Extremes (2015) 18:241–271
DOI 10.1007/s10687-014-0212-y

Tail correlation functions of max-stable processes

Construction principles, recovery and diversity of some mixing max-stable processes with identical TCF

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Received: 19 February 2014 / Revised: 21 November 2014 / Accepted: 1 December 2014 /
Published online: 25 January 2015

Abstract The tail correlation function (TCF) is a popular bivariate extremal dependence measure to summarize data in the domain of attraction of a max-stable process. For the class of TCFs, being largely unexplored so far, several aspects are contributed: (i) generalization of some mixing max-stable processes (ii) transfer of two geostatistical construction principles to max-stable processes, including the turning bands operator (iii) identification of subclasses of TCFs, including M3 processes based on radial monotone shapes (iv) recovery of subclasses of max-stable processes from TCFs (v) parametric classes (iv) diversity of max-stable processes sharing an identical TCF. We conclude that caution should be exercised when using TCFs for statistical inference.

Keywords Brown-Resnick · Extremal coefficient · Mixed moving maxima · Poisson storm · Stationary truncation · Tail dependence · Turning bands

AMS 2000 Subject Classification Primary–60G70; Secondary–60G60

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1 Introduction

The *tail correlation function (TCF)* χ of a stationary process X on $\mathbb{R}^d$ is defined through the following limit provided that it exists

$$\chi(t) = \lim_{\tau \uparrow \tau_{\mathrm{up}}} \mathbb{P}(X_t > \tau \mid X_o > \tau), \qquad t \in \mathbb{R}^d.$$

Here, $o \in \mathbb{R}^d$ denotes the origin and τ_{up} is the upper endpoint of X_o. If $\chi(t) = 0$, the variables X_t and X_o are called *asymptotically independent*. Otherwise $\chi(t)$ expresses the strength of *asymptotic dependence*. Dating back to Geffroy (1958/1959), Sibuya (1960) and Tiago de Oliveira (1962/63) the bivariate summary statistic $\chi(t)$ is one of the most popular extremal dependence measures that has entered the literature under various names including *(upper) tail dependence coefficient* (Beirlant et al. 2004; Davis and Mikosch 2009; Falk 2005) or $\chi - measure$ (Coles et al. 1999). We chose the name "tail correlation function" in order to emphasize the spatial character of χ and that it is a symmetric positive definite function. Indeed, the TCF χ (or equivalently $\theta := 2 - \chi$, see below) was proposed as an extreme value analogue to the usual correlation function (Schlather and Tawn 2003; Smith 1990).

If the process X is max-stable, the TCF χ is equivalent to the *extremal coefficient function (ECF)*

$$\theta(t) = \log \mathbb{P}(X_t \leq \tau, X_o \leq \tau) / \log \mathbb{P}(X_o \leq \tau), \qquad t \in \mathbb{R}^d,$$

since $\chi(t) = 2 - \theta(t)$, see also Eq. 3 below. Note that in the max-stable case the expression on the right-hand side is indeed independent of the threshold τ. Estimators for θ (and, thus, also for χ) can be found for instance in Smith (1990), Schlather and Tawn (2003), Cooley et al (2006) and Naveau et al (2009). Parametric subclasses of max-stable processes have been fitted to environmental spatial data and the ECF θ (that is equivalent to the TCF χ) is usually considered in order to assess the goodness of fit (Blanchet and Davison 2011; Engelke et al. 2012b; Davison and Gholamrezaee 2012; Davison et al. 2012; Schlather and Tawn 2003; Thibaud and Opitz 2014). All these references contain plots comparing non-parametric estimates of extremal coefficients to the theoretical ECFs.

While continuous correlation functions can be characterized by means of Bochner's theorem as Fourier transforms of probability measures, no such characterization is available for the subclass of (continuous) TCFs. At least, Fiebig et al. (2014) show that the set of TCFs on an arbitrary space T is closed under convex combinations, products and pointwise limits and provide necessary conditions for a function to be a tail correlation function. In particular, χ cannot be differentiable except when χ is constant, cf. Schlather and Tawn (2003). Some attempts to recover a max-stable random vector from a prescribed tail correlation matrix can be found in Falk (2005) and Ferreira (2012).

The text is structured as follows. Section 2 gives an overview over well-known classes of mixing max-stable processes and their TCFs. In Section 3 we transfer two construction principles from geostatistics to max-stable processes and their TCFs – a turning bands operator and a stationary truncation. In Section 4 some classes of radial monotonous TCFs and their relations are identified. Sharp bounds for some parametric families of positive definite functions are derived as well. Subsequently, Section 5 deals with the recovery of some max-stable processes from a prescribed TCF. Additionally, we reassess the stationary truncation of variance-mixed Brown-Resnick processes as possibly useful models with tractable TCFs. Finally, Section 6 provides a concrete example for the non-uniqueness of a max-stable process with a prescribed TCF. Since Sections 4 and 5 depend heavily on results in Gneiting (1999c), for convenience, some of them are recalled in the notation of the present setup. Appendix B provides some background information on monotonicity properties of continuous functions. We close the text with some remarks (Section 7) indicating that our results may be relevant beyond the max-stable setting. All proofs are postponed to Appendix A.

Some notation Let ν_d be the Lebesgue measure on the Borel σ-algebra $\mathcal{B}^d$ of $\mathbb{R}^d$ and $\|\cdot\|$ the Euclidean norm. We denote $B_r^d := \left\{h \in \mathbb{R}^d : \|h\| \leq r\right\}$ the d-dimensional ball of radius r centred at the origin $o \in \mathbb{R}^d$. The constant

$$\kappa_d := \nu_d\left(B_1^d\right) = \pi^{d/2}/\Gamma\left(1 + d/2\right)$$

is the volume of the d-dimensional unit ball. When a function on $\mathbb{R}^d$ depends on the Euclidean norm only, we will usually treat it as a function on $[0, \infty)$. The expression cdf abbreviates "cumulative distribution function". When we consider a cdf G on $(0, \infty)$, it is always meant that $G(0+) = 0$. The Laplace transform of a cdf G on $[0, \infty)$ is $\mathcal{L}(G)(x) = \int_0^\infty \exp(-xt)\, \mathrm{d}G(t)$. The function

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-y^2}\, \mathrm{d}y$$

is the complementary error function, while $\operatorname{erf}(x) = 1 - \operatorname{erfc}(x)$ is the error function and $\mathbf{1}_A$ is the indicator function of A. By $a \wedge b$ we denote the minimum between a and b, whereas $\bigvee_{i \in I} a_i$ is the supremum over the a_i. We set $a_+ := \max(a, 0)$. Finally, an integral of the form $\int f(x)\, \mathrm{d}F(x)$, where F is a monotone function, is always meant in the Riemann-Stieltjes sense.

2 Max-stable processes and their TCFs

A stochastic process $X = \{X_t\}_{t \in \mathbb{R}^d}$ is called *max-stable* if all its finite-dimensional distributions are max-stable, that is, for each $m, n \in \mathbb{N}$, $t_1, \ldots, t_m \in \mathbb{R}^d$ and n independent copies $\left(Y^{(i)}\right)_{i=1}^n$ of the random vector $Y = \left(X_{t_1}, \ldots, X_{t_m}\right)$ the componentwise maximum $\bigvee_{i=1}^n Y^{(i)}$ is distributed as the random vector $a_n Y + b_n$ for suitable norming vectors $a_n \in (0, \infty)^m$ and $b_n \in \mathbb{R}^m$. Henceforth, we will consider only *stationary* max-stable processes with *standard Fréchet marginals*, i.e. $\mathbb{P}(X_t \leq x) = e^{-1/x}$ for $t \in \mathbb{R}^d$ and $x > 0$. Note that the TCF χ is invariant under eventually continuous order-preserving marginal transformations.

