Some Analytical Techniques for Partial Differential Equations on Periodic Structures and their Applications to the Study of Metamaterials

Ph.D.

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DECLARATION

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Summary

The work presented in this thesis is a study of homogenisation problems in electromagnetics and elasticity with potential applications to the development of metamaterials.

In Chapter 1, I study the leading order frequency approximations of the quasi-static Maxwell equations on the torus. A higher-order asymptotic regime is used to derive a higher-order homogenised equation for the solution of an elliptic second-order partial differential equation. The equivalent variational approach to this problem is studied which leads to an equivalent higher-order homogenised equation. Finally, the derivation of higher-order constitutive laws relating the fields to their inductions is presented.

In Chapter 2, I study the governing equations of linearised elasticity where the periodic composite material of interest is made up of a "critically" scaled "stiff" rod framework with the voids in between filled in with a "soft" material which is in high-contrast with the stiff material. Using results from two-scale convergence theory, a well posed homogenised model is presented with features reminiscent of both high-contrast and thin structure homogenised models with the additional feature of a linking relation of Wentzell type. The spectrum of the limiting operator is investigated and the establishment of the convergence of spectra from the initial problem is derived.

In the final chapter, I investigate briefly three additional homogenisation problems. In the first problem, I study a periodic dielectric composite and show that there exists a critical scaling between the material parameter of the soft inclusion and the period of the composite. In the second problem, I use of two-scale convergence theory to derive a homogenised model for Maxwell's equations on thin rod structures and in the final problem I study Maxwell's equations in $\mathbb{R}^3$ under a chiral transformation of the coordinates and derive a homogenised model in this special geometry.

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Contents

Introduction

0.1 Motivation

The motivation for the work presented in this thesis is the relatively new innovation of metamaterial science. Where conventional materials usually possess properties due to the molecules from which they are constructed, metamaterials owe their properties to the man-made structures which replace the role of the molecules. Depending on the purpose of the metamaterial, these structures can range from being a few millimeters to a few nanometers in length (Pendry [57]). The hope in creating metamaterials is to develop materials with properties not usually found in nature. A property of particular interest is that of negative refractive index. From Maxwell's equations (Jackson [39]) it can be seen that the refractive index n is given by the formula $n = \pm\sqrt{\hat{\epsilon}(\omega)\hat{\mu}(\omega)}$ where $\hat{\epsilon}$ is the electric permittivity, $\hat{\mu}$ is the magnetic permeability and ω is the propagation frequency. The sign of n is determined by the signs of the real parts of the permittivity and permeability. If both $\mathrm{Re}(\hat{\epsilon}) < 0$ and $\mathrm{Re}(\hat{\mu}) < 0$ then materials will exhibit a negative refractive index (Demetriadou & Pendry [27]).

Materials possessing a negative refractive index are desirable since it has been shown through experiments that materials possessing such a property can exhibit characteristics of cloaking (Demetriadou & Pendry [27]). Originally motivated by the work of Pendry, Schurig & Smith [58] and Leonhardt [46], the cloaking of a region of space is (generally) done in one of two ways; a material with prescribed properties "guides" waves around it (passive cloaking) or wave sources (either inside or outside the cloaking region) negate the field which is scattered from this region. For example, Parnell & Shearer [56] discuss the passive cloaking of materials from so called antiplane elastic waves by means of a spatial transformation. Another example of passive cloaking can be seen in DJ Colquitt, et al. [25] where the authors consider also a spatial transformation in two dimensions for a square domain. Further theory regarding the construction and developments of metamaterials possessing cloaking properties can additionally be found in Milton, Briane & Willis [49].

One way in which the mathematical theory of metamaterials is being developed further is through the theory of homogenisation of Maxwell's equations under a 'chiral' transformation of the variables. To paraphrase Lord Kelvin [42], a figure or group of points is said to be chiral if its image in a plane mirror cannot be brought to coincide with itself. It has been suggested (Pendry & Demetriadou [27]) that electromagnetic waves propagating through a domain with non-symmetry (chirality) will lead to a negative refractive index. This avenue of research is briefly explored in the work presented. The study of twisted electrostatic problems has been considered by Nicolet, Movchan, Guenneau & Zolla [52] and Nicolet, Zolla & Guenneau [53] where the latter work is motivated by the study of photonic crystals. Photonic crystals are periodic dielectric nanostructures which can effect the propagation of electromagnetic waves (see Yablonovitch [80]). The use of homogenisation theory in the study of photonic crystals has been an active research area for some time now (see for example Bouchitté, Guenneau & Zolla [15] and Guenneau, Zolla & Nicolet [38]) and now due to the recent developments, authors have been using homogenisation theory to gain insight into the effective (or averaged) properties of metamaterials also. The small scale periodic nature of metamaterials makes the use of homogenisation theory a practical tool for their study. For example, see Ouchetto, Ouchetto, Zhoudi & Sekkaki [55] and Martini, Sardi & Maci [47] for some recent developments in the use of homogenisation theory for the study of metamaterials.

Homogenisation theory is a well grounded subject and a short review of this rather broad subject will now be presented.

0.2 A Brief Review of Homogenisation Theory

This brief discussion on the origins and methods of homogenisation theory are based mostly on the sources and works which have been of particular relevance and inspiration to the completion of the presented thesis. It is acknowledged that homogenisation theory as a subject is both vast and diverse and so the review here should be regarded in no way thorough and will be predominantly confined to the periodic setting.

It is regarded by most that homogenisation theory has its origins in the work of De Giorgi and Spagnolo [36] wherein the authors proved a result on passing to the limit in partial differential equations which contain a uniformly elliptic, rapidly oscillating coefficient. This result was proven using the technical notion of G-convergence which was introduced and examined by Spagnolo [71] in the context of symmetric linear partial differential equations. It was not too long after this publication that the method of compensated compactness was used to directly prove the homogenisation theorem. Indeed, it was Tartar [72] who initially used the so-called energy method which, in conjunction with the classical compactness lemma, takes a special class of chosen test functions in the variational formulation of the problem in question before then passing to the limit.

The solution of homogenisation problems via the method of asymptotic expansions is a well established theory. In 1974, Sanchez-Palencia [63] made use of a two-scale asymptotic expansion to construct a homogenised model for his equation but it was in 1975 that Bakhvalov [5] showed that the limit function was indeed the limit of the solutions to the original heterogeneous problems as the period went to zero. Asymptotic theory is a powerful tool in its own right (see for example Erdélyi [28] for its wide reaching applications) but in the context of homogenisation, the use of asymptotic methods in dealing with problems involving multiple scales is invaluable. Using asymptotic methods to solve problems in homogenisation can not only provide valuable insight into the structure of solution but also the whole method can often be made rigorous through the bounding of the relative error incurred between the true solution and a truncated asymptotic solution by a small parameter (often the period of the material of study). Numerous applications of the theory to a variety of physical problems can be found in the books of Bakhvalov & Panasenko [6], Benoussan, Lions & Papanicolaou [9], Sanchez-Palencia [64] and, more recently, in the book of Chechkin, Piatnitski & Shamaev [19].

A major development of the homogenisation theory is the introduction of the method of two-scale convergence. Originally introduced by Nguetseng [51] and then further developed by Allaire [1], two-scale convergence methods acknowledge the existence of a non-classical limit as the solution to classes of homogenisation problems. Indeed, utilisng a variation of the two-scale compactness lemma, a function which depends on a "microscopic" variable as well as on a "macroscopic" variable can be shown to be the limit of some homogenisation problems. Moreover, this scheme in turn recovers previously established results when applied to equivalent variational formulations of partial differential equations containing a rapidly oscillating coefficient.

The class of homogenisation problems for which two-scale analysis can be used with is vast. In particular, high-contrast problems which exhibit a loss of uniform ellipticity may be dealt with. A composite material is said to be in high-contrast if there is a sufficient contrast between the material parameters in the constitutive material components. A further class of problems which two-scale convergence handles well are problems on thin structures with junctions. It was Zhikov [81] who extended the method of two-scale convergence to a broader setting where more general spaces with arbitrary measures could be considered. This approach has since been applied to problems on thin networks with junctions by Zhikov [82] and by Zhikov & Pastukhova [84].

The treatment of homogenisation theory has been applied to a vast number physical problems, most often in the mathematical literature to elasticity theory (see also Oleinik, Shamaev & Yosifian [54]) but in more recent history, it has become a growing trend to apply this theory to the governing equations of electromagnetics. Homogenisation of Maxwell's equations with periodic rapidly-oscillating coefficients has been discussed in brief in Bakvalov & Panasenko [6] and Sanchez-Palencia [64] but not in any extensive detail. It might be argued that the application of homogenisation theory to a periodic dielectric medium is analogous to the application of homogenisation theory to a periodic composite structure, however, a number of significant technicalities arise in the analysis of Maxwell's equations which are not necessarily found in elasticity. One such difference in the study of Maxwell's equations is the nontrivial kernel of the **curl** operator in domains which are not simply connected. The dependence of the kernel of the **curl** operator on the geometry of bounded non-simply connected domains is addressed in Dautray & Lions [26, Chapter IX] where it was shown that elements of this kernel will solve a particular problem on domains called"cuts". These cuts can be potentially complicated surfaces to identify which can make finding a solution to this problem on the cuts difficult to find. Moreover, this theory does not address the case of unbounded non-simply connected domains which may also be of interest.

In the classical literature available [6, 9, 40, 64], the discussions on the homogenisation of Maxwell's equations have been relatively brief when compared with the extensive literature available on the homogenisation of the governing equations of elasticity theory. However, in recent years, thanks to the development of the theory of photonic crystals and the development of metamaterial science, a deeper and broader theory is being developed for the homogenisation of Maxwell's equations. In Wellander [75], the method of two-scale convergence as introduced by Nguetseng [51] is extended to homogenise Maxwell's equations with inhomogeneous initial conditions and prove corrector type results. Following on from this paper, Wellander [76] used similar techniques to homogenise Maxwell's equations in a heterogeneous medium where the electric conductivity is nonlinear. This leads to the development of new compactness results and corrector results are proved which are important to numerical implementation. Other avenues of investigation into the homogenisation of Maxwell's equations include: the study of bianisotropic materials (Barbitis & Stratis [7], and Bossavit, Griso & Miara [12, 13]); the study of random elliptic systems (Barbatis, Stratis & Yannacopoulos [8]) and the study of different techniques to homogenise Maxwell's equations including the use of Bloch waves (Sjöberg, *et. al.* [67]), and the use of a singular value decomposition (Sjöberg [66]). More recently, Cherednichenko & Cooper [23] proved results with regards a high-contrast problem wherein, the technicalities regarding Maxwell's equations are made even more apparent with regards the kernel of the operator of study. Moreover, the application of homogenisation theory to problems involving metamaterials has been explored recently by Fernandes, Ottonello & Raffetto [31] and Serrano [65].

0.3 An Overview of the Work Presented

In the broadest sense, the work presented in this thesis is a selection of problems from the theories of electromagnetism and elasticity where a variety of techniques from homogenisation theory are used to further understand and solve the problems in question. The first chapter regards the derivation of "higher-order" homogenised equations for the quasistatic system of Maxwell equations in a periodic medium via the method of two-scale asymptotic expansions. The second chapter concerns the two-scale analysis of a high-contrast elasticity problem on so-called "critically" scaled thin structures. In the final chapter, a selection of problems are presented which all concern the homogenisation of Maxwell's equations; one as an extension to the work discussed in the first chapter of this thesis and two other problems in which the geomtery plays an important role in the analysis. The treatment of such problems requires deep mathematical notions including, e.g. homogenisation theory, variational methods, tensor analysis, multiple-scale convergence methods, spectral theory, *etc.*

In Chapter 1, higher-order homogenised equations are derived for Maxwell's equations on a three-dimensional torus $\mathbb{T} = [0, T]^3$, $T > 0$ with period cells Q of size ε such that T/ε is a positive integer.

In Section 1.1, the vector problem is formulated and a two-scale asymptotic expansion is given a priori for the solution of this problem. Initially, considerations are restricted to a medium where the electric permittivity is constant and the magnetic permeability is rapidly oscillating. The governing equation describes the behaviour of the electric field in a periodic medium subject to some known external source of current density $\boldsymbol{f}$. The problem studied is to find a $\mathbb{T}$-periodic solution to the second-order elliptic equation

$$\mathbf{curl}\left(\hat{A}^{\varepsilon}(\boldsymbol{x})\mathbf{curl}\,\boldsymbol{u}^{\varepsilon}(\boldsymbol{x})\right) = \boldsymbol{f}(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{T},$$

$$\operatorname{div} \boldsymbol{u}^{\varepsilon} = 0, \quad \int_{\mathbb{T}} \boldsymbol{u}^{\varepsilon}(\boldsymbol{x})\ \mathrm{d}\boldsymbol{x} = \mathbf{0}.$$

The 3×3 matrix $\hat{A}^{\varepsilon}(\boldsymbol{x}) = A(\boldsymbol{x}/\varepsilon)$ represents the inverse of the magnetic permeability of the medium. The permeability matrix is symmetric, Q-periodic and uniformly elliptic where $Q = [0, 1]^3$ is the associated unit cell of the problem. It is assumed that there is some external source of current density $\boldsymbol{f}$ which is divergence free, has zero average over $\mathbb{T}$ and is $\mathbb{T}$-periodic.

The asymptotic expansion used is

$$\boldsymbol{u}^{\varepsilon}(\boldsymbol{x}) = \boldsymbol{v}(\boldsymbol{x},\varepsilon) + \sum_{j=1}^{\infty} \varepsilon^{j} \Big\{ \nabla_y \Big\{ K^{(j)}(\boldsymbol{y}) \nabla_x^{j} \boldsymbol{v}(\boldsymbol{x},\varepsilon) \Big\} + \nabla_x \Big\{ K^{(j-1)}(\boldsymbol{y}) \nabla_x^{j-1} \boldsymbol{v}(\boldsymbol{x},\varepsilon) \Big\} + N^{(j)}(\boldsymbol{y}) \nabla_x^{j-1} \mathbf{curl}_x \boldsymbol{v}(\boldsymbol{x},\varepsilon) \Big\} \Big|_{\boldsymbol{y}=\frac{\boldsymbol{x}}{\varepsilon}}, \tag{0.3.1}$$

where

$$\boldsymbol{v}(\boldsymbol{x},\varepsilon) = \boldsymbol{v}_0(\boldsymbol{x}) + \sum_{k=1}^{\infty} \varepsilon^{k} \boldsymbol{v}_k(\boldsymbol{x}). \tag{0.3.2}$$

This expansion is similar to the expansions used for the classical homogenisation of the governing equations of linearised elasticity (see Bakhvalov & Panasenko [6]) with the noticeable difference that there are additional "gradient" terms added on. These extra terms appear due to the non-trivial kernel associated with the **curl** operator on $\mathbb{T}$. The derivation of the homogenised equation is achieved using well established means seen in, for example, Sanchez-Palencia [64]. The derivation of the higher-order homogenised equation follows a similar path to that of Smyshlyaev & Cherednichenko [69]. The homogenised equation of infinite-order takes the form

$$\mathbf{curl}\, \hat{h}^{(2)} \mathbf{curl}\, \boldsymbol{v} + \sum_{j=1}^{\infty} \varepsilon^{j} \mathbf{curl}\, \hat{h}^{(j+2)} \nabla^{j} \mathbf{curl}\, \boldsymbol{v} = \boldsymbol{f},$$

where $\hat{h}^{(j+2)},\ j = 1, 2, \ldots$ are the higher-order homogenised coefficients. It is also shown that a truncation of (0.3.1) to order $O(\varepsilon^{K})$, denoted $\boldsymbol{u}_K$, is a good approximation to the solution of the original problem as $\varepsilon \to 0$. This is done via a rigorous justification which involves taking $\boldsymbol{u}_K$ with a truncation of series (0.3.2) to order $O(\varepsilon^{K})$ substituted in and bounding the error incurred between this series and the true solution by a constant times ε^{K-1}. This section is concluded with an example where the matrix A is piecewise constant and the results of the calculation of the first higher-order term (which is of order $O(\varepsilon^{2})$) of the higher-order homogenised equation are given.

In Section 1.2 the same problem is considered from a variational stand point. The perturbative nature in which the higher-order homogenised equations are constructed via the method of asymptotic expansions means that, should the expansion be truncated at some finite-order, then the resulting operator may not be elliptic which can be an issue with regards to numerical implementation. This difficulty is overcome however by using a combination of asymptotic methods and variational techniques. The equivalent variational formulation considered is

$$I(\varepsilon, \boldsymbol{f}) = \min_{\boldsymbol{u}} \int_{\mathbb{T}} \Big(\frac{1}{2} \hat{A}^{\varepsilon} \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{u} - \boldsymbol{f} \cdot \boldsymbol{u} \Big), \tag{0.3.3}$$

where the minimisation is taken over those $\mathbb{T}$-periodic functions $\boldsymbol{u}$ with zero average and zero divergence over $\mathbb{T}$. Problem (0.3.3) possesses the same unique solution as the original problem which leads to an alternative expression for $I(\varepsilon, \boldsymbol{f})$ only involving an integral of

$\boldsymbol{u}^{\varepsilon} \cdot \boldsymbol{f}$. Variational asymptotics are established by substituting the asymptotic expansions (0.3.1) and (0.3.2) into this new expression. Motivated by the want to remove the effect of the rapid oscillations from the asymptotic expansion (0.3.1), consideration is given to a new family problems, equivalent to the original problem but with the matrix $\hat{A}^{\varepsilon}$ replaced by the matrix $\hat{A}^{\varepsilon}_{\boldsymbol{\zeta}}(\cdot) := \hat{A}^{\varepsilon}(\cdot + \boldsymbol{\zeta})$. This in turn leads to the consideration of the equivalent family of variational problems where making use of averaging properties with respect to the variable $\boldsymbol{\zeta}$, brings about another homogenised equation of infinite order but this time achieved via variational means. The final part of Section 1.2 is then dedicated to illustrating that the two homogenised equations of infinite-order coincide to all orders via a tensor symmetrisation process which is picked with the purpose of maintaining the structure of the "**curl curl**" operator.

In Section 1.3 a brief overview of Maxwell's equations is given and its relation to the work presented in the first two sections is outlined. The section then goes on to explore the constitutive relations and develop homogenised expansions of infinite-order for said constitutive relations. To conclude the chapter, consideration is then given to the homogenisation of the full system of Maxwell equations where details of the changes needed to be made to the already established theory are documented.

In the second chapter of this thesis, the analysis of a periodic composite is presented where the constitutive components of the composite comprise of a "stiff" thin structure with the gaps inbetween filled in with a different "soft" material. The periodic rod framework in question comprises of rods of thickness $a = a(\varepsilon) > 0$ where $\varepsilon > 0$ is the period of the rod framework and, moreover, $a \to 0$ as $\varepsilon \to 0$. The rod framework in question is scaled such that $a/\varepsilon^2 \to \theta > 0$. This scaling is referred to as "critical" in the literature (Zhikov [82]). Moreover, it is assumed that the ratio of the material stiffness of the soft component to the stiff component is of order $O(\varepsilon^2)$. For the purposes of the analysis of Chapter 2, it is more convenient to deal with the scaled rod length $h := a/\varepsilon$ so that $h \to 0$ and $h/\varepsilon \to \theta$ as $\varepsilon \to 0$.

In Section 2.1, the problem of study is introduced along with the tools essential to the derivation of the associated homogenised problem. The domain of consideration Ω is a bounded Lipschitz domain in $\mathbb{R}^2$ and is split into two subdomains $\Omega_1^{\varepsilon,h}$ and $\Omega_0^{\varepsilon,h}$ where $\Omega_1^{\varepsilon,h}$ is the set of all stiff inclusions (rods) contained in Ω and $\Omega_0^{\varepsilon,h}$ is the set of all soft inclusions contained in Ω. For each $\varepsilon, h > 0$ and each $\boldsymbol{f}^{h,\varepsilon} \in [L^2(\Omega, \mathrm{d}\mu^h_\varepsilon)]^2$, the problem studied is to find $\boldsymbol{u}_\varepsilon^h \in [H_0^1(\Omega)]^2$ such that the following integral equation is satisfied:

$$\int_{\Omega_1^{\varepsilon,h}} A_1 e(\boldsymbol{u}_\varepsilon^h) \cdot e(\boldsymbol{\varphi}) \, \mathrm{d}\mu_\varepsilon^h + \varepsilon^2 \int_{\Omega_0^{\varepsilon,h}} A_0 e(\boldsymbol{u}_\varepsilon^h) \cdot e(\boldsymbol{\varphi}) \, \mathrm{d}\mu_\varepsilon^h +$$

$$+ \int_\Omega \boldsymbol{u}_\varepsilon^h \cdot \boldsymbol{\varphi} \, \mathrm{d}\mu_\varepsilon^h = \int_\Omega \boldsymbol{f}^{h,\varepsilon} \cdot \boldsymbol{\varphi} \, \mathrm{d}\mu_\varepsilon^h, \quad \forall \boldsymbol{\varphi} \in [H_0^1(\Omega)]^2.$$

Here $e(\cdot)$ denotes the symmetric gradient, A_1 and A_0 are constant, positive definite matrices and $\mathrm{d}\mu_\varepsilon^h$ is a composite measure comprised of the Lebesgue measure and a periodic normalised measure concentrated on the stiff component. For each fixed ε and each fixed h, the above problem is shown to posses a unique solution. Following the establishment of the problem, details of the main definitions and results from the theory of two-scale analysis are presented, including a result on two-scale compactness (see Allaire [1] or Zhikov [81]) which is key to proving the existence (up to possibly taking a subsequence) of a weak two-scale limit. This is then briefly followed by a subsection containing definitions pertaining to the theory of periodic rigid displacements which play a key role in the analysis.

In Section 2.2, it is shown that the sequences $\boldsymbol{u}_\varepsilon^h$, $e(\boldsymbol{u}_\varepsilon^h)$, $\varepsilon e(\boldsymbol{u}_\varepsilon^h)$ are bounded in $[L^2(\Omega, \mathrm{d}\mu_\varepsilon^h)]^2$, $[L^2(\Omega_1^{\varepsilon,h}, \mathrm{d}\mu_\varepsilon^h)]^3$ and $[L^2(\Omega_0^{\varepsilon,h}, \mathrm{d}\mu_\varepsilon^h)]^3$ respectively and therefore by the two-scale compactness lemma, they possess weakly two-scale convergent subsequences. The remainder of the section is devoted to proving results pertaining to the structure of the two-scale limits. Amongst these results, it is shown that the weak two-scale limit of the sequence $\boldsymbol{u}_\varepsilon^h$ when restricted to the stiff component is in fact the trace of the two-scale limit of $\boldsymbol{u}_\varepsilon^h$. This is a new result and moreover, when combined with the already known results presented in Zhikov [82] and Zhikov & Pastukhova [84], it is shown further that this trace satisfies a fourth-order differential equation on each link of the limiting singular structure. This type of condition is referred to as a Ventcel' (Wentzell) boundary condition (Ventcel' [74]).

In Section 2.3, the homogenised problem for the high-contrast, critically scaled model is derived which uniquely determines all the unknowns of the problem. This homogenised problem can be expressed as a system of three partial differential equations; a macroscopic equation, an equation on the period cell and an equation on the trace function. Following this, the main homogenisation theorem is proven by taking the limit of various integral functionals with different test functions which exhaust the so called energy space. This proof involves several technicalities which take into account the shrinking rod structure to the singular structure as well as the expanding soft regions.

In Section 2.4 a full description of the spectrum of the limiting problem is given and the convergence of spectra is established which once again takes into account the intricacies involved in the dimension reduction of the rod structure.

In the final chapter of this thesis, results from three separate problems are presented which all pertain to the homogenisation of Maxwell's equations.

In Section 3.1, a modification of the problem presented in Chapter 1 is demonstrated where the unit cell Q now comprises two given materials on two non-intersecting subdomains: the stiff matrix Q_1 and the soft inclusion Q_0. The problem is to examine this periodic dielectric medium when the magnetic permeability is of the form

$$A(\boldsymbol{y}) := A^{\delta}(\boldsymbol{y}) = \begin{cases} \delta I, & \text{if } \boldsymbol{y} \in Q_0, \\ I, & \text{if } \boldsymbol{y} \in Q_1, \end{cases} \qquad \delta > 0.$$

In this study, it is shown that there exists a "critical" scaling between the contrast parameter δ and the period ε. This scaling is considered to be critical since the behaviour of the asymptotic expansion for the solution of the problem of consideration is significantly different from the behaviour of the asymptotic expansion for the solution of the problem under consideration when any other scaling is chosen. The results of this section are obtained using a combination of the asymptotic methods described in Chapter 1 and additionally expanding the unit cell solutions $N^{(j)},\ j = 1, 2, \ldots$ described in Section 1.1.2 and Section 1.1.3 in asymptotic series of the contrast parameter δ.

In Section 3.2, Maxwell's equations on rod structures are studied and the homogenised equation in the case when the rods are "thin" is given. This work makes use of the two-scale convergence theory described in Chapter 2 to prove the existence of weak two-scale limits to sequences bounded in particular L^2-spaces. Forming a suitable energy space, the homogenised equation is obtained by taking the limit of various integral identities against suitable test functions. At the end of this section, the many possible extensions of this work are briefly discussed.

In Section 3.3, Maxwell's equations are studied under a transformation of coordinates and a system of homogenised equations are derived. This problems separates itself from the other problems studied in this thesis as it is the only one to be studied on an unbounded, multiply connected domain which makes certain aspects of the analysis more tricky to deal with and the results more complicated.

Chapter 1

Full Two-Scale Asymptotic Expansion and Higher-Order Constitutive Laws in the Homogenisation of the System of Quasi-static Maxwell Equations

Introduction

In the following chapter, a higher-order homogenised system of equations for the solutions to periodic problems with period $\varepsilon > 0$ in a dielectric medium are derived. The homogenisation theory of second-order periodic elliptic equations (see Bakhvalov & Panasenko [6], Bensoussan, Lions & Papanicolaou [9], Sanchez-Palencia [64]), has amongst its crowning results the derivation of the homogenised (or "averaged") equation which captures the effective behaviour of the original problem independently of the small scale structure intrinsic to the original problem. A higher-order homogenised equation, while retaining in some sense the homogenised equation structure to leading order, has higher-order terms present in increasing orders of magnitude of the period size. It has been suggested in the theory of elasticity (see Smyshlyaev & Fleck [70]) that these higher-order effects or strain-gradient effects can account for a variety of scale effects observed in multiply scaled media when said scales are not "too widely separated" (Cherednichenko [21]). The role of strain gradients in the theory of elasticity has been explored by the likes of Fleck & Willis [33], Smyshlyaev & Fleck [70] and Smyshlyaev & Cherednichenko [69]. As a part of this work, the strain-gradient theory associated with the homogenised elasticity problems discussed in the aforementioned works will be explored in the context of a dielectric medium.

Homogenisation via the method of asymptotic expansions has been used to explore the role of higher-order effects in periodic media by Boutin [16] and Triantafyllidis & Bardenhagen [73] wherein, the authors expect the solution in the form of a two-scale asymptotic expansion of the small parameter ε which is a ratio of the period cell size to the domain size. The unknown function coefficients in the expansion are sought to depend on the macroscopic (slow) variable, usually denoted x and the microscopic (fast) variable, usually denoted $y := x/\varepsilon$. Using some formal "averaging" process, a homogenised equation is derived with this process made rigorous by bounding the difference between the true solution and some finite truncation of the asymptotic expansion for the solution by some power of ε. Using this theory, Boutin [16] and Triantafyllidis & Bardenhagen [73] justify asymptotic expansions for higher-order stress-strain relations and note that this process may breakdown close to the boundary of the domain under consideration. However, this issue is rectified by considering an infinite periodic medium with a fixed large period T such that the ratio T to the unit cell size ε is a positive integer and, moreover, provided the body force $\boldsymbol{f}$ is T-periodic, the theory of Bakhvalov & Panasenko [6] can be applied throughout the domain with remainder estimates guaranteed everywhere. The work carried out in this chapter considers such a geometrical setup and indeed yields remainder estimates on the whole domain.

The asymptotic approach has the advantage that it provides a rigorous way (in the sense that the error incurred is small when ε is small) in which to gain insight into the effect of strain gradients. However, there are two notable disadvantages which arise when considering the problem from the perspective of potential applications. First of all, the higher-order expansions for the homogenised solutions and higher-order constitutive relations are "perturbed" in the sense that by the addition of terms in increasing powers of the period ε, the expansion is only expected to be accurate if ε is sufficiently small and so can't be expected to retain the same level of accuracy if a larger period cell is desired. One other disadvantage of the asymptotic procedure lies in the fact that a truncation of the higher-order constitutive relations at some order of magnitude of the period may lead to a loss of ellipticity of the corresponding operator obtained. This can be a complication from both the point of view of numerical simulations and from the view point that the truncated problem may even be ill-posed. Although for the Maxwell problem no example is presented illustrating this loss of ellipticity, in Cherednichenko [21] when the elasticity tensor takes the form $A(\boldsymbol{y}) = a(y_2)I$ where $a(y_2) = 2 + \sin(2\pi y_2)$ it was found that the fourth-order truncation led to a loss of ellipticity of the governing operator. It has been confirmed that if the same matrix is taken for the inverse of the permittivity in the

Maxwell system, there too is a loss of ellipticity.

One way to overcome these difficulties is via a combination of asymptotics and variational calculus. Inspired by the similar considerations of Smyshlyaev & Cherednichenko [69] where the authors derived higher-order homogenised solutions for an elastic body under an "anti-plane" shear force, both an asymptotic approach and variational approach are considered in combination during the analysis of a periodic composite. The use of variational asymptotics allows them to construct higher-order homogenised equations which do retain their ellipticity and, moreover, the authors show that the homogenised solution is "close" to the true solution in some variational sense. A further acknowledgment the authors make is the fact that, by considering the energy functional for the problem, they can show that not only does this functional converge to the homogenised energy (as $\varepsilon \to 0$) but also that the higher-order asymptotics for the energy functional are determined solely by the higher-order terms in the homogenised solution. All these observations carry over to the equivalent considerations for the homogenisation of Maxwell's equations.

Another consideration of the aforementioned authors is to cancel the effect of the rapid oscillations in higher-order terms via the (ensemble) averaging of another family of problems. This family of problems is given via a shift in the fast variable of the original problem by a parameter which, in other words, is all possible realisations of the periodic medium. This process does indeed remove all the rapid oscillations from terms of all orders and moreover yields a higher-order homogenised solution whose truncation provides rigorous asymptotics for the homogenised solution obtained through the asymptotic approach. This all carries over to the considerations of this work wherein Maxwell's equations are studied. It is noted however that more care must be taken with the manipulation of the formulae derived and higher-order homogenised tensors.

The solution of the problem of consideration is unique and moreover, the asymptotic and variational methods presented in this chapter are both rigorous and equivalent. However, proving that the higher-order homogenised solutions derived via these two approaches are the same is not so straight forward. Indeed, a more involved tensor analysis is required in order to establish this equality. In terms of differences between the homogenisation procedure carried out by Smyshlyaev & Cherednichenko [69] and the work presented forthcoming, the full system of Maxwell equations is a much more involved system than the linearised elasticity equation and constitutive stress-strain relation. Indeed, the system of Maxwell equations considered in this work consists of six coupled first-order equations; two field (**curl**) equations, two continuity (div) equations and two linear constitutive laws which are then expressed (after substitution) as two second-order equations for the fields plus the two continuity equations. Section 1.1-Section 1.2 are devoted solely to the consideration of one pair of equations which govern the behaviour of the electric field when the electric permittivity is constant. The same consideration can be made for the remaining pair of equations for the magnetic field intensity (now with the electric permittivity rapidly oscillating and the magnetic permeability constant) however, a complication arises in formulating the second-order equation. Indeed, the current density term on the right-hand side of the equation now rapidly oscillates also meaning extra care must be taken to yield a higher-order homogenised equation (this is detailed in Section 1.3.4). The consideration of the full system of Maxwell equations (now with both the electric permittivity and magnetic permeability rapidly oscillating) is then carried out requiring a handful of modifications to be made to the analysis laid out for the individual systems. The system of equations separates completely into two systems for the electric field and the magnetic field intensity respectively as might be expected meaning that the procedure carried out can be applied to a wide variety of problems, e.g. problems in the absence of electric fields but still in the presence of a rapidly oscillating electric permittivity and a rapidly oscillating magnetic permeability.

The system of Maxwell equations studied are restricted to the time harmonic formulation and then further considered in the quasistatic approximation. In Carey & O'Brien [18], the quasi-static limit (the limit as the frequency goes to zero) is described as being "the physically interesting situation in which a wave process degenerates into a diffusive process". For the purposes of the work presented, if the electric field $\boldsymbol{\mathcal{E}} = \boldsymbol{\mathcal{E}}(\boldsymbol{x}, t)$, is written in time harmonic form and the quasistatic approximation is then applied, it is assumed that

$$\boldsymbol{\mathcal{E}}(\boldsymbol{x}, t) = \boldsymbol{E}^{\omega}(\boldsymbol{x})\mathrm{e}^{\mathrm{i}\omega t}, \quad \text{where} \quad \boldsymbol{E}^{\omega}(\boldsymbol{x}) = \boldsymbol{E}_0(\boldsymbol{x}) + \mathrm{i}\omega\boldsymbol{E}_1(\boldsymbol{x}) + (\mathrm{i}\omega)^2\boldsymbol{E}_2(\boldsymbol{x}) + \ldots.$$

The quasistatic approximation entails considering the amplitudes/phasors of the respective fields being expanded in asymptotic expansions of the small but fixed frequency ω. In essence, taking the quasistatic approximation means that it is assumed that the sources in the problem vary sufficiently slowly such that the problem can be considered to be static. Quasistatic theory can also be considered as an intermediary theory between the static theory and the full Maxwell system (Larsson [43]). Rapetti & Rousseaux [59], have considered quasistatic models of Maxwell's equations in their work, however, where as Rapetti & Rousseaux consider the dependence of the non-dimensional Maxwell equations on the "small parameters" involved for some physical examples, the quasistatic approximation presented here considers the leading order governing equations obtained when the frequency is small but finite.

With regards to notation, throughout Chapter 1 vectors will be written in boldface italics. All vectors will be three-dimensional and all higher-order tensors of order K will contain 3^K entries. The three-dimensional periodic torus is denoted $\mathbb{T}$ and the period cell is denoted $Q := [0,1]^3$. Averages over $\mathbb{T}$ and Q are denoted $\langle \cdot \rangle_{\mathbb{T}}$ and $\langle \cdot \rangle$ respectively. Throughout Section 1.1 and Section 1.2, $\boldsymbol{u}^\varepsilon$ is used to denote the electric field, $\hat{A}^\varepsilon(\cdot) := A(\cdot/\varepsilon)$ will denote the inverse of the magnetic permeability and $\boldsymbol{f}$ will denote the source of current density. In Section 1.3 the standard notation for the vector fields for Maxwell's equations will be written in a calligraphic font where as the amplitudes (or phasors) will be written in usual boldface roman lettering. With regards the notation for the function spaces, they will be denote in the form $[X^k(D)]^{3^n}$ where n denotes the tensor order of the object in question. Predominantly, n will be equal to 1. Note however that the norms will be denoted $\| \cdot \|_{X^k(D)}$ without reference to the order of the tensor.

1.1 Solution of the Problem by Asymptotic Expansion

In the following chapter, all vectors and domains will be three dimensional. Consideration is given solely to the second-order linear partial differential equation which must be satisfied by the electric field (denoted in this section as $\boldsymbol{u}^\varepsilon$) in the absence any magnetic fluxes.

The highlights of this section include the formulation of the problem of interest, the statement of a full two-scale expansion for the solution and the derivation of an infinite-order homogenised equation. The analysis will then be justified by means of asymptotically bounding the error incurred between the true solution and the truncation of the infinite-order solution.

1.1.1 Formulation of the Problem

It is first noted that no regard is given to boundary effects in this text and so the domain of consideration is that of a three-dimensional torus denoted $\mathbb{T} = [0,T]^3$, $T > 0$ where the cell of periodicity is denoted $Q = [0,1]^3$. Let the following notational conventions be introduced for the averages over $\mathbb{T}$ and Q respectively:

$$\langle \boldsymbol{f} \rangle_{\mathbb{T}} := \frac{1}{|\mathbb{T}|} \int_{\mathbb{T}} \boldsymbol{f}(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}, \quad \langle \boldsymbol{g} \rangle := \frac{1}{|Q|} \int_Q \boldsymbol{g}(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y}.$$

Consider the following vector equation:

$$(L_\varepsilon \boldsymbol{u}^\varepsilon)(\boldsymbol{x}) \equiv \mathbf{curl}\, A\Big(\frac{\boldsymbol{x}}{\varepsilon}\Big) \mathbf{curl}\, \boldsymbol{u}^\varepsilon(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{T}, \quad \varepsilon > 0, \quad T/\varepsilon \in \mathbb{N}. \tag{1.1.1}$$

The coefficient matrix A, with entries denoted α_{ij} is Q-periodic, symmetric:

$$\alpha_{ij}(\boldsymbol{y}) = \alpha_{ji}(\boldsymbol{y}), \quad \forall \boldsymbol{y} \in Q, \quad i,j = 1,2,3,$$

bounded and uniformly elliptic:

$$\exists\nu > 0 \ : \quad \nu|\boldsymbol{\xi}|^2 \le \alpha_{ij}(\boldsymbol{y})\xi_i\xi_j \le \nu^{-1}|\boldsymbol{\xi}|^2, \quad \forall\boldsymbol{\xi}\in\mathbb{R}^3,\ \boldsymbol{y}\in Q.$$

The notation $\hat{A}^\varepsilon(\cdot) := A(\cdot/\varepsilon)$ will be used to denote matrix functions with the above properties. Right-hand sides $\boldsymbol{f} \in [C^\infty_{\text{per}}(\mathbb{T})]^3$ are assumed to have zero average on $\mathbb{T}$ and to be divergence free. Equation (1.1.1) is a second-order equation satisfied by the electric field in the quasistatic approximation (see Section 1.3). Here, the quantities A, $\boldsymbol{u}^\varepsilon$ and $\boldsymbol{f}$ represent the inverse of the magnetic permeability $\hat{\mu}$, the electric field $\boldsymbol{E}_1^\varepsilon$ and the current density $-\boldsymbol{J}_0$ respectively at each point $\boldsymbol{x}\in\mathbb{R}^3(\text{mod } \mathbb{T})$.

Weak solutions $\boldsymbol{u}^\varepsilon$ to equation (1.1.1) belong to the space[1] $X(\mathbb{T}) \subset H^1_{\mathbf{curl}}(\mathbb{T})$, where

$$X(\mathbb{T}) = \left\{\boldsymbol{u}\in[L^2(\mathbb{T})]^3 \mid \operatorname{div}\boldsymbol{u} = 0,\ \langle\boldsymbol{u}\rangle_{\mathbb{T}} = \mathbf{0}\right\} \cap H^1_{\mathbf{curl}}(\mathbb{T}).$$

For $\boldsymbol{u}^\varepsilon\in X(\mathbb{T})$, the following identity holds:

$$\int_{\mathbb{T}} \hat{A}^\varepsilon \mathbf{curl}\,\boldsymbol{u}^\varepsilon \cdot \mathbf{curl}\,\boldsymbol{\varphi} = \int_{\mathbb{T}} \boldsymbol{f}\cdot\boldsymbol{\varphi}, \qquad \forall\boldsymbol{\varphi}\in[C^\infty(\mathbb{T})]^3. \tag{1.1.2}$$

When $X(\mathbb{T})$ is equipped with the $H^1_{\mathbf{curl}}$-norm

$$\|\boldsymbol{u}\|_{H^1_{\mathbf{curl}}(\mathbb{T})} = \|\boldsymbol{u}\|_{L^2(\mathbb{T})} + \|\mathbf{curl}\,\boldsymbol{u}\|_{L^2(\mathbb{T})}, \tag{1.1.3}$$

it is a Sobolev space.

Consider the following problem: given $\boldsymbol{f}\in[C^\infty_{\text{per}}(\mathbb{T})]^3\cap X(\mathbb{T})$, a solution is sought to the problem

$$(L_\varepsilon\boldsymbol{u}^\varepsilon)(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x}), \quad \boldsymbol{u}^\varepsilon\in X(\mathbb{T}). \tag{1.1.4}$$

For all $\varepsilon > 0$, problem (1.1.4) is well posed as shown by the following theorem.

Theorem 1.1.1. *For all $\boldsymbol{f}\in[C^\infty(\mathbb{T})]^3\cap X(\mathbb{T})$, there exists a unique solution of problem (1.1.4).*

Proof. Define a bilinear form $B(\boldsymbol{u},\boldsymbol{\varphi})$ on $X(\mathbb{T})$ by the formula

$$B(\boldsymbol{u},\boldsymbol{\varphi}) = \int_{\mathbb{T}} \hat{A}^\varepsilon\mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{\varphi}, \quad \boldsymbol{u},\boldsymbol{\varphi}\in X(\mathbb{T}).$$

A unique solution to the problem

$$B(\boldsymbol{u}^\varepsilon,\boldsymbol{\varphi}) = \int_{\mathbb{T}}\boldsymbol{f}\cdot\boldsymbol{\varphi}, \quad \forall\boldsymbol{\varphi}\in X(\mathbb{T}),$$

[1] $H^1_{\mathbf{curl}}(\mathbb{T})$ is defined as the closure of the space of continuously differentiable functions $[C^\infty(\mathbb{T})]^3$ with respect to the norm defined by (1.1.3). More concisely, $H^1_{\mathbf{curl}}(\mathbb{T}) = \{\boldsymbol{u}\in[L^2(\mathbb{T})]^3 \mid \mathbf{curl}\,\boldsymbol{u}\in[L^2(\mathbb{T})]^3\}$ (see for example Jikov, Kozlov & Oleinik [40]).

exists by the Lax-Milgram Lemma (see Gilbarg & Trudinger [35, p. 83]). Indeed, coercivity of B follows by the ellipticity of the matrix A as follows:

$$B(\boldsymbol{u},\boldsymbol{u}) = \int_{\mathbb{T}} \hat{A}^{\varepsilon}\mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{u} \geq \int_{\mathbb{T}} \nu|\mathbf{curl}\,\boldsymbol{u}|^2 \geq \frac{1}{2}\nu\int_{\mathbb{T}}|\mathbf{curl}\,\boldsymbol{u}^{\varepsilon}|^2\mathrm{d}\boldsymbol{x}+\frac{1}{2}\nu\int_{\mathbb{T}}|\mathbf{curl}\,\boldsymbol{u}^{\varepsilon}|^2\mathrm{d}\boldsymbol{x}.$$

Using a Maxwell inequality (see Neff, Pauly & Witsch [50] or Appendix 1.A for further details) and noting that $\operatorname{div}\boldsymbol{u}^{\varepsilon} = 0$, for some positive constant $C \in \mathbb{R}$ which depends on $|\mathbb{T}|$

$$B(\boldsymbol{u}^{\varepsilon},\boldsymbol{u}^{\varepsilon}) \geq \frac{1}{2}\nu\int_{\mathbb{T}}|\mathbf{curl}\,\boldsymbol{u}^{\varepsilon}|^2\mathrm{d}\boldsymbol{x} + \frac{1}{2}\nu C\int_{\mathbb{T}}|\boldsymbol{u}^{\varepsilon}|^2\mathrm{d}\boldsymbol{x} \geq C_1\|\boldsymbol{u}^{\varepsilon}\|^2_{H^1_{\mathbf{curl}}(\mathbb{T})},$$

where $C_1 = \min\{\nu/2,\ \nu C/2\}$.

The continuity of $B(\cdot,\cdot)$ follows by the boundedness of A and the Cauchy-Schwarz inequality:

$$\begin{aligned}
B(\boldsymbol{u},\boldsymbol{\varphi}) &= \int_{\mathbb{T}} \hat{A}^{\varepsilon}\mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\\
&\leq \Big(\int_{\mathbb{T}}|\hat{A}^{\varepsilon}\mathbf{curl}\,\boldsymbol{u}|^2\Big)^{1/2}\Big(\int_{\mathbb{T}}|\mathbf{curl}\,\boldsymbol{\varphi}|^2\Big)^{1/2}\\
&\leq \hat{C}\nu^{-1}\Big(\int_{\mathbb{T}}|\mathbf{curl}\,\boldsymbol{u}|^2\Big)^{1/2}\Big(\int_{\mathbb{T}}|\mathbf{curl}\,\boldsymbol{\varphi}|^2\Big)^{1/2}\\
&\leq \hat{C}\nu^{-1}\|\boldsymbol{u}^{\varepsilon}\|_{H^1_{\mathbf{curl}}(\mathbb{T})}\|\boldsymbol{\varphi}\|_{H^1_{\mathbf{curl}}(\mathbb{T})},
\end{aligned}$$

for $\hat{C} > 0$. □

Remark. Note that in some circumstances, the following equation may want to be studied as opposed to equation (1.1.1):

$$\mathbf{curl}\hat{A}^{\varepsilon}\mathbf{curl}\,\boldsymbol{u}^{\varepsilon} + \lambda\boldsymbol{u}^{\varepsilon} = \boldsymbol{f}, \quad \lambda > 0.$$

This equation arises if consideration is given to the frequency dependent Maxwell equations as opposed to the quasistatically approximated equation being studied here (see Section 1.3.1. It is further noted that in the consideration of the equation above, it suffices to consider $\boldsymbol{u}^{\varepsilon} \in H^1_{\mathbf{curl}}(\mathbb{T})$.

1.1.2 Asymptotic Expansion of the Solution of Equation (1.1.1)

A solution of equation (1.1.1) is sought in the form of a two-scale power series in powers of ε, i.e., a solution is sought in the form

$$\boldsymbol{u}^{\varepsilon}(\boldsymbol{x}) = \sum_{j=0}^{\infty}\varepsilon^j\boldsymbol{u}_j\Big(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}\Big), \quad \boldsymbol{x} \in \mathbb{T}.$$

A particular form in the coefficients $\boldsymbol{u}_j$ arises upon substitution of the above into (1.1.1). Namely:

$$\boldsymbol{u}^\varepsilon(\boldsymbol{x}) = \boldsymbol{v}(\boldsymbol{x},\varepsilon) + \sum_{j=1}^{\infty} \varepsilon^j \Big\{ \nabla_y\big(K^{(j)}(\boldsymbol{y})\nabla_x^j \boldsymbol{v}(\boldsymbol{x},\varepsilon)\big) + \nabla_x\big(K^{(j-1)}(\boldsymbol{y})\nabla_x^{j-1}\boldsymbol{v}(\boldsymbol{x},\varepsilon)\big) + \\ + N^{(j)}(\boldsymbol{y})\nabla_x^{j-1}\mathbf{curl}_x \boldsymbol{v}(\boldsymbol{x},\varepsilon)\Big\}\Big|_{\boldsymbol{y}=\boldsymbol{x}/\varepsilon}. \qquad (1.1.5)$$

The divergence-free vector field $\boldsymbol{v} \in [C^\infty_{\text{per}}(\mathbb{T})]^3$ is sought in a power series:

$$\boldsymbol{v}(\boldsymbol{x},\varepsilon) = \boldsymbol{v}_0(\boldsymbol{x}) + \sum_{k=1}^{\infty} \varepsilon^k \boldsymbol{v}_k(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{T}, \qquad (1.1.6)$$

where the $\boldsymbol{v}_k$ do not depend on ε, $k = 1, 2, \ldots$. The coefficients $K^{(j)}$ are tensors of order $(j+1)$, $j = 0, 1, \ldots$ whose components belong to the space $\{\, g \in H^2_{\text{per}}(Q) \mid \langle g\rangle = 0\,\}$ and the coefficients $N^{(j)}$ are divergence-free tensors of order $(j+1)$, $j = 1, 2, \ldots$ whose components belong to the space $\{\, g \in H^1_{\text{per}}(Q) \mid \langle g\rangle = 0,\,\}$. Note that the coefficient $K^{(0)}$ is assumed to be constant.

Remark. The following remark lists some technicalities regarding the tensors described above.

1. The tensor products in (1.1.5) are evaluated in the following way:

$$K^{(j)}\nabla_x^j \boldsymbol{v} = K^{(j)}_{i_1 i_2 \ldots i_{j+1}} v_{i_{j+1}, i_1 \ldots i_j},$$ [2]

$$N^{(j)}\nabla_x^{j-1}\mathbf{curl}_x \boldsymbol{v} = N^{(j)}_{i_1 i_2 \ldots i_{j+1}} (\mathbf{curl}_x \boldsymbol{v})_{i_{j+1}, i_2 \ldots i_j},$$

   for $i_k = 1,2,3,\ k = 1,2,\ldots,j+1$, where the Einstein summation convention is being used for repeated indices and indices following a comma denote differentiation.

2. The divergence of a tensor of order $(j+1)$ is a tensor of order j and is evaluated in the following way:

$$(\operatorname{div} N^{(j)})_{i_1 i_2 \ldots i_j} = N^{(j)}_{s i_1 i_2 \ldots i_j, s}.$$

   Similarly, the **curl** of a tensor of order $(j+1)$ is a tensor of order $(j+1)$ and is evaluated in the following way:

$$(\mathbf{curl}\, N^{(j)})_{i_1 i_2 \ldots i_{j+1}} = \epsilon_{i_1 s t} N^{(j)}_{t i_2 \ldots i_{j+1}, s}.$$

In what follows, series (1.1.5) is formally substituted into equation (1.1.1) where the slow variables $\boldsymbol{x} = (x_1, x_2, x_3)$ and the fast variables $\boldsymbol{y} = (y_1, y_2, y_3)$ are treated independently,

[2] The notation ∇^j is used to represent the repeated use of the gradient operator: $\nabla^j_x = D^\alpha_x = \partial^{|\alpha|}/\partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}\partial^{\alpha_3}_{x_3}$ where $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3_0$ is a multi-index and $|\alpha| := \alpha_1 + \alpha_2 + \alpha_3 = j$.

so that in particular $\nabla = \nabla_x + \varepsilon^{-1}\nabla_y$, $\quad$ $\mathbf{curl} = \mathbf{curl}_x + \varepsilon^{-1}\mathbf{curl}_y$. Note also the following identities

$$\mathbf{curl}_x(\nabla_x(\cdot)) = \mathbf{curl}_y(\nabla_y(\cdot)) = \mathbf{0}, \quad \mathbf{curl}_x(\nabla_y(\cdot)) = -\mathbf{curl}_y(\nabla_x(\cdot)). \tag{1.1.7}$$

The arguments $\boldsymbol{x}$ and $\boldsymbol{y}$ will be omitted from the subsequent manipulations for brevity. Performing the afore mentioned substitution yields

$$\begin{aligned}\mathbf{curl}_x A\,\mathbf{curl}_x \boldsymbol{v} + \varepsilon^{-1}\mathbf{curl}_y A\,\mathbf{curl}_x \boldsymbol{v} + \sum_{j=1}^{\infty}\varepsilon^j\Big\{\mathbf{curl}_x A\,\mathbf{curl}_x(N^{(j)}\nabla_x^{j-1}\mathbf{curl}_x\boldsymbol{v})\Big\}\\ + \sum_{j=1}^{\infty}\varepsilon^{j-1}\Big\{\mathbf{curl}_y A\,\mathbf{curl}_x(N^{(j)}\nabla_x^{j-1}\mathbf{curl}_x\boldsymbol{v}) + \mathbf{curl}_x A\,\mathbf{curl}_y(N^{(j)}\nabla_x^{j-1}\mathbf{curl}_x\boldsymbol{v})\Big\}\\ + \sum_{j=1}^{\infty}\varepsilon^{j-2}\Big\{\mathbf{curl}_y A\,\mathbf{curl}_y(N^{(j)}\nabla_x^{j-1}\mathbf{curl}_x\boldsymbol{v})\Big\} = \boldsymbol{f}.\end{aligned} \tag{1.1.8}$$

All terms involving the tensors $K^{(j)}$ have cancelled with one another as a result of identities (1.1.7) except for one term $\mathbf{curl}_y\nabla_x(K^{(0)}(\boldsymbol{y})\boldsymbol{v}(\boldsymbol{x},\varepsilon))$. This term vanishes by observing that the vector $K^{(0)}$ is constant. In equation (1.1.8), the coefficients of ε are equated and hence

$$O(\varepsilon^{-1}): \qquad \mathbf{curl}_y A\Big\{\mathbf{curl}_y(N^{(1)}\mathbf{curl}_x\boldsymbol{v}) + \mathbf{curl}_x\boldsymbol{v}\Big\} = \mathbf{0}. \tag{1.1.9}$$

$$\begin{aligned}O(\varepsilon^{0}): \qquad \mathbf{curl}_y A\,\mathbf{curl}_y(N^{(2)}\nabla_x\mathbf{curl}_x\boldsymbol{v}) &= \boldsymbol{f} - \mathbf{curl}_x A\,\mathbf{curl}_x\boldsymbol{v}\\ &- \mathbf{curl}_y A\,\mathbf{curl}_x(N^{(1)}\mathbf{curl}_x\boldsymbol{v}) - \mathbf{curl}_x A\,\mathbf{curl}_y(N^{(1)}\mathbf{curl}_x\boldsymbol{v}).\end{aligned} \tag{1.1.10}$$

Equation (1.1.9) should be satisfied for all admissible vectors $\boldsymbol{v}$ and hence $N^{(1)}$ satisfies the so-called unit cell equation

$$\mathbf{curl}\,A\mathbf{curl}\,N^{(1)} = -\mathbf{curl}\,A, \quad \iff \quad \Big\langle A\Big(\mathbf{curl}\,N^{(1)} + I\Big)\,\mathbf{curl}\,\phi\Big\rangle = 0, \quad \forall\phi \in [C^\infty_{\text{per}}(Q)]^3.$$

The matrix $N^{(1)}$ is determined uniquely under the conditions that it is Q-periodic and has zero average.

Viewing (1.1.10) as an equation for $N^{(2)}$, applying the condition of solvability (see Lemma 1.1.1 for full details) yields

$$\mathbf{curl}\,\hat{h}^{(2)}\mathbf{curl}\,\boldsymbol{v} = \boldsymbol{f}, \quad \boldsymbol{v} \in X(\mathbb{T}), \qquad \hat{h}^{(2)} = \Big\langle A\Big(\mathbf{curl}\,N^{(1)} + I\Big)\Big\rangle. \tag{1.1.11}$$

Equation (1.1.11) is referred to as the homogenised equation (c.f. Bakhvalov & Panasenko [6] for the scalar homogenised equation and Wellander [75] for the Maxwell homogenised equation) and $\hat{h}^{(2)}$ is referred to as the homogenised matrix. The homogenised equation captures properties of the original problem (1.1.4) in the limit as $\varepsilon \to 0$. In the next section, the infinite-order homogenised equation will be derived. This equation will describe the original problem in the case when a small but finite value for ε is considered.

1.1.3 Infinite-Order Homogenised Equation

Denote by $\boldsymbol{H}_j = \boldsymbol{H}_j(\boldsymbol{x}, \boldsymbol{y})$ the expression for the coefficient of ε^j, $j = 0, 1, \dots$ in the expansion (1.1.8). Explicitly

$$\begin{aligned}\boldsymbol{H}_0 = \mathbf{curl}_x A\, \mathbf{curl}_x \boldsymbol{v} + \mathbf{curl}_y A\, \mathbf{curl}_x (N^{(1)} \mathbf{curl}_x \boldsymbol{v}) + \\ + \mathbf{curl}_x A\, \mathbf{curl}_y (N^{(1)} \mathbf{curl}_x \boldsymbol{v}) + \mathbf{curl}_y A\, \mathbf{curl}_y (N^{(2)} \nabla_x \mathbf{curl}_x \boldsymbol{v}),\end{aligned} \qquad (1.1.12)$$

$$\begin{aligned}\boldsymbol{H}_j = \mathbf{curl}_x A\, \mathbf{curl}_x (N^{(j)} \nabla_x^{j-1} \mathbf{curl}_x \boldsymbol{v}) + \mathbf{curl}_y A\, \mathbf{curl}_x (N^{(j+1)} \nabla_x^{j} \mathbf{curl}_x \boldsymbol{v}) + \\ + \mathbf{curl}_x A\, \mathbf{curl}_y (N^{(j+1)} \nabla_x^{j} \mathbf{curl}_x \boldsymbol{v}) + \mathbf{curl}_y A\, \mathbf{curl}_y (N^{(j+2)} \nabla_x^{j+1} \mathbf{curl}_x \boldsymbol{v}), \quad j = 1, 2, \dots.\end{aligned} \qquad (1.1.13)$$

The aim is to write $\boldsymbol{H}_j(\boldsymbol{x}, \boldsymbol{y})$ in the form $h^{(j+2)}(\boldsymbol{y}) \nabla_x^{j+1} \mathbf{curl}_x \boldsymbol{v}(\boldsymbol{x})$ by commuting all $\boldsymbol{x}$-derivatives through to the right-hand sides of the tensors $N^{(j)}$ in expressions (1.1.12)-(1.1.13). This is accomplished by introducing tensors $M^{(j)} = M^{(j)}(\boldsymbol{y})$ and $L^{(j)} = L^{(j)}(\boldsymbol{y})$ of order $(j+1)$, such that the following operator identities hold for all $j = 2, 3, \dots$:

$$M^{(1)} = I, \qquad M^{(j)}_{i_1 \dots i_{j+1}} = \left(\epsilon N^{(j-1)}\right)_{i_1 \dots i_{j+1}} = \epsilon_{i_1 i_2 s} N^{(j-1)}_{s i_3 \dots i_{j+1}}, \qquad (1.1.14)$$

$$L^{(j)}_{i_1 \dots i_{j+1}} = \left(\epsilon A \big\{ \mathbf{curl}\, N^{(j-1)} + M^{(j-1)} \big\}\right)_{i_1 \dots i_{j+1}} = \epsilon_{i_1 i_2 s} A_{st} \big\{ \mathbf{curl}\, N^{(j-1)} + M^{(j-1)} \big\}_{t i_3 \dots i_{j+1}}. \qquad (1.1.15)$$

In essence, these tensors commute derivatives in $\boldsymbol{x}$ through the tensors $N^{(j)}$, i.e.

$$M^{(j)} \nabla_x = \mathbf{curl}_x N^{(j-1)}, \quad L^{(j)} \nabla_x = \mathbf{curl}_x A \big\{ \mathbf{curl}\, N^{(j-1)} + M^{(j-1)} \big\}, \quad \forall j = 2, 3, \dots.$$

Utilising definitions (1.1.14)-(1.1.15), the expression for $\boldsymbol{H}_j$, $j = 0, 1, \dots$ is rewritten as

$$\boldsymbol{H}_j(\boldsymbol{x}, \boldsymbol{y}) = h^{(j+2)}(\boldsymbol{y}) \nabla_x^{j+1} \mathbf{curl}_x \boldsymbol{v}(\boldsymbol{x}),$$

where

$$h^{(j+2)} = \mathbf{curl} A\, \mathbf{curl}\, N^{(j+2)} + \mathbf{curl} A M^{(j+2)} + L^{(j+2)}. \qquad (1.1.16)$$

Formally, the left-hand side of equation (1.1.1) may be written as

$$(L_\varepsilon \boldsymbol{u}^\varepsilon)(\boldsymbol{x}) = \sum_{j=0}^{\infty} \varepsilon^j h^{(j+2)}(\boldsymbol{y}) \nabla_x^{j+1} \mathbf{curl}_x \boldsymbol{v}(\boldsymbol{x}).$$

By analogy with the matrix $\hat{h}^{(2)}$, the tensors $h^{(j+2)}$ are asked to be independent of the fast variable $\boldsymbol{y}$. The system of equations (1.1.16) can be used to recursively determine the tensors $N^{(j+2)}, h^{(j+2)}$ $j = 0, 1, \dots,$ uniquely as shown by the following lemma.

Lemma 1.1.1. *Let $F^{(j)}$ be a tensor field of order $(j+1)$ whose components are differentiable and Q-periodic. Furthermore, assume that A is a positive definite, Q-periodic* *matrix. The divergence-free, Q-periodic tensor field $N^{(j)}$ with zero average over Q solves the equation*

$$\mathbf{curl}\, A\mathbf{curl}\, N^{(j)} = F^{(j)}, \tag{1.1.17}$$

if and only if

$$\operatorname{div} F^{(j)} = 0, \quad \langle F^{(j)} \rangle = 0. \tag{1.1.18}$$

Proof. Taking the divergence and the average of both sides of equation (1.1.17) and noting that both A and $N^{(j)}$ are Q-periodic yields one half of the desired result. Since the operator $\mathbf{curl}\, A\mathbf{curl}$ on $X(Q)$ is self-adjoint, then it suffices that the right-hand side of (1.1.17) is $L^2(Q)$-orthogonal to all elements of the kernel of the left-hand side, namely[3]

$$\int_Q F^{(j)}(\boldsymbol{y}) \cdot \nabla w^{(j-1)}(\boldsymbol{x}, \boldsymbol{y})\, \mathrm{d}\boldsymbol{y} = 0, \quad \int_Q F^{(j)}(\boldsymbol{y}) \cdot c^{(j)}(\boldsymbol{x})\, \mathrm{d}\boldsymbol{y} = 0,$$

for all smooth tensor fields $w^{(j-1)}$ of order $(j-1)$ and for all constant tensor fields $c^{(j)}$ of order j. The result immediately follows by (1.1.18). $\square$

Therefore, by Lemma 1.1.1, the following equation must hold:

$$\Big\langle -\mathbf{curl}\, AM^{(j+2)} - L^{(j+2)} + h^{(j+2)} \Big\rangle = 0 \qquad \Rightarrow \qquad h^{(j+2)} = \Big\langle L^{(j+2)} \Big\rangle.$$

Hence, it follows that

$$\begin{aligned} h^{(j+2)} \nabla^{j+1} \mathbf{curl}\, \boldsymbol{v}(\boldsymbol{x}, \varepsilon) &= \Big\langle L^{(j+2)} \Big\rangle \nabla^{j+1} \mathbf{curl}\, \boldsymbol{v}(\boldsymbol{x}, \varepsilon) \\ &= \mathbf{curl} \left\langle A \Big\{ \mathbf{curl}\, N^{(j+1)} + M^{(j+1)} \Big\} \right\rangle \nabla^{j} \mathbf{curl}\, \boldsymbol{v}(\boldsymbol{x}, \varepsilon). \end{aligned}$$

The "infinite-order homogenised equation" for $\boldsymbol{u}^\varepsilon$ takes the form

$$\mathbf{curl}\, \hat{h}^{(2)} \mathbf{curl}\, \boldsymbol{v}(\boldsymbol{x}, \varepsilon) + \sum_{j=1}^{\infty} \varepsilon^j \mathbf{curl}\, \hat{h}^{(j+2)} \nabla^j \mathbf{curl}\, \boldsymbol{v}(\boldsymbol{x}, \varepsilon) = \boldsymbol{f}(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{T}, \tag{1.1.19}$$

where

$$\hat{h}^{(j+2)} = \left\langle A \Big\{ \mathbf{curl}\, N^{(j+1)} + M^{(j+1)} \Big\} \right\rangle, \tag{1.1.20}$$

are the higher-order homogenised coefficients. These higher-order coefficients $\hat{h}^{(j+2)}$ are related to the tensors $h^{(j+2)}$ via the relation $\mathbf{curl}\, \hat{h}^{(j+2)} = h^{(j+2)} \nabla$.

All the calculations so far have been formal and it will be shown in Section 1.1.5 that the asymptotic expansion for the solution of (1.1.4) is "close" to the true solution in a certain sense. Before doing so, a brief discussion on the system of equations which determine the tensors $K^{(j)}$ will follow. These equations arise as a result of taking the divergence of the asymptotic expansion (1.1.5).

[3]The operation of convolution denoted $\cdot$ is defined as follows; if $F^{(j)}$ and $G^{(j)}$ are two tensors of order j, then $F^{(j)} \cdot G^{(j)} = (F^{(j)})_{i_1 i_2 \dots i_j} (G^{(j)})_{i_1 i_2 \dots i_j}$.

1.1.4 Recurrence Relations for $K^{(j)}, j = 2, 3, \ldots$

The tensors $K^{(j)}$, $j = 0, 1, \ldots$ are determined by a system of recurrent relations which arise in the consideration of the solvability condition that $\operatorname{div} \boldsymbol{u}^\varepsilon = 0$. Indeed, taking the formal divergence of the asymptotic expansion (1.1.5) yields

$$\operatorname{div} \boldsymbol{u}^\varepsilon = \sum_{j=1}^{\infty} \varepsilon^j \Big\{ \Delta_{xy}(K^{(j)} \nabla_x^j \boldsymbol{v}) + \Delta_x (K^{(j-1)} \nabla_x^{j-1} \boldsymbol{v}) + \operatorname{div}_x (N^{(j)} \nabla_x^{j-1} \mathbf{curl}_x \boldsymbol{v}) \Big\} +$$
$$+ \sum_{j=1}^{\infty} \varepsilon^{j-1} \Big\{ \Delta_y (K^{(j)} \nabla_x^j \boldsymbol{v}) + \Delta_{xy} (K^{(j-1)} \nabla_x^{j-1} \boldsymbol{v}) \Big\} = 0,$$

where
$$\Delta_x = \frac{\partial^2}{\partial x_i \partial x_i}, \quad \Delta_y = \frac{\partial^2}{\partial y_i \partial y_i}, \quad \Delta_{xy} = \frac{\partial^2}{\partial x_i \partial y_i}.$$

Comparing terms with equal powers of ε, in the above expression yields the following system of recurrence relations

$$\Delta_y (K^{(1)} \nabla_x \boldsymbol{v}) = 0, \tag{1.1.21}$$

$$\Delta_y (K^{(l+1)} \nabla_x^{l+1} \boldsymbol{v}) + 2\Delta_{xy} (K^{(l)} \nabla_x^l \boldsymbol{v}) + \Delta_x (K^{(l-1)} \nabla_x^{l-1} \boldsymbol{v}) + \operatorname{div}_x (N^{(l)} \nabla_x^{l-1} \mathbf{curl}_x \boldsymbol{v}) = 0, \tag{1.1.22}$$

for $l = 1, 2, \ldots$. The Q-periodic solution $K^{(1)}$ of equation (1.1.21) which has zero average over Q is identically zero. Hence the first non-trivial tensor is the third-order tensor $K^{(2)}$ which satisfies the equation

$$\Delta_y (K^{(2)} \nabla_x^2 \boldsymbol{v}) = -\operatorname{div}_x (N^{(1)} \mathbf{curl}_x \boldsymbol{v}),$$

for admissible vector $\boldsymbol{v}$. Substituting series (1.1.6) into the system of equations (1.1.22) above yields

$$\sum_{\substack{j+k=l \\ j \in \mathbb{N},\ k \in \mathbb{N}_0}} \Big\{ \Delta_y (K^{(j+1)} \nabla_x^{j+1} \boldsymbol{v}_k) + 2\Delta_{xy} (K^{(j)} \nabla_x^j \boldsymbol{v}_k) + \Delta_x (K^{(j-1)} \nabla_x^{j-1} \boldsymbol{v}_k) +$$
$$+ \operatorname{div}_x (N^{(j)} \nabla_x^{j-1} \mathbf{curl}_x \boldsymbol{v}_k) \Big\} = 0, \quad l = 1, 2, \ldots. \tag{1.1.23}$$

This system of equations will be pertinent to establishing a bound on $\operatorname{div} \boldsymbol{u}^\varepsilon$ and in turn, a bound on the remainder of the asymptotic series (1.1.5).

