S_\infty] = \exp\left(\int_0^\infty (\exp(\mathrm{i}u\theta) - 1 - \mathrm{i}u\theta)\, \mu_{\alpha/d}(du)\right), \tag{101}$$

where $\mu_{\alpha/d}$ is supported on $(0,\infty)$ having density

$$q_{\alpha/d}(u) = \frac{1}{2u}\sum_{k=1}^{\infty}\exp\left(-\frac{u}{2\lambda_k(S_\infty)}\right), \quad u > 0. \tag{102}$$

Furthermore, $q_{\alpha/d}$ has the following asymptotics as $u \longrightarrow 0^+$ and $u \longrightarrow \infty$,

$$\begin{aligned} q_{\alpha/d}(u) &\sim \frac{[\widetilde{c}(d,\alpha)|D|^{(d-\alpha)/d}]^{1/(1-\alpha/d)}\Gamma\left(\frac{1}{1-\alpha/d}\right)\left(\frac{u}{2}\right)^{-1/(1-\alpha/d)}}{2u[(1-\alpha/d)]} \\ &= \frac{2^{\frac{\alpha/d}{1-\alpha/d}}[\widetilde{c}(d,\alpha)|D|^{(d-\alpha)/d}]^{1/(1-\alpha/d)}\Gamma\left(\frac{1}{1-\alpha/d}\right)u^{\frac{(\alpha/d)-2}{(1-\alpha/d)}}}{[(1-\alpha/d)]} \quad \text{as } u \longrightarrow 0^+, \\ q_{\alpha/d}(u) &\sim \frac{1}{2u}\exp(-u/2\lambda_1(S_\infty)), \quad \text{as } u \longrightarrow \infty, \end{aligned} \tag{103}$$

where $\widetilde{c}(d,\alpha)$ is defined as in equation (34).

(ii) *$S_\infty \in \mathcal{SD}(\mathbb{R})$, and hence it has a bounded density.*

(iii) *$S_\infty \in T(\mathbb{R})$, with Thorin measure given by*

$$U(dx) = \frac{1}{2}\sum_{k=1}^{\infty}\delta_{\frac{1}{2\lambda_k(S_\infty)}}(x),$$

where $\delta_a(x)$ is the Dirac delta-function at point a.

(iv) *S_∞ admits the integral representation*

$$S_\infty \underset{d}{=} \int_0^\infty \exp(-u)\, d\left(\sum_{k=1}^{\infty}\lambda_k(S_\infty)A^{(k)}(u)\right) \underset{d}{=} \int_0^\infty \exp(-u)\, dZ(u), \tag{104}$$

where

$$Z(t) = \sum_{k=1}^{\infty}\lambda_k(S_\infty)A^{(k)}(t), \quad t \geq 0, \tag{105}$$

with $A^{(k)}$, $k \geq 1$, being independent copies of a Lévy process.

Proof. (i) The proof follows from Theorem 3.1, equation (33), Corollary 4.2, and Lemma 6.1 below (see Appendix B), in a similar way to Theorem 4.2 of Veillette and Taqqu (2013). Specifically, let us first consider a truncated version of the random series representation (93)

$$S_\infty^{(M)} = \sum_{k=1}^{M}\lambda_k(S_\infty)(\varepsilon_k^2 - 1),$$

with $S_\infty^M \underset{d}{\longrightarrow} S_\infty$, as M tends to infinity. From the Lévy-Khintchine representation of the chi-square distribution (see, for instance, Applebaum, 2004, Example 1.3.22),

$$\begin{aligned} E\left[\exp(\mathrm{i}\theta S_\infty^{(M)})\right] &= \prod_{k=1}^{M} E\left[\exp\left(\mathrm{i}\theta\lambda_k(S_\infty)(\varepsilon_k^2-1)\right)\right] \\ &= \prod_{k=1}^{M}\exp\left(-\mathrm{i}\theta\lambda_k(S_\infty) + \int_0^\infty (\exp(\mathrm{i}\theta u)-1)\left[\frac{\exp(-u/(2\lambda_k(S_\infty)))}{2u}\right]du\right) \\ &= \prod_{k=1}^{M}\exp\left(\int_0^\infty (\exp(\mathrm{i}\theta u)-1-\mathrm{i}\theta u)\left[\frac{\exp(-u/2\lambda_k(S_\infty))}{2u}\right]du\right) \\ &= \exp\left(\int_0^\infty (\exp(\mathrm{i}\theta u)-1-\mathrm{i}\theta u)\left[\frac{1}{2u}G_{\lambda(\alpha/d)}^{(M)}(\exp(-u/2))\right]du\right), \end{aligned} \tag{106}$$