Spectral representation Max-stable processes that are separable in probability allow for a *spectral representation* of the form (de Haan 1984; Kabluchko 2009; Stoev and Taqqu 2005)

$$\{X_t\}_{t\in\mathbb{R}^d} \stackrel{\mathcal{D}}{=} \left\{ \bigvee_{n=1}^{\infty} U_n V_t(e_n) \right\}_{t\in\mathbb{R}^d}. \tag{1}$$

Here, $\stackrel{\mathcal{D}}{=}$ means equality in distribution and $\{(U_n, e_n)\}_n$ is a Poisson point process on $\mathbb{R}_+ \times E$ with intensity $u^{-2}\,\mathrm{d}u \times \mu(\,\mathrm{d}e)$ for some Polish measure space $(E, \mathcal{E}, \mu)$, and the functions $V_t : E \to \mathbb{R}_+$, called *spectral functions*, are measurable with $\int_E V_t(e)\mu(\,\mathrm{d}e) = 1$ for each $t \in \mathbb{R}^d$. Of course, any process X of the form (1) is max-stable and has standard Fréchet marginals. In particular, the finite-dimensional distributions of X are given through

$$-\log \mathbb{P}(X(t_k) \leq x_k;\ k = 1, \ldots, m) = \int_E \bigvee_{k=1}^{m} \frac{V_{t_k}(e)}{x_k}\, \mu(\,\mathrm{d}e) \tag{2}$$

and the TCF χ of the max-stable process X may be expressed as

$$\begin{aligned}\chi(t) &= 2 - \lim_{\tau\to\infty} \frac{1 - \mathbb{P}\,(X_t \leq \tau, X_o \leq \tau)}{1 - \mathbb{P}(X_o \leq \tau)} = 2 - \frac{\log \mathbb{P}\,(X_t \leq \tau, X_o \leq \tau)}{\log \mathbb{P}(X_o \leq \tau)} \\ &= 2 - \int_E V_t(e) \vee V_o(e)\, \mu(\,\mathrm{d}e) = \int_E V_t(e) \wedge V_o(e)\, \mu(\,\mathrm{d}e). \end{aligned} \tag{3}$$

A max-stable process X with spectral representation (1) is *mixing* if and only if its TCF $\chi(t)$ converges to 0 as t tends to ∞ (Kabluchko and Schlather 2010; Stoev 2008), while it is *ergodic* if and only if its TCF $\chi(t)$ converges to 0 in a Cesàro sense as t tends to ∞ (Wang et al. 2013, Theorem 5.3).

If the measure space $(E, \mathcal{E}, \mu)$ is a probability space, the spectral functions $\{V_t\}_{t\in\mathbb{R}^d}$ themselves form a stochastic process on $\mathbb{R}^d$, which we call *spectral process*. Note that the stationarity of the spectral process V is a sufficient but not a necessary condition for X being stationary (Kabluchko et al. 2009; Molchanov and Stucki 2013).

2.1 Subclasses of mixing max-stable processes and their TCFs

(Mixed) Moving maxima and subclasses (M3r, M2r and M3b) Slightly different notions for M3 processes are given in the literature, cf. Kabluchko and Stoev (2012), Segers (2006), Smith (1990), Stoev (2008), Stoev and Taqqu (2005) and Zhang and Smith (2004), for example. We consider the following version: Let $(\Omega, \mathcal{A}, \nu)$ be a probability space and $f : \mathbb{R}^d \times \Omega \to [0, \infty]$ be measurable, such that

$$\int_\Omega \int_{\mathbb{R}^d} f(t, \omega)\ \mathrm{d}t\, \nu(\,\mathrm{d}\omega) = 1. \tag{4}$$

We refer to the assignment $\omega \mapsto (t \mapsto f(t, \omega))$ which maps each element from the probability space to its sample path on $\mathbb{R}^d$ as *(random) shape function*. A process X

with spectral representation on $(E, \mathcal{E}, \mu) = \left(\mathbb{R}^d \times \Omega, \mathcal{B}^d \otimes \mathcal{A}, \nu_d \times \nu\right)$ and spectral functions

$$V_t((z, \omega)) := f(t - z, \omega), \qquad (z, \omega) \in \mathbb{R}^d \times \Omega, \qquad t \in \mathbb{R}^d,$$

will be called *Mixed Moving Maxima process (M3 process)*, or *Moving Maxima process (M2 process)* if the random shape function is deterministic, i.e., if ν charges only one point $\omega_0 \in \Omega$. In this case, we simply treat f as a deterministic shape function on $\mathbb{R}^d$ without a second argument $\omega \in \Omega$.

We put particular emphasis on such random shapes, where each realization $f(\cdot, \omega)$ is *radially symmetric* around the origin $o \in \mathbb{R}^d$ and *non-increasing* as the radius grows, and refer to this class as *M3r processes*, or *M2r processes* if the random shape is deterministic. Moreover, we will also consider the subclass of *M3b processes* that have as shape functions only *normalized indicator functions of balls* B_R^d, i.e.

$$f(t, \omega) = \nu_d \left(B^d_{R(\omega)}\right)^{-1} \mathbf{1}_{B^d_{R(\omega)}}(t) = \kappa_d^{-1} R(\omega)^{-d} \mathbf{1}_{B^d_{R(\omega)}(t)},$$

where $R\colon (\Omega, \mathcal{A}, \nu) \to ((0, \infty), \mathcal{B}((0, \infty)), \mathbb{P}_R)$ is a random radius. Clearly, M3r, M2r and M3b processes are stationary and isotropic and both M2r processes and M3b processes each form a proper subclass of M3r processes.

Mixed poisson storm processes (MPS) We consider a mixed version of the Poisson storm process introduced in Lantuéjoul et al. (2011). The construction is similar to the construction of an M3b process, where the ball with random radius R is replaced by the typical cell of a Poisson hyperplane mosaic with random intensity β. To this end, let us consider some facts from stochastic geometry based on Schneider and Weil (2008). Any pair of polar coordinates $(S, r) \in S^{d-1} \times \mathbb{R}_+$ determines a hyperplane $H(S, r) = \left\{t \in \mathbb{R}^d \mid \langle t, S\rangle = r\right\}$ in $\mathbb{R}^d$. The hyperplanes that arise from a homogeneous Poisson point process $\{(S_n, r_n)\}_n$ on $S^{d-1} \times \mathbb{R}_+$ split $\mathbb{R}^d$ into polytopes called cells. The collection of these cells forms a *Poisson hyperplane mosaic* and can be seen as a stationary process of convex particles, whose intensity measure uniquely determines an intensity $\beta > 0$ and a grain distribution $\mathbb{Q}_\beta$ (Schneider and Weil 2008, p. 101, Theorem 4.1.1). A random set that is distributed according to $\mathbb{Q}_\beta$ is called *typical cell*. Now, if $C \sim \mathbb{Q}_1$ and $\beta > 0$, then $\beta^{-1}C = \{x : \beta x \in C\} \sim \mathbb{Q}_\beta$ and $\beta^{-1}C$ has expected volume

$$\mathbb{E}\left(\nu_d\left(\beta^{-1}C\right)\right) = d^d \kappa_{d-1}^{-d} \kappa_d^{d-1} \beta^{-d} =: \mu_d(\beta)$$

(Schneider and Weil 2008, Eqs. 10.4 and 10.4.6). Now, let (β, C) be a random element on a probability space $(\Omega, \mathcal{A}, \nu)$ where β is distributed according to a cdf G_β on $(0, \infty)$ with $G_\beta(0+) = 0$ and, independently, $C \sim \mathbb{Q}_1$. Let

$$f(t, \omega) := \mu_d(\beta(\omega))^{-1} \mathbf{1}_{\beta(\omega)^{-1}C(\omega)}(t), \qquad t \in \mathbb{R}^d.$$

Conditioning on β, the function f satisfies (4) and, thus, defines an isotropic M3 process X. We call this process *Mixed Poisson storm process (MPS process)* with *intensity mixing distribution* G_β.