1.1.5 Rigorous Justification

The infinite-order homogenised equation (1.1.19) was obtained via a formal calculation in which the infinite-order series (1.1.5) was substituted in to equation (1.1.1). It will now be shown that series (1.1.5) considered is indeed close in the $L^2(\mathbb{T})$ sense to the exact solution of problem (1.1.4).

Theorem 1.1.2. *Let K be a positive integer and define a remainder $\boldsymbol{R}_K$ by the relation*

$$\boldsymbol{R}_K(\boldsymbol{x},\varepsilon) = \boldsymbol{u}^{\varepsilon}(\boldsymbol{x}) - \boldsymbol{u}^{(K)}(\boldsymbol{x},\varepsilon), \tag{1.1.24}$$

where

$$\boldsymbol{u}^{(K)}(\boldsymbol{x},\varepsilon) = \boldsymbol{v}^{(K)}(\boldsymbol{x},\varepsilon) + \sum_{j=1}^{K} \varepsilon^j \Big\{ \nabla_y (K^{(j)}(\boldsymbol{y}) \nabla_x^j \boldsymbol{v}^{(K)}(\boldsymbol{x},\varepsilon)) + \nabla_x (K^{(j-1)}(\boldsymbol{y}) \nabla_x^{j-1} \boldsymbol{v}^{(K)}(\boldsymbol{x},\varepsilon)) +$$
$$+ N^{(j)}(\boldsymbol{y}) \nabla_x^{j-1} (\mathbf{curl}_x \boldsymbol{v}^{(K)}(\boldsymbol{x},\varepsilon)) \Big\}\Big|_{\boldsymbol{y}=\boldsymbol{x}/\varepsilon}, \tag{1.1.25}$$

and

$$\boldsymbol{v}^{(K)}(\boldsymbol{x},\varepsilon) = \boldsymbol{v}_0(\boldsymbol{x}) + \sum_{k=1}^{K} \varepsilon^k \boldsymbol{v}_k(\boldsymbol{x}). \tag{1.1.26}$$

Then

(i) $\|\mathbf{curl}\, \boldsymbol{R}_K\|_{L^2(\mathbb{T})} \le C_1^{(K)} \varepsilon^{K-1}$,

(ii) $\|\mathrm{div}\, \boldsymbol{R}_K\|_{H^{-1}(\mathbb{T})} \le C_2^{(K)} \varepsilon^{K}$,

(iii) $\forall M,\ |\langle \boldsymbol{R}_K \rangle_{\mathbb{T}}| \le C_M^{(K)} \varepsilon^{M}$,

where the constants $C_1^{(K)},\ C_2^{(K)},\ C_M^{(K)}$ are independent of ε but may depend on $\boldsymbol{f}$.

Proof. (i) Substitution of series (1.1.6) into the homogenised equation (1.1.19) yields the following sequence of recurrence relations on the coefficients on series (1.1.6):

$$\mathbf{curl}\, \hat{h}^{(2)} \mathbf{curl}\, \boldsymbol{v}_0 = \boldsymbol{f},$$
$$\mathbf{curl}\, \hat{h}^{(2)} \mathbf{curl}\, \boldsymbol{v}_l + \sum_{\substack{j+k=l \\ j\in\mathbb{N},\ k\in\mathbb{N}_0}} \mathbf{curl}\, \hat{h}^{(j+2)} \nabla^j \mathbf{curl}\, \boldsymbol{v}_k = \mathbf{0}, \quad l = 1,2,\dots \tag{1.1.27}$$
$$\boldsymbol{v}_k \in X(\mathbb{T}), \qquad k = 0,1,\dots. \tag{1.1.28}$$

Lemma 1.1.2. *The matrix $\hat{h}^{(2)}$ is symmetric and positive definite. Hence for any given right-hand sides $\boldsymbol{f} \in [C^{\infty}_{\mathrm{per}}(\mathbb{T})]^3 \cap X(\mathbb{T})$, there exists a unique solution sequence $\boldsymbol{v}_k,\ k = 0,1,\dots$ to equations (1.1.27)-(1.1.28).*

Proof. Recall that $\hat{h}^{(2)}$ is given by the formula in (1.1.11). By equation (1.1.9), it follows by the periodicity of A that

$$\Big\langle \Big(A \mathbf{curl}\, N^{(1)} + A \Big) \mathbf{curl}\, \boldsymbol{\phi} \Big\rangle = \mathbf{0}, \quad \forall \boldsymbol{\phi}.$$

In particular, for $\boldsymbol{\phi} = N^{(1)}$, it follows that

$$\hat{h}^{(2)} = \Big\langle A \Big\{ \mathbf{curl}\, N^{(1)} + I \Big\} + \Big(A \mathbf{curl}\, N^{(1)} + A \Big) \mathbf{curl}\, N^{(1)} \Big\rangle$$
$$= \Big\langle A \Big\{ \Big(\mathbf{curl}\, N^{(1)} + I \Big) \Big(\mathbf{curl}\, N^{(1)} + I \Big) \Big\} \Big\rangle.$$

Since A is symmetric and positive definite, so is $\hat{h}^{(2)}$. Applying the Lax-Milgram Lemma to each problem (1.1.27)-(1.1.28) yields the result. □

Evaluating the expression for $L_\varepsilon \boldsymbol{u}^{(K)}$ explicitly yields

$$L_\varepsilon \boldsymbol{u}^{(K)} = \mathbf{curl}_x \hat{h}^{(2)} \mathbf{curl}_x \boldsymbol{v}^{(K)} + \sum_{j=1}^{K-2} \varepsilon^j \mathbf{curl}_x \hat{h}^{(j+2)} \mathbf{curl}_x \boldsymbol{v}^{(K)} + \varepsilon^{K-1} \boldsymbol{\theta}_1(\boldsymbol{v}^{(K)}; \varepsilon, K), \tag{1.1.29}$$

where

$$\begin{aligned}\boldsymbol{\theta}_1(\boldsymbol{\cdot}; \varepsilon, K) = \varepsilon \mathbf{curl}_x A\, \mathbf{curl}_x \left(\nabla_y \left(K^{(K)} \nabla_x^K \boldsymbol{\cdot}\right) + N^{(K)} \nabla_x^{K-1} \mathbf{curl}_x \boldsymbol{\cdot}\right) + \\ + \mathbf{curl}_y A\, \mathbf{curl}_x \left(\nabla_y \left(K^{(K)} \nabla_x^K \boldsymbol{\cdot}\right)\right) + \left(\mathbf{curl}_y A M^{(K+1)} + L^{(K+1)}\right) \nabla_x^K \mathbf{curl}_x \boldsymbol{\cdot}\, .\end{aligned}$$

Substituting the expansion (1.1.26) into equation (1.1.29) yields

$$\begin{aligned}L_\varepsilon \boldsymbol{u}^{(K)} = \mathbf{curl}_x \hat{h}^{(2)} \mathbf{curl}_x \boldsymbol{v}_0 + \sum_{k=1}^{K} \varepsilon^k \mathbf{curl}_x \hat{h}^{(2)} \mathbf{curl}_x \boldsymbol{v}_k + \\ + \sum_{j=1}^{K-2} \sum_{k=0}^{K} \varepsilon^{j+k} \mathbf{curl}_x \hat{h}^{(j+2)} \nabla_x^j \mathbf{curl}_x \boldsymbol{v}_k + \varepsilon^{K-1} \sum_{k=0}^{K} \varepsilon^k \boldsymbol{\theta}_1(\boldsymbol{v}_k; \varepsilon, K),\end{aligned}$$

In view of the set of equations (1.1.27), the above can be shown to reduce to

$$L_\varepsilon \boldsymbol{u}^{(K)} = \boldsymbol{f} + \varepsilon^{K-1} \boldsymbol{\theta}_2(\boldsymbol{x}, \varepsilon, K),$$

where

$$\boldsymbol{\theta}_2(\boldsymbol{x}, \varepsilon, K) := \sum_{j+k=K-1}^{2K-2} \varepsilon^{j+k-K+1} \mathbf{curl}_x \hat{h}^{(j+2)} \nabla_x^j \mathbf{curl}_x \boldsymbol{v}_k + \sum_{k=0}^{K} \varepsilon^k \boldsymbol{\theta}_1(\boldsymbol{v}_k; \varepsilon, K),$$

for $0 \le j \le K-2,\ 0 \le k \le K$. It can be shown that $|\boldsymbol{\theta}_2(\boldsymbol{x}, \varepsilon, K)| \le c_1(\boldsymbol{f})$ for all $\boldsymbol{x} \in \mathbb{T}$ where c_1 is independent of ε. Noting that $\boldsymbol{u}^\varepsilon$ satisfies the relation $L_\varepsilon \boldsymbol{u}^\varepsilon = \boldsymbol{f}$ yields

$$L_\varepsilon \boldsymbol{R}_K(\boldsymbol{x}, \varepsilon) = -\varepsilon^{K-1} \boldsymbol{\theta}_2(\boldsymbol{x}, \varepsilon, K). \tag{1.1.30}$$

Taking the scalar product of both sides of equation (1.1.30) with $\boldsymbol{R}_K$ and integrating over $\mathbb{T}$ yields

$$\int_{\mathbb{T}} \mathbf{curl}\, A\Big(\frac{\boldsymbol{x}}{\varepsilon}\Big) \mathbf{curl}\, \boldsymbol{R}_K \cdot \boldsymbol{R}_K \, \mathrm{d}\boldsymbol{x} = -\varepsilon^{K-1} \int_{\mathbb{T}} \boldsymbol{\theta}_2 \cdot \boldsymbol{R}_K \, \mathrm{d}\boldsymbol{x}.$$

Integrating by parts on the left-hand side and using the Cauchy-Schwarz inequality on the right-hand side leads to the following inequality

$$\begin{aligned}\int_{\mathbb{T}} A\Big(\frac{\boldsymbol{x}}{\varepsilon}\Big) \mathbf{curl}\, \boldsymbol{R}_K \cdot \mathbf{curl}\, \boldsymbol{R}_K \, \mathrm{d}\boldsymbol{x} \le \varepsilon^{K-1} \|\boldsymbol{\theta}_2\|_{L^2(\mathbb{T})} \|\boldsymbol{R}_K\|_{L^2(\mathbb{T})} \\ \Rightarrow \nu \|\mathbf{curl}\, \boldsymbol{R}_K\|^2_{L^2(\mathbb{T})} \le \varepsilon^{K-1} \|\boldsymbol{\theta}_2\|_{L^2(\mathbb{T})} \|\boldsymbol{R}_K\|_{L^2(\mathbb{T})}.\end{aligned}$$

Using a Maxwell inequality yields

$$\nu \|\mathbf{curl}\, \boldsymbol{R}_K\|^2_{L^2(\mathbb{T})} \le \varepsilon^{K-1} C(\mathbb{T}) \|\boldsymbol{\theta}_2\|_{L^2(\mathbb{T})} \Big(\|\mathbf{curl}\, \boldsymbol{R}_K\|_{L^2(\mathbb{T})} + \|\mathrm{div}\, \boldsymbol{R}_K\|_{L^2(\mathbb{T})}\Big),$$

where $C(\mathbb{T}) > 0$. It will be shown in the proof of (ii) that $\|\mathrm{div}\, \boldsymbol{R}_K\|_{L^2(\mathbb{T})} \leq C_2\varepsilon^K$. Hence

$$\begin{aligned}\|\mathbf{curl}\, \boldsymbol{R}_K\|^2_{L^2(\mathbb{T})} \leq& \frac{C(\mathbb{T})\|\boldsymbol{\theta}_2\|_{L^2(\mathbb{T})}}{\nu}\varepsilon^{K-1}\|\mathbf{curl}\, \boldsymbol{R}_K\|_{L^2(\mathbb{T})} + O(\varepsilon^{2K-1}),\\ \Rightarrow \|\mathbf{curl}\, \boldsymbol{R}_K\|_{L^2(\mathbb{T})} \leq& C_1^{(K)}\varepsilon^{K-1},\end{aligned}$$

as required.

(ii) Note that $\mathrm{div}\, \boldsymbol{u}^\varepsilon = 0$ and so $\mathrm{div}\, \boldsymbol{R}_K = -\mathrm{div}\, \boldsymbol{u}^{(K)}$. Denote by $\boldsymbol{U}_l$ the coefficient of ε^l in the asymptotic expansion (1.1.25) once the series (1.1.26) has been substituted in, i.e.

$$\boldsymbol{U}_0 = \boldsymbol{v}_0, \quad \boldsymbol{U}_l = \sum_{\substack{j+k=l\\1\leq j\leq K\\0\leq k\leq K}} \Big\{\nabla_y(K^{(j)}\nabla_x^j\boldsymbol{v}_k) + \nabla_x(K^{(j-1)}\nabla_x^{j-1}\boldsymbol{v}_k) + N^{(j)}\nabla_x^{j-1}\mathbf{curl}_x\boldsymbol{v}_k\Big\},$$

$l = 1, 2, \ldots$. Hence

$$-\mathrm{div}\, \boldsymbol{R}_K = \sum_{l=1}^{2K}\Big\{\varepsilon^l\mathrm{div}_x\boldsymbol{U}_l + \varepsilon^{l-1}\mathrm{div}_y\boldsymbol{U}_l\Big\}\Big|_{\boldsymbol{y}=\boldsymbol{x}/\varepsilon}.$$

The vectors $\boldsymbol{v}_k$ satisfy the system of equations (1.1.23) and thus

$$-\mathrm{div}\, \boldsymbol{R}_K = \varepsilon^K\theta_3(\boldsymbol{x},\varepsilon,K), \quad \theta_3(\boldsymbol{x},\varepsilon,K) = \left\{\sum_{l=K}^{2K}\varepsilon^{l-K}\mathrm{div}_x\boldsymbol{U}_l + \sum_{l=K+1}^{2K}\varepsilon^{l-K-1}\mathrm{div}_y\boldsymbol{U}_l\right\}\Bigg|_{\boldsymbol{y}=\boldsymbol{x}/\varepsilon}.$$

Notice that $\boldsymbol{U}_l$ is a finite sum of terms of the form $U(\boldsymbol{y})V(\boldsymbol{x})$ for some tensors U with elements in $H^1(Q)$ and tensors V with elements in $C^\infty_{\mathrm{per}}(\mathbb{T})$. Furthermore, since

$$\mathrm{div}_y\boldsymbol{U}_l|_{\boldsymbol{y}=\boldsymbol{x}/\varepsilon} = \varepsilon\mathrm{div}(\boldsymbol{U}_l|_{\boldsymbol{y}=\boldsymbol{x}/\varepsilon}) - \varepsilon\mathrm{div}_x\boldsymbol{U}_l|_{\boldsymbol{y}=\boldsymbol{x}/\varepsilon},$$

it follows that θ_3 is a finite sum of terms of the form $\tilde{U}(\boldsymbol{y})\tilde{V}(\boldsymbol{x})$ for some tensors $\tilde{U}$ with elements in $L^2(Q)$ and $\tilde{V}$ with elements in $C^\infty_{\mathrm{per}}(\mathbb{T})$. Using a version of the theorem proven in Smyshlyaev & Cherednichenko [69], (see Appendix 1.B) it follows that the $L^2(\mathbb{T})$-norm of θ_3 is bounded by a constant independent of ε, $C_2^{(K)}$ say. Hence

$$\|\mathrm{div}\, \boldsymbol{R}_K\|_{L^2(\mathbb{T})} \leq \varepsilon^K\|\theta_3\|_{L^2(\mathbb{T})} \leq C_2\varepsilon^K.$$

(iii) Noting the Q-periodicity of $K^{(j)}$ and the $\mathbb{T}$-perioddicty of $\boldsymbol{v}$, integrating (1.1.25) over $\mathbb{T}$ yields

$$\int_{\mathbb{T}}\boldsymbol{u}^{(K)}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{T}}\Big\{\sum_{j=1}^{K}\varepsilon^jN^{(j)}(\boldsymbol{x}/\varepsilon)\nabla_x^{j-1}(\mathbf{curl}_x\boldsymbol{v}^{(K)}(\boldsymbol{x}))\Big\}\,\mathrm{d}\boldsymbol{x}.$$

The result follows by once again using the theorem in Appendix 1.B.

□

Corollary 1.1.1. *For $\boldsymbol{R}_K$ as defined by relation (1.1.24), the following estimate holds:*

$$\|\boldsymbol{R}_K\|_{L^2(\mathbb{T})} \leq C_3^{(K)}\varepsilon^{K-1}, \quad \|\boldsymbol{R}_K\|_{H^1_{\mathbf{curl}}(\mathbb{T})} \leq C_4^{(K)}\varepsilon^{K-1},$$

where $C_3^{(K)}, C_4^{(K)}$ are independent of ε but may depend on the function $\boldsymbol{f}$.

Proof. This result is an immediate consequence of the previous theorem and the Maxwell inequality (1.A.1). □

This concludes the construction of the infinite-order homogenised equation which has now been made rigorous. It is worth noting however that truncating of (1.1.19) at some finite order of ε may not be the most suitable choice for the higher-order homogenised equation as the corresponding operator may not be elliptic. One way in which to avoid this loss of ellipticity is via the use of a variational approach which will be discussed in Section 1.2.

The theory described above will now be illustrated with an example where the matrix A is given by the formula $A(\boldsymbol{y}) = \alpha(y_2)I$ and where α is piecewise constant.

1.1.6 Example in a Two-Layered Medium

Consider the matrix $A(\boldsymbol{y})$ defined as

$$A(\boldsymbol{y}) = \alpha(y_2)I, \quad \alpha(y_2) = \begin{cases} \alpha_1, & 0 \leq y_2 \leq l_1, \\ \alpha_2, & l_1 < y_2 \leq 1, \end{cases} \quad 0 < l_1 < 1, \quad \alpha_1, \alpha_2 \in \mathbb{R}_+.$$

A higher-order homogenised expansion for the equation (1.1.1) will now be derived. As already seen, this higher-order expansion takes the form of (1.1.19):

$$L_\varepsilon \boldsymbol{u}^\varepsilon = \mathbf{curl}\, \hat{h}^{(2)}\mathbf{curl}\, \boldsymbol{v} + \varepsilon\mathbf{curl}\, \hat{h}^{(3)}\nabla\mathbf{curl}\, \boldsymbol{v} + \varepsilon^2\mathbf{curl}\, \hat{h}^{(4)}\nabla^2\mathbf{curl}\, \boldsymbol{v} + O(\varepsilon^3).$$

The first three tensors $\hat{h}^{(j)}$, $j = 2, 3, 4$ will now be derived for the two-layered medium. For the term of order $O(1)$, the matrices of interest are calculated as

$$N^{(1)} = \begin{pmatrix} 0 & 0 & -N \\ 0 & 0 & 0 \\ N & 0 & 0 \end{pmatrix}, \quad \hat{h}^{(2)} = \begin{pmatrix} \langle\alpha^{-1}\rangle^{-1} & 0 & 0 \\ 0 & \langle\alpha\rangle & 0 \\ 0 & 0 & \langle\alpha^{-1}\rangle^{-1} \end{pmatrix},$$

where $N = N(y_2)$ satisfies the differential equation $-(\alpha N')' = \alpha'$ and the prime notation denotes differentiation with respect to y_2. Furthermore, it is straightforward to see that

$$\langle\alpha^{-1}\rangle^{-1} = \{\alpha_1^{-1}l_1 + \alpha_2^{-1}(1 - l_1)\}^{-1}, \quad \langle\alpha\rangle = (\alpha_1 l_1 + \alpha_2(1 - l_1)).$$

For the term of order $O(\varepsilon)$, the related tensors are calculated as

$$N_{ijk}^{(2)} = \begin{cases} M, & ijk = \{123\}, \\ -M, & ijk = \{321\}, \\ L, & ijk = \{132\}, \\ -L, & ijk = \{312\}, \\ 0, & \text{otherwise}, \end{cases} \qquad \hat{h}_{ijk}^{(3)} = \begin{cases} a, & ijk = \{112,\ 332\}, \\ b, & ijk = \{211,\ 233\}, \\ 0, & \text{otherwise}, \end{cases}$$

where $M = M(y_2)$ satisfies the ODE $(\alpha M')' = (\alpha N)'$, $L = L(y_2)$ satisfies the ODE $(\alpha L')' = \langle \alpha \rangle - \alpha$, and $a = -\langle \alpha L' \rangle,\ b = -\langle \alpha N \rangle$.

It can be confirmed that $a = -b$ (see Appendix 1.C) and as a consequence the term of order $O(\varepsilon)$ in the higher-order expansion vanishes. Thus, the first non-trivial higher-order term in the infinite-order homogenised equation is the term of order $O(\varepsilon^2)$ where the relevant tensors in this case are calculated to be

$$N_{ijkl}^{(3)} = \begin{cases} P, & ijkl = \{1232\}, \\ -P, & ijkl = \{3212\}, \\ Q, & ijkl = \{1223\}, \\ -Q, & ijkl = \{3221\}, \\ R, & ijkl = \{1311, 1333\}, \\ -R, & ijkl = \{3111, 3133\}, \\ 0, & \text{otherwise}, \end{cases} \qquad \hat{h}_{ijkl}^{(4)} = \begin{cases} c, & ijkl = \{1212,\ 3232\}, \\ d, & ijkl = \{2121,\ 2323\}, \\ e, & ijkl = \{2112,\ 2332\}, \\ f, & ijkl = \begin{Bmatrix} 1111, 1133 \\ 3311, 3333 \end{Bmatrix}, \\ 0, & \text{otherwise}, \end{cases}$$

where $P = P(y_2)$ satisfies the ODE $-(\alpha P')' = a + \alpha L' + (\alpha L)'$, $Q = Q(y_2)$ satisfies the ODE $-(\alpha Q')' = (\alpha M)'$, $R = R(y_2)$ satisfies the ODE $(\alpha R')' = b + \alpha N$ and $c = -\langle \alpha P' + \alpha L \rangle,\ d = \langle \alpha Q' \rangle,\ e = \langle \alpha L \rangle,\ f = -\langle \alpha R' \rangle$.

It can be shown that the constants c, d, e and f are given by the formulae

$$\begin{aligned} c =& \frac{1}{12} l_1^2 l_2^2 [\alpha_1^{-1} l_1 + \alpha_2^{-1} l_2]^{-1} (1 - \beta_1)(\beta_2 - 1), \\ d =& \frac{1}{12} l_1^2 l_2^2 [\alpha_1^{-1} l_1 + \alpha_2^{-1} l_2]^{-1} (\beta_1 - 1)(\beta_2 - 1), \\ e =& \frac{1}{12} l_1^2 l_2^2 (\alpha_1^{-1} l_1 + \alpha_2^{-1} l_2)(\alpha_1 - \alpha_2)^2, \\ f =& \frac{1}{12} l_1^2 l_2^2 [\alpha_1^{-1} l_1 + \alpha_2^{-1} l_2]^{-2} (1 - \beta_1)(1 - \beta_2)(\alpha_2^{-1} l_1 + \alpha_1^{-1} l_2), \end{aligned}$$

where $l_1 + l_2 = 1$, and $\beta_1 = \frac{\alpha_1}{\alpha_2} = \beta_2^{-1}$. Note that it can also be shown (see Appendix 1.C) that $c = -d$ in the case when $A(\boldsymbol{y}) = \alpha(y_2)I$. Hence the infinite-order homogenised equation takes the form

$$L_\varepsilon \boldsymbol{u}^\varepsilon = \begin{pmatrix} \langle\alpha^{-1}\rangle^{-1}(v_{2,12}-v_{1,22}) - \langle\alpha\rangle(v_{1,33}-v_{3,13}) \\ \langle\alpha^{-1}\rangle^{-1}(v_{3,23}-v_{2,33}-v_{2,11}+v_{1,12}) \\ \langle\alpha\rangle(v_{1,13}-v_{3,11}) - \langle\alpha^{-1}\rangle^{-1}(v_{3,22}-v_{2,23}) \end{pmatrix} +$$

$$+\,\varepsilon^2 \begin{pmatrix} f(v_{3,1223}-v_{1,2233}) - e(v_{1,1133}-v_{3,1113}+v_{1,3333}-v_{3,1333}) \\ 0 \\ -f(v_{3,1122}-v_{1,1223}) + e(v_{1,1113}-v_{3,1111}+v_{1,1333}-v_{3,1133}) \end{pmatrix} + O(\varepsilon^3).$$

Remark. It was shown in Smyshlyaev & Cherednichenko [69], that in the case of a scalar equation, all terms with odd powers of ε are absent from the corresponding infinite-order homogenised equation. The above two-layered case provides an example where, in particular, there is a non-trivial term of order $O(\varepsilon^3)$ in the homogenisation procedure for the Maxwell system. The relevant tensors for the term of order $O(\varepsilon^3)$ are calculated as

$$N^{(4)}_{ijklm} = \begin{cases} N_1, & ijklm = \{12232\}, \\ -N_1, & ijklm = \{32212\}, \\ N_2, & ijklm = \{13121, 13323\}, \\ -N_2, & ijklm = \{31121, 31323\}, \\ N_3, & ijklm = \{13332, 13112\}, \\ -N_3, & ijklm = \{31332, 31112\}, \\ N_4, & ijklm = \{12311, 12333\}, \\ -N_4, & ijklm = \{32111, 32133\}, \\ N_5, & ijklm = \{12223\}, \\ -N_5, & ijklm = \{32221\}, \\ N_6, & ijklm = \{23212\}, \\ -N_6, & ijklm = \{21232\}, \\ N_7, & ijklm = \{23111, 23133\}, \\ -N_7, & ijklm = \{21311, 21333\}, \\ 0, & \text{otherwise}, \end{cases} \quad \hat{h}^{(5)}_{ijklm} = \begin{cases} h_1, & ijklm = \{12212, 32232\}, \\ h_2, & ijklm = \left\{\begin{matrix} 11121, & 11323 \\ 33121, & 33323 \end{matrix}\right\}, \\ h_3, & ijklm = \left\{\begin{matrix} 11112, & 11332 \\ 33112, & 33332 \end{matrix}\right\}, \\ h_4, & ijklm = \{21212, 23232\}, \\ h_5, & ijklm = \{21221, 23223\}, \\ h_6, & ijklm = \left\{\begin{matrix} 21111, & 21133 \\ 23311, & 23333 \end{matrix}\right\}, \\ h_7, & ijklm = \left\{\begin{matrix} 12111, & 12133 \\ 32311, & 32333 \end{matrix}\right\}, \\ 0, & \text{otherwise.} \end{cases}$$

The functions $N_i = N_i(y_2),\ i = 1, \dots, 7$, satisfy the ODEs

$$-(\alpha N_1')' = (\alpha P)' + \alpha P' + \alpha L + c, \quad -(\alpha N_2')' = \alpha M + d, \quad -(\alpha N_3')' = \alpha L - e,$$

$$-(\alpha N_4')' = (\alpha R)' + \alpha R' + f, \quad -(\alpha N_5')' = (\alpha Q)', \quad -(\alpha N_6')' = \alpha P' + \alpha L + c,$$

$$-(\alpha N_7')' = \alpha R' + f,$$

and the constants $h_i,\ i = 1, \dots, 7$ are

$$h_1 = -\langle \alpha N_1' + \alpha P \rangle, \quad h_2 = -\langle \alpha N_2' \rangle, \quad h_3 = -\langle \alpha N_3' \rangle,$$

$$h_4 = \langle \alpha P \rangle, \quad h_5 = \langle \alpha Q \rangle, \quad h_6 = \langle \alpha R \rangle, \quad h_7 = -\langle \alpha N_4' + \alpha R \rangle.$$

It can be shown that $h_1 = -h_5$, $h_2 = -h_7$ and $h_3 = -h_6$ and hence the term of order $O(\varepsilon^3)$ in the infinite-order homogenised equation for a two-layered medium is

$$\begin{pmatrix} -h_4(v_{1,11233} - v_{3,11123} + v_{1,23333} - v_{3,12333}) \\ 0 \\ h_4(v_{1,12333} - v_{3,11233} + v_{1,11123} - v_{3,11112}) \end{pmatrix},$$

where $h_4 \neq 0$.

1.2 Variational Approach

The work in the following section is driven towards the minimisation of the functional $E_\varepsilon(\boldsymbol{u}, \boldsymbol{f})$ over all admissible functions $\boldsymbol{u}$. The energy functional of interest is defined as

$$E_\varepsilon(\boldsymbol{u}, \boldsymbol{f}) := \int_{\mathbb{T}} \Big(\frac{1}{2}\hat{A}^\varepsilon \mathbf{curl}\,\boldsymbol{u} \cdot \mathbf{curl}\,\boldsymbol{u} - \boldsymbol{f} \cdot \boldsymbol{u}\Big),$$

and the class of admissible functions are those $\mathbb{T}$-periodic functions with zero average over $\mathbb{T}$ and zero divergence. Consider the following minimisation problem:

$$I(\varepsilon, \boldsymbol{f}) = \min_{\boldsymbol{u}} E_\varepsilon(\boldsymbol{u}, \boldsymbol{f}), \tag{1.2.1}$$

Equation (1.1.1) is the Euler-Lagrange equation for the minimisation problem (1.2.1) and therefore there exists a unique solution $\boldsymbol{u}^\varepsilon \in X(\mathbb{T})$ which coincides with the solution found in Section 1.1. Hence, since $\boldsymbol{u}^\varepsilon$ satisfies equation (1.1.2), it follows that

$$I(\varepsilon, \boldsymbol{f}) = -\frac{1}{2}\int_{\mathbb{T}} \hat{A}^\varepsilon \mathbf{curl}\,\boldsymbol{u}^\varepsilon \cdot \mathbf{curl}\,\boldsymbol{u}^\varepsilon = -\frac{1}{2}\int_{\mathbb{T}} \boldsymbol{f} \cdot \boldsymbol{u}^\varepsilon. \tag{1.2.2}$$

1.2.1 Variational Asymptotics

In light of equation (1.2.2), variational asymptotics for $I(\varepsilon, \boldsymbol{f})$ will be established. Similarly to (1.2.1), the solution of the homogenised equation (1.1.11) also minimises the functional

$$E_0(\boldsymbol{v}, \boldsymbol{f}) := \int_{\mathbb{T}} \Big(\frac{1}{2}\hat{h}^{(2)} \mathbf{curl}\,\boldsymbol{v} \cdot \mathbf{curl}\,\boldsymbol{v} - \boldsymbol{f} \cdot \boldsymbol{v}\Big),$$

over all divergence-free, $\mathbb{T}$-periodic functions with zero average over $\mathbb{T}$. Define

$$I_0(\boldsymbol{f}) := \min_{\boldsymbol{v}(\boldsymbol{x})} E_0(\boldsymbol{v}, \boldsymbol{f}). \tag{1.2.3}$$

It is well known (see for example Jikov, Kozlov & Oleinik [40]) that in scalar homogenisation theory, the energy functional converges to the homogenised energy functional as $\varepsilon \to 0$. This is true too for the vector homogenisation presented here, i.e., as $\varepsilon \to 0$, for any function $\boldsymbol{f}$ it follows that $I(\varepsilon, \boldsymbol{f}) \to I_0(\boldsymbol{f})$. This result will now be extended to all finite orders of ε.

Remark firstly that substitution of $\boldsymbol{u}^\varepsilon = \boldsymbol{u}^{(K)} + \boldsymbol{R}_K$ into (1.2.2) yields

$$I(\varepsilon, \boldsymbol{f}) = -\frac{1}{2}\int_{\mathbb{T}} \boldsymbol{f} \cdot \boldsymbol{u}^{(K)}\,\mathrm{d}\boldsymbol{x} + \tilde{\boldsymbol{R}}_K(\boldsymbol{x}, \varepsilon), \qquad |\tilde{\boldsymbol{R}}_K| \leq \tilde{C}_K(\boldsymbol{f})\varepsilon^K. \tag{1.2.4}$$

Proposition 1.2.1. *For any* $\mathbb{T}$*-periodic function* $\boldsymbol{f} \in [C^\infty(\mathbb{T})]^3$ *and any positive integer* K*, there exists a constant* $\widehat{C}_K(\boldsymbol{f})$ *such that*

$$\left| I(\varepsilon, \boldsymbol{f}) + \frac{1}{2}\int_{\mathbb{T}} \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{v}^{(K)}(\boldsymbol{x}, \varepsilon)\,\mathrm{d}\boldsymbol{x} \right| \leq \widehat{C}_K(\boldsymbol{f})\varepsilon^K.$$

Proof. Substituting series (1.1.25) into (1.2.4) yields

$$I(\varepsilon, \boldsymbol{f}) = -\frac{1}{2}\int_{\mathbb{T}} \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{v}^{(K)}(\boldsymbol{x}, \varepsilon)\,\mathrm{d}\boldsymbol{x} + \sum_{\substack{1\leq j\leq K\\ 0\leq k\leq K}} \varepsilon^{j+k}\int_{\mathbb{T}} N^{(j)}(\boldsymbol{x}/\varepsilon)F_{jk}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} + \tilde{\boldsymbol{R}}_K(\boldsymbol{x}, \varepsilon), \tag{1.2.5}$$

where $F_{jk} = \nabla_x^{j-1}\mathbf{curl}_x \boldsymbol{v}_k * \boldsymbol{f}$ are infinitely smooth and $\mathbb{T}$-periodic. As the $N^{(j)}$ are Q-periodic and have zero mean, the second term in (1.2.5) will decay as $\varepsilon \to 0$ faster than any power of ε (see Appendix 1.B), i.e.

$$\left| \int_{\mathbb{T}} N^{(j)}(\boldsymbol{x}/\varepsilon)F_{jk}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \right| \leq C_{jk}^K \varepsilon^K,$$

for any K, j and k and for some constants C_{jk}^K. Hence the result. □

As a result of the above proposition, the following asymptotic expansion for the energy holds:

$$I(\varepsilon, \boldsymbol{f}) \sim I_0(\boldsymbol{f}) + \sum_{k=1}^{\infty} \varepsilon^k I_k(\boldsymbol{f}), \quad (\varepsilon \to 0), \quad I_k(\boldsymbol{f}) := -\frac{1}{2}\int_{\mathbb{T}} \boldsymbol{f} \cdot \boldsymbol{v}_k, \quad k = 0, 1, \ldots. \tag{1.2.6}$$

1.2.2 Infinite-Order Homogenised Solution

In this section, the effect of the rapid oscillations in the tensors $N^{(j)}$ in the asymptotic expansion (1.1.5) is removed by considering the solutions of another family of problems. More precisely, for all $\boldsymbol{\zeta} \in Q$, define $A_\zeta(\boldsymbol{y}) = A(\boldsymbol{y} + \boldsymbol{\zeta})$, $\boldsymbol{y} \in Q$ and consider the family of equations

$$(L_\varepsilon^\zeta \boldsymbol{u})(\boldsymbol{x}) \equiv \mathbf{curl}\,\hat{A}_\zeta^\varepsilon(\boldsymbol{x})\mathbf{curl}\,\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x}), \qquad \boldsymbol{x} \in \mathbb{T}, \tag{1.2.7}$$

under the same assumptions on $\boldsymbol{f}$ and subject to the same conditions on the solution $\boldsymbol{u}$ seen in Section 1.1. For any $\boldsymbol{\zeta} \in Q$, (1.2.7) is equivalent to (1.1.1) and when $\boldsymbol{\zeta}$ is fixed, equation (1.2.7) admits a unique solution in $X(\mathbb{T})$ which will be denoted $\boldsymbol{u}_\zeta^\varepsilon$. Denote by $\langle\,\cdot\,\rangle_\zeta$ the average over Q with respect to $\boldsymbol{\zeta}$, i.e.

$$\bar{\boldsymbol{u}}^\varepsilon(\boldsymbol{x}) := \langle \boldsymbol{u}_\zeta^\varepsilon \rangle_\zeta = \frac{1}{|Q|}\int_Q \boldsymbol{u}_\zeta^\varepsilon(\boldsymbol{x})\,\mathrm{d}\boldsymbol{\zeta}, \qquad \boldsymbol{x} \in \mathbb{T}.$$

This averaging process with respect to $\boldsymbol{\zeta}$ is analogous to "ensemble averaging" in probability, when the underlying probability measure is uniform over Q. The following result illustrates how this translation averaging eliminates the oscillations due to the tensors $N^{(j)}$ and in fact shows that the asymptotics for $\bar{\boldsymbol{u}}^\varepsilon$ are given by series (1.1.6).

Proposition 1.2.2. *For a given $\mathbb{T}$-periodic function $\boldsymbol{f}$, the series (1.1.6) provides asymptotics for $\bar{\boldsymbol{u}}^\varepsilon$ in the sense that for any positive integer K, there exists a positive constant $C_K(\boldsymbol{f})$ such that*

$$\int_{\mathbb{T}} |\bar{\boldsymbol{u}}^\varepsilon - \boldsymbol{v}^{(K)}|^2 \mathrm{d}\boldsymbol{x} \leq C_K(\boldsymbol{f})\varepsilon^{2K}.$$

Proof. It is first noted that for all $\boldsymbol{\zeta} \in Q$, the solution $\boldsymbol{u}^\varepsilon_\zeta$ of (1.2.7) can be written as an analogous asymptotic expansion to that of the series given by (1.1.5) with the tensors $K^{(j)}$ and $N^{(j)}$ replaced by the translated tensors $K^{(j)}_\zeta(\cdot) = K^{(j)}(\cdot + \boldsymbol{\zeta})$ and $N^{(j)}_\zeta(\cdot) = N^{(j)}(\cdot + \boldsymbol{\zeta})$ respectively. Moreover, remainder estimates similar to those seen in Section 1.2.1 will all hold. In particular,

$$\|\boldsymbol{u}^\varepsilon_\zeta - \boldsymbol{u}^{(K)}_\zeta\|_{L^2(\mathbb{T})} \leq \bar{C}_K(\boldsymbol{f})\varepsilon^K,$$

where $\bar{C}_K(\boldsymbol{f})$ is independent of $\boldsymbol{\zeta}$ and $\boldsymbol{u}^{(K)}_\zeta$ denotes the Kth-order truncated asymptotic expansion of $\boldsymbol{u}^\varepsilon_\zeta$ similar to (1.1.25). Since $K^{(j)}_\zeta$ and $N^{(j)}_\zeta$ are Q-periodic and have zero average over Q, it follows that

$$\bar{\boldsymbol{u}}^\varepsilon(\boldsymbol{x}) - \boldsymbol{v}^{(K)}(\boldsymbol{x},\varepsilon) = \int_Q \Big(\boldsymbol{u}^\varepsilon_\zeta(\boldsymbol{x}) - \boldsymbol{u}^{(K)}_\zeta(\boldsymbol{x},\varepsilon)\Big)\mathrm{d}\boldsymbol{\zeta}.$$

Therefore

$$\begin{aligned}\int_{\mathbb{T}} \left|\bar{\boldsymbol{u}}^\varepsilon(\boldsymbol{x}) - \boldsymbol{v}^{(K)}(\boldsymbol{x},\varepsilon)\right|^2 \mathrm{d}\boldsymbol{x} &\leq \int_{\mathbb{T}} \left(\int_Q \left|\boldsymbol{u}^\varepsilon_\zeta(\boldsymbol{x}) - \boldsymbol{u}^{(K)}_\zeta(\boldsymbol{x},\varepsilon)\right|\mathrm{d}\boldsymbol{\zeta}\right)^2 \mathrm{d}\boldsymbol{x}\\ &\leq \int_Q\int_{\mathbb{T}} \left|\boldsymbol{u}^\varepsilon_\zeta(\boldsymbol{x}) - \boldsymbol{u}^{(K)}_\zeta(\boldsymbol{x},\varepsilon)\right|^2 \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{\zeta} \leq C_K(\boldsymbol{f})\varepsilon^{2K},\end{aligned}$$

as required. □

1.2.3 Higher-Order Variational Problems

In this section, consideration is given to the equivalent family of variational formulations for the family of equations (1.2.7). By analogy with Section 1.2, the “translated” energy functional is defined as:

$$E_{\varepsilon,\zeta}(\boldsymbol{u},\boldsymbol{f}) = \int_{\mathbb{T}} \Big(\frac{1}{2}\hat{A}^\varepsilon_\zeta \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{u} - \boldsymbol{f}\cdot\boldsymbol{u}\Big). \tag{1.2.8}$$

Define also

$$I_\zeta(\varepsilon,\boldsymbol{f}) := \min_{\boldsymbol{u}(\boldsymbol{x})} E_{\varepsilon,\zeta}(\boldsymbol{u},\boldsymbol{f}).$$

Obtained in exactly the same way, it can be shown that the asymptotics for $I_\zeta(\varepsilon,\boldsymbol{f})$ are analogous to the asymptotics (1.2.6) and that these asymptotics are independent of the parameter $\boldsymbol{\zeta}$.

Define the averaged functional to be $\bar{I}(\varepsilon, \boldsymbol{f}) := \langle I_\zeta(\varepsilon, \boldsymbol{f}) \rangle_\zeta$. Hence

$$\bar{I}(\varepsilon, \boldsymbol{f}) = \int_Q \Big(\min_{\boldsymbol{u}(\boldsymbol{x})} E_{\varepsilon,\zeta}(\boldsymbol{u}, \boldsymbol{f}) \Big) \mathrm{d}\boldsymbol{\zeta} = \min_{\boldsymbol{u}(\boldsymbol{x},\boldsymbol{\zeta})} \bar{E}_\varepsilon(\boldsymbol{u}, \boldsymbol{f}), \tag{1.2.9}$$

where

$$\bar{E}_\varepsilon(\boldsymbol{u}, \boldsymbol{f}) = \int_Q E_{\varepsilon,\zeta}(\boldsymbol{u}, \boldsymbol{f})\, \mathrm{d}\boldsymbol{\zeta}. \tag{1.2.10}$$

It is obvious that $\bar{I}(\varepsilon, \boldsymbol{f})$ also retains the asymptotics given in (1.2.6).

The variational problem (1.2.9) has as its minimiser the function $\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{\zeta}) = \boldsymbol{u}^\varepsilon_\zeta(\boldsymbol{x})$ where $\boldsymbol{u}^\varepsilon_\zeta(\boldsymbol{x})$ is the solution of (1.2.7). Recall that $\boldsymbol{u}^\varepsilon_\zeta$ can be expanded asymptotically in a series analogous to (1.1.5). The idea now is to construct a higher-order homogenised variational problem in light of such a series expansion for $\boldsymbol{u}^\varepsilon_\zeta$. Fixing $\varepsilon > 0$, define U_K to be the set of trial fields $\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{\zeta})$:

$$\begin{aligned} U_K = \Bigg\{ \boldsymbol{u}(\boldsymbol{x}, \boldsymbol{\zeta}) \;\Big|\; \boldsymbol{u}(\boldsymbol{x}, \boldsymbol{\zeta}) = \boldsymbol{v}(\boldsymbol{x}) + \sum_{j=1}^{K} \varepsilon^j \Big\{ \nabla_y (K^{(j)}_\zeta(\boldsymbol{y}) \nabla^j_x \boldsymbol{v}(\boldsymbol{x})) + \nabla_x (K^{(j-1)}_\zeta(\boldsymbol{y}) \nabla^{j-1}_x \boldsymbol{v}(\boldsymbol{x})) + \\ + N^{(j)}_\zeta(\boldsymbol{y}) \nabla^{j-1}_x (\mathbf{curl}_x \boldsymbol{v}(\boldsymbol{x})) \Big\} \Big|_{\boldsymbol{y}=\boldsymbol{x}/\varepsilon} \text{ for some } \boldsymbol{v} \Bigg\}. \end{aligned} \tag{1.2.11}$$

Here, $\boldsymbol{v}$ belongs to the set of smooth, $\mathbb{T}$-periodic, divergence-free vector fields with zero average over $\mathbb{T}$. Consideration is now given to the same minimisation problem (1.2.9) but over the restricted set U_K. Directly substituting a trial field from the set (1.2.11) into equation (1.2.10) yields

$$\bar{E}_\varepsilon(\boldsymbol{u}, \boldsymbol{f}) = \int_{\mathbb{T}} \Bigg\{ \sum_{j=0}^{K} \sum_{k=0}^{K} \frac{1}{2} \varepsilon^{j+k} \tilde{h}^{jk} \nabla^j (\mathbf{curl}\, \boldsymbol{v}) \nabla^k (\mathbf{curl}\, \boldsymbol{v}) - \boldsymbol{f} \cdot \boldsymbol{v} \Bigg\}, \tag{1.2.12}$$

where $\tilde{h}^{jk}$ is a tensor of order $(j + k + 2)$ given explicitly by the formula

$$\tilde{h}^{jk} = \Big\langle A \big(\mathbf{curl}\, N^{(j+1)} + M^{(j+1)} \big) \big(\mathbf{curl}\, N^{(k+1)} + M^{(k+1)} \big) \Big\rangle \tag{1.2.13}$$

Whenever $\boldsymbol{u}$ and $\boldsymbol{v}$ are related by the expression in U_K, set

$$\bar{E}_\varepsilon(\boldsymbol{u}, \boldsymbol{f}) =: E_K(\boldsymbol{v}, \boldsymbol{f}, \varepsilon),$$

so that

$$I_K(\boldsymbol{f}, \varepsilon) := \min_{\boldsymbol{v}} E_K(\boldsymbol{v}, \boldsymbol{f}, \varepsilon) = \min_{\boldsymbol{u}(\boldsymbol{x},\boldsymbol{\zeta}) \in U_K} \bar{E}_\varepsilon(\boldsymbol{u}, \boldsymbol{f}), \quad \boldsymbol{v} \in [C^\infty_{\mathrm{per}}(\mathbb{T})]^3 \cap X(\mathbb{T}).$$

Moreover, for all ε, any $\boldsymbol{f}$ and for any finite K, the functional $E_K(\cdot, \boldsymbol{f}, \varepsilon)$ is convex with respect to $\boldsymbol{v}$ as a result of the $\boldsymbol{\zeta}$-averaging of the convex functional $E_{\varepsilon,\zeta}$.

Denote by $\boldsymbol{v}_K(\boldsymbol{x}, \varepsilon)$[4] the minimiser of $I_K(\boldsymbol{f}, \varepsilon)$. The following proposition demonstrates that $\boldsymbol{v}_K$ is in fact the best choice of a truncated approximation in a variational sense.

[4]The minimiser $\boldsymbol{v}_K$ can be shown to exist and to be unique up to an arbitrary constant provided $\boldsymbol{f} \in [C^\infty(\mathbb{T})]^3$. This result can be shown by, for example, using the Fourier transformation along with the equation for E_K. A proof for the scalar homogenisation can be found in Cherednichenko [21].

Proposition 1.2.3. *For $K \geq 2$ and all functions $\boldsymbol{v}$ it follows that*

$$E_K(\boldsymbol{v}, \boldsymbol{f}, \varepsilon) \geq E_K(\boldsymbol{v}_K, \boldsymbol{f}, \varepsilon) \geq \bar{I}(\varepsilon, \boldsymbol{f}). \tag{1.2.14}$$

Moreover

$$E_K(\boldsymbol{v}_K, \boldsymbol{f}, \varepsilon) - \bar{I}(\varepsilon, \boldsymbol{f}) \leq c_K(\boldsymbol{f})\varepsilon^{2K}, \tag{1.2.15}$$

for some positive constant $c_K(\boldsymbol{f})$.

Proof. Inequalities (1.2.14) follow by observing that $\boldsymbol{v}_K$ is the minimiser of $I_K(\boldsymbol{f}, \varepsilon)$ and $\bar{I}_K(\varepsilon, \boldsymbol{f}) \geq \bar{I}(\varepsilon, \boldsymbol{f})$ where the latter inequality follows since $\bar{I}_K(\varepsilon, \boldsymbol{f})$ is considered over a restricted set. To obtain inequality (1.2.15), substitute $\boldsymbol{u}^{\varepsilon}_{\zeta}(\boldsymbol{x}) = \boldsymbol{u}^{(K)}_{\zeta}(\boldsymbol{x}, \varepsilon) + \boldsymbol{R}_K(\boldsymbol{x}, \boldsymbol{\zeta}, \varepsilon)$ into (1.2.8) and integrate by parts. The remainder $\boldsymbol{R}_K(\boldsymbol{x}, \boldsymbol{\zeta}, \varepsilon)$ satisfies estimates analogous to those proven for $\boldsymbol{R}_K$ in Theorem 1.1.2 and hence

$$0 \leq E_{\varepsilon,\zeta}(\boldsymbol{u}^{(K)}_{\zeta}, \boldsymbol{f}) - I_{\zeta}(\varepsilon, \boldsymbol{f}) \leq c_K(\boldsymbol{f})\varepsilon^{2K}.$$

Averaging the last inequality over Q with respect to $\boldsymbol{\zeta}$ and once again using the fact that $\boldsymbol{v}_K$ is a minimiser yields the desired results. □

Remark. In formula (1.2.13), the expression for the tensor is evaluated as follows:

$$\tilde{h}^{jk}_{i_1 \dots i_{j+k+2}} = \Big\langle A_{st} \big(\mathbf{curl}\, N^{(j+1)} + M^{(j+1)}\big)_{s i_1 \dots i_{j+1}} \big(\mathbf{curl}\, N^{(k+1)} + M^{(k+1)}\big)_{t i_{j+2} \dots i_{j+k+2}} \Big\rangle.$$

Note also that $\tilde{h}^{jk}$ can be represented in an alternative way. Using integration by parts, for $j = 1, 2, \dots$, the following formula can be shown to hold:

$$\Big\langle A\big(\mathbf{curl}\, N^{(j+1)} + M^{(j+1)}\big)\mathbf{curl}\, \boldsymbol{\phi} \Big\rangle = -\Big\langle L^{(j+1)}\boldsymbol{\phi}\Big\rangle, \qquad \forall \boldsymbol{\phi}, \text{ such that } \langle \boldsymbol{\phi} \rangle = 0.$$

Hence

$$\tilde{h}^{jk} = \Big\langle A\big(\mathbf{curl}\, N^{(j+1)} + M^{(j+1)}\big) M^{(k+1)} \Big\rangle - \Big\langle L^{(j+1)} N^{(k+1)} \Big\rangle, \tag{1.2.16}$$

where

$$\Big(L^{(j+1)} N^{(k+1)}\Big)_{i_1 \dots i_{j+k+2}} = L^{(j+1)}_{s i_1 \dots i_{j+1}} N^{(k+1)}_{s i_{j+2} \dots i_{j+k+2}}.$$

1.2.4 Infinite-Order Variational Homogenised Equation

For each non-negative integer K, the functional E_K admits a unique minimiser $\boldsymbol{v}_K$. Constructing the Euler-Lagrange equation for functional (1.2.12) yields

$$\int_{\mathbb{T}} \left\{ \frac{1}{2} \sum_{j,k=0}^{K} \varepsilon^{j+k} \tilde{h}^{jk} \Big(\nabla^j (\mathbf{curl}\, \boldsymbol{\varphi}) \nabla^k (\mathbf{curl}\, \boldsymbol{v}_K) + \nabla^j (\mathbf{curl}\, \boldsymbol{v}_K) \nabla^k (\mathbf{curl}\, \boldsymbol{\varphi}) \Big) - \boldsymbol{f} \cdot \boldsymbol{\varphi} \right\} = 0, \tag{1.2.17}$$

for all $\boldsymbol{\varphi} \in C^\infty_{\rm per}(\mathbb{T})^3$. When $K = 0$, the following identity is obtained:

$$\int_{\mathbb{T}} \left(\frac{1}{2} \tilde{h}^{00} \mathbf{curl}\, \boldsymbol{\varphi} \cdot \mathbf{curl}\, \boldsymbol{v}_0 + \frac{1}{2} \tilde{h}^{00} \mathbf{curl}\, \boldsymbol{v}_0 \cdot \mathbf{curl}\, \boldsymbol{\varphi} - \boldsymbol{f} \cdot \boldsymbol{\varphi} \right) = 0, \quad \forall \boldsymbol{\varphi} \in [C^\infty_{\rm per}(\mathbb{T})]^3.$$

Writing the above in a differential formulation yields

$$\mathbf{curl}\, \tilde{h}^{00} \mathbf{curl}\, \boldsymbol{v}_0 = \boldsymbol{f}.$$

By considering the formula (1.2.13) for h^{jk}, it is seen that $\tilde{h}^{00}$ coincides with $\hat{h}^{(2)}$. Furthermore, the minimiser $\boldsymbol{v}_0$ coincides with the leading order homogenised solution also denoted $\boldsymbol{v}_0$.

Consider now when $K = 1$. In this case, the Euler-Lagrange equation reads

$$\mathbf{curl}\, \tilde{h}^{00} \mathbf{curl}\, \boldsymbol{v}_1 + \varepsilon \mathbf{curl}\, \tilde{\tilde{h}}^1 \nabla \mathbf{curl}\, \boldsymbol{v}_1 + \varepsilon^2 \mathbf{curl}\, \bar{h}^2 \nabla^2 \mathbf{curl}\, \boldsymbol{v}_1 = \boldsymbol{f},$$

where

$$\tilde{\tilde{h}}^1_{ijk} = \frac{1}{2} \Big\{ \tilde{h}^{01}_{ijk} - \tilde{h}^{01}_{kji} - \tilde{h}^{10}_{jik} + \tilde{h}^{10}_{jki} \Big\}, \quad \bar{h}^2_{ijkl} = -\Big\{ h^{11}_{jikl} + h^{11}_{ljki} \Big\}. \tag{1.2.18}$$

In Section 1.2.5, it will be shown that the third-order tensor in (1.2.18) coincides with the third-order tensor $\hat{h}^{(3)}$ seen in the infinite-order homogenised equation (1.1.19). The fourth-order tensor is in a sense "incomplete" since it does not include all the fourth-order tensors h^{jk} in its formula. When values of K are chosen, these tensors will be included.

Remark. For higher-order expressions, it is possible to have two different constant tensors producing the same term at equal orders, i.e., in the case of third-order tensors, it can happen that

$$\mathbf{curl}\, h_1 \nabla \mathbf{curl}\, \boldsymbol{v} = \mathbf{curl}\, h_2 \nabla \mathbf{curl}\, \boldsymbol{v}, \qquad \forall \boldsymbol{v},$$

where $h_1 \neq h_2$. A simple example where this happens is

$$(h_1)_{ijk} = \begin{cases} 1, & ijk = \{121, 323\}, \\ 0, & \text{otherwise}, \end{cases} \qquad (h_2)_{ijk} = \begin{cases} -1, & ijk = \{222\}, \\ 0, & \text{otherwise}. \end{cases}$$

This will be considered further in Appendix 1.D.

Consider formally when $K = \infty$ in equation (1.2.17) and denote the corresponding minimiser by $\boldsymbol{v}^{(\infty)}$. Remark that

$$\min_{\boldsymbol{v}} E_\infty(\boldsymbol{v}, \boldsymbol{f}, \varepsilon) = \min_{\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{\zeta})} E_\varepsilon(\boldsymbol{u}, \boldsymbol{f}) = \bar{I}(\varepsilon, \boldsymbol{f}),$$

where the minimiser $\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{\zeta}) = \boldsymbol{u}^\varepsilon_{\zeta}(\boldsymbol{x})$. Hence $\boldsymbol{v}^{(\infty)} = \bar{\boldsymbol{u}}^\varepsilon$. After a series of manipulations, it follows that the Euler-Lagrange equation takes the form

$$\mathbf{curl}\, \tilde{h}^{00} \mathbf{curl}\, \boldsymbol{v}^{(\infty)} + \sum_{n=1}^{\infty} \varepsilon^n \mathbf{curl}\, \tilde{\tilde{h}}^n \nabla^n \mathbf{curl}\, \boldsymbol{v}^{(\infty)} = \boldsymbol{f}, \tag{1.2.19}$$

where

$$\tilde{\tilde{h}}^{n}_{i_1\dots i_{n+2}} = \frac{1}{2}\sum_{\substack{j+k=n,\\ j,k\in\mathbb{N}_0}} \Big\{(-1)^j \tilde{h}^{jk}_{i_{j+1}i_2\dots i_j i_1 i_{j+2}\dots i_{j+k+2}} + (-1)^k \tilde{h}^{jk}_{i_{j+1}i_2\dots i_j i_{j+k+2} i_{j+2}\dots i_{j+k+1} i_1}\Big\}. \tag{1.2.20}$$

The infinite-order homogenised equation (1.1.19) will be shown to coincide with the variational infinite-order homogenised equation (1.2.19) in Section 1.2.5.

Remark. In order to interpret the change of indices in equation (1.2.20) correctly, it should be observed that the $n+2 = j+k+2$ indices are split into two groups of length $j+1$ and $k+1$ respectively:

$$\underbrace{i_1 i_2 \dots i_{j+1}}_{j+1 \text{ indices}}\underbrace{i_{j+2}\dots i_{j+k+2}}_{k+1 \text{ indices}}.$$

So, for example, the term multiplying the factor $(-1)^j$ has had indices i_1 and i_{j+1} swapped and this should be interpreted as swapping the first index from the first group of $(j+1)$ indices with the last index from the first group of $(j+1)$ indices.

The concluding proposition concerns showing that the minimiser $\boldsymbol{v}_K$ of $I_K(\varepsilon, \boldsymbol{f})$ approximates the infinite-order homogenised solution $\boldsymbol{v}^{(\infty)}$ to all orders in ε.

Proposition 1.2.4. *Let $K \geq 2$ be a positive integer. Then for any function $\boldsymbol{f}$, there exists a positive constant $\hat{c}_K$ such that*

$$\int_{\mathbb{T}} \big|\boldsymbol{v}^{(\infty)}(\boldsymbol{x},\varepsilon) - \boldsymbol{v}_K(\boldsymbol{x},\varepsilon)\big|^2 \mathrm{d}\boldsymbol{x} \leq \hat{c}_K \varepsilon^{2K}.$$

Proof. Let $\boldsymbol{u}^{(K)}_\zeta$ denote the vector with associated $\boldsymbol{v}^{(K)}$ from the set U_K given in formula (1.2.11). Define $\boldsymbol{R}_K(\boldsymbol{x},\boldsymbol{\zeta})$ to be the remainder

$$\boldsymbol{R}_K(\boldsymbol{x},\boldsymbol{\zeta}) = \boldsymbol{u}^{\varepsilon}_\zeta(\boldsymbol{x}) - \boldsymbol{u}^{(K)}_\zeta(\boldsymbol{x},\varepsilon).$$

Clearly

$$\begin{aligned} E_K(\boldsymbol{v}_K, \boldsymbol{f}, \varepsilon) &= \bar{E}_\varepsilon(\boldsymbol{u}^{(K)}_\zeta, \boldsymbol{f}) = \int_Q E_{\varepsilon,\zeta}(\boldsymbol{u}^{(K)}_\zeta, \boldsymbol{f})\,\mathrm{d}\boldsymbol{\zeta} \\ &= \int_Q\int_{\mathbb{T}} \Big(\frac{1}{2}\hat{A}^{\varepsilon}_\zeta \mathbf{curl}\,\boldsymbol{u}^{(K)}_\zeta \cdot \mathbf{curl}\,\boldsymbol{u}^{(K)}_\zeta - \boldsymbol{f}\cdot\boldsymbol{u}^{(K)}_\zeta\Big)\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{\zeta} \end{aligned}$$

$$\begin{aligned} = \int_Q\int_{\mathbb{T}} \Big(&\frac{1}{2}\hat{A}^{\varepsilon}_\zeta \mathbf{curl}\,\boldsymbol{u}^{\varepsilon}_\zeta \cdot \mathbf{curl}\,\boldsymbol{u}^{\varepsilon}_\zeta - \frac{1}{2}\hat{A}^{\varepsilon}_\zeta \mathbf{curl}\,\boldsymbol{u}^{\varepsilon}_\zeta \cdot \mathbf{curl}\,\boldsymbol{R}_K - \\ &- \frac{1}{2}\hat{A}^{\varepsilon}_\zeta \mathbf{curl}\,\boldsymbol{R}_K \cdot \mathbf{curl}\,\boldsymbol{u}^{\varepsilon}_\zeta + \frac{1}{2}\hat{A}^{\varepsilon}_\zeta \mathbf{curl}\,\boldsymbol{R}_K \cdot \mathbf{curl}\,\boldsymbol{R}_K - \boldsymbol{f}\cdot\boldsymbol{u}^{\varepsilon}_\zeta + \boldsymbol{f}\cdot\boldsymbol{R}_K\Big)\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{\zeta}. \end{aligned}$$

Since $\boldsymbol{u}^{\varepsilon}_\zeta$ solves (1.2.7), by integrating by parts, it can be seen that

$$E_K(\boldsymbol{v}_K, \boldsymbol{f}, \varepsilon) = \bar{I}(\varepsilon, \boldsymbol{f}) + \int_Q \Big(\int_{\mathbb{T}} \frac{1}{2}\hat{A}^{\varepsilon}_\zeta \mathbf{curl}\,\boldsymbol{R}_K \cdot \mathbf{curl}\,\boldsymbol{R}_K\,\mathrm{d}\boldsymbol{x}\Big)\mathrm{d}\boldsymbol{\zeta}.$$

Using the positive definiteness of the matrix A_ζ

$$\int_Q \left(\int_{\mathbb{T}} \frac{1}{2} \hat{A}_\zeta^\varepsilon \mathbf{curl}\, \boldsymbol{R}_K \cdot \mathbf{curl}\, \boldsymbol{R}_K \, \mathrm{d}\boldsymbol{x} \right) \mathrm{d}\boldsymbol{\zeta} \geq \frac{\nu}{2} \int_Q \left(\int_{\mathbb{T}} |\mathbf{curl}\, \boldsymbol{R}_K|^2 \, \mathrm{d}\boldsymbol{x} \right) \mathrm{d}\boldsymbol{\zeta}$$
$$\geq \frac{\nu}{2} \int_{\mathbb{T}} \left(\int_Q |\mathbf{curl}\, \boldsymbol{R}_K|^2 \mathrm{d}\boldsymbol{\zeta} \right) \mathrm{d}\boldsymbol{x}$$
$$\geq \frac{\nu}{2} \int_{\mathbb{T}} |\mathbf{curl}\, (\bar{\boldsymbol{u}}^\varepsilon - \boldsymbol{v}_K)|^2 \mathrm{d}\boldsymbol{x}.$$

Invoking the remainder estimate (1.2.15) from Proposition 1.2.3 yields

$$\int_{\mathbb{T}} |\mathbf{curl}\, (\bar{\boldsymbol{u}}^\varepsilon - \boldsymbol{v}_K)|^2 \, \mathrm{d}\boldsymbol{x} \leq \frac{2c_K}{\nu} \varepsilon^{2K}.$$

Making use of a Maxwell inequality (and noting that $\operatorname{div} \bar{\boldsymbol{u}}^\varepsilon = \operatorname{div} \boldsymbol{v}_K = 0$) yields the result. □

The last two propositions establish that the order K homogenised solution $\boldsymbol{v}_K$ which minimises $I_K(\varepsilon, \boldsymbol{f})$ is also the best approximation of the infinite-order homogenised solution $\boldsymbol{v}$ in the sense of the functional asymptotics (Proposition 1.2.3) and furthermore recovers the asymptotics for $\boldsymbol{v}$ to all orders of ε for sufficiently large K.

The final part of this section looks to address the equality of the two series expansions for the two infinite-order homogenised equations (1.1.19) and (1.2.19). It was already shown that the solution to the original problem is unique and so these series must coincide but proving so requires the introduction of a symmetrisation procedure individual to the operator in question.

1.2.5 Tensor Analysis of the Infinite-Order Homogenised Equations

Before proceeding to show that $\hat{h}^{(n+2)}$ and $\tilde{\tilde{h}}^n$ are equal after applying a symmetrisation procedure, it will be confirmed that the two third-order tensors labeled $\hat{h}^{(3)}$ and $\tilde{\tilde{h}}^1$ appearing in the respective infinite-order homogenised equations coincide. Recall the following formulae:

$$\tilde{\tilde{h}}^1_{ijk} = \frac{1}{2} \left\{ \tilde{h}^{01}_{ijk} - \tilde{h}^{01}_{kji} - \tilde{h}^{10}_{jik} + \tilde{h}^{10}_{jki} \right\}, \quad \hat{h}^{(3)}_{ijk} = \langle A_{is} (\mathbf{curl}\, N^{(2)} + M^{(2)})_{sjk} \rangle,$$

where

$$\tilde{h}^{01}_{ijk} = \langle A_{st} (\mathbf{curl}\, N^{(1)} + I)_{si} (\mathbf{curl}\, N^{(2)} + M^{(2)})_{tjk} \rangle,$$
$$\tilde{h}^{10}_{ijk} = \langle A_{st} (\mathbf{curl}\, N^{(2)} + M^{(2)})_{sij} (\mathbf{curl}\, N^{(1)} + I)_{tk} \rangle.$$

The latter two tensors satisfy the symmetry property $h^{10}_{ijk} = h^{01}_{kij}$, and moreover it can be shown that $\hat{h}^{(3)}_{ijk}$ is expressible in terms of $\tilde{h}^{01}_{ijk}$ and $\tilde{h}^{10}_{ijk}$. Indeed, using the definitions (1.1.14) and (1.1.15) and relation (1.2.16) yields

$$\hat{h}^{(3)}_{ijk} = \tilde{h}^{01}_{ijk} - \tilde{h}^{10}_{jik}.$$

Hence,

$$(\hat{h}^{(3)} - \tilde{\tilde{h}}^1)_{ijk} = \tilde{h}^{01}_{ijk} - \tilde{h}^{10}_{jik} - \frac{1}{2}\Big\{\tilde{h}^{01}_{ijk} - \tilde{h}^{01}_{kji} - \tilde{h}^{10}_{jik} + \tilde{h}^{10}_{jki}\Big\}$$
$$= \tilde{h}^{01}_{ijk} - \tilde{h}^{01}_{kji} - \frac{1}{2}\Big\{\tilde{h}^{01}_{ijk} - \tilde{h}^{01}_{kji} - \tilde{h}^{01}_{kji} + \tilde{h}^{01}_{ijk}\Big\} = 0.$$

A symmetrisation process will now be introduced in order to prove that the two infinite-order homogenised equations obtained are equivalent. A standard symmetrisation procedure can be seen in Wrede [79, p.309] however, this method of symmetrisation proved unsuccessful in proving the desired result in this work.

Definition 1.2.1. For $n \geq 2$, the partial symmetrisation of a tensor $h_{ik_1\dots k_n j}$, denoted $h_{i(k_1\dots k_n)j}$, is defined by the relation:

$$h_{i(k_1\dots k_n)j} := \frac{1}{n!}\sum_{(k_1,k_2,\dots,k_n)} h_{ik_1\dots k_n j},$$

where the summation is over all permutations of the indices $k_1, k_2, \dots, k_n$. The indices i and j are known as the fixed indices.

Partial symmetrisation arises under consideration of the operator $\mathbf{curl}\,\hat{h}^{(n+2)}\nabla^n\mathbf{curl}$ which appears in the infinite-order homogenised equations. Indeed, if the partial symmetrisation of this operator is considered, then the structure of the operator remains unchanged.