where $G^{(M)}_{\lambda(\alpha/d)}(x) = \sum_{k=1}^{M} x^{[\lambda_k(S_\infty)]^{-1}}$. To apply the Dominated Convergence Theorem, the following upper bound is used:

$$
\begin{aligned}
\left|(\exp(\mathrm{i}\theta u) - 1 - \mathrm{i}\theta u)\left[\frac{1}{2u}G^{(M)}_{\lambda(\alpha/d)}(\exp(-u/2))\right]\right| &\leq \frac{\theta^2}{4}uG^{(M)}_{\lambda(\alpha/d)}(\exp(-u/2)) \\
&\leq \frac{\theta^2}{4}uG_{\lambda(\alpha/d)}(\exp(-u/2)),
\end{aligned}
\tag{107}
$$

where, as indicated in Veillette and Taqqu (2013), we have applied the inequality $|\exp(\mathrm{i}z)-1-z| \leq \frac{z^2}{2}$, for $z \in \mathbb{R}$. The right-hand side of (107) is continuous, for $0 < u < \infty$, and from Theorem 3.1, equation (33), Corollary 4.2, and Lemma 6.1 in Appendix B, we obtain

$$
\begin{aligned}
uG_{\lambda(\alpha/d)}(\exp(-u/2)) &\sim u\exp(-u/2\lambda_1(S_\infty)), \quad \text{as } u \longrightarrow \infty \\
uG_{\lambda(\alpha/d)}(\exp(-u/2)) &\sim [\widetilde{c}(d,\alpha)|D|^{1-\alpha/d}]^{1/1-\alpha/d}\frac{u}{(1-\alpha/d)} \\
&\quad \Gamma\left(\frac{1}{1-\alpha/d}\right)(1-\exp(-u/2))^{-1/(1-\alpha/d)} \\
&\sim Cu^{-\frac{\alpha/d}{1-\alpha/d}} \quad \text{as } u \longrightarrow 0,
\end{aligned}
\tag{108}
$$

for some constant C. Since $0 < \frac{\alpha/d}{1-\alpha/d} < 1$, equation (108) implies that the right-hand side of (107), which does not depend on M, is integrable on $(0,\infty)$. Hence, by the Dominated Convergence Theorem, as $M \to \infty$,

$$
\begin{aligned}
&E\left[\exp(\mathrm{i}\theta S^{(M)}_\infty)\right] \longrightarrow E\left[\exp(\mathrm{i}\theta S_\infty)\right] \\
&= \exp\left(\int_0^\infty (\exp(\mathrm{i}\theta u) - 1 - \mathrm{i}\theta u)\left[\frac{1}{2u}G_{\lambda(\alpha/d)}(\exp(-u/2))\right]du\right),
\end{aligned}
\tag{109}
$$

which proves that equations (101) and (102) hold.

Again, from Theorem 3.1, equation (33), Corollary 4.2, and Lemma 6.1 below,

$$
\begin{aligned}
&\frac{1}{2u}G_{\lambda(\alpha/d)}(\exp(-u/2)) \sim [\widetilde{c}(d,\alpha)|D|^{1-\alpha/d}]^{1/(1-\alpha/d)}\frac{\Gamma\left(\frac{1}{1-\alpha/d}\right)\left(\frac{u}{2}\right)^{-1/(1-\alpha/d)}}{2u[(1-\alpha/d)]} \\
&= \frac{2^{\frac{\alpha/d}{1-\alpha/d}}[\widetilde{c}(d,\alpha)|D|^{1-\alpha/d}]^{1/(1-\alpha/d)}\Gamma\left(\frac{1}{1-\alpha/d}\right)u^{\frac{(\alpha/d)-2}{(1-\alpha/d)}}}{[(1-\alpha/d)]} \qquad \text{as } u \longrightarrow 0 \\
&\frac{1}{2u}G_{\lambda(\alpha/d)}(\exp(-u/2)) \sim \frac{1}{2u}\exp(-u/2\lambda_1(S_\infty)) \quad \text{as } u \longrightarrow \infty.
\end{aligned}
\tag{110}
$$

Thus, equation (110) provides the asymptotic orders given in (103).