(Variance-mixed) Brown-Resnick processes (BR and VBR)
Let $\{W_t\}_{t \in \mathbb{R}^d}$ be a Gaussian process with stationary increments (meaning that the law of $\{W_{t+h} - W_h\}_{t \in \mathbb{R}^d}$ does not depend on $h \in \mathbb{R}^d$) and variance $\sigma^2(t) = \mathrm{Var}(W_t)$. Independently, let S be a random variable on $(0, \infty)$ with cdf G_S. Then we call the process X with spectral process

$$V_t = \exp\left(SW_t - S^2\sigma^2(t)/2\right), \qquad t \in \mathbb{R}^d,$$

variance-mixed Brown-Resnick process (VBR process) with *variance mixing distribution* G_S. The law of X is stationary and depends on the variogram $\gamma(t) = \mathbb{E}(W_t - W_o)^2$ and the cdf G_S only (Kabluchko et al. 2009, Theorem 2). If $S = 1$ almost surely, V is the usual *Brown-Resnick process (BR process)*. We shall assume throughout the text that the variogram $\gamma(t)$ tends to ∞ as $t \to \infty$, that is we treat only mixing VBR processes, cf. their TCFs in Table 1. If the variogram tends to ∞ fast enough, a BR process may even be representable as an M3 process (Kabluchko et al. 2009, Theorem 14).

Remark 1 A related construction as in the case of a VBR process can be found in Engelke et al. (2012a), where the BR process is mixed in its scale instead of its variance. This yields the same class of processes in the most prominent example when W_t is a fractional Brownian motion.

Subclasses of TCFs The TCFs of the above processes are listed in Table 1, the formulae therein being easily derived from the indicated references. In the sequel we will identify relations between the classes of TCFs arising from the processes above. To this end, we use the notation

$$T^d_{model} = \left\{ \chi : \mathbb{R}^d \to [0,1] \;\middle|\; \begin{array}{l} \chi \text{ TCF of a process } X \text{ on } \mathbb{R}^d \\ \text{from the process class } model \end{array} \right\} \tag{5}$$

when referring to the set of TCFs arising from processes on $\mathbb{R}^d$ of the class *model*. For instance, T^d_{M3} is the set of TCFs of M3 processes on $\mathbb{R}^d$. By

$$T^d = \left\{ \chi : \mathbb{R}^d \to [0,1] \;\middle|\; \chi \text{ TCF on } \mathbb{R}^d \right\}$$

we denote the set of *all* TCFs on $\mathbb{R}^d$.

3 Construction principles for stationary max-stable processes

Two well-known construction principles for correlation functions in a geostatistical context also yield valid operations on the set of TCFs. First, also inspired by the work of Kabluchko and Stoev (2012), the *turning bands operator* can be transferred as an operator from lower to higher dimensions in the context of isotropic max-stable processes. Second, the *stationary truncation* generalizes a construction described in Schlather (2002, p. 39) and corresponds to the multiplication of a given TCF with another TCF that has compact support. It can shorten the range of tail dependence, e.g. to a compact set, a feature which is of interest for modelling purposes, cf.

Table 1 Tail correlation functions χ of max-stable processes introduced in Section 2.1. Here erfc denotes the complementary error function and $\mathcal{L}(G)$ the Laplace transform of a cdf G

Process model		Parameter	TCF $\chi(t)$ for $t \in \mathbb{R}^d$	Reference
M3r	M3 of radial non-increas. shapes	non-increasing random shape $f \geq 0$ on $[0,\infty) \times \Omega$ s.t. (4) holds	$\int_\Omega \int_{\mathbb{R}^d} f(\|z\|, \omega) \ldots \wedge f(\|z-t\|, \omega)\ \mathrm{d}z\, \nu(\mathrm{d}\omega)$	Eq. 3
M2r	M2 of radial non-increas. shapes	non-increasing determ. shape $f \geq 0$ on $[0,\infty)$ s.t. $\|f\|_{L^1(\mathbb{R}^d)} = 1$	$\int_{\mathbb{R}^d} f(\|z\|) \wedge f(\|z-t\|)\ \mathrm{d}z$	ibid.
M3b	M3 of ball indicators	random radius R on $(0,\infty)$	$\mathbb{E}_R \int_{\mathbb{R}^d} \frac{\mathbf{1}_{\|z\| \leq R} \wedge \mathbf{1}_{\|z-t\| \leq R}}{\kappa_d R^d}\ \mathrm{d}z$	ibid.
MPS	Mixed Poisson Storm	cdf G_β on $(0,\infty)$	$\mathcal{L}(G_\beta)\left(\frac{2\kappa_{d-1}}{d\kappa_d}\|t\|\right)$	Lantuéjoul et al. (2011), Prop.4
BR	Brown-Resnick	variogram γ increasing to ∞	$\operatorname{erfc}\left(\sqrt{\gamma(t)/8}\right)$	Kabluchko et al. (2009), Remark 25
VBR	Var-mixed Brown-Resnick	variogram γ increasing to ∞, cdf G_S on $(0,\infty)$	$\int_0^\infty \operatorname{erfc}\left(s\sqrt{\gamma(t)/8}\right)\ \mathrm{d}G_S(s)$	ibid.

Section 5 for an example. In the geostatistics literature the multiplication with a compactly supported covariance function is known as tapering high-dimensional data, cf. Furrer et al (2006).

3.1 Turning bands

The turning bands operator Let $k, d \in \mathbb{N}$ with $1 \leq k \leq d$. The set of ordered tuples $(x_1, \ldots, x_k)$ of k orthonormal vectors in $\mathbb{R}^d$ is known as the *Stiefel manifold of orthonormal k-frames in* $\mathbb{R}^d$ (Nachbin 1976, p. 131), and is denoted by $\mathcal{V}_k\left(\mathbb{R}^d\right)$. Interpreting the vectors $x_1, \ldots, x_k$ as columns of a matrix, we identify

$$\mathcal{V}_k\left(\mathbb{R}^d\right) = \left\{A \in \mathbb{R}^{d \times k} \,:\, A^{\mathrm{T}} A = \mathbb{1}_{k \times k}\right\}, \tag{6}$$

where A^{T} denotes the transpose of A and $\mathbb{1}_{k \times k}$ the identity matrix in $\mathbb{R}^{k \times k}$. A matrix $A \in \mathcal{V}_k\left(\mathbb{R}^d\right)$ embeds $\mathbb{R}^k$ linearly and isometrically into $\mathbb{R}^d$, whereas A^{T} applied to a vector $t \in \mathbb{R}^d$ is a vector in $\mathbb{R}^k$ whose coordinates can be interpreted as the coordinates of the projection of t onto $A\left(\mathbb{R}^k\right)$ with respect to the orthonormal frame defined by the columns of A. For $k = 1$ the Stiefel manifold is simply the sphere $\mathcal{V}_1\left(\mathbb{R}^d\right) = S^{d-1}$, and for $k = d$ the orthogonal group $\mathcal{V}_d\left(\mathbb{R}^d\right) = \mathrm{O}(d)$. In view of Eq. 6 the Stiefel manifold $\mathcal{V}_k\left(\mathbb{R}^d\right)$ is a compact submanifold of $\mathbb{R}^{d\times k}$. The action of the orthogononal group $\mathrm{O}(d)$ (from the left) exhibits $\mathcal{V}_k\left(\mathbb{R}^d\right)$ as a locally compact homogeneous space on which a unique normalized left invariant *Haar measure* σ_k^d can be defined (Nachbin 1976, p. 142, Example 4), which we call *uniform distribution* (Jupp and Mardia 1979; Mardia and Khatri 1977).