Recall the expression for the tensor appearing in the infinite-order homogenised expansion derived via the variational approach:

$$\tilde{\tilde{h}}^n_{ik_1\dots k_n j} = \frac{1}{2}\sum_{\substack{p+q=n,\\ p,q\in\mathbb{N}_0}}\Big\{(-1)^p\tilde{h}^{pq}_{k_pk_1\dots k_{p-1}ik_{p+1}\dots k_{p+q}j} + (-1)^q\tilde{h}^{pq}_{k_pk_1\dots k_{p-1}jk_{p+1}\dots k_{p+q}i}\Big\}. \quad (1.2.21)$$

Partially symmetrising the above expression gives

$$\tilde{\tilde{h}}^n_{i(k_1\dots k_n)j} = \frac{1}{2(n!)}\sum_{(k_1,\dots,k_n)}\sum_{\substack{p+q=n,\\ p,q\in\mathbb{N}_0}}\Big\{(-1)^p\tilde{h}^{pq}_{k_pk_1\dots k_{p-1}ik_{p+1}\dots k_{p+q}j} + (-1)^q\tilde{h}^{pq}_{k_pk_1\dots k_{p-1}jk_{p+1}\dots k_{p+q}i}\Big\}. \quad (1.2.22)$$

In order to show that (1.2.22) is equal to the tensor obtained via the asymptotic approach when symmetrised, the following lemma is needed.

Lemma 1.2.1. *Let $\tilde{h}^{pq}$ be a tensor of order $(p+q+2)$ satisfying the symmetry property*

$$\tilde{h}^{pq}_{ik_1\dots k_pk_{p+1}\dots k_{p+q}j} = \tilde{h}^{qp}_{k_{p+1}\dots k_{p+q}jik_1\dots k_p}. \quad (1.2.23)$$

Then

$$\frac{1}{n!}\sum_{(k_1,\dots,k_n)}\sum_{\substack{p+q=n,\\ p,q\in\mathbb{N}_0}}(-1)^p\tilde{h}^{pq}_{k_pk_1\dots k_{p-1}ik_{p+1}\dots k_{p+q}j} = \frac{1}{n!}\sum_{(k_1,\dots,k_n)}\sum_{\substack{p+q=n,\\ p,q\in\mathbb{N}_0}}(-1)^q\tilde{h}^{pq}_{k_pk_1\dots k_{p-1}jk_{p+1}\dots k_{p+q}i}.$$

Hence

$$\tilde{\tilde{h}}^n_{i(k_1\ldots k_n)j} = \frac{1}{n!}\sum_{(k_1,\ldots,k_n)}\sum_{\substack{p+q=n,\\ p,q\in\mathbb{N}_0}}(-1)^p\tilde{h}^{pq}_{k_pk_1\ldots k_{p-1}ik_{p+1}\ldots i_{p+q}j}.$$

Proof. Consider the four terms appearing in equation (1.2.21) with upper indices pq and qp:

$$(-1)^p\tilde{h}^{pq}_{k_pk_1\ldots k_{p-1}ik_{p+1}\ldots k_{p+q}j},\ (-1)^q\tilde{h}^{pq}_{k_pk_1\ldots k_{p-1}jk_{p+1}\ldots k_{p+q}i},$$
$$(-1)^q\tilde{h}^{qp}_{k_qk_1\ldots k_{q-1}ik_{q+1}\ldots k_{p+q}j},\ (-1)^p\tilde{h}^{qp}_{k_qk_1\ldots k_{q-1}jk_{q+1}\ldots k_{p+q}i}.$$

Making use of the symmetry property (1.2.23), the four terms above can be written as

$$(-1)^p\tilde{h}^{pq}_{k_pk_1\ldots k_{p-1}ik_{p+1}\ldots k_{p+q}j},\ (-1)^q\tilde{h}^{pq}_{k_pk_1\ldots k_{p-1}jk_{p+1}\ldots k_{p+q}i},$$
$$(-1)^q\tilde{h}^{pq}_{k_{q+1}\ldots k_{p+q}jk_qk_1\ldots k_{q-1}i},\ (-1)^p\tilde{h}^{pq}_{k_{q+1}\ldots k_{p+q}ik_qk_1\ldots k_{q-1}j}.$$

Comparing the positions of the fixed indices in the above, the terms with i and j in the same positions can be "matched". Hence, when (1.2.21) is symmetrised, a summation over all index permutations will occur and so all terms in the first half of the sum will "double up" with those terms in the second half of the sum. Hence the result. □

It was shown that the third-order tensor labeled $\hat{h}^{(3)}$ could be expressed in terms of a sum of the third-order tensors $\tilde{h}^{01}$ and $\tilde{h}^{10}$ in some combination. In much the same way, it can be shown that $\hat{h}^{(n+2)}$ can be expressed in terms of the $(n+2)$th order tensors $\tilde{h}^{0,n}, \tilde{h}^{1,n-1}, \ldots, \tilde{h}^{n,0}$. The construction requires the use of a recurrent procedure making use of the definitions of these tensors, equations (1.1.14)-(1.1.15) and relation (1.2.16). Omitting the details, it can be shown that

$$\hat{h}^{(n+2)}_{ik_1\ldots k_nj} = \tilde{h}^{n,0}_{k_1\ldots k_nji} - \tilde{h}^{n-1,1}_{k_2\ldots k_njk_1i} + \tilde{h}^{n-2,2}_{k_3\ldots k_njk_2k_1i} - \tilde{h}^{n-3,3}_{k_4\ldots k_njk_3k_2k_1i} + \cdots -$$
$$- (-1)^n\tilde{h}^{1,n-1}_{k_njk_{n-1}k_{n-2}\ldots k_1i} + (-1)^n\tilde{h}^{0,n}_{jk_nk_{n-1}\ldots k_1i}.$$

Partially symmetrising the above and using the symmetry properties (1.2.23) yields

$$\begin{aligned}\hat{h}^{(n+2)}_{i(k_1\ldots k_n)j} &= \frac{1}{n!}\sum_{(k_1,\ldots,k_n)}\left\{\tilde{h}^{n,0}_{k_1\ldots k_nji} - \tilde{h}^{n-1,1}_{k_2\ldots k_njk_1i} + \cdots + (-1)^n\tilde{h}^{0,n}_{jk_nk_{n-1}\ldots k_1i}\right\}\\ &= \frac{1}{n!}\sum_{(k_1,\ldots,k_n)}\left\{\tilde{h}^{0,n}_{ik_1\ldots k_nj} - \tilde{h}^{1,n-1}_{k_1ik_2\ldots k_nj} + \cdots + (-1)^n\tilde{h}^{n,0}_{k_nk_{n-1}\ldots k_1ij}\right\}\\ &= \frac{1}{n!}\sum_{(k_1,\ldots,k_n)}\sum_{\substack{p+q=n,\\ p,q\in\mathbb{N}_0}}(-1)^p\tilde{h}^{pq}_{k_pk_1\ldots k_{p-1}ik_{p+1}\ldots i_{p+q}j} = \tilde{\tilde{h}}^n_{i(k_1\ldots k_n)j}\end{aligned}$$

Hence the following result is established (*c.f.*[p. 1357][69], for the scalar case):

Theorem 1.2.1. *Let $\hat{h}^{(n+2)}$ and $\tilde{\tilde{h}}^n$ be the tensors given by expressions (1.1.20) and (1.2.20) respectively. Then*

$$\hat{h}^{(n+2)}_{i(k_1\ldots k_n)j} = \tilde{\tilde{h}}^n_{i(k_1\ldots k_n)j}.$$

Furthermore, since the infinite-order homogenised equations (1.1.19) and (1.2.19) obtained via the asymptotic and variational approaches respectively are invariant under partial symmetrisation, the two series expansions coincide.

The following section is devoted to discussions on Maxwell's equations in general and the derivation of higher-order constitutive laws. This work will then be complemented by a discussion on the homogenisation of the full system of Maxwell equations.

1.3 Higher-Order Homogenised Constitutive Laws for the System of Maxwell Equations

1.3.1 Maxwell's Equations and the Quasistatic Approximation

The well known system of equations which govern the effects of electromagnetics are Maxwell's equations (see Jackson [39] for full details). Maxwell's equations are derived from conservation laws and can be formulated in two different ways; either in an integral formulation or, after applying Gauss' Divergence Theorem and Stokes' Theorem (see Widder [77, Chapter 7]) to the integral formulation, in a differential formulation. In their differential form, assuming the charge density (denoted ρ) is zero, Maxwell's equations are given by the following system:

$$\mathbf{curl}\,\boldsymbol{\mathcal{E}}(\boldsymbol{x},t) = -\frac{\partial \boldsymbol{\mathcal{B}}}{\partial t}(\boldsymbol{x},t), \qquad \mathbf{curl}\,\boldsymbol{\mathcal{H}}(\boldsymbol{x},t) = \frac{\partial \boldsymbol{\mathcal{D}}}{\partial t}(\boldsymbol{x},t) + \boldsymbol{\mathcal{J}}(\boldsymbol{x},t), \tag{1.3.1}$$

$$\operatorname{div}\left(\frac{\partial}{\partial t}\boldsymbol{\mathcal{D}}(\boldsymbol{x},t)\right) = -\operatorname{div}\boldsymbol{\mathcal{J}}(\boldsymbol{x},t), \qquad \operatorname{div}\boldsymbol{\mathcal{B}}(\boldsymbol{x},t) = 0, \tag{1.3.2}$$

$$\boldsymbol{\mathcal{B}}(\boldsymbol{x},t) = \hat{\mu}(\boldsymbol{x})\boldsymbol{\mathcal{H}}(\boldsymbol{x},t), \qquad \boldsymbol{\mathcal{D}}(\boldsymbol{x},t) = \hat{\epsilon}(\boldsymbol{x})\boldsymbol{\mathcal{E}}(\boldsymbol{x},t). \tag{1.3.3}$$

The vector fields $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{B}}$ denote the electric field and magnetic field respectively while the vector fields $\boldsymbol{\mathcal{D}}$ and $\boldsymbol{\mathcal{H}}$ denote the electric field displacement and the magnetic field intensity respectively. The matrices $\hat{\mu}$ and $\hat{\epsilon}$ represent the magnetic permeability and the electric permittivity respectively and the vector field $\boldsymbol{\mathcal{J}}$ represents a source of current density. The equations themselves represent a variety of physical laws also. Equations (1.3.1) represent Faraday's law of induction and Ampère's law of circuitry respectively, equations (1.3.2) are consistency equations known also as Gauss' law and Gauss' law for magnetism respectively and equations (1.3.3) are referred too as the constitutive laws which relate the fields to their respective field intensities.

The following assumptions will now be made:

- The unknowns $\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{B}}, \boldsymbol{\mathcal{D}}, \boldsymbol{\mathcal{H}}$ in Maxwell's equations depend on the domain period $\varepsilon > 0$. These unknowns will now be denoted $\boldsymbol{\mathcal{E}}^\varepsilon, \boldsymbol{\mathcal{B}}^\varepsilon, \boldsymbol{\mathcal{D}}^\varepsilon, \boldsymbol{\mathcal{H}}^\varepsilon$,

- The matrices $\hat{\mu}$ and $\hat{\epsilon}$ will be symmetric, uniformly elliptic and depend periodically on the variable $\boldsymbol{x}/\varepsilon$. The permeability and the permittivity will now be denoted $\hat{\mu}^\varepsilon$ and $\hat{\epsilon}^\varepsilon$ respectively,

- For $\boldsymbol{\mathcal{A}}^\varepsilon \in \{\boldsymbol{\mathcal{E}}^\varepsilon, \boldsymbol{\mathcal{B}}^\varepsilon, \boldsymbol{\mathcal{D}}^\varepsilon, \boldsymbol{\mathcal{H}}^\varepsilon\}$ and some frequency $\omega \in \mathbb{R}$,

$$\boldsymbol{\mathcal{A}}^\varepsilon(\boldsymbol{x}, t) = \boldsymbol{A}^{\omega,\varepsilon}(\boldsymbol{x})\mathrm{e}^{\mathrm{i}\omega t},$$

- (The Quasistatic Approximation) Each amplitude $\boldsymbol{A}^{\omega,\varepsilon}$ can be expanded in a power series of the form

$$\boldsymbol{A}^{\omega,\varepsilon}(\boldsymbol{x}) = \boldsymbol{A}_0^\varepsilon(\boldsymbol{x}) + (\mathrm{i}\omega)\boldsymbol{A}_1^\varepsilon(\boldsymbol{x}) + (\mathrm{i}\omega)^2\boldsymbol{A}_2^\varepsilon(\boldsymbol{x}) + \ldots, \tag{1.3.4}$$

where $\boldsymbol{A}_i^\varepsilon : \mathbb{T} \to \mathbb{R}^3$, for all $i \in \mathbb{N}_0$.

- The current density $\boldsymbol{\mathcal{J}}$ is assumed independent of ε but expressible in a time harmonic form with amplitude denoted $\boldsymbol{J}^\omega$. This amplitude may be expanded in an analogous series to series (1.3.4) with it's coefficients independent of ε.

Written in time harmonic form, Maxwell's equations are given by the following system:

$$\mathbf{curl}\,\boldsymbol{E}^{\omega,\varepsilon} = -\mathrm{i}\omega\boldsymbol{B}^{\omega,\varepsilon}, \qquad \mathbf{curl}\,\boldsymbol{H}^{\omega,\varepsilon} = \mathrm{i}\omega\boldsymbol{D}^{\omega,\varepsilon} + \boldsymbol{J}^\omega, \tag{1.3.5}$$

$$\mathrm{i}\omega\mathrm{div}\,\boldsymbol{D}^{\omega,\varepsilon} = -\mathrm{div}\,\boldsymbol{J}^\omega, \qquad \mathrm{div}\,\boldsymbol{B}^{\omega,\varepsilon} = 0, \tag{1.3.6}$$

$$\boldsymbol{B}^{\omega,\varepsilon} = \hat{\mu}^\varepsilon\boldsymbol{H}^{\omega,\varepsilon}, \qquad \boldsymbol{D}^{\omega,\varepsilon} = \hat{\epsilon}^\varepsilon\boldsymbol{E}^{\omega,\varepsilon}. \tag{1.3.7}$$

The following pair of second-order equations can be derived from system (1.3.5)-(1.3.7):

$$\mathbf{curl}\,(\hat{\mu}^\varepsilon)^{-1}\,\mathbf{curl}\,\boldsymbol{E}^{\omega,\varepsilon} = \omega^2\hat{\epsilon}^\varepsilon\boldsymbol{E}^{\omega,\varepsilon} - \mathrm{i}\omega\boldsymbol{J}^\omega, \tag{1.3.8}$$

$$\mathbf{curl}\,(\hat{\epsilon}^\varepsilon)^{-1}\,\mathbf{curl}\,\boldsymbol{H}^{\omega,\varepsilon} = \omega^2\hat{\mu}^\varepsilon\boldsymbol{H}^{\omega,\varepsilon} + \mathbf{curl}((\hat{\epsilon}^\varepsilon)^{-1}\,\boldsymbol{J}^\omega). \tag{1.3.9}$$

Substituting in the frequency expansions (1.3.4) into equations (1.3.8)-(1.3.9), terms of equal powers of ω, are compared. The system is said to satisfy the quasistatic approximation if equations of equal orders of ω are satisfied. For small frequencies ω, the leading order equations are

$$\begin{cases} \mathbf{curl}\,(\hat{\mu}^\varepsilon)^{-1}\,\mathbf{curl}\,\boldsymbol{E}_1^\varepsilon = -\boldsymbol{J}_0, \\ \mathrm{div}(\hat{\epsilon}^\varepsilon\boldsymbol{E}_1^\varepsilon) = -\mathrm{div}\,\boldsymbol{J}_2, \end{cases} \qquad \begin{cases} \mathbf{curl}\,(\hat{\epsilon}^\varepsilon)^{-1}\,\mathbf{curl}\,\boldsymbol{H}_1^\varepsilon = \mathbf{curl}((\hat{\epsilon}^\varepsilon)^{-1}\,\boldsymbol{J}_1), \\ \mathrm{div}(\hat{\mu}^\varepsilon\boldsymbol{H}_1^\varepsilon) = 0. \end{cases} \tag{1.3.10}$$

Note that the following first-order equations are satisfied:

$$\mathbf{curl}\,\boldsymbol{E}_0^\varepsilon = \mathbf{0}, \quad \mathbf{curl}\,\boldsymbol{H}_0^\varepsilon = \boldsymbol{J}_0, \quad \mathrm{div}\,\boldsymbol{J}_i = 0, \qquad \text{for } i = 0, 1 \ldots.$$

The zero divergence condition on the current density coefficients $\boldsymbol{J}_i$ is justified by the continuity equation $\operatorname{div} \boldsymbol{J}^\omega = -\frac{\partial \rho}{\partial t}$. Since the charge density ρ was assumed to be zero, $\operatorname{div} \boldsymbol{J}^\omega = 0$ and hence $\operatorname{div} \boldsymbol{J_i} = 0,\ i = 1, 2, \ldots$.

The main focus of the work presented in Sections 1.1 and 1.2 is the first equation in the system (1.3.10) where $\boldsymbol{E}_1^\varepsilon = \boldsymbol{u}^\varepsilon$, $(\hat{\mu}^\varepsilon)^{-1} = \hat{A}^\varepsilon$ and $-\boldsymbol{J}_0 = \boldsymbol{f}$. Note also that under the assumption that $\hat{\epsilon}^\varepsilon = I$, it follows that $\operatorname{div} \boldsymbol{u}^\varepsilon = 0$.

Remark. The form of the second-order equation for the magnetic field intensity is somewhat different to that of the second-order equation for the electric field. Indeed, the dependence of the right-hand side on the permittivity matrix $\hat{\epsilon}^\varepsilon$ means dependence on ε. A modified approach which deals with magnetic field intensity equation is described in Section 1.3.4.

1.3.2 Energy Considerations

For a more detailed account about electromagnetic energy, including a discussion on Poynting's Theorem, see Bleaney & Bleaney [11, p.232-234].

For the system of equations (1.3.5)-(1.3.7), the electric and magnetic energies are defined as

$$u_{\text{elec}}^{\omega,\varepsilon} := \frac{1}{2} \int_{\mathbb{T}} \big(\boldsymbol{E}^{\omega,\varepsilon} \cdot (\boldsymbol{D}^{\omega,\varepsilon})^* \big), \quad u_{\text{mag}}^{\omega,\varepsilon} := \frac{1}{2} \int_{\mathbb{T}} \big(\boldsymbol{B}^{\omega,\varepsilon} \cdot (\boldsymbol{H}^{\omega,\varepsilon})^* \big), \tag{1.3.11}$$

where the star notation denotes complex conjugation and the total energy $u^{\omega,\varepsilon}$ is defined as the sum of the electric energy and the magnetic energy. Substituting equations (1.3.5) and equations (1.3.7) into expressions (1.3.11) yields the following expressions for the electric and magnetic energies:

$$u_{\text{elec}}^{\omega,\varepsilon} = \frac{1}{2\omega^2} \int_{\mathbb{T}} \big((\hat{\epsilon}^\varepsilon)^{-1} \big(\mathbf{curl}\, \boldsymbol{H}^{\omega,\varepsilon} - \boldsymbol{J}^\omega \big) \cdot \big(\mathbf{curl}\, \boldsymbol{H}^{\omega,\varepsilon} - \boldsymbol{J}^\omega \big)^* \big), \tag{1.3.12}$$

$$u_{\text{mag}}^{\omega,\varepsilon} = \frac{1}{2\omega^2} \int_{\mathbb{T}} \big((\hat{\mu}^\varepsilon)^{-1} \mathbf{curl}\, \boldsymbol{E}^{\omega,\varepsilon} \cdot (\mathbf{curl}\, \boldsymbol{E}^{\omega,\varepsilon})^* \big). \tag{1.3.13}$$

Substituting the formal power series for $\boldsymbol{E}^{\omega,\varepsilon}$ into the magnetic energy integral (1.3.13) yields the following:

$$\begin{aligned} \frac{1}{2} \int_{\mathbb{T}} \boldsymbol{B}_0^\varepsilon \cdot \boldsymbol{H}_0^\varepsilon + O(\omega^2) = \frac{1}{2\omega^2} \int_{\mathbb{T}} (\hat{\mu}^\varepsilon)^{-1} \mathbf{curl}\, \boldsymbol{E}_0^\varepsilon \cdot \mathbf{curl}\, \boldsymbol{E}_0^\varepsilon + \\ + \frac{1}{2} \int_{\mathbb{T}} (\hat{\mu}^\varepsilon)^{-1} \mathbf{curl}\, \boldsymbol{E}_1^\varepsilon \cdot \mathbf{curl}\, \boldsymbol{E}_1^\varepsilon + O(\omega^2). \end{aligned}$$

Noting that $\mathbf{curl}\, \boldsymbol{E}_0^\varepsilon = \mathbf{0}$, the leading order energy from the quasistatic approximation is

$$\frac{1}{2} \int_{\mathbb{T}} \boldsymbol{B}_0^\varepsilon \cdot \boldsymbol{H}_0^\varepsilon = \frac{1}{2} \int_{\mathbb{T}} (\hat{\mu}^\varepsilon)^{-1} \mathbf{curl}\, \boldsymbol{E}_1^\varepsilon \cdot \mathbf{curl}\, \boldsymbol{E}_1^\varepsilon. \tag{1.3.14}$$

Hence, the energy functional which was minimised in Section 1.2 was a representation of the leading order magnetic energy in the quasistatic approximation.

The next section will see the discussions on Maxwell's equations in the last two sections used in conjunction with the work derived in Section 1.1 and Section 1.2 to develop a higher-order constitutive relation between $\boldsymbol{H}_0^\varepsilon$ and $\boldsymbol{B}_0^\varepsilon$. This relation will be derived in two separate ways which will be shown to coincide using the partial symmetrisation procedure developed in Section 1.2.5.

1.3.3 Higher-Order Constitutive Relations

In Section 1.3.1, it was noted that the constitutive laws (1.3.3) relate the electric field and magnetic field to the electric field displacement and magnetic field intensity respectively. In this section, a higher-order expression between the $\boldsymbol{\zeta}$-averaged magnetic field intensity $\bar{\boldsymbol{H}}_0^\varepsilon$ and the $\boldsymbol{\zeta}$-averaged magnetic field $\bar{\boldsymbol{B}}_0^\varepsilon$ will be derived.

Let $\hat{\mu}_\varepsilon$ be a Q-periodic, symmetric, elliptic matrix. Consider the following equation

$$\mathbf{curl}\,(\hat{\mu}^\varepsilon)^{-1}\mathbf{curl}\,\boldsymbol{E}_1^\varepsilon = \boldsymbol{f},$$

where $\boldsymbol{f} = -\boldsymbol{J}_0 \in [C^\infty_{\text{per}}(\mathbb{T})]^3 \cap X(\mathbb{T})$. Seeking a solution $\boldsymbol{E}_1^\varepsilon \in X(\mathbb{T})$ leads to the consideration of series (1.1.5). In turn, this yields a homogenised equation akin to (1.1.19).

Recall that $\boldsymbol{B}_0^\varepsilon$ satisfies the equation $\boldsymbol{B}_0^\varepsilon = -\mathbf{curl}\,\boldsymbol{E}_1^\varepsilon$. Hence the magnetic field can formally be expressed as

$$\boldsymbol{B}_0^\varepsilon(\boldsymbol{x}) = -\sum_{j=0}^{\infty} \varepsilon^j \Big\{\mathbf{curl}_y N^{(j+1)}(\boldsymbol{y}) + M^{(j+1)}(\boldsymbol{y})\Big\}\Big|_{\boldsymbol{y}=\boldsymbol{x}/\varepsilon} \nabla_x^j \mathbf{curl}_x \boldsymbol{v}(\boldsymbol{x}).$$

Consider the family of problems seen in Section 1.2.2. To this end, let the parameter $\boldsymbol{\zeta}$ represent a shift in the microscopic variable and a suitable change in notation be reflected by this shift, $\hat{\mu}_\zeta^\varepsilon$, $\boldsymbol{E}_1^{\varepsilon,\zeta}$, *etc.* Note the following averages with respect to $\boldsymbol{\zeta}$:

$$\bar{\boldsymbol{D}}_1^\varepsilon = \bar{\boldsymbol{E}}_1^\varepsilon = \boldsymbol{v}, \quad \langle \mathbf{curl}\,\boldsymbol{E}_1^{\varepsilon,\zeta}\rangle_\zeta = \mathbf{curl}_x \boldsymbol{v}. \tag{1.3.15}$$

where the bar notation once more represents averaging over Q with respect to $\boldsymbol{\zeta}$. The main result of this chapter is now presented where a higher-order constitutive law for the $\boldsymbol{\zeta}$-averaged magnetic field intensity $\bar{\boldsymbol{H}}_0^{\varepsilon,\zeta}$ is derived.

$$\begin{aligned}
-\bar{\boldsymbol{H}}_0^\varepsilon := -\langle \boldsymbol{H}_0^{\varepsilon,\zeta}\rangle_\zeta &= \langle (\hat{\mu}_\zeta^\varepsilon)^{-1}\mathbf{curl}\,\boldsymbol{E}_1^{\varepsilon,\zeta}\rangle_\zeta \quad \text{(by Maxwell's equations)}\\
&= \sum_{j=0}^{\infty} \varepsilon^j \hat{h}^{(j+2)} \nabla^j \mathbf{curl}\,\boldsymbol{v} \quad \text{(making use of expansion (1.1.5))}\\
&= \sum_{j=0}^{\infty} \varepsilon^j \hat{h}^{(j+2)} \nabla^j \langle \mathbf{curl}\,\boldsymbol{E}_1^{\varepsilon,\zeta}\rangle_\zeta \quad \text{(by (1.3.15))}\\
&= \sum_{j=0}^{\infty} \varepsilon^j \hat{h}^{(j+2)} \nabla^j \langle -\boldsymbol{B}_0^{\varepsilon,\zeta}\rangle_\zeta \quad \text{(by Maxwell's equations).}
\end{aligned}$$

Hence

$$\bar{\boldsymbol{H}}_0^\varepsilon = \sum_{j=0}^{\infty} \varepsilon^j \hat{h}^{(j+2)} \nabla^j \bar{\boldsymbol{B}}_0^\varepsilon. \tag{1.3.16}$$

The higher-order constitutive law (1.3.16) is of the form $\bar{\boldsymbol{H}}_0^\varepsilon = (\mu^{\text{eff}})^{-1}\bar{\boldsymbol{B}}_0^\varepsilon$, where μ^{eff} is the "effective permeability operator".

An alternative expression for the higher-order constitutive law will be derived using the energy integral (1.3.14). All of the results of Section 1.2 hold but in particular when substituting a trial field into (1.3.14), the averaged magnetic energy functional is given by the following formula

$$\bar{u}_{\text{mag}}^\varepsilon = \frac{1}{2}\int_{\mathbb{T}} \sum_{j,k=0}^{\infty} \varepsilon^{j+k} \tilde{h}^{jk} \nabla^j (\mathbf{curl}\, \boldsymbol{v}) \nabla^k (\mathbf{curl}\, \boldsymbol{v}), \tag{1.3.17}$$

where $\tilde{h}^{jk}$ is the same tensor given by expression (1.2.13) with $A = \hat{\mu}^{-1}$. Hence, integrating the expression given in (1.3.17) by parts and using an appropriate rearrangement of indices yields

$$\bar{u}_{\text{mag}}^\varepsilon = \frac{1}{2}\int_{\mathbb{T}} \bar{\mathfrak{H}}_0^\varepsilon \cdot \bar{\boldsymbol{B}}_0^\varepsilon, \qquad \bar{\mathfrak{H}}_0^\varepsilon := \sum_{\substack{j+k=n \\ j,k\in\mathbb{N}_0}} \varepsilon^n (-1)^k \bar{h}^{jk} \nabla^n \bar{\boldsymbol{B}}_0^\varepsilon, \tag{1.3.18}$$

where

$$\bar{h}^{jk}_{i_1 i_2 \dots i_{j+k+2}} := \tilde{h}^{jk}_{i_{j+1} i_2 \dots i_j i_{j+k+2} i_{j+2} \dots i_{j+k+1} i_1}.$$

It can be shown by making use of the partial symmetrisation procedure described in Section 1.2.5 along with Lemma 1.2.1, that the two expressions (1.3.16) and (1.3.18) obtained for the magnetic field intensity coincide to all orders.

Similar results will now be illustrated for the second-order equation which governs the magnetic field intensity with the amendments which have to be made being highlighted.

1.3.4 The Magnetic Field Intensity Equation

Assume that $\hat{\mu}^\varepsilon \equiv I$ and that $\hat{\epsilon}^\varepsilon$ is Q-periodic, symmetric and uniformly elliptic. For $\boldsymbol{J}_1 \in [C^\infty_{\text{per}}(\mathbb{T})]^3 \cap X(\mathbb{T})$, a solution is sought of the following:

$$\mathbf{curl}(\hat{\epsilon}^\varepsilon)^{-1}\mathbf{curl}\, \boldsymbol{H}_1^\varepsilon = \mathbf{curl}\big((\hat{\epsilon}^\varepsilon)^{-1}\boldsymbol{J}_1\big), \quad \boldsymbol{H}_1^\varepsilon \in X(\mathbb{T}). \tag{1.3.19}$$

Most of what is discussed in Section 1.1 is still valid, subject to a modification to the asymptotic expansion (1.1.5). The expansion of consideration for $\boldsymbol{H}_1^\varepsilon$ is the following:

$$\begin{aligned}\boldsymbol{H}_1^\varepsilon(\boldsymbol{x}) = \boldsymbol{w}(\boldsymbol{x},\varepsilon) + \sum_{j=1}^{\infty} \varepsilon^j \Big\{ &\nabla_y (S^{(j)}(\boldsymbol{y}) \nabla_x^j \boldsymbol{w}(\boldsymbol{x},\varepsilon)) + \nabla_x (S^{(j-1)}(\boldsymbol{y}) \nabla_x^{j-1} \boldsymbol{w}(\boldsymbol{x},\varepsilon)) + \\ &+ T^{(j)}(\boldsymbol{y}) \nabla_x^{j-1} \Big(\mathbf{curl}_x \boldsymbol{w}(\boldsymbol{x},\varepsilon) - \boldsymbol{J}_1(\boldsymbol{x})\Big)\Big\}\Big|_{\boldsymbol{y}=\boldsymbol{x}/\varepsilon},\end{aligned}$$

where $\boldsymbol{w} \in [C^\infty_{\text{per}}(\mathbb{T})]^3$ is divergence-free and may be expanded in an asymptotic series:

$$\boldsymbol{w}(\boldsymbol{x}, \varepsilon) = \boldsymbol{w}_0(\boldsymbol{x}) + \sum_{k=1}^{\infty} \varepsilon^k \boldsymbol{w}_k(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{T}.$$

Where $S^{(j)}$ plays the role of $K^{(j)}$ and $T^{(j)}$ plays the role of $N^{(j)}$, all the same properties as seen in Section 1.1.2 are satisfied by these tensors. Applying the same procedure seen in Section 1.1 the homogenised equation is found to be

$$\mathbf{curl}\, \hat{k}^{(2)} \mathbf{curl}\, \boldsymbol{w} = \mathbf{curl}\, \hat{k}^{(2)} \boldsymbol{J}_1, \tag{1.3.20}$$

and where the infinite-order homogenised equation is found to be

$$\sum_{j=0}^{\infty} \varepsilon^j \mathbf{curl}\, \hat{k}^{(j+2)} \nabla^j \mathbf{curl}\, \boldsymbol{w}(\boldsymbol{x}, \varepsilon) = \sum_{j=0}^{\infty} \varepsilon^j \mathbf{curl}\Big(\hat{k}^{(j+2)} \nabla^j \boldsymbol{J}_1(\boldsymbol{x})\Big),$$

where

$$\hat{k}^{(j+2)} = \left\langle \hat{\epsilon}^{-1} \left\{ \mathbf{curl}\, T^{(j+1)} + R^{(j+1)} \right\} \right\rangle,$$

and $R^{(j)}$ plays the role of $M^{(j)}$. A similar theorem to that of Theorem 1.1.2 regarding the rigorous justification of the asymptotic procedure can be proven with analogous error estimates.

The analogous variational considerations of Section 1.2 will now be presented. Recall the expression for the electric energy given by formula (1.3.11). Upon applying the quasistatic approximation, the leading order energy is given by the expression

$$u^\varepsilon_{\text{elec}} := \frac{1}{2} \int_{\mathbb{T}} (\boldsymbol{E}^\varepsilon_0 \cdot \boldsymbol{D}^\varepsilon_0) = \frac{1}{2} \int_{\mathbb{T}} (\hat{\epsilon}^\varepsilon)^{-1} \big(\mathbf{curl}\, \boldsymbol{H}^\varepsilon_1 - \boldsymbol{J}_1\big) \cdot \big(\mathbf{curl}\, \boldsymbol{H}^\varepsilon_1 - \boldsymbol{J}_1\big).$$

Expanding the dot product under the integral on the right-hand side yields

$$u^\varepsilon_{\text{elec}} = \int_{\mathbb{T}} \Big(\frac{1}{2} (\hat{\epsilon}^\varepsilon)^{-1} \mathbf{curl}\, \boldsymbol{H}^\varepsilon_1 \cdot \mathbf{curl}\, \boldsymbol{H}^\varepsilon_1 - \mathbf{curl}((\hat{\epsilon}^\varepsilon)^{-1} \boldsymbol{J}_1) \cdot \boldsymbol{H}^\varepsilon_1\Big) + \\ + \frac{1}{2} \int_{\mathbb{T}} (\hat{\epsilon}^\varepsilon)^{-1} \boldsymbol{J}_1 \cdot \boldsymbol{J}_1. \tag{1.3.21}$$

The first integral in equation (1.3.21) is the functional whose Euler-Lagrange equation is (1.3.19) where as the integral on the right of equation (1.3.21) is something extra which is independent of $\boldsymbol{H}^\varepsilon_1$. Hence, the minimiser of the functional $E_\varepsilon(\boldsymbol{H}, \boldsymbol{J}_1)$ defined as

$$E_\varepsilon(\boldsymbol{H}, \boldsymbol{J}_1) = \frac{1}{2} \int_{\mathbb{T}} (\hat{\epsilon}^\varepsilon)^{-1} \big(\mathbf{curl}\, \boldsymbol{H} - \boldsymbol{J}_1\big) \cdot \big(\mathbf{curl}\, \boldsymbol{H} - \boldsymbol{J}_1\big), \tag{1.3.22}$$

will coincide with $\boldsymbol{H}^\varepsilon_1 \in X(\mathbb{T})$, the solution of (1.3.19). Hence, all the same conclusions made in Section 1.2 will hold and furthermore an infinite-order homogenised equation can be obtained via these variational considerations. This infinite-order homogenised equation is obtained by considering the minimisation of functional (1.3.22) over restricted trial fields, i.e., by considering the minimisation of functionals of the kind

$$\bar{E}_K(\boldsymbol{w}, \boldsymbol{J}_1, \varepsilon) := \int_{\mathbb{T}} \left\{ \sum_{j,l=0}^{K} \frac{1}{2} \varepsilon^{j+l} \tilde{k}^{jl} \nabla^j (\mathbf{curl}\, \boldsymbol{w} - \boldsymbol{J}_1) \nabla^l (\mathbf{curl}\, \boldsymbol{w} - \boldsymbol{J}_1) \right\},$$

where $\tilde{k}^{jl}$ is a tensor of order $(j + l + 2)$ defined by the following formula:

$$\tilde{k}^{jl} = \left\langle \hat{\epsilon}^{-1} \big(\mathbf{curl}\, T^{(j+1)} + R^{(j+1)} \big) \big(\mathbf{curl}\, T^{(l+1)} + R^{(l+1)} \big) \right\rangle.$$

Taking different values of K yields higher-order homogenised equations as seen in Section 1.2.4. In particular, when $K = 0$, it can be shown that

$$\mathbf{curl}\, \tilde{k}^{00} \mathbf{curl}\, \boldsymbol{w}^{(0)} = \mathbf{curl}\, \tilde{k}^{00} \boldsymbol{J}_1.$$

This equation coincides with the homogenised equation (1.3.20). Considering the case when $K = \infty$ leads to the infinite-order homogenised equation

$$\mathbf{curl}\, \tilde{k}^{00} \mathbf{curl}\, \boldsymbol{w}^{(\infty)} + \sum_{n=1}^{\infty} \varepsilon^n \mathbf{curl}\, \tilde{\tilde{k}}^n \nabla^n \mathbf{curl}\, \boldsymbol{w}^{(\infty)} = \sum_{n=0}^{\infty} \varepsilon^n \mathbf{curl}\, \tilde{\tilde{k}}^n \nabla^n \boldsymbol{J}_1,$$

where

$$\tilde{\tilde{k}}^n_{i_1 \ldots i_{n+2}} = \frac{1}{2} \sum_{\substack{j+l=n, \\ j,l \in \mathbb{N}_0}} \left\{ (-1)^j \tilde{k}^{jl}_{i_{j+1} i_2 \ldots i_j i_1 i_{j+2} \ldots i_{j+l+2}} + (-1)^l \tilde{k}^{jl}_{i_{j+1} i_2 \ldots i_j i_{j+l+2} i_{j+2} \ldots i_{j+l+1} i_1} \right\}.$$

Applying the partial symmetrisation procedure of Section 1.2.5 once more yields that the infinite-order expansions coincide to all orders.

The final consideration of this section concerns the derivation of a higher-order constitutive law between $\boldsymbol{E}_0^\varepsilon$ and $\boldsymbol{D}_0^\varepsilon$. See Section 1.3.3 for full details but by considering the $\boldsymbol{\zeta}$-shifted family of problems and averaging over Q with respect to $\boldsymbol{\zeta}$ yields a higher-order constitutive law of the following form:

$$\bar{\boldsymbol{E}}_0^\varepsilon = \sum_{j=0}^{\infty} \varepsilon^j \hat{k}^{(j+2)} \nabla^j \bar{\boldsymbol{D}}_0^\varepsilon.$$

Applying then the $\boldsymbol{\zeta}$-averaging to the equivalent family of $\boldsymbol{\zeta}$-shifted variational problems also yields a higher-order homogenised equation:

$$\bar{\boldsymbol{\mathfrak{E}}}_0^\varepsilon := \sum_{\substack{j+l=n \\ j,l \in \mathbb{N}_0}} \varepsilon^n (-1)^l \bar{k}^{jl} \nabla^n \bar{\boldsymbol{D}}_0^\varepsilon, \qquad \bar{k}^{jl}_{i_1 i_2 \ldots i_{j+l+2}} := \tilde{k}^{jl}_{i_{j+1} i_2 \ldots i_j i_{j+l+2} i_{j+2} \ldots i_{j+l+1} i_1}.$$

Note that $\bar{\boldsymbol{D}}_0^\varepsilon = \mathbf{curl}\, \boldsymbol{w} - \boldsymbol{J}_1$. Once more, these two higher-order constitutive laws are shown to coincide to all orders by making use of the partial symmetrisation procedure of Section 1.2.5 and Lemma 1.2.1.

In the concluding section of this chapter, it will be shown that the full system of Maxwell equations can be homogenised and in turn that infinite-order homogenised equations for the electric field and magnetic field intensity can be derived simultaneously.

1.3.5 Homogenisation of the Full System of Maxwell Equations

Assume that both the permittivity $\hat{\epsilon}^\varepsilon$ and the permeability $\hat{\mu}^\varepsilon$ are periodic, symmetric and uniformly elliptic. Moreover, consider the following space:

$$V_{\hat{A}^\varepsilon}(\mathbb{T}) := \left\{ \boldsymbol{\varphi} \in [L^2_{\text{per}}(\mathbb{T})]^3 \;\middle|\; \text{div}_y \hat{A}(\boldsymbol{y})\nabla_y \boldsymbol{\varphi}(\boldsymbol{y}) = -\text{div}_y \hat{A}(\boldsymbol{y}),\ \boldsymbol{y} = \frac{\boldsymbol{x}}{\varepsilon} \right\}, \quad \hat{A} \in \{\hat{\epsilon}, \hat{\mu}\}.$$

Given $\boldsymbol{f}, \boldsymbol{J}_1 \in [C^\infty(\mathbb{T})]^3 \cap X(\mathbb{T})$, a solution is sought of

$$\begin{cases} \mathbf{curl}(\hat{\mu}^\varepsilon)^{-1}\mathbf{curl}\,\boldsymbol{E}_1^\varepsilon = \boldsymbol{f}, \\ \text{div}\big(\hat{\epsilon}^\varepsilon \boldsymbol{E}_1^\varepsilon\big) = 0, \end{cases} \qquad \begin{cases} \mathbf{curl}(\hat{\epsilon}^\varepsilon)^{-1}\mathbf{curl}\,\boldsymbol{H}_1^\varepsilon = \mathbf{curl}((\hat{\epsilon}^\varepsilon)^{-1}\boldsymbol{J}_1), \\ \text{div}\big(\hat{\mu}^\varepsilon \boldsymbol{H}_1^\varepsilon\big) = 0, \end{cases}$$

where

$$\boldsymbol{E}_1^\varepsilon, \in \{\boldsymbol{u} \in [L^2_{\text{per}}(\mathbb{T})]^3 \mid \mathbf{curl}\,\boldsymbol{u} \in [L^2(\mathbb{T})]^3,\ \boldsymbol{u} \perp V_{\hat{\epsilon}^\varepsilon}(\mathbb{T})\},$$

$$\boldsymbol{H}_1^\varepsilon \in \{\boldsymbol{u} \in [L^2_{\text{per}}(\mathbb{T})]^3 \mid \mathbf{curl}\,\boldsymbol{u} \in [L^2(\mathbb{T})]^3,\ \boldsymbol{u} \perp V_{\hat{\mu}^\varepsilon}(\mathbb{T})\}.$$

For the problem presented, the asymptotic expansions that should be considered for the electric field and magnetic field intensity respectively are

$$\begin{aligned} \boldsymbol{E}_1^\varepsilon(\boldsymbol{x}) = \Big(\nabla_y K^{(0)}(\boldsymbol{y}) + I\Big)\Big|_{\boldsymbol{y}=\boldsymbol{x}/\varepsilon} \boldsymbol{v}(\boldsymbol{x},\varepsilon) + \sum_{j=1}^\infty \varepsilon^j \Big\{ \nabla_y (K^{(j)}(\boldsymbol{y})\nabla_x^j \boldsymbol{v}(\boldsymbol{x},\varepsilon)) + \\ + \nabla_x (K^{(j-1)}(\boldsymbol{y})\nabla_x^{j-1}\boldsymbol{v}(\boldsymbol{x},\varepsilon)) + N^{(j)}(\boldsymbol{y})\nabla_x^{j-1}(\mathbf{curl}_x \boldsymbol{v}(\boldsymbol{x},\varepsilon))\Big\}\Big|_{\boldsymbol{y}=\boldsymbol{x}/\varepsilon}, \end{aligned}$$

$$\begin{aligned} \boldsymbol{H}_1^\varepsilon(\boldsymbol{x}) = \Big(\nabla_y S^{(0)}(\boldsymbol{y}) + I\Big)\Big|_{\boldsymbol{y}=\boldsymbol{x}/\varepsilon} \boldsymbol{w}(\boldsymbol{x},\varepsilon) + \sum_{j=1}^\infty \varepsilon^j \Big\{ \nabla_y (S^{(j)}(\boldsymbol{y})\nabla_x^j \boldsymbol{w}(\boldsymbol{x},\varepsilon)) + \\ + \nabla_x (S^{(j-1)}(\boldsymbol{y})\nabla_x^{j-1}\boldsymbol{w}(\boldsymbol{x},\varepsilon)) + T^{(j)}(\boldsymbol{y})\nabla_x^{j-1}\Big(\mathbf{curl}_x \boldsymbol{w}(\boldsymbol{x},\varepsilon) - \boldsymbol{J}_1(\boldsymbol{x})\Big)\Big\}\Big|_{\boldsymbol{y}=\boldsymbol{x}/\varepsilon}. \end{aligned}$$

The following amendments are made to the assumptions presented in Section 1.1:

1. The tensors $K^{(0)}$ and $S^{(0)}$ satisfy the equations

$$\text{div}\,\hat{\varepsilon}\nabla K^{(0)} = -\text{div}\,\hat{\varepsilon}, \quad \text{div}\,\hat{\mu}\nabla S^{(0)} = -\text{div}\,\hat{\mu}. \tag{1.3.23}$$

2. For all $j \in \mathbb{N}_0$, the elements of the tensors $K^{(j)},\ S^{(j)}$ belong to $H^1_{\text{per}}(Q)$.

Noting the above amendments, the majority of the work presented in the previous sections holds. Amongst the differences is the work in Section 1.1.4 where the equations satisfied by the tensors $K^{(j)}$ will now be slightly modified to accommodate the amended divergence conditions. It is also noted that it is not the Maxwell inequality (1.A.1) that is used to establish uniqueness of the solution of the full system of equations but the more general Maxwell inequality (1.A.2).

Carrying out a procedure analogous to that seen in Section 1.1, the following homogenised system is obtained for the full system of Maxwell equations:

$$\begin{cases} \mathbf{curl}\, \hat{h}_{-1}^{(2)} \mathbf{curl}\, \boldsymbol{v} + \sum_{j=1}^{\infty} \varepsilon^j \mathbf{curl}\, \hat{h}_{-1}^{(j+2)} \nabla^j \mathbf{curl}\, \boldsymbol{v} = \boldsymbol{f}, \\ \operatorname{div}(\hat{k}^{(2)} \boldsymbol{v}) + \sum_{j=1}^{\infty} \varepsilon^j \operatorname{div}(\hat{k}^{(j+2)} \nabla^j \boldsymbol{v}) = 0, \end{cases} \tag{1.3.24}$$

$$\begin{cases} \mathbf{curl}\, \hat{k}_{-1}^{(2)} \mathbf{curl}\, \boldsymbol{w} + \sum_{j=1}^{\infty} \varepsilon^j \mathbf{curl}\, \hat{k}_{-1}^{(j+2)} \nabla^j \mathbf{curl}\, \boldsymbol{w} = \sum_{j=0}^{\infty} \mathbf{curl}\, (\hat{k}_{-1}^{(j+2)} \nabla^j \boldsymbol{J}_1), \\ \operatorname{div}(\hat{h}^{(2)} \boldsymbol{w}) + \sum_{j=1}^{\infty} \varepsilon^j \operatorname{div}(\hat{h}^{(j+2)} \nabla^j \boldsymbol{w}) = 0, \end{cases} \tag{1.3.25}$$

where

$$\hat{h}_{-1}^{(j+2)} = \left\langle \hat{\mu}^{-1} \left(\mathbf{curl}\, N^{(j+1)} + M^{(j+1)} \right) \right\rangle, \quad \hat{k}_{-1}^{(j+2)} = \left\langle \hat{\epsilon}^{-1} \left(\mathbf{curl}\, T^{(j+1)} + R^{(j+1)} \right) \right\rangle,$$

$$\hat{h}^{(2)} = \left\langle \hat{\mu} \left(\nabla S^{(0)} + I \right) \right\rangle, \quad \hat{h}^{(j+2)} = \left\langle \hat{\mu} \left(\nabla S^{(j)} + \tilde{R}^{(j+1)} \right) + Q^{(j+1)} \right\rangle,$$

$$\hat{k}^{(2)} = \left\langle \hat{\epsilon} \left(\nabla K^{(0)} + I \right) \right\rangle, \quad \hat{k}^{(j+2)} = \left\langle \hat{\epsilon} \left(\nabla K^{(j)} + \tilde{M}^{(j+1)} \right) + P^{(j+1)} \right\rangle.$$

Here

$$P_{i_1 i_2 \ldots i_{j+2}}^{(j+1)} = (\hat{\epsilon} * K^{(j-1)})_{i_1 i_2 \ldots i_{j+2}} = \hat{\epsilon}_{i_1 i_2} K_{i_3 \ldots i_{j+2}}^{(j-1)}, \quad Q_{i_1 i_2 \ldots i_{j+2}}^{(j+1)} = (\hat{\mu} * S^{(j-1)})_{i_1 i_2 \ldots i_{j+2}} = \hat{\mu}_{i_1 i_2} S_{i_3 \ldots i_{j+2}}^{(j-1)}.$$

$$\tilde{M}_{i_1 i_2 \ldots i_{j+2}}^{(j+1)} = N_{i_1 \ldots i_j s}^{(j)} \epsilon_{s i_{j+1} i_{j+2}}, \quad \tilde{R}_{i_1 i_2 \ldots i_{j+2}}^{(j+1)} = T_{i_1 \ldots i_j s}^{(j)} \epsilon_{s i_{j+1} i_{j+2}}.$$

Proposition 1.3.1. *The four homogenised tensors $\hat{h}^{(2)}$, $\hat{k}^{(2)}$, $\hat{h}_{-1}^{(2)}$, $\hat{k}_{-1}^{(2)}$ satisfy the relations*

$$\hat{h}^{(2)} = \left(\hat{h}_{-1}^{(2)} \right)^{-1}, \quad \hat{k}^{(2)} = \left(\hat{k}_{-1}^{(2)} \right)^{-1}.$$

Proof. Recall that $N^{(1)}$ satisfies equation (1.1.9). Hence, it follows that

$$\hat{\mu}^{-1} \left(\mathbf{curl}\, N^{(1)} + I \right) = \nabla \boldsymbol{\psi} + C,$$

where $\boldsymbol{\psi}$ is periodic and satisfies the equation $\operatorname{div}[\hat{\mu}(\nabla \boldsymbol{\psi} + C)] = 0$. It is also clear that $C = \hat{h}_{-1}^{(2)}$. Multiplying the above equation on both sides by $\hat{\mu}$ and taking the average yields

$$I = \left\langle \hat{\mu} \left(\nabla \boldsymbol{\psi} + \hat{h}_{-1}^{(2)} \right) \right\rangle.$$

By comparing the equation satisfied by $\boldsymbol{\psi}$ with the equation satisfied by $S^{(0)}$ in equation (1.3.23), it is shown that $\boldsymbol{\psi} = \hat{h}_{-1}^{(2)} S^{(0)}$. Hence, noting that $\hat{h}_{-1}^{(2)}$ is symmetric yields

$$I = \left\langle \hat{\mu} \left(\nabla S^{(0)} + I \right) \right\rangle \hat{h}_{-1}^{(2)} = \hat{h}^{(2)} \hat{h}_{-1}^{(2)},$$

as required. An analogous proof holds for the other pair of homogenised tensors. $\square$

This concludes discussions on the higher-order homogenisation of Maxwell's equations. In comparison with the homogenisation of scalar elliptic equations via the method of asymptotic expansions, homogenisation of Maxwell's equations can be handled in much the same way but it is clear that care must be given if true physical phenomena want to be understood. In particular, if considerations are given to the second-order equation for the magnetic field intensity, then the right-hand side of this equation must take an appropriate form. Moreover, due to the more complicated form of the operator, attention must be given to the two different constructions of the infinite-order homogenised equations and the associated tensors involved in both cases.

Appendices

1.A Maxwell Inequality

In order to prove that the solution of equation (1.1.4) was unique, the following inequality was used:

$$\|\boldsymbol{v}\|^2_{L^2(\mathbb{T})} \leq C(\mathbb{T})(\|\mathbf{curl}\,\boldsymbol{v}\|^2_{L^2(\mathbb{T})} + \|\mathrm{div}\,\boldsymbol{v}\|^2_{L^2(\mathbb{T})}). \tag{1.A.1}$$

where $C(\mathbb{T})$ is a constant which depends on the size of the domain $\mathbb{T}$. It is noted that inequality (1.A.1) is only required in the context of periodic domains but that more complicated domains can be considered and Maxwell inequalities derived in for these domains (see Neff, Pauly & Witsch [50]).

Lemma 1.A.1. *Let $\boldsymbol{v} \in [L^2_{\mathrm{per}}(\mathbb{T})]^3$. If $\mathbf{curl}\,\boldsymbol{v} \in [L^2(\mathbb{T})]^3$, $\mathrm{div}\,\boldsymbol{v} \in L^2(\mathbb{T})$ and $\langle \boldsymbol{v} \rangle_{\mathbb{T}} = 0$ then inequality (1.A.1) holds.*

Proof. As $\boldsymbol{v} \in [L^2(\mathbb{T})]^3$, $\langle \boldsymbol{v} \rangle_{\mathbb{T}}$ it can be written in a Fourier series:

$$\boldsymbol{v}(\boldsymbol{x}) = \sum_{\boldsymbol{k} \in \mathbb{Z}^3 \setminus \{\boldsymbol{0}\}} \boldsymbol{c}_{\boldsymbol{k}} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}},$$

where $\boldsymbol{c}_{\boldsymbol{k}} = (c_{k_1}, c_{k_2}, c_{k_3}) \in \mathbb{C}^3$ are the Fourier coefficients and $\boldsymbol{k} = (k_1, k_2, k_3) \in \mathbb{Z}^3$. Note that for a constant vector $\boldsymbol{k}$, it can be shown that

$$\mathbf{curl}(\boldsymbol{c}_{\boldsymbol{k}} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}) = \mathrm{i}(\boldsymbol{k} \times \boldsymbol{c}_{\boldsymbol{k}})\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}, \quad \mathrm{div}(\boldsymbol{c}_{\boldsymbol{k}} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}) = \mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{c}_{\boldsymbol{k}})\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}.$$

Assuming that $\mathbf{curl}\,\boldsymbol{v} \in [L^2(\mathbb{T})]^3, \mathrm{div}\,\boldsymbol{v} \in L^2(\mathbb{T})$, by Parseval's Identity (see Gasquet & Witomski [34, Chapter 4]) and Lagrange's Identity[5] it follows that

$$\begin{aligned} \|\boldsymbol{v}\|^2_{L^2(\mathbb{T})} &= C(\mathbb{T}) \sum_{\boldsymbol{k} \in \mathbb{Z}^3 \setminus \{\boldsymbol{0}\}} |\boldsymbol{c}_{\boldsymbol{k}}|^2 \\ &\leq C(\mathbb{T}) \sum_{\boldsymbol{k} \in \mathbb{Z}^3 \setminus \{\boldsymbol{0}\}} |\boldsymbol{k}|^2 |\boldsymbol{c}_{\boldsymbol{k}}|^2 \\ &= C(\mathbb{T}) \sum_{\boldsymbol{k} \in \mathbb{Z}^3 \setminus \{\boldsymbol{0}\}} (|\boldsymbol{k} \times \boldsymbol{c}_{\boldsymbol{k}}|^2 + |\boldsymbol{k} \cdot \boldsymbol{c}_{\boldsymbol{k}}|^2) \\ &= C(\mathbb{T})(\|\mathbf{curl}\,\boldsymbol{v}\|^2_{L^2(\mathbb{T})} + \|\mathrm{div}\,\boldsymbol{v}\|^2_{L^2(\mathbb{T})}), \end{aligned}$$

[5] Any two vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^3$ satisfy Lagrange's Identity: $|\boldsymbol{u}|^2|\boldsymbol{v}|^2 = |\boldsymbol{u} \times \boldsymbol{v}|^2 + |\boldsymbol{u} \cdot \boldsymbol{v}|^2$.

as required. $\square$

The following result is a Maxwell inequality pertinent to the study of the full system of Maxwell equations in the quasistatic limit.

Lemma 1.A.2. *Let* $\boldsymbol{v} \in [L^2_{\text{per}}(\mathbb{T})]^3$ *have zero average on* $\mathbb{T}$ *and let* A *be some bounded, measurable, uniformly elliptic matrix field. If* $\mathbf{curl}\,\boldsymbol{v} \in [L^2(\mathbb{T})]^3$, $\operatorname{div}(A\boldsymbol{v}) \in L^2(\mathbb{T})$ *then the following inequality holds:*

$$\|\boldsymbol{v}\|^2_{L^2(\mathbb{T})} \le C(\mathbb{T})(\|\mathbf{curl}\,\boldsymbol{v}\|^2_{L^2(\mathbb{T})} + \|\operatorname{div}(A\boldsymbol{v})\|^2_{L^2(\mathbb{T})}). \tag{1.A.2}$$

Proof. Let $\boldsymbol{v}$ be expressed as a sum $\boldsymbol{v}_0 + \boldsymbol{v}_1 + \boldsymbol{v}_2$ where $\boldsymbol{v}_0 \in \mathbb{C}^3$, $\boldsymbol{v}_1$ is curl free and $A\boldsymbol{v}_2$ is divergence free. The ability to write $\boldsymbol{v}$ as such a sum is guaranteed by a version of de Rham's theorem (see Amrouche, Ciarlet & Mardare [3] for further details). Note that since $\langle \boldsymbol{v} \rangle_{\mathbb{T}} = \mathbf{0}$ it immediately follows that $\boldsymbol{v}_0 = \mathbf{0}$.

Consider the $\mathbb{T}$-periodic functions $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ which solve the following problems:

$$\begin{cases} \mathbf{curl} A^{-1} \mathbf{curl}\,\boldsymbol{\varphi} = \mathbf{curl}\,\boldsymbol{v}, \\ \operatorname{div}\boldsymbol{\varphi} = 0, \end{cases} \qquad \begin{cases} \operatorname{div} A \nabla \boldsymbol{\psi} = \operatorname{div}(A\boldsymbol{v}), \\ \langle \boldsymbol{\psi} \rangle = \mathbf{0}, \end{cases}$$

Considering the above equations in their weak form and using ellipticity estimates and the Cauchy-Schwarz inequality, the following estimates are obtained:

$$\|A^{-1}\mathbf{curl}\,\boldsymbol{\varphi}\|_{L^2(\mathbb{T})} \le C_1 \|\mathbf{curl}\,\boldsymbol{v}\|_{L^2(\mathbb{T})}, \quad \|\nabla\boldsymbol{\psi}\|_{L^2(\mathbb{T})} \le C_2 \|\operatorname{div}(A\boldsymbol{v})\|_{L^2(\mathbb{T})},$$

where the constants $C_1,\ C_2$ depend on the size of $\mathbb{T}$ and the ellipticity and boundedness constants for the matrix field A. Hence,

$$\begin{aligned} \|\boldsymbol{v}\|^2_{L^2(\mathbb{T})} \le \|\boldsymbol{v}_1\|^2_{L^2(\mathbb{T})} + \|\boldsymbol{v}_2\|^2_{L^2(\mathbb{T})} &= \|\nabla\boldsymbol{\psi}\|_{L^2(\mathbb{T})} + \|A^{-1}\mathbf{curl}\,\boldsymbol{\varphi}\|_{L^2(\mathbb{T})} \\ &\le C(\mathbb{T}) \left(\|\mathbf{curl}\,\boldsymbol{v}\|^2_{L^2(\mathbb{T})} + \|\operatorname{div}(A\boldsymbol{v})\|^2_{L^2(\mathbb{T})} \right), \end{aligned}$$

as required. $\square$

1.B Proof of Theorem 1.1.2 (iii)

The result presented in this appendix completes the proof presented in Section 1.1.5. An analogous proof of this result can be found in Smyshlayaev & Cherednichenko [69] for the scalar homogenisation.

Theorem 1.B.1. *Let* $M(\boldsymbol{x}/\varepsilon) \in [L^2(\mathbb{T})]^{j^3}$ *be a* Q*-periodic tensor of order* j *whose components have zero average over* Q *and let* $g(\boldsymbol{x})$ *be a smooth,* $\mathbb{T}$*-periodic tensor of order* $j-1$*. Then there exists a constant* C_r *such that*

$$\left| \int_{\mathbb{T}} M(\boldsymbol{x}/\varepsilon) g(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \right| \le C_r \varepsilon^r, \qquad \forall r \in \mathbb{N}. \tag{1.B.1}$$

where

$$\big(M(\boldsymbol{x}/\varepsilon)g(\boldsymbol{x})\big)_{i_1} = M(\boldsymbol{x}/\varepsilon)_{i_1 i_2 \ldots i_j} g(\boldsymbol{x})_{i_2 i_3 \ldots i_j}.$$

Proof. Each element of $M(\boldsymbol{x}/\varepsilon)$ can be represented as a convergent Fourier series, i.e., for $N \in \mathbb{N}$, as $N \to \infty$

$$\big(M^N(\boldsymbol{x}/\varepsilon)\big)_{i_1 i_2 \ldots i_j} := \sum_{|k_1|=0}^{N} \sum_{|k_2|=0}^{N} \sum_{|k_3|=0}^{N} (c_{\boldsymbol{k}})_{i_1 i_2 \ldots i_j} \exp(\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{x}/\varepsilon) \to \big(M(\boldsymbol{x}/\varepsilon)\big)_{i_1 i_2 \ldots i_j},$$

where $\boldsymbol{k} = (k_1, k_2, k_3) \in \mathbb{Z}^3$ and $c_{\boldsymbol{k}}$ are the Fourier coefficients. Note that since M has zero average over Q it follows that $c_{\boldsymbol{0}} = 0$. Let Q_ε^p, $p = 1, \ldots, (T/\varepsilon)^3$, denote the period cells of $\mathbb{T}$. Hence for all p and any $\varepsilon > 0$, all elements of $M^N(\boldsymbol{x}/\varepsilon)$ converge:

$$\int_{Q_\varepsilon^p} M^N(\boldsymbol{x}/\varepsilon) g(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \to \int_{Q_\varepsilon^p} M(\boldsymbol{x}/\varepsilon) g(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}, \qquad (N \to \infty).$$

Omitting all fixed subscript indices, fixing all repeated indices of M and g and fixing $N \in \mathbb{N}$, p and $\varepsilon > 0$ yields

$$\begin{aligned}
\left|\int_{Q_\varepsilon^p} M^N(\boldsymbol{x}/\varepsilon) g(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}\right| = \Bigg| &\sum_{|k_1|=1}^{N} c_{(k_1,0,0)} \int_{Q_\varepsilon^p} g(\boldsymbol{x}) \exp(\mathrm{i}k_1 x_1/\varepsilon)\mathrm{d}\boldsymbol{x} + \\
&+ \sum_{|k_2|=1}^{N} c_{(0,k_2,0)} \int_{Q_\varepsilon^p} g(\boldsymbol{x}) \exp(\mathrm{i}k_2 x_2/\varepsilon)\mathrm{d}\boldsymbol{x} + \sum_{|k_3|=1}^{N} c_{(0,0,k_3)} \int_{Q_\varepsilon^p} g(\boldsymbol{x}) \exp(\mathrm{i}k_3 x_3/\varepsilon)\mathrm{d}\boldsymbol{x} + \\
&+ \sum_{|k_1|,|k_2|=1}^{N} c_{(k_1,k_2,0)} \int_{Q_\varepsilon^p} g(\boldsymbol{x}) \exp(\mathrm{i}(k_1 x_1 + k_2 x_2)/\varepsilon)\mathrm{d}\boldsymbol{x} + \\
&+ \sum_{|k_1|,|k_3|=1}^{N} c_{(k_1,0,k_3)} \int_{Q_\varepsilon^p} g(\boldsymbol{x}) \exp(\mathrm{i}(k_1 x_1 + k_3 x_3)/\varepsilon)\mathrm{d}\boldsymbol{x} + \\
&+ \sum_{|k_2|,|k_3|=1}^{N} c_{(0,k_2,k_3)} \int_{Q_\varepsilon^p} g(\boldsymbol{x}) \exp(\mathrm{i}(k_2 x_2 + k_3 x_3)/\varepsilon)\mathrm{d}\boldsymbol{x} + \\
&+ \sum_{|k_1|=1}^{N} \sum_{|k_2|=1}^{N} \sum_{|k_3|=1}^{N} c_{\boldsymbol{k}} \int_{Q_\varepsilon^p} g(\boldsymbol{x}) \exp(\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{x}/\varepsilon)\mathrm{d}\boldsymbol{x}\Bigg|.
\end{aligned}$$

Integrating by parts an appropriate number of times in the above and adopting the notation $\partial^{r,s,t}g(\boldsymbol{x}) = \frac{\partial^{r+s+t}g(\boldsymbol{x})}{\partial^r x_1 \partial^s x_2 \partial^t x_3}$, for any $r \in \mathbb{N}$, it follows that

$$
\begin{aligned}
\left|\int_{Q_\varepsilon^p} M^N(\boldsymbol{x}/\varepsilon)g(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right| &= \left|\sum_{|k_1|=1}^{N} c_{(k_1,0,0)}\frac{(\mathrm{i}\varepsilon)^r}{k_1^r}\int_{Q_\varepsilon^p}\partial^{r,0,0}g(\boldsymbol{x})\exp(\mathrm{i}k_1x_1/\varepsilon)\mathrm{d}\boldsymbol{x}+\right.\\
&+\sum_{|k_2|=1}^{N} c_{(0,k_2,0)}\frac{(\mathrm{i}\varepsilon)^r}{k_2^r}\int_{Q_\varepsilon^p}\partial^{0,r,0}g(\boldsymbol{x})\exp(\mathrm{i}k_2x_2/\varepsilon)\mathrm{d}\boldsymbol{x}+\sum_{|k_3|=1}^{N} c_{(0,0,k_3)}\frac{(\mathrm{i}\varepsilon)^r}{k_3^r}\int_{Q_\varepsilon^p}\partial^{0,0,r}g(\boldsymbol{x})\exp(\mathrm{i}k_3x_3/\varepsilon)\mathrm{d}\boldsymbol{x}+\\
&+\sum_{|k_1|,|k_2|=1}^{N} c_{(k_1,k_2,0)}\frac{(\mathrm{i}\varepsilon)^{2r}}{k_1^rk_2^r}\int_{Q_\varepsilon^p}\partial^{r,r,0}g(\boldsymbol{x})\exp(\mathrm{i}(k_1x_1+k_2x_2)/\varepsilon)\mathrm{d}\boldsymbol{x}+\\
&+\sum_{|k_1|,|k_3|=1}^{N} c_{(k_1,0,k_3)}\frac{(\mathrm{i}\varepsilon)^{2r}}{k_1^rk_3^r}\int_{Q_\varepsilon^p}\partial^{r,0,r}g(\boldsymbol{x})\exp(\mathrm{i}(k_1x_1+k_3x_3)/\varepsilon)\mathrm{d}\boldsymbol{x}+\\
&+\sum_{|k_2|,|k_3|=1}^{N} c_{(0,k_2,k_3)}\frac{(\mathrm{i}\varepsilon)^{2r}}{k_2^rk_3^r}\int_{Q_\varepsilon^p}\partial^{0,r,r}g(\boldsymbol{x})\exp(\mathrm{i}(k_2x_2+k_3x_3)/\varepsilon)\mathrm{d}\boldsymbol{x}+\\
&\left.+\sum_{|k_1|=1}^{N}\sum_{|k_2|=1}^{N}\sum_{|k_3|=1}^{N} c_{\boldsymbol{k}}\frac{(\mathrm{i}\varepsilon)^{3r}}{k_1^rk_2^rk_3^r}\int_{Q_\varepsilon^p}\partial^{r,r,r}g(\boldsymbol{x})\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}/\varepsilon)\mathrm{d}\boldsymbol{x}\right|
\end{aligned}
$$

$$
\begin{aligned}
&\le \varepsilon^r\Big(\sum_{\boldsymbol{k}\in\mathbb{Z}^3} c_{\boldsymbol{k}}^2\Big)^{1/2}\Big(\sum_{i=1}^{3}\sum_{k_i\in\mathbb{Z}}\frac{1}{k_i^{2r}}+\frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{3}\sum_{k_i,k_j\in\mathbb{Z}}\frac{1}{k_i^{2r}k_j^{2r}}+\sum_{\boldsymbol{k}\in\mathbb{Z}^3}\frac{1}{k_1^{2r}k_2^{2r}k_3^{2r}}\Big)^{1/2}\times\\
&\quad\times\Big(\max_{\boldsymbol{x}\in\mathbb{T}}|\partial^{r,r,r}g(\boldsymbol{x})|+\max_{\boldsymbol{x}\in\mathbb{T}}|\partial^{r,r,0}g(\boldsymbol{x})|+\max_{\boldsymbol{x}\in\mathbb{T}}|\partial^{r,0,r}g(\boldsymbol{x})|+\max_{\boldsymbol{x}\in\mathbb{T}}|\partial^{0,r,r}g(\boldsymbol{x})|+\\
&\quad+\max_{\boldsymbol{x}\in\mathbb{T}}|\partial^{r,0,0}g(\boldsymbol{x})|+\max_{\boldsymbol{x}\in\mathbb{T}}|\partial^{0,r,0}g(\boldsymbol{x})|+\max_{\boldsymbol{x}\in\mathbb{T}}|\partial^{0,0,r}g(\boldsymbol{x})|\Big)|Q_\varepsilon^p|\\
&\le C_r\varepsilon^{r+3},
\end{aligned}
$$

for any p, $\varepsilon > 0$ and any $r \in \mathbb{N}$. Hence

$$
\left|\int_{\mathbb{T}} M(\boldsymbol{x}/\varepsilon)g(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right| \le \sum_{p=1}^{(T/\varepsilon)^3}\left|\int_{Q_\varepsilon^p} M(\boldsymbol{x}/\varepsilon)g(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right| \le C_r\varepsilon^r,
$$

as required. □

1.C Example in Section 1.1.6

It will now be shown that $a = -b$ where $a,\ b$ are the constants from the example considered in Section 1.1.6.