(ii) From (i), it follows that $S_\infty \in \mathcal{SD}(\mathbb{R})$, and hence it has a bounded density (see Bondesson, 1992, Example 27.8 of Sato, 1999 and Yamazato, 1978). Note that an alternative proof of the boundedness of the probability density of S_∞ is provided in Appendix C, where an upper bound is also obtained.

(iii) In view of (100) and (102), $S_\infty \in T(\mathbb{R})$ with Thorin measure given by

$$
U(dx) = \frac{1}{2}\sum_{k=1}^{\infty}\delta_{\frac{1}{2\lambda_k(S_\infty)}}(x),
\tag{111}
$$

where $\delta_a(x)$ is the Dirac delta-function at point a. From Theorem 3.1, Corollaries 4.1 and Proposition 4.1 (see also Corollary 4.2), the number of terms in the sum (111) is infinite. Hence, the Thorin measure $U(dx)$, as a counting measure, has infinite total mass. The form of Thorin measure is a direct consequence of (100) and (102).

(iv) As in Maejima and Tudor (2013), we consider a gamma subordinator $\gamma_\lambda(t)$, $t \geq 0$, with parameter $\lambda > 0$, that is, a Lévy process such that $\gamma_\lambda(0) = 0$, and $P\{\gamma_\lambda(t) \in dx\} = \lambda^{-t}\Gamma^{-1}(t)\exp(-x\lambda)x^{t-1}dx$, $x > 0$, and a homogeneous Poisson process $N(t)$, $t \geq 0$, with unit rate. Assume that the two processes are independent. Then (see Aoyama *et al.*, 2011), for any $c > 0$, and $\lambda > 0$, the Jurek representation (98) can be specified as follows:

$$\gamma_\lambda(c) \underset{d}{=} \int_0^\infty \exp(-t)\, d\gamma_\lambda(N(ct)).$$

The process $A(t) = \gamma_{1/2}(N(t/2)) - t$, $t \geq 0$, is a Lévy process.

For $k \geq 1$, let us consider $\gamma_{\frac{1}{2}}^{(k)}\left(\frac{1}{2}\right)$ and $A^{(k)}(t)$ to be independent copies of $\gamma_{\frac{1}{2}}\left(\frac{1}{2}\right)$ and $A(t)$, respectively. Then, we have

$$\varepsilon_k^2 - 1 \underset{d}{=} \gamma_{\frac{1}{2}}^{(k)}\left(\frac{1}{2}\right) \underset{d}{=} \int_0^\infty \exp(-u)\, dA^{(k)}(u),$$

where ε_k are independent and identically distributed standard normal random variables as given in the series expansion (93). Then, for $\lambda_k(S_\infty)$, $k \geq 1$, being the eigenvalues appearing in such a series expansion, arranged into a decreasing order of their magnitudes, we obtain that the distribution of S_∞ admits the integral representation (104), with, A^k, $k \geq 1$, in equation (105) being independent copies of the Lévy process $A(t) = \gamma_{1/2}(N(t/2)) - t$, $t \geq 0$.

■

For any $0 < \alpha/d < 1/2$, the Lévy measure $\mu_{\alpha/d}$ satisfies

$$\int_0^\infty u^2 \mu_{\alpha/d}(du) = \mathrm{E}[S_\infty^2] = [a_d(D)]^2.$$

Furthermore, when $\alpha/d \longrightarrow 1/2$, since $(\exp(\mathrm{i}\theta u) - 1 - \mathrm{i}\theta u) \to (-1/2)\theta^2$ (see Veillette and Taqqu, 2013), we have

$$\phi(\theta) = \exp\left(\int_0^\infty \frac{\exp(\mathrm{i}\theta u) - 1 - \mathrm{i}\theta u}{u^2} u^2 \mu_{\alpha/d}(du)\right) \longrightarrow \exp\left(-\frac{1}{2}\theta^2\right),$$

which means that $S_\infty \longrightarrow N(0,1)$.