By $C\left(\mathbb{R}^d\right)$ we denote the set of real-valued continuous functions on $\mathbb{R}^d$. Since $\mathcal{V}_k\left(\mathbb{R}^d\right)$ is compact, the so-called *turning bands operator*

$$\mathrm{TB}_k^d : C\left(\mathbb{R}^k\right) \to C\left(\mathbb{R}^d\right), \qquad \mathrm{TB}_k^d(f)(t) = \int_{\mathcal{V}_k(\mathbb{R}^d)} f\left(A^{\mathrm{T}}(t)\right)\, \sigma_k^d(\,\mathrm{d}A).$$

is well-defined. Moreover, it is compatible with compositions (see Lemma 18 in Appendix A)

$$\mathrm{TB}_{k_2}^{k_3} \circ \mathrm{TB}_{k_1}^{k_2} = \mathrm{TB}_{k_1}^{k_3} \qquad \text{for } k_1 \leq k_2 \leq k_3. \tag{7}$$

Turning bands in the Gaussian case The turning bands operator TB_1^d is a familiar operator on positive definite functions, see Gneiting (1999a), Gneiting (1999b), Gneiting and Z. Sasvári (1999), Lantuéjoul (2002), Matheron (1973), Schlather (2012) and zu Castell (2002), where explicit formulae and recurrence relations are provided. For convenience, we recall some of them here: Let

$$\Phi_d = \left\{\rho_d : [0,\infty) \to [0,1] \;\mid\; \rho_d(\|\cdot\|) \text{ continuous correlation function on } \mathbb{R}^d\right\}.$$

Schoenberg (1938) showed that a function ρ_d belongs to the class Φ_d if and only if there exists a cdf F on $[0,\infty)$ such that ρ_d can be represented as a scale mixture

$$\rho_d(t) = \int_{[0,\infty)} \Omega_d(ts)\, \mathrm{d}F(s), \qquad t \geq 0, \tag{8}$$

with $\Omega_d(t) = \Gamma(d/2)\,(2/t)^{(d-2)/2}\, J_{(d-2)/2}(t)$, where J denotes a Bessel function of the first kind. For instance, $\Omega_1(t) = \cos(t)$ and $\Omega_3(t) = \sin(t)/t$. This relation provides a bijection of Φ_d with the set of cdfs on $[0,\infty)$ and hence a bijection of Φ_d and Φ_k for any $k \geq 1$. In particular, the mapping of ρ_1 to ρ_d may be expressed as (cf. Gneiting, 1999a, Eq. 6),

$$\rho_d(t) = \frac{2\,\Gamma(d/2)}{\sqrt{\pi}\,\Gamma((d-1)/2)} \int_0^1 \rho_1(tw)\left(1-w^2\right)^{(d-3)/2} \mathrm{d}w, \qquad t \geq 0, \tag{9}$$

and is known as turning bands operator (Gneiting 1999a; Matheron 1973). In our notation $\mathrm{TB}_1^d(\rho_1) = \rho_d(\|\cdot\|)$. In view of Eq. 7 this implies that TB_k^d is a bijection between Φ_k and Φ_d. In fact $\mathrm{TB}_k^d(\rho_k(\|\cdot\|)) = \rho_d(\|\cdot\|)$ (where the norms are taken in $\mathbb{R}^k$ and $\mathbb{R}^d$, respectively). Recurrence relations between the basis functions Ω_d and Ω_k immediately lead to recurrence relations between ρ_d and ρ_k (zu Castell 2002). For instance, the recursive relation

$$\rho_d(t) = \rho_{d+2}(t) + \frac{t}{d}\rho'_{d+2}(t), \qquad t \geq 0,$$

and its inverse

$$\mathrm{TB}_d^{d+2}(\rho_d(\|\cdot\|))(te) = \rho_{d+2}(t) = \frac{d}{t^d}\int_0^t u^{d-1}\rho_d(u)\,\mathrm{d}u, \qquad t \geq 0,\ e \in S^{d+1}, \tag{10}$$

hold true. Because $\mathrm{TB}_k^d\colon \Phi_k \to \Phi_d$ is a bijection, the *turning bands method* is an important tool for simulating stationary isotropic Gaussian processes. Given a correlation function $\rho_d \in \Phi_d$ with $\rho_d = \mathrm{TB}_1^d(\rho_1)$, one may approximate a Gaussian random field on $\mathbb{R}^d$ with correlation function ρ_d through

$$Z(t) = n^{-1/2}\sum_{i=1}^{n} Y_i(\langle t, S_i\rangle), \qquad t \in \mathbb{R}^d,$$

for sufficiently high $n \in \mathbb{N}$, an i.i.d. sequence $S_i \in S^{d-1}$ and independent copies Y_i of a random field Y on $\mathbb{R}$ with correlation function ρ_1 (Matheron 1973).

Turning bands in the max-stable case In the context of max-stable processes and their TCFs the situation transfers to the following extent. Let X be a stochastically continuous simple max-stable process on $\mathbb{R}^k$. Then the process X has a spectral representation as in Eq. 1

$$X_t = \bigvee_{n=1}^{\infty} U_n V_t(e_n), \qquad t \in \mathbb{R}^k, \tag{11}$$

where $\{(U_n, e_n)\}_n$ denotes a Poisson point process on $\mathbb{R}_+ \times E$ with intensity $u^{-2}\,\mathrm{d}u\,\mu(\,\mathrm{d}e)$ and the spectral function $V_t(e)$ is jointly measurable in the variables $t \in \mathbb{R}^k$ and $e \in E$ (Wang and Stoev 2010, Proposition 4.1). Based on this representation we define another simple max-stable process Y on $\mathbb{R}^d$ with $d \geq k$ as follows. Let $\{(U_n, e_n, A_n)\}_n$ be a Poisson point process on $\mathbb{R}_+ \times E \times \mathcal{V}_k\left(\mathbb{R}^d\right)$ of intensity $u^{-2}\,\mathrm{d}u\,\mu(\,\mathrm{d}e)\,\sigma_k^d(\,\mathrm{d}A)$, where $\sigma_k^d(\,\mathrm{d}A)$ is the uniform distribution on the Stiefel manifold $\mathcal{V}_k\left(\mathbb{R}^d\right)$ and let

$$Y_t = \bigvee_{n=1}^{\infty} U_n V_{A_n^{\mathrm{T}}(t)}(e_n), \qquad t \in \mathbb{R}^d. \tag{12}$$

Lemma 2 *Let X and Y be max-stable processes as given by Eqs. 11 and 12, respectively.*

a) *If X is stationary, then Y is stationary.*
b) *For any $M \in O(d)$ the law of $\left\{Y_{M(t)}\right\}_{t\in\mathbb{R}^d}$ and the law of Y coincide, i.e., Y is isotropic.*
c) *Let X be stationary. The (radial) TCF $\chi^{(Y)}$ of the stationary isotropic process Y can be expressed in terms of the TCF $\chi^{(X)}$ of X by*

$$\chi^{(Y)} = TB_k^d\left(\chi^{(X)}\right).$$

Proposition 3 *If χ is a continuous TCF on $\mathbb{R}^k$ and $k \leq d$, then $TB_k^d(\chi)$ is a TCF on $\mathbb{R}^d$ (which is also continuous).*

Remark 4 The function $\chi(t) = e^{-\|t\|}$ is an admissible radial TCF on $\mathbb{R}^d$ for any $d \geq 1$, see e.g. Table 2. Therefore, the radial function $\mathrm{TB}_1^3(\chi)(t) = \left(1 - e^{-\|t\|}\right)/\|t\|$ is a radial TCF on $\mathbb{R}^3$ by Eq. 10. However, contrary to correlation functions, not all radial continuous TCFs on $\mathbb{R}^d$ arise as $\mathrm{TB}_k^d(\chi)$ for some TCF χ on $\mathbb{R}^k$. As a counterexample consider the identity

$$e^{-\|t\|} = \mathrm{TB}_1^3(f)(t), \quad t \in \mathbb{R}^3 \quad \text{with} \quad f(t) = (1 - \|t\|)\, e^{-\|t\|}, \quad t \in \mathbb{R}^1.$$

While $e^{-\|t\|}$ is a valid radial TCF on $\mathbb{R}^3$, f cannot be a TCF on $\mathbb{R}^1$ since f attains negative values.

Remark 5 The turning bands method is compatible with iterations in the following sense: Let $q \geq d$ and construct a process Z on $\mathbb{R}^q$ from the spectral representation of Y on $\mathbb{R}^d$ by

$$Z_t = \bigvee_{n=1}^{\infty} U_n V_{B_n^{\mathrm{T}} \circ A_n^{\mathrm{T}}(t)}(e_n) = \bigvee_{n=1}^{\infty} U_n V_{(A_n \circ B_n)^{\mathrm{T}}(t)}(e_n), \qquad t \in \mathbb{R}^q,$$

where $\{(U_n, e_n, A_n, B_n)\}_n$ is a Poisson point process on $\mathbb{R}_+ \times E \times \mathcal{V}_k\left(\mathbb{R}^d\right) \times \mathcal{V}_d\left(\mathbb{R}^q\right)$ with intensity $u^{-2}\, \mathrm{d}u\, \mu(\mathrm{d}e)\, \sigma_k^d(\mathrm{d}A)\, \sigma_d^q(\mathrm{d}B)$. Then Z has the same law as

$$\widetilde{Z}_t = \bigvee_{n=1}^{\infty} U_n V_{C_n^{\mathrm{T}}(t)}(e_n), \qquad t \in \mathbb{R}^q,$$

where $\{(U_n, e_n, C_n)\}_n$ is a Poisson point process of intensity $u^{-2}\, \mathrm{d}u\, \mu(\mathrm{d}e)\, \sigma_k^q(\mathrm{d}C)$ (see Lemma 18 in Appendix A). Thus, Z can be constructed directly from the spectral representation of X without involving Y as an intermediate step.