Consider the differential equation and solvability conditions for $L = L(y_2)$:

$$
-(\alpha L')' = \alpha - \langle\alpha\rangle, \quad L(0) = L(1), \quad \langle L\rangle = 0.
$$

Solving explicitly yields

$$
\begin{aligned}
L(y) = &\int_0^y \alpha^{-1}(t)\int_0^t(\langle\alpha\rangle-\alpha(s))\mathrm{d}s\mathrm{d}t-\\
&-\langle\alpha^{-1}\rangle^{-1}\Big(\int_0^y\alpha^{-1}(t)\mathrm{d}t\Big)\Big(\int_0^1\frac{1}{\alpha(t)}\int_0^t(\langle\alpha\rangle-\alpha(s))\mathrm{d}s\mathrm{d}t\Big)+\mathrm{const},
\end{aligned}
$$

where the constant is chosen such that $\langle L \rangle = 0$. Therefore

$$a = -\langle \alpha L' \rangle = \int_0^1 \int_0^y (\alpha(t) - \langle \alpha \rangle)\mathrm{d}t\mathrm{d}y - \langle \alpha^{-1} \rangle^{-1} \int_0^1 \frac{1}{\alpha(y)} \int_0^y (\alpha(t) - \langle \alpha \rangle)\mathrm{d}t\mathrm{d}y$$

$$= \int_0^1 (1 - \langle \alpha^{-1} \rangle^{-1} \alpha^{-1}(y)) \int_0^y (\alpha(t) - \langle \alpha \rangle)\mathrm{d}t\mathrm{d}y. \tag{1.C.1}$$

Consider now the differential equation and solvability conditions for $M = M(y_2)$:

$$(\alpha M')' = (\alpha N)', \quad M(0) = M(1), \quad \langle M \rangle = 0.$$

Again, solving explicitly yields

$$M(y) = \int_0^y N(t)\mathrm{d}t - \int_0^1 \int_0^y N(t)\mathrm{d}t\mathrm{d}y,$$

and, moreover, using the explicit formula for N, it can be shown that

$$b = -\langle \alpha M' \rangle = \int_0^1 1 - \alpha(y) \int_0^y ((\langle \alpha^{-1} \rangle^{-1} \alpha^{-1}(t))\mathrm{d}t\mathrm{d}y - \langle \alpha \rangle \int_0^1 \int_0^y (1 - \langle \alpha^{-1} \rangle^{-1} \alpha^{-1}(t))\mathrm{d}t\mathrm{d}y$$

$$= \int_0^1 (\alpha(y) - \langle \alpha \rangle) \int_0^y (1 - \langle \alpha^{-1} \rangle^{-1} \alpha^{-1}(t))\mathrm{d}t\mathrm{d}y.$$

Changing the order of integration in (1.C.1) and renaming t as y and vice versa yields

$$a = \int_0^1 (\alpha(y) - \langle \alpha \rangle) \int_y^1 (1 - \langle \alpha^{-1} \rangle^{-1} \alpha^{-1}(t))\mathrm{d}t\mathrm{d}y.$$

Hence

$$a + b = \int_0^1 (\alpha(y) - \langle \alpha \rangle) \int_0^1 (1 - \langle \alpha^{-1} \rangle^{-1} \alpha^{-1}(t))\mathrm{d}t\mathrm{d}y = 0.$$

Hence $a = -b$ as required.

It was also stated in Section 1.1.6 that the two constants labeled c and d were related via the equality $c = -d$. This will also now be illustrated for completeness. Recall that

$$c = -\langle \alpha P' + \alpha L \rangle, \quad d = \langle \alpha Q' \rangle = -\langle \alpha M \rangle.$$

Recall also that the following differential equations and solvability conditions are satisfied:

$$-(\alpha N')' = \alpha', \qquad N(0) = N(1), \quad \langle N \rangle = 0$$

$$-(\alpha P')' = a + \alpha L' + (\alpha L)', \qquad P(0) = P(1), \quad \langle P \rangle = 0.$$

Consider the differential equation for $P = P(y_2)$. Both sides of this equation will be multiplied by N and then both sides will be averaged over $[0, 1]$. Using integration by parts multiple times and noting the 1-periodicity and zero average of N, M, L, and P

yields the following argument:

$$
\begin{aligned}
& -\langle(\alpha P')'N + (\alpha L)'N\rangle && = \langle \alpha N L'\rangle && \\
\Longleftrightarrow \quad & \langle(\alpha P')N' + (\alpha N')L\rangle && = -\langle(\alpha N)'L\rangle && \text{(Using the ODE for } M\text{)} \\
\Longleftrightarrow \quad & -\langle P(\alpha N')' - (\alpha N')L\rangle && = -\langle(\alpha M')'L\rangle && \text{(Using the ODE for } N\text{)} \\
\Longleftrightarrow \quad & \langle \alpha' P + (\langle \alpha^{-1}\rangle^{-1} - \alpha)L\rangle && = -\langle(\alpha L')'M\rangle && \text{(Using the ODE for } L\text{)} \\
\Longleftrightarrow \quad & -\langle \alpha P' + \alpha L\rangle && = \langle(\alpha - \langle\alpha\rangle)M\rangle && \\
\Longleftrightarrow \quad & c && = \langle \alpha M\rangle = -d. &&
\end{aligned}
$$

1.D Zero Class of Order n for Equations of Maxwell Type

As an alternative approach to the symmetrisation procedure described in Section 1.2.5, a discussion on the zero class of order n will be laid out with the goal in mind of showing that the infinite-order expansions (1.1.19) and (1.2.19) are equal to all orders. The work which follows was originally motivated by the the absence of the $O(\varepsilon)$ term from the higher-order homogenised equation presented in the example in Section 1.1.6.

Consider those non-zero partially symmetrised tensors $h^{(n)}$ which appear in the kernel of the operator

$$\mathcal{H}\boldsymbol{v} \equiv \mathbf{curl}\, h^{(n)} \nabla^{n-2} \mathbf{curl}\, \boldsymbol{v},$$

for fixed vector $\boldsymbol{v}$. Two non-trivial tensors h_1 and h_2 of order n are equivalent, denoted $h_1 \sim h_2$, if

$$\mathbf{curl}\, h_1 \nabla^{n-2} \mathbf{curl}\, \boldsymbol{v} = \mathbf{curl}\, h_2 \nabla^{n-2} \mathbf{curl}\, \boldsymbol{v}, \qquad \forall \boldsymbol{v}.$$

Equivalently, this may be written $\mathbf{curl}\, h^{(n)} \nabla^{n-2} \mathbf{curl}\, \boldsymbol{v} = \mathbf{0}$, where $h^{(n)} = h_1 - h_2$. To this end, the idea is to show that the homogenised tensors $\hat{h}^{(n)}$ and $\tilde{\tilde{h}}^{n-2}$ are equivalent.

Consider those tensors $h^{(n)}$ which belong to the set

$$\sigma_n := \left\{ \; h^{(n)} \in \mathbb{R}^{n^3} \; \middle| \; \mathbf{curl}\, h^{(n)} \nabla^{n-2} \mathbf{curl}\, \boldsymbol{v} = \mathbf{0}, \;\; \forall \boldsymbol{v} \right\}. \tag{1.D.1}$$

This set is called the zero class of order n. When $n = 3$, it has already been shown that σ_3 is non-trivial (see Section 1.2.4) but recall that there equality of the tensors $\hat{h}^{(3)}$ and $\tilde{\tilde{h}}^{1}$. A further interest is to highlight the general structure of a constant tensor belonging to σ_3 and in particular determine how many independent constants they depend on. Consider the equation

$$\mathbf{curl}\, h^{(3)} \nabla \mathbf{curl}\, \boldsymbol{v} = \mathbf{0}, \qquad \Longleftrightarrow \qquad (\epsilon_{pi_1 s} h^{(3)}_{si_2 t} \epsilon_{ti_3 q}) \partial_{i_1} \partial_{i_2} \partial_{i_3} v_q = 0_p, \quad \forall p.$$

Taking the Fourier transform of the above expression over $\mathbb{T}$ and integrating by parts yields

$$(-\mathrm{i})^3 H_{pi_1 i_2 i_3 q} \xi_{i_1} \xi_{i_2} \xi_{i_3} (F v_q)(\boldsymbol{\xi}) = 0_p, \tag{1.D.2}$$

where $H_{pi_1i_2i_3q} := \epsilon_{pi_1s}h_{si_2t}\epsilon_{ti_3q}$ and

$$(Fv_q)(\boldsymbol{\xi}) = \int_{\mathbb{T}} v_q(\boldsymbol{x})\mathrm{e}^{-\mathrm{i}\boldsymbol{\xi}\cdot\boldsymbol{x}}\ \mathrm{d}\boldsymbol{x},$$

Hence, equation (1.D.2) is satisfied if and only if $H_{pi_1i_2i_3q}\xi_{i_1}\xi_{i_2}\xi_{i_3} = 0$, for all p, q. For fixed p, q, this expression may be treated as a non-commuting polynomial in the variables ξ_1, ξ_2, ξ_3, i.e., for example, where $\xi_1\xi_2\xi_1$ and $\xi_1^2\xi_2$ are treated as different objects. Only polynomials where the total power in the product is 3 will be considered, i.e., polynomials of the form $\xi_{i_1}^{\alpha}\xi_{i_2}^{\beta}\xi_{i_3}^{\gamma}$ where $\alpha+\beta+\gamma = 3$, $\alpha, \beta, \gamma \in \mathbb{N}_0$ are considered.

Denote the space of non-commuting polynomials in three variables by V_{nc}^3 and denote the space of commuting polynomials in three variables by V_c^3. Define a mapping $L : V_{nc}^3 \to V_c^3$ in the natural way. Hence

$$L(H_{pi_1i_2i_3q}\xi_{i_1}\xi_{i_2}\xi_{i_3}) = \tilde{H}_{p(i_1i_2i_3)q}\xi_1^{\alpha_1}\xi_2^{\alpha_2}\xi_3^{\alpha_3},$$

where $\tilde{H}_{p(i_1i_2i_3)q}$ denotes the summation over all tensor entries with α_1 1's, α_2 2's, α_3 3's and $\alpha_1+\alpha_2+\alpha_3 = 3$. Hence, it may be concluded that equation (1.D.2) is satisfied if and only if

$$\tilde{H}_{p(i_1i_2i_3)q} = 0, \quad \forall p, q, \quad \text{and } \forall(\alpha_1, \alpha_2, \alpha_3) \text{ such that } \alpha_1+\alpha_2+\alpha_3 = 3.$$

In order to find the structure of those tensors for which (1.D.2) is satisfied, the above argument demonstrates that a system of linear equations on the tensor entries must be fulfilled. In the case when the tensors are third-order, solving this system leads to the following structure:

$$h_{ijk}^{(3)} = \begin{cases} a_1, & ijk = \{122, 133\}, \\ a_2, & ijk = \{221, 331\}, \\ a_3, & ijk = \{211, 233\}, \\ a_4, & ijk = \{112, 332\}, \\ a_5, & ijk = \{311, 322\}, \\ a_6, & ijk = \{113, 223\}, \\ a_1+a_2, & ijk = \{111\}, \\ a_3+a_4, & ijk = \{222\}, \\ a_5+a_6, & ijk = \{333\}, \\ 0, & \text{otherwise}, \end{cases}$$

where $a_i,\ i = 1, \ldots 6$ are independent constants.

Remark. It is noted that by the dimension theorem of linear algebra:

$$\dim(V_{nc}^n) = \dim(\ker(L)) + \dim(\mathrm{Img}(L)).$$

Since the mapping L is surjective $\dim(\mathrm{Img}(L)) = \dim(V_c^n)$, and hence

$$\dim(\ker(L)) = 3^n - \begin{pmatrix} n+2 \\ 2 \end{pmatrix} = 3^n - \frac{1}{2}(n+1)(n+2).$$

The formula above gives some guidance as to the structure for the formula desired to describe how many constants a tensor in the set (1.D.1) depends on.

Developing the idea of the zero class to the next order it can be shown that $\hat{h}^{(4)} \sim \tilde{\tilde{h}}^2$, i.e., $(\hat{h}^{(4)} - \tilde{\tilde{h}}^2)$ belongs to σ_4. Moreover, where tensors from σ_3 were found to depend on six independent constants, it can be shown (directly via calculation) that partially symmetrised tensors from σ_4 depend on seventeen independent constants. Without providing all the details, the relevant equations satisfied by the tensor entries in the 4th order zero class are

$$\begin{cases} h_{ijji} = h_{ijjk} = h_{i(jk)i} = 0, \\ 2h_{i(jk)j} = h_{ikkk}, \quad 2h_{i(ji)k} = h_{jjjk}, \\ h_{ijjj} + h_{jjji} = 2h_{i(ij)i}, & i,j,k \in [3], \quad i \neq j \neq k. \\ 2h_{i(ji)j} + 2h_{j(ji)i} = h_{iiii} + h_{jjjj}, \\ h_{iiii} + 2h_{j(jk)k} = 2h_{i(ik)k} + 2h_{j(ji)i} \end{cases}$$

Likewise, it can be shown that symmetrised tensors of order 5 which belong to σ_5 depend on thirty-three independent constants and satisfy the following system of equations:

$$\begin{cases} h_{ijjji} = h_{ijjjk} = h_{i(jjk)i} = 0, \\ h_{ijjjj} = 3h_{i(jjk)k}, \quad h_{iiiij} = 3h_{k(iik)j}, \\ h_{ijjjj} + h_{jjjji} = 3h_{i(ijj)i}, \\ h_{i(jjk)j} = h_{i(jkk)k}, \quad h_{i(iij)k} = h_{j(jji)k}, \\ 3h_{i(ijj)i} = h_{ijjjj} + 3h_{k(jjk)i}, \quad 3h_{i(ijj)i} = h_{jjjji} + 3h_{i(kjj)k}, \\ h_{i(ijj)i} = h_{i(kjj)k} + h_{k(kjj)i}, & i,j,k \in [3], \; i \neq j \neq k. \\ 3h_{i(ijj)j} + 3h_{j(ijj)i} = h_{jjjjj} + 3h_{i(jii)i}, \\ h_{i(jkk)k} + h_{k(jkk)i} = 2h_{i(ijk)i}, \quad h_{i(jkk)k} + h_{j(kjj)i} = 2h_{i(ijk)i}, \\ h_{iiiii} + 6h_{j(ijk)k} = 3h_{i(iik)k} + 3h_{j(iij)i}, \\ h_{i(iij)i} + h_{j(jjk)k} = 2h_{i(ijk)k} + h_{j(ijj)i}, \quad h_{i(iij)i} + h_{k(kjj)j} = 2h_{k(ijk)i} + h_{i(ijj)j}, \\ 2h_{i(ijk)k} + 2h_{k(ijk)i} = h_{i(jii)i} + h_{k(jkk)k}, \end{cases}$$

It is of interest to generalise the considerations to the zero class of order n to try and determine the number of independent constants which tensors from σ_n depend on. Although confirming that the difference between the tensors $\hat{h}^{(n)}$ and $\tilde{\tilde{h}}^{n-2}$ belongs to σ_n is not necessarily possible from this viewpoint, investigation into this point is still of interest.

What is understood about the n-th order zero class σ_n will now be presented. To determine the elements of the class, consider the equation in the set in (1.D.1) in index notation:

$$(\epsilon_{pi_1t}h^{(n)}_{t(i_2\dots i_{n-1})s}\epsilon_{si_nq})\partial_{i_1}\partial_{i_2}\dots\partial_{i_{n-1}}\partial_{i_n}v_q = 0 \tag{1.D.3}$$

As before, conditions are sought on the constant elements of the now n-th order tensor such that the above equation is satisfied for all vectors $\boldsymbol{v}$. Even for the third and fourth order cases, this is a computationally challenge feat, however a number of things may be determined in general by reposing the problem combinatorially.

Let $\boldsymbol{i} = (i_1, i_2, \dots, i_n) \in [3]^n$ be an ordered n-tuple where $[3] = \{1, 2, 3\}$. Furthermore, define an ordered pair (2-tuple) $\boldsymbol{j} = (p, q) \in [3]^2$ with the obvious use of notation used to coincide with the index notation used in (1.D.3). In the following, only tuples where $i_1 \neq p$ and $i_n \neq q$ will be considered. This constraint is necessary since the permutation symbol ϵ_{ijk} takes value zero whenever any two of its indices are the same. It is not too difficult to see that there are $4 \times 3^{n-2}$ possible tuples of consideration. A further observation is made; for any n-th order tensor $h^{(n)}$ satisfying $\mathbf{curl}\, h^{(n)}\nabla^{n-2}\mathbf{curl}\, \boldsymbol{v} = \mathbf{0}$, $\forall \boldsymbol{v}$, the twelve entries

$$h^{(n)}_{ij\dots ji}, \quad i \neq j, \quad h^{(n)}_{ij\dots jk}, \quad i \neq j \neq k, \quad n \geq 2,$$

are zero. This follows by choosing all entries of $\boldsymbol{i}$ equal and then considering the four different valid pairs $\boldsymbol{j}$.

The following proposition addresses the construction of a formula for $P^n_{\boldsymbol{j}}(\boldsymbol{i}\,; \alpha_1, \alpha_2, \alpha_3)$, the number of permutations of $\boldsymbol{i}$ such that $i_1 \neq p$, $i_n \neq q$ and such that $\boldsymbol{i}$ contains α_1 1's, α_2 2's and α_3 3's. Note that the cases $\boldsymbol{j} = (p, p)$ and $\boldsymbol{j} = (p, q)$, $p \neq q$ are considered separately.

Proposition 1.D.1. *Let $\boldsymbol{i} = (i_1, \dots, i_n) \in [3]^n$ be an n-tuple ($n \geq 3$) containing α_p p's, α_q q's and α_r r's with $p \neq q \neq r$. Denote by $P^n_{\boldsymbol{j}}(\boldsymbol{i}\,; \alpha_p, \alpha_q, \alpha_r)$ the number permutations of $\boldsymbol{i}$ such that $\boldsymbol{j} \neq (i_1, i_n)$. Then*

1. *if $\boldsymbol{j} = (p, p)$*

$$P^n_{\boldsymbol{j}}(\boldsymbol{i}\,; \alpha_p, \alpha_q, \alpha_r) = (n - \alpha_p)(n - \alpha_p - 1)\frac{(n-2)!}{\alpha_p!\alpha_q!\alpha_r!},$$

2. *if $\boldsymbol{j} = (p, q)$*

$$P^n_{\boldsymbol{j}}(\boldsymbol{i}\,; \alpha_p, \alpha_q, \alpha_r) = [\alpha_r(n-1) + \alpha_p\alpha_q]\frac{(n-2)!}{\alpha_p!\alpha_q!\alpha_r!}.$$

Proof. It is well known that the total number of permutations of an n-tuple containing α_1 1's, α_2 2's and α_3 3's, $\alpha_1+\alpha_2+\alpha_3=n$ is given by the multinomial coefficient

$$\begin{pmatrix} n \\ \alpha_1,\alpha_2,\alpha_3 \end{pmatrix} = \frac{n!}{\alpha_1!\alpha_2!\alpha_3!}.$$

To then incorporate the constraint that $i_1 \neq p,\ i_n \neq q$, the number of permutations which start with a p or end with a q are subtracted off the above multinomial coefficient.

1. When $\boldsymbol{j}=(p,p)$, the number of permutations starting or ending with p is given as

$$(2n-\alpha_p-1)\frac{(n-2)!}{(\alpha_p-1)!\alpha_q!\alpha_r!}, \quad \alpha_p \geq 1.$$

   Note that the case when $\alpha_p=0$ is excluded here but it is clear that the end result works for all $\alpha_p \in [n]$. Hence

$$\begin{aligned} P^n_{\boldsymbol{j}}(\boldsymbol{i}\ ;\alpha_p,\alpha_q,\alpha_r) &= \frac{n!}{\alpha_p!\alpha_q!\alpha_r!} - (2n-\alpha_p-1)\frac{(n-2)!}{(\alpha_p-1)!\alpha_q!\alpha_r!} \\ &= \Big(n(n-1)-\alpha_p(2n-\alpha_p-1)\Big)\frac{(n-2)!}{\alpha_p!\alpha_q!\alpha_r!} \\ &= (n-\alpha_p)(n-\alpha_p-1)\frac{(n-2)!}{\alpha_p!\alpha_q!\alpha_r!}, \end{aligned}$$

   as required.

2. For the case when $\boldsymbol{j}=(p,q),\ p\neq q$, the same idea is applied; the number of permutations which start with p or end with a q is

$$\Big((\alpha_p+\alpha_q)(n-1)-\alpha_p\alpha_q\Big)\frac{(n-2)!}{\alpha_p!\alpha_q!\alpha_r!}.$$

   Hence

$$\begin{aligned} P^n_{\boldsymbol{j}}(\boldsymbol{i}\ ;\alpha_p,\alpha_q,\alpha_r) &= \frac{n!}{\alpha_p!\alpha_q!\alpha_r!} - \Big((\alpha_p+\alpha_q)(n-1)-\alpha_p\alpha_q\Big)\frac{(n-2)!}{\alpha_p!\alpha_q!\alpha_r!} \\ &= [n(n-1)-(\alpha_p+\alpha_q)(n-1)+\alpha_p\alpha_q]\frac{(n-2)!}{\alpha_p!\alpha_q!\alpha_r!} \\ &= [(n-1)(n-\alpha_p-\alpha_q)+\alpha_p\alpha_q]\frac{(n-2)!}{\alpha_p!\alpha_q!\alpha_r!} \\ &= [\alpha_r(n-1)+\alpha_p\alpha_q]\frac{(n-2)!}{\alpha_p!\alpha_q!\alpha_r!} \end{aligned}$$

   as required.

□

The final result that can be proven regards the number of equations which will turn up in the consideration of this problem.

Proposition 1.D.2. *A symmetrised tensor depends on* $\dfrac{9}{2}n(n-1)$ *entries and the number of linear equations* N_n *associated with the equation* $\mathbf{curl}\, h^{(n)}\nabla^{n-2}\mathbf{curl}\,\boldsymbol{v} = \mathbf{0}$ *is*

$$N_n = 9W(n,3) - 21 = \frac{3}{2}(3n^2 + 9n - 8), \quad n \geq 3,$$

where $W(n,3)$ *denotes the number of weak compositions*[6] *of* n *into 3 parts.*

Proof. Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ be a 3-tuple which denotes the number of 1's, 2's and 3's respectively in the n-tuple $\boldsymbol{i}$. Clearly $\boldsymbol{\alpha}$ represents a weak composition of n into 3 parts. Hence there will be a number of equations equal to the number of weak compositions of n into 3 parts minus the number of cases which are excluded by the constraint that $i_1 \neq p,\ i_n \neq q$. When $p = q$ there are 3 cases to be excluded and that when $p \neq q$ there are 2 cases to be excluded. Denoting the number of weak compositions of n into 3 parts by $W_n(3)$, the total number of equations N_n is

$$N_n = 6(W(n,3) - 2) + 3(W(n,3) - 3) = 9W(n,3) - 21.$$

The total number of weak compositions of n into m parts is given by

$$W(n,m) = \binom{n+m-1}{n}.$$

Hence

$$N_n = 9 \times \binom{n+2}{n} - 21 = \frac{9}{2}(n+2)(n+1) - 21 = \frac{3}{2}(3n^2 + 9n - 8),$$

as required. □

The determination of how many of the N_n equations are linearly independent remains unclear. The following conjecture is proposed:

Conjecture. Let $h^{(n)} \in \sigma_n$ be an n-th order symmetrised tensor. Then the number of linearly independent equations L_n associated with obtaining the general structure of $h^{(n)}$ increases quadratically in n. Hence

$$L_n = 2n^2 + 2n - 3$$

Moreover, tensors belonging to σ_n depend on at most C_n constants where

$$C_n = \frac{1}{2}(5n^2 - 13n + 6) = \frac{1}{2}(5n-3)(n-2).$$

[6]A composition of a positive integer n into m parts is an m-tuple $(a_1, a_2, \ldots, a_m)$ such that $a_i \in \mathbb{N},\ i = 1, 2, \ldots, m$ and $a_1 + a_2 + \ldots a_m = n$. A weak compoisition has the same definition except with the alteration $a_i \in \mathbb{N}_0$. For more details, see Riordan [60].

The work of this appendix is superfluous to the considerations discussed on partial symmetrisation, however, the work has presented an alternative viewpoint to the symmetrisation procedure which has in turn led to an overlap with an otherwise unrelated field. It is hypothesised that the there is a link between the proof of the above conjecture and the number of integer partitions of n of length less than or equal to 3. An integer partition of n is a multiset such that the elements of the set sum to n. As of yet, no closed formula is known for the number of integer partitions of an arbitrary integer n but it is hypothesised that the answer to the above conjecture may lead to a partial result for the number of integer partitions.

Chapter 2

Homogenisation of High-Contrast Composites with Periodic Frameworks of Critical Thickness

Introduction

According to Milton [48], composites are "materials that have inhomogeneities on length scales that are much larger than the atomic scale [...] but which are essentially homogeneous at macroscopic length scales". As the name might suggest, composite materials constitute two or more material components with the overall material possessing properties inherent to the constitutive materials. An example of such a composite is reinforced concrete with the constitutive materials being concrete which is poured around a thick steel wire frame. Reinforced concrete is primarily made up of the cheaper and lighter concrete but with the strength of steel. The study of composite materials is of interest in the mathematical sciences due to such desirable properties.

In the study of periodic composite materials, the theory of two-scale convergence is a useful tool. In 1989 Nguetseng [51] proposed the multi-scale extension of the notion of the weak L^2-limit. In Allaire [1] a theorem on the two-scale compactness of L^2-bounded sequences was proven which in turn led to the establishment of a corrector-type result for the uniformly elliptic periodic homogenisation problem. Not only do two-scale convergence methods have the ability to recover "classical" homogenisation results but also these methods have the ability to deal with homogenisation problems of composite media with more complicated geometries on the period cell. Indeed, in some homogenisation problems where there is no strong L^2-limit, e.g., problems with degeneracies, see Smyshlyaev [68], two-scale convergence techniques will capture the multi-scale limit structure by providing a suitable notion of strong convergence. As opposed to the uniformly elliptic case where the limit function only depends on the macroscopic variable and which is the solution of a single boundary-value problem, the multi-scale limit for degenerate homogenization problems satisfy a system of coupled equations for the macroscopic and microscopic parts of the limit solution. This is the case when this theory is applied to thin structures.

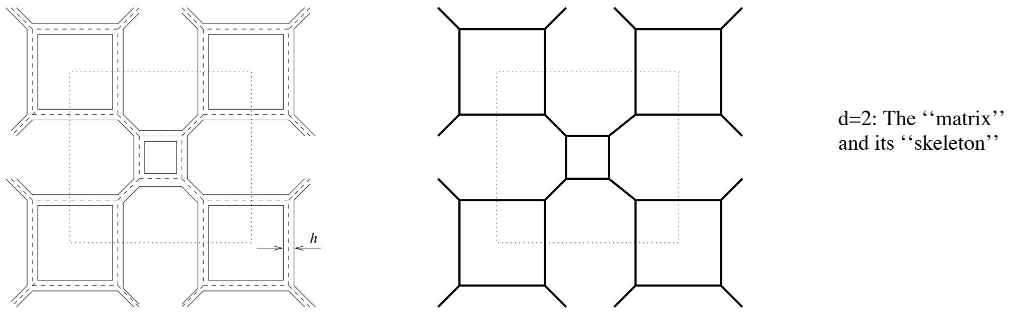

Figure 2.1: Example of a periodic network and unit cell.

A thin structure is defined as an arrangement of rods of thickness $a > 0$ which meet at junctions points ("nodes") as seen in Figure 2.1. This figure shows not only the thin structure on the left-hand side but also the limiting "singular" structure on the right which is obtained by taking the midlines of the rods. In the literature, the study of the equations of elasticity on rod frameworks has been considered when the rod thickness a is treated as a parameter depending on some typical rod length. In the context of periodic homogenisation problems, rod frameworks are arranged periodically with period ε and the limit behaviour of the structure is studied as $\varepsilon \to 0$ by studying the various two-scale limits of the sequences involved.

The use of two-scale techniques for the study of thin structures has been proposed by Zhikov [81, 82] and Bouchite & Fragala [14]. In [81], the work of Nguetseng [51] and Allaire [1] on two-scale analysis was extended to the setting of general Borel measures with the conditions on the measure necessary and sufficient for passing to the two-scale limit determined. Additionally, it was shown that the spectrum of the "double-porosity model" where the components of the composite have contrasting properties, is close to a band spectrum, the complement of which is an infinite set of disjoint intervals ("gaps"). This property is possessed by the homogenised operator derived in the work presented. In [82] it was established that if the thickness of the rods is $a = a(\varepsilon)$, and $a \to 0$ as $\varepsilon \to 0$, then the limit of the homogenisation problem depends on the assumptions made on the ratio a/ε^2. In particular, in the case when $a/\varepsilon^2 \to \theta > 0$, the sequence of symmetric gradients are, in general, not compact with respect to strong two-scale convergence. Consequently, the equation describing the limit energy balance is no longer obtained by setting the the test function equal to the solution of the homogenised equation for the corresponding singular structure found by considering the midlines of the rods with the measure induced by the thin structure. This problem was rectified in 2003 by Zhikov & Pastukhova [84] where the correct form of the missing part of the energy equality was determined and in turn the limiting system of equations which describe the homogenised problem was derived. A number of works follows the afore mentioned publication which included works on the analysis of Sobolev spaces for a variable measure [83, 84], work on Korn inequalities for periodic frames [86] and a study of the gaps in the spectrum of the elasticity operator on a high-contrast periodic structure [87] with non-vanishing volume fraction of the components as $\varepsilon \to 0$. In [87], viewed as an extension of the results of Zhikov [81], the band-gap nature of the spectrum of this operator is analysed and, moreover, it is shown that their is convergence of the spectra of the heterogeneous problems to the limit spectrum. It was first observed in Zhikov & Pastukhova [85] that the spectrum of the limit problem for thin structures in the critically scaled regime bears a remarkable similarity to the limit spectrum of the high-contrast, fixed-volume fraction case of the problem presented in [87]. Reasons for this similarity have been found in the more recent work of Cherednichenko & Kiselev [24] where operator-theoretic tools are used to show that the resolvents of both models are close in the operator-norm sense to a limit Konnig-Penney model of the so called "δ-type".

In the work presented in this chapter, the works of Zhikov & Pastukhova [84, 87] are extended for the study of an elasticity problem on a two-component periodic composite where the region occupied by the main material ("matrix") has the form of a framework with $a/\varepsilon^2 \to \theta > 0$, and the complementary part of the space, consisting of disjoint "inclusions", is filled with a less rigid material so that the ratio between the stiffness of the inclusions and the matrix is of order $O(\varepsilon^2)$. In other words, in addition to the assumption of high-contrast, it is assumed that the stiff component is a thin structure constructed such that its volume fraction is of order $O(\varepsilon)$. While the analysis presented uses elements of multi-scale approaches to thin structures as utiliised in [81, 84] and elements of high-contrast composite theory as seen in [81, 87], new tools are developed to prove the main results (the homogenisation theorem (Theorem 2.3.1) and spectral convergence (Theorem 2.4.1) which link the behaviour of the solutions of the original sequence of problems with rapidly oscillating coefficients on the stiff component and on the inclusions. Moreover, the limit functions for the restrictions of the solutions to each of the two components are coupled together (see Section 2.3.1) and lead to a new kind of homogenised system of equations.

With regards notation, all vectors will again be denoted in bold face text and will be

2-dimensional. Matrices will all be 2×2 and symmetric and therefore have 3 distinct elements which will be reflected in the notation of the function spaces they belong to. Moreover, $\langle \cdot \rangle$ will be used to denote the average over Q with respect to the composite measure μ and $\langle \cdot \rangle_\lambda$ will be used to denote the average over Q with respect to the singular measure λ.

2.1 Analytical Tools and Two-Scale Convergence

2.1.1 The Problem of Consideration

Consider a periodic rod framework (the "stiff" component of the composite) filled in with a different material ("soft" component). An example of such a structure can be seen in Figure 2.2. It will be assumed that the rod thickness $a = a(\varepsilon) > 0$ is a function of the period $\varepsilon > 0$ in the setting where $\lim_{\varepsilon\to 0} a/\varepsilon^2 = \theta > 0$. This particular scaling is referred to as "critical" in the literature. It is also assumed that the ratio of the elastic moduli of the soft and stiff components is of order $O(\varepsilon^2)$. Let F_1^h denote the domain occupied by the scaled rods of thickness $h = h(\varepsilon) := a/\varepsilon$ in the scaled structure of period 1 and let F_1 denote the corresponding thin structure obtained as $h \to 0$. The periodicity cell is denoted by $Q := [0,1)^2$ where $Q_1 := Q \cap F_1$ and $Q_0 := Q \backslash Q_1$. Define further the "contraction" $F_1^{h,\varepsilon} := \varepsilon F_1^h$ of the framework F_1^h. The soft component $\mathbb{R}^2 \backslash F_1^h$ and its contraction $\varepsilon(\mathbb{R}^2 \backslash F_1^h)$ are denoted F_0^h and $F_0^{h,\varepsilon}$ respectively. Denote by χ_1, χ_1^h, $\chi_1^{h,\varepsilon}$ and χ_0, χ_0^h, $\chi_0^{h,\varepsilon}$ the characteristic functions on the sets F_1, F_1^h, $F_1^{h,\varepsilon}$ and $(\mathbb{R}^2 \backslash F_1), F_0^h$, $F_0^{h,\varepsilon}$ respectively.

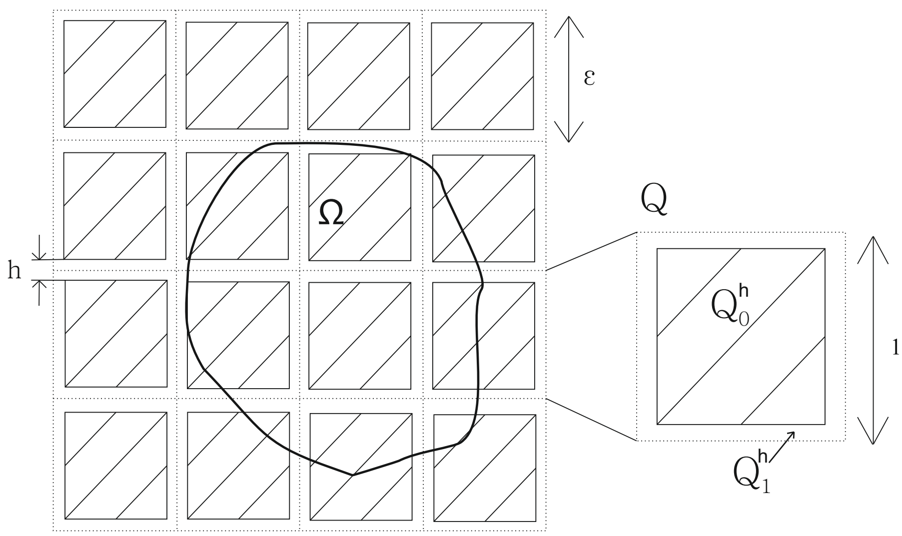

Figure 2.2: Periodic network with high-contrast. Note that $Q_1^h = Q \cap F_1^h$ and $Q_0^h = Q \cap F_0^h$.

Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and denote by $\Omega_1^{\varepsilon,h} := \Omega \cap F_1^{h,\varepsilon}$ the set of all stiff inclusions contained in Ω and denote by $\Omega_0^{\varepsilon,h} := \Omega \cap F_0^{h,\varepsilon}$ the set of all soft inclusions contained in Ω.

In what follows, the equations of two-dimensional elasticity in $\mathbb{R}^2$ are considered where these equations arise from the full three-dimensional system of linearised elasticity equations when there is a direction, say x_3, along which material properties are constant and when it is assumed that the displacement does not depend on x_3. At each point $\boldsymbol{x} \in \mathbb{R}^2$, the fourth-order tensor for the elastic moduli of the medium is given by

$$A^\varepsilon = \varepsilon^2 \chi_0^h(\cdot/\varepsilon) A_0 + \chi_1^h(\cdot/\varepsilon) A_1$$

where A_0 and A_1 are constant positive definite matrices[1]:

$$c_j \xi^2 \leq A_j \xi \cdot \xi \leq c_j^{-1} \xi^2, \qquad c_j > 0,\ j = 0, 1, \quad \forall \xi \in \mathrm{Sym}_2,$$

where Sym_2 is the set of all 2×2 symmetric matrices. Consider the measures λ, λ^h defined by

$$\lambda(B) = \frac{\mathcal{H}^1(B \cap F_1)}{\mathcal{H}^1(Q \cap F_1)}, \quad \lambda^h(B) = \frac{\mathcal{H}^2(B \cap F_1^h)}{\mathcal{H}^2(Q \cap F_1^h)}, \tag{2.1.1}$$

for all Borel sets $B \subset Q$, where $\mathcal{H}^1$, $\mathcal{H}^2$ are the one-dimensional and two-dimensional Hausdorff measures, see Evans & Gariepy [30]. These measures are extended to $\mathbb{R}^2$ by Q-periodicity. Moreover, define the composite measures $\mu := \frac{1}{2}\mathrm{d}\boldsymbol{x} + \frac{1}{2}\lambda$ and $\mu^h := \frac{1}{2}\mathrm{d}\boldsymbol{x} + \frac{1}{2}\lambda^h$ where $\mathrm{d}\boldsymbol{x}$ is the plane Lebesgue measure[2]. Finally, the "scaled" measures λ_ε^h and $\mu_\varepsilon^h := \frac{1}{2}\mathrm{d}\boldsymbol{x} + \frac{1}{2}\lambda_\varepsilon^h$ are introduced such that $\lambda_\varepsilon^h(B) = \varepsilon^2 \lambda^h(\varepsilon^{-1}B)$ for all Borel sets $B \subset \mathbb{R}^2$. Moreover, there are convergences $\lambda^h \rightharpoonup \lambda$ as $h \to 0$ (hence $\mu^h \rightharpoonup \mu$ as $h \to 0$) and $\mu_\varepsilon^h \rightharpoonup \mathrm{d}\boldsymbol{x}$ as $\varepsilon \to 0$ in the sense that

$$\lim_{h \to 0} \int_Q \boldsymbol{\varphi}\, \mathrm{d}\lambda^h = \int_Q \boldsymbol{\varphi}\, \mathrm{d}\lambda, \quad \forall \boldsymbol{\varphi} \in [C_{\mathrm{per}}^\infty(Q)]^2,$$

$$\lim_{\varepsilon \to 0} \int_\Omega \boldsymbol{\varphi}\, \mathrm{d}\mu_\varepsilon^h = \int_\Omega \boldsymbol{\varphi}\, \mathrm{d}\boldsymbol{x}, \quad \forall \boldsymbol{\varphi} \in [C_0^\infty(\Omega)]^2.$$

The first of these convergences can be shown by taking a Taylor's expansion of $\boldsymbol{\varphi}$ in a neighbourhood of F_1^h and acknowledging that higher-order terms are of order $O(h)$. The second result follows by a similar arguement to that presented in Appendix 2.B, taking a Bloch transform of $\boldsymbol{\varphi}$ and then using the same argument as for the first convergence above.

[1]The scalar product of two symmetric matrices $\xi = \{\xi_{ij}\}$ and $\eta = \{\eta_{ij}\}$ is defined by $\xi \cdot \eta = \xi_{ij}\eta_{ij}$. In particular, $\xi^2 = \xi \cdot \xi$. The product of the fourth-order elasticity tensor A with a symmetric matrix ξ is defined as $A\xi = a_{ijkl}\xi_{kl}$ and thus $A\xi \cdot \xi = a_{ijkl}\xi_{ij}\xi_{kl}$.

[2]Here, $\mathrm{d}\boldsymbol{x}$ is used to denote the plane Lebesgue measure as well as the element of integration.

The space $[H_0^1(\Omega)]^2$ is understood to be the closure of the space $[C_0^\infty(\Omega)]^2$ with respect to the norm $\left(\int_\Omega \left(|\boldsymbol{\varphi}|^2 + e(\boldsymbol{\varphi})^2\right) \mathrm{d}\mu_\varepsilon^h\right)^{1/2}$ where the matrix $e(\boldsymbol{\varphi}) := \frac{1}{2}\left\{\nabla\boldsymbol{\varphi} + (\nabla\boldsymbol{\varphi})^T\right\}$ is referred to as the symmetric gradient.

For $\varepsilon, h > 0$ and $\boldsymbol{f}^{h,\varepsilon} \in [L^2(\Omega, \mathrm{d}\mu_\varepsilon^h)]^2$, the problem at hand is to find a vector-valued function $\boldsymbol{u}_\varepsilon^h \in [H_0^1(\Omega)]^2$ satisfying the identity

$$\int_{\Omega_1^{\varepsilon,h}} A_1 e(\boldsymbol{u}_\varepsilon^h) \cdot e(\boldsymbol{\varphi}) \,\mathrm{d}\mu_\varepsilon^h + \varepsilon^2 \int_{\Omega_0^{\varepsilon,h}} A_0 e(\boldsymbol{u}_\varepsilon^h) \cdot e(\boldsymbol{\varphi}) \,\mathrm{d}\mu_\varepsilon^h + \\ + \int_\Omega \boldsymbol{u}_\varepsilon^h \cdot \boldsymbol{\varphi} \,\mathrm{d}\mu_\varepsilon^h = \int_\Omega \boldsymbol{f}^{h,\varepsilon} \cdot \boldsymbol{\varphi} \,\mathrm{d}\mu_\varepsilon^h, \quad \forall \boldsymbol{\varphi} \in [H_0^1(\Omega)]^2. \quad (2.1.2)$$

Theorem 2.1.1. *Define a bilinear form $B_\varepsilon^h(\cdot,\cdot)$ and a linear form $L_\varepsilon^h(\cdot)$ by the following relations:*

$$B_\varepsilon^h(\boldsymbol{u}, \boldsymbol{v}) := \int_{\Omega_1^{\varepsilon,h}} A_1 e(\boldsymbol{u}) \cdot e(\boldsymbol{v}) \,\mathrm{d}\mu_\varepsilon^h + \varepsilon^2 \int_{\Omega_0^{\varepsilon,h}} A_0 e(\boldsymbol{u}) \cdot e(\boldsymbol{v}) \,\mathrm{d}\mu_\varepsilon^h + \int_\Omega \boldsymbol{u} \cdot \boldsymbol{v} \,\mathrm{d}\mu_\varepsilon^h,$$

$$L_\varepsilon^h(\boldsymbol{v}) = \int_\Omega \boldsymbol{f}^{h,\varepsilon} \cdot \boldsymbol{v} \,\mathrm{d}\mu_\varepsilon^h.$$

The bilinear form $B_\varepsilon^h(\cdot,\cdot)$ is both coercive and continuous and the linear form $L_\varepsilon^h(\cdot)$ is continuous on $[H_0^1(\Omega)]^2$. Hence, by the Lax-Milgram Lemma (see Evans [29, Chapter 6]), a unique solution $\boldsymbol{u}_\varepsilon^h \in [H_0^1(\Omega)]^2$ exists to problem (2.1.2).

The proof of this result is a routine use of ellipticity estimates and the Cauchy-Schwarz inequality as seen in Chapter 1, Section 1.1.1.

In the work which follows, a description of the structure of the limiting problem for the weak two-scale limit of the function $\boldsymbol{u}_\varepsilon^h$ as $\varepsilon \to 0$ will be derived. In general, the structure of the homogenised problem depends on the way in which the ratio h/ε goes to zero as $\varepsilon \to 0$. However, from the general theory of homogenisation on periodic rod structures (see Zhikov [82], Zhikov & Pastukhova [84]), when A_0 is formally replaced by zero in (2.1.2) and Theorem 2.1.1 the following results hold regardless of the limit of the ratio $h(\varepsilon)/\varepsilon$:

1. There exists a vector function $\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{y}) \in [L^2(\Omega, L^2(Q, \mathrm{d}\lambda))]^2$ such that for all $\boldsymbol{\varphi} \in [L^2(\Omega, L^2(Q, \mathrm{d}\mu))]^2$:
   (a) $\displaystyle\frac{1}{|\Omega_1^{h,\varepsilon}|}\int_{\Omega_1^{h,\varepsilon}} \boldsymbol{u}_\varepsilon^h(\boldsymbol{x}) \cdot \boldsymbol{\varphi}(\boldsymbol{x}, \boldsymbol{x}/\varepsilon) \,\mathrm{d}\boldsymbol{x} \to \int_\Omega\int_Q \boldsymbol{u}(\boldsymbol{x}, \boldsymbol{y}) \cdot \boldsymbol{\varphi}(\boldsymbol{x}, \boldsymbol{y}) \,\mathrm{d}\lambda(\boldsymbol{y})\mathrm{d}\boldsymbol{x}, \quad (\varepsilon \to 0),$
   (b) $\displaystyle\frac{1}{|\Omega_1^{h,\varepsilon}|}\int_{\Omega_1^{h,\varepsilon}} |\boldsymbol{u}_\varepsilon^h(\boldsymbol{x})|^2 \,\mathrm{d}\boldsymbol{x} \to \int_\Omega\int_Q |\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{y})|^2 \,\mathrm{d}\lambda(\boldsymbol{y})\mathrm{d}\boldsymbol{x}, \quad (\varepsilon \to 0).$
2. The vector $\boldsymbol{u}(\boldsymbol{x}, \cdot)$ is a "periodic rigid displacement" (see Definition 2.1.4):
   $$\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{u}_0(\boldsymbol{x}) + \boldsymbol{\chi}(\boldsymbol{x}, \boldsymbol{y}), \quad \text{a.e. } \boldsymbol{x} \in \Omega, \quad \lambda\text{-a.e. } \boldsymbol{y} \in Q \cap F_1,$$
   where $\boldsymbol{u}_0 \in [H_0^1(\Omega)]^2$ and $\boldsymbol{\chi}(\boldsymbol{x}, \cdot)$ is a periodic transverse displacement (see equation 2.1.13).

3. The "macroscopic" equation

$$-\mathrm{div}(A_\lambda^{\mathrm{hom}} e(\boldsymbol{u}_0)) + \langle u\rangle_\lambda = \langle \boldsymbol{f}\rangle_\lambda,$$

is satisfied where A_λ^{hom} is the "λ-homogenised tensor" (see (2.2.9)) and where $\boldsymbol{f}$ is the weak two-scale limit of the sequence $\boldsymbol{f}^{h,\varepsilon}$ (see Section 2.1.2)

The main result of this chapter, Theorem 2.3.1, illustrates that in the case of a critical scaling between the rod thickness a and the period ε, as $\varepsilon \to 0$, the solutions $\boldsymbol{u}_\varepsilon^h$ of problem (2.1.2) where $h = a/\varepsilon$ converge in the appropriate sense of two-scale convergence (see Section 2.1.2) to a function $\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{y})$, $\boldsymbol{x} \in \Omega$, $\boldsymbol{y} \in Q$. Moreover, it will be shown that the trace of $\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{y})$ on $Q \cap F_1$ is a periodic rigid displacement and, in addition, the function $\boldsymbol{u}(\boldsymbol{x}, \cdot) - \boldsymbol{u}_0(\boldsymbol{x}) =: \boldsymbol{U}(\boldsymbol{x}, \cdot)$, $\boldsymbol{x} \in \Omega$ belongs to the space $[H^1_{\mathrm{per}}(Q)]^2$ for a.e. $\boldsymbol{x} \in \Omega$ and is the solution to an elliptic equation which couples the equation for $\boldsymbol{u}_0$.

Let $\boldsymbol{\tau}$, $\boldsymbol{\nu}$ be unit tangent and unit normal vectors respectively to any link I of the thin network F_1 such that they form a positively orientated system. Any vector $\boldsymbol{v} \in \mathbb{R}^2$ can be written in the form $\boldsymbol{v} = v^{(\tau)}\boldsymbol{\tau} + v^{(\nu)}\boldsymbol{\nu}$ where $v^{(\tau)} = (\boldsymbol{v} \cdot \boldsymbol{\tau})$ and $v^{(\nu)} = (\boldsymbol{v} \cdot \boldsymbol{\nu})$. Then, where the trace of $\boldsymbol{u}$ is denoted $\boldsymbol{\chi}$, the vectors $\boldsymbol{U}$ and $\boldsymbol{\chi}$ are shown to satisfy differential equations of the form

$$\mathcal{A}_0\boldsymbol{U} + \boldsymbol{u} = \boldsymbol{f}, \qquad \mathcal{L}_\tau\chi^{(\nu)} + \mathcal{T}_\nu U^{(\nu)} + u^{(\nu)} = f^{(\nu)}, \tag{2.1.3}$$

where $\mathcal{A}_0$ is a second-order differential operator in Q expressed in terms of the tensor A_0 only, $\mathcal{L}_\tau$ is a fourth-order differential operator "acting" in the tangential direction $\boldsymbol{\tau}$ and $\mathcal{T}_\nu$ is a first-order differential operator "acting" in the transversal direction $\boldsymbol{\nu}$ corresponding to each link I. In particular, the system of equations is obtained when the tensor A_0 is isotropic is demonstrated in Section 2.3.1. By isotropic tensor, it is understood that for a symmetric matrix ξ,

$$A_0\xi = 2M_0\xi + L_0(\mathrm{tr}\ \xi)I, \tag{2.1.4}$$

where, $M_0, L_0 > 0$ are the Lamé constants. Note further that the Lamé constants are denoted this way rather than with λ and μ to save confusion with the measures used in this chapter.

2.1.2 Overview of Two-Scale Convergence

In problems involving interacting length scales, the theory of two-scale convergence is a natural tool for the analysis of such problems since these techniques can capture the "non-classical" limiting behaviour of the weak two-scale limit as a function of two variables. The results presented below are those necessary to carry out the analysis of equation (2.1.2).

It will be assumed throughout that the sequence $\boldsymbol{u}_\varepsilon^h$ is bounded in $[L^2(\Omega, \mathrm{d}\mu_\varepsilon^h)]^2$:

$$\limsup_{\varepsilon \to 0} \int_\Omega |\boldsymbol{u}_\varepsilon^h|^2 \, \mathrm{d}\mu_\varepsilon^h < \infty. \tag{2.1.5}$$

Before introducing the definition of weak two-scale convergence, the definitions of weak and strong convergence are recalled along with the so called mean value property of weak convergence.

Definition 2.1.1 (Weak & Strong Convergence)**.** Let $\boldsymbol{u}^h$ be a bounded sequence in $[L^2(Q, \mathrm{d}\mu^h)]^2$. The sequence $\boldsymbol{u}^h$ converges:

1. weakly to $\boldsymbol{u} \in [L^2(Q, \mathrm{d}\mu)]^2$, denoted $\boldsymbol{u}^h \rightharpoonup \boldsymbol{u}$, if

$$\lim_{h \to 0} \int_Q \boldsymbol{u}^h \cdot \boldsymbol{\varphi} \, \mathrm{d}\mu^h = \int_Q \boldsymbol{u} \cdot \boldsymbol{\varphi} \, \mathrm{d}\mu, \quad \text{for each } \boldsymbol{\varphi} \in [C^\infty_{\mathrm{per}}(Q)]^2. \tag{2.1.6}$$

2. strongly to $\boldsymbol{u} \in [L^2(Q, \mathrm{d}\mu)]^2$, denoted $\boldsymbol{u}^h \to \boldsymbol{u}$, if

$$\lim_{h \to 0} \int_Q \boldsymbol{u}^h \cdot \boldsymbol{v}^h \, \mathrm{d}\mu^h = \int_Q \boldsymbol{u} \cdot \boldsymbol{v} \, \mathrm{d}\mu. \tag{2.1.7}$$

   for any sequence $\boldsymbol{v}^h \in [L^2(Q, \mathrm{d}\mu^h)]^2$ converging weakly to $\boldsymbol{v} \in [L^2(Q, \mathrm{d}\mu)]^2$.

Lemma 2.1.1 (Mean Value Property)**.** *Let Ω be a Jordan measurable set and let $\boldsymbol{u}^h \rightharpoonup \boldsymbol{u}$ in $[L^2(Q, \mathrm{d}\mu^h)]^2$. Then*

$$\lim_{\varepsilon \to 0} \int_\Omega \boldsymbol{\varphi}(\boldsymbol{x}) \cdot \boldsymbol{u}^h(\boldsymbol{x}/\varepsilon) \, \mathrm{d}\mu_\varepsilon^h = \int_\Omega \int_Q \boldsymbol{\varphi}(\boldsymbol{x}) \cdot \boldsymbol{u}(\boldsymbol{y}) \, \mathrm{d}\mu \mathrm{d}\boldsymbol{x}, \quad \forall \boldsymbol{\varphi} \in [C(\overline{\Omega})]^2.$$

Definition 2.1.2 (Weak Two-Scale Convergence)**.** Let $\boldsymbol{u}_\varepsilon^h \in [L^2(\Omega, \mathrm{d}\mu_\varepsilon^h)]^2$ be a bounded sequence. The function $\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{y}) \in [L^2(\Omega \times Q, \mathrm{d}\boldsymbol{x} \times \mathrm{d}\mu)]^2$ is said to be the weak two-scale limit of $\boldsymbol{u}_\varepsilon^h$, denoted $\boldsymbol{u}_\varepsilon^h \overset{2}{\rightharpoonup} \boldsymbol{u}$, if

$$\lim_{\varepsilon \to 0} \int_\Omega \boldsymbol{\Phi}(\boldsymbol{x}, \boldsymbol{x}/\varepsilon) \cdot \boldsymbol{u}_\varepsilon^h(\boldsymbol{x}) \, \mathrm{d}\mu_\varepsilon^h = \int_\Omega \int_Q \boldsymbol{\Phi}(\boldsymbol{x}, \boldsymbol{y}) \cdot \boldsymbol{u}(\boldsymbol{x}, \boldsymbol{y}) \, \mathrm{d}\mu \mathrm{d}\boldsymbol{x}, \quad \forall \boldsymbol{\Phi} \in [L^2(\Omega, C_{\mathrm{per}}(Q))]^2. \tag{2.1.8}$$

The following proposition on two-scale compactness forms an essential part of the theory of two-scale convergence. With this result, it is possible (up to possibly taking a subsequence) to pass to the limit of a bounded sequence.

Proposition 2.1.1. *If a sequence $\boldsymbol{u}_\varepsilon^h$ is bounded in $[L^2(\Omega, \mathrm{d}\mu_\varepsilon^h)]^2$, then it is relatively compact with respect to weak two-scale convergence.*

The proof of this result can be found in, for example Allaire [1] but the proof presented here is a version of the proof found in Zhikov [81].

Proof. Define Γ to be a set of test functions which is a dense countable subset of functions of the form

$$\boldsymbol{\Phi}(\boldsymbol{x},\boldsymbol{y}) = \varphi(\boldsymbol{x})\boldsymbol{b}(\boldsymbol{y}), \qquad \varphi \in C_0^\infty(\Omega), \quad \boldsymbol{b} \in [C_{\mathrm{per}}^\infty(Q)]^2.$$

Then a subsequence $\varepsilon_k \to 0$ is extracted such that

$$\lim_{\varepsilon_k\to 0}\int_\Omega \boldsymbol{\Phi}(\boldsymbol{x},\boldsymbol{x}/\varepsilon)\cdot \boldsymbol{u}_\varepsilon^h(\boldsymbol{x})\ \mathrm{d}\mu_\varepsilon^h =: L_\varepsilon(\boldsymbol{\Phi}), \tag{2.1.9}$$

exists for any $\boldsymbol{\Phi} \in \Gamma$. This limit is defined and by the Cauchy-Schwarz inequality,

$$\Big(\int_\Omega \boldsymbol{\Phi}(\boldsymbol{x},\boldsymbol{x}/\varepsilon)\cdot \boldsymbol{u}_\varepsilon^h(\boldsymbol{x})\ \mathrm{d}\mu_\varepsilon^h\Big)^2 \le \int_\Omega |\boldsymbol{u}_\varepsilon^h|^2\ \mathrm{d}\mu_\varepsilon^h \int_\Omega |\boldsymbol{\Phi}|^2\ \mathrm{d}\mu_\varepsilon^h \le C\int_\Omega |\boldsymbol{\Phi}|^2\ \mathrm{d}\mu_\varepsilon^h.$$

Since L_ε is bounded, by the Banach-Alaoglu Theorem (see Rudin [61]), it possesses a weak-$*$ convergent subsequence and limit L. Moreover, making use of the mean value property

$$|L(\boldsymbol{\Phi})|^2 \le C\int_\Omega\int_Q |\boldsymbol{\Phi}|^2\ \mathrm{d}\mu\mathrm{d}\boldsymbol{x}.$$

By the Riesz Representation Theorem (see Brezis [17]), the linear functional $L(\boldsymbol{\Phi})$ has a representation of the form

$$L(\boldsymbol{\Phi}) = \int_\Omega\int_Q \boldsymbol{\Phi}(\boldsymbol{x},\boldsymbol{y})\cdot\boldsymbol{u}(\boldsymbol{x},\boldsymbol{y})\ \mathrm{d}\mu\mathrm{d}\boldsymbol{x}, \quad [\boldsymbol{u}\in L^2(\Omega\times Q, \mathrm{d}\boldsymbol{x}\times\mathrm{d}\mu)]^2.$$

The linear span of test functions is a dense set in $[L^2(\Omega\times Q,\mathrm{d}\boldsymbol{x}\times\mathrm{d}\mu)]^2$. Hence, the function $\boldsymbol{u}$ is uniquely defined. $\square$

The following is a statement on lower semi-continuity.

Proposition 2.1.2. *Let $\boldsymbol{u}_\varepsilon^h$ be a bounded sequence in $[L^2(\Omega,\mathrm{d}\mu_\varepsilon^h)]^2$ and let $\boldsymbol{u}_\varepsilon^h \overset{2}{\rightharpoonup} \boldsymbol{u}$ where $\boldsymbol{u}\in[L^2(\Omega\times Q,\mathrm{d}\boldsymbol{x}\times\mathrm{d}\mu)]^2$. Then*

$$\liminf_{\varepsilon\to 0}\int_\Omega |\boldsymbol{u}_\varepsilon^h|^2\ \mathrm{d}\mu_\varepsilon^h \ge \int_\Omega\int_Q |\boldsymbol{u}|^2\,\mathrm{d}\mu\mathrm{d}\boldsymbol{x}. \tag{2.1.10}$$

Analogous to the notion of strong convergence, there is a notion of strong two-scale convergence.

Definition 2.1.3 (Strong Two-Scale Convergence)**.** Let $\boldsymbol{u}_\varepsilon^h$ be a bounded sequence in $[L^2(\Omega,\mathrm{d}\mu_\varepsilon^h)]^2$ and let $\boldsymbol{v}_\varepsilon^h$ be such that $\boldsymbol{v}_\varepsilon^h \overset{2}{\rightharpoonup} \boldsymbol{v}$. The function $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x},\boldsymbol{y}) \in [L^2(\Omega\times Q,\mathrm{d}\boldsymbol{x}\times\mathrm{d}\mu)]^2$ is called the strong two-scale limit of $\boldsymbol{u}_\varepsilon^h$, denoted $\boldsymbol{u}_\varepsilon^h \overset{2}{\to} \boldsymbol{u}$, if

$$\lim_{\varepsilon\to 0}\int_\Omega \boldsymbol{u}_\varepsilon^h(\boldsymbol{x})\cdot\boldsymbol{v}_\varepsilon^h(\boldsymbol{x})\ \mathrm{d}\mu_\varepsilon^h = \int_\Omega\int_Q \boldsymbol{u}(\boldsymbol{x},\boldsymbol{y})\cdot\boldsymbol{v}(\boldsymbol{x},\boldsymbol{y})\ \mathrm{d}\mu(\boldsymbol{y})\mathrm{d}\boldsymbol{x}. \tag{2.1.11}$$

Strong convergence implies weak convergence (simply take $\boldsymbol{v}_\varepsilon^h(x) = \boldsymbol{\Phi}(\boldsymbol{x},\boldsymbol{x}/\varepsilon)$ in the above definition) and moreover setting $\boldsymbol{v}_\varepsilon^h = \boldsymbol{u}_\varepsilon^h$ yields

$$\lim_{\varepsilon\to 0}\int_\Omega |\boldsymbol{u}_\varepsilon^h|^2\ \mathrm{d}\mu_\varepsilon^h = \int_\Omega\int_Q |\boldsymbol{u}|^2\ \mathrm{d}\mu\mathrm{d}\boldsymbol{x}. \tag{2.1.12}$$

The next proposition, also proven in Zhikov & Pastukhova [84], shows that the converse also holds.

Proposition 2.1.3. *Let $\boldsymbol{u}_\varepsilon^h$ be a bounded sequence in $[L^2(\Omega, \mathrm{d}\mu_\varepsilon^h)]^2$ and $\boldsymbol{u} \in [L^2(\Omega \times Q, \mathrm{d}\boldsymbol{x} \times \mathrm{d}\mu)]^2$. If $\boldsymbol{u}_\varepsilon^h \overset{2}{\rightharpoonup} \boldsymbol{u}$ and convergence (2.1.12) holds, then $\boldsymbol{u}_\varepsilon^h \overset{2}{\to} \boldsymbol{u}$.*

In the following section, further tools commonly associated with the analysis of thin structures will be acknowledged.

2.1.3 Some Auxiliary Tools for Thin Structures

Of particular importance to the study of thin structures, the concepts of periodic rigid displacement and transverse displacement will now be introduced.

Definition 2.1.4 (Periodic Rigid Displacements)**.** A vector function $\boldsymbol{u} \in [L^2(Q, \mathrm{d}\lambda)]^2$ is called a periodic rigid displacement (with respect to the measure λ) if there exists a smooth sequence $\{\boldsymbol{u}_n\}_{n\in\mathbb{N}} \in [C^\infty_{\text{per}}(Q)]^2$ such that

$$\boldsymbol{u}_n \to \boldsymbol{u}, \quad \text{in } L^2(Q, \mathrm{d}\lambda)^2, \quad e(\boldsymbol{u}_n) \to 0, \quad \text{in } [L^2(Q, \mathrm{d}\lambda)]^3.$$

The set of all periodic rigid displacements is denoted $\mathcal{R}$ and is equipped with the $[L^2(Q, \mathrm{d}\lambda)]^2$-norm.

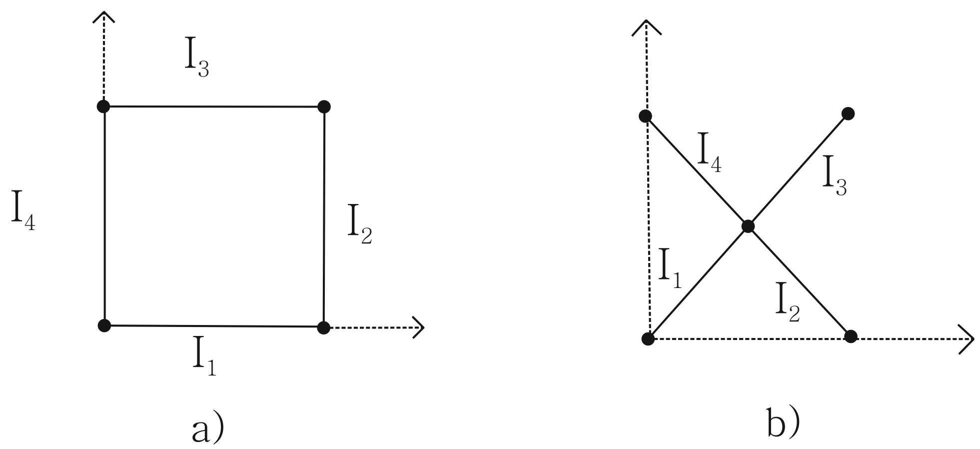

Figure 2.3: a) Square framework; b) Diagonal framework

Example. Examples of periodic rigid displacements will be demonstrated.

Consider firstly a square framework (see Figure 2.3a)) with links aligned with the axes and bottom left node at the origin. Solving the equation $e(\boldsymbol{u}) = 0$, is not simple since the link in question plays a key role. For example, on I_1,

$$\boldsymbol{u}(\boldsymbol{y}) = \begin{pmatrix} y_2\alpha'(y_1) \\ -\alpha(y_1) \end{pmatrix},$$

with $\alpha \in H^2_{\text{per}}(I_1)$ and $\alpha(0) = 0$ satisfies the equation $e(\boldsymbol{u}) = 0$ as $y_2 = 0$ on I_1. Vectors of a similar form can be found which are periodic rigid displacements on each of the links.

For the diagonal framework (see Figure 2.3b)) with incline from the y_1-axis of $\pi/4$, in a similar fashion it can be shown that on the link I_1 that

$$\boldsymbol{u}(\boldsymbol{y}) = \begin{pmatrix} (\frac{1}{2}y_1^2 - y_2)\alpha(y_2) \\ (y_1 - \frac{1}{2}y_2^2)\alpha(y_1) \end{pmatrix}$$

with $\alpha \in H^1_{\text{per}}(I_1)$ is a periodic rigid displacement. Similar expressions can be found on the remaining links.