In addition, from Theorem 5.1, it can be proved, in a similar way to Corollary 4.3 and 4.4 of Veillette and Taqqu (2013), that, for $0 < \alpha/d < 1/2$, the probability density function of S_∞ is infinitely differentiable with all derivatives tending to 0 as $|x| \longrightarrow \infty$. Also, the following inequality holds

$$P[S_\infty < -x] \leq \exp\left(-\frac{1}{2}x^2\right), \quad x > 0.$$

We also note that, for $\epsilon > 0$,

$$\lim_{u\to\infty} \frac{P[S_\infty > u + \epsilon]}{P[S_\infty > u]} = \exp\left(-\frac{\epsilon}{2\lambda_1(S_\infty)}\right).$$

Remark 5.1. In view of the integral representation (104), one can construct an Ornstein-Uhlenbeck type process

$$dS(t) = -\lambda S(t) + dL(\lambda t), \quad t \geq 0, \quad \lambda > 0,$$

driven by a Lévy process $L(t)$, $t \geq 0$, and with marginal Rosenblatt distribution S_∞. The driving process $L(t)$ is referred to as the background Lévy process, and it is introduced in (105).

6 Appendices

Appendix A

Proof of Corollary 4.1

From condition (82), operator $\mathcal{H}$ is a Hilbert-Schmidt operator from $L^2_E(\mathbb{R}^d, G_\alpha)$ into $L^2_E(\mathbb{R}^d, G_\alpha)$, which admits a spectral decomposition, in terms of a sequence of eigenvalues $\{\mu_{\mathbf{n}}(\mathcal{H}),\ \mathbf{n} \in \mathbb{N}^d_*\}$, and a complete orthonormal system of eigenvectors $\{\varphi_{\mathbf{n}},\ \mathbf{n} \in \mathbb{N}^d_*\}$ of $L^2_E(\mathbb{R}^d, G_\alpha)$, as follows:

$$H_1(\mathbf{x}-\mathbf{y}) = H(\mathbf{x},\mathbf{y}) = \sum_{\mathbf{n}\in\mathbb{N}^d_*} \mu_{\mathbf{n}}(\mathcal{H})\varphi_{\mathbf{n}}(\mathbf{x})\overline{\varphi_{\mathbf{n}}(\mathbf{y})}, \tag{112}$$

where convergence holds in the $L^2_E(\mathbb{R}^d, G_\alpha) \otimes L^2_E(\mathbb{R}^d, G_\alpha)$ sense, i.e.,

$$\left\| H(\mathbf{x},\mathbf{y}) - \sum_{\mathbf{n}\in\mathbb{N}^d_*} \mu_{\mathbf{n}}(\mathcal{H})\varphi_{\mathbf{n}} \otimes \overline{\varphi}_{\mathbf{n}} \right\|^2_{L^2_E(\mathbb{R}^d,G_\alpha)\otimes L^2_E(\mathbb{R}^d,G_\alpha)} = 0. \tag{113}$$

We can establish the following isometry $\widehat{\mathcal{I}}_2$ between the Hilbert space $L^2_E(\mathbb{R}^d, G_\alpha)\otimes L^2_E(\mathbb{R}^d, G_\alpha)$, and the two-Wiener chaos of the isonormal process X on $H = L^2_E(\mathbb{R}^d, G_\alpha)$, given by

$$X : h \in L^2_E(\mathbb{R}^d, G_\alpha) \longrightarrow X(h) = \int'_{\mathbb{R}^d} h(\mathbf{x}) \frac{Z\,(d\boldsymbol{x})}{\|\mathbf{x}\|^{(d-\alpha)/2}} \tag{114}$$