Table 2 Parametric families of continuous radially symmetric functions on $\mathbb{R}^d$ and their sharp parameter bounds for being a correlation function (CF) and for being a tail correlation function (TCF) on $\mathbb{R}^d$, respectively

Parametric family of cts. radial functions on $\mathbb{R}^d$	CF for	TCF for
powered exponential $\exp(-r^\nu)$	$0 < \nu \leq 2$	$0 < \nu \leq 1$
Whittle-Matérn $2^{1-\nu}\, \Gamma(\nu)^{-1}\, r^\nu\, K_\nu(r)$	$0 < \nu$	$0 < \nu \leq 0.5$
Cauchy $(1 + r^\nu)^{-\beta}$ $\quad \beta > 0$	$0 < \nu \leq 2$	$0 < \nu \leq 1$
powered error function* $\mathrm{erfc}(r^\nu)$	$0 < \nu \leq 1$	$0 < \nu \leq 1$
truncated power function* $(1 - r)_+^\nu$	$\nu \geq (d+1)/2$	$\nu \geq \lfloor d/2 \rfloor + 1$

There are two exceptions: The TCF bound for the truncated power function is sharp for odd dimensions and the CF bound for the powered error function is sharp if we require validity of the model for all dimensions

3.2 Stationary truncation

Let X be a stochastically continuous max-stable process on $\mathbb{R}^d$ with spectral representation as in Eq. 11 with $k = d$ and let $\{B(t)\}_{t\in\mathbb{R}^d}$ be a measurable process on $\mathbb{R}^d$ taking values in $\{0, 1\}$. We denote the probability space corresponding to B by $(\Omega_B, \mathcal{A}_B, \mathbb{P}_B)$ and expectation w.r.t. $\mathbb{P}_B$ by $\mathbb{E}_B$. Further, we require that $c_B := \int_{\mathbb{R}^d} B(t)\ \mathrm{d}t \in (0,\infty)$ holds $\mathbb{P}_B$-almost surely. Based on these two processes X and B we define another max-stable process Y on $\mathbb{R}^d$ by

$$Y_t = \bigvee_{n=1}^{\infty} U_n \frac{B_n(t-z_n)}{c_{B_n}} V_t(e_n), \qquad t \in \mathbb{R}^d, \tag{13}$$

where $\{(U_n, e_n, z_n, B_n)\}_n$ is a Poisson point process on $\mathbb{R}_+ \times E \times \mathbb{R}^d \times \Omega_B$ with intensity $u^{-2}\ \mathrm{d}u \times \mu \times \nu_d \times \mathbb{P}_B$.

Lemma 6 *Let X and Y be simple max-stable processes as given by Eq. 11 for $k = d$ and Eq. 13, respectively.*

a) *If X is stationary, then Y is stationary.*
b) *Let X be stationary. Then the TCF $\chi^{(Y)}$ of the stationary process Y is given by the product*

$$\chi^{(Y)}(t) = \chi^{(X)}(t) \cdot \chi^{(B)}(t), \qquad t \in \mathbb{R}^d,$$

with $\chi^{(X)}$ being the TCF of X and

$$\chi^{(B)}(t) = \mathbb{E}_B\left[\frac{\int_{\mathbb{R}^d} B(z)B(z-t)\, dz}{\int_{\mathbb{R}^d} B(z)\, dz}\right], \qquad t \in \mathbb{R}^d.$$

Example 7 If the process B on $\mathbb{R}^d$ is chosen to be the indicator function $B(t) = \mathbf{1}_{\|t\|\le R}$ of the ball B_R^d for a random radius $R \in (0,\infty)$, then

$$\chi^{(B)}(t) = \mathbb{E}_R \int_{\mathbb{R}^d} \frac{\mathbf{1}_{\|z\|\le R} \wedge \mathbf{1}_{\|z-t\|\le R}}{\kappa_d R^d}\ \mathrm{d}z, \qquad t \in \mathbb{R}^d,$$

which means that the functions $\chi^{(B)}$ build the class T^d_{M3b}, cf. Eq. 5 and the entry on M3b processes in Table 1.

4 Identification of classes of TCFs and their relations

Some relations between the subclasses of TCFs that arise from the subclasses of max-stable processes introduced in Section 2.1 follow immediately from their definition, e.g. $T^d_{\mathrm{M2r}} \subset T^d_{\mathrm{M3r}}$ and $T^d_{\mathrm{M3b}} \subset T^d_{\mathrm{M3r}}$. The aim of this section is to identify more sophisticated relations between these subclasses and to provide necessary and sufficient conditions for a given function χ to belong to such a class. We conclude with some parametric families of TCFs and their affiliation to the respective subclasses.

The subsequent considerations rely on certain monotonicity properties of functions. We review the required notions *α-times monotone* and *completely monotone* in Appendix B and focus on a clear statement of the relations and conditions in this section, while the proofs are postponed to Appendix A. We start with sharpening the inclusions $T^d_{\mathrm{M2r}} \subset T^d_{\mathrm{M3r}}$ and $T^d_{\mathrm{M3b}} \subset T^d_{\mathrm{M3r}}$. Since elements of these classes are functions that depend on the Euclidean norm only, we will identify them with the respective functions on $[0, \infty)$ henceforth.

Proposition 8 *For all $d \geq 1$ we have $T^d_{M3r} = T^d_{M2r} = T^d_{M3b}$.*

In fact, the class $T^d_{\mathrm{M3r}} = T^d_{\mathrm{M2r}} = T^d_{\mathrm{M3b}}$ is well-known in geostatistics and has been intensively studied in Gneiting (1999c), therein called H_d. Gneiting (1999c) defines H_d as the class of scale mixtures of the function $h_d(t) = \tilde{h}_d(t)/\tilde{h}_d(0)$ where $\tilde{h}_d$ is the self-convolution of the ball indicator function $\mathbf{1}_{B^d_{0.5}}$ viewed as a radial function, i.e.

$$H_d = \left\{ \varphi(t) = \int_{(0,\infty)} h_d(st) \; \mathrm{d}G(s) \;\middle|\; Gcdf on (0, \infty) \right\}, \qquad \text{where}$$

$$h_d(t) = \frac{d\,\Gamma(d/2)}{\sqrt{\pi}\,\Gamma((d+1)/2)} \int_t^1 \left(1 - v^2\right)_+^{(d-1)/2} \mathrm{d}v. \tag{14}$$

For $d = 1, 2, 3$ the function h_d is given by

$$\begin{aligned} h_1(t) &= 7(1-t)_+ \\ h_2(t) &= 2\pi^{-1} \left(\arccos(t) - t\sqrt{1-t^2} \right) \mathbf{1}_{t \leq 1} \\ h_3(t) &= (1-t)_+^2 (2+t)/2. \end{aligned}$$