It is shown, see e.g. Zhikov [82], that any vector $\boldsymbol{u} \in \mathcal{R}$ has a unique representation of the form

$$\boldsymbol{u}(\boldsymbol{y}) = \boldsymbol{c} + \boldsymbol{\chi}(\boldsymbol{y}), \quad \boldsymbol{y} \in Q, \tag{2.1.13}$$

where $\boldsymbol{c} \in \mathbb{R}^2$ and $\boldsymbol{\chi}$ is a periodic transverse displacement, i.e., on each link of the singular network F_1, it is orthogonal to the link. Denoting the set of all transverse displacements by $\widehat{\mathcal{R}}$ it follows that $\mathcal{R} = \mathbb{R}^2 \oplus \widehat{\mathcal{R}}$. The following definition characterises the class of transverse displacements which appear in the study of rod networks which are critically scaled.

Definition 2.1.5. Let $\boldsymbol{\nu}$, $\boldsymbol{\tau}$ be normal and tangent to each link I of the periodic network F_1 such that they form a positively orientated frame and denote by $I_1, \dots, I_n$ the links which share an arbitrary node $\mathcal{O}$. Then the set $\widehat{\mathcal{R}}^0 \subset \widehat{\mathcal{R}}$ denotes the set of all transverse displacements $\boldsymbol{\chi}$ satisfying the following conditions:

1. The function $\chi^{(\nu_j)}|_{I_j}$, $j = 1, 2, \dots, n$, is square integrable and has square integrable first and second derivatives on I_j, denoted $\chi^{(\nu_j)} \in H^2(I_j)$.

2. Equality of the first derivatives at each node[3]:

$$(\chi^{(\nu_1)})'|_{\mathcal{O}} = (\chi^{(\nu_2)})'|_{\mathcal{O}} = \cdots = (\chi^{(\nu_n)})'|_{\mathcal{O}},$$

3. Fastening at each node: $\boldsymbol{\chi}|_{\mathcal{O}} = \boldsymbol{0}$.

The norm in $\widehat{\mathcal{R}}^0$ is defined as the sum of the H^2-norms of $\chi^{(\nu)}$ over all the links.

Definition 2.1.6 (Potential & Solenoidal Matrices)**.** Let κ be a Borel measure on Q. The space of κ-potential matrices, denoted V^κ_{pot}, is defined to be the closure of the set $\{\, e(\boldsymbol{u}) \mid \boldsymbol{u} \in C^\infty_{\text{per}}(Q)^2 \,\}$ in $[L^2(Q, \mathrm{d}\kappa)]^3$. A symmetric matrix $v \in [L^2(Q, \mathrm{d}\kappa)]^3$ is said to be κ-solenoidal if

$$\int_Q \big(v \cdot e(\boldsymbol{u})\big)\, \mathrm{d}\kappa = 0, \quad \text{for all } \boldsymbol{u} \in [C^\infty_{\text{per}}(Q)]^2.$$

The set of κ-solenoidal matrices is denoted V^κ_{sol} and moreover $[L^2(Q, \mathrm{d}\kappa)]^3 = V^\kappa_{\text{pot}} \oplus V^\kappa_{\text{sol}}$.

[3] In what follows, $(\chi^{(\nu)})'$, denotes the derivative in the tangential direction: $(\chi^{(\nu)})' := \frac{\mathrm{d}\chi^{(\nu)}}{\mathrm{d}\boldsymbol{\tau}} = (\boldsymbol{\tau}\cdot\nabla)\chi^{(\nu)}$.

Denote by $L^2(\Omega, V_{\text{pot}}^\kappa)$ the closure in $[L^2(\Omega\times Q, \mathrm{d}\boldsymbol{x}\times\mathrm{d}\kappa)]^3$ of the linear span of matrices $we(\boldsymbol{u})$, $w \in C_0^\infty(\Omega)$, $\boldsymbol{u} \in [C_{\text{per}}^\infty(Q)]^2$ and denote by $L^2(\Omega, V_{\text{sol}}^\kappa)$ the closure in $[L^2(\Omega \times Q, \mathrm{d}\boldsymbol{x} \times \mathrm{d}\kappa)]^3$ of the linear span of matrices wv, $w \in C_0^\infty(\Omega)$, $v \in V_{\text{sol}}^\kappa$. Then, it is shown (see Zhikov [82]) that $[L^2(\Omega \times Q, \mathrm{d}\boldsymbol{x} \times \mathrm{d}\kappa)]^3 = L^2(\Omega, V_{\text{pot}}^\kappa) \oplus L^2(\Omega, V_{\text{sol}}^\kappa)$. When κ is the Lebesgue measure on Q these spaces are denoted V_{pot} and V_{sol}.

Definition 2.1.7 (Natural Extension)**.** Let $b \in L^2(Q, \mathrm{d}\lambda)$ be periodic and let $I_1, \dots, I_n$ be the links of the singular network F_1 which meet at a single node $\mathcal{O}$. Denote the rod of thickness h with midline I_j by I_j^h, $j = 1, 2, \dots, n$. On each link I_j, define $b_j^h(\boldsymbol{y}) = b(\boldsymbol{y}^*)$ whenever $\boldsymbol{y}$ is within a h-neighbourhood of I_j and $|\boldsymbol{y} - \boldsymbol{y}^*| = \operatorname{dist}(\boldsymbol{y}, I_j)$, $\boldsymbol{y}^* \in I_j$ and $b_j^h(\boldsymbol{y}) = 0$ otherwise. The sum over all rods meeting at the node $\mathcal{O}$ is called the natural extension of b and is denoted $[b]^h$.

2.2 Two-Scale Structure of Solution Sequences

In this section, the structure of the various two-scale limits on the stiff and soft components will be established. To achieve this, the limit as $\varepsilon \to 0$ will be passed to in the integrals seen in (2.1.2) with suitably chosen test functions $\boldsymbol{\varphi}$.

2.2.1 Two-Scale Compactness of solutions to (2.1.2)

Consider equation (2.1.2) with $\boldsymbol{\varphi} = \boldsymbol{u}_\varepsilon^h$:

$$\int_{\Omega_1^{\varepsilon,h}} A_1(\cdot/\varepsilon)e(\boldsymbol{u}_\varepsilon^h) \cdot e(\boldsymbol{u}_\varepsilon^h)\,\mathrm{d}\mu_\varepsilon^h + \varepsilon^2 \int_{\Omega_0^{\varepsilon,h}} A_0(\cdot/\varepsilon)e(\boldsymbol{u}_\varepsilon^h) \cdot e(\boldsymbol{u}_\varepsilon^h)\,\mathrm{d}\mu_\varepsilon^h + \\ + \int_\Omega |\boldsymbol{u}_\varepsilon^h|^2\,\mathrm{d}\mu_\varepsilon^h = \int_\Omega \boldsymbol{f}^{h,\varepsilon} \cdot \boldsymbol{u}_\varepsilon^h\,\mathrm{d}\mu_\varepsilon^h.$$

Using ellipticity estimates on the left-hand side and the Cauchy-Schwarz inequality on the right-hand side followed by the inequality

$$\sqrt{ab} \leq \frac{1}{2\alpha}a + \frac{\alpha}{2}b, \qquad \forall \alpha \in \mathbb{R}_+, \tag{2.2.1}$$

it follows that

$$c_1\|e(\boldsymbol{u}_\varepsilon^h)\|^2_{L^2(\Omega_1^{\varepsilon,h},\mathrm{d}\mu_\varepsilon^h)} + c_0\varepsilon^2\|e(\boldsymbol{u}_\varepsilon^h)\|^2_{L^2(\Omega_0^{\varepsilon,h},\mathrm{d}\mu_\varepsilon^h)} + \frac{1}{2}\|\boldsymbol{u}_\varepsilon^h\|^2_{L^2(\Omega,\mathrm{d}\mu_\varepsilon^h)} \leq \frac{1}{2}\|\boldsymbol{f}^{h,\varepsilon}\|^2_{L^2(\Omega,\mathrm{d}\mu_\varepsilon^h)}.$$

Hence the following a priori bounds hold:

Proposition 2.2.1. *Let $\boldsymbol{u}_\varepsilon^h$ be a sequence in $[L^2(\Omega, \mathrm{d}\mu_\varepsilon^h)]^2$ of solutions of equation (2.1.2). Then there exists $C > 0$ such that the following bounds hold:*

$$\|\boldsymbol{u}_\varepsilon^h\|_{L^2(\Omega,\mathrm{d}\mu_\varepsilon^h)} \leq C, \quad \|e(\boldsymbol{u}_\varepsilon^h)\|_{L^2(\Omega_1^{\varepsilon,h},\mathrm{d}\mu_\varepsilon^h)} \leq C, \quad \varepsilon\|e(\boldsymbol{u}_\varepsilon^h)\|_{L^2(\Omega_0^{\varepsilon,h},\mathrm{d}\mu_\varepsilon^h)} \leq C.$$

By Proposition 2.1.1, the following weak two-scale convergences hold up to the consideration of a suitably chosen subsequence:

$$\boldsymbol{u}_\varepsilon^h \xrightharpoonup{2} \boldsymbol{u}(\boldsymbol{x},\boldsymbol{y}) \in L^2(\Omega\times Q, \mathrm{d}\boldsymbol{x}\times\mathrm{d}\mu)^2, \quad \text{in} \quad [L^2(\Omega,\mathrm{d}\mu_\varepsilon^h)]^2, \tag{2.2.2}$$

$$\chi_1^{h,\varepsilon}\boldsymbol{u}_\varepsilon^h \xrightharpoonup{2} \widehat{\boldsymbol{u}}(\boldsymbol{x},\boldsymbol{y}) \in L^2(\Omega\times Q, \mathrm{d}\boldsymbol{x}\times\mathrm{d}\lambda)^2, \quad \text{in} \quad [L^2(\Omega_1^{\varepsilon,h},\mathrm{d}\lambda_\varepsilon^h)]^2, \tag{2.2.3}$$

$$\chi_1^{h,\varepsilon}e(\boldsymbol{u}_\varepsilon^h) \xrightharpoonup{2} p(\boldsymbol{x},\boldsymbol{y}) \in L^2(\Omega\times Q, \mathrm{d}\boldsymbol{x}\times\mathrm{d}\lambda)^3, \quad \text{in} \quad [L^2(\Omega_1^{\varepsilon,h},\mathrm{d}\lambda_\varepsilon^h)]^3, \tag{2.2.4}$$

$$\varepsilon\chi_0^{h,\varepsilon}e(\boldsymbol{u}_\varepsilon^h) \xrightharpoonup{2} \tilde{p}(\boldsymbol{x},\boldsymbol{y}) \in L^2(\Omega\times Q, \mathrm{d}\boldsymbol{x}\times\mathrm{d}\boldsymbol{y})^3, \quad \text{in} \quad [L^2(\Omega_0^{\varepsilon,h},\mathrm{d}\mu_\varepsilon^h)]^3. \tag{2.2.5}$$

Note that $[L^2(\Omega\times Q,\mathrm{d}\boldsymbol{x}\times\mathrm{d}\boldsymbol{y})]^3$ is treated as a subspace of $[L^2(\Omega\times Q,\mathrm{d}\boldsymbol{x}\times\mathrm{d}\mu)]^3$.

2.2.2 Convergence on the Stiff Component

The first investigation yields a relationship between $\boldsymbol{u}(\boldsymbol{x},\boldsymbol{y})$ and $\widehat{\boldsymbol{u}}(\boldsymbol{x},\boldsymbol{y})$. Convergence on the unit cell of functions dependent on the rod thickness h is now introduced.

Definition 2.2.1. Let the sequence $\boldsymbol{\psi}_\varepsilon^h := \boldsymbol{\psi}^h(\cdot/\varepsilon)$ where $\boldsymbol{\psi}^h \in [L^2(Q,\mathrm{d}\mu^h)]^2$ be extended to $\mathbb{R}^2$ by Q-periodicity. Then for some bounded domain Ω, it is said that

1. the sequence $\boldsymbol{\psi}_\varepsilon^h$ weakly converges to $\boldsymbol{\psi} \in [L^2(Q,\mathrm{d}\mu)]^2$, denoted $\boldsymbol{\psi}_\varepsilon^h \xrightharpoonup{\mu_\varepsilon^h} \boldsymbol{\psi}$, if

$$\frac{1}{|\Omega|}\int_\Omega \boldsymbol{\psi}_\varepsilon^h\cdot\boldsymbol{\xi}\Big(\frac{\cdot}{\varepsilon}\Big)\,\mathrm{d}\mu_\varepsilon^h \to \int_Q \boldsymbol{\psi}\cdot\boldsymbol{\xi}\,\mathrm{d}\mu, \quad \forall\boldsymbol{\xi}\in[C_{\mathrm{per}}^\infty(Q)]^2,$$

   where $\boldsymbol{\xi}$ is extended to $\mathbb{R}^2$ by Q-periodicity.

2. the sequence $\boldsymbol{\psi}_\varepsilon^h$ strongly converges to a function $\boldsymbol{\psi} \in [L^2(Q,\mathrm{d}\mu)]^2$, denoted $\boldsymbol{\psi}_\varepsilon^h \xrightarrow{\mu_\varepsilon^h} \boldsymbol{\psi}$ if

$$\frac{1}{|\Omega|}\int_\Omega \boldsymbol{\psi}_\varepsilon^h\cdot\boldsymbol{\xi}^h\Big(\frac{\cdot}{\varepsilon}\Big)\,\mathrm{d}\mu_\varepsilon^h \to \int_Q \boldsymbol{\psi}\cdot\boldsymbol{\xi}\,\mathrm{d}\mu, \quad \text{if and only if} \quad \boldsymbol{\xi}_\varepsilon^h \xrightharpoonup{\mu_\varepsilon^h} \boldsymbol{\xi}.$$

Remark. Note that if such an Ω exists then it can be replaced by any other such domain Ω and therefore the definition above is independent of the choice of Ω.

Proposition 2.2.2. *Assume that $\boldsymbol{u}_\varepsilon^h(\boldsymbol{x}) \xrightharpoonup{2} \boldsymbol{u}(\boldsymbol{x},\boldsymbol{y})$ in $[L^2(\Omega,\mathrm{d}\mu_\varepsilon^h)]^2$ and that $\boldsymbol{\psi}_\varepsilon^h \xrightarrow{\mu_\varepsilon^h} \boldsymbol{\psi}$. Then*

$$\int_\Omega \boldsymbol{u}_\varepsilon^h\cdot\boldsymbol{\psi}_\varepsilon^h\varphi\,\mathrm{d}\mu_\varepsilon^h \to \int_\Omega\int_Q \boldsymbol{u}(\boldsymbol{x},\boldsymbol{y})\cdot\boldsymbol{\psi}(\boldsymbol{y})\varphi(\boldsymbol{x})\,\mathrm{d}\mu(\boldsymbol{y})\mathrm{d}\boldsymbol{x}, \quad \forall\varphi\in C_0^\infty(\Omega).$$

Proof. Since $\boldsymbol{\psi}_\varepsilon^h \xrightarrow{\mu_\varepsilon^h} \boldsymbol{\psi}$, it follows that for all $\boldsymbol{\zeta}\in[C_{\mathrm{per}}(Q)]^2$

$$\lim_{h\to0}\int_Q\left|\boldsymbol{\psi}^h-\boldsymbol{\zeta}\right|^2\mathrm{d}\mu^h = \int_Q|\boldsymbol{\psi}-\boldsymbol{\zeta}|^2\,\mathrm{d}\mu \quad \overset{\text{Lemma 2.1.1}}{\Longleftrightarrow} \quad \lim_{\varepsilon\to0}\int_\Omega\left|\boldsymbol{\psi}_\varepsilon^h-\boldsymbol{\zeta}\left(\frac{\cdot}{\varepsilon}\right)\right|^2\mathrm{d}\mu_\varepsilon^h = |\Omega|\int_Q|\boldsymbol{\psi}-\boldsymbol{\zeta}|^2\,\mathrm{d}\mu. \tag{2.2.6}$$

Moreover, by Hölder's inequality, it follows that

$$\left|\int_\Omega \boldsymbol{u}_\varepsilon^h\cdot\left(\boldsymbol{\psi}_\varepsilon^h-\boldsymbol{\zeta}\left(\frac{\cdot}{\varepsilon}\right)\right)\varphi\,\mathrm{d}\mu_\varepsilon^h\right| \le \max_\Omega|\varphi|\|\boldsymbol{u}_\varepsilon^h\|_{L^2(\Omega,\mathrm{d}\mu_\varepsilon^h)}\Big(\int_\Omega\left|\boldsymbol{\psi}_\varepsilon^h-\boldsymbol{\zeta}\left(\frac{\cdot}{\varepsilon}\right)\right|^2\mathrm{d}\mu_\varepsilon^h\Big)^{\frac{1}{2}}.$$

Hence, by the weak two-scale convergence of $\boldsymbol{u}_\varepsilon^h$ and relation (2.2.6), it is concluded that

$$\limsup_{\varepsilon\to 0}\Big|\int_\Omega \boldsymbol{u}_\varepsilon^h\cdot\boldsymbol{\psi}_\varepsilon^h\varphi\,\mathrm{d}\mu_\varepsilon^h-\int_\Omega\int_Q \boldsymbol{u}(\boldsymbol{x},\boldsymbol{y})\cdot\boldsymbol{\zeta}(\boldsymbol{y})\varphi(\boldsymbol{x})\,\mathrm{d}\mu(\boldsymbol{y})\mathrm{d}\boldsymbol{x}\Big|$$

$$=\limsup_{\varepsilon\to 0}\Big|\int_\Omega \boldsymbol{u}_\varepsilon^h\cdot\boldsymbol{\psi}_\varepsilon^h\varphi\,\mathrm{d}\mu_\varepsilon^h-\int_\Omega \boldsymbol{u}_\varepsilon^h\cdot\boldsymbol{\zeta}\left(\frac{\cdot}{\varepsilon}\right)\varphi\,\mathrm{d}\mu_\varepsilon^h\Big|\le C\Big(\int_Q|\boldsymbol{\psi}-\boldsymbol{\zeta}|^2\,\mathrm{d}\mu\Big)^{\frac{1}{2}},\quad\forall\boldsymbol{\zeta}\in[C_{\mathrm{per}}(Q)]^2.$$

Choosing a suitable approximation sequence $\boldsymbol{\zeta}_k$ such that $\boldsymbol{\zeta}_k\to\boldsymbol{\psi}$ in $[L^2(Q,\mathrm{d}\mu)]^2$ yields the result. □

Theorem 2.2.1. *The function $\widehat{\boldsymbol{u}}$ is the trace of $\boldsymbol{u}$ on F_1, in the sense that $\boldsymbol{u}(\boldsymbol{x},\boldsymbol{y})=\widehat{\boldsymbol{u}}(\boldsymbol{x},\boldsymbol{y})$,* a.e. $\boldsymbol{x}\in\Omega$, λ-a.e. $\boldsymbol{y}\in F_1$, *i.e.,*

$$\chi_1(\boldsymbol{y})\boldsymbol{u}(\boldsymbol{x},\boldsymbol{y})=\widehat{\boldsymbol{u}}(\boldsymbol{x},\boldsymbol{y})\quad\text{a.e.. }\boldsymbol{y}\in F_1.$$

Proof. The set $F_1\cap Q$, $(F_1^h\cap Q)$ consists of a finite number of links I_j, (rods of thickness h, I_j^h), $j=1,2,\dots,m$. Let $\widehat{\boldsymbol{\psi}}\in[L^2(Q,\mathrm{d}\lambda)]^2$ and for any such $\widehat{\boldsymbol{\psi}}$, define

$$\boldsymbol{\psi}(\boldsymbol{y}):=\begin{cases}\widehat{\boldsymbol{\psi}}(\boldsymbol{y}), & \boldsymbol{y}\in F_1\cap Q,\\ \boldsymbol{0}, & \boldsymbol{y}\in Q\backslash F_1,\end{cases}\qquad [\boldsymbol{\psi}]^h(\boldsymbol{y}):=\begin{cases}\sum\limits_{j=1}^m\boldsymbol{\psi}_j^h(\boldsymbol{y}), & \boldsymbol{y}\in F_1^h\cap Q,\\ \boldsymbol{0}, & \boldsymbol{y}\in Q\backslash F_1^h,\end{cases}\tag{2.2.7}$$

where $[\boldsymbol{\psi}]^h$ is the natural extension of $\widehat{\boldsymbol{\psi}}(\boldsymbol{y})$ on the thin structure with the functions $\boldsymbol{\psi}_j^h$ defined as in Definition 2.1.7. Observe that for all $\varphi\in C_0^\infty(\Omega)$

$$\int_\Omega \boldsymbol{u}_\varepsilon^h\cdot[\boldsymbol{\psi}]_\varepsilon^h\varphi\,\mathrm{d}\mu_\varepsilon^h=\int_\Omega \boldsymbol{u}_\varepsilon^h\chi_1^h\Big(\frac{\cdot}{\varepsilon}\Big)\cdot[\boldsymbol{\psi}]_\varepsilon^h\varphi\,\mathrm{d}\mu_\varepsilon^h+\int_\Omega \boldsymbol{u}_\varepsilon^h\chi_0^h\Big(\frac{\cdot}{\varepsilon}\Big)\cdot[\boldsymbol{\psi}]_\varepsilon^h\varphi\,\mathrm{d}\mu_\varepsilon^h.\tag{2.2.8}$$

Lemma 2.2.1. *There is convergence $[\boldsymbol{\psi}]_\varepsilon^h\xrightarrow{\mu_\varepsilon^h}\boldsymbol{\psi}$.*

Proof. Let the periodic cell Q contain M nodes $\mathcal{O}_i$, $i=1,2,\dots,M$. For sufficiently small $h>0$, some subset of the m rods whose associated links share a common node $\mathcal{O}_i$ will"overlap" in a small δ_h-neighbourhood of $\mathcal{O}_i$ where it is clear that as $h\to0$, $\delta_h\to0$. Denote this region of overlap by $B_{\delta_h}(\mathcal{O}_i)$ for each $i=1,2,\dots,M$. Consider the following integral:

$$\Psi^{h,\varepsilon}:=\int_Q[\boldsymbol{\psi}]^h\Big(\frac{\boldsymbol{x}}{\varepsilon}\Big)\cdot\boldsymbol{\xi}^h\Big(\frac{\boldsymbol{x}}{\varepsilon}\Big)\,\mathrm{d}\mu_\varepsilon^h,\qquad \boldsymbol{\xi}^h(\cdot/\varepsilon)\xrightarrow{\mu_\varepsilon^h}\boldsymbol{\xi}.$$

Since $\boldsymbol{\psi}^h$ is defined only on the rod structure in Q, $\Psi^{h,\varepsilon}$ may be rewritten as

$$\Psi^{h,\varepsilon}=\frac12\sum_{i=1}^M\int_{Q\cap B_{\delta_h}(\mathcal{O}_i)}[\boldsymbol{\psi}]^h\Big(\frac{\boldsymbol{x}}{\varepsilon}\Big)\cdot\boldsymbol{\xi}^h\Big(\frac{\boldsymbol{x}}{\varepsilon}\Big)\,\mathrm{d}\lambda_\varepsilon^h+\frac12\sum_{j=1}^m\int_{\widetilde{I}_j^h}[\boldsymbol{\psi}]^h\Big(\frac{\boldsymbol{x}}{\varepsilon}\Big)\cdot\boldsymbol{\xi}^h\Big(\frac{\boldsymbol{x}}{\varepsilon}\Big)\,\mathrm{d}\lambda_\varepsilon^h,$$

where $\widetilde{I}_j^h:=I_j^h\backslash\bigcup_{i=1}^M B_{\delta_h}(\mathcal{O}_i)$. Since $|\boldsymbol{\psi}^h|\le\max\limits_{\boldsymbol{y}\in B_{\delta_h}(\mathcal{O}_i)}|\widehat{\boldsymbol{\psi}}(\boldsymbol{y})|$, the first integral can be bounded above by $C\delta_h$ where the constant C is independent of h and hence this integral vanishes in the limit as $h \to 0$. For the second integral, noting the definition of $\boldsymbol{\psi}^h$ given by (2.2.7), the following is obtained

$$\Psi^{h,\varepsilon} = \frac{1}{2} \sum_{j,k=1}^{m} \int_{-h}^{h} \left\{ \int_{\widetilde{I}_j} \widehat{\boldsymbol{\psi}}\Big(\frac{\boldsymbol{x}}{\varepsilon}\Big) \cdot \boldsymbol{\xi}^h\Big(\frac{\boldsymbol{x}}{\varepsilon}\Big) \, \mathrm{d}\tau_k \right\} \mathrm{d}\nu_k + O(\delta_h),$$

where $\widetilde{I}_j = I_j \backslash \bigcup_{i=1}^{M} B_{\delta_h}(\mathcal{O}_i)$ and $\mathrm{d}\tau_k = \mathrm{d}(\boldsymbol{\tau}_k \cdot \boldsymbol{y})$, $\mathrm{d}\nu_k = \mathrm{d}(\boldsymbol{\nu}_k \cdot \boldsymbol{y})$. Passing to the limit as $\varepsilon \to 0$, it is clear that by Definition 2.2.1, the following convergence holds:

$$\lim_{\varepsilon \to 0} \Psi^{h,\varepsilon} = \frac{1}{2} \int_{Q \cap F_1} \widehat{\boldsymbol{\psi}} \cdot \boldsymbol{\xi} \, \mathrm{d}\lambda = \int_Q \boldsymbol{\psi} \cdot \boldsymbol{\xi} \, \mathrm{d}\mu,$$

as required. $\square$

By the above lemma, in the limit as $\varepsilon \to 0$, the left-hand side converges and it's limit takes the form

$$\int_\Omega \boldsymbol{u}_\varepsilon^h \cdot [\boldsymbol{\psi}]_\varepsilon^h \varphi \, \mathrm{d}\mu_\varepsilon^h \to \int_\Omega \int_Q \boldsymbol{u}(\boldsymbol{x},\boldsymbol{y}) \cdot \boldsymbol{\psi}(\boldsymbol{y}) \varphi(\boldsymbol{x}) \mathrm{d}\mu(\boldsymbol{y}) \mathrm{d}\boldsymbol{x} = \frac{1}{2} \int_\Omega \int_Q \boldsymbol{u}(\boldsymbol{x},\boldsymbol{y}) \cdot \widehat{\boldsymbol{\psi}}(\boldsymbol{y}) \varphi(\boldsymbol{x}) \, \mathrm{d}\lambda \mathrm{d}\boldsymbol{x}.$$

Similarly for the first integral on the right-hand side of equation (2.2.8), it follows that

$$\int_\Omega \boldsymbol{u}_\varepsilon^h \chi_1^h\Big(\frac{\cdot}{\varepsilon}\Big) \cdot [\boldsymbol{\psi}]_\varepsilon^h \varphi \, \mathrm{d}\mu_\varepsilon^h \to \int_\Omega \int_Q \widehat{\boldsymbol{u}}(\boldsymbol{x},\boldsymbol{y}) \cdot \widehat{\boldsymbol{\psi}}(\boldsymbol{y}) \varphi(\boldsymbol{x}) \, \mathrm{d}\lambda \mathrm{d}\boldsymbol{x}.$$

Hence, it only remains to be shown that the second integral on the right of equation (2.2.8) goes to zero as $\varepsilon \to 0$. This follows immediately since $\boldsymbol{u}_\varepsilon^h \chi_0^h(\cdot/\varepsilon)) \stackrel{2}{\rightharpoonup} \boldsymbol{u}(\boldsymbol{x},\boldsymbol{y})\chi_0(\boldsymbol{y})$. Hence the result. $\square$

The proofs of the following two theorems are omitted since analogous theorems are found [82] with similar proofs up to the notation the results are presented in. The first theorem concerns the structure of the two-scale limit of the function $\boldsymbol{u}_\varepsilon^h$ on the stiff component and the second theorem concerns the structure of the limit of the sequence $\chi_1^{h,\varepsilon} e(\boldsymbol{u}_\varepsilon^h)$.

Theorem 2.2.2. *Suppose that* $\chi_1^{h,\varepsilon} \boldsymbol{u}_\varepsilon^h(\boldsymbol{x}) \stackrel{2}{\rightharpoonup} \widehat{\boldsymbol{u}}(\boldsymbol{x},\boldsymbol{y})$, *and that* $\varepsilon \chi_1^{h,\varepsilon} e(\boldsymbol{u}_\varepsilon^h) \to 0$. *Then* $\widehat{\boldsymbol{u}}(\boldsymbol{x},\boldsymbol{y}) \in [L^2(\Omega, \mathcal{R})]^2$.

Theorem 2.2.3. *Define the* λ*-homogenised tensor* A_λ^{hom} *by the relation*

$$A_\lambda^{\mathrm{hom}} \xi \cdot \xi = \min_{v \in V_{\mathrm{pot}}^\lambda} \int_Q A_1(\xi + v) \cdot (\xi + v) \, \mathrm{d}\lambda, \quad \forall \xi \in \mathrm{Sym}_2, \tag{2.2.9}$$

Suppose that the homogenised tensor is non-degenerate, periodic rigid displacements take the form (2.1.13) and that $\chi_1^{h,\varepsilon} \boldsymbol{u}_\varepsilon^h(\boldsymbol{x}) \stackrel{2}{\rightharpoonup} \widehat{\boldsymbol{u}}(\boldsymbol{x},\boldsymbol{y})$. *Then*

1. $\widehat{\boldsymbol{u}}(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{u}_0(\boldsymbol{x}) + \boldsymbol{\chi}(\boldsymbol{x},\boldsymbol{y})$, $\forall \boldsymbol{x} \in \Omega$, λ-a.e. $\boldsymbol{y} \in F_1$ *where* $\boldsymbol{u}_0 \in [H_0^1(\Omega)]^2$ *and* $\boldsymbol{\chi} \in [L^2(\Omega, \widehat{\mathcal{R}})]^2$.

2. $\chi_1^{h,\varepsilon} e(\boldsymbol{u}_\varepsilon^h) \stackrel{2}{\rightharpoonup} e(\boldsymbol{u}_0(\boldsymbol{x})) + v(\boldsymbol{x},\boldsymbol{y}), \quad v \in L^2(\Omega, V_{\mathrm{pot}}^\lambda)$.

3. *Suppose that*

$$\lim_{\varepsilon\to 0}\int_\Omega A_1 e(\boldsymbol{u}_\varepsilon^h(\boldsymbol{x}))\cdot e_y(\boldsymbol{w}(\boldsymbol{x}/\varepsilon))\varphi(\boldsymbol{x})\,\mathrm{d}\lambda_\varepsilon^h = 0, \quad \forall\varphi\in C_0^\infty(\Omega), \quad \boldsymbol{w}\in[C_{\mathrm{per}}^\infty(Q)]^2.$$

Then $\chi_1^{h,\varepsilon}A_1(\cdot/\varepsilon)e(\boldsymbol{u}_\varepsilon^h) \rightharpoonup \langle A_1\{e(\boldsymbol{u}_0(\boldsymbol{x}))+v(\boldsymbol{x},\boldsymbol{y})\}\rangle_\lambda = A_\lambda^{\mathrm{hom}}e(\boldsymbol{u}_0)$ *where* $A_1\{e(\boldsymbol{u}_0)+v\}\in L^2(\Omega,V_{\mathrm{sol}}^\lambda)$.

Proof. 1. By the decomposition (2.1.13), it immediately follows that $\widehat{\boldsymbol{u}}(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{u}_0(\boldsymbol{x})+\boldsymbol{\chi}(\boldsymbol{x},\boldsymbol{y})$, $\forall\boldsymbol{x}\in\Omega$, λ-a.e. $\boldsymbol{y}\in F_1$ with $\boldsymbol{\chi}\in[L^2(\Omega,\widehat{\mathcal{R}})]^2$. The fact that $\boldsymbol{u}_0\in[H_0^1(\Omega)]^2$ follows by part 2).

2. See Zhikov [82, Theorem 9.5] for full details.

3. See Zhikov [82, Theorem 9.6] for full details.

□

Remark. The unknown potential matrix v is determined via the following problem

$$v\in V_{\mathrm{pot}}^\lambda, \quad \int_Q A_1(e(\boldsymbol{u}_0)+v)\cdot\varphi\,\mathrm{d}\lambda = 0, \quad \forall\varphi\in V_{\mathrm{pot}}^\lambda, \quad \Longleftrightarrow \quad \operatorname{div}(A_1(e(\boldsymbol{u}_0)+v)) = 0.$$

From this last equality, it follows by definition that $A_1(e(\boldsymbol{u}_0)+v)\in V_{\mathrm{sol}}^\lambda$.

2.2.3 Convergence in the Soft Component

The next structure theorem regards the form of the two-scale limit of the sequence $\varepsilon\chi_0^{h,\varepsilon}e(\boldsymbol{u}_\varepsilon^h)$, denoted $\tilde{p}(\boldsymbol{x},\boldsymbol{y})$, on the soft inclusions.

Theorem 2.2.4. *Let* $\{\boldsymbol{u}_\varepsilon^h\}\subset[H^1(\Omega)]^2$ *such that* $\boldsymbol{u}_\varepsilon^h \xrightharpoonup{2} \boldsymbol{u}(\boldsymbol{x},\boldsymbol{y})$ *and* $\varepsilon\chi_0^{h,\varepsilon}e(\boldsymbol{u}_\varepsilon^h)\xrightharpoonup{2}\tilde{p}(\boldsymbol{x},\boldsymbol{y})$ *in* $[L^2(\Omega_0^{\varepsilon,h},\mathrm{d}\mu_\varepsilon^h)]^3$. *Then* $\boldsymbol{u}\in[L^2(\Omega,H^1(Q))]^2$ *and* $\tilde{p}(\boldsymbol{x},\boldsymbol{y}) = e_y(\boldsymbol{u}(\boldsymbol{x},\boldsymbol{y}))$ *for* a.e. $\boldsymbol{x}\in\Omega$, $\boldsymbol{y}\in Q$.

Proof. For each $\delta>0$, let Q_δ be a C^∞-domain such that $Q\cap F_0^{2\delta}\subset Q_\delta\subset Q\cap F_0^\delta$. Define $\mathcal{X}_\delta$ to be the set of all matrices $b\in C^\infty(Q_\delta)^3$ such that $b\boldsymbol{n}|_{\partial Q_\delta}=\boldsymbol{0}$ where $\boldsymbol{n}$ is the unit outward pointing normal to ∂Q_δ. For $b\in\mathcal{X}_\delta$, $\boldsymbol{a}=\operatorname{div}b$ in Q_δ and consider the functions

$$\tilde{\boldsymbol{a}}(\boldsymbol{y}) := \begin{cases}\boldsymbol{a}(\boldsymbol{y}), & \boldsymbol{y}\in Q_\delta,\\ \boldsymbol{0}, & \boldsymbol{y}\in Q\backslash Q_\delta,\end{cases} \qquad \tilde{b}(\boldsymbol{y}) := \begin{cases} b(\boldsymbol{y}), & \boldsymbol{y}\in Q_\delta,\\ 0, & \boldsymbol{y}\in Q\backslash Q_\delta,\end{cases}$$

which are extended to $\mathbb{R}^2$ by Q-periodicity. Picking $\varepsilon>0$ to be sufficiently small (recalling that $h\to 0$ as $\varepsilon\to 0$), the following identity holds:

$$\varepsilon\int_{\Omega_0^{\varepsilon,h}}\tilde{b}\left(\frac{\cdot}{\varepsilon}\right)\cdot e(\boldsymbol{\psi})\,\mathrm{d}\mu_\varepsilon^h = -\int_{\Omega_0^{\varepsilon,h}}\tilde{\boldsymbol{a}}\left(\frac{\cdot}{\varepsilon}\right)\cdot\boldsymbol{\psi}\,\mathrm{d}\mu_\varepsilon^h, \quad \forall\boldsymbol{\psi}\in[H_0^1(\Omega)]^2.$$

Setting $\boldsymbol{\psi} = \varphi \boldsymbol{u}^h_\varepsilon$, $\varphi \in C_0^\infty(\Omega)$, yields

$$\varepsilon \int_{\Omega_0^{\varepsilon,h}} \tilde{b}\left(\frac{\cdot}{\varepsilon}\right) \varphi \cdot e(\boldsymbol{u}^h_\varepsilon)\, \mathrm{d}\mu^h_\varepsilon + \varepsilon \int_{\Omega_0^{\varepsilon,h}} \tilde{b}\left(\frac{\cdot}{\varepsilon}\right) \cdot \frac{1}{2}(\boldsymbol{u}^h_\varepsilon \otimes \nabla\varphi + \nabla\varphi \otimes \boldsymbol{u}^h_\varepsilon)\, \mathrm{d}\mu^h_\varepsilon = -\int_{\Omega_0^{\varepsilon,h}} \tilde{\boldsymbol{a}}\left(\frac{\cdot}{\varepsilon}\right) \cdot \varphi \boldsymbol{u}^h_\varepsilon\, \mathrm{d}\mu^h_\varepsilon.$$

Passing to the limit as $\varepsilon \to 0$ and noting that $\tilde{\boldsymbol{a}}$ and $\tilde{b}$ vanish on $Q \backslash Q_\delta$, the following equality is obtained:

$$\int_\Omega \int_{Q_\delta} \tilde{p}(\boldsymbol{x}, \boldsymbol{y}) \varphi(\boldsymbol{x}) \cdot b(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y} \mathrm{d}\boldsymbol{x} = -\int_\Omega \int_{Q_\delta} \boldsymbol{u}(\boldsymbol{x}, \boldsymbol{y}) \varphi(\boldsymbol{x}) \cdot \boldsymbol{a}(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y} \mathrm{d}\boldsymbol{x}.$$

As $\varphi \in C_0^\infty(\Omega)$ is arbitrary, it follows that

$$\int_{Q_\delta} \tilde{p}(\boldsymbol{x}, \boldsymbol{y}) \cdot b(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y} = -\int_{Q_\delta} \boldsymbol{u}(\boldsymbol{x}, \boldsymbol{y}) \cdot \boldsymbol{a}(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y}, \quad \text{a.e. } \boldsymbol{x} \in \Omega.$$

For divergence-free fields $b \in \mathfrak{X}_\delta$, it is deduced that (see Dautray & Lions [26]) that there exists $\boldsymbol{v} \in [L^2(\Omega, H^1(Q_\delta))]^2$ such that $\tilde{p}(\boldsymbol{x}, \boldsymbol{y}) = e_y(\boldsymbol{v}(\boldsymbol{x}, \boldsymbol{y}))$, $\boldsymbol{y} \in Q_\delta$. This in turn implies that

$$\int_{Q_\delta} \boldsymbol{u}(\boldsymbol{x}, \boldsymbol{y}) \cdot \boldsymbol{a}(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y} = \int_{Q_\delta} \boldsymbol{v}(\boldsymbol{x}, \boldsymbol{y}) \cdot \boldsymbol{a}(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y}, \quad \text{a.e. } \boldsymbol{x} \in \Omega,$$

where the latter equality holds for all vectors $\boldsymbol{a}$ in the set

$$\{\operatorname{div} b \mid b \in \mathfrak{X}_\delta\} = \left\{\boldsymbol{a} \in C^\infty(Q_\delta) \;\middle|\; \int_{Q_\delta} \boldsymbol{a} = 0\right\}.$$

Since vector functions $\boldsymbol{a}$ with the above representation are dense in $[L^2(Q_\delta)]^2$ with zero mean, $\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{y})$ and $\boldsymbol{v}(\boldsymbol{x}, \boldsymbol{y})$ differ by a constant for $\boldsymbol{y} \in Q_\delta$ and hence $\tilde{p} = e_y(\boldsymbol{v}) = e_y(\boldsymbol{u})$ for a.e. $\boldsymbol{y} \in Q_\delta$. Since $\delta > 0$ was arbitrary, it is concluded that $\tilde{p} = e_y(\boldsymbol{u})$ for a.e. $\boldsymbol{y} \in Q$ as required. □

By the theorems presented in Section 2.2.2 and Section 2.2.3, it may be concluded that the difference $\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{y}) - \boldsymbol{u}_0(\boldsymbol{x}) =: \boldsymbol{U}(\boldsymbol{x}, \boldsymbol{y})$ is in $[L^2(\Omega, H^1(Q))]^2$ with $\boldsymbol{u}_0 \in [H_0^1(\Omega)]^2$. Moreover $\tilde{p}(\boldsymbol{x}, \boldsymbol{y}) = e_y(\boldsymbol{U}(\boldsymbol{x}, \boldsymbol{y}))$ and $\boldsymbol{U}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{\chi}(\boldsymbol{x}, \boldsymbol{y})$, a.e. $\boldsymbol{x} \in \Omega$, λ-a.e. $\boldsymbol{y} \in Q \cap F_1$, $\boldsymbol{\chi} \in [L^2(\Omega, \widehat{\mathfrak{R}})]^2$.

It can also be concluded that $\boldsymbol{U}(\boldsymbol{x}, \cdot) \in [H^1_{\mathrm{per}}(Q)]^2$ by a similar argument to the one presented in [81, Theorem 4.5]. In order to show that $\boldsymbol{\chi} \in [L^2(\Omega, \widehat{\mathfrak{R}}^0)]^2$, several results presented in [84, Section 3] need to be used. These results are presented (without proof) in Appendix 2.A. Using these results, the following theorem holds:

Theorem 2.2.5. *In the formula $\widehat{\boldsymbol{u}}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{u}_0(\boldsymbol{x}) + \boldsymbol{\chi}(\boldsymbol{x}, \boldsymbol{y})$, the periodic transverse displacement $\boldsymbol{\chi}$ is an element of the space $[L^2(\Omega, \widehat{\mathfrak{R}}^0)]^2$.*

2.3 Homogenisation Theorem

In this section, the proof of the homogenisation theorem for a critically scaled, high-contrast thin structure is presented.

2.3.1 Homogenised System of Equations

The space of limiting functions, also known as the "energy space" is denoted V and is defined as follows:

Definition 2.3.1. Define V to be the energy space which consists of all vectors of the form

$$\boldsymbol{u}(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{u}_0(\boldsymbol{x}) + \boldsymbol{U}(\boldsymbol{x},\boldsymbol{y}), \quad \boldsymbol{u}_0 \in [H_0^1(\Omega)]^2, \quad \boldsymbol{U} \in [L^2(\Omega, H^1_{\text{per}}(Q))]^2,$$

$$\boldsymbol{U}(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{\chi}(\boldsymbol{x},\boldsymbol{y}), \quad \text{a.e. } \boldsymbol{x} \in \Omega, \ \lambda\text{-a.e. } \boldsymbol{y} \in Q \cap F_1, \quad \boldsymbol{\chi} \in [L^2(\Omega, \widehat{\mathfrak{R}}^0)]^2.$$

The vector $\boldsymbol{u} \in V$ is called the solution of the homogenised problem if the following integral identity is satisfied for all test functions $\boldsymbol{\varphi}(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{\varphi}_0(\boldsymbol{x}) + \boldsymbol{\Phi}(\boldsymbol{x},\boldsymbol{y}) \in V$:

$$\int_\Omega A_\lambda^{\text{hom}} e(\boldsymbol{u}_0) \cdot e(\boldsymbol{\varphi}_0)\,\mathrm{d}\boldsymbol{x} + \frac{\theta^2}{6}\int_\Omega\int_Q K_1 \boldsymbol{\chi}'' \cdot \boldsymbol{\Phi}''\,\mathrm{d}\lambda \mathrm{d}\boldsymbol{x} + \frac{1}{2}\int_\Omega\int_Q A_0 e_y(\boldsymbol{U}) \cdot e_y(\boldsymbol{\Phi})\,\mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{x} +$$
$$+ \int_\Omega\int_Q \big(\boldsymbol{u}_0 + \boldsymbol{U}\big) \cdot (\boldsymbol{\varphi}_0 + \boldsymbol{\Phi})\,\mathrm{d}\mu\mathrm{d}\boldsymbol{x} = \int_\Omega\int_Q \boldsymbol{f} \cdot (\boldsymbol{\varphi}_0 + \boldsymbol{\Phi})\,\mathrm{d}\mu\mathrm{d}\boldsymbol{x}, \quad (2.3.1)$$

where A_λ^{hom} is given by equation (2.2.9), where $K_1 = (A_1^{-1}\eta \cdot \eta)^{-1}$, $\eta = -\boldsymbol{\tau} \otimes \boldsymbol{\tau}$ and $\boldsymbol{\tau}$ is a direction along the link.

The homogenised equation (2.3.1) is equivalent to a system of partial differential equations which are obtained by considering various classes of the test functions $\boldsymbol{\varphi}$ in (2.3.1). For simplicity, it will be assumed that the tensor A_0 is isotropic (see equation (2.1.4)). By considering test functions of the form $\boldsymbol{\varphi}(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{\varphi}_0(\boldsymbol{x})$, the following macroscopic equation is obtained:

$$-\text{div} A_\lambda^{\text{hom}} e(\boldsymbol{u}_0) + \boldsymbol{u}_0 + \langle \boldsymbol{U} \rangle = \langle \boldsymbol{f} \rangle, \quad \boldsymbol{u}_0 \in [H_0^1(\Omega)]^2. \quad (2.3.2)$$

In fact, by a standard result (see Evans [29]), $\boldsymbol{u}_0 \in [H_0^2(\Omega)]^2$. Two other restrictions of the test function $\boldsymbol{\varphi}$ are now considered. Consider the test functions of the form $\boldsymbol{\varphi}(\boldsymbol{x},\boldsymbol{y}) = \varphi(\boldsymbol{x})\boldsymbol{\Psi}(\boldsymbol{y})$ (i.e. $\boldsymbol{\varphi}_0 \equiv \boldsymbol{0}$) where $\varphi \in C_0^\infty(\Omega)$, $\boldsymbol{\Psi} \in \widetilde{V}$ where the space $\widetilde{V}$ consists of functions in $[H^1_{\text{per}}(Q)]^2$ whose trace on $F_1 \cap Q$ coincides with a rigid body motion λ-a.e. (see Definition 2.1.4). Taking first $\boldsymbol{\Psi} \in [C_0^\infty(F_0 \cap Q)]^2$ yields

$$-\Big\{M_0 \boldsymbol{\Delta U} + (M_0 + L_0)\nabla(\text{div}\,\boldsymbol{U})\Big\} + \boldsymbol{u} = P_{\mathfrak{v}}\boldsymbol{f}, \quad \boldsymbol{U}(\boldsymbol{x},\cdot) \in [H^1_{\text{per}}(Q)]^2, \ \boldsymbol{x} \in \Omega, \quad (2.3.3)$$

$$\boldsymbol{U}(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{\chi}(\boldsymbol{x},\boldsymbol{y}) \in [L^2(\Omega, \widehat{\mathfrak{R}}^0)]^2, \ \ \boldsymbol{x} \in \Omega, \ \lambda\text{-a.e. } \boldsymbol{y} \in Q \cap F_1, \quad (2.3.4)$$

where $P_{\mathfrak{v}}$ is the orthogonal projection from $[L^2(\Omega \times Q, \mathrm{d}\boldsymbol{x} \times \mathrm{d}\mu)]^2$ onto $[L^2(\Omega \times Q, \mathrm{d}\boldsymbol{x} \times \mathrm{d}\boldsymbol{y})]^2$. Furthermore, if an arbitrary function $\boldsymbol{\Psi} \in \widetilde{V}$ is chosen, an additional equation which couples the framework $F_1 \cap Q$ and the inclusion $F_0 \cap Q$ is obtained. Indeed, for example, on those links parallel to the y_2-axis, the "linking" equation is (see the paragraph containing equation (2.1.3) for notation)

$$\frac{\theta^2 K_1}{3}\partial_2^4\chi^{(e_1)} + (L_0 + 2M_0)\partial_1 U^{(e_1)} + \left(u_0^{(e_1)} + \chi^{(e_1)}\right) = P_{\mathfrak{v}^\perp} f^{(e_1)}, \tag{2.3.5}$$

where $P_{\mathfrak{v}^\perp}$ is the orthogonal projection from $[L^2(\Omega \times Q, \mathrm{d}\boldsymbol{x} \times \mathrm{d}\mu)]^2$ onto $[L^2(\Omega \times Q, \mathrm{d}\boldsymbol{x} \times \mathrm{d}\lambda)]^2$. For a general periodic framework F_1 where to each link I there is an associated pair of positively orientated vectors $\boldsymbol{\tau}$, $\boldsymbol{\nu}$ with $\boldsymbol{\tau}$ pointing along the link and $\boldsymbol{\nu}$ orthogonal to the link, equation (2.3.5) is replaced by

$$\frac{\theta^2 K_1}{3}\partial_\tau^4\chi^{(\nu)} + (L_0 + 2M_0)\partial_\nu U^{(\nu)} + u^{(\nu)} = f^{(\nu)}, \tag{2.3.6}$$

where ∂_τ denotes differentiation along the links and ∂_ν denotes differentiation in the direction normal to the links. See Appendix 2.C for more details.

Before proving the main result (Theorem 2.3.1 below), some auxiliary results pertinent to the proof of the homogenisation theorem will be presented.

2.3.2 Extensions of functions defined on F_1 to functions defined on F_1^h

A description of a class of functions and several results which were first demonstrated in [84, Section 4.1] is now given which extend periodic rigid displacements in $\widehat{\mathcal{R}}^0$ on the framework F_1 to the rod network F_1^h.

Definition 2.3.2. Let D denote the space of functions $\boldsymbol{g} \in \widehat{\mathcal{R}}^0$ such that

1. $\boldsymbol{g}$ is infinitely smooth outside a neighbourhood of the nodes of the network F_1,

2. In a neighbourhood $B_\delta(\mathcal{O}) := \{\boldsymbol{y} \,|\, |\boldsymbol{y} - \mathcal{O}| < \delta\}$ of each node $\mathcal{O}$, $\boldsymbol{g}$ takes the form $\boldsymbol{g}(\boldsymbol{y}) = C(\boldsymbol{\omega}(\boldsymbol{y}) - \boldsymbol{\omega}(\mathcal{O}))$, $\boldsymbol{y} \in F_1$ where C is a constant and $\boldsymbol{\omega}(\boldsymbol{y}) = (-y_2, y_1)$.

The above equality actually means that $\boldsymbol{g}(\boldsymbol{y}) \cdot \boldsymbol{\nu}_i = C(\boldsymbol{\omega}(\boldsymbol{y}) - \boldsymbol{\omega}(\mathcal{O})) \cdot \boldsymbol{\nu}_i$ for each normal $\boldsymbol{\nu}_i$ orthogonal to the links meeting at the node $\mathcal{O}$ with the constant C fixed. The following result shows in fact that functions in $\widehat{\mathcal{R}}^0$ are approximable by functions in D.

Proposition 2.3.1. *The set D is dense in $\widehat{\mathcal{R}}^0$ with respect to the norm of $[L^2(Q, \mathrm{d}\lambda)]^2$.*

The next result concerns the extension functions needed to prove the homogenisation theorem at the end of this section.

Lemma 2.3.1. *For each $\boldsymbol{g} \in D$, there exists an extension $\boldsymbol{g}^h = \boldsymbol{g}^h(\boldsymbol{y})$ to the network F_1^h such that*

1. *The symmetric gradient $e_y(\boldsymbol{g}^h)$ is zero in a neighbourhood $B_{\delta_j}(\mathcal{O}_j)$ of each node $\mathcal{O}_j$ of $F_1 \cap Q$.*

2. *The following asymptotic expansion in h holds on $(F_1^h \cap Q) \backslash (\cup_j B_{\delta_j}(\mathcal{O}_j))$*

$$A_1 e_y(\boldsymbol{g}^h) = h[(\boldsymbol{g}\cdot\boldsymbol{\nu})'' K_1]^h \sigma^h + O(h^2), \quad h \to 0, \tag{2.3.7}$$

where for each $h > 0$ and for each link I_k of $F_1 \cap Q$, σ^h is defined by relation

$$\sigma^h(\boldsymbol{y}) = (\boldsymbol{\tau}\otimes\boldsymbol{\tau})\frac{\boldsymbol{\nu}\cdot(\mathcal{O}-\boldsymbol{y})}{h}, \qquad \boldsymbol{y} \in I_k^h \backslash \left(\cup_j B_{\delta_j}(\mathcal{O}_j)\right),$$

where $\boldsymbol{\tau}$, $\boldsymbol{\nu}$ are unit tangent and unit normal to the link I_k, I_k^h is the h-neighbourhood of I_k and $\mathcal{O}$ is either node of the link in question. Note further that $[(\boldsymbol{g}\cdot\boldsymbol{\nu})'' K_1]^h$ is the natural extension of the function $(\boldsymbol{g}\cdot\boldsymbol{\nu})'' K_1$ on F_1^h.

3. *There exists a constant $C > 0$ independent of h such that $\|\boldsymbol{g}^h - \boldsymbol{g}\|_{L^2(Q,\mathrm{d}\mu^h)} \leq Ch$.*

2.3.3 Convergence of Solutions

The main result is now presented.

Theorem 2.3.1. *Let $\boldsymbol{u}_\varepsilon^h$ solve the integral identity (2.1.2) and suppose that $h/\varepsilon \to \theta > 0$ as $\varepsilon \to 0$.*

1. *If $\boldsymbol{f}^{h,\varepsilon} \overset{2}{\rightharpoonup} \boldsymbol{f}$ then up to a suitable subsequence $\boldsymbol{u}_\varepsilon^h \overset{2}{\rightharpoonup} \boldsymbol{u}$ and $\boldsymbol{u}$ satisfies the homogenised equation (2.3.1).*

2. *If $\boldsymbol{f}^{h,\varepsilon} \overset{2}{\to} \boldsymbol{f}$ then up to a suitable subsequence $\boldsymbol{u}_\varepsilon^h \overset{2}{\to} \boldsymbol{u}$ and in addition there is also convergence of the elastic energies:*

$$\lim_{\varepsilon\to 0}\left\{\int_{\Omega_1^{\varepsilon,h}} A_1 e(\boldsymbol{u}_\varepsilon^h)\cdot e(\boldsymbol{u}_\varepsilon^h)\,\mathrm{d}\mu_\varepsilon^h + \varepsilon^2\int_{\Omega_0^{\varepsilon,h}} A_0 e(\boldsymbol{u}_\varepsilon^h)\cdot e(\boldsymbol{u}_\varepsilon^h)\,\mathrm{d}\mu_\varepsilon^h\right\} =$$
$$= \int_\Omega A_\lambda^{\hom} e(\boldsymbol{u}_0)\cdot e(\boldsymbol{u}_0)\,\mathrm{d}\boldsymbol{x} + \frac{\theta^2}{6}\int_\Omega\int_Q K_1\boldsymbol{\chi}''\cdot\boldsymbol{\chi}''\,\mathrm{d}\lambda\mathrm{d}\boldsymbol{x} + \frac{1}{2}\int_\Omega\int_Q A_0 e_y(\boldsymbol{U})\cdot e_y(\boldsymbol{U})\,\mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{x}.$$

Proof. 1) Consider the integral identity (2.1.2) restated here for convenience:

$$\int_{\Omega_1^{\varepsilon,h}} A_1 e(\boldsymbol{u}_\varepsilon^h)\cdot e(\boldsymbol{\varphi})\,\mathrm{d}\mu_\varepsilon^h + \varepsilon^2\int_{\Omega_0^{\varepsilon,h}} A_0 e(\boldsymbol{u}_\varepsilon^h)\cdot e(\boldsymbol{\varphi})\,\mathrm{d}\mu_\varepsilon^h + \int_\Omega \boldsymbol{u}_\varepsilon^h\cdot\boldsymbol{\varphi}\,\mathrm{d}\mu_\varepsilon^h = \int_\Omega \boldsymbol{f}^{h,\varepsilon}\cdot\boldsymbol{\varphi}\,\mathrm{d}\mu_\varepsilon^h, \tag{2.3.8}$$

which is satisfied for all $\boldsymbol{\varphi} \in V$. Setting $\boldsymbol{\varphi} = \boldsymbol{\varphi}_0(\boldsymbol{x}) \in [H_0^1(\Omega)]^2$ and using Theorems 2.2.3 and 2.2.4 yield the following as $\varepsilon \to 0$:

$$\int_\Omega A_\lambda^{\hom} e(\boldsymbol{u}_0)\cdot e(\boldsymbol{\varphi}_0)\,\mathrm{d}\boldsymbol{x} + \int_\Omega\int_Q \boldsymbol{u}\cdot\boldsymbol{\varphi}_0\,\mathrm{d}\mu\mathrm{d}\boldsymbol{x} = \int_\Omega\int_Q \boldsymbol{f}\cdot\boldsymbol{\varphi}_0\,\mathrm{d}\mu\mathrm{d}\boldsymbol{x}. \tag{2.3.9}$$

The second integral in (2.3.8) vanishes in the limit as $\varepsilon \to 0$ since $\boldsymbol{\varphi}_0$ is independent of the microscopic variable.

Let $\boldsymbol{G} \in [C_{\mathrm{per}}^\infty(Q)]^2$, and $\boldsymbol{g} \in D$ be such that $\boldsymbol{G}(\boldsymbol{y}) = \boldsymbol{g}(\boldsymbol{y})$ for λ-a.e. $\boldsymbol{y} \in F_1 \cap Q$. Then $\boldsymbol{G}$ is approximated by a sequence of functions $\boldsymbol{G}^h \in [C_{\mathrm{per}}^\infty(Q)]^2$ such that $\boldsymbol{G}^h = \boldsymbol{g}^h$ on

$F_1^h \cap Q$ and where $\boldsymbol{g}^h$ is the extension described in Lemma 2.3.1. This approximation is achieved, for example, by setting $\boldsymbol{G}^h = \boldsymbol{G}\chi_h + \bar{\boldsymbol{g}}^h(1-\chi_h)$ where

$$\bar{\boldsymbol{g}}^h = \begin{cases} \boldsymbol{0}, & \text{in } F_0^h \cap Q, \\ \boldsymbol{g}^{2h}, & \text{in } F_1^{2h} \cap Q, \end{cases}, \quad \chi_h = \begin{cases} 1, & \text{on } F_0^{2h} \cap Q, \\ 0, & \text{on } F_1^h \cap Q, \\ \varphi_h, & \text{on } (F_1^{2h} \backslash F_1^h) \cap Q. \end{cases}$$

The function χ_h is a cut-off function on Q where $\varphi_h \in C^\infty(F_1^{2h}\backslash F_1^h)$ and φ_h is constructed via a suitable sum of convolutions of the charateristic functions $\chi_0^{3h/2}$ and $(1-\chi_1^{3h/2})$ with a smooth function $v : \mathbb{R}^2 \to \mathbb{R}$ with $\operatorname{supp} v \subset \{\boldsymbol{x} \mid |\boldsymbol{x}| \le \frac{1}{8}\}$. It will be shown that $\|\boldsymbol{G}^h - \boldsymbol{G}\|_{H^1(Q)} \to 0$.

Consider those test functions

$$\boldsymbol{\varphi} = \boldsymbol{\varphi}_\varepsilon^h = w\boldsymbol{G}^h\left(\frac{\cdot}{\varepsilon}\right), \quad w \in C_0^\infty(\Omega). \tag{2.3.10}$$

Hence plugging a test function of the form (2.3.10) into equation (2.3.8) yields after simplification:

$$\begin{aligned} &\varepsilon^{-1}\int_{\Omega_1^{\varepsilon,h}} A_1 e(\boldsymbol{u}_\varepsilon^h) \cdot e_y(\boldsymbol{g}^h)\left(\frac{\cdot}{\varepsilon}\right) w \, \mathrm{d}\mu_\varepsilon^h + \int_{\Omega_1^{\varepsilon,h}} A_1 e(\boldsymbol{u}_\varepsilon^h) \cdot (\boldsymbol{G}^h\left(\frac{\cdot}{\varepsilon}\right) \otimes \nabla w) \, \mathrm{d}\mu_\varepsilon^h + \\ &+ \varepsilon \int_{\Omega_0^{\varepsilon,h}} A_0 e(\boldsymbol{u}_\varepsilon^h) \cdot e_y(\boldsymbol{G}^h)\left(\frac{\cdot}{\varepsilon}\right) w \, \mathrm{d}\mu_\varepsilon^h + \varepsilon^2 \int_{\Omega_0^{\varepsilon,h}} A_0 e(\boldsymbol{u}_\varepsilon^h) \cdot (\boldsymbol{G}^h\left(\frac{\cdot}{\varepsilon}\right) \otimes \nabla w) \, \mathrm{d}\mu_\varepsilon^h = \\ &= \int_\Omega (\boldsymbol{f}^{h,\varepsilon} - \boldsymbol{u}_\varepsilon^h) \cdot \boldsymbol{G}^h w \, \mathrm{d}\mu_\varepsilon^h, \end{aligned} \tag{2.3.11}$$

Let the four integrals on the left-hand side of equation (2.3.11) be labeled as $I_1(\varepsilon), \ldots, I_4(\varepsilon)$ respectively. By the fact that $A_1(e(\boldsymbol{u}_0) + v)$ is pointwise orthogonal to the matrix $\boldsymbol{G}^h \otimes \nabla w$ (see [82, Lemma 5.3]) and the L^2-boundedness of the sequence $\varepsilon e(\boldsymbol{u}_\varepsilon^h)$, it follows that integrals $I_2(\varepsilon)$ and $I_4(\varepsilon)$ tend to zero as $\varepsilon \to 0$. By Theorem 2.2.4, it follows that

$$\begin{aligned} \lim_{\varepsilon\to 0} \varepsilon I_3(\varepsilon) &= \lim_{\varepsilon\to 0} \frac{1}{2}\int_{\Omega_0^{\varepsilon,h}} A_0 \left(\varepsilon e(\boldsymbol{u}_\varepsilon^h)\right) \cdot e_y(\boldsymbol{G}) w \, \mathrm{d}\boldsymbol{y} + \lim_{\varepsilon\to 0}\frac{1}{2}\int_{\Omega_0^{\varepsilon,h}} A_0\left(\varepsilon e(\boldsymbol{u}_\varepsilon^h)\right) \cdot e_y(\boldsymbol{G}^h - \boldsymbol{G}) w \, \mathrm{d}\boldsymbol{y} = \\ &= \frac{1}{2}\int_\Omega \int_Q A_0 e_y(\boldsymbol{u}) \cdot e_y(\boldsymbol{G}) w \, \mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{x}. \end{aligned}$$

For the integral $I_1(\varepsilon)$, it follows by the properties of the extension functions $\boldsymbol{g}^h$ and results in Appendix 2.A that

$$\lim_{\varepsilon\to 0} I_1(\varepsilon) = \frac{\theta^2}{6}\int_\Omega \int_Q K_1 \boldsymbol{\chi}'' \cdot \boldsymbol{g}'' w \, \mathrm{d}\lambda \mathrm{d}\boldsymbol{x}. \tag{2.3.12}$$

To complete the proof, consider the following lemma:

Lemma 2.3.2. *There is convergence $\boldsymbol{G}^h \to \boldsymbol{G}$ in $[H^1(Q)]^2$ as $\varepsilon \to 0$.*

Proof. Recall that $\boldsymbol{G}^h = \boldsymbol{G}$ on F_0^{2h}. Note further that since $\boldsymbol{G}, \boldsymbol{G}^h, \boldsymbol{g}^h \in [C^\infty_{\mathrm{per}}(Q)]^2$ and $\boldsymbol{g} \in D$, these vector functions are bounded in L^2-norm by constants and therefore on F_1^h and $F_1^{2h} \backslash F_1^h$, their norms are of order $O(h)$ as $h \to 0$. Hence, $\|\boldsymbol{G}^h - \boldsymbol{G}\|_{L^2(Q)} \to 0$ as $h \to 0$.

Consider now $\|e(\boldsymbol{G}^h - \boldsymbol{G})\|_{L^2(Q)}$. Since $\boldsymbol{G}^h - \boldsymbol{G} = (\boldsymbol{G} - \bar{\boldsymbol{g}}^h)(\chi_h - 1)$ and the vectors a sufficiently smooth, $\|(\chi_h - 1)e(\boldsymbol{G} - \bar{\boldsymbol{g}}^h)\|_{L^2(Q)} \to 0$ as $h \to 0$ by the same argument as above. The difficulty is in the term $\|(\boldsymbol{G} - \bar{\boldsymbol{g}}^h)e(\chi_h - 1)\|_{L^2(Q)}$ since on $F_1^{2h} \backslash F_1^h$ $e(\chi_h) = O(h^{-1})$. Firstly

$$\|(\boldsymbol{G} - \bar{\boldsymbol{g}}^h)e(\chi_h - 1)\|_{L^2(Q)} = \|(\boldsymbol{G} - \bar{\boldsymbol{g}}^h)e(\chi_h - 1)\|_{L^2((F_1^{2h} \backslash F_1^h) \cap Q)},$$

since $\chi_h = 0$ on F_1^h. Noting that for sufficiently small h, $\boldsymbol{G}(\boldsymbol{y}) \sim \boldsymbol{g}^{2h}(\boldsymbol{y})$ for any $\boldsymbol{y} \in F_1^{2h}$, it follows that by expanding $\boldsymbol{G}$ in a Taylor's expansion about a point $\boldsymbol{y}' \in (F_1^{2h} \backslash F_1^h) \cap Q)$ that

$$\boldsymbol{G}(\boldsymbol{y}) \approx \boldsymbol{g}^{2h}(\boldsymbol{y}') + \nabla \boldsymbol{G}(\boldsymbol{y}')(\boldsymbol{y} - \boldsymbol{y}'), \quad (h \to 0).$$

Since for $\boldsymbol{y}, \boldsymbol{y}' \in F_1^{2h} \backslash F_1^h$, $|\boldsymbol{y} - \boldsymbol{y}'| = O(h)$ it therefore it follows that

$$\begin{aligned}\|(\boldsymbol{G} - \bar{\boldsymbol{g}}^h)e(\chi_h - 1)\|_{L^2((F_1^{2h} \backslash F_1^h) \cap Q)} \leq C h^{-1} \|\boldsymbol{g}^{2h}(\boldsymbol{y}') - \boldsymbol{g}^{2h}(\boldsymbol{y})\|_{L^2((F_1^{2h} \backslash F_1^h) \cap Q)} + \\ + C\|\nabla \boldsymbol{G}(\boldsymbol{y}')\|_{L^2((F_1^{2h} \backslash F_1^h) \cap Q))}.\end{aligned}$$

Hence by boundedness $e(\boldsymbol{g}^{2h})$ and $\nabla \boldsymbol{G}$, the result follows. □

Hence passing to the limit in (2.3.11), the following equation holds:

$$\frac{\theta^2}{6} \int_\Omega \int_Q K_1 \boldsymbol{\chi}'' \cdot \boldsymbol{g}'' w \, \mathrm{d}\lambda \mathrm{d}\boldsymbol{x} + \frac{1}{2} \int_\Omega \int_Q A_0 e_y(\boldsymbol{u}) \cdot e_y(\boldsymbol{G}) w \, \mathrm{d}\boldsymbol{y} \mathrm{d}\boldsymbol{x} = \int_\Omega \int_Q (\boldsymbol{f} - \boldsymbol{u}) \cdot \boldsymbol{G} w \, \mathrm{d}\mu \mathrm{d}\boldsymbol{x}, \tag{2.3.13}$$

Define $\boldsymbol{\Phi} = \boldsymbol{G}w$ and note that the linear span of test functions of the form $\boldsymbol{G}(\boldsymbol{y})w(\boldsymbol{x})$ is dense in $[L^2(\Omega, H^1_{\mathrm{per}}(Q))]^2$. Hence, adding the two integral identitiess (2.3.9) and (2.3.13) together and denoting by $\boldsymbol{\varphi}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{\varphi}_0(\boldsymbol{x}) + \boldsymbol{\Phi}(\boldsymbol{x}, \boldsymbol{y})$, it follows that the homogenised equation (2.3.1) holds.