(see Peccati and Taqqu, 2011, Chapter 9), considering the following identification between orthonormal bases of both spaces: For a given orthonormal basis $\{\varphi_{\mathbf{n}}\otimes\overline{\varphi_{\mathbf{n}}},\ \mathbf{n} \in \mathbb{N}^d_*\}$ of $L^2_E(\mathbb{R}^d, G_\alpha)\otimes L^2_E(\mathbb{R}^d, G_\alpha)$, its image by such an isometry $\widehat{\mathcal{I}}_2$ is defined as

$$\widehat{\mathcal{I}}_2(\varphi_{\mathbf{n}} \otimes \overline{\varphi_{\mathbf{n}}}) = \int''_{\mathbb{R}^{2d}} \left[\varphi_{\mathbf{n}}(\mathbf{x}_1)\overline{\varphi_{\mathbf{n}}(\mathbf{x}_2)}\right] \frac{Z\,(d\mathbf{x}_1)}{\|\mathbf{x}_1\|^{(d-\alpha)/2}} \frac{Z\,(d\mathbf{x}_2)}{\|\mathbf{x}_2\|^{(d-\alpha)/2}}, \tag{115}$$

which also defines an orthonormal basis in the two-Wiener chaos of the isonormal process X in (114).

From equation (113), by Parseval identity, for the orthonormal basis $\varphi_n \otimes \overline{\varphi}_n$ of $L^2_E(\mathbb{R}^d, G_\alpha)\otimes L^2_E(\mathbb{R}^d, G_\alpha)$, constructed from the eigenvectors of integral operator $\mathcal{H}$ with kernel H, we obtain

$$\begin{aligned}\langle H, \varphi_{\mathbf{k}} \otimes \overline{\varphi}_{\mathbf{k}}\rangle_{L^2_E(\mathbb{R}^d,G_\alpha)\otimes L^2_E(\mathbb{R}^d,G_\alpha)} &= \left\langle \sum_{\mathbf{n}\in\mathbb{N}^d_*} \mu_{\mathbf{n}}(\mathcal{H})\varphi_{\mathbf{n}} \otimes \overline{\varphi}_{\mathbf{n}}, \varphi_{\mathbf{k}} \otimes \overline{\varphi_{\mathbf{k}}} \right\rangle_{L^2_E(\mathbb{R}^d,G_\alpha)\otimes L^2_E(\mathbb{R}^d,G_\alpha)} \\ &= \mu_{\mathbf{k}}(\mathcal{H}), \quad \forall \mathbf{k} \in \mathbb{N}^d_*. \end{aligned} \tag{116}$$

From equation (116), using isometry $\widehat{\mathcal{I}}_2$ in (115)

$$\begin{aligned} &\int''_{\mathbb{R}^{2d}} H(\mathbf{x}_1,\mathbf{x}_2)\frac{Z\,(d\mathbf{x}_1)}{\|\mathbf{x}_1\|^{(d-\alpha)/2}} \frac{Z\,(d\mathbf{x}_2)}{\|\mathbf{x}_2\|^{(d-\alpha)/2}} \\ = &\sum_{\mathbf{n}\in\mathbb{N}^d_*} \mu_{\mathbf{n}}(\mathcal{H}) \int''_{\mathbb{R}^{2d}} \left[\varphi_{\mathbf{n}}(\mathbf{x}_1)\overline{\varphi_{\mathbf{n}}(\mathbf{x}_2)}\right] \frac{Z\,(d\mathbf{x}_1)}{\|\mathbf{x}_1\|^{(d-\alpha)/2}} \frac{Z\,(d\mathbf{x}_2)}{\|\mathbf{x}_2\|^{(d-\alpha)/2}} \\ = &\sum_{\mathbf{n}\in\mathbb{N}^d_*} \mu_{\mathbf{n}}(\mathcal{H}) H_2\left(\int_{\mathbb{R}^d} \varphi_{\mathbf{n}}(\mathbf{x})\frac{Z\,(d\mathbf{x})}{\|\mathbf{x}\|^{(d-\alpha)/2}}\right), \end{aligned} \tag{117}$$

where H_2 denotes, as before, the second Hermite polynomial. Note that summation and integration can be swapped, in view of the convergence of the series (112) in the space $L^2_E(\mathbb{R}^d, G_\alpha)\otimes$

$L^2_E(\mathbb{R}^d, G_\alpha)$, and the referred isometry between $L^2_E(\mathbb{R}^d, G_\alpha) \otimes L^2_E(\mathbb{R}^d, G_\alpha)$ and the two-Wiener chaos of isonormal process X introduced in (114) (see also equations (113)–(116)).