From the definition of H_d it is apparent that $H_d = T^d_{\mathrm{M3b}}$, since the minimum in the following expression is in fact a multiplication. Indeed, we may rewrite the M3b entry in Table 1 as a scale mixture of h_d

$$\begin{aligned} &\mathbb{E}_R \left[\left(\kappa_d R^d \right)^{-1} \int_{\mathbb{R}^d} \mathbf{1}_{\|z\| \leq R} \wedge \mathbf{1}_{\|z-t\| \leq R} \; \mathrm{d}z \right] \\ &= \tilde{h}_d(0)^{-1} \mathbb{E}_R \left[\int_{\mathbb{R}^d} \mathbf{1}_{\|z\| \leq 0.5} \, \mathbf{1}_{\|z - t/(2R)\| \leq 0.5} \; \mathrm{d}z \right] \\ &= \mathbb{E}_R \left[\frac{\tilde{h}_d\left(\|t\|/(2R)\right)}{\tilde{h}_d(0)} \right] = \int_{(0,\infty)} h_d(s\|t\|) \; \mathrm{d}G_{1/(2R)}(s) \end{aligned} \tag{15}$$

if $G_{1/(2R)}$ is the distribution function of $1/(2R) \in (0, \infty)$ and vice versa. Another way to perform the integration (one may think of full balls as foliated by spheres) leads to the coincidence of H_d with the *Mittal-Berman class* V_d (Gneiting (1999c, Eq. 40) and Mittal (1976)), which in turn is easily connected to T^d_{M2r}, see also Proof of Proposition 8 in Appendix A. A crucial observation in Gneiting (1999c) is that the first derivative of h_d applied to the square root is proportional to Askey's function with exponent $(d+1)/2 - 1$, that is

$$-h'_d\left(\sqrt{t}\right) \sim (1-t)_+^{(d+1)/2-1}.$$

Now, scale mixtures of this function are precisely $(d+1)/2$-times monotone functions (cf. Appendix B), which entails the characterization

$$H_d = \left\{ \varphi : [0,\infty) \to [0,1] \;\middle|\; \begin{array}{l} \varphi \text{ continuous}, \varphi(0) = 1, \lim_{t\to\infty} \varphi(t) = 0, \\ -\varphi'\left(\sqrt{\cdot}\right) \text{ is } (d+1)/2\text{-times monotone on } (0,\infty) \end{array} \right\}$$

Gneiting (1999c, Proof on Theorem 3.1, Theorem 3.2, and Criterion 1.2. on p. 103). A simplified version of this statement, which is easier to handle, is that

$$\begin{aligned} T^1_{\mathrm{M3r}} &= T^1_{\mathrm{M2r}} = T^1_{\mathrm{M3b}} = H_1 \\ &= \left\{ \varphi : [0,\infty) \to [0,1] \;\middle|\; \begin{array}{l} \varphi \text{ continuous, convex,} \\ \varphi(0) = 1, \lim_{t\to\infty} \varphi(t) = 0 \end{array} \right\}, \end{aligned} \tag{16}$$

and for $d \geq 2$

$$\begin{aligned} T^d_{\mathrm{M3r}} &= T^d_{\mathrm{M2r}} = T^d_{\mathrm{M3b}} = H_d \\ &\supseteq \left\{ \varphi : [0,\infty) \to [0,1] \;\middle|\; \begin{array}{l} \varphi \text{ continuous}, \varphi(0) = 1, \lim_{t\to\infty} \varphi(t) = 0, \\ (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}t^k} \left[-\varphi'\left(\sqrt{t}\right)\right] \text{ convex for } k = \lfloor (d-2)/2 \rfloor \end{array} \right\} \end{aligned} \tag{17}$$

where $\lfloor (d-2)/2 \rfloor$ denotes the greatest integer less than or equal to $(d-2)/2$. The inclusion (17) is in fact an equality if and only if d is odd (Gneiting 1999c, Theorem 3.1., Theorem 3.3., Criterion 1.2). The classes H_d are all nested, i.e. $H_1 \supset H_2 \supset H_d \supset \ldots$ Gneiting (1999c), Theorems 3.7 and 3.8, also characterizes the class

$$H_\infty = \bigcap_{d=1}^{\infty} H_d. \tag{18}$$

as scale mixtures of the complementary error function

$$\begin{aligned} H_\infty &= \left\{ \varphi(t) = \int_{(0,\infty)} \operatorname{erfc}(st)\ \mathrm{d}G(s) \;\middle|\; G \text{ cdf on } (0,\infty) \right\} \\ &= \left\{ \varphi : [0,\infty) \to [0,1] \;\middle|\; \begin{array}{l} \varphi \text{ continuous}, \varphi(0) = 1, \lim_{t\to\infty} \varphi(t) = 0, \\ -\varphi'\left(\sqrt{\cdot}\right) \text{ completely monotone on } (0,\infty) \end{array} \right\}. \end{aligned} \tag{19}$$

This characterization of H_∞ is not too surprising in view of the proportionality $-\operatorname{erfc}'(\sqrt{t}) \sim e^{-t}$, which corresponds to Bernstein's Theorem that can be seen as the limiting case of Williamson's Theorem as $n \to \infty$ (cf. Appendix B). It is astonishing, however, that characterization (19) of H_∞ provides a direct link between the TCFs of VBR processes and the TCFs of M3r processes: From the VBR entry in Table 1 and Eq. 19 (replacing $\gamma/8$ by $\tilde{\gamma}$ and G corresponding to the variance mixing distribution G_S), we see that

$$\begin{aligned} T^d_{\mathrm{VBR}} &= \left\{ \int_{(0,\infty)} \operatorname{erfc}(s\sqrt{\tilde{\gamma}})\ \mathrm{d}G(s) \;\middle|\; \tilde{\gamma} \text{ variogram on } \mathbb{R}^d \text{ and } G \text{ cdf on } (0,\infty) \right\} \\ &= \left\{ \varphi\left(\sqrt{\tilde{\gamma}}\right) \;\middle|\; \tilde{\gamma} \text{ variogram on } \mathbb{R}^d \text{ and } \varphi \in H_\infty \right\}. \end{aligned} \tag{20}$$

On the other hand the equalities in Proposition 8 carry over. For instance, a function belongs to the class H_∞ if and only if it is a TCF of an M3r process on $\mathbb{R}^d$ for any dimension $d \geq 1$. In particular both classes, T^d_{VBR} and T^d_{M3r}, comprise the class H_∞ in any dimension $d \geq 1$, see Proposition 9 below.

Finally, we observe from the MPS entry in Table 1 that in every dimension $d \geq 1$ the class of TCFs arising from MPS processes is given by Laplace transforms of cdfs on $(0, \infty)$ and, thus, coincides with

$$T^d_{\text{MPS}} = \left\{ \varphi(t) = \int_{(0,\infty)} e^{-st} \, \mathrm{d}G(s) \;\middle|\; G \text{ cdf on } (0,\infty) \right\} \tag{21}$$
$$= \left\{ \varphi : [0,\infty) \to [0,1] \;\middle|\; \begin{array}{l} \varphi \text{ continuous}, \varphi(0) = 1, \lim_{t\to\infty} \varphi(t) = 0, \\ \varphi \text{ completely monotone on } (0,\infty) \end{array} \right\}.$$

Here the cdf G is related to the intensity mixing distribution G_β via $G(s) = G_\beta\ (s/c_d)$ with $c_d = 2\kappa_{d-1}/(d\kappa_d)$, cf. Table 1. In particular, the class T^d_{MPS} does not depend on the specific dimension d, even though the involved factor c_d does. These observations lead to the following relations between the classes of TCFs arising from the considered mixing processes, which are also illustrated in Fig. 1.

Proposition 9 *The following relations hold for all dimensions $d \geq 1$:*

a) $T^d_{MPS} \subset H_\infty \subset H_d = T^d_{M3r} = T^d_{M2r} = T^d_{M3b}$.
b) $T^d_{BR} \cup H_\infty \subset T^d_{VBR}$.
c) $erfc(t^\alpha) \in T^d_{BR} \Leftrightarrow \alpha \in (0,1]$ *and* $erfc(t^\alpha) \in T^d_{MPS} \Leftrightarrow \alpha \in (0, 0.5]$. *In particular,* $T^d_{BR} \setminus T^d_{MPS} \neq \emptyset$ *and* $T^d_{BR} \cap T^d_{MPS} \neq \emptyset$.
d) *While H_d contains functions with compact support, the class T^d_{VBR} does not contain such functions. In particular $H_d \setminus T^d_{VBR} \neq \emptyset$.*
e) *$H_\infty \setminus T^d_{BR} \neq \emptyset$ for all $d \geq d_0$ and some fixed dimension $d_0 \in \mathbb{N}$.*