2) Assume that $\boldsymbol{f}^{h,\varepsilon} \stackrel{2}{\to} \boldsymbol{f}$. Let $\boldsymbol{v}_\varepsilon^h \in [H_0^1(\Omega)]^2$ solve the following problem: $\boldsymbol{g}^{h,\varepsilon}$, i.e., consider the problem

$$\begin{aligned}\int_{\Omega_1^{\varepsilon,h}} A_1 e(\boldsymbol{v}_\varepsilon^h) \cdot e(\boldsymbol{\varphi}) \, \mathrm{d}\mu_\varepsilon^h + \varepsilon^2 \int_{\Omega_0^{\varepsilon,h}} A_0 e(\boldsymbol{v}_\varepsilon^h) \cdot e(\boldsymbol{\varphi}) \, \mathrm{d}\mu_\varepsilon^h + \\ + \int_\Omega \boldsymbol{v}_\varepsilon^h \cdot \boldsymbol{\varphi} \, \mathrm{d}\mu_\varepsilon^h = \int_\Omega \boldsymbol{g}^{h,\varepsilon} \cdot \boldsymbol{\varphi} \, \mathrm{d}\mu_\varepsilon^h, \quad \forall \boldsymbol{\varphi} \in H_0^1(\Omega)^2, \quad (2.3.14)\end{aligned}$$

where $\boldsymbol{g}^{h,\varepsilon} \in [L^2(\Omega, \mathrm{d}\mu_\varepsilon^h)]^2$ is an arbitrary sequence with weak two-scale limit $\boldsymbol{g}$. Setting $\boldsymbol{\varphi} = \boldsymbol{u}_\varepsilon^h$ in the above and $\boldsymbol{\varphi} = \boldsymbol{v}_\varepsilon^h$ in (2.3.8) and then subtracting one from the other yields

$$\int_\Omega \boldsymbol{u}_\varepsilon^h \cdot \boldsymbol{g}^{h,\varepsilon} \, \mathrm{d}\mu_\varepsilon^h = \int_\Omega \boldsymbol{v}_\varepsilon^h \cdot \boldsymbol{f}^{h,\varepsilon} \, \mathrm{d}\mu_\varepsilon^h. \tag{2.3.15}$$

As $\boldsymbol{g}^{h,\varepsilon} \overset{2}{\rightharpoonup} \boldsymbol{g}$, it follows that $\boldsymbol{v}_\varepsilon^h \overset{2}{\rightharpoonup} \boldsymbol{v}$ where $\boldsymbol{v}$ solves the homogenised equation with right-hand side $\boldsymbol{g}$. Hence, it follows that

$$\lim_{\varepsilon \to 0} \int_\Omega \boldsymbol{u}_\varepsilon^h \cdot \boldsymbol{g}^{h,\varepsilon} \, \mathrm{d}\mu_\varepsilon^h = \lim_{\varepsilon \to 0} \int \boldsymbol{v}_\varepsilon^h \cdot \boldsymbol{f}^{h,\varepsilon} \, \mathrm{d}\mu_\varepsilon^h = \int_\Omega \int_Q \boldsymbol{v} \cdot \boldsymbol{f} \, \mathrm{d}\mu \mathrm{d}\boldsymbol{x} = \int_\Omega \int_Q \boldsymbol{u} \cdot \boldsymbol{g} \, \mathrm{d}\mu \mathrm{d}\boldsymbol{x}.$$

Since $\boldsymbol{g}^{h,\varepsilon}$ is arbitrary, the strong two-scale convergence $\boldsymbol{u}_\varepsilon^h \overset{2}{\to} \boldsymbol{u}$ is established.

To show the convergence of energies, making use of the two-scale convergence result (2.1.12), it is seen that

$$\lim_{\varepsilon \to 0} \left\{ \int_{\Omega_1^{\varepsilon,h}} A_1 e(\boldsymbol{u}_\varepsilon^h) \cdot e(\boldsymbol{u}_\varepsilon^h) \, \mathrm{d}\mu_\varepsilon^h + \varepsilon^2 \int_{\Omega_0^{\varepsilon,h}} A_0 e(\boldsymbol{u}_\varepsilon^h) \cdot e(\boldsymbol{u}_\varepsilon^h) \, \mathrm{d}\mu_\varepsilon^h \right\} = \int_\Omega \int_Q |\boldsymbol{f}|^2 - |\boldsymbol{u}|^2 \, \mathrm{d}\mu \mathrm{d}\boldsymbol{x}.$$

$$= \int_\Omega A^{\mathrm{hom}} e(\boldsymbol{u}_0) \cdot e(\boldsymbol{u}_0) \, \mathrm{d}\boldsymbol{x} + \frac{\theta^2}{6} \int_\Omega \int_Q K_1 \boldsymbol{\chi}'' \cdot \boldsymbol{\chi}'' \, \mathrm{d}\lambda \mathrm{d}\boldsymbol{x} + \frac{1}{2} \int_\Omega \int_Q A_0 e_y(\boldsymbol{U}) \cdot e_y(\boldsymbol{U}) \, \mathrm{d}\boldsymbol{y} \mathrm{d}\boldsymbol{x}$$

as required.

$\square$

2.4 Convergence of Spectra

In the following section the convergence of the spectrum of the operators associated with problem (2.1.2) to the spectrum given by the limit problem (2.3.1) will be established.

2.4.1 Spectrum of the Limit Operator

Consider the bilinear forms (*c.f.* (2.3.1))

$$B_{\mathrm{macro}}(\boldsymbol{u}_0, \boldsymbol{\varphi}_0) = \int_\Omega A^{\mathrm{hom}} e(\boldsymbol{u}_0) \cdot e(\boldsymbol{\varphi}_0) \, \mathrm{d}\boldsymbol{x}, \quad \boldsymbol{u}_0, \boldsymbol{\varphi}_0 \in [H_0^1(Q)]^2, \tag{2.4.1}$$

$$B_{\mathrm{micro}}(\boldsymbol{U}, \boldsymbol{\Phi}) = \frac{\theta^2}{6} \int_Q K_1 \boldsymbol{\chi}'' \cdot \boldsymbol{\Phi}'' \, \mathrm{d}\lambda + \frac{1}{2} \int_Q A_0 e_y(\boldsymbol{U}) \cdot e_y(\boldsymbol{\Phi}) \, \mathrm{d}\boldsymbol{y}, \quad \boldsymbol{U}, \boldsymbol{\Phi} \in \widetilde{V}, \tag{2.4.2}$$

where, recall that, $\widetilde{V}$ is the space of $[H^1_{\mathrm{per}}(Q)]^2$ functions whose trace on $Q \cap F_1$ coincides with a rigid-body motion λ-a.e. The associated spectral problem for (2.3.1) can be written as

$$\begin{cases} B_{\mathrm{macro}}(\boldsymbol{u}_0, \boldsymbol{\varphi}_0) & = s\langle \boldsymbol{u}_0 + \langle \boldsymbol{U} \rangle, \boldsymbol{\varphi}_0 \rangle_{L^2(\Omega)}, \quad \forall \boldsymbol{\varphi}_0 \in [H_0^1(\Omega)]^2, \\ B_{\mathrm{micro}}(\boldsymbol{U}, \boldsymbol{\Phi}) & = s\langle \boldsymbol{u}_0 + \boldsymbol{U}, \boldsymbol{\Phi} \rangle_{L^2(Q, \mathrm{d}\mu)}, \qquad \forall \boldsymbol{\Phi} \in \widetilde{V}. \end{cases} \tag{2.4.3}$$

Let $\{\boldsymbol{\phi}_n\}_{n\in\mathbb{N}}$ be an orthonormal set of eigenvectors with non-zero average for the bilinear form B_{micro} with corresponding set of eigenvalues $\{\omega_n\}_{n\in\mathbb{N}}$:

$$B_{\text{micro}}(\boldsymbol{\phi}_n, \boldsymbol{\Phi}) = \omega_n \left\langle \boldsymbol{\phi}_n, \boldsymbol{\Phi} \right\rangle_{L^2(Q,\mathrm{d}\mu)}, \quad \forall \boldsymbol{\Phi} \in \widetilde{V}. \tag{2.4.4}$$

The existence of such a set of eigenvalues and eigenfunctions is guaranteed by Lax [44, Chapter 28, Theorem 3] which essentially states that a compact, symmetric operator mapping a Hilbert space into itself has a discrete spectrum and an orthonormal set of eigenfunctions. Note that B_{micro} may also have a set of eigenvalues $\{\omega_n'\}_{n\in\mathbb{N}}$ corresponding to those eigenfunctions $\{\boldsymbol{\phi}_n'\}_{n\in\mathbb{N}}$ with zero average.

Using the above observations, the function $\boldsymbol{U}(\boldsymbol{x}, \boldsymbol{y})$ will now be expressed as a linear combination of the eigenfunctions $\{\boldsymbol{\phi}_n\}_{n\in\mathbb{N}}$. Note that the 2×2 identity matrix, denoted I, may be written as the following expansion of the eigenfunctions $\{\boldsymbol{\phi}_n\}_{n\in\mathbb{N}}$:

$$I = \sum_{n=1}^{\infty} (\boldsymbol{\phi}_n \otimes \boldsymbol{c}_n), \quad \boldsymbol{c}_n := \langle \boldsymbol{\phi}_n \rangle.$$

Let $b = b(\boldsymbol{y}, s)$ be the matrix such that the vector $\hat{\boldsymbol{b}} := b\boldsymbol{u}_0$ satisfies the equation

$$B_{\text{micro}}(\hat{\boldsymbol{b}}, \boldsymbol{\Phi}) = s\langle \hat{\boldsymbol{b}}, \boldsymbol{\Phi} \rangle_{L^2(Q,\mathrm{d}\mu)} + \langle \boldsymbol{u}_0, \boldsymbol{\Phi} \rangle_{L^2(Q,\mathrm{d}\mu)}. \tag{2.4.5}$$

Then b has representation

$$b(\boldsymbol{y}, s) = \sum_{n=1}^{\infty} \frac{\boldsymbol{\phi}_n \otimes \boldsymbol{c}_n}{\omega_n - s}. \tag{2.4.6}$$

Indeed, assume formally that the vector $\hat{\boldsymbol{b}}$ has the following expansion:

$$\hat{\boldsymbol{b}}(\boldsymbol{x}, \boldsymbol{y}) = \sum_{n=1}^{\infty} b_n(\boldsymbol{x}) \boldsymbol{\phi}_n(\boldsymbol{y}).$$

Substituting this expansion and the expansion for the identity matrix into equation (2.4.5) yields

$$\sum_{n=1}^{\infty} b_n B_{\text{micro}}(\boldsymbol{\phi}_n, \boldsymbol{\Phi}) = \sum_{n=1}^{\infty} \left\{ s b_n \langle \boldsymbol{\phi}_n, \boldsymbol{\Phi} \rangle_{L^2(Q,\mathrm{d}\mu)} + \langle (\boldsymbol{\phi}_n \otimes \boldsymbol{c}_n) \boldsymbol{u}_0, \boldsymbol{\Phi} \rangle_{L^2(Q,\mathrm{d}\mu)} \right\}$$

$$\iff \sum_{n=1}^{\infty} \{ b_n \omega_n - s b_n - (\boldsymbol{c}_n \cdot \boldsymbol{u}_0) \} \langle \boldsymbol{\phi}_n, \boldsymbol{\Phi} \rangle_{L^2(Q,\mathrm{d}\mu)} = 0.$$

Hence, for $s \notin \sigma(B_{\text{micro}})$

$$b_n(\boldsymbol{x}) = \frac{\boldsymbol{c}_n \cdot \boldsymbol{u}_0(\boldsymbol{x})}{\omega_n - s}.$$

Plugging this expression back into the expansion for $\hat{\boldsymbol{b}}$ yields

$$\hat{\boldsymbol{b}} = \sum_{n=1}^{\infty} \frac{\boldsymbol{c}_n \cdot \boldsymbol{u}_0}{\omega_n - s} \boldsymbol{\phi}_n = \sum_{n=1}^{\infty} \frac{\boldsymbol{\phi}_n \otimes \boldsymbol{c}_n}{\omega_n - s} \boldsymbol{u}_0,$$

and thus the result. Therefore, $\boldsymbol{U}$ admits a representation of the form

$$\boldsymbol{U}(\boldsymbol{x}, \boldsymbol{y}) = s \sum_{n=1}^{\infty} \frac{\boldsymbol{c}_n \cdot \boldsymbol{u}_0(\boldsymbol{x})}{\omega_n - s} \boldsymbol{\phi}_n(\boldsymbol{y})$$

Substituting the expansion for $\boldsymbol{U}(\boldsymbol{x},\boldsymbol{y})$ into the first equation given in (2.4.3) yields

$$B_{\text{macro}}(\boldsymbol{u}_0,\boldsymbol{\varphi}_0) = \langle \beta(s)\boldsymbol{u}_0,\boldsymbol{\varphi}_0\rangle_{L^2(\Omega)}\,, \tag{2.4.7}$$

where $\beta(s)$ is defined as

$$\beta(s) := s\left(I + s\sum_{n=1}^{\infty}\frac{(\boldsymbol{c}_n\otimes\boldsymbol{c}_n)}{\omega_n - s}\right). \tag{2.4.8}$$

Versions of the function β can be found in the studies of scalar Zhikov [81] and vector (Smyshlyaev [68], Zhikov & Pastukhova [85, 87]) homogenisation problems. The following is a straightforward modification of a result seen in [85]:

Lemma 2.4.1. *Denote by $\mathfrak{V}$ the closure in $[L^2(\Omega\times Q, \mathrm{d}\boldsymbol{x}\times\mathrm{d}\mu)]^2$ of the energy space V and consider the operator $\mathfrak{U}$ whose domain consists of all solution pairs $(\boldsymbol{u}_0,\boldsymbol{U})$ to the identity*

$$B_{\text{macro}}(\boldsymbol{u}_0,\boldsymbol{\varphi}_0) + B_{\text{micro}}(\boldsymbol{U},\boldsymbol{\Phi}) = \langle \boldsymbol{f},\boldsymbol{\varphi}_0+\boldsymbol{\Phi}\rangle_{L^2(\Omega\times Q,\mathrm{d}\boldsymbol{x}\times\mathrm{d}\mu)},\quad \forall\boldsymbol{\varphi}_0+\boldsymbol{\Phi}\in V, \tag{2.4.9}$$

as the right-hand side $\boldsymbol{f}$ runs over all elements of $\mathfrak{V}$ and defined by the relation $\boldsymbol{f} = \mathfrak{U}(\boldsymbol{u}_0+\boldsymbol{U})$ holds if and only if (2.4.9) holds.

Then the resolvent set $\rho(\mathfrak{U})$ of the operator $\mathfrak{U}$, is given by the set

$$\rho(\mathcal{A}) = \rho(B_{\text{micro}}) \cap \{\ s \mid \text{all eigenvalues of } \beta(s) \text{ belong to } \rho(B_{\text{macro}})\ \}, \tag{2.4.10}$$

where $\rho(B_{\text{micro}})$ denotes the resolvent set of the operator generated by the form B_{micro} in the closure[4] *of $\widetilde{V}$ in $[L^2(Q)]^2$ and $\rho(B_{\text{macro}})$ is the resolvent set of the operator generated by the form B_{macro}.*

Proof. Suppose that s belongs to the set on the right-hand side of (2.4.10). Consider the problem

$$\begin{cases} B_{\text{macro}}(\boldsymbol{u}_0,\boldsymbol{\varphi}_0) - s\langle \boldsymbol{u}_0 + \langle\boldsymbol{U}\rangle,\boldsymbol{\varphi}_0\rangle_{L^2(\Omega)} &= \langle \boldsymbol{f},\boldsymbol{\varphi}_0\rangle_{L^2(\Omega)},\\ B_{\text{micro}}(\boldsymbol{U},\boldsymbol{\Phi}) \quad - s\langle \boldsymbol{u}_0+\boldsymbol{U},\boldsymbol{\Phi}\rangle_{L^2(Q,\mathrm{d}\mu)} &= \langle \boldsymbol{f},\boldsymbol{\Phi}\rangle_{L^2(Q,\mathrm{d}\mu)}. \end{cases} \tag{2.4.11}$$

Then this problem has a solution for every $\boldsymbol{f}\in\mathfrak{V}$. Indeed, Since $s\notin\sigma(B_{\text{micro}})$, it follows that $\boldsymbol{U} = b\,(s\boldsymbol{u}_0+\boldsymbol{f})$ solves the second equation in (2.4.11) where b is the matrix given by equation (2.4.6). Substituting this into the first equation given in system (2.4.11) yields

$$B_{\text{macro}}(\boldsymbol{u}_0,\boldsymbol{\varphi}_0) - \langle\beta(s)\boldsymbol{u}_0,\boldsymbol{\varphi}_0\rangle_{L^2(\Omega)} = \langle (I+s\langle b\rangle)\boldsymbol{f},\boldsymbol{\varphi}_0\rangle_{L^2(\Omega)}, \tag{2.4.12}$$

Since all the eigenvalues of $\beta(s)$ belong to $\rho(B_{\text{macro}})$, the operator induced by the bilinear form on the left-hand side is invertible and thus (2.4.12) is uniquely solvable.

[4] Note that the domain of this operator is dense in this closure

To prove the converse, note that $\rho(\mathfrak{U}) \subset \rho(B_{\mathrm{macro}})$ and assume that $s \in \rho(\mathfrak{U})$. In this case, $\beta(s)$ has no eigenvalues in $\sigma(B_{\mathrm{micro}})$ for otherwise problem (2.4.11) would not be uniquely solvable for every $\boldsymbol{f} \in \mathfrak{V}$. $\square$

It is noted (see [87]) that all points of nontrivial spectrum for the periodic problem induced by the bilinear form B_{micro} are at those points s where the matrix $\beta(s)$ is singular. The trivial spectral points are $\omega = 0$ which corresponds to constant eigenfunctions and those $\omega' \in \sigma(B_{\mathrm{micro}})$ such that the corresponding eigenfunctions have zero average. Let $\{\omega_n'\}_{n\in\mathbb{N}}$ be the set of such eigenvalues.

It is noted that when the periodic framework F_1^h is the model framework from Figure 2.2, then the matrix β is proportional to the identity matrix I. Indeed, if the set $F_1 \cap Q$ is invariant with respect to a rotation R, i.e. $F_1 \cap Q = \{\, R\boldsymbol{y} \mid \boldsymbol{y} \in F_1 \cap Q \,\}$, then for the eigenfunction $\boldsymbol{\phi}$ of the bilinear form B_{micro}, the vector $R\boldsymbol{\phi}$ is an eigenvector with the same eigenvalue and hence, in view of the definition of the matrix β, $R\beta(s)R^{-1} = \beta(s)$. Taking R to be a rotation through $\pi/2$ yields the claim that $\beta(s) = b(s)I$ for some scalar function b.

Let $\{\gamma_n\}_{n\in\mathbb{N}}$ denote the increasing sequence of zeros of the function b and let $\{\delta_n\}_{n\in\mathbb{N}}$ denote the increasing sequence of eigenvalues in the set $\{\omega_n\}_{n\in\mathbb{N}}$ which counts multiple eigenvalues only once. Hence, the spectrum of the limit operator $\mathfrak{U}$ has the "band" form:

$$\sigma(\mathfrak{U}) = \left(\bigcup_{n\in\mathbb{N}} \{ s \in (\gamma_n, \delta_n) \mid b(s) \in \sigma(B_{\mathrm{macro}}) \} \right) \cup \{\delta_n\}_{n\in\mathbb{N}} \cup \{\omega_n'\}_{n\in\mathbb{N}},$$

The intervals (δ_n, γ_{n+1}), $n \in \mathbb{N}$ are "gaps" in the spectrum, which do not have common points with $\sigma(\mathfrak{U})$, except, possibly, for elements of the set $\{\omega_n'\}_{n\in\mathbb{N}}$.

2.4.2 Proof of Spectral Convergence

It will be shown that the spectra of the operators associated with the original problem (2.1.2) converges in the sense of Hausdorff to the spectra of the limit problem (2.3.1).

Definition 2.4.1 (Hausdorff Convergence)**.** A sequence of sets $\mathcal{X}_\varepsilon \subset \mathbb{R}$, $\varepsilon > 0$ converge in the sense of Hausdorff to $\mathcal{X} \subset \mathbb{R}$ if the following two statements hold:

(H1) for $\omega \in \mathcal{X}$, there exists $\omega_\varepsilon \in \mathcal{X}_\varepsilon$ such that $\omega_\varepsilon \to \omega$,

(H2) for $\omega_\varepsilon \in \mathcal{X}_\varepsilon$ such that $\omega_\varepsilon \to \omega \in \mathbb{R}$, it follows that $\omega \in \mathcal{X}$.

Definition 2.4.2 (Strong Two-scale Resolvent Convergence)**.** The family of operators $\mathcal{A}_\varepsilon$ in $[L^2(\Omega, \mathrm{d}\mu_\varepsilon^h)]^2$ is said to be strongly two-scale resolvent convergent as $\varepsilon \to 0$ to the operator $\mathcal{A}$ in $[L^2(\Omega \times Q, \mathrm{d}\boldsymbol{x} \times \mathrm{d}\mu)]^2$, denoted $\mathcal{A}_\varepsilon \xrightarrow{2} \mathcal{A}$, if for all $\boldsymbol{f}$ in the range $R(\mathcal{A})$ of the operator $\mathcal{A}$ and for all sequences $\boldsymbol{f}_\varepsilon^h \in [L^2(\Omega, \mathrm{d}\mu_\varepsilon^h)]^2$ such that $\boldsymbol{f}_\varepsilon^h \overset{2}{\to} \boldsymbol{f}$, the two- scale convergence $(\mathcal{A}_\varepsilon + I)^{-1}\boldsymbol{f}_\varepsilon^h \overset{2}{\to} (\mathcal{A}+I)^{-1}\boldsymbol{f}$ holds.

The following argument shows that given the strong two-scale resolvent convergence of operators, then property (H1) holds.

Let $T_\varepsilon = (\mathcal{A}_\varepsilon + I)^{-1}$ and let $T = (\mathcal{A}+I)^{-1}$. If $s \in \sigma(\mathcal{A})$ then $t = (1+s)^{-1} \in \sigma(T)$. Therefore, for any $\delta > 0$, there exists a vector $\boldsymbol{f} \in R(\mathcal{A})$ such that

$$\|\boldsymbol{f}\|_{L^2(\Omega\times Q, \mathrm{d}\boldsymbol{x}\times\mathrm{d}\mu)} = 1, \quad \|(T-t)\boldsymbol{f}\|_{L^2(\Omega\times Q, \mathrm{d}\boldsymbol{x}\times\mathrm{d}\mu)} \leq \frac{\delta}{4}.$$

Consider a sequence $\boldsymbol{f}_\varepsilon^h \in [L^2(\Omega, \mathrm{d}\mu_\varepsilon^h)]^2$ such that $\boldsymbol{f}_\varepsilon^h \overset{2}{\to} \boldsymbol{f}$. Hence, using the definition of strong two-scale resolvent convergence, it follows that

$$\lim_{\varepsilon\to 0} \|(T_\varepsilon - t)\boldsymbol{f}_\varepsilon^h\|_{L^2(\Omega, \mathrm{d}\mu_\varepsilon^h)} = \|(T-t)\boldsymbol{f}\|_{L^2(\Omega\times Q, \mathrm{d}\boldsymbol{x}\times\mathrm{d}\mu)} \leq \frac{\delta}{4}.$$

Hence, $\|(T_\varepsilon - t)\boldsymbol{f}_\varepsilon^h\|_{L^2(\Omega, \mathrm{d}\mu_\varepsilon^h)} \leq \frac{\delta}{2}$ and $\|\boldsymbol{f}_\varepsilon^h\|_{L^2(\Omega, \mathrm{d}\mu_\varepsilon^h)} \geq \frac{1}{2}$ for sufficiently small ε. Therefore, the interval $(-\delta + t, \delta + t)$ contains a point of the spectrum of the operator T_ε. Moreover, every interval centered at s contains a point of the spectrum of the operator $\mathcal{A}_\varepsilon$ for small enough ε. Hence the following result is proven:

Proposition 2.4.1. *If $\mathcal{A}_\varepsilon \xrightarrow{2} \mathcal{A}$, then property (H1) holds with $\mathcal{X}_\varepsilon = \sigma(\mathcal{A}_\varepsilon)$ and $\mathcal{X} = \sigma(\mathcal{A})$.*

Corollary 2.4.1. *For the operators $\mathfrak{U}_\varepsilon^h$ defined by the identity*

$$B_\varepsilon^h(\boldsymbol{u}, \boldsymbol{v}) = L_\varepsilon^h(\boldsymbol{v}),$$

where B_ε^h and L_ε^h are defined in Theorem 2.1.1, $\boldsymbol{f} = \mathfrak{U}_\varepsilon^h\boldsymbol{u}$ and the operator $\mathfrak{U}$ is defined in Proposition 2.4.1, the property (H1) holds with $\mathcal{X}_\varepsilon = \sigma(\mathfrak{U}_\varepsilon^h)$, $\mathcal{X} = \sigma(\mathfrak{U})$ and $h = h(\varepsilon)$.

The property (H2) of the Hausdorff convergence does not hold for the spectra $\sigma(\mathfrak{U}_\varepsilon^h)$ in general. This is due to the fact that the soft component may have non-empty intersections with the boundary of Ω. Additionally, sequences of eigenfunctions of $\sigma(\mathfrak{U}_\varepsilon^h)$ may converge to the $\varkappa$-quasiperiodic eigenfunctions, $\varkappa \in [0, 2\pi)^2$ of the "Bloch spectrum" associated with the bilinear form (2.4.2). However, a suitable version of (H2) holds for a modified operator family where the corresponding elements of the soft component are replaced by the stiff material. To be precise, for each ε, h, denote by $\widehat{\mathfrak{U}}_\varepsilon^h$ the operator defined in the same way as $\mathfrak{U}_\varepsilon^h$ but with $\Omega_0^{\varepsilon,h}$ and $\Omega_1^{\varepsilon,h}$ in (2.1.2) replaced by $\widehat{\Omega}_0^{h,\varepsilon}$ and $\Omega\backslash\widehat{\Omega}_0^{h,\varepsilon}$ respectively. Here $\widehat{\Omega}_0^{h,\varepsilon}$ is the union over all $\boldsymbol{k} \in \mathbb{Z}^2$ of sets $\varepsilon(Q \cap F_0^h + \boldsymbol{k})$ such that $\varepsilon(Q + \boldsymbol{k}) \subset \Omega$.

Theorem 2.4.1. *Suppose that for all ε, h the vector fields $\boldsymbol{u}_\varepsilon^h \in [H_0^1(\Omega)]^2$ are eigenfunctions of $\widehat{\mathfrak{U}}_\varepsilon^h$ with unit L^2-norm:*

$$\widehat{\mathfrak{U}}_\varepsilon^h \boldsymbol{u}_\varepsilon^h = \omega_\varepsilon \boldsymbol{u}_\varepsilon^h, \quad \|\boldsymbol{u}_\varepsilon^h\|_{L^2(\Omega, \mathrm{d}\mu_\varepsilon^h)} = 1. \tag{2.4.13}$$

If $\omega_\varepsilon \to \omega \notin \sigma(B_{\rm micro})$, then the eigenfunction sequence $\{\boldsymbol{u}_\varepsilon^h\}$ is compact with respect to strong two-scale convergence in $[L^2(\Omega, \mathrm{d}\mu_\varepsilon^h)]^2$.

Proof. The eigenvalue problem (2.4.13) is understood in the sense that the identity

$$\int_{\Omega\backslash\widehat{\Omega}_0^{h,\varepsilon}} A_1 e(\boldsymbol{u}_\varepsilon^h)\cdot e(\boldsymbol{\varphi})\,\mathrm{d}\mu_\varepsilon^h + \varepsilon^2\int_{\widehat{\Omega}_0^{h,\varepsilon}} A_0 e(\boldsymbol{u}_\varepsilon^h)\cdot e(\boldsymbol{\varphi})\,\mathrm{d}\mu_\varepsilon^h = \omega_\varepsilon\int_\Omega \boldsymbol{u}_\varepsilon^h\cdot\boldsymbol{\varphi}\,\mathrm{d}\mu_\varepsilon^h, \quad \forall\boldsymbol{\varphi}\in[H_0^1(\Omega)]^2, \tag{2.4.14}$$

holds and in particular that

$$\int_{\Omega\backslash\widehat{\Omega}_0^{h,\varepsilon}} A_1 e(\boldsymbol{u}_\varepsilon^h)\cdot e(\boldsymbol{u}_\varepsilon^h)\,\mathrm{d}\mu_\varepsilon^h + \varepsilon^2\int_{\widehat{\Omega}_0^{h,\varepsilon}} A_0 e(\boldsymbol{u}_\varepsilon^h)\cdot e(\boldsymbol{u}_\varepsilon^h)\,\mathrm{d}\mu_\varepsilon^h = \omega_\varepsilon.$$

Hence, $\|e(\boldsymbol{u}_\varepsilon^h)\|_{L^2(\widehat{\Omega}_1^{h,\varepsilon},\mathrm{d}\mu_\varepsilon^h)}$ are uniformly bounded where $\widehat{\Omega}_1^{h,\varepsilon}$ denotes the union over all $\boldsymbol{k}\in\mathbb{Z}^2$ of sets $\varepsilon(Q\cap F_1^h+\boldsymbol{k})$ such that $\varepsilon(Q+\boldsymbol{k})\subset\Omega$. The claim is that for all ε and h there exists $\widetilde{\boldsymbol{u}}_\varepsilon^h$ such that

$$e(\boldsymbol{u}_\varepsilon^h) = e(\widetilde{\boldsymbol{u}}_\varepsilon^h) \ \text{ on } \ \widehat{\Omega}_1^{h,\varepsilon}, \quad \widetilde{\boldsymbol{u}}_\varepsilon^h\in[H_0^1(\Omega)]^2, \quad \|e(\widetilde{\boldsymbol{u}}_\varepsilon^h)\|_{L^2(\widehat{\Omega}_0^{h,\varepsilon},\mathrm{d}\mu_\varepsilon^h)} \le \|e(\boldsymbol{u}_\varepsilon^h)\|_{L^2(\widehat{\Omega}_1^{h,\varepsilon},\mathrm{d}\mu_\varepsilon^h)}, \tag{2.4.15}$$

$$\int_{\widehat{\Omega}_0^{h,\varepsilon}} A_0 e(\widetilde{\boldsymbol{u}}_\varepsilon^h)\cdot e(\boldsymbol{\varphi})\,\mathrm{d}\mu_\varepsilon^h = 0, \quad \forall\boldsymbol{\varphi}\in[H_0^1(\Omega)]^2 \text{ such that } e(\boldsymbol{\varphi})=0 \text{ in } \widehat{\Omega}_1^{h,\varepsilon}, \tag{2.4.16}$$

where the constant $C>0$ is independent of $\varepsilon,\ h$. To prove this claim, consider $\widetilde{\boldsymbol{u}}_\varepsilon^h$ such that $\boldsymbol{z}_\varepsilon^h := \boldsymbol{u}_\varepsilon^h - \widetilde{\boldsymbol{u}}_\varepsilon^h$ solves the minimisation problem

$$\min\left\{\frac{1}{2}\int_{\widehat{\Omega}_0^{h,\varepsilon}} A_0 e(\boldsymbol{v})\cdot e(\boldsymbol{v})\,\mathrm{d}\mu_\varepsilon^h - \int_{\widehat{\Omega}_0^{h,\varepsilon}} A_0 e(\boldsymbol{u}_\varepsilon^h)\cdot e(\boldsymbol{v})\,\mathrm{d}\mu_\varepsilon^h\right\}, \tag{2.4.17}$$

where the minimisation is taken over all functions $\boldsymbol{v}\in[H_0^1(\Omega)]^2$ whose restriction to $\widehat{\Omega}_1^{h,\varepsilon}$ is a rigid-body motion with respect to the Lebesgue measure, i.e., $e(\boldsymbol{v})=0$ in $\widehat{\Omega}_1^{h,\varepsilon}$. It is clear that $e(\boldsymbol{z}_\varepsilon^h)=0$ in $\widehat{\Omega}_1^{h,\varepsilon}$ and moreover

$$\int_{\Omega\backslash(\widehat{\Omega}_0^{h,\varepsilon}\cup\widehat{\Omega}_1^{h,\varepsilon})} A_1 e(\boldsymbol{u}_\varepsilon^h)\cdot e(\boldsymbol{\varphi})\,\mathrm{d}\mu_\varepsilon^h + \int_{\widehat{\Omega}_1^{h,\varepsilon}} A_1 e(\boldsymbol{z}_\varepsilon^h)\cdot e(\boldsymbol{\varphi})\,\mathrm{d}\mu_\varepsilon^h + \varepsilon^2\int_{\widehat{\Omega}_0^{h,\varepsilon}} A_0 e(\boldsymbol{z}_\varepsilon^h)\cdot e(\boldsymbol{\varphi})\,\mathrm{d}\mu_\varepsilon^h -$$
$$-\,\omega_\varepsilon\int_\Omega \boldsymbol{z}_\varepsilon^h\cdot\boldsymbol{\varphi}\,\mathrm{d}\mu_\varepsilon^h = \omega_\varepsilon\int_\Omega \widetilde{\boldsymbol{u}}_\varepsilon^h\cdot\boldsymbol{\varphi}\,\mathrm{d}\mu_\varepsilon^h, \qquad \forall\boldsymbol{\varphi}\in[H_0^1(\Omega)]^2, \ \ e(\boldsymbol{\varphi})=0 \text{ in } \widehat{\Omega}_1^{h,\varepsilon}. \tag{2.4.18}$$

The last equality is found by combining (2.4.14), (2.4.16) and the Euler-Lagrange equation for (2.4.17). Using the bound (2.4.15), it follows that $\widetilde{\boldsymbol{u}}_\varepsilon^h$ is compact with respect to strong convergence in $[L^2(\Omega,\mathrm{d}\mu_\varepsilon^h)]^2$, i.e., there exists $\widetilde{\boldsymbol{u}}=\widetilde{\boldsymbol{u}}(\boldsymbol{x})$ such that, up to picking a suitable subsequence, $\widetilde{\boldsymbol{u}}_\varepsilon^h \longrightarrow \widetilde{\boldsymbol{u}}$ in $[L^2(\Omega,\mathrm{d}\mu_\varepsilon^h)]^2$.

Lemma 2.4.2. *Suppose that for each $\varepsilon,\ h$ the function $\boldsymbol{f}_\varepsilon^h$ belongs to the closure in $[L^2(\Omega)]^2$ of the set of smooth functions whose restriction to $\widehat{\Omega}_1^{h,\varepsilon}$ are rigid-body motions with respect to the Lebesgue measure. Suppose further that $\boldsymbol{f}_\varepsilon^h \stackrel{2}{\rightarrow} \boldsymbol{f}\in\mathfrak{V}$ (see Lemma 2.4.1).*

For all ε and h consider the vector field $\boldsymbol{v}_\varepsilon^h \in [H_0^1(\Omega)]^2$ such that $e(\boldsymbol{v}_\varepsilon^h) = 0$ in $\widehat{\Omega}_1^{h,\varepsilon}$ and the following resolvent identity holds (see (2.4.18)):

$$\int_{\Omega\setminus(\widehat{\Omega}_0^{h,\varepsilon}\cup\widehat{\Omega}_1^{h,\varepsilon})} A_1 e(\boldsymbol{u}_\varepsilon^h)\cdot e(\boldsymbol{\varphi})\,\mathrm{d}\mu_\varepsilon^h + \int_{\widehat{\Omega}_1^{h,\varepsilon}} A_1 e(\boldsymbol{v}_\varepsilon^h)\cdot e(\boldsymbol{\varphi})\,\mathrm{d}\mu_\varepsilon^h + \varepsilon^2 \int_{\widehat{\Omega}_0^{h,\varepsilon}} A_0 e(\boldsymbol{v}_\varepsilon^h)\cdot e(\boldsymbol{\varphi})\,\mathrm{d}\mu_\varepsilon^h -$$
$$- \omega_\varepsilon \int_\Omega \boldsymbol{v}_\varepsilon^h\cdot\boldsymbol{\varphi}\,\mathrm{d}\mu_\varepsilon^h = \int_\Omega \boldsymbol{f}_\varepsilon^h\cdot\boldsymbol{\varphi}\,\mathrm{d}\mu_\varepsilon^h, \qquad \forall\boldsymbol{\varphi}\in[H_0^1(\Omega)]^2,\ \ e(\boldsymbol{\varphi}) = 0 \text{ in } \widehat{\Omega}_1^{h,\varepsilon}. \quad (2.4.19)$$

Then $\boldsymbol{v}_\varepsilon^h \stackrel{2}{\rightharpoonup} \boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x},\boldsymbol{y}) \in [L^2(\Omega,\widetilde{V})]^2$ and

$$\frac{\theta^2}{6}\int_\Omega\int_Q K_1\boldsymbol{\chi}''\cdot\boldsymbol{\Phi}''\,\mathrm{d}\lambda(\boldsymbol{y})\mathrm{d}\boldsymbol{x} + \frac{1}{2}\int_\Omega\int_Q A_0 e_y(\boldsymbol{v})\cdot e_y(\boldsymbol{\varphi})\,\mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{x} - \omega\int_\Omega\int_Q \boldsymbol{v}\cdot\boldsymbol{\varphi}\,\mathrm{d}\mu(\boldsymbol{y})\mathrm{d}\boldsymbol{x} =$$
$$= \int_\Omega\int_Q \boldsymbol{f}\cdot\boldsymbol{\varphi}\,\mathrm{d}\mu(\boldsymbol{y})\mathrm{d}\boldsymbol{x}, \quad \forall\boldsymbol{\varphi}\in[L^2(\Omega,\widetilde{V})]^2, \quad \boldsymbol{\varphi}(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{\Phi}(\boldsymbol{x},\boldsymbol{y}) \text{ a.e. } \boldsymbol{x}\in\Omega,\ \lambda\text{-a.e. } \boldsymbol{y}\in\partial Q,$$
$$(2.4.20)$$

where $\boldsymbol{\chi}(\boldsymbol{x},\cdot)$ is there trace of $\boldsymbol{v}(\boldsymbol{x},\cdot)$ on $F_1\cap Q$ for a.e. $\boldsymbol{x}\in\Omega$.

Proof. It is first shown that the spectra of the operators $\widehat{\mathfrak{A}}_\varepsilon^0$ defined via the bilinear forms (cf. 2.4.19)

$$\widehat{b}_\varepsilon^0(\boldsymbol{v},\boldsymbol{\varphi}) = \int_{\Omega\setminus(\widehat{\Omega}_0^{h,\varepsilon}\cup\widehat{\Omega}_1^{h,\varepsilon})} A_1 e(\boldsymbol{u}_\varepsilon^h)\cdot e(\boldsymbol{\varphi})\,\mathrm{d}\mu_\varepsilon^h + \int_{\widehat{\Omega}_1^{h,\varepsilon}} A_1 e(\boldsymbol{v}_\varepsilon^h)\cdot e(\boldsymbol{\varphi})\,\mathrm{d}\mu_\varepsilon^h +$$
$$+ \varepsilon^2\int_{\widehat{\Omega}_0^{h,\varepsilon}} A_0 e(\boldsymbol{v}_\varepsilon^h)\cdot e(\boldsymbol{\varphi})\,\mathrm{d}\mu_\varepsilon^h, \quad \boldsymbol{v},\boldsymbol{\varphi}\in H_0^1(\Omega)^2, \quad e(\boldsymbol{v}), e(\boldsymbol{\varphi}) = 0 \text{ in } \widehat{\Omega}_1^{h,\varepsilon},$$

converges in the sense of Hausdorff as $\varepsilon\to 0$ to $\sigma(B_{\text{micro}})$. It is indeed the case that $\widehat{\mathfrak{A}}_\varepsilon^0 \xrightarrow{2} \widehat{\mathfrak{A}}^0$ where the operator $\widehat{\mathfrak{A}}^0$ is associated with the bilinear form

$$\widehat{b}^0(\boldsymbol{v},\boldsymbol{\varphi}) = \frac{\theta^2}{6}\int_Q K_1\boldsymbol{\chi}''\cdot\boldsymbol{\Phi}''\,\mathrm{d}\lambda + \int_{Q\cap F_0} A_0 e(\boldsymbol{v})\cdot e(\boldsymbol{\varphi})\,\mathrm{d}\mu, \qquad \boldsymbol{v},\boldsymbol{\varphi}\in[L^2(\Omega,\widetilde{V})]^2,$$
$$\boldsymbol{v}(\boldsymbol{y}) = \boldsymbol{\chi}(\boldsymbol{y}), \qquad \boldsymbol{\varphi}(\boldsymbol{y}) = \boldsymbol{\Phi}(\boldsymbol{y}), \qquad \lambda\text{-a.e. } \boldsymbol{y}\in\partial Q.$$

Hence $\sigma(\widehat{\mathfrak{A}}^0) \subset \lim_{\varepsilon\to 0}\sigma(\widehat{\mathfrak{A}}_\varepsilon^0)$ by Proposition 2.4.1. On the other hand any sequence of L^2-normalised eigenfunctions of $\widehat{\mathfrak{A}}_\varepsilon^0$ whose eigenvalues ω_ε^0 converge to $\omega^0\in\mathbb{R}$ is compact in the sense of two-scale convergence (see [82, Theorem 12.2]) and therefore $\omega\in\sigma(\widehat{\mathfrak{A}}^0)$. Thus $\sigma(\widehat{\mathfrak{A}}^0) = \sigma(B_{\text{micro}})$.

Whenever ω_ε in (2.4.19) converge to a point outside $\sigma(B_{\text{micro}})$, the identity (2.4.19) does not have non-zero solutions $\boldsymbol{v}_\varepsilon^h$ for $\boldsymbol{f}_\varepsilon^h = \boldsymbol{0}$ and ω_ε can be replaced by any value in some finite neighbourhood of the set $\{\omega_\varepsilon\}_{\varepsilon<\varepsilon_0}$ for some $\varepsilon_0 > 0$. Hence for an L^2-bounded sequence of right-hand sides $\boldsymbol{f}_\varepsilon^h$, the function $\boldsymbol{v}_\varepsilon^h$ that satisfy (2.4.19) are uniformly bounded in $[L^2(\Omega,\mathrm{d}\mu_\varepsilon^h)]^2$ for $\varepsilon<\varepsilon_0$.

Moreover, setting $\boldsymbol{\varphi} = \boldsymbol{v}_\varepsilon^h$ in (2.4.19) and noting that A_0 is positive definite yields the uniform estimate

$$\varepsilon\|\chi_0^{h,\varepsilon} e(\boldsymbol{v}_\varepsilon^h)\|_{L^2(\Omega_0^{\varepsilon,h},\mathrm{d}\mu_\varepsilon^h)} \le C,$$

for some positive constant C. Making use of the results established in Section 2.2.1 and noting that $\widehat{\Omega}_0^{h,\varepsilon} \cup \widehat{\Omega}_1^{h,\varepsilon} \to \Omega$ as $\varepsilon \to 0$, a subsequence is extracted from $\boldsymbol{v}_\varepsilon^h$ which weakly two-scale converges to $\boldsymbol{v} \in [L^2(\Omega, \widetilde{V})]^2$ and such that $\chi_0^{h,\varepsilon} e(\boldsymbol{v}_\varepsilon^h) \xrightharpoonup{2} e_y(\boldsymbol{v})$ in $[L^2(\Omega, \mathrm{d}\mu_\varepsilon^h)]^2$.

Finally, in the limit as $\varepsilon \to 0$ in identity (2.4.19), equation (2.4.20) is obtained. By uniqueness, the whole sequence $\boldsymbol{v}_\varepsilon^h$ weakly two-scale converges to $\boldsymbol{v}$. □

The preceding lemma implies that the sequence $\boldsymbol{z}_\varepsilon^h$ is compact with respect to the weak two-scale convergence and its two-scale limit $\boldsymbol{z} = \boldsymbol{z}(\boldsymbol{x}, \boldsymbol{y})$ is a rigid-body motion on F_1 and satisfies the weak problem

$$\frac{\theta^2}{6} \int_\Omega \int_Q K_1 \boldsymbol{q}'' \cdot \boldsymbol{\Phi}'' \, \mathrm{d}\lambda(\boldsymbol{y}) \mathrm{d}\boldsymbol{x} + \frac{1}{2} \int_\Omega \int_Q A_0 e_y(\boldsymbol{z}) \cdot e_y(\boldsymbol{\varphi}) \, \mathrm{d}\boldsymbol{y} \mathrm{d}\boldsymbol{x} - \omega \int_\Omega \int_Q \boldsymbol{z} \cdot \boldsymbol{\varphi} \, \mathrm{d}\mu(\boldsymbol{y}) \mathrm{d}\boldsymbol{x} =$$
$$= \omega \int_\Omega \int_Q \widetilde{\boldsymbol{u}} \cdot \boldsymbol{\varphi} \, \mathrm{d}\mu(\boldsymbol{y}) \mathrm{d}\boldsymbol{x},$$
$$\forall \boldsymbol{\varphi} \in [L^2(\Omega, \widetilde{V})]^2, \quad \boldsymbol{z}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{q}(\boldsymbol{x}, \boldsymbol{y}), \quad \boldsymbol{\varphi}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{\Phi}(\boldsymbol{x}, \boldsymbol{y}) \ \text{ a.e. } \boldsymbol{x} \in \Omega, \ \ \lambda\text{-a.e. } \boldsymbol{y} \in \partial Q, \tag{2.4.21}$$

Setting $\boldsymbol{\varphi} = \boldsymbol{v}_\varepsilon^h$ in identity (2.4.18) and $\boldsymbol{\varphi} = \boldsymbol{z}_\varepsilon^h$ in (2.4.19) yields

$$\int_\Omega \boldsymbol{z}_\varepsilon^h \cdot \boldsymbol{f}_\varepsilon^h \, \mathrm{d}\mu_\varepsilon^h = \omega_\varepsilon \int_\Omega \boldsymbol{v}_\varepsilon^h \cdot \widetilde{\boldsymbol{u}}_\varepsilon^h \, \mathrm{d}\mu_\varepsilon^h, \quad \forall \varepsilon, h. \tag{2.4.22}$$

Taking the limit of both sides of (2.4.22) and using the two-scale convergence properties of $\boldsymbol{v}_\varepsilon^h$ and $\widetilde{\boldsymbol{u}}_\varepsilon^h$ yields

$$\lim_{\varepsilon \to 0} \int_\Omega \boldsymbol{z}_\varepsilon^h \cdot \boldsymbol{f}_\varepsilon^h \, \mathrm{d}\mu_\varepsilon^h = \omega \int_\Omega \int_Q \boldsymbol{v}(\boldsymbol{x}, \boldsymbol{y}) \cdot \widetilde{\boldsymbol{u}}(\boldsymbol{x}) \, \mathrm{d}\mu(\boldsymbol{y}) \mathrm{d}\boldsymbol{x}. \tag{2.4.23}$$

Furthermore, by equation (2.4.20) with $\boldsymbol{\varphi} = \boldsymbol{z}$ and equation (2.4.21) with $\boldsymbol{\varphi} = \boldsymbol{v}$ yields

$$\omega \int_\Omega \int_Q \boldsymbol{v}(\boldsymbol{x}, \boldsymbol{y}) \cdot \widetilde{\boldsymbol{u}}(\boldsymbol{x}) \, \mathrm{d}\mu(\boldsymbol{y}) \mathrm{d}\boldsymbol{x} = \int_\Omega \int_Q \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}) \cdot \boldsymbol{z}(\boldsymbol{x}, \boldsymbol{y}) \, \mathrm{d}\mu(\boldsymbol{y}) \mathrm{d}\boldsymbol{x}. \tag{2.4.24}$$

Finally, setting $\boldsymbol{f}_\varepsilon^h = \boldsymbol{z}_\varepsilon^h$ in (2.4.23) and using (2.4.24), it is seen that the convergence $\|\boldsymbol{z}_\varepsilon^h\|_{L^2(\Omega, \mathrm{d}\mu_\varepsilon^h)} \to \|\boldsymbol{z}\|_{L^2(\Omega \times Q, \mathrm{d}\boldsymbol{x} \times \mathrm{d}\mu)}$. Therefore the sequence $\boldsymbol{z}_\varepsilon^h$ strongly two-scale converges to $\boldsymbol{z}$. □

Appendices

2.A Technical Lemmas

The following appendix details several technical lemmas which are required to prove a variety of results. In particular, the results presented pertain to proving the two-scale convergence 2.3.12 and that $\boldsymbol{\chi} \in [L^2(\Omega, \widehat{\mathfrak{R}}^0)]^2$. All these results are found in Zhikov & Pastukhova [84] and have only been modified as far as the notation they have been presented in this thesis.

Let I be a line segment in the periodic network F_1 which may contain a node. Assume without loss of generality that this segment lies completely within a periodic cell Q. The following lemma is restricted to the case when I lies on the y_1-axis.

Lemma 2.A.1. *Define $J(\varepsilon)$ to be the following functional*

$$J(\varepsilon) := \frac{h}{\varepsilon^2}\int_\Omega \boldsymbol{u}^h_\varepsilon(\boldsymbol{x}) \cdot \boldsymbol{b}\Big(\frac{\boldsymbol{x}}{\varepsilon}\Big)\alpha\Big(\frac{x_1}{\varepsilon}\Big)\varphi(\boldsymbol{x})\,\mathrm{d}\mu^h_\varepsilon = \frac{h}{\varepsilon^2}\int_\Omega (u^h_\varepsilon)_1(\boldsymbol{x})\beta_0\Big(\frac{x_2}{\varepsilon}\Big)\alpha\Big(\frac{x_1}{\varepsilon}\Big)\varphi(\boldsymbol{x})\,\mathrm{d}\mu^h_\varepsilon, \tag{2.A.1}$$

where $\boldsymbol{b}(\boldsymbol{y}) = (\beta_0(y_2), 0)$, $\alpha(y_1) \in C^\infty_0(I)$ is extended periodically into $\mathbb{R}^2$, $\varphi \in C^\infty_0(\Omega)$ and

$$\beta_0(t) = \begin{cases} \dfrac{t}{h}, & |t| \leq h, \\ 0, & |t| > h. \end{cases}$$

Then

$$\lim_{\varepsilon\to 0} J(\varepsilon) = \frac{\theta^2}{3}\int_\Omega\int_{Q\cap I} u_2(\boldsymbol{x},\boldsymbol{y})\alpha'(y_1)\varphi(\boldsymbol{x})\,\mathrm{d}\lambda\mathrm{d}\boldsymbol{x}.$$

The above result is extended to the case when the interval I in question is in an arbitrary direction $\boldsymbol{\tau}$. Let $\boldsymbol{\nu}$ be normal to I^h such that $\boldsymbol{\tau}$, $\boldsymbol{\nu}$ forms a positively orientated frame and where I^h is the strip of thickness $2h$ with mid line I. Consider the function β defined on Q given as

$$\beta(\boldsymbol{y}) = \begin{cases} \frac{1}{h}\boldsymbol{\nu}\cdot(\boldsymbol{y}-\boldsymbol{y}_0), & \text{on } I^h, \\ 0, & \text{on } Q\backslash I^h, \end{cases} \tag{2.A.2}$$

where $\boldsymbol{y}_0 \in I$ and the function is continued periodically onto $\mathbb{R}^2$. Let $\alpha(\boldsymbol{y})$ be a smooth function defined on I^h which depends only on the longitudinal component and vanishes in the neighbourhood of the ends of the interval. Hence

$$\lim_{\varepsilon\to 0}\frac{h}{\varepsilon^2}\int_\Omega \boldsymbol{u}^h_\varepsilon(\boldsymbol{x})\cdot\boldsymbol{\tau}\beta\Big(\frac{\boldsymbol{x}}{\varepsilon}\Big)\alpha\Big(\frac{\boldsymbol{x}}{\varepsilon}\Big)\varphi(\boldsymbol{x})\,\mathrm{d}\mu^h_\varepsilon=\frac{\theta^2}{3}\int_\Omega\int_{Q\cap I}\boldsymbol{u}(\boldsymbol{x},\boldsymbol{y})\cdot\boldsymbol{\nu}\frac{\partial\alpha}{\partial\boldsymbol{\tau}}(\boldsymbol{y})\varphi(\boldsymbol{x})\,\mathrm{d}\lambda\mathrm{d}\boldsymbol{x},$$

for all $\boldsymbol{\varphi}\in C^\infty_0(\Omega)$. In the case that the interval I lies on the y_2-axis, $\boldsymbol{\tau}=(0,1)$, $\boldsymbol{\nu}=(-1,0)$ and the above reduces to the result of Lemma 2.A.1.

The following results are needed to show that the limit function $\widehat{\boldsymbol{u}}$ satisfies the requirements of Definition 2.1.5 so that $\boldsymbol{\chi}(\boldsymbol{x},\cdot)\in\widehat{\mathfrak{R}}^0$.

Lemma 2.A.2. *On each interval I of the network F_1, the normal component $\widehat{\boldsymbol{u}}$, denoted $u^{(\nu)}$, is an element of $H^2(I)$.*

The result is proven by considering the limit of the following integral:

$$T(\varepsilon)=-\frac{h}{\varepsilon}\int_\Omega e(\boldsymbol{u}^h_\varepsilon)\cdot(\boldsymbol{\tau}\otimes\boldsymbol{\tau})\beta\Big(\frac{\boldsymbol{x}}{\varepsilon}\Big)\alpha\Big(\frac{\boldsymbol{x}}{\varepsilon}\Big)\varphi(\boldsymbol{x})\,\mathrm{d}\mu^h_\varepsilon.$$

Moreover, For an interval of general position, denote by w the weak two-scale limit of the sequence

$$w_\varepsilon:=-\frac{h}{\varepsilon}e(\boldsymbol{u}^h_\varepsilon)\cdot(\boldsymbol{\tau}\otimes\boldsymbol{\tau})\beta\Big(\frac{\boldsymbol{x}}{\varepsilon}\Big).$$

Consider the network F_1 and on each link fix a point $\boldsymbol{y}_0$, a direction $\boldsymbol{\tau}$, and a normal $\boldsymbol{\nu}$. Then, on F^h_1, the matrix $\sigma(\boldsymbol{y})$ is defined by setting

$$\sigma(\boldsymbol{y})=-(\boldsymbol{\tau}\otimes\boldsymbol{\tau})\beta(\boldsymbol{y}),\quad\text{where }\beta(\boldsymbol{y})=\frac{\boldsymbol{\nu}\cdot(\boldsymbol{y}-\boldsymbol{y}_0)}{h}\quad\text{on }I^h.$$

Lemma 2.A.3. *The following two-scale convergence holds:*

$$\frac{h}{\varepsilon}\chi^{h,\varepsilon}_1 e(\boldsymbol{u}^h_\varepsilon)\cdot\sigma(\boldsymbol{y})\overset{2}{\rightharpoonup}\frac{\theta^2}{3}\chi_1(\boldsymbol{y})(\boldsymbol{\chi}\cdot\boldsymbol{\nu})''. \tag{2.A.3}$$

The next results are used to show that properties 2) and 3) of Definition 2.1.5 are satisfied by $\boldsymbol{\chi}$. Without loss of generality, consider two orthogonal links denoted I_1 and I_2 aligned with the horizontal and vertical axes respectively and meeting at a node denoted $\mathcal{O}$ situated at the origin with each link of length 1/4. To avoid cumbersome notation, it will be understood that if $\boldsymbol{u}$ is on I_1 then $\boldsymbol{u}=\boldsymbol{u}(\boldsymbol{x},y_1)$ and similarly that if $\boldsymbol{u}$ is on I_2 then $\boldsymbol{u}=\boldsymbol{u}(\boldsymbol{x},y_2)$

Proposition 2.A.1. *Consider the linear functional $l(\gamma)$ defined as*

$$l(\gamma)=\frac{\theta^2}{3}\int_\Omega\int_{Q\cap I_1}u_2(\boldsymbol{x},\boldsymbol{y})\gamma''(y_1)\,\varphi(\boldsymbol{x})\,\mathrm{d}\lambda\mathrm{d}\boldsymbol{x}+\frac{\theta^2}{3}\int_\Omega\int_{Q\cap I_2}u_1(\boldsymbol{x},\boldsymbol{y})\gamma''(y_2)\,\varphi(\boldsymbol{x})\,\mathrm{d}\lambda\mathrm{d}\boldsymbol{x},$$

for a fixed function $\varphi\in C^\infty_0(\Omega)$. Then

$$|l|^2\le c(\varphi)\int_0^{\frac14}|\gamma|^2\,\mathrm{d}t.$$

Theorem 2.A.1. *Let F_1 be a connected singular network on the period torus. For $h(\varepsilon) \to 0$ in an arbitrary manner, consider the sequence v_ε^h satisfying*

$$\varepsilon^2 \int_\Omega |\nabla v_\varepsilon^h|^2 \, \mathrm{d}\mu_\varepsilon^h + \int_\Omega |v_\varepsilon^h|^2 \, \mathrm{d}\mu_\varepsilon^h \leq const < \infty. \tag{2.A.4}$$

Then, after a possible transition to a subsequence, it follows that

$$v_\varepsilon^h(\boldsymbol{x}) \stackrel{2}{\rightharpoonup} v(\boldsymbol{x}, \boldsymbol{y}), \quad v(\boldsymbol{x}, \cdot) \in H^1_{\text{per}}(Q, \mathrm{d}\lambda) \quad \text{a.e. } \boldsymbol{x} \in \Omega, \quad \varepsilon \nabla v_\varepsilon^h \stackrel{2}{\rightharpoonup} \nabla_y v(\boldsymbol{x}, \boldsymbol{y}). \tag{2.A.5}$$

In the application of this theorem, it is required that each component of the sequence vector $\boldsymbol{u}_\varepsilon^h$ satisfies inequality (2.A.4) where this inequality is a consequence of the results presented in [84, Section 5]. With this result, it is concluded that each component of $\boldsymbol{\chi}$ belongs to the Sobolev space $H^1_{\text{per}}(Q, \mathrm{d}\lambda)$ and moreover that $\boldsymbol{\chi}|_{\mathcal{O}} = \boldsymbol{0}$ for any node $\mathcal{O}$.

2.B Additional Results Pertaining to the Derivation of the Homogenised Equation

In this appendix, additional results will be presented for completeness of the work presented in this text.

In the scalar homogenisation [81], it is the property of ergodicity of the measure μ which is essential to deriving an approximation theorem. However, it is the representation (2.1.13) which plays the equivalent role in the vector homogenisation. For $\boldsymbol{a} \in [L^2(Q, \mathrm{d}\lambda)]^2$ and $b \in L^2(Q, \mathrm{d}\lambda)^3$, the expression $\boldsymbol{a} = \operatorname{div} b$ is understood in the following sense:

$$\int_Q \big(b \cdot e(\boldsymbol{u})\big) \, \mathrm{d}\lambda = - \int_Q (\boldsymbol{a} \cdot \boldsymbol{u}) \, \mathrm{d}\lambda, \quad \text{for all } \boldsymbol{u} \in [C^\infty_{\text{per}}(Q)]^2. \tag{2.B.1}$$

Theorem 2.B.1 (Approximation Lemma)**.** *Consider the set S of all vectors $\boldsymbol{a} \in [L^2(Q, \mathrm{d}\lambda)]^2$ which admit a representation $\boldsymbol{a} = \operatorname{div} b$ for some $b \in [L^2(Q, \mathrm{d}\lambda)]^3$. Then S is dense in $\mathcal{R}^\perp$.*

The natural extension (see Definition 2.1.7) is used in proving several results on the stiff component of the network. Several properties are preserved when considering the natural extension, in particular if $[b]^h \in L^2(Q, \mathrm{d}\lambda^h)$ then

$$\int_Q [b]^h \, \mathrm{d}\lambda^h = \int_Q b \, \mathrm{d}\lambda.$$

Moreover, it can be shown that if $[b]^h \rightharpoonup b$ then

$$\lim_{h \to 0} \int_Q |[b]^h|^2 \, \mathrm{d}\lambda^h = \int_Q |b|^2 \, \mathrm{d}\lambda,$$

and hence $[b]^h \longrightarrow b$, in $L^2(Q, \mathrm{d}\lambda^h)$.

Proposition 2.B.1. *Let $b \in V_{\text{sol}}^{\lambda}$ be a λ-solenoidal matrix. Then the natural extension $[b]^h$ is a λ^h-solenoidal matrix. Moreover, if $[\boldsymbol{a} \in L^2(Q, \mathrm{d}\lambda)]^2$ and $\boldsymbol{a} \in \mathcal{R}^{\perp}$, then the exists a symmetric matrix $b \in [L^2(Q, \mathrm{d}\lambda)]^3$ such that $\boldsymbol{a} = \operatorname{div} b$ and the natural extensions of b and $\boldsymbol{a}$ satisfy $[\boldsymbol{a}]^h = \operatorname{div}[b]^h$ in the sense of the measure λ^h:*

$$\int_Q [\boldsymbol{a}]^h \cdot \boldsymbol{\varphi} \, \mathrm{d}\lambda^h = -\int_Q [b]^h \cdot e(\boldsymbol{\varphi}) \, \mathrm{d}\lambda^h, \quad \forall \boldsymbol{\varphi} \in [C_{\text{per}}^{\infty}(Q)]^2. \tag{2.B.2}$$

It is also noted that a relation similar to (2.B.2) holds for test functions with compact support on $\mathbb{R}^2$. Indeed the following identity holds:

$$\int_{\mathbb{R}^2} [\boldsymbol{a}]^h \left(\frac{\boldsymbol{x}}{\varepsilon}\right) \cdot \boldsymbol{\varphi}(\boldsymbol{x}) \, \mathrm{d}\lambda_{\varepsilon}^h = -\varepsilon \int_{\mathbb{R}^2} [b]^h \left(\frac{\boldsymbol{x}}{\varepsilon}\right) \cdot e\left((\boldsymbol{\varphi}(\boldsymbol{x})\right) \mathrm{d}\lambda_{\varepsilon}^h, \quad \forall \boldsymbol{\varphi} \in [C_0^{\infty}(\mathbb{R}^2)]^2.$$

To obtain the above result, divide $\mathbb{R}^2$ up into squares of side length ε and denote them by εQ_j where $Q_j = [0,1)^2 + \boldsymbol{j}$, $\boldsymbol{j} \in \mathbb{Z}^2$. Then

$$\begin{aligned}
\varepsilon \int_{\mathbb{R}^2} [b]^h \left(\frac{\boldsymbol{x}}{\varepsilon}\right) \cdot e\left((\boldsymbol{\varphi}(\boldsymbol{x})\right) \mathrm{d}\lambda_{\varepsilon}^h &= \varepsilon \sum_{j=1}^{\infty} \int_{\varepsilon Q_j} [b]^h \left(\frac{\boldsymbol{x}}{\varepsilon}\right) \cdot e\left((\boldsymbol{\varphi}(\boldsymbol{x})\right) \mathrm{d}\lambda_{\varepsilon}^h \\
&= \varepsilon \sum_{j=1}^{\infty} \int_{\varepsilon Q} [b]^h \left(\frac{\boldsymbol{x} + \varepsilon \boldsymbol{j}}{\varepsilon}\right) \cdot e\left((\boldsymbol{\varphi}(\boldsymbol{x} + \varepsilon \boldsymbol{j})\right) \mathrm{d}\lambda_{\varepsilon}^h.
\end{aligned}$$

Note that $[b]^h$ is 1-periodic and denote $\tilde{\boldsymbol{\varphi}}(\boldsymbol{y}) = \sum_{j=1}^{\infty} \boldsymbol{\varphi}(\varepsilon(\boldsymbol{y} + \boldsymbol{j}))$. Hence

$$\varepsilon \sum_{j=1}^{\infty} \int_{\varepsilon Q} [b]^h \left(\frac{\boldsymbol{x} + \varepsilon \boldsymbol{j}}{\varepsilon}\right) \cdot e\left((\boldsymbol{\varphi}(\boldsymbol{x} + \varepsilon \boldsymbol{j})\right) \mathrm{d}\lambda_{\varepsilon}^h(\boldsymbol{x}) = \int_Q [b]^h(\boldsymbol{y}) \cdot e_y\left(\tilde{\boldsymbol{\varphi}}(\boldsymbol{y})\right) \mathrm{d}\lambda^h(\boldsymbol{y}).$$

The function $\tilde{\boldsymbol{\varphi}}$ is periodic and hence relation (2.B.2) can be used to yield

$$\int_Q [b]^h(\boldsymbol{y}) \cdot e_y\left(\tilde{\boldsymbol{\varphi}}(\boldsymbol{y})\right) \mathrm{d}\lambda^h(\boldsymbol{y}) = -\int_Q [\boldsymbol{a}]^h(\boldsymbol{y}) \cdot \tilde{\boldsymbol{\varphi}}(\boldsymbol{y}) \, \mathrm{d}\lambda^h(\boldsymbol{y}).$$

Reversing the procedure described above yields the result.

The following construction gives one possible way of extending solenoidal vectors defined on the singular structure to solenoidal vectors defined on the rod structure. This construction in particular makes clear what is done in a small neighbourhood of the nodes where "overlap" of the rods may be a factor.

It can be shown that for $\boldsymbol{b} \in [V_{\text{sol}}(Q, \mathrm{d}\lambda)]^2$, on each link $I_1, I_2, \ldots, I_m$ adjoining a node $\mathcal{O}$, there are constants $b_1, b_2, \ldots, b_m$ such that

$$b|_{I_j} = b_j \boldsymbol{\tau}_j, \quad \sum_{j=1}^{m} b_j = 0,$$

where $\boldsymbol{\tau}_j$ are unit vectors pointing along I_j away from the node. To construct a vector $\boldsymbol{b}^h \in V_{\text{sol}}(Q, \mathrm{d}\lambda^h)$, an auxilliary problem is solved in a neighbourhood of each node $\mathcal{O}$. Define Q_h to be the union of the disc of radius h centered at $\mathcal{O}$ and the m strips of width $2h$ and length $4h$ with midline I_j. The outer ends of these strips will be denoted $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$ respectively. Consider the Neumann problem

$$\operatorname{div}(\nabla w) = 0, \quad \text{in } Q_h, \quad \frac{\partial w}{\partial n}\Big|_{\Gamma_j} = b_j, \quad j = 1, \ldots, m,$$

$$\frac{\partial w}{\partial n} = 0 \quad \text{on the remainder of the boundary of } Q_h,$$

where n is the unit outward pointing normal to Q_h. This problem is soluable and moreover it can be shown that

$$\int_{Q_h} |\nabla w|^2 \,\mathrm{d}\boldsymbol{x} \leq ch^2.$$

Thus

$$\boldsymbol{b}^h := \begin{cases} \nabla w, & \text{in } Q_h, \\ [\boldsymbol{b}]^h, & \text{outside } Q_h, \end{cases}$$

where $[\boldsymbol{b}]^h$ is the natural extension of $\boldsymbol{b}$. This definition of $\boldsymbol{b}^h$ satisfies the requirements that $\boldsymbol{b}^h \in V_{\mathrm{sol}}(Q, \mathrm{d}\lambda^h)$ and that $\boldsymbol{b}^h \to \boldsymbol{b}$ in $L^2(Q, \mathrm{d}\lambda^h)$.