The random variables

$$\int_{\mathbb{R}^{2d}}^{\prime\prime} \varphi_{\mathbf{n}}(\mathbf{x}) \frac{Z(d\boldsymbol{x})}{\|\mathbf{x}\|^{(d-\alpha)/2}}, \quad \mathbf{n} \in \mathbb{N}^d_*,$$

with zero mean and variance $\int_{\mathbb{R}^{2d}} |\varphi_{\mathbf{n}}(\mathbf{x})|^2 G_\alpha(d\mathbf{x})$ are jointly Gaussian and are independent, due to the orthogonality of the functions $\varphi_{\mathbf{n}}$, $\mathbf{n} \in \mathbb{N}^d_*$, in the space $L^2_E(\mathbb{R}^d, G_\alpha)$. From equations (84) and (117),

$$S_\infty \underset{d}{=} c(d,\alpha)|D| \sum_{\mathbf{n} \in \mathbb{N}_*} \mu_{\mathbf{n}}(\mathcal{H})(\varepsilon^2_{\mathbf{n}} - 1).$$

Equation (93) is then obtained by setting $\lambda_{\mathbf{n}}(S_\infty) = c(d,\alpha)|D|\mu_{\mathbf{n}}(\mathcal{H})$.

Proof of Proposition 4.1

Let us consider $\mathcal{F}$ and $\mathcal{F}^{-1}$ the Fourier and inverse Fourier transforms respectively defined on $L^1(\mathbb{R}^d)$ and $L^2(\mathbb{R}^d)$. Consider an eigenpair (μ, h) of the operator $\mathcal{A}_\alpha$, we have that $\int_{\mathbb{R}^d} |h(\mathbf{y})|^2 \frac{1}{\|\mathbf{y}\|^{d-\alpha}} < \infty$. Applying the inverse Fourier transform $\mathcal{F}$ to both sides of the identity

$$\mu h = \mathcal{A}_\alpha h,$$

we get

$$\mu \mathcal{F}^{-1}(h) = \mathcal{F}^{-1}(\mathcal{A}_\alpha h) = c(d,\alpha)\mathcal{F}^{-1}(H_1 * H_2),$$

where, as before,

$$H_1(\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2) = H(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2),$$

with kernel H being defined in equation (85), and $H_2(\mathbf{y}) = \|\mathbf{y}\|^{-d+\alpha} h(\mathbf{y})$. In the computation of this inverse Fourier transform, we note that $H_1 \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. In order to apply the convolution theorem, we first perform the following decomposition:

$$H_2(\mathbf{y}) = \|\mathbf{y}\|^{-d+\alpha} h(\mathbf{y}) \mathbf{1}_{\mathcal{B}_1(\mathbf{0})}(\mathbf{y}) + \|\mathbf{y}\|^{-d+\alpha} h(\mathbf{y}) \mathbf{1}_{\mathbb{R}^d \setminus \mathcal{B}_1(\mathbf{0})}(\mathbf{y}) := H_2^-(\mathbf{y}) + H_2^+(\mathbf{y}),$$

where $\mathcal{B}_1(\mathbf{0})$ denotes, as before, the ball with center zero and radius one in $\mathbb{R}^d$. Since

$$\int_{\mathbb{R}^d} h^2(\mathbf{y}) \|\mathbf{y}\|^{-d+\alpha} d\mathbf{y} < \infty,$$

$H_2^- \in L^1(\mathbb{R}^d)$, and $H_2^+ \in L^2(\mathbb{R}^d)$. Applying the linearity of the convolution and Fourier transform, the convolution theorem for both L^1 and L^2 functions (see Triebel, 1978, and Stade, 2005) leads to

$$\begin{aligned} \mu \mathcal{F}^{-1}(h) &= c(d,\alpha)\mathcal{F}^{-1}(H_1 * H_2) = c(d,\alpha)\left[\mathcal{F}^{-1}(H_1 * H_2^-) + \mathcal{F}^{-1}(H_1 * H_2^+)\right] \\ &= c(d,\alpha)|D|^{-1}\mathbf{1}_D(\mathcal{F}^{-1}(H_2^- + H_2^+)) = c(d,\alpha)|D|^{-1}\mathbf{1}_D \mathcal{F}^{-1} H_2, \end{aligned} \tag{118}$$

where we have considered equations (83) and (85). From (118), we can see that the support of $\mathcal{F}^{-1}(h)$ is contained in D, for any eigenfunction h of $\mathcal{A}_\alpha$. The convolution theorem for generalized functions (see Triebel, 1978) can be applied again to H_2, since h has compact support. By (95), $G_\alpha(d\mathbf{x}) = g_\alpha(\mathbf{x})d\mathbf{x}$, with $g_\alpha(\mathbf{x}) = \|\mathbf{x}\|^{-d+\alpha}$. Then,

$$h(\mathbf{y})\|\mathbf{y}\|^{-d+\alpha} = \mathcal{F}\left(\mathcal{F}^{-1}(h) * \mathcal{F}^{-1}(g_\alpha)\right)(\mathbf{y}).$$