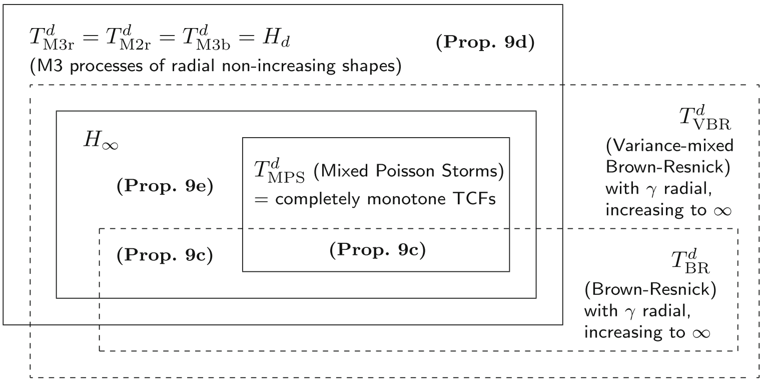

Fig. 1 Inclusions and intersection of sets of tail correlation functions arising from mixing max-stable processes, cf. Proposition 9. The expression "(**Prop. ...**)" provides the reference for the indicated region to be non-empty. See also Eqs. 14, 19, 20 and 21 for characterizations of these classes and scale mixture representations

One might assume the impression that any continuous radial TCF on $\mathbb{R}^d$ that is non-increasing and convex on $[0, \infty)$ and that vanishes at ∞ belongs already to the class T^d_{M3r} or at least appears already in Fig. 1. This is true for $d = 1$. By means of the operations from Section 3, however, we may construct counterexamples in higher dimensions. Let us denote

$$T_r^d := \left\{ \chi : [0,\infty) \to [0,1] \;\middle|\; \begin{array}{l} \chi \text{ continuous radial TCF on } \mathbb{R}^d \\ \text{that is convex in the radius} \\ \text{and vanishes at } \infty \end{array} \right\} = T^d \cap H_1.$$

First, we provide for each $d \geq 3$ an example of a TCF $\varphi_d \in T_r^d \setminus H_d$. To this end, we consider the *tent function* $h_1(t) = (1-t)_+$, which is the basis function of H_1, see Eq. 14. If we apply the turning bands operator, we obtain the radial TCF φ_d on $\mathbb{R}^d$ (cf. Proposition 3)

$$\varphi_d(t) := \mathrm{TB}_1^d(h_1)(t), \qquad t \geq 0. \tag{22}$$

Proposition 10 a) *For $d \geq 1$ we have $\varphi_d \in T_r^d = T^d \cap H_1$.*

b) *For $d \geq 1$ and $k \geq 3$ we have $\varphi_d \notin H_k$.*

c) *For $d = 1$ and $d \geq 6$ we have $\varphi_d \notin H_2$.*

Remark 11 The TCF φ_d from Eq. 22 decreases linearly on the interval $[0, 1]$, cf. Eq. 32 in Appendix A.

$$\varphi_d(t) = 1 - \beta_d\, t, \qquad t \in [0,1], \qquad \text{where} \qquad \beta_d = \frac{\Gamma(d/2)}{\sqrt{\pi}\,\Gamma((d+1)/2)}. \tag{23}$$

Therefore, the radial function $\chi_\beta(t) := 1 - \beta t$ is an admissible radial TCF on the d-dimensional ball B_r^d of radius r if $\beta \in [0, \beta_d/r]$. This complements results in Gneiting (1999a), where it is shown that $\varphi(t) = 1 - \alpha t$ is positive definite on B_r^d if and only if $\alpha \in [0, 2\beta_d/r]$. We assume that the bound β_d/r is sharp for $\chi_\beta(t)$ to be a TCF on B_r^d.

Secondly, combining the turning bands operator and the stationary truncation leads to an example of a TCF $\chi_3 \in T_r^3$ that is not contained in any of the classes given in Fig. 1 for $d = 3$, and we conjecture that our example χ_d satisfies this property also for any other dimension $d \geq 2$. With φ_d from Eq. 22 we consider the function

$$\chi_d(t) := \varphi_d(2t)\, h_d(t), \qquad t \geq 0. \tag{24}$$

Proposition 12 a) *For $d \geq 1$ we have $\chi_d \in T_r^d \setminus T_{VBR}^d$.*

b) *For $d = 3$ we have $\chi_d \in T_r^d \setminus \left(T_{VBR}^d \cup H_d\right)$.*

Parametric families The considerations above also lead to sharp bounds for some well-known parametric families of positive definite functions to be a TCF, see Table 2.

The first three families (*powered exponential*, *Whittle-Matérn*, *Cauchy*) are completely monotone for the parameters given in Table 2 (Miller and Samko 2001, Eqs. 1.2, 1.6 and 2.32 for example), and thus they can be realized by either an MPS process, an M3 process of non-increasing shapes (e.g. M2r or M3b) or by a VBR process (in all cases in any dimension). The *powered error function* is not completely monotone but a member of the class H_∞. That means it can be realized by an M3 process of non-increasing shapes or by a VBR process (both in any dimension), but not by an MPS process. In all of these cases, we may exclude bigger parameters ν because the (right-hand) derivative at 0 vanishes for bigger ν (which would entail the differentiability of the respective function when viewed as a function on $\mathbb{R}^d$).

The *truncated power function* is an example of a TCF with compact support. Hence, the valid model parameter depends on the dimension. The function belongs to H_d (Gneiting 1999c, Theorem 6.3), and thus can be realized by an M3 process of non-increasing shapes on $\mathbb{R}^d$ (e.g. M2r or M3b). Because of its compact support the function cannot belong to any of the other classes presented in Fig. 1. The bound in Table 2 is valid in any dimension and sharp in odd dimensions, cf. (Golubov 1981, Theorem 1 and p. 165). For even dimensions ν has to satisfy at least $\nu \geq (d+1)/2$ in order to ensure positive definiteness.

Remark 13 Davison and Gholamrezaee (2012), p. 590, provide some examples of $\chi^{(B)}$ from Lemma 6 when $d = 2$, e.g., the function h_2 (from below Eq. 14) corresponding to a deterministic radius in Eq. 15 is computed and the approximation $\alpha(t) = (1 - \|t\|)_+$ proposed. However, note that $\alpha(t)$ is not admissible for $d = 2$, since $\nu \geq 1.5$ is needed for $\alpha_\nu(t) = (1 - \|t\|)_+^\nu$ to be at least positive definite.

5 Recovery of some subclasses of max-stable processes from TCFs

For some subclasses of max-stable processes the recovery of the process from its TCF is mathematically trivial. Indeed the formulae in Table 1 give one-to-one relations between the underlying variogram γ of a BR process and its TCF χ and the underlying intensity mixing distribution G_β of an MPS process and its TCF χ. Up to the dimension specific scaling constant $c_d = 2\kappa_{d-1}/(d\kappa_d)$ the cdf G_β is the inverse

Table 3 Recovery expressions for the defining quantities of an M2r and an M3b process from a prescribed TCF $\chi \in T^d_{\mathrm{M2r}} = T^d_{\mathrm{M3b}} = H_d$ in dimensions $d = 1, 2, 3$ (cf. Proposition 14): (i) the deterministic shape function f of an M2r process and (ii) the density g_{2R} of $2R$, where R is the random radius that defines an M3b process (if the density g_{2R} exists)

	d=1	d=2	d=3
f(u)	$-\chi'(2u)$	$\frac{4u}{\pi}\int_0^{1/(2u)}((2ut)^{-2}-1)^{1/2}\ \mathrm{d}\lambda_\chi(t)$	$\chi''(2u)/(\pi\ u)$
$g_{2R}(s)$	$s\ \chi''(s)$	$\frac{s^2}{2}\int_0^{1/s}\left((s/t)^2-1\right)^{-1/2}\ \mathrm{d}\lambda_\chi(t)$	$\frac{s}{3}\left(\chi''(s) - s\chi'''(s)\right)$

The function f may have a pole at 0 and g_{2R} may have other poles as well. We abbreviate $\lambda_\chi(t) := t\ \chi''(1/t)$

Laplace transform of χ, i.e. $G_\beta(s) = \mathcal{L}^{-1}(\chi)(c_d s)$. We address the remaining processes of type M2r, M3b and VBR in this section and close with connections to the stationary truncation of a VBR process (cf. Section 3).