2.C Homogenised Equation for a General Periodic Framework

In this appendix, the construction of limiting system of partial differential equations for the homogenised equation will given in the case of a general periodic framework. In particular, it will be shown that for a general periodic framework and an isotropic tensor A_0 that the equation on the singular structure is indeed given by equation (2.3.6).

Let $\boldsymbol{\tau}$ and $\boldsymbol{\nu}$ be the unit direction and unit normal respectively to any link of the thin network F_1 such that they form a positively orientated system. It can be shown (since the geometry is two-dimensional) that if $\boldsymbol{\nu} = (\nu_1, \nu_2)$, then $\boldsymbol{\tau} = (\nu_2, -\nu_1)$.

Let the domain Q be divided up into sub-domains Q_n, $n = 1, \ldots, m$, such that each Q_n is the maximal open connected set into which Q may be split by the singular network F_1. In other words,

$$Q \backslash F_1 = \bigcup_{n=1}^{m} Q_n.$$

Recall the bilinear form

$$B_{\mathrm{micro}}(\boldsymbol{U}, \boldsymbol{\Phi}) = \frac{\theta^2}{6} \int_Q K_1 \boldsymbol{\chi}'' \cdot \boldsymbol{\Phi}'' \,\mathrm{d}\lambda + \frac{1}{2} \int_Q A_0 e_y(\boldsymbol{U}) \cdot e_y(\boldsymbol{\Phi}) \,\mathrm{d}\boldsymbol{y}.$$

Integration by parts will be implemented in the second integral which leads to a boundary integral over the singular structure being accumulated. Firstly note that for transverse displacements $\boldsymbol{U} = (U_1, U_2)$, the following simplifications are made:

$$\boldsymbol{U} = U^{(\nu)} \boldsymbol{\nu}, \quad U_1 = U^{(\nu)} \nu_1, \quad U_2 = U^{(\nu)} \nu_2,$$

$$U_{i,j} = \nu_i(\tau_j \partial_\tau U^{(\nu)} + \nu_j \partial_\nu U^{(\nu)}), \quad i,j \in \{1,2\}. \tag{2.C.1}$$

Note that $\boldsymbol{\tau}$ and $\boldsymbol{\nu}$ will depend on which link of the network is being considered but that on each individual link, $\boldsymbol{\tau}$ and $\boldsymbol{\nu}$ are constant. Consider the integral

$$\int_Q U_{i,j}\Phi_{k,l}\ \mathrm{d}\boldsymbol{y} = \sum_{n=1}^{m} \int_{Q_n} U_{i,j}\Phi_{k,l}\ \mathrm{d}\boldsymbol{y}.$$

Integrating by parts in the above by implementing the Gauss-Green formula (See Evans [29, Appendix C]) yields

$$\int_Q U_{i,j}\Phi_{k,l}\,\mathrm{d}\boldsymbol{y} = \sum_{n=1}^{m} \left(\int_{\partial Q_n} U_{i,j}\Phi_k \nu_l^{(n)}\,\mathrm{d}\lambda - \int_{Q_n} U_{i,jl}\Phi_k\,\mathrm{d}\boldsymbol{y} \right),$$

where $\boldsymbol{\nu}^{(n)} = (\nu_1^{(n)}, \nu_2^{(n)})$ is the outward pointing normal to the sub-domain Q_n and $\partial Q_n \subset F_1$ is the thin network which forms the boundary of Q_n. Hence, substituting in the relations (2.C.1), the integral takes the form

$$\int_Q U_{i,j}\Phi_{k,l}\,\mathrm{d}\boldsymbol{y} = \sum_{n=1}^{m} \left(\int_{\partial Q_n} \left(\nu_j^{(n)} \partial_\nu U^{(\nu)} + \tau_j^{(n)} \partial_\tau U^{(\nu)} \right) \Phi^{(\nu)} \nu_i^{(n)} \nu_k^{(n)} \nu_l^{(n)}\,\mathrm{d}\lambda - \int_{Q_n} U_{i,jl}\Phi_k\,\mathrm{d}\boldsymbol{y} \right). \tag{2.C.2}$$

The second integral on the right-hand side of B_{micro} is a linear combination of integrals of the form (2.C.2) and hence, it can be shown that upon integration by parts, the following equation holds true:

$$\int_Q A_0 e_y(\boldsymbol{U}) \cdot e_y(\boldsymbol{\Phi})\,\mathrm{d}\boldsymbol{y} = (L_0 + 2M_0) \sum_{n=1}^{m} \left\{ \int_{\partial Q_n} \partial_\nu U^{(\nu)} \Phi^{(\nu)}\,\mathrm{d}\lambda \right\} - \\ - \int_Q \left(M_0 \Delta \boldsymbol{U} + (M_0 + L_0) \nabla (\mathrm{div}\ \boldsymbol{U}) \right) \cdot \boldsymbol{\Phi}\,\mathrm{d}\boldsymbol{y}.$$

Hence

$$B_{\text{micro}}(\boldsymbol{U}, \boldsymbol{\varphi}) = \int_{F_1 \cap Q} \left(\frac{\theta^2}{6} K_1 \partial_\tau^4 U^{(\nu)} + (L_0 + 2M_0) \partial_\nu U^{(\nu)} \right) \Phi^{(\nu)}\,\mathrm{d}\lambda - \\ - \frac{1}{2} \int_Q \left(M_0 \Delta \boldsymbol{U} + (M_0 + L_0) \nabla (\mathrm{div}\ \boldsymbol{U}) \right) \cdot \boldsymbol{\Phi}\,\mathrm{d}\boldsymbol{y}.$$

Thus the derivation of the system of partial differential equations in the setting of a general periodic framework is achieved by a suitable restriction of test functions.

Chapter 3

Further Aspects of Homogenisation of Maxwell's Equations

Introduction

In the concluding chapter of this thesis, three additional problems related to homogenisation theory will be presented along with relevant results. The problems in question all involve the study of Maxwell's equations in different geometries with the first problem being the study of a high-contrast periodic dielectric medium, the second problem being the study of Maxwell's equations on "thin" rods and the final problem being an examination of a dielectric medium which has been "twisted" by a change of coordinates and the effective properties possessed by this medium.

The study of high-contrast media has been of interest to the homogenisation community for a number of years now (see Figotin & Kuchment [32], Zhikov[81], Cherdantsev & Cherednichenko [20], Zhikov & Pastukhova [87], Cherednichenko & Cooper [22, 23]). For a given periodic problem with associated family of operators, the problem is said to be in high-contrast if there is the loss of uniform ellipticity due to the ellipticity coefficients of the operators vanishing as the period of the medium tends to zero. Such behaviour is seen in the operators associated with the study of photonic crystals (Russell [62] and Guenneau & Zolla [37]) and such behaviour is desirable due to the "band-gap" nature of the spectrum associated with the corresponding operator families which can lead to the development of "photonic crystal fibres" which allow for the propagation of electromagnetic waves of a specific frequency to travel in specific directions with little to no interference from the outside.

The high-contrast terminology refers to the fact that there is contrast between the different material components which constitute the composite dielectric material where the contrast increases as the period becomes smaller. In literature on high-contrast periodic media, if the period is $\varepsilon > 0$ and the contrast parameter is $\delta > 0$, then the high-contrast scaling is usually $\delta = \varepsilon^2$. This scaling is critical in the sense that problems with this scaling behave significantly different asymptotically to problems where an alternative scaling is chosen. In the work of Section 3.1, it will be shown that this critical scaling is indeed the case when studying Maxwell's equations on a three-dimensional torus with high-contrast periodic cells comprising a stiff region with a soft inclusion.

Following on from the work of Chapter 2, a study of rod networks will be presented now in the context of Maxwell's equations and in the case when the scaling between the rod thickness $a > 0$ and the period $\varepsilon > 0$ is such that $a/\varepsilon^2 \to 0$ (as $\varepsilon \to 0$). The study of the governing equations of linearised elasticity on rod structures has been studied fully in [81, 82, 84] but the equivalent study for Maxwell's equations has yet to be detailed fully. The desire for such a study stems from studies of the spectrum of so-called "quantum-graphs", structures of "zero thickness" (see Berkolaiko & Kuchment [10]). Using the thoery of two-scale convergence (*c.f.* Section 2.2.1), a homogenised equation is found which captures the effective behaviour of the model in the limit as the period tends to zero.

The work presented on thin rod structures is just a small piece of research which can be extended to many possible avenues of investigation including the study of rod structure with alternative scalings between the rod thickness and the period and the study of high-contrast problems.

The final piece of work of this thesis sees the study of a chiral transformation of Maxwell's equation in $\mathbb{R}^3$. This study is motivated by the significant developments made in the physics (see Pendry [57], Pendry, Schurig & Smith [58], Demetriadou & Pendry[27]) and mathematical (see Nicolet, Zolla & Guenneau [53], Nicolet, Movchan, Guenneau & Zolla [52], Willis [78]) communities on metamaterial science. In the paper of Demetriadou & Pendry [27], a link is made between metamaterials exhibiting cloaking properties and the "swiss roll" like structure of the material of study. The geometry of this problem is what prompted the idea to study Maxwell's equations under a change of coordinates which "twists" the domain. Given the invariance of structure of Maxwell's equations under general transformations of coordinates (see [41], the study revolves around finding appropriate solutions to the usual system of Maxwell of equations but with unbounded electric permittivity and unbounded magnetic permeability. The goal was to find a homogenised system of equations and analyse the results to try and assess whether there were any links to the properties exhibited by metamaterials, e.g., a desirable property would be a negative effective permeability and/or negative effective permittivity.

While this work has not succeeded in its initial goal, results which have been found are acknowledged and presented with the possibility for this work to be continued and extended.

3.1 The Existence of a Critical Scaling in the Study of Maxwell's Equations in a High-Contrast Periodic Dielectric Medium

3.1.1 Formulation of the Problem

In engineering communities, composite materials are of great interest for the reasons explained in the introduction to Chapter 2. It is important for engineers to understand how the constitutive components of the material being constructed behave in relation to one another as this will govern many properties of the composite in question. It is the ratio of the material parameters of the constitutive components of the composite which indicate the behaviour of the overall material and is a principle reason for the study of the problem to follow. From the mathematical point of view, the study of the asymptotic behaviour of the solution of some periodic elliptic partial differential equations (elasticity system, Maxwell's equations, *etc.*) can be siginificantly different depending on the behaviour of the material parameters compared with the period of the composite. As backed by numerical evidence, the leading order asymptotic behaviour of the solution can be shown to depend not only on a function of a macroscopic variable but also on the derivatives of the same function. This behaviour will be made apparant for a periodic dielectric medium where the unit period cell is made of two contrasting materials when a particular scaling of the material parameter is chosen.

In the following work, the problem (1.1.1) will be considered again but when the periodic cell Q is now divided into two non-intersecting regions Q_0 and Q_1 which are in contrast. It will then be shown that there is a "critical scaling" between the contrast parameter $\delta > 0$ and the period scale $\varepsilon > 0$ which leads to an asymptotic expansion for the solution which behaves differently from the solution when no such contrast is present.

As in Chapter 1, consider the problem

$$\mathbf{curl}\,\hat{A}^{\varepsilon}\mathbf{curl}\,\boldsymbol{u}^{\varepsilon} = \boldsymbol{f}, \quad \boldsymbol{u}^{\varepsilon} \in X(\mathbb{T}), \tag{3.1.1}$$

where all the same notation and functions spaces as described in Section 1.1 applies. Hence, a unique solution of this problem exists (Theorem 1.1.1) and such a solution can be found in an asymptotic expansion of the period $\varepsilon > 0$ (as given in equation (1.1.5)).

Let the unit cell $Q := [-\frac{1}{2}, \frac{1}{2})^3$ be divided into two simply connected domains Q_1 and Q_0 such that $Q_0 \cap \partial Q = \emptyset$. Let Γ denote the interface between the regions Q_1 and Q_0. Denote by χ_1 and χ_0 the charaterstic functions on Q_1 and Q_0 respectively and denote by $\langle \cdot \rangle_{Q_1}$ and $\langle \cdot \rangle_{Q_0}$ the integrals over the domains Q_1 and Q_0 respectively. The matrix $\hat{A}^\varepsilon(\cdot) = A(\cdot/\varepsilon)$ is redefined via the following equality:

$$A(\boldsymbol{y}) := A^\delta(\boldsymbol{y}) = \begin{cases} \delta I, & \text{if } \boldsymbol{y} \in Q_0, \\ I, & \text{if } \boldsymbol{y} \in Q_1, \end{cases} \qquad \delta > 0. \tag{3.1.2}$$

We now seek a solution to problem (3.1.1) but with $\hat{A}^\varepsilon$ replaced with A^δ. While the

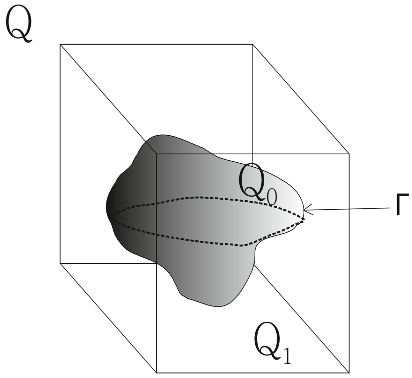

Figure 3.1: Example of unit cell Q with inclusion Q_0.

solution of this problem will still be a two-scale expansion of the form (1.1.5), the tensors $N^{(j)}$, $j = 1, 2, \ldots$, may now be dependent on the parameter δ. To this end, redefine the $N^{(j)}$ in all relevant equations seen in Chapter 1 as $N_\delta^{(j)}$. In turn, the tensors $M^{(j)}$ and $L^{(j)}$, as defined in Section 1.1.3, are also now denoted $M_\delta^{(j)}$ and $L_\delta^{(j)}$ respectively. In addition, suitable interface conditions are required on the surface Γ and to this end the following jump conditions are introduced:

$$\begin{cases} \left[\boldsymbol{n} \cdot N_\delta^{(j)} \right]_\Gamma = 0, \\ \left[\boldsymbol{n} \times A^\delta \left(\mathbf{curl}\, N_\delta^{(j)} + M_\delta^{(j)} \right) \right]_\Gamma = 0, \end{cases} \qquad j = 1, 2, 3, \ldots, \tag{3.1.3}$$

where $\boldsymbol{n}$ denotes the unit outward pointing normal. Note that throughout this work, the normal on Γ will be denoted $\boldsymbol{n}_1$ if the associated problem of consideration is on Q_1 and denoted $\boldsymbol{n}_0$ otherwise where at each point on the surface Γ, $\boldsymbol{n}_1 = -\boldsymbol{n}_0$.

3.1.2 Asymptotic Behaviour of $N_\delta^{(j)}$, $j = 1, 2, \ldots$

The asymptotic behaviour of the tensors $N_\delta^{(j)}$ $j = 1, 2, \ldots$ will be examined as $\delta \to 0$. The matrix $N_\delta^{(1)}$ solves the following problem (*c.f.* equation (1.1.9)) where it is recalled that this equation is understood columnwise:

$$\begin{cases} \mathbf{curl}\, A^{\delta}\Big(\mathbf{curl}\, N_{\delta}^{(1)} + I\Big) = 0, \\ N_{\delta}^{(1)} \text{ } Q\text{-periodic}, \quad \langle N_{\delta}^{(1)} \rangle = 0, \quad \operatorname{div} N_{\delta}^{(1)} = 0, \\ \Big[\boldsymbol{n} \cdot N_{\delta}^{(1)}\Big]_{\Gamma} = 0, \quad \Big[\boldsymbol{n} \times A^{\delta}\left(\mathbf{curl}\, N_{\delta}^{(1)} + I\right)\Big]_{\Gamma} = 0. \end{cases} \tag{3.1.4}$$

Consider the following asymptotic series for $N_{\delta}^{(1)}$:

$$N_{\delta}^{(1)} = N_0^{(1)} + \delta N_1^{(1)} + O(\delta^2), \quad \delta \to 0. \tag{3.1.5}$$

Substituting this series expansion into problem (3.1.4) and equating equal powers of the parameter δ, the following set of equations and boundary conditions become apparant:

$$\mathbf{curl}\,(\mathbf{curl}(\chi_1 N_0^{(1)}) + I) = 0, \quad \text{in } Q_1, \quad \boldsymbol{n}_1 \times \Big(\mathbf{curl}\,(\chi_1 N_0^{(1)}) + I\Big) = 0, \quad \text{on } \Gamma,$$

$$\mathbf{curl}\,(\mathbf{curl}(\chi_0 N_j^{(1)}) + I) + \mathbf{curl}\,(\mathbf{curl}(\chi_1 N_{j+1}^{(1)}) + I) = 0, \quad \forall j = 0, 1, 2, \ldots.$$

$$\chi_1 N_j^{(1)} \text{ } Q\text{-periodic}, \quad \langle \chi_1 N_j^{(1)} \rangle_{Q_1} + \langle \chi_0 N_j^{(1)} \rangle_{Q_0} = 0, \quad \operatorname{div}\,(\chi_1 N_j^{(1)}) + \operatorname{div}\,(\chi_0 N_j^{(1)}) = 0,$$

$$\boldsymbol{n}_0 \times \Big(\mathbf{curl}\,(\chi_0 N_j^{(1)}) + I\Big) = \boldsymbol{n}_1 \times \Big(\mathbf{curl}\,(\chi_1 N_{j+1}^{(1)}) + I\Big), \quad \boldsymbol{n}_1 \cdot (\chi_1 N_j^{(1)}) = \boldsymbol{n}_0 \cdot (\chi_0 N_j^{(1)}), \quad \text{on } \Gamma.$$

Firstly, define $N_{i,j}^{(1)} := \chi_i N_j^{(1)}$ to simplify the notation to follow. The first problem to be solved is the following:

$$\begin{cases} \mathbf{curl}\,(\mathbf{curl}\, N_{1,0}^{(1)} + I) = 0, & \text{in } Q_1, \\ \operatorname{div} N_{1,0}^{(1)} = 0, & \text{in } Q_1, \quad N_{1,0}^{(1)} \text{ } Q\text{-periodic}, \\ \boldsymbol{n}_1 \times \mathbf{curl}\, N_{1,0}^{(1)} = -\boldsymbol{n}_1 \times I, & \text{on } \Gamma. \end{cases}$$

The solution to this problem can be found and is unique up to an unknown constant by the results in Arfken [4, p.95-96] and in Zhou [88]. The next problem to be solved involves the unknowns $N_{0,0}^{(1)}$ and $N_{1,1}^{(1)}$ and is a system of equations which must be solved simultaneously. The system is the following:

$$\begin{cases} \mathbf{curl}\,(\mathbf{curl}\, N_{0,0}^{(1)} + I) = 0, & \text{in } Q_0, \\ \operatorname{div} N_{0,0}^{(1)} = 0, & \text{in } Q_0, \end{cases} \qquad \begin{cases} \mathbf{curl}\,(\mathbf{curl}\, N_{1,1}^{(1)} + I) = 0, & \text{in } Q_1, \\ \operatorname{div} N_{1,1}^{(1)} = 0, & \text{in } Q_1, \end{cases}$$

$$N_{1,1}^{(1)} \text{ } Q\text{-periodic}, \quad \langle N_{0,0}^{(1)} \rangle_{Q_0} = -\langle N_{1,0}^{(1)} \rangle_{Q_1},$$

$$\boldsymbol{n}_0 \cdot (N_{0,0}^{(1)}) = \boldsymbol{n}_1 \cdot (N_{1,0}^{(1)}), \quad \boldsymbol{n}_0 \times \Big(\mathbf{curl}\,(N_{0,0}^{(1)}) + I\Big) = \boldsymbol{n}_1 \times \Big(\mathbf{curl}\,(N_{1,1}^{(1)}) + I\Big), \quad \text{on } \Gamma.$$

This system determines $N_{0,0}^{(1)}$ and determines $N_{1,1}^{(1)}$ up to a constant. This unknown constant can then be determined by solving the next system of equations which arises and so on. Importantly, to leading order, the solution $N_{\delta}^{(1)} = O(1)$, $\delta \to 0$.

Recall the problem for the third-order tensor $N_\delta^{(2)}$:

$$\begin{cases} \mathbf{curl}\, A^\delta \Big(\mathbf{curl}\, N_\delta^{(2)} + M_\delta^{(2)} \Big) = \langle L_\delta^{(2)} \rangle - L_\delta^{(2)}, \\ N_\delta^{(2)} \ Q\text{-periodic}, \quad \langle N_\delta^{(2)} \rangle = 0, \quad \operatorname{div} N_\delta^{(2)} = 0, \\ \Big[\boldsymbol{n} \cdot N_\delta^{(2)} \Big]_\Gamma = 0, \quad \Big[\boldsymbol{n} \times A^\delta \Big(\mathbf{curl}\, N_\delta^{(2)} + M_\delta^{(2)} \Big) \Big]_\Gamma = 0. \end{cases} \tag{3.1.6}$$

where $M_\delta^{(2)} = \epsilon N_\delta^{(1)}$, $L_\delta^{(2)} = \epsilon A^\delta (\mathbf{curl}\, N_\delta^{(1)} + I)$ and ϵ is the permutation tensor:

$$\epsilon_{ijk} = \begin{cases} 1, & \text{if } ijk = \{123, 231, 312\}, \\ -1, & \text{if } ijk = \{132, 213, 321\}, \\ 0, & \text{otherwise.} \end{cases}$$

Note that

$$\langle L_\delta^{(2)} \rangle = \langle \epsilon (\mathbf{curl}\, (\chi_1 N_\delta^{(1)}) + I) \rangle_{Q_1} + \delta \langle \epsilon (\mathbf{curl}\, (\chi_0 N_\delta^{(1)}) + I) \rangle_{Q_0}.$$

Hence it is clear by the δ-asymptotics for $N_\delta^{(1)}$ that $\langle L_\delta^{(2)} \rangle = \tilde{h}_{1,0}^{(2)} + \delta(\tilde{h}_{0,0}^{(2)} + \tilde{h}_{1,1}^{(2)}) + O(\delta^2)$ where $\tilde{h}_{i,j}^{(2)} = \langle \mathbf{curl}\, N_{i,j}^{(1)} + I \rangle_{Q_i}$. Let the notation $T_{i,j}^{(k)} := \chi_i T_j^{(k)}$ be adopted for any tensor $T_j^{(k)} \in \{N_j^{(k)}, M_j^{(k)}, L_j^{(k)}\}$ with the index $i \in \{0,1\}$ and $j \in \mathbb{N}$. Before writing $N_\delta^{(2)}$ in an asymptotic expansion, observe that the following equations are satisfied on each of the domains Q_1 and Q_0 respectively:

$$\mathbf{curl}\, \Big(\mathbf{curl}\, (\chi_1 N_\delta^{(2)}) + (M_{1,0}^{(2)} + \delta M_{1,1}^{(2)}) \Big) + O(\delta^2) = \tilde{h}_{1,0}^{(2)} + \delta(\tilde{h}_{0,0}^{(2)} + \tilde{h}_{1,1}^{(2)}) - (L_{1,0}^{(2)} + \delta L_{1,1}^{(2)}) + O(\delta^2) \tag{3.1.7}$$

$$\delta \mathbf{curl}\, \Big(\mathbf{curl}\, (\chi_0 N_\delta^{(2)}) + M_{0,0}^{(2)} \Big) + O(\delta^2) = \tilde{h}_{1,0}^{(2)} + \delta(\tilde{h}_{0,0}^{(2)} + \tilde{h}_{1,1}^{(2)}) - \delta L_{0,0}^{(2)} + O(\delta^2) \tag{3.1.8}$$

From equation (3.1.8), it is observed that dividing through by δ, leaves the right-hand side of order $O(\delta^{-1})$. In light of this observation, write $N_\delta^{(2)}$ in δ-asymptotic expansion of the form

$$N_\delta^{(2)} := \delta^{-1} N_{-1}^{(2)} + N_0^{(2)} + O(\delta).$$

Hence, substituting this expansion into equations (3.1.7)-(3.1.8) and the relevant boundary conditions leads to a system of problems which are solved systematically and in tandem, the first few of which a presented below:

$$\begin{cases} \mathbf{curl}\,\mathbf{curl}\, N_{1,-1}^{(2)} = 0, & \text{on } Q_1, \\ \operatorname{div} N_{1,-1}^{(2)} = 0, & \text{on } Q_1, \quad N_{1,-1}^{(2)} \ Q\text{-periodic}, \\ \boldsymbol{n}_1 \times \mathbf{curl}\, N_{1,-1}^{(2)} = 0, & \text{on } \Gamma, \end{cases} \tag{3.1.9}$$

$$\begin{cases} \mathbf{curl}\,\mathbf{curl}\, N_{0,-1}^{(2)} = \tilde{h}_{1,0}^{(2)}, & \text{on } Q_0, \\ \operatorname{div} N_{0,-1}^{(2)} = 0, & \text{on } Q_0, \quad \langle N_{0,-1}^{(2)} \rangle_{Q_0} = -\langle N_{1,-1}^{(2)} \rangle, \\ \boldsymbol{n}_0 \cdot N_{0,-1}^{(2)} = \boldsymbol{n}_1 \cdot N_{1,-1}^{(2)}, & \text{on } \Gamma, \end{cases} \tag{3.1.10}$$

$$\begin{cases} \mathbf{curl}\,(\mathbf{curl}\,N_{1,0}^{(2)} + M_{1,0}^{(2)}) = \tilde{h}_{1,0}^{(2)} - L_{1,0}^{(2)}, & \text{on } Q_1, \\ \operatorname{div} N_{1,0}^{(2)} = 0, & \text{on } Q_1, \quad N_{1,0}^{(2)} \ Q\text{-periodic}, \\ \boldsymbol{n}_1 \times (\mathbf{curl}\,N_{1,0}^{(2)} + M_{1,0}^{(2)}) = \boldsymbol{n}_0 \times \mathbf{curl}\,N_{0,-1}^{(2)}, & \text{on } \Gamma, \end{cases} \tag{3.1.11}$$

The solution to problem (3.1.9) is unique up to a constant and takes the form

$$N_{1,-1}^{(2)} = \nabla\varphi_{1,-1} + C, \tag{3.1.12}$$

where $\varphi_{1,-1} \in [H^1_{\text{per}}(Q)]^{3^2}$ is harmonic on Q_1 and C is a constant tensor. Problems (3.1.10)-(3.1.11) are solved simultaneously leading to the conclusion that $N_\delta^{(2)} = O(\delta^{-1})$ as $\delta \to 0$.

Recall the problem for $N_\delta^{(3)}$:

$$\begin{cases} \mathbf{curl}\,A^\delta\Big(\mathbf{curl}\,N_\delta^{(3)} + M_\delta^{(3)}\Big) = \langle L_\delta^{(3)}\rangle - L_\delta^{(3)}, \\ N_\delta^{(3)} \ Q\text{-periodic}, \quad \langle N_\delta^{(3)}\rangle = 0, \quad \operatorname{div} N_\delta^{(3)} = 0, \\ \Big[\boldsymbol{n}\cdot N_\delta^{(3)}\Big]_\Gamma = 0, \quad \Big[\boldsymbol{n}\times A^\delta\Big(\mathbf{curl}\,N_\delta^{(3)} + M_\delta^{(3)}\Big)\Big]_\Gamma = 0. \end{cases} \tag{3.1.13}$$

This problem is solved in an analogous way as problem (3.1.6) is solved. Note that since

$$\langle L_\delta^{(3)}\rangle = \langle\epsilon(\mathbf{curl}\,(\delta^{-1}N_{1,-1}^{(2)} + N_{1,0}^{(2)}) + M_{1,0}^{(2)})\rangle_{Q_1} + \delta\langle\epsilon(\mathbf{curl}\,(\delta^{-1}N_{0,-1}^{(2)}))\rangle_{Q_0} + O(\delta),$$

and $\mathbf{curl}\,N_{1,-1}^{(2)} = 0$, it follows that $L_\delta^{(3)} = O(1)$ and hence $N_\delta^{(3)} = O(\delta^{-1})$.

The procedure above is carried out for all higher-order tensors $N_\delta^{(k)}$ and as a result, the following asymptotic relations are obtained

$$N_\delta^{(2j)} = N_\delta^{(2j+1)} = O(\delta^{-j}), \quad \forall j = 1, 2, \ldots.$$

Recall the asymptotic expansion for $\boldsymbol{u}^\varepsilon$ (*c.f.* (1.1.5)). Neglecting the "gradient" terms in the expansion, (it will be shown in Appendix 3.A that they have no impact on the expansion below), the asymptotics for $\boldsymbol{u}^\varepsilon$ are of the form

$$\boldsymbol{u}^\varepsilon \sim \boldsymbol{v} + \varepsilon N^{(1)}\mathbf{curl}\,\boldsymbol{v} + \varepsilon^2\delta^{-1}\tilde{N}^{(2)}\nabla\mathbf{curl}\,\boldsymbol{v} + \varepsilon^3\delta^{-1}\tilde{N}^{(3)}\nabla^2\mathbf{curl}\,\boldsymbol{v} + \varepsilon^4\delta^{-2}\tilde{N}^{(4)}\nabla^3\mathbf{curl}\,\boldsymbol{v} + \ldots.$$

A particular scaling between the contrast parameter δ and the period ε is observed from the above which results in a "different kind" of asymptotic expansion. Indeed, for the scaling $\delta = \varepsilon^2$, the above expansion will consist of terms of order $O(1)$ and order $O(\varepsilon)$ only. This observation justifies the phrase "critical scaling" for the case when $\delta = \varepsilon^2$ since it leads to a deviation from the classical asymptotics for the solution.

Further Discussions

The derivation of the homogenised equation in the case of a high-contrast periodic dielectric medium was given for Maxwell's equations in Cherenichenko & Cooper [23]. A further study of the work of this section would be to derive the homogenised equation in the case of the critical scaling via asymptotic methods and compare the results with those found in this paper, not only to check that the results agree, but also to find the higher-order terms in the higher-order homogenised expansion.

3.2 Homogenisation of Maxwell's Equations on Thin Structures

3.2.1 Formulation of the Problem and Tools

In this section, the limiting homogenised problem for Maxwell's equations on a rod structure will be outlined in the regime when the rods are "sufficiently thin", i.e., when the rod thickness $a = a(\varepsilon) > 0$ and the period $\varepsilon > 0$ are such that $a/\varepsilon^2 \to 0$ (as $\varepsilon \to 0$). The work of this section will follow a similar vain to the work of Chapter 2 with a lot of the same notation used.

Let F be a connected, 3-dimensional periodic singular structure. Denote by F^h the rod framework formed by replacing the line segments in F by cylindrical rods of diameter $h := a/\varepsilon$ wherein, the original line segments form the medians of the rods of F^h. Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and let $Q = [0,1)^3$ denote the period cell for the network. Define λ and λ^h by the relations (*c.f.* (2.1.1)):

$$\lambda(B) = \frac{\mathcal{H}^1(B \cap F)}{\mathcal{H}^1(Q \cap F)}, \quad \lambda^h(B) = \frac{\mathcal{H}^3(B \cap F^h)}{\mathcal{H}^3(Q \cap F^h)},$$

for all Borel sets $B \subset Q$. Once more there is weak convergence $\lambda^h \rightharpoonup \lambda$. Furthermore, the measure λ^h_ε is introduced where λ^h_ε is concentrated on the contracted network $F^{\varepsilon,h} = \varepsilon F^h$ and is defined by the relation $\lambda^h_\varepsilon(B) = \varepsilon^3 \lambda^h(\varepsilon^{-1}B)$, for every Borel set $B \subset \mathbb{R}^3$. Moreover, the measure λ^h_ε, has period ε and

$$\int_{\varepsilon Q} \mathrm{d}\lambda^h_\varepsilon = \varepsilon^3 \int_Q \mathrm{d}\lambda^h = \varepsilon^3.$$

Note there is convergence $\lambda^h_\varepsilon \rightharpoonup \mathrm{d}\boldsymbol{x}$ as $\varepsilon \to 0$ also. The parameters h and ε are connected via the relation $h(\varepsilon) \to 0,\ \varepsilon \to 0$. Throughout this work, it will be assumed that $\boldsymbol{u}^h_\varepsilon$ is a bounded sequence in $[L^2(\Omega, \mathrm{d}\lambda^h_\varepsilon)]^3$:

$$\limsup_{\varepsilon \to 0} \int_\Omega |\boldsymbol{u}^h_\varepsilon|^2 \, \mathrm{d}\lambda^h_\varepsilon < \infty.$$

The following subset of the space $H^1_{\mathbf{curl}}(\Omega, \mathrm{d}\lambda^h_\varepsilon)$ is introduced:

$$H^1_{\mathbf{curl},0}(\Omega, \mathrm{d}\lambda^h_\varepsilon) := \{\boldsymbol{u} \in [L^2(\Omega, \mathrm{d}\lambda^h_\varepsilon)]^3 \mid \mathbf{curl}\,\boldsymbol{u} \in [L^2(\Omega, \mathrm{d}\lambda^h_\varepsilon)]^3,\ A^\varepsilon \mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} \text{ on } \partial\Omega\},$$

where A^ε is a uniformly elliptic, symmetric, periodic, 3×3 matrix and $\boldsymbol{n}$ is the unit, outward pointing normal to the boundary of Ω.

The following equation is derived from Maxwell's equations (*c.f.* Section 1.3.1) and the problem is to find $\boldsymbol{u}^h_\varepsilon \in H^1_{\mathbf{curl},0}(\Omega, \mathrm{d}\lambda^h_\varepsilon)$ such that

$$\mathbf{curl}\, A^\varepsilon \mathbf{curl}\, \boldsymbol{u}^h_\varepsilon - \boldsymbol{u}^h_\varepsilon = \boldsymbol{f}_\varepsilon. \tag{3.2.1}$$

The vector field $\boldsymbol{f}_\varepsilon \in [L^2(\Omega, \mathrm{d}\lambda^h_\varepsilon)]^3$. Remark also that $A^\varepsilon(\cdot) = A(\cdot/\varepsilon)$, $\boldsymbol{f}_\varepsilon(\cdot) = \boldsymbol{f}(\cdot, \cdot/\varepsilon)$. Equivalently, equation (3.2.1) can be written in the weak form as

$$\int_\Omega A^\varepsilon \mathbf{curl}\, \boldsymbol{u}^h_\varepsilon \cdot \mathbf{curl}\, \boldsymbol{\varphi} \,\mathrm{d}\lambda^h_\varepsilon - \int_\Omega \boldsymbol{u}^h_\varepsilon \cdot \boldsymbol{\varphi} \,\mathrm{d}\lambda^h_\varepsilon = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi} \,\mathrm{d}\lambda^h_\varepsilon, \quad \forall \boldsymbol{\varphi} \in [C_0^\infty(\Omega)]^3 \qquad (3.2.2)$$

The behaviour of $\boldsymbol{u}^h_\varepsilon$ will be examined when the thickness $h = h(\varepsilon)$ is such that $h/\varepsilon \to 0$ as $\varepsilon \to 0$. In this scaling, the rods are described as being "sufficiently thin".

In addition to the theory of two-scale analysis (*c.f.* Section 2.1.2), additional characterisations of potential vectors and solenoidal vectors (*c.f.* Section 2.1.3) will be needed for the analysis of this section.

Definition 3.2.1 (Potential & Solenoidal Vectors)**.** The space of potential vectors (curl free vectors), denoted V_{pot} is the closure of the set $\{\nabla\varphi \mid \varphi \in C^\infty_{\text{per}}(Q)\}$ in $[L^2(Q, \mathrm{d}\lambda)]^3$. A vector $\boldsymbol{a} \in [L^2(Q, \mathrm{d}\lambda)]^3$ is said to be solenoidal (or divergence free) if

$$\int_Q (\boldsymbol{a} \cdot \boldsymbol{b}) \,\mathrm{d}\lambda = 0, \quad \text{for all } \boldsymbol{b} \in V_{\text{pot}}.$$

The set of all solenoidal vectors is denoted V_{sol} and moreover, the following orthogonal decomposition holds $[L^2(Q, \mathrm{d}\lambda)]^3 = V_{\text{pot}} \oplus V_{\text{sol}}$.

It is also true that $[L^2(\Omega \times Q, \mathrm{d}\boldsymbol{x} \times \mathrm{d}\lambda)]^3 = L^2(\Omega, V_{\text{pot}}) \oplus L^2(\Omega, V_{\text{sol}})$. It has been shown (see Zhikov [82]) that $L^2(\Omega, V_{\text{pot}})$ is the closure in $[L^2(\Omega \times Q, \mathrm{d}\boldsymbol{x} \times \mathrm{d}\lambda)]^3$ of the linear span of vectors $w(\boldsymbol{x})\nabla\varphi$, $w \in C_0^\infty(\Omega)$, $\varphi \in C^\infty_{\text{per}}(Q)$. Moreover, $L^2(\Omega, V_{\text{sol}})$ is the closure in $[L^2(\Omega \times Q, \mathrm{d}\boldsymbol{x} \times \mathrm{d}\lambda)]^3$ of the linear span of vectors $w(\boldsymbol{x})\boldsymbol{a}(\boldsymbol{y})$, $w \in C_0^\infty(\Omega)$, $\boldsymbol{a} \in V_{\text{sol}}$.

Alternatively (see Jikov, Kozlov & Oleinik [40]), a vector $\boldsymbol{v}(\boldsymbol{x}, \boldsymbol{y}) \in L^2(\Omega, V_{\text{pot}})$ if and only if

$$\boldsymbol{v}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{\kappa}(\boldsymbol{x}) + \boldsymbol{\pi}(\boldsymbol{x}, \boldsymbol{y}), \;\; \boldsymbol{\kappa} \in [L^2(\Omega, \mathrm{d}\lambda)]^3, \;\; \boldsymbol{\pi}(\boldsymbol{x}, \cdot) \in L^2_{\text{pot}}(Q), \qquad (3.2.3)$$

where $L^2_{\text{pot}}(Q) := \{\boldsymbol{u} \in [L^2(Q, \mathrm{d}\lambda)]^3 \mid \mathbf{curl}\, \boldsymbol{u} = \mathbf{0}\}$.

Let $\boldsymbol{a}, \boldsymbol{b} \in [L^2(Q, \mathrm{d}\lambda)]^3$. It is said that $\boldsymbol{a} = \mathbf{curl}\, \boldsymbol{b}$ if the following identity holds:

$$\int_Q (\boldsymbol{a} \cdot \boldsymbol{\psi}) \,\mathrm{d}\lambda = \int_Q (\boldsymbol{b} \cdot \mathbf{curl}\, \boldsymbol{\psi}) \,\mathrm{d}\lambda, \quad \text{for all } \boldsymbol{\psi} \in [C^\infty_{\text{per}}(Q)]^3.$$

Theorem 3.2.1 (Approximation Lemma)**.** *Let $\mathcal{T}$ denote set of vectors $\boldsymbol{a} \in [L^2(Q, \mathrm{d}\lambda)]^3$ admitting the representation $\boldsymbol{a} = \mathbf{curl}\, \boldsymbol{b}$, where $\boldsymbol{b} \in [L^2(Q, \mathrm{d}\lambda)]^3$. Then $\mathcal{T}$ is dense in V_{sol}.*

Proof. Assume that there exists a vector $\boldsymbol{h} \in V_{\text{sol}} \cap \mathcal{T}^\perp$. Consider the following problem for periodic $\boldsymbol{u} \in [H^1_{\mathbf{curl}}(Q)]^3$:

$$\int_Q (\mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{\varphi} + \boldsymbol{u} \cdot \boldsymbol{\varphi}) \,\mathrm{d}\lambda = \int_Q \boldsymbol{h} \cdot \boldsymbol{\varphi} \,\mathrm{d}\lambda, \quad \forall \boldsymbol{\varphi} \in [C^\infty_{\text{per}}(Q)]^3. \qquad (3.2.4)$$

Setting $\boldsymbol{\varphi} = \boldsymbol{u}$ yields

$$\int_Q \left(|\mathbf{curl}\,\boldsymbol{u}|^2 + |\boldsymbol{u}|^2\right) \mathrm{d}\lambda = \int_Q \boldsymbol{h} \cdot \boldsymbol{u} \,\mathrm{d}\lambda.$$

By the Cauchy-Schwartz inequality and inequality (2.2.1), it follows that

$$\int_Q |\boldsymbol{u}|^2 \,\mathrm{d}\lambda \leq \int_Q \left(|\mathbf{curl}\,\boldsymbol{u}|^2 + |\boldsymbol{u}|^2\right) \mathrm{d}\lambda = \int_Q \boldsymbol{h} \cdot \boldsymbol{u} \,\mathrm{d}\lambda \leq \frac{1}{2} \int_Q |\boldsymbol{h}|^2 \,\mathrm{d}\lambda + \frac{1}{2} \int_Q |\boldsymbol{u}|^2 \,\mathrm{d}\lambda.$$

Hence,

$$\int_Q |\boldsymbol{u}|^2 \,\mathrm{d}\lambda \leq \int_Q |\boldsymbol{h}|^2 \,\mathrm{d}\lambda. \tag{3.2.5}$$

From equation (3.2.4), clearly $(\boldsymbol{h} - \boldsymbol{u}) \in \mathcal{T}$ and hence that $(\boldsymbol{h} - \boldsymbol{u})$ is orthogonal to $\boldsymbol{h}$. Thus, once again by the Cauchy-Schwarz inequality and inequality (2.2.1), it follows that

$$\int_Q |\boldsymbol{h}|^2 \,\mathrm{d}\lambda = \int_Q \boldsymbol{h} \cdot \boldsymbol{u} \,\mathrm{d}\lambda \leq \int_Q |\boldsymbol{u}|^2 \,\mathrm{d}\lambda. \tag{3.2.6}$$

Hence, by equations (3.2.5) and (3.2.6) it follows that

$$\int_Q |\boldsymbol{h}|^2 \,\mathrm{d}\lambda = \int_Q |\boldsymbol{u}|^2 \,\mathrm{d}\lambda.$$

Thus

$$\int_Q \left(|\mathbf{curl}\,\boldsymbol{u}|^2 + |\boldsymbol{u}|^2\right) \mathrm{d}\lambda = \int_Q \boldsymbol{h} \cdot \boldsymbol{u} \,\mathrm{d}\lambda = \int_Q |\boldsymbol{h}|^2 \,\mathrm{d}\lambda, \quad \Rightarrow \quad \int_Q |\mathbf{curl}\,\boldsymbol{u}|^2 \,\mathrm{d}\lambda = 0.$$

Therefore $\boldsymbol{u} \in V_{\text{pot}}$ and moreover, $\boldsymbol{u} = \boldsymbol{h} \in V_{\text{pot}}$ and hence $\boldsymbol{h} = \mathbf{0}$. □

The extension of a function defined on the limiting structure F to the support of the rod framework F^h will be now be established. If I is a link of the singular structure F, then the corresponding rod on F^h is $I^h := I \times D_h$ where D_h is the disk of radius h. The approximation of potential vectors outside a neighbourhood of the nodes of the singular structure F is of particular interest. Let $D \subset V_{\text{pot}}$ be a dense subset which contains vectors compactly supported outside a neighbourhood of the nodes of F. Consider a link $I = [0, l] \times \{0\} \times \{0\}$ of F which lies on the horizontal axis with the corresponding rod of F^h denoted $I^h = [0, l] \times D_h(0)$. Next consider a smooth vector defined as

$$\boldsymbol{g}(\boldsymbol{y}) = (a_1'(y_1), 0, 0), \quad y_1 \in [\delta, l - \delta], \quad 0 < \delta < l.$$

Extending the vector $\boldsymbol{g}$ to the support of the measure λ^h on the rod $[\delta, l - \delta] \times D_h(0)$ is achieved by the following

$$\boldsymbol{g}_h(\boldsymbol{y}) = \Big((1 + y_2 + y_3)a_1', y_2 a_1, y_3 a_1\Big),$$

and outside this rod $\boldsymbol{g}_h(\boldsymbol{y}) = \mathbf{0}$. Hence, on the rod I^h

$$\mathbf{curl}\,\boldsymbol{g}_h = \Big(0, (1 - y_2)a_1'(y_1), (y_3 - 1)a_1'(y_1)\Big),$$

and $\mathbf{curl}\,\boldsymbol{g}_h = \mathbf{0}$ otherwise. Moreover, it can be shown that

$$\int_Q |\mathbf{curl}\,\boldsymbol{g}_h|^2 \,\mathrm{d}\lambda^h = O(h^2), \quad \int_Q |\boldsymbol{g}_h - [\boldsymbol{g}]^h|^2 \,\mathrm{d}\lambda^h = O(h^2), \tag{3.2.7}$$

where $[\boldsymbol{g}]^h(\boldsymbol{y})$ is the natural extension (*c.f.* Definition 2.1.7) of $\boldsymbol{g}_h(\boldsymbol{y})$. Therefore,

$$\varepsilon^{-2} \int_\Omega |\mathbf{curl}\,\boldsymbol{g}_h(\boldsymbol{x}/\varepsilon)|^2 \,\mathrm{d}\lambda^h_\varepsilon = O\left(\left(\frac{h}{\varepsilon}\right)^2\right), \tag{3.2.8}$$

$$\int_\Omega |\boldsymbol{g}_h(\boldsymbol{x}/\varepsilon) - [\boldsymbol{g}]^h(\boldsymbol{x}/\varepsilon)|^2 \,\mathrm{d}\lambda^h_\varepsilon = O\left(\left(\frac{h}{\varepsilon}\right)^2\right), \tag{3.2.9}$$

This approximation can be easily applied to an arbitrary rod of the network F. These results will be used to prove the homogenisation theorem in Section 3.2.2 in the case when the scaling $h/\varepsilon \to 0$ is considered.

3.2.2 Two-Scale Limits and Homogenisation Theorem

Note that it can be shown in much the same way as shown in Section 2.2 that the sequences $\boldsymbol{u}^h_\varepsilon$ and $\mathbf{curl}\,\boldsymbol{u}^h_\varepsilon$ are bounded in $[L^2(\Omega, \mathrm{d}\lambda^h_\varepsilon)]^3$ and hence by the compactness lemma (*c.f.* Proposition 2.1.1), they possess (upto possibly taking a suitable subsequence) a weak two-scale limit.

Theorem 3.2.2. *Suppose that*

$$\begin{cases} \boldsymbol{u}^h_\varepsilon(\boldsymbol{x}) \stackrel{2}{\rightharpoonup} \boldsymbol{u}(\boldsymbol{x},\boldsymbol{y}), & \textit{in } [L^2(\Omega, \mathrm{d}\lambda^h_\varepsilon)]^3, \\ \varepsilon\mathbf{curl}\,\boldsymbol{u}^h_\varepsilon \to \mathbf{0}, & \textit{in } [L^2(\Omega, \mathrm{d}\lambda^h_\varepsilon)]^3. \end{cases} \tag{3.2.10}$$

Then $\boldsymbol{u}(\boldsymbol{x},\boldsymbol{y}) \in L^2(\Omega, V_{\mathrm{pot}})$.

Proof. Let $\boldsymbol{a} \in [L^2(Q, \mathrm{d}\lambda)]^3$ and $\boldsymbol{b} \in [L^2(Q, \mathrm{d}\lambda)]^3$ be such that $\boldsymbol{a} = \mathbf{curl}\,\boldsymbol{b}$. Then by definition the following relation holds:

$$\varepsilon \int_\Omega \boldsymbol{b}\Big(\frac{\boldsymbol{x}}{\varepsilon}\Big) \cdot \mathbf{curl}\,\boldsymbol{\psi} \,\mathrm{d}\lambda^h_\varepsilon = \int_\Omega \boldsymbol{a}\Big(\frac{\boldsymbol{x}}{\varepsilon}\Big) \cdot \boldsymbol{\psi}(\boldsymbol{x}) \,\mathrm{d}\lambda^h_\varepsilon, \quad \forall \boldsymbol{\psi} \in [C_0^\infty(\Omega)]^3.$$

Consider those vectors $\boldsymbol{\psi}(\boldsymbol{x}) = \varphi(\boldsymbol{x})\boldsymbol{u}^h_\varepsilon(\boldsymbol{x})$ where $\varphi \in C_0^\infty(\Omega)$. Hence

$$\varepsilon \int_\Omega \boldsymbol{b}\Big(\frac{\boldsymbol{x}}{\varepsilon}\Big)\varphi(\boldsymbol{x}) \cdot \mathbf{curl}\,\boldsymbol{u}^h_\varepsilon \,\mathrm{d}\lambda^h_\varepsilon + \varepsilon \int_\Omega \boldsymbol{b}\Big(\frac{\boldsymbol{x}}{\varepsilon}\Big) \cdot (\nabla\varphi \times \boldsymbol{u}^h_\varepsilon) \,\mathrm{d}\lambda^h_\varepsilon = \int_\Omega \boldsymbol{a}\Big(\frac{\boldsymbol{x}}{\varepsilon}\Big) \cdot \varphi(\boldsymbol{x})\boldsymbol{u}^h_\varepsilon(\boldsymbol{x}) \,\mathrm{d}\lambda^h_\varepsilon, \tag{3.2.11}$$

where the relation $\mathbf{curl}(\varphi\boldsymbol{u}^h_\varepsilon) = \varphi\mathbf{curl}\,\boldsymbol{u}^h_\varepsilon + \nabla\varphi \times \boldsymbol{u}^h_\varepsilon$, has been utilised. Hence, passing to the limit as $\varepsilon \to 0$ on both sides and making use of the hypotheses of the theorem, it can be seen

$$\int_\Omega \int_Q \varphi(\boldsymbol{x})\boldsymbol{a}(\boldsymbol{y}) \cdot \boldsymbol{u}(\boldsymbol{x},\boldsymbol{y}) \,\mathrm{d}\lambda \mathrm{d}\boldsymbol{x} = 0.$$

By the Approximation Lemma (Theorem 3.2.1), since $\boldsymbol{a}$ belongs to a set dense in V_{sol} it follows that $\boldsymbol{u}(\boldsymbol{x},\boldsymbol{y}) \in L^2(\Omega, V_{\mathrm{pot}})$ and hence the result follows. □

In what follows, the structure of the limit of $\mathbf{curl}\, \boldsymbol{u}_\varepsilon^h$ will be established.

Theorem 3.2.3. *Suppose potential vectors satisfy the decomposition (3.2.3) and that* $\mathbf{curl}\, \boldsymbol{u}_\varepsilon^h(\boldsymbol{x}) \overset{2}{\rightharpoonup} \boldsymbol{p}(\boldsymbol{x},\boldsymbol{y})$ *in* $[L^2(\Omega, \mathrm{d}\lambda_\varepsilon^h)]^3$. *Then*

1. $\boldsymbol{u}(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{u}_0(\boldsymbol{x}) + \boldsymbol{\Pi}(\boldsymbol{x},\boldsymbol{y})$, *where* $\boldsymbol{u}_0 \in H^1_{\mathbf{curl}}(\Omega)$ *and* $\boldsymbol{\Pi} \in L^2(\Omega, V_{\mathrm{pot}})$.

2. *where* $\boldsymbol{v}(\boldsymbol{x},\boldsymbol{y}) \in L^2(\Omega, V_{\mathrm{sol}})$, *it follows that*

$$\boldsymbol{p}(\boldsymbol{x},\boldsymbol{y}) = \mathbf{curl}_x \boldsymbol{u}_0(\boldsymbol{x}) + \boldsymbol{v}(\boldsymbol{x},\boldsymbol{y}), \tag{3.2.12}$$

Moreover, if $\boldsymbol{u}_\varepsilon^h \in H^1_{\mathbf{curl},0}(\Omega)$ *then* $\boldsymbol{u}_0 \in H^1_{\mathbf{curl},0}(\Omega)$.

Proof. The fact that $\boldsymbol{u}(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{u}_0(\boldsymbol{x}) + \boldsymbol{\Pi}(\boldsymbol{x},\boldsymbol{y})$ follows by decomposition (3.2.3) with $\boldsymbol{u}_0 \in [L^2(\Omega, \mathrm{d}\lambda)]^3$ and $\boldsymbol{\Pi} \in L^2(\Omega, V_{\mathrm{pot}})$. It will be established that $\mathbf{curl}_x \boldsymbol{u}_0 \in [L^2(\Omega, \mathrm{d}\lambda)]^3$ and hence that $\boldsymbol{u}_0 \in H^1_{\mathbf{curl}}(\Omega)$ in the following argument. For the proof of part 2), let $\boldsymbol{b} \in V_{\mathrm{pot}}$, $\varphi \in C_0^\infty(\Omega)$. Then

$$\int_\Omega \mathbf{curl}\, \boldsymbol{u}_\varepsilon^h \cdot \varphi(\boldsymbol{x}) \boldsymbol{b}\Big(\frac{\boldsymbol{x}}{\varepsilon}\Big)\, \mathrm{d}\lambda_\varepsilon^h = -\int_\Omega \big[\nabla\varphi \times \boldsymbol{u}_\varepsilon^h\big] \cdot \boldsymbol{b}\Big(\frac{\boldsymbol{x}}{\varepsilon}\Big)\, \mathrm{d}\lambda_\varepsilon^h.$$

Taking the limit as $\varepsilon \to 0$ and applying the theory of weak two-scale convergence yields

$$\int_\Omega \int_Q \boldsymbol{p}(\boldsymbol{x},\boldsymbol{y}) \cdot \varphi(\boldsymbol{x}) \boldsymbol{b}(\boldsymbol{y})\, \mathrm{d}\lambda \mathrm{d}\boldsymbol{x} = -\int_\Omega \int_Q \big[\nabla\varphi(\boldsymbol{x}) \times \boldsymbol{u}(\boldsymbol{x},\boldsymbol{y})\big] \cdot \boldsymbol{b}(\boldsymbol{y})\, \mathrm{d}\lambda \mathrm{d}\boldsymbol{x}.$$

Writing $\boldsymbol{u}(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{u}_0(\boldsymbol{x}) + \boldsymbol{\Pi}(\boldsymbol{x},\boldsymbol{y})$, and noting that $\langle [\nabla\varphi(\boldsymbol{x}) \times \boldsymbol{\Pi}(\boldsymbol{x},\boldsymbol{y})] \cdot \boldsymbol{b}(\boldsymbol{y}) \rangle = 0$ (see Appendix 3.B), it follows that

$$\int_\Omega \int_Q \boldsymbol{p}(\boldsymbol{x},\boldsymbol{y}) \cdot \varphi(\boldsymbol{x}) \boldsymbol{b}(\boldsymbol{y})\, \mathrm{d}\lambda \mathrm{d}\boldsymbol{x} = -\int_\Omega \int_Q \big[\nabla\varphi(\boldsymbol{x}) \times \boldsymbol{u}_0(\boldsymbol{x})\big] \cdot \boldsymbol{b}(\boldsymbol{y})\, \mathrm{d}\lambda \mathrm{d}\boldsymbol{x}.$$

Using the fact that $\mathbf{curl}_x \boldsymbol{u}_0 \cdot \langle \boldsymbol{b} \rangle \in L^2(\Omega)$ in the distributional sense, it follows that

$$\int_\Omega \int_Q \boldsymbol{p}(\boldsymbol{x},\boldsymbol{y}) \cdot \varphi(\boldsymbol{x}) \boldsymbol{b}(\boldsymbol{y})\, \mathrm{d}\lambda \mathrm{d}\boldsymbol{x} = \int_\Omega \mathbf{curl}_x \boldsymbol{u}_0 \cdot \varphi(\boldsymbol{x}) \langle \boldsymbol{b} \rangle\, \mathrm{d}\boldsymbol{x}.$$

Hence, by a similar argument to the argument presented in Zhikov [82, Theorem 9.5], $\mathbf{curl}_x \boldsymbol{u}_0 \in [L^2(\Omega)]^3$ and therefore $\boldsymbol{u}_0 \in H^1_{\mathbf{curl}}(\Omega)$. Rewriting the above in the form

$$\int_\Omega \int_Q \big[\boldsymbol{p}(\boldsymbol{x},\boldsymbol{y}) - \mathbf{curl}_x \boldsymbol{u}_0\big] \cdot \varphi(\boldsymbol{x}) \boldsymbol{b}(\boldsymbol{y})\, \mathrm{d}\lambda \mathrm{d}\boldsymbol{x} = 0,$$

as vectors of the form $\varphi(\boldsymbol{x}) \boldsymbol{b}(\boldsymbol{y})$ are dense in $L^2(\Omega, V_{\mathrm{pot}})$ it follows that

$$\boldsymbol{p}(\boldsymbol{x},\boldsymbol{y}) = \mathbf{curl}_x \boldsymbol{u}_0(\boldsymbol{x}) + \boldsymbol{v}(\boldsymbol{x},\boldsymbol{y}), \quad \boldsymbol{v}(\boldsymbol{x},\cdot) \in V_{\mathrm{sol}}, \tag{3.2.13}$$

as required.

Since it can be shown that both sequences $\boldsymbol{u}_\varepsilon^h$ and $\mathbf{curl}\, \boldsymbol{u}_\varepsilon^h$ are bounded in $[L^2(\Omega, \mathrm{d}\lambda_\varepsilon^h)]^3$, it follows that $\boldsymbol{u}_\varepsilon^h \in [H^1_{\mathbf{curl},0}(\Omega)]^3$. Let $\tilde{\boldsymbol{u}}_\varepsilon^h \in [C_0^\infty(\Omega)]^3$, be a sequence which approximates

$\boldsymbol{u}_\varepsilon^h$ and extend this approximation by zero to some larger domain $\widehat{\Omega} \supset \overline{\Omega}$. It is obvious that the two-scale limit of $\boldsymbol{u}_\varepsilon^h$ in $\widehat{\Omega}$ coincides with the extension by zero of the limit in Ω. Thus, outside Ω, $\boldsymbol{u}_0$ is zero and $\boldsymbol{u}_0 \in H^1_{\mathbf{curl}}(\Omega)$. By the assumption that Ω has a Lipschitz boundary, it follows that $\boldsymbol{u}_0 \in H^1_{\mathbf{curl},0}(\Omega)$ (See Dautry & Lions [26, Page 204]). $\square$

One final convergence result of interest is the so called convergence of momenta. Define the homogenised tensor (*c.f.* equation (2.2.9)) by the following minimisation problem:

$$A^{\mathrm{hom}}\boldsymbol{\xi}\cdot\boldsymbol{\xi} = \min_{\boldsymbol{v}\in V_{\mathrm{sol}}} \int_Q A(\boldsymbol{\xi}+\boldsymbol{v})\cdot(\boldsymbol{\xi}+\boldsymbol{v})\,\mathrm{d}\lambda. \tag{3.2.14}$$

Note that alternative formulations for the homogenised tensor can be given in the case of Maxwell's equations (see Cherednichenko & Cooper [23]). The Euler-Lagrange equation for the above minimisation problem is given as

$$\boldsymbol{v}\in V_{\mathrm{sol}}, \qquad \int_Q A(\boldsymbol{\xi}+\boldsymbol{v})\cdot\boldsymbol{\varphi}\,\mathrm{d}\lambda = 0, \quad \forall\boldsymbol{\varphi}\in V_{\mathrm{sol}}.$$

Equivalently:

$$\mathbf{curl}\,(A(\boldsymbol{\xi}+\boldsymbol{v})) = \mathbf{0}. \tag{3.2.15}$$

Hence, $A(\boldsymbol{\xi}+\boldsymbol{v}) \in V_{\mathrm{pot}}$ and moreover, for $\boldsymbol{v}$ the solution to (3.2.15), it follows that

$$A^{\mathrm{hom}}\boldsymbol{\xi}\cdot\boldsymbol{\xi} = \int_Q A(\boldsymbol{\xi}+\boldsymbol{v})\cdot(\boldsymbol{\xi}+\boldsymbol{v})\,\mathrm{d}\lambda = \left(\int_Q A(\boldsymbol{\xi}+\boldsymbol{v})\,\mathrm{d}\lambda\right)\cdot\boldsymbol{\xi},$$

and so in conclusion,

$$A^{\mathrm{hom}}\boldsymbol{\xi} = \int_Q A(\boldsymbol{\xi}+\boldsymbol{v})\,\mathrm{d}\lambda. \tag{3.2.16}$$

Consider the following theorem.

Theorem 3.2.4. *Suppose that*

$$\lim_{\varepsilon\to 0}\int_\Omega A^\varepsilon(\boldsymbol{x})\mathbf{curl}\ \boldsymbol{u}_\varepsilon^h(\boldsymbol{x})\cdot\mathbf{curl}_y\boldsymbol{w}\Big(\frac{\boldsymbol{x}}{\varepsilon}\Big)\varphi(\boldsymbol{x})\,\mathrm{d}\lambda_\varepsilon^h = 0, \quad \varphi\in C_0^\infty(\Omega), \quad \boldsymbol{w}\in[C^\infty_{\mathrm{per}}(Q)]^3. \tag{3.2.17}$$

Then the following weak convergence follows:

$$A^\varepsilon\mathbf{curl}\,\boldsymbol{u}_\varepsilon^h \rightharpoonup A^{\mathrm{hom}}\mathbf{curl}_x\boldsymbol{u}_0 \quad \text{in} \quad [L^2(\Omega,\mathrm{d}\lambda_\varepsilon^h)]^3.$$

Moreover, the vector $\boldsymbol{v}$ seen in equation (3.2.13) is the solution to the periodic problem (3.2.15) for $\boldsymbol{\xi} = \mathbf{curl}_x\boldsymbol{u}_0$. That is

$$\mathbf{curl}_y\,(A(\boldsymbol{y})(\mathbf{curl}_x\boldsymbol{u}_0(\boldsymbol{x}) + \boldsymbol{v}(\boldsymbol{x},\boldsymbol{y})) = \mathbf{0}.$$

Proof. From the general theory of weak two-scale convergence, it immediately follows that

$$\begin{aligned} A^\varepsilon\mathbf{curl}\,\boldsymbol{u}_\varepsilon^h &\overset{2}{\rightharpoonup} A(\boldsymbol{y})[\mathbf{curl}_x\boldsymbol{u}_0(\boldsymbol{x}) + \boldsymbol{v}(\boldsymbol{x},\boldsymbol{y})],\\ A^\varepsilon\mathbf{curl}\,\boldsymbol{u}_\varepsilon^h &\rightharpoonup \int_Q A(\boldsymbol{y})[\mathbf{curl}_x\boldsymbol{u}_0(\boldsymbol{x}) + \boldsymbol{v}(\boldsymbol{x},\boldsymbol{y})]\,\mathrm{d}\lambda. \end{aligned}$$

In particular, taking the weak two-scale limit of equation (3.2.17) yields

$$\int_\Omega \int_Q A(\boldsymbol{y})[\mathbf{curl}_x \boldsymbol{u}_0(\boldsymbol{x}) + \boldsymbol{v}(\boldsymbol{x},\boldsymbol{y})]\varphi(\boldsymbol{x}) \cdot \mathbf{curl}_y \boldsymbol{w}(\boldsymbol{y}) \,\mathrm{d}\lambda \mathrm{d}\boldsymbol{x} = 0.$$

As φ is arbitrary in a space dense in $L^2(\Omega)$, it follows that

$$\int_Q A(\boldsymbol{y})[\mathbf{curl}_x \boldsymbol{u}_0(\boldsymbol{x}) + \boldsymbol{v}(\boldsymbol{x},\boldsymbol{y})] \cdot \mathbf{curl}_y \boldsymbol{w}(\boldsymbol{y}) \,\mathrm{d}\lambda = 0.$$

Thus, $\boldsymbol{v}(\boldsymbol{x},\cdot)$ solves the periodic problem (3.2.15) for $\boldsymbol{\xi} = \mathbf{curl}_x \boldsymbol{u}_0$ and moreover, equation (3.2.16) implies that

$$\int_Q A(\boldsymbol{y})[\mathbf{curl}_x \boldsymbol{u}_0(\boldsymbol{x}) + \boldsymbol{v}(\boldsymbol{x},\boldsymbol{y})] \,\mathrm{d}\lambda = A^{\mathrm{hom}} \mathbf{curl}_x \boldsymbol{u}_0.$$

Thus by the convergence above, the theorem is proved. □

Remark. Note that if the solution $\boldsymbol{v}$ to the problem (3.2.15) above is written in the form $\boldsymbol{v}(\boldsymbol{x},\boldsymbol{y}) = N(\boldsymbol{y})\mathbf{curl}_x \boldsymbol{u}_0(\boldsymbol{x})$ where $N(\boldsymbol{y})$ is a 3×3 solenoidal matrix, then these results would coincide with the analysis of Chapter 1. Moreover, since N is solenoidal, the existence of a matrix $N^{(1)}$ is infered such that $N(\boldsymbol{y}) = \mathbf{curl}_y N^{(1)}(\boldsymbol{y})$ which leads to the solution of the problem above exactly coinciding with the solution of equation (1.1.9).

The derivation of the homogenised equation will be presented but firstly, the energy space V is defined.

Definition 3.2.2. The set V contains all those function $\boldsymbol{\varphi} = \boldsymbol{\varphi}(\boldsymbol{x},\boldsymbol{y})$ such that

$$\boldsymbol{\varphi}(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{\varphi}_0(\boldsymbol{x}) + \boldsymbol{\Psi}(\boldsymbol{x},\boldsymbol{y}), \quad \boldsymbol{\varphi}_0 \in [H^1_{\mathbf{curl},0}(\Omega)]^3, \quad \boldsymbol{\Psi}(\boldsymbol{x},\boldsymbol{y}) \in L^2(\Omega, V_{\mathrm{pot}}).$$

The following is the two-scale homogenised equation which captures the problem described by (3.2.1):

$$\int_\Omega A^{\mathrm{hom}} \mathbf{curl}\, \boldsymbol{u}_0 \cdot \mathbf{curl}\, \boldsymbol{\varphi} \,\mathrm{d}\boldsymbol{x} - \int_\Omega \int_Q (\boldsymbol{u}_0 + \boldsymbol{\Pi}) \cdot \boldsymbol{\varphi} \,\mathrm{d}\lambda \mathrm{d}\boldsymbol{x} = \int_\Omega \int_Q \boldsymbol{f} \cdot \boldsymbol{\varphi} \,\mathrm{d}\lambda \mathrm{d}\boldsymbol{x}. \qquad (3.2.18)$$

In differential form, equation (3.2.18) may be written as the following system:

$$\mathbf{curl}\, A^{\mathrm{hom}} \mathbf{curl}\, \boldsymbol{u}_0 - \omega \langle \boldsymbol{u}_0 + \boldsymbol{\Pi} \rangle = \langle \boldsymbol{f} \rangle, \qquad (3.2.19)$$

$$\boldsymbol{u}_0 \in [H^1_{\mathbf{curl},0}(\Omega)]^3, \quad \boldsymbol{\Pi}(\boldsymbol{x},\cdot) \in V_{\mathrm{pot}}, \quad \mathrm{div}_y(\boldsymbol{\Pi}(\boldsymbol{x},\cdot) + \boldsymbol{f}(\boldsymbol{x},\cdot)) = 0. \qquad (3.2.20)$$

It will now be proven that this is indeed the two-scale limit of the original problem.