Therefore, in equation (118), we obtain

$$\begin{aligned} \mu \mathcal{F}^{-1}(h) &= c(d,\alpha)|D|^{-1}\mathbf{1}_D \mathcal{F}^{-1}\left[\mathcal{F}\left(\mathcal{F}^{-1}(h) * \mathcal{F}^{-1}(g_\alpha)\right)\right] \\ &= c(d,\alpha)|D|^{-1}\mathbf{1}_D\left(\mathcal{F}^{-1}(h) * \mathcal{F}^{-1}(g_\alpha)\right). \end{aligned} \tag{119}$$

The inverse Fourier transform $\mathcal{F}^{-1}$ of $g_\alpha(\mathbf{y}) = \|\mathbf{y}\|^{-d+\alpha}$ is obtained from equation (25) (see, Lemma 3.1, or Lemma 1 in p.117 of Stein, 1970):

$$\mathcal{F}^{-1}(g_\alpha)(\mathbf{z}) = \frac{1}{c(d,\alpha)\|\mathbf{z}\|^{\alpha}} = \frac{\pi^{d/2}2^{\alpha}\Gamma(\alpha/2)}{\Gamma\left(\frac{d-\alpha}{2}\right)}\|\mathbf{z}\|^{-\alpha}.$$

Applying (119) and this last relation, we finally obtain that, for an eigenpair (μ, h) of $\mathcal{A}_\alpha$, the following identities hold:

$$\mu\mathcal{F}^{-1}(h)(\mathbf{z}) = |D|^{-1}\mathbf{1}_D(\mathbf{z})\int_D \|\mathbf{z}-\mathbf{y}\|^{-\alpha}\mathcal{F}^{-1}(h)(\mathbf{y})d\mathbf{y}, \qquad (120)$$

since, as commented before, $\mathcal{F}^{-1}(h)$ is supported on D. Thus, if (μ, h) is an eigenpair of $\mathcal{A}_\alpha$, then $(\mu, \mathcal{F}^{-1}(h))$ is and eigenpair for $|D|^{-1}\mathcal{K}_\alpha$ on $L^2(D)$. The converse assertion also holds, and, hence, there exists a one-to-one correspondence between eigenpairs of $\mathcal{A}_\alpha$ and $|D|^{-1}\mathcal{K}_\alpha$, which preserves the eigenvalues. Therefore, these operators have the same eigenvalues, and this fact completes the proof.

Appendix B

The proof of Theorem 5.1 is based on the following lemma, Lemma 4.1 of Veillette and Taqqu (2013).

Lemma 6.1. *Define the function* $G_c(x) = \sum_{k=1}^{\infty} x^{c_k}$, *with* $c = \{c_n\}$ *being a positive strictly increasing sequence such that* $c_n \sim \beta n^{\alpha}$, *as* $n \longrightarrow \infty$, *for some* $1/2 < \alpha < 1$, *and constant* $\beta > 0$. *Then,*

$$\begin{aligned} G_c(x) &\sim x^{c_1}, \quad \text{as } x \longrightarrow 0 \\ G_c(x) &\sim \frac{1}{\alpha\beta^{1/\alpha}}\Gamma\left(\frac{1}{\alpha}\right)(1-x)^{-1/\alpha}, \quad \text{as } x \longrightarrow 1. \end{aligned} \qquad (121)$$

Appendix C

An alternative proof of the boundedness of the probability density of S_∞, based on the series representation given in Corollary 4.1 is derived, and an upper bound is also provided.