Theorem 3.2.5. *Let $\boldsymbol{u}^h_\varepsilon$ solve the Dirichlet problem (3.2.1) and let $\boldsymbol{f}_\varepsilon$ be bounded in $[L^2(\Omega, \mathrm{d}\lambda^h_\varepsilon)]^3$ such that $\boldsymbol{f}_\varepsilon(\boldsymbol{x}) \overset{2}{\rightharpoonup} \boldsymbol{f}(\boldsymbol{x},\boldsymbol{y})$ ($\boldsymbol{f}_\varepsilon(\boldsymbol{x}) \overset{2}{\to} \boldsymbol{f}(\boldsymbol{x},\boldsymbol{y})$). Then*

$$\boldsymbol{u}^h_\varepsilon \overset{2}{\rightharpoonup} \boldsymbol{u}(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{u}_0 + \boldsymbol{\Pi}(\boldsymbol{x},\boldsymbol{y}), \qquad (\boldsymbol{u}^h_\varepsilon \overset{2}{\to} \boldsymbol{u}_0 + \boldsymbol{\Pi}(\boldsymbol{x},\boldsymbol{y})),$$

$$\mathbf{curl}\, \boldsymbol{u}^h_\varepsilon \overset{2}{\rightharpoonup} \mathbf{curl}_x \boldsymbol{u}_0 + \boldsymbol{v}(\boldsymbol{x},\boldsymbol{y}), \quad (\mathbf{curl}\, \boldsymbol{u}^h_\varepsilon \overset{2}{\to} \mathbf{curl}_x \boldsymbol{u}_0 + \boldsymbol{v}(\boldsymbol{x},\boldsymbol{y})),$$

$$\boldsymbol{u}_0 \in [H^1_{\mathbf{curl},0}(\Omega)]^3, \quad \boldsymbol{\Pi}(\boldsymbol{x},\cdot) \in V_{\text{pot}}.$$

The vector field $\boldsymbol{u}$ *solves (3.2.19) and the vector* $\boldsymbol{v}$ *is the periodic solution to the cell problem (3.2.15) with* $\boldsymbol{\xi} = \mathbf{curl}_x \boldsymbol{u}_0$*. Moreover, if* $h/\varepsilon \to 0$ *as* $\varepsilon \to 0$ *then relation (3.2.20) also holds*

Proof. Consider the weak formulation (3.2.2). Using standard inequalities implies that the sequences $\boldsymbol{u}^h_\varepsilon$ and $\mathbf{curl}\,\boldsymbol{u}^h_\varepsilon$ are bounded in $[L^2(\Omega, \mathrm{d}\lambda^h_\varepsilon)]^3$. Hence, without loss of generality, it can be assumed that the convergences (3.2.10) and (3.2.12) hold. Take as test functions from the energy space V, firstly $\boldsymbol{\varphi}(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{\varphi}_0(\boldsymbol{x})$. Hence, in the limit as $\varepsilon \to 0$, equation (3.2.2) tends to

$$\int_\Omega \int_Q A(\boldsymbol{y})\,(\mathbf{curl}_x \boldsymbol{u}_0 + \boldsymbol{v}(\boldsymbol{x},\boldsymbol{y}))\cdot \mathbf{curl}_x \boldsymbol{\varphi}_0(\boldsymbol{x})\,\mathrm{d}\lambda \mathrm{d}\boldsymbol{x} - \int_\Omega \int_Q \boldsymbol{u}\cdot\boldsymbol{\varphi}_0\,\mathrm{d}\lambda\mathrm{d}\boldsymbol{x} = \int_\Omega\int_Q \boldsymbol{f}\cdot\boldsymbol{\varphi}_0\,\mathrm{d}\lambda\mathrm{d}\boldsymbol{x}.$$

By the convergence of momenta (which can be seen by taking test functions of the form $\boldsymbol{\psi}(\boldsymbol{x}) = \varepsilon\varphi(\boldsymbol{x})\boldsymbol{w}(\boldsymbol{x}/\varepsilon)$, with $\varphi \in C^\infty_0(\Omega)$ and $\boldsymbol{w} \in [C^\infty_{\text{per}}(Q)]^3$ in equation (3.2.2)), it follows that

$$\int_\Omega A^{\text{hom}}\mathbf{curl}\,\boldsymbol{u}_0 \cdot \mathbf{curl}\,\boldsymbol{\varphi}_0\,\mathrm{d}\boldsymbol{x} - \int_\Omega\int_Q \boldsymbol{u}\cdot\boldsymbol{\varphi}_0\,\mathrm{d}\lambda\mathrm{d}\boldsymbol{x} = \int_\Omega\int_Q \boldsymbol{f}\cdot\boldsymbol{\varphi}_0\,\mathrm{d}\lambda\mathrm{d}\boldsymbol{x}.$$

To obtain the relation (3.2.20), test functions of the form $\boldsymbol{\varphi}(\boldsymbol{x},\boldsymbol{y}) = \psi(\boldsymbol{x})\boldsymbol{g}_h(\boldsymbol{y})$ will be taken where $\psi \in [C^\infty_0(\Omega)]^3$ and $\boldsymbol{g}_h$ is a sequence as seen in (3.2.7) which approximates a potential vector $\boldsymbol{g}$. Hence equation (3.2.2) now takes the form:

$$\int_\Omega A^\varepsilon \mathbf{curl}\,\boldsymbol{u}^h_\varepsilon\cdot(\mathbf{curl}\,\boldsymbol{g}_h)\psi\,\mathrm{d}\lambda^h_\varepsilon + \int_\Omega A^\varepsilon\mathbf{curl}\,\boldsymbol{u}^h_\varepsilon\cdot(\nabla\psi\times\boldsymbol{g}_h)\,\mathrm{d}\lambda^h_\varepsilon = \int_\Omega(\boldsymbol{u}^h_\varepsilon + \boldsymbol{f}_\varepsilon)\cdot\boldsymbol{g}_h\psi\,\mathrm{d}\lambda^h_\varepsilon,$$

The first integral will vanish in the limit as $\varepsilon \to 0$ by the approximation (3.2.8). The second integral vanishes in the limit as $\varepsilon \to 0$ since by convergence (3.2.9) and the fact that the the natural extension $[\boldsymbol{g}]^h$ converges to $\boldsymbol{g}$, by a result in Zhikov [82, Lemma 5.3], $A(\cdot)(\mathbf{curl}_x\boldsymbol{u}_0(\boldsymbol{x}) + \boldsymbol{v}(\boldsymbol{x},\cdot)) \perp (\boldsymbol{g}(\cdot)\times\nabla\varphi(\boldsymbol{x}))$. Therefore, in the limit as $\varepsilon \to 0$ equation (3.2.2) takes the form

$$\int_\Omega\int_Q(\boldsymbol{u}(\boldsymbol{x},\boldsymbol{y}) + \boldsymbol{f}(\boldsymbol{x},\boldsymbol{y}))\cdot\boldsymbol{g}(\boldsymbol{y})\psi(\boldsymbol{x})\,\mathrm{d}\lambda\mathrm{d}\boldsymbol{x} = 0.$$

Hence the result.

Assume now that there is strong two-scale convergence $\boldsymbol{f}_\varepsilon \xrightarrow{2} \boldsymbol{f}$. Substituting $\boldsymbol{\varphi} = \boldsymbol{u}^h_\varepsilon$ into the weak formulation (3.2.2) yields

$$\int_\Omega A^\varepsilon\mathbf{curl}\,\boldsymbol{u}^h_\varepsilon\cdot\mathbf{curl}\,\boldsymbol{u}^h_\varepsilon\,\mathrm{d}\lambda^h_\varepsilon - \int_\Omega|\boldsymbol{u}^h_\varepsilon|^2\,\mathrm{d}\lambda^h_\varepsilon = \int_\Omega\boldsymbol{f}\cdot\boldsymbol{u}^h_\varepsilon\,\mathrm{d}\lambda^h_\varepsilon.$$

Using the fact that there is weak two-scale convergence $\boldsymbol{u}^h_\varepsilon \overset{2}{\rightharpoonup} \boldsymbol{u}$ and Proposition (2.1.2) it follows that:

$$\int_\Omega \int_Q A(\boldsymbol{y})\,(\mathbf{curl}_x \boldsymbol{u}_0 + \boldsymbol{v}(\boldsymbol{x},\boldsymbol{y})) \cdot (\mathbf{curl}_x \boldsymbol{u}_0 + \boldsymbol{v}(\boldsymbol{x},\boldsymbol{y}))\,\mathrm{d}\lambda \mathrm{d}\boldsymbol{x} - \int_\Omega \int_Q |\boldsymbol{u}|^2\,\mathrm{d}\lambda \mathrm{d}\boldsymbol{x} =$$

$$= \int_\Omega \int_Q \boldsymbol{f}\cdot\boldsymbol{u}\,\mathrm{d}\lambda \mathrm{d}\boldsymbol{x} = \lim_{\varepsilon\to 0} \int_\Omega \boldsymbol{f}_\varepsilon \cdot \boldsymbol{u}_\varepsilon^h\,\mathrm{d}\lambda_\varepsilon^h = \lim_{\varepsilon\to 0} \left\{ \int_\Omega A^\varepsilon \mathbf{curl}\,\boldsymbol{u}_\varepsilon^h \cdot \mathbf{curl}\,\boldsymbol{u}_\varepsilon^h\,\mathrm{d}\lambda_\varepsilon^h - \int_\Omega |\boldsymbol{u}_\varepsilon^h|^2\,\mathrm{d}\lambda_\varepsilon^h \right\} \geq$$

$$\geq \int_\Omega \int_Q A(\boldsymbol{y})\,(\mathbf{curl}_x \boldsymbol{u}_0 + \boldsymbol{v}(\boldsymbol{x},\boldsymbol{y})) \cdot (\mathbf{curl}_x \boldsymbol{u}_0 + \boldsymbol{v}(\boldsymbol{x},\boldsymbol{y}))\,\mathrm{d}\lambda \mathrm{d}\boldsymbol{x} - \int_\Omega \int_Q |\boldsymbol{u}|^2\,\mathrm{d}\lambda \mathrm{d}\boldsymbol{x}.$$

Therefore, by the Squeeze Theorem

$$\lim_{\varepsilon\to 0} \int_\Omega |\boldsymbol{u}_\varepsilon^h|^2\,\mathrm{d}\lambda_\varepsilon^h = \int_\Omega \int_Q |\boldsymbol{u}|^2\,\mathrm{d}\lambda \mathrm{d}\boldsymbol{x}, \tag{3.2.21}$$

$$\lim_{\varepsilon\to 0} \int_\Omega A^\varepsilon \mathbf{curl}\,\boldsymbol{u}_\varepsilon^h \cdot \mathbf{curl}\,\boldsymbol{u}_\varepsilon^h\,\mathrm{d}\lambda_\varepsilon^h = \int_\Omega \int_Q A(\boldsymbol{y})\,(\mathbf{curl}_x \boldsymbol{u}_0 + \boldsymbol{v}(\boldsymbol{x},\boldsymbol{y}))\cdot(\mathbf{curl}_x \boldsymbol{u}_0 + \boldsymbol{v}(\boldsymbol{x},\boldsymbol{y}))\,\mathrm{d}\lambda \mathrm{d}\boldsymbol{x}. \tag{3.2.22}$$

Weak two-scale convergence plus equality (3.2.21) imply the strong two-scale convergence $\boldsymbol{u}_\varepsilon^h(\boldsymbol{x}) \xrightarrow{2} \boldsymbol{u}(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{u}_0(\boldsymbol{x}) + \boldsymbol{\Pi}(\boldsymbol{x},\boldsymbol{y})$. Finally, in light of the weak two-scale convergence $\mathbf{curl}\,\boldsymbol{u}_\varepsilon^h \xrightharpoonup{2} \mathbf{curl}_x \boldsymbol{u}_0 + \boldsymbol{v}$, there is weak two-scale convergence

$$(A^\varepsilon)^{1/2}(\boldsymbol{x})\mathbf{curl}\,\boldsymbol{u}_\varepsilon^h(\boldsymbol{x}) \xrightharpoonup{2} A^{1/2}(\boldsymbol{y})(\mathbf{curl}_x \boldsymbol{u}_0(\boldsymbol{x}) + \boldsymbol{v}(\boldsymbol{x},\boldsymbol{y})).$$

This convergence is made strong in light of the convergence (3.2.22) and hence by the definition of strong two-scale convergence (see Definition 2.1.3), there is strong two-scale convergence $\mathbf{curl}\,\boldsymbol{u}_\varepsilon^h \xrightarrow{2} \mathbf{curl}_x \boldsymbol{u}_0 + \boldsymbol{v}$ as required. □

Remark. In the case of classical homogenisation of Maxwell's equations, that is, the homogenisation of Maxwell's equations on a bounded domain with periodic structure but no rod framework, the same conclusions can be made with regards the structure of the homogenised equation.

Further Discussions

Many possible extensions of the above work can be carried out. Firstly, it may be of interest to carry out the relevant analysis for rod structures which are sufficiently thick ($\lim_{\varepsilon\to 0} a/\varepsilon^2 \to \infty$) and for rod structures which are critically thick ($\lim_{\varepsilon\to 0} a/\varepsilon^2 \to \theta > 0$, *c.f.* Chapter 2) and derive the homogenised models for these problems. It is suspected that these results will not be too different from the results presented in [82] and [84] for the elasticity case.

A study of the spectrum for the $\mathbf{curl}\,\hat{A}^\varepsilon\mathbf{curl}$ operator on rod structures (of all cases of $\lim_{\varepsilon\to 0} a/\varepsilon^2$) is also a problem to be examined. Once again, it is suspected that the analysis should follow similar lines to that of [81] but the non-trivial kernel of the **curl** operator may provide minor hurdles to the analysis.

Extending the rod structure problem further to a setting where the space between the rods is filled with a soft material which is in high-contrast is also work to be carried out. In [23], the authors investigate the nature of an electromagnetics problem for a high-contrast, periodic dielectric medium but with no thin structure. It would be of interest to see what similarities lie between the results of this work and also the corresponding results for rod structures.

Electrodynamics problems depend sensitively on the geometry of the domain. In particular, the analysis of a stiff rod structure with soft inclusions will be different depending on whether or not the rod structure leaves the soft inclusion simply connected or not and it would be expected to yield a homogenised equation of a different nature in the case of a multiply connected soft inclusion.

3.3 Homogenisation of Maxwell's Equations under a Chiral Transformation

3.3.1 Change of Coordinates and Governing Equations

In this section, work related to the motivation of this thesis is presented. It has been suggested that there is a connection between materials possessing a negative refractive index and materials which a chiral geometry. In the work to follow, a chiral transformation of the system of Maxwell equations will be followed by a homogenisation of the transformed system to try and gain insight into this suggested link.

Let $\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 \backslash \{x_1 = 0, x_2 = 0\} =: \widetilde{\mathbb{R}}^3$ denote the Cartesian coordinates and let $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3) \in \widetilde{\mathbb{R}}^3$ be the chiral coordinates which are given by the following transformation:

$$\begin{cases} \xi_1 = x_1 \cos(\alpha x_3) - x_2 \sin(\alpha x_3), \\ \xi_2 = x_1 \sin(\alpha x_3) + x_2 \cos(\alpha x_3), \\ \xi_3 = x_3, \end{cases} \qquad \begin{cases} x_1 = \quad \xi_1 \cos(\alpha \xi_3) + \xi_2 \sin(\alpha \xi_3), \\ x_2 = -\xi_1 \sin(\alpha \xi_3) + \xi_2 \cos(\alpha \xi_3), \\ x_3 = \xi_3, \end{cases}$$

where α is a parameter which characterises torsion. Note that $x_1^2 + x_2^2 = \xi_1^2 + \xi_2^2$ and distance is preserved under this coordinate change. The associated Jacobian matrices for the above coordinate changes are

$$J_\xi := \begin{pmatrix} \cos(\alpha\xi_3) & -\sin(\alpha\xi_3) & -\alpha\xi_2 \\ \sin(\alpha\xi_3) & \cos(\alpha\xi_3) & \alpha\xi_1 \\ 0 & 0 & 1 \end{pmatrix}, \quad J_x := \begin{pmatrix} \cos(\alpha x_3) & \sin(\alpha x_3) & \alpha x_2 \\ -\sin(\alpha x_3) & \cos(\alpha x_3) & -\alpha x_1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let ∇_x and ∇_ξ denote the gradients with respect to the variables $\boldsymbol{x}$ and $\boldsymbol{\xi}$ respectively. By the chain rule, the following relations are derived between the gradients ∇_x and ∇_ξ:

$$\nabla_\xi = J_x^T \nabla_x, \quad \nabla_x = J_\xi^T \nabla_\xi.$$

Denote by T_ξ^α and T_x^α the transformation matrices defined by the following relations:

$$T_\xi^\alpha := \frac{J_\xi J_\xi^T}{\det J_\xi} = \begin{pmatrix} 1+\alpha^2\xi_2^2 & -\alpha^2\xi_1\xi_2 & -\alpha\xi_2 \\ -\alpha^2\xi_1\xi_2 & 1+\alpha^2\xi_1^2 & \alpha\xi_1 \\ -\alpha\xi_2 & \alpha\xi_1 & 1 \end{pmatrix}, \quad T_x^\alpha := \frac{J_x J_x^T}{\det J_x} = \begin{pmatrix} 1+\alpha^2 x_2^2 & -\alpha^2 x_1 x_2 & \alpha x_2 \\ -\alpha^2 x_1 x_2 & 1+\alpha^2 x_1^2 & -\alpha x_1 \\ \alpha x_2 & -\alpha x_1 & 1 \end{pmatrix}.$$

Note that $\det J_\xi = \det J_x = 1$.

Remark. Note that the transformation matrix T_ξ^α depends only on the variable $\boldsymbol{y} := \alpha\boldsymbol{\xi}$, i.e., $T_\xi^\alpha = T(\boldsymbol{y})$. Moreover, T_ξ^α is independent of the variable ξ_3.

A well known property of Maxwell's equations is the fact that they are invariant under coordinate transformations (see Johnson [41] for full details).

Theorem 3.3.1. *Consider the system of Maxwell equations:*

$$\mathbf{curl}\,\boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t}, \quad \mathbf{curl}\,\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \tag{3.3.1}$$

$$\boldsymbol{D} = \varepsilon \boldsymbol{E}, \quad \boldsymbol{B} = \mu \boldsymbol{H}, \tag{3.3.2}$$

$$\operatorname{div} \boldsymbol{D} = 0, \quad \operatorname{div} \boldsymbol{B} = 0. \tag{3.3.3}$$

Let $\boldsymbol{x} \mapsto \boldsymbol{x}' := (x_1', x_2', x_3')$ *be a coordinate transformation with Jacobian matrix* $J_{ij} = \partial x_i'/\partial x_j$*. Then the system of Maxwell equations in the new coordinates is given as*

$$\mathbf{curl}'\,\boldsymbol{H}' = \frac{\partial \boldsymbol{D}'}{\partial t}, \quad \mathbf{curl}'\,\boldsymbol{E}' = -\frac{\partial \boldsymbol{B}'}{\partial t}, \tag{3.3.4}$$

$$\boldsymbol{B}' = \mu' \boldsymbol{H}', \quad \boldsymbol{D}' = \varepsilon' \boldsymbol{E}', \tag{3.3.5}$$

$$\operatorname{div}' \boldsymbol{D}' = 0, \quad \operatorname{div}' \boldsymbol{B}' = 0, \tag{3.3.6}$$

where

$$\boldsymbol{H}' = (J^T)^{-1}\boldsymbol{H}, \quad \boldsymbol{E}' = (J^T)^{-1}\boldsymbol{E}, \quad \boldsymbol{B}' = \frac{J\boldsymbol{B}}{\det J}, \quad \boldsymbol{D}' = \frac{J\boldsymbol{D}}{\det J}, \quad \varepsilon' = \frac{J\varepsilon J^T}{\det J}, \; \mu' = \frac{J\mu J^T}{\det J},$$

and where $\mathbf{curl}'$ *and* div' *are the curl and divergence with respect to the variable* $\boldsymbol{x}'$

Proof. Equation (3.3.4) will be shown only as showing the other equations is similar. Consider equation (3.3.1) written in index notation:

$$\epsilon_{abc}\partial_a H_b = \varepsilon_{cd}\frac{\partial E_d}{\partial t},$$

where ϵ_{abc} is the usual Levi-Civita symbol. Under the change in coordinates, $\boldsymbol{x} \mapsto \boldsymbol{x}'$, $\partial_a = J_{ba}\partial_b'$, and hence

$$\epsilon_{abc}J_{ia}\partial_i' J_{jb}H_j' = \varepsilon_{cd}J_{ld}\frac{\partial E_l'}{\partial t}.$$

Noting that $\epsilon_{abc}J_{ia}\partial_i' J_{jb} = \epsilon_{abc}\partial_a J_{jb} = 0$, since $\partial_a J_{jb} = \partial_b J_{ja}$, after multiplying both side of the above by J_{kc} yields

$$\epsilon_{abc}J_{kc}J_{jb}J_{ia}\partial_i' H_j' = J_{kc}\varepsilon_{cd}J_{ld}\frac{\partial E_l'}{\partial t}.$$

Hence, noticing that $\epsilon_{abc}J_{kc}J_{jb}J_{ia} = \epsilon_{ijk}\det J$, back in vector notation, the above equation may be written in the following way

$$\mathbf{curl}'\,\boldsymbol{H}' = \frac{J\varepsilon J^T}{\det J}\frac{\partial \boldsymbol{E}'}{\partial t},$$

as required. □

3.3.2 Formulation of the Problem and Solution by Asymptotic Expansion

Consider equations (3.3.4)-(3.3.6) with the prime notation dropped for convenience and where the transformation is $\boldsymbol{x} \mapsto \boldsymbol{\xi}$. It is assumed that the vectors $\boldsymbol{E}$ and $\boldsymbol{H}$ can be written in time harmonic form, i.e.

$$\boldsymbol{E}(\boldsymbol{\xi}, t) = \widetilde{\boldsymbol{E}}(\boldsymbol{\xi})\mathrm{e}^{\mathrm{i}\omega t}, \quad \boldsymbol{H}(\boldsymbol{\xi}, t) = \widetilde{\boldsymbol{H}}(\boldsymbol{\xi})\mathrm{e}^{\mathrm{i}\omega t}, \quad \omega \in \mathbb{R}.$$

Hence the equations of consideration are

$$\mathbf{curl}\, \widetilde{\boldsymbol{H}} = \mathrm{i}\omega\varepsilon\widetilde{\boldsymbol{E}}, \quad \mathbf{curl}\, \widetilde{\boldsymbol{E}} = -\mathrm{i}\omega\mu\widetilde{\boldsymbol{H}}, \tag{3.3.7}$$

$$\mathrm{div}(\varepsilon\widetilde{\boldsymbol{E}}) = 0, \quad \mathrm{div}(\mu\widetilde{\boldsymbol{H}}) = 0. \tag{3.3.8}$$

It is assumed further that in the original Cartesian coordinates that $\varepsilon = \varepsilon_0(\alpha x_3)I$, $\mu = \mu_0(\alpha x_3)I$ where ε_0, μ_0 are 1-periodic functions. Hence in the chiral coordinates the permittivity and permeability matrices (still denoted ε and μ respectively) are given by the formulae

$$\varepsilon = \varepsilon_0(y_3)T(\boldsymbol{y}), \quad \mu = \mu_0(y_3)T(\boldsymbol{y}),$$

where $T(\boldsymbol{y})$ is the transformation matrix seen in Section 3.3.1. Note that the constitutive equations (3.3.5) have been substituted into equations (3.3.4), (3.3.6).

Assume that the vector fields $\widetilde{\boldsymbol{H}}$, $\widetilde{\boldsymbol{E}}$, now denoted $\boldsymbol{H}^\alpha$, $\boldsymbol{E}^\alpha$, can be expanded in asymptotic expansions of the torsion parameter α:

$$\boldsymbol{H}^\alpha(\boldsymbol{\xi}) = \boldsymbol{H}_0(\boldsymbol{\xi}, \alpha\boldsymbol{\xi}) + \alpha^{-1}\boldsymbol{H}_1(\boldsymbol{\xi}, \alpha\boldsymbol{\xi}) + O(\alpha^{-1}), \tag{3.3.9}$$

$$\boldsymbol{E}^\alpha(\boldsymbol{\xi}) = \boldsymbol{E}_0(\boldsymbol{\xi}, \alpha\boldsymbol{\xi}) + \alpha^{-1}\boldsymbol{E}_1(\boldsymbol{\xi}, \alpha\boldsymbol{\xi}) + O(\alpha^{-1}). \tag{3.3.10}$$

Here, the variable $\alpha\boldsymbol{\xi} =: \boldsymbol{y}$ will be defined on the unit cell $Q := (\mathbb{R}^2 \times \mathbb{S}^1)\backslash\{y_1 = 0, y_2 = 0\}$ where $\mathbb{S}^1$ denotes the unit circle, i.e., the interval $[0, 1]$ with it's end points 'glued' together. In other words, functions of $\boldsymbol{y}$ will be 1-periodic in the y_3-direction.

The variables $\boldsymbol{\xi}$ and $\boldsymbol{y}$ are treated indepedently, and hence the curl and divergence operators are written

$$\mathbf{curl} = \mathbf{curl}_\xi + \alpha\mathbf{curl}_y, \quad \mathrm{div} = \mathrm{div}_\xi + \alpha\mathrm{div}_y.$$

Hence, formally substituting the asymptotic series (3.3.9) and (3.3.10), into (3.3.7) and (3.3.8) yields

$$\alpha\mathbf{curl}_y\, \boldsymbol{H}_0 + \{\mathbf{curl}_y\, \boldsymbol{H}_1 + \mathbf{curl}_\xi\, \boldsymbol{H}_0\} + O(\alpha^{-1}) = \mathrm{i}\omega\varepsilon(\boldsymbol{E}_0 + \alpha^{-1}\boldsymbol{E}_1) + O(\alpha^{-1}), \tag{3.3.11}$$

$$\alpha\mathbf{curl}_y\, \boldsymbol{E}_0 + \{\mathbf{curl}_y\, \boldsymbol{E}_1 + \mathbf{curl}_\xi\, \boldsymbol{E}_0\} + O(\alpha^{-1}) = -\mathrm{i}\omega\mu(\boldsymbol{H}_0 + \alpha^{-1}\boldsymbol{H}_1) + O(\alpha^{-1}). \tag{3.3.12}$$

$$\alpha\operatorname{div}_y(\mu\boldsymbol{H}_0) + \{\operatorname{div}_y(\mu\boldsymbol{H}_1) + \operatorname{div}_\xi(\mu\boldsymbol{H}_0)\} + O(\alpha^{-1}) = 0 \tag{3.3.13}$$

$$\alpha\operatorname{div}_y(\varepsilon\boldsymbol{E}_0) + \{\operatorname{div}_y(\varepsilon\boldsymbol{E}_1) + \operatorname{div}_\xi(\varepsilon\boldsymbol{E}_0)\} + O(\alpha^{-1}) = 0 \tag{3.3.14}$$

Hence the leading order equations from (3.3.11) and (3.3.12) are given as

$$O(\alpha) : \begin{cases} \mathbf{curl}_y \boldsymbol{H}_0(\boldsymbol{\xi}, \boldsymbol{y}) = \mathbf{0}, \\ \mathbf{curl}_y \boldsymbol{E}_0(\boldsymbol{\xi}, \boldsymbol{y}) = \mathbf{0}, \end{cases} \qquad (\boldsymbol{\xi}, \boldsymbol{y}) \in \widetilde{\mathbb{R}}^3 \times Q, \tag{3.3.15}$$

$$O(1) : \begin{cases} \mathbf{curl}_y \boldsymbol{H}_1(\boldsymbol{\xi}, \boldsymbol{y}) = -\mathbf{curl}_\xi \boldsymbol{H}_0(\boldsymbol{\xi}, \boldsymbol{y}) + \mathrm{i}\omega\varepsilon(\boldsymbol{y})\boldsymbol{E}_0(\boldsymbol{\xi}, \boldsymbol{y}), \\ \mathbf{curl}_y \boldsymbol{E}_1(\boldsymbol{\xi}, \boldsymbol{y}) = -\mathbf{curl}_\xi \boldsymbol{E}_0(\boldsymbol{\xi}, \boldsymbol{y}) - \mathrm{i}\omega\mu(\boldsymbol{y})\boldsymbol{H}_0(\boldsymbol{\xi}, \boldsymbol{y}). \end{cases} \qquad (\boldsymbol{\xi}, \boldsymbol{y}) \in \widetilde{\mathbb{R}}^3 \times Q. \tag{3.3.16}$$

The solution of the first equation in (3.3.15) on the domain Q can be shown (see Appendix 3.C) to take the form

$$\boldsymbol{H}_0(\boldsymbol{\xi}, \boldsymbol{y}) = \alpha_H(\boldsymbol{\xi})\boldsymbol{q}(y_1, y_2) + \beta_H(\boldsymbol{\xi})\boldsymbol{e}_3 + \nabla_y\varphi_H(\boldsymbol{\xi}, \boldsymbol{y}), \tag{3.3.17}$$

where $\varphi_H \in H^1(\widetilde{\mathbb{R}}^3 \times Q)$, $\alpha_H, \beta_H \in H^1(\widetilde{\mathbb{R}}^3)$ are to be determined and

$$\boldsymbol{q} := \begin{pmatrix} q_1 \\ q_2 \\ 0 \end{pmatrix}, \quad \begin{cases} q_1 = \dfrac{\partial q}{\partial y_1}, \\ q_2 = \dfrac{\partial q}{\partial y_2}, \end{cases} \quad q(y_1, y_2) = \begin{cases} \tan^{-1}\left(\dfrac{y_2}{y_1}\right), & y_1 > 0, y_2 > 0, \\ \pi/2, & y_1 = 0, y_2 > 0, \\ \pi + \tan^{-1}\left(\dfrac{y_2}{y_1}\right), & y_1 < 0, \\ 3\pi/2, & y_1 = 0, y_2 < 0, \\ 2\pi + \tan^{-1}\left(\dfrac{y_2}{y_1}\right), & y_1 > 0, y_2 < 0. \end{cases}$$

Notice that the function q is curl-free and divergence-free and hence also harmonic on $\mathbb{R}^2\backslash\{y_2 = 0, y_1 > 0\}$ as well.

The function φ_H is determined via the following problem which is obtained from the leading order equation in (3.3.13):

$$-\operatorname{div}_y\left(\mu\nabla_y\varphi_H\right) = (\alpha_H + \beta_H)\mu_0', \tag{3.3.18}$$

where the prime notation denotes differentiation with respect to the y_3 variable. Clearly

$$\varphi_H(\boldsymbol{\xi}, \boldsymbol{y}) = (\alpha_H(\boldsymbol{\xi}) + \beta_H(\boldsymbol{\xi}))\Phi_H(\boldsymbol{y}) + \sigma_H(\boldsymbol{\xi})\psi_H(\boldsymbol{y})$$

where $\Phi_H, \psi_H \in H^1(Q)$ and solve the equations

$$-\operatorname{div}_y\left(\mu\nabla_y\Phi_H\right) = \mu_0', \quad -\operatorname{div}_y\left(\mu\nabla_y\psi_H\right) = 0. \tag{3.3.19}$$

These equations will be discussed further in Section 3.3.3.

Remark. Note that the following results also hold for the second equation in (3.3.15):

$$\boldsymbol{E}_0(\boldsymbol{\xi},\boldsymbol{y}) = \alpha_E(\boldsymbol{\xi})\boldsymbol{q}(y_1,y_2)+\beta_E(\boldsymbol{\xi})\boldsymbol{e}_3+\nabla_y\varphi_E(\boldsymbol{\xi},\boldsymbol{y}), \quad \varphi_E \in H^1(\widetilde{\mathbb{R}}^3\times Q), \quad \alpha_E,\beta_E \in H^1(\widetilde{\mathbb{R}}^3), \tag{3.3.20}$$

where φ_E solves the equation

$$-\mathrm{div}_y\,(\varepsilon\nabla_y\varphi_E) = (\alpha_E+\beta_E)\varepsilon_0', \tag{3.3.21}$$

and

$$\varphi_E(\boldsymbol{\xi},\boldsymbol{y}) = (\alpha_E(\boldsymbol{\xi})+\beta_E(\boldsymbol{\xi}))\Phi_E(\boldsymbol{y}) + \sigma_E(\boldsymbol{\xi})\psi_E(\boldsymbol{y}), \quad \Phi_E,\psi_E \in H^1(Q),$$

$$-\mathrm{div}_y\,(\varepsilon\nabla_y\Phi_E) = \varepsilon_0', \quad -\mathrm{div}_y\,(\varepsilon\nabla_y\psi_E) = 0. \tag{3.3.22}$$

3.3.3 Homogenised System of Equations

Observe that the equations (3.3.16) are solvable if and only if the right-hand side of each equation is orthogonal in the $L^2(Q)$ sense to all elements of the kernel of the left-hand side. Hence, in light of equation (3.3.17), the following must hold:

$$\int_Q \{-\mathbf{curl}_\xi\boldsymbol{H}_0 + \mathrm{i}\omega\varepsilon\boldsymbol{E}_0\}\cdot\underbrace{(a\boldsymbol{q}+b\boldsymbol{e}_3+c\nabla_y\boldsymbol{\psi})}_{=\boldsymbol{K}}\,\mathrm{d}\boldsymbol{y} = 0,$$

where $a,b,c\in\mathbb{R}$ are constants and $\boldsymbol{\psi}\in H^1(Q)$. Notice that in light of the identity $\mathbf{curl}_\xi(f(\boldsymbol{\xi})\boldsymbol{v}(\boldsymbol{y})) = \nabla_\xi f(\boldsymbol{\xi})\times\boldsymbol{v}(\boldsymbol{y})$ and the relations (3.3.17) and (3.3.20), the above integral may be represented as

$$\int_Q\Big\{-\left(\nabla_\xi\alpha_H\times\boldsymbol{q}+\nabla_\xi\beta_H\times\boldsymbol{e}_3+\nabla_\xi(\alpha_H+\beta_H)\times\nabla_y\Phi_H+\nabla_\xi\sigma_H\times\nabla_y\psi_H\right)+$$
$$+\mathrm{i}\omega\Big(\alpha_E\varepsilon_0\left((1+|\hat{\boldsymbol{y}}|^2)\boldsymbol{q}+\boldsymbol{e}_3\right)+\beta_E\varepsilon_0(|\hat{\boldsymbol{y}}|^2\boldsymbol{q}+\boldsymbol{e}_3)+(\alpha_E+\beta_E)\varepsilon\nabla_y\Phi_E+\sigma_E\varepsilon\nabla_y\psi_E\Big)\Big\}\cdot\boldsymbol{K}\,\mathrm{d}\boldsymbol{y} = 0, \tag{3.3.23}$$

where $\hat{\boldsymbol{y}} = (y_1,y_2,0)$. If $a=b=0$ then the above equation is satisfied automatically in light of the leading order equation in (3.3.14) and the identity $\mathrm{div}_y\mathbf{curl}_\xi = -\mathrm{div}_\xi\mathbf{curl}_y$.

Let $a=1$ $b=c=0$. Note that for any two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, the identity $\boldsymbol{u}\cdot(\boldsymbol{u}\times\boldsymbol{v})=0$ holds and moreover it is understood that

$$\int_{\mathbb{R}^2\setminus\{\boldsymbol{0}\}}\boldsymbol{q}\,\mathrm{d}y_1\mathrm{d}y_2 = \boldsymbol{0},$$

in the sense that

$$\lim_{l\to\infty}\left(\int_{-l}^{l}\int_{-l}^{l}\frac{y_1}{y_1^2+y_2^2}\,\mathrm{d}y_1\mathrm{d}y_2\right) = \lim_{l\to\infty}\left(\int_{-l}^{l}\int_{-l}^{l}\frac{-y_2}{y_1^2+y_2^2}\,\mathrm{d}y_1\mathrm{d}y_2\right) = 0.$$

Hence (3.3.23) reduces to the following

$$\underbrace{\left(\int_Q \nabla_y \Phi_H \times \boldsymbol{q}\,\mathrm{d}\boldsymbol{y}\right)}_{:=I^{(1)}_{\boldsymbol{q}}(\Phi_H)} \cdot \nabla_\xi(\alpha_H+\beta_H) + \underbrace{\left(\int_Q \nabla_y \psi_H \times \boldsymbol{q}\,\mathrm{d}\boldsymbol{y}\right)}_{:=I^{(1)}_{\boldsymbol{q}}(\psi_H)} \cdot \nabla_\xi \sigma_H =$$
$$= \mathrm{i}\omega \underbrace{\left(\int_Q \varepsilon_0(T\nabla_y \Phi_E \cdot \boldsymbol{q}+1)\,\mathrm{d}\boldsymbol{y}\right)}_{:=I^{(2)}_{\boldsymbol{q}}(\Phi_E)}(\alpha_E+\beta_E) + \mathrm{i}\omega \underbrace{\int_Q \varepsilon_0\left(\sigma_E T\nabla_y\psi_E\cdot\boldsymbol{q}+\frac{\alpha_E}{|\hat{\boldsymbol{y}}|^2}\right)\mathrm{d}\boldsymbol{y}}_{:=I^{(3)}_{\boldsymbol{q}}(\psi_E)}, \tag{3.3.24}$$

Similarly, setting $b=1$ and $a=c=0$ yields

$$\underbrace{\left(\int_Q \nabla_y \Phi_H \times \boldsymbol{e}_3\,\mathrm{d}\boldsymbol{y}\right)}_{:=I^{(1)}_{\boldsymbol{e}_3}(\Phi_H)} \cdot \nabla_\xi(\alpha_H+\beta_H) + \underbrace{\left(\int_Q \nabla_y \psi_H \times \boldsymbol{e}_3\,\mathrm{d}\boldsymbol{y}\right)}_{:=I^{(1)}_{\boldsymbol{e}_3}(\psi_H)} \cdot \nabla_\xi \sigma_H =$$
$$= \mathrm{i}\omega \underbrace{\left(\int_Q \varepsilon_0(T\nabla_y \Phi_E + \boldsymbol{e}_3)\cdot \boldsymbol{e}_3\,\mathrm{d}\boldsymbol{y}\right)}_{:=I^{(2)}_{\boldsymbol{e}_3}(\Phi_E)}(\alpha_E+\beta_E) + \mathrm{i}\omega \underbrace{\left(\int_Q \varepsilon\nabla_y\psi_E\cdot\boldsymbol{e}_3\,\mathrm{d}\boldsymbol{y}\right)}_{:=I^{(3)}_{\boldsymbol{e}_3}(\psi_E)}\sigma_E, \tag{3.3.25}$$

If in equations (3.3.24) and (3.3.25) $H \mapsto E$, $E \mapsto H$, $\varepsilon_0 \mapsto \mu_0$ and $\mu_0 \mapsto \varepsilon_0$ then these equations become the solvability conditions for the second equation in (3.3.16). It is first noted that if the integral denoted $I^{(3)}_{\boldsymbol{q}}(\psi_E)$ is to be finite then $\sigma_E \propto \alpha_E$. Hence, $\sigma_E = C_E\alpha_E$ ($\sigma_H = C_H\alpha_H$) where C_E (C_H) is a constant.

The equations above will form a homogenised system of equations provided solutions $\Phi_H, \Phi_E, \psi_H, \psi_E$ can be found to equations (3.3.18) and (3.3.21) such that the integrals in equations (3.3.24) and (3.3.25) are finite.

The aim now is to describe the class of solutions of the equations (3.3.19) such that Φ_H, ψ_H are 1-periodic in the y_3 variable and such that all the relevant integrals in expressions (3.3.24) and (3.3.25) are finite along with those integrals which come from the other system of solvability conditions.

Consider the change of variables $\boldsymbol{y} \overset{\alpha=1}{\mapsto} \boldsymbol{\xi} \mapsto \boldsymbol{x}$. Then equations (3.3.19) can be rewritten as

$$-\mathrm{div}_x\left(\mu_0(x_3)\left\{\nabla_x\widetilde{\Phi}_H+\boldsymbol{e}_3\right\}\right)=0, \quad -\mathrm{div}_x\left(\mu_0(x_3)\nabla_x\widetilde{\psi}_H\right)=0, \quad \text{on } Q,$$

where $\widetilde{\Phi}_H(\boldsymbol{x}) = \Phi_H(\boldsymbol{y}(\boldsymbol{x}))$, $\widetilde{\psi}_H(\boldsymbol{x}) = \psi_H(\boldsymbol{y}(\boldsymbol{x}))$. The kernel of the divergence operator on unbounded, multiply connected domains is discussed in Appendix 3.C. Hence

$$\mu_0\left\{\nabla_x\widetilde{\Phi}_H+\boldsymbol{e}_3\right\} = a_1\boldsymbol{e}_3+b_1\boldsymbol{q}+c_1\mathbf{curl}_x\boldsymbol{u}_H, \quad \mu_0\nabla_x\widetilde{\psi}_H = a_2\boldsymbol{e}_3+b_2\boldsymbol{q}+c_2\mathbf{curl}_x\boldsymbol{v}_H, \tag{3.3.26}$$

where $a_i, b_i, c_i \in \mathbb{R}$, $i=1,2$ are constants and $\boldsymbol{u}_H, \boldsymbol{v}_H \in H^1_{\mathbf{curl}}(Q)$.

Consider the integrals $I^{(2)}_{\boldsymbol{q}}(\Phi_H)$, $I^{(3)}_{\boldsymbol{q}}(\psi_E)$, $I^{(2)}_{\boldsymbol{e}_3}(\Phi_H)$, $I^{(3)}_{\boldsymbol{e}_3}(\psi_H)$. Making the same change of variables in these integrals it is found that the following integrals need to be finite:

$$I^{(2)}_{\boldsymbol{q}}(\Phi_H) = \int_Q \left\{ \frac{b_1}{|\hat{\boldsymbol{x}}|^2} + c_1 \mathbf{curl}_x \boldsymbol{u}_H \cdot \boldsymbol{q} \right\} \mathrm{d}\boldsymbol{x}, \quad I^{(2)}_{\boldsymbol{e}_3}(\Phi_H) = \int_Q \{a_1 + c_1 \mathbf{curl}_x \boldsymbol{u}_H \cdot \boldsymbol{e}_3\} \,\mathrm{d}\boldsymbol{x}, \tag{3.3.27}$$

$$I^{(3)}_{\boldsymbol{q}}(\psi_H) = \int_Q \left\{ \frac{\mu_0 C_H + b_2}{|\hat{\boldsymbol{x}}|^2} + c_2 \mathbf{curl}_x \boldsymbol{v}_H \cdot \boldsymbol{q} \right\} \mathrm{d}\boldsymbol{x}, \quad I^{(3)}_{\boldsymbol{e}_3}(\psi_H) = \int_Q \{a_2 + c_2 \mathbf{curl}_x \boldsymbol{v}_H \cdot \boldsymbol{e}_3\} \,\mathrm{d}\boldsymbol{x}, \tag{3.3.28}$$

Further expressing the first integral in (3.3.27) in polar coordinates and simplifying the expression in the second integral yields

$$\int_Q \left\{ \frac{b_1}{r} + c_1 \frac{\partial u_3}{\partial r} \right\} \mathrm{d}\boldsymbol{r}, \quad \int_Q \left\{ a_1 + c_1 \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right) \right\} \mathrm{d}\boldsymbol{x}, \tag{3.3.29}$$

where $\mathrm{d}\boldsymbol{r} = \mathrm{d}r\mathrm{d}\theta\mathrm{d}x_3$ and $\tilde{\tilde{\Phi}}_H(\boldsymbol{r}) = \tilde{\Phi}(\boldsymbol{r}(\boldsymbol{x}))$. Given that the terms a_1 and b_1/r will not give finite integrals, it is concluded that $a_1 = b_1 = 0$ otherwise

$$\boldsymbol{u}_H = \boldsymbol{U}_H + \frac{1}{2c_1} \begin{pmatrix} a_1 x_2 \\ -a_1 x_1 \\ -b_1 \ln(x_1^2 + x_2^2) \end{pmatrix},$$

where $\boldsymbol{U}_H \in H^1_{\mathbf{curl}}(Q)$. However, this cannot be since the vector on the right-hand side is not in $H^1(Q)$. For the second set of integrals (3.3.28), it is concluded that $a_2 = 0$ but it suffices that $b_2 = -C_H \langle \mu_0 \rangle$ for finiteness of the first integral in (3.3.28).

The matrices $M_{\boldsymbol{q}} := J_x^T \times \boldsymbol{q}$ and $M_{\boldsymbol{e}_3} := J_x^T \times \boldsymbol{e}_3$ are introduced and given explicitly by the following expressions:

$$M_{\boldsymbol{q}} = \frac{1}{|\hat{\boldsymbol{x}}|^2} \begin{pmatrix} x_2(-x_1\cos(x_3) + x_2\sin(x_3)) & -x_1(-x_1\cos(x_3) + x_2\sin(x_3)) & -x_1\cos(x_3) + x_2\sin(x_3) \\ -x_2(x_1\sin(x_3) + x_2\cos(x_3)) & x_1(x_1\sin(x_3) + x_2\cos(x_3)) & -(x_1\sin(x_3) + x_2\cos(x_3)) \\ x_1 & x_2 & 0 \end{pmatrix}, \tag{3.3.30}$$

$$M_{\boldsymbol{e}_3} = \begin{pmatrix} \sin(x_3) & \cos(x_3) & 0 \\ -\cos(x_3) & \sin(x_3) & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.3.31}$$

Noting that $\langle M_{\boldsymbol{q}} \boldsymbol{q} \rangle = \langle M_{\boldsymbol{e}_3} \boldsymbol{q} \rangle = \mathbf{0}$ and that $\langle M_{\boldsymbol{q}} \boldsymbol{e}_3 \rangle = \langle M_{\boldsymbol{e}_3} \boldsymbol{e}_3 \rangle = \mathbf{0}$, it can be shown that

$$I^{(1)}_{\boldsymbol{q}}(\Phi_H) = c_1 \int_Q \frac{\mu_0^{-1}}{|\hat{\boldsymbol{x}}|^2} M_{\boldsymbol{q}} \mathbf{curl}\, \boldsymbol{u}_H \,\mathrm{d}\boldsymbol{x}, \quad I^{(1)}_{\boldsymbol{e}_3}(\Phi_H) = c_1 \int_Q \mu_0^{-1} M_{\boldsymbol{e}_3} \mathbf{curl}\, \boldsymbol{u}_H \,\mathrm{d}\boldsymbol{x}, \tag{3.3.32}$$

$$I^{(1)}_{\boldsymbol{q}}(\psi_H) = c_2 \int_Q \frac{\mu_0^{-1}}{|\hat{\boldsymbol{x}}|^2} M_{\boldsymbol{q}} \mathbf{curl}\, \boldsymbol{v}_H \,\mathrm{d}\boldsymbol{x}, \quad I^{(1)}_{\boldsymbol{e}_3}(\psi_H) = c_2 \int_Q \mu_0^{-1} M_{\boldsymbol{e}_3} \mathbf{curl}\, \boldsymbol{v}_H \,\mathrm{d}\boldsymbol{x}. \tag{3.3.33}$$

Note that these integrals are also vectors and therefore it is required that all three components of each vector need to be finite. Consider the integrals $I^{(1)}_{\boldsymbol{q}}(\Phi_H)$ and $I^{(1)}_{\boldsymbol{e}_3}(\Phi_H)$.

Expressing these integrals in polar coordnates, it can be shown that the following integrals need to be finite:

$$I_{\boldsymbol{q}}^{(1)}(\Phi_H)\cdot\boldsymbol{e}_1 = c_1\int_Q \mu_0^{-1}(x_3)\cos(\theta+x_3)\left\{\frac{\partial\tilde{u}_1}{\partial r}-\frac{1}{r}\frac{\partial\tilde{u}_2}{\partial\theta}+r\frac{\partial\tilde{u}_2}{\partial x_3}+\frac{1}{r}\tilde{u}_1+r\frac{\partial u_3}{\partial r}\right\}\mathrm{d}\boldsymbol{r},$$

$$I_{\boldsymbol{q}}^{(1)}(\Phi_H)\cdot\boldsymbol{e}_2 = c_1\int_Q \mu_0^{-1}(x_3)\sin(\theta+x_3)\left\{\frac{\partial\tilde{u}_1}{\partial r}-\frac{1}{r}\frac{\partial\tilde{u}_2}{\partial\theta}+r\frac{\partial\tilde{u}_2}{\partial x_3}+\frac{1}{r}\tilde{u}_1+r\frac{\partial u_3}{\partial r}\right\}\mathrm{d}\boldsymbol{r},$$

$$I_{\boldsymbol{q}}^{(1)}(\Phi_H)\cdot\boldsymbol{e}_3 = \int_Q -\mu_0^{-1}(x_3)\frac{\partial\tilde{u}_1}{\partial x_3}\,\mathrm{d}\boldsymbol{r},$$

$$I_{\boldsymbol{e}_3}^{(1)}(\Phi_H)\cdot\boldsymbol{e}_1 = \int_Q \mu_0^{-1}(x_3)\left\{-\cos(\theta+x_3)\frac{\partial u_3}{\partial r}+\frac{\sin(\theta+x_3)}{r}\frac{\partial u_3}{\partial\theta}+\cos(x_3)\frac{\partial u_1}{\partial x_3}-\sin(x_3)\frac{\partial u_2}{\partial x_3}\right\}\mathrm{d}\boldsymbol{r},$$

$$I_{\boldsymbol{e}_3}^{(1)}(\Phi_H)\cdot\boldsymbol{e}_2 = \int_Q \mu_0^{-1}(x_3)\left\{-\sin(\theta+x_3)\frac{\partial u_3}{\partial r}-\frac{\cos(\theta+x_3)}{r}\frac{\partial u_3}{\partial\theta}+\sin(x_3)\frac{\partial u_1}{\partial x_3}+\cos(x_3)\frac{\partial u_2}{\partial x_3}\right\}\mathrm{d}\boldsymbol{r},$$

$$I_{\boldsymbol{e}_3}^{(1)}(\Phi_H)\cdot\boldsymbol{e}_3 = 0,$$

where $\tilde{u}_1 = -\sin\theta\, u_1 + \cos\theta\, u_2$ and $\tilde{u}_2 = \cos\theta\, u_1 + \sin\theta\, u_2$. Hence, functions Φ_H, Φ_E, ψ_H, ψ_E are considered such that the integrals above (and their similar counterparts)are finite. Moreover, define $\gamma_H = \alpha_H + \beta_H$, $\gamma_E = \alpha_E + \beta_E$ and define by $D_{\boldsymbol{a}}^{\boldsymbol{b}}$ the following operator:

$$D_{\boldsymbol{a}}^{\boldsymbol{b}} \equiv \left(\int_Q \nabla_y\boldsymbol{b}\times\boldsymbol{a}\,\mathrm{d}\boldsymbol{y}\right)\cdot\nabla_\xi.$$

Then a homogenised system of equations is obtained in the form

$$D_{\boldsymbol{q}}^{\boldsymbol{\Phi}_H}\gamma_H + D_{\boldsymbol{q}}^{\boldsymbol{\psi}_H}\alpha_H = \mathrm{i}\omega(\varepsilon_{\boldsymbol{q}}^{\mathrm{hom}}\gamma_E + j_{\boldsymbol{q}}^E\alpha_E),\quad D_{\boldsymbol{e}_3}\gamma_H = \mathrm{i}\omega(\varepsilon_{\boldsymbol{e}_3}^{\mathrm{hom}}\gamma_E + j_{\boldsymbol{e}_3}^E\alpha_E) \tag{3.3.34}$$

$$D_{\boldsymbol{q}}\gamma_E = -\mathrm{i}\omega(\mu_{\boldsymbol{q}}^{\mathrm{hom}}\gamma_H + j_{\boldsymbol{q}}^H\alpha_H),\quad D_{\boldsymbol{e}_3}\gamma_E = -\mathrm{i}\omega(\mu_{\boldsymbol{e}_3}^{\mathrm{hom}}\gamma_H + j_{\boldsymbol{e}_3}^H\alpha_H) \tag{3.3.35}$$

where

$$\varepsilon_{\boldsymbol{a}}^{\mathrm{hom}} := \int_Q \varepsilon(\nabla_y\Phi_E + \boldsymbol{e}_3)\cdot\boldsymbol{a}\,\mathrm{d}\boldsymbol{y},\qquad \mu_{\boldsymbol{a}}^{\mathrm{hom}} := \int_Q \mu(\nabla_y\Phi_H + \boldsymbol{e}_3)\cdot\boldsymbol{a}\,\mathrm{d}\boldsymbol{y}, \tag{3.3.36}$$

$$j_{\boldsymbol{a}}^E := \int_Q \varepsilon\left(\boldsymbol{q}+\nabla_y\psi_E\right)\cdot\boldsymbol{a}\,\mathrm{d}\boldsymbol{y},\qquad j_{\boldsymbol{a}}^H := \int_Q \mu\left(\boldsymbol{q}+\nabla_y\psi_H\right)\cdot\boldsymbol{a}\,\mathrm{d}\boldsymbol{y}. \tag{3.3.37}$$

Note that the above set of equations is a scalar system as opposed to being a vector system of equations.

Further Discussions

The work described in the section above was motivated by recent developments in metamaterial science. Links have been made between materials which possess a negative refractive index and chirality and the initial goal of this project was to make a chiral change of coordinates in the governing equations and to see if a negative effective permeability and/or a negative electric permittivity could be found. A further extension of the work carried out above could be to take a bounded domain and appropriate boundary conditions to see if any further relations between chirality and metamaterials can be deduced.

Another possible extension is to use the same coordinate transformation of this chapter but for the so called Drude-Born-Fedorov model of the Maxwell system of equations which has been investigated with regards to metamaterials already (see Guenneau & Zolla [37]).

Appendices

3.A Asymptotics of the tensors $K_\delta^{(j)}$, $j = 1, 2, \ldots$

The asymptotics for the tensors $K_\delta^{(j)}$ will now be analysed. Recall that the tensors $K_\delta^{(j)}$, $j = 1, 2, \ldots$ satisfy a system of equations which derive from the divergence condition $\operatorname{div} \boldsymbol{u}^\varepsilon = 0$. The system of equations is (*c.f.* equation 1.1.22)

$$\Delta_y(K_\delta^{(j+1)}\nabla_x^{j+1}\boldsymbol{v})+2\Delta_{xy}(K_\delta^{(j)}\nabla_x^{j}\boldsymbol{v})+\Delta_x(K_\delta^{(j-1)}\nabla_x^{j-1}\boldsymbol{v})+\operatorname{div}_x(N_\delta^{(j)}\nabla_x^{j-1}\mathbf{curl}_x\boldsymbol{v}) = 0, \tag{3.A.1}$$

for $j = 1, 2, \ldots$.Together with the conditions $K_\delta^{(j)}$ is Q-periodic and $\langle K_\delta^{(j)}\rangle = 0$, these tensors are determined uniquely.

The δ-asymptotic behaviour of the tensor $K_\delta^{(j)}$ is dependent on the δ-asymptotic behaviour of the tensor $N_\delta^{(j-1)}$, $j = 1, 2, \ldots$ which can be observed from the equations above. Noting once again that $K_\delta^{(1)} = 0$, the first non-trivial tensor $K_\delta^{(2)} = (K_\delta^{(2)})_{ijk}$ satisfies the following (index) equation:

$$\partial_i\partial_i(K_\delta^{(2)})_{jkl} = -(N_\delta^{(1)})_{jm}\epsilon_{mkl}.$$

It becomes apparent that because $N_\delta^{(1)} = O(1)$ that $K_\delta^{(2)} = O(1)$. Advancing this procedure, it can also be shown that

$$K_\delta^{(2j+1)} = K_\delta^{(2j+2)} = O(\delta^{-j}), \quad j = 1, 2, \ldots.$$

Therefore, substituting these δ-asymptotics for the tensors $K_\delta^{(j)}$ into the asymptotic expansion for the solution $\boldsymbol{u}^\varepsilon$ along with the asymptotics for $N_\delta^{(j)}$ yields

$$\begin{aligned}\boldsymbol{u}^\varepsilon \sim \boldsymbol{v} + \varepsilon N^{(1)}\mathbf{curl}_x\boldsymbol{v} + \varepsilon^2\left\{\delta^{-1}\tilde{N}^{(2)}\nabla_x\mathbf{curl}_y\boldsymbol{v} + \nabla_y\left(\tilde{K}^{(2)}\nabla_x^2\boldsymbol{v}\right)\right\} + \\ + \varepsilon^3\left\{\delta^{-1}\tilde{N}^{(3)}\nabla_x^2\mathbf{curl}_x\boldsymbol{v} + \delta^{-1}\nabla_y\left(\tilde{K}^{(3)}\nabla_x^3\boldsymbol{v}\right) + \nabla_x\left(\tilde{K}^{(2)}\nabla_x^2\boldsymbol{v}\right)\right\} + \\ + \varepsilon^4\left\{\delta^{-2}\tilde{N}^{(4)}\nabla_x^3\mathbf{curl}_x\boldsymbol{v} + \delta^{-1}\nabla_y\left(\tilde{K}^{(4)}\nabla_x^4\boldsymbol{v}\right) + \delta^{-1}\nabla_x\left(\tilde{K}^{(3)}\nabla_x^3\boldsymbol{v}\right)\right\} + \ldots\end{aligned}$$

This expansion reveals no further scalings that could be considered as "critical" other than the scaling $\delta = \varepsilon^2$.

3.B A Result on Potential Vectors

Lemma 3.B.1. *Let $\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y}) \in L^2(\Omega, V_{\text{pot}})$. Then for any $\boldsymbol{b}(\boldsymbol{y}) \in V_{\text{pot}}$ and any constant vector $\boldsymbol{c}(\boldsymbol{x})$,*

$$\int_Q (\boldsymbol{c}(\boldsymbol{x}) \times \boldsymbol{P}(\boldsymbol{x},\boldsymbol{y})) \cdot \boldsymbol{b}(\boldsymbol{y})\,\mathrm{d}\lambda(\boldsymbol{y}) = 0.$$

Proof. Note that since vectors of the form $\nabla\psi$, $\psi \in C^\infty_{\text{per}}(Q)$ are dense in V_{pot}, it is sufficient to prove the result for $\boldsymbol{b} = \nabla\psi$. Hence

$$\int_Q (\boldsymbol{c} \times \boldsymbol{P}) \cdot \nabla\psi\,\mathrm{d}\lambda = \int_{\partial Q} (\boldsymbol{c} \times \boldsymbol{P})\,\psi \cdot \boldsymbol{n}\,\mathrm{d}\lambda - \int_Q \operatorname{div}(\boldsymbol{c} \times \boldsymbol{P})\,\psi\,\mathrm{d}\lambda,$$

where $\boldsymbol{n}$ is the unit outward pointing normal to the boundary of Q. Since Q is a unit cube and both $\boldsymbol{P}$ and ψ are periodic, the boundary integral vanishes. Moreover, using the identity $\operatorname{div}(\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{a} \cdot \mathbf{curl}\,\boldsymbol{b} - \boldsymbol{b} \cdot \mathbf{curl}\,\boldsymbol{a}$, it follows that

$$\int_Q \operatorname{div}(\boldsymbol{c} \times \boldsymbol{P})\,\psi\,\mathrm{d}\lambda = \int_Q (\boldsymbol{c} \cdot \mathbf{curl}\,\boldsymbol{P} - \boldsymbol{P} \cdot \mathbf{curl}\,\boldsymbol{c})\,\psi\,\mathrm{d}\lambda = 0.$$

The last equality is true since $\boldsymbol{P}$ is potential and $\boldsymbol{c}$ is constant. □

3.C Kernel of the curl and div Operators on Multiply Connected, Unbounded Domains

Let $\Omega \subset \mathbb{R}^3$. When Ω is bounded and simply connected, the solutions $\boldsymbol{v} \in [L^2(\Omega)]^3$ of the problem

$$\mathbf{curl}\,\boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{0}, \quad \boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0} \text{ on } \partial\Omega, \quad \boldsymbol{x} \in \Omega,$$

are elements of the set

$$\ker \mathbf{curl} := \ker\{\mathbf{curl},\ L^2(\Omega)\} = \nabla H^1(\Omega) := \{\boldsymbol{v}(x) = \nabla w(x) \mid w \in H^1(\Omega)\}.$$

The extension of this result to unbounded, simply connected domains is achieved by imposing a sufficient decay at infinity (see Ledger & Zaglmayr[45]).

The case when Ω is bounded and multiply connected will now be outlined. For full details, see Dautray & Lions [26, Chapter IX]. Let Ω be a 3-dimensional, bounded connected domain which lies wholly on one side of it's boundary $\partial\Omega$ which has dimension 2 and is furthermore of the class C^k where $k \geq 2$. Let $\partial\Omega$ have a finite number of connected components denoted $\Gamma_0, \ldots, \Gamma_n$ where Γ_0 denotes the boundary of the infinite connected component of $\Omega' = \mathbb{R}^3 \backslash \overline{\Omega}$.

Definition 3.C.1. Let $\Sigma_1, \ldots, \Sigma_m$ be a collection of smooth surfaces such that

1. $\Sigma_1, \ldots, \Sigma_m$ are open subsets of smooth manifolds $\mathcal{M}_1, \ldots, \mathcal{M}_m$ respectively,

2. the boundary of each Σ_i is contained in $\partial\Omega$,

3. the intersection of any two cuts is empty, i.e., $\Sigma_i \cap \Sigma_j = \emptyset$, $i \neq j$,

4. the open set $\Omega' = \Omega \backslash \bigcup_{i=1}^{m} \Sigma_i$ is simply connected and pseudo-Lipschitz[1].

Then we call $\Sigma_1, \ldots, \Sigma_m$ a collection of cuts for the domain Ω.

The purpose of these cuts is to make every curl-free field on Ω' the gradient of some scalar field.

Proposition 3.C.1. *Let Ω be an open set which is bounded and multiply connected in $\mathbb{R}^3$ such that there are cuts satisfying the assumptions of Defintion 3.C.1. Then the kernel of the* **curl** *operator in $[L^2(\Omega)]^3$ is the sum of two orthogonal spaces; $\nabla H^1(\Omega)$ and $\mathbb{H}_1(\Omega)$ where the latter is the vector space of dimension m (the number of cuts required to make Ω simply connected) defined by*

$$\mathbb{H}_1(\Omega) = \left\{ \boldsymbol{v} \in [L^2(\Omega)]^3 \;\middle|\; \boldsymbol{v} = \nabla w,\ \textit{in the classical sense}^2 \textit{ in } \Omega \textit{ with } w \in H^1(\Omega') \textit{ a solution of (3.C.1)} \right\},$$

where the problem in question is

$$\begin{cases} \Delta w = 0, \quad \textit{in } \Omega', \\ \left.\dfrac{\partial w}{\partial n}\right|_{\Gamma} = 0, \\ [w]_{\Sigma_i} = \textit{constant}, \quad i = 1, \ldots, m, \\ \left[\dfrac{\partial w}{\partial n}\right]_{\Sigma_i} = 0, \quad i = 1, \ldots, m. \end{cases} \tag{3.C.1}$$

where $[w]_{\Sigma_i} = w|_{\Sigma_i^+} - w|_{\Sigma_i^-}$.

Elements of the kernel of the divergence operator on a multiply connected domain Ω can be shown to be of the form

$$\boldsymbol{u} = \boldsymbol{h}_1 + \mathbf{curl}\, \boldsymbol{w}, \qquad \boldsymbol{w} \times \boldsymbol{n}|_{\Gamma} = 0, \tag{3.C.2}$$

where $\boldsymbol{h}_1 \in \mathbb{H}_1(\Omega)$ and $\boldsymbol{w} \in [H^1(\Omega)]^3$.

The domain $Q := (\mathbb{R}^2 \backslash \{\mathbf{0}\}) \times \mathcal{S}^1$ is multiply connected but is also unbounded and hence the above result cannot be applied right away since the condition $\left.\dfrac{\partial w}{\partial n}\right|_{\Gamma} = 0$ in (3.C.1) and the condition $\boldsymbol{w} \times \boldsymbol{n}|_{\Gamma} = 0$ in (3.C.2) make no sense. These conditions are replaced by the conditions that the limits

$$\lim_{R \to \infty} \left\{ \left.\frac{\partial w}{\partial n}\right|_{y_1^2 + y_2^2 = R^2} \right\} = 0, \qquad \forall y_3 \in \mathcal{S}^1, \tag{3.C.3}$$

[1]For the definition of pseudo-Lipschitz, see Amrouche, Bernadi, Dauge & Girault [2]

[2]in the sense of distributions on Ω'

$$\lim_{R\to\infty}\left\{\boldsymbol{w}\times\boldsymbol{n}\big|_{y_1^2+y_2^2=R^2}\right\}=0,\qquad \forall y_3\in\mathbb{S}^1. \tag{3.C.4}$$

Hence the kernel of the **curl** operator on the domain Q is determined. Two cuts Σ_1 and Σ_2 are made such that every curl-free field becomes the gradient of a scalar function and, without loss of generality, the first cut Σ_1 is made in the $y_3=0$ plane and the second cut Σ_2 is made in the $y_1>0, y_2=0$ plane. It can be confirmed that the solution of problem (3.C.1) with the condition (3.C.3) replacing the requirement that $\frac{\partial w}{\partial n}|_\Gamma=0$ is a linear combination of the functions

$$w_1(\boldsymbol{y})=y_3,\quad w_2(\boldsymbol{y})=\tan^{-1}\left(\frac{y_2}{y_1}\right), \tag{3.C.5}$$

Hence, the kernel of the $\mathbf{curl}_y$ operator on $\widetilde{\mathbb{R}}^3\times Q$ is given by vector fields of the form

$$\boldsymbol{v}(\boldsymbol{\xi},\boldsymbol{y})=\nabla_y\varphi_H(\boldsymbol{\xi},\boldsymbol{y})+\alpha(\boldsymbol{\xi})\nabla_y w_1(\boldsymbol{y})+\beta(\boldsymbol{\xi})\nabla_y w_2(\boldsymbol{y})=\nabla_y\varphi_H(\boldsymbol{\xi},\boldsymbol{y})+\alpha(\boldsymbol{\xi})\boldsymbol{e}_3+\beta(\boldsymbol{\xi})\boldsymbol{q}(y_1,y_2),$$

as required. Moreover, functions in the kernel of the div operator on $\widetilde{\mathbb{R}}^3\times Q$ take the form

$$\boldsymbol{u}(\boldsymbol{\xi},\boldsymbol{y})=\alpha(\boldsymbol{\xi})\boldsymbol{e}_3+\beta(\boldsymbol{\xi})\boldsymbol{q}(y_1,y_2)+\mathbf{curl}_y\boldsymbol{w}(\boldsymbol{\xi},\boldsymbol{y}), \tag{3.C.6}$$

Note that the vector field $\boldsymbol{w}$ can be found uniquely provided

$$\operatorname{div}\boldsymbol{w}=0,\qquad \lim_{R\to\infty}\int_{B(\mathbf{0},R)}\boldsymbol{w}\cdot\boldsymbol{n}\,\mathrm{d}y_1\mathrm{d}y_2=0,\qquad \forall y_3\in\mathbb{S}^1,\ \ \forall\boldsymbol{\xi}\in\widetilde{\mathbb{R}}^3.$$

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