Proof of boundedness of the probability density of S_∞

From Corollary 4.2, there exist two indexes $\mathbf{k}_0$ and $\mathbf{k}_1$ such that $\lambda_{\mathbf{k}_0}(S_\infty) > \lambda_{\mathbf{k}_1}(S_\infty)$. Then,

$$S_\infty = \sum_{\mathbf{k}\in\mathbb{N}_*^d} \lambda_{\mathbf{k}}(S_\infty)\left(\varepsilon_{\mathbf{k}}^2 - 1\right) = \lambda_{\mathbf{k}_0}(S_\infty)(\varepsilon_{\mathbf{k}_0}^2 - 1) + \lambda_{\mathbf{k}_1}(S_\infty)(\varepsilon_{\mathbf{k}_1}^2 - 1) + \eta.$$

where

$$\eta = \sum_{\mathbf{k}\in\mathbb{N}_*^d, \mathbf{k}\neq\mathbf{k}_0,\mathbf{k}_1} \lambda_{\mathbf{k}}(S_\infty)\left(\varepsilon_{\mathbf{k}}^2 - 1\right).$$

Thus,

$$S_\infty = \lambda_{\mathbf{k}_1}(S_\infty)(\beta\varepsilon_{\mathbf{k}_0}^2 + \varepsilon_{\mathbf{k}_1}^2) - (\lambda_{\mathbf{k}_0}(S_\infty) + \lambda_{\mathbf{k}_1}(S_\infty)) + \eta_2,$$

where $\beta = \lambda_{\mathbf{k}_0}(S_\infty)/\lambda_{\mathbf{k}_1}(S_\infty)$.

The random variables $\varepsilon_{\mathbf{k}_0}^2$ and $\varepsilon_{\mathbf{k}_1}^2$ are independent. Since the density of $\varepsilon_{\mathbf{k}_1}^2$ is of the form

$$f_{\varepsilon_{\mathbf{k}_1}^2}(x) = \frac{1}{\Gamma(\frac{1}{2})\sqrt{2}}x^{-1/2}e^{-x/2}, \; x > 0,$$

and the density of $\beta\varepsilon^2_{\mathbf{k}_0}$ is given by

$$f_{\beta\varepsilon^2_{\mathbf{k}_0}}(x) = \frac{1}{\beta\Gamma(\frac{1}{2})\sqrt{2}}(x/\beta)^{-1/2}e^{-x/2\beta},\ x > 0,$$

noting that $\beta = \frac{\lambda_{\mathbf{k}_0}(S_\infty)}{\lambda_{\mathbf{k}_1}(S_\infty)} > 1$, then the density of $\varsigma = \beta\varepsilon^2_{\mathbf{k}_0} + \varepsilon^2_{\mathbf{k}_1}$ satisfies

$$\begin{aligned}
f_\varsigma(u) &= \int_0^u f_{\varepsilon^2_{\mathbf{k}_1}}(u-x) f_{\beta\varepsilon^2_{\mathbf{k}_0}}(x)dx \\
&= \frac{e^{-u/2}}{2\Gamma^2(\frac{1}{2})\sqrt{\beta}} \int_0^u (u-x)^{-1/2} e^{\frac{x}{2}} x^{-1/2} e^{-\frac{x}{2\beta}} dx = \\
&\quad [1 - \frac{1}{\beta} > 0] \\
&= \frac{e^{-u/2}}{2\Gamma^2(\frac{1}{2})\sqrt{\beta}} \int_0^u (u-x)^{-1/2} e^{\frac{x}{2}\left(1-\frac{1}{\beta}\right)} x^{-1/2} dx \\
&\leq \frac{e^{-u/2} e^{\frac{u}{2}\left(1-\frac{1}{\beta}\right)}}{2\Gamma^2(\frac{1}{2})\sqrt{\beta}} \int_0^u (u-x)^{-1/2} x^{-1/2} dx \\
&\leq e^{-\frac{u}{2\beta}} \frac{B(\frac{1}{2},\frac{1}{2})}{2\Gamma^2(\frac{1}{2})\sqrt{\beta}} \leq \frac{1}{2\sqrt{\beta}} = \frac{1}{2\sqrt{\frac{\lambda_{\mathbf{k}_0}(S_\infty)}{\lambda_{\mathbf{k}_1}(S_\infty)}}} \leq \frac{1}{2}.
\end{aligned} \tag{122}$$

As the convolution of a bounded density with other is bounded, we then obtain the desired result.

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