Recovery of M2r and M3b processes We know already that the set of TCFs arising from M2r or M3b processes coincides with the Gneiting class H_d, cf. Proposition 8. For a prescibed TCF $\chi \in H_d$, the following proposition essentially restates recovery results from Gneiting (1999c). Explicit expressions in dimensions $d = 1, 2, 3$ are given in Table 3.

Proposition 14 *Let $\chi \in T^d_{M2r} = T^d_{M3b} = H_d$. For odd $d \geq 1$ set $k = (d-1)/2$ and define the (right-hand) derivative*

$$\lambda(t) = (-1)^k \frac{d^k}{dt^k}\left[-\chi'\left(\sqrt{t}\right)\right].$$

For even $d \geq 2$ set $k = d/2$ and define the (right-hand) derivative

$$\mu(t) = \frac{d^k}{dt^k}\left[-\int_0^t \frac{v^{(d-1)/2}\chi'\left(1/\sqrt{v}\right)}{\sqrt{\pi}(t-v)^{1/2}}\, dv\right].$$

a) *The monotone shape function f of an M2r process with TCF χ is given by*

$$f(u) = \begin{cases} \left(2/\sqrt{\pi}\right)^{d-1}\lambda(4u^2) & d \geq 1 \text{ odd}, \\ \int_0^{1/(2u)} \left(2s/\sqrt{\pi}\right)^{d-1}\, d\mu(s^2) & d \geq 2 \text{ even}. \end{cases}$$

b) *The cdf $G_{1/(2R)}$ of $1/(2R)$, where R is the random radius of an M3b process with TCF χ, is given by*

$$G_{1/(2R)}(s) = \begin{cases} \sqrt{\pi}(d\Gamma(d/2))^{-1}\int_0^s t^{-d}\, d\lambda\left(t^{-2}\right) & d \geq 1 \text{ odd}, \\ \sqrt{\pi}(d\Gamma(d/2))^{-1}\int_0^s t^{-1}\, d\mu(t^2) & d \geq 2 \text{ even}. \end{cases}$$

c) *The monotone shape function f and the cdf $G_{1/(2R)}$ can be recovered from each other by*

$$f(u) = \int_0^{1/(2u)} \frac{(2s)^d}{\kappa_d}\, dG_{1/(2R)}(s) \text{ and } G_{1/(2R)}(s) = \int_0^s \frac{\kappa_d}{(2u)^d}\, d\left[f\left(\frac{1}{2u}\right)\right]. \tag{25}$$

Recovery of VBR processes The TCF of a VBR processes is given by $\chi = \varphi\left(\sqrt{\tilde{\gamma}}\right)$, where $\tilde{\gamma}$ is a variogram and φ is a scale mixture of the complementary error function, that is $\varphi \in H_\infty$, cf. (20). If the variance mixing distribution G_S that determines φ is fixed, the TCF of a VBR process uniquely determines the law of the VBR process, since the variogram $\gamma = 8\tilde{\gamma}$ can be recovered from χ. The following lemma can be useful in order to detect pairs φ and G, such that indeed $\varphi(t) = \int_{(0,\infty)} \operatorname{erfc}(st)\, \mathrm{d}G(s)$ holds. In Table 4 we give some examples of corresponding pairs φ and cdfs G. They include the Whittle-Matérn family (ii), the arctan model (iii) and the Dagum model (iv), cf. Berg et al (2008).

Table 4 Members φ of the class H_∞ as scale mixtures of the complementary error function

	Distribution function $G(s)$ or $g(s) = G'(s)$	$\varphi(t) = \int_0^\infty \operatorname{erfc}(st)\ \mathrm{d}G(s)$	
(i)	$G(s) = e^{-1/(as)^2}$	$e^{-2t/a}$	$a > 0$
(ii)	$g(s) = \frac{\sqrt{\pi}}{\Gamma(\nu)\Gamma(\frac{1}{2}-\nu)} \int_0^s \frac{x^{2\nu-3}e^{-1/(4x^2)}}{(s^2-x^2)^{\nu+1/2}}\ \mathrm{d}x$	$\frac{2^{1-\nu}}{\Gamma(\nu)} t^\nu K_\nu(t)$	$\nu \in (0, 1/2)$
(iii)	$G(s) = \operatorname{erf}(as)$	$1 - \frac{2}{\pi} \arctan(t/a)$	$a > 0$
(iv)	$G(s) = 1 - e^{-(as)^2}$	$1 - \left(1 + (t/a)^{-2}\right)^{-1/2}$	$a > 0$

Lemma 15 *Let $g(s) = \sqrt{\pi}\, f(s^2)$ be a probability density on $(0, \infty)$ and let $\varphi : [0, \infty) \to [0, 1]$ with $\varphi(0) = 1$ be such that $-\varphi'\left(\sqrt{\cdot}\right)$ is the Laplace transform of f in the sense that $-\varphi'\left(\sqrt{t}\right) = \int_0^\infty e^{-rt} f(r)\ dr$. Then*

$$\varphi(t) = \int_0^\infty \mathit{erfc}\,(st)\, g(s)\ ds.$$

Stationary truncated VBR processes It is well-known that BR processes do not allow for a restricted range of asymptotic dependence, i.e., their TCF cannot have compact support, which also holds true for their generalized version of VBR processes (cf. Proposition 9d)). However, this feature may be incorporated by stationary truncation, i.e.

$$X_t := \bigvee_{n=1}^{\infty} \frac{U_n \mathbf{1}_{\|t-z_n\| \le R_n}}{\kappa_d\,(R_n)^d} \exp\left(S_n W_t^{(n)} - \frac{(S_n)^2 \sigma^2(t)}{2}\right), \qquad t \in \mathbb{R}^d, \tag{26}$$

where (U_n, z_n) is a Poisson process on $\mathbb{R}_+ \times \mathbb{R}^d$ with intensity $u^{-2}\,\mathrm{d}u \times \nu_d$ and, independently, R_n, S_n, $W^{(n)}$ are mutually independent i.i.d. random elements, such that

- $1/(2R_n)$ is drawn from a cdf $G_{1/(2R)}$ on $(0, \infty)$,
- S_n is drawn from a cdf G_S on $(0, \infty)$,
- $W^{(n)}$ is a realization of a Gaussian process on $\mathbb{R}^d$ with stationary increments with variogram $\gamma(t) = \mathbb{E}(W_t - W_o)^2$ and variance $\sigma^2(t) = \mathrm{Var}(W_t)$.

We call X a *stationary truncated VBR processes*. Any stationary truncated process X might be still attractive for statistical inference of high-dimensional data, even though in many cases full or partial likelihoods cannot be computed anymore, as the TCF X is still tractable, cf. Lemma 6. Here, we have

$$\chi(t) = \varphi(\sqrt{\gamma(t)/8})\, \psi(\|t\|),$$

where $\varphi \in H_\infty$ and $\psi \in H_d$ with

$$\varphi(t) = \int_{(0,\infty)} \operatorname{erfc}(st)\ \mathrm{d}G_S(s) \qquad \text{and} \qquad \psi(t) = \int_{(0,\infty)} h_d(st)\ \mathrm{d}G_{1/(2R)}(s).$$

Example 16 If both $G_S(s) = \mathbf{1}_{[1,\infty)}(as)$ and $G_{1/(2R)}(s) = \mathbf{1}_{[1,\infty)}(bs)$ are cdfs of a deterministic distribution with total mass at $1/a > 0$ and $1/b > 0$, respectively, then we obtain the TCF of a stationary truncated BR process as

$$\chi(t) = \operatorname{erfc}\left(\sqrt{\gamma(t)/(8a^2)}\right) h_d\left(\|t\|/b\right), \qquad t \in \mathbb{R}^d,$$

which is compactly supported on the ball of radius b in $\mathbb{R}^d$, see Eq. 14 for h_d.

6 Example of max-stable processes with an identical TCF

Although it has been a commonplace that the TCF of a max-stable process (with standardized margins) does not uniquely determine the process, the diversity of the processes that share an identical TCF seems to be remarkably large. Here, we illustrate this diversity with a concrete example. Since the recovery in odd dimensions is computationally easier to handle (cf. Proposition 14 and Table 3), we consider only the two-dimensional sections of M2r and M3b processes on $\mathbb{R}^3$ instead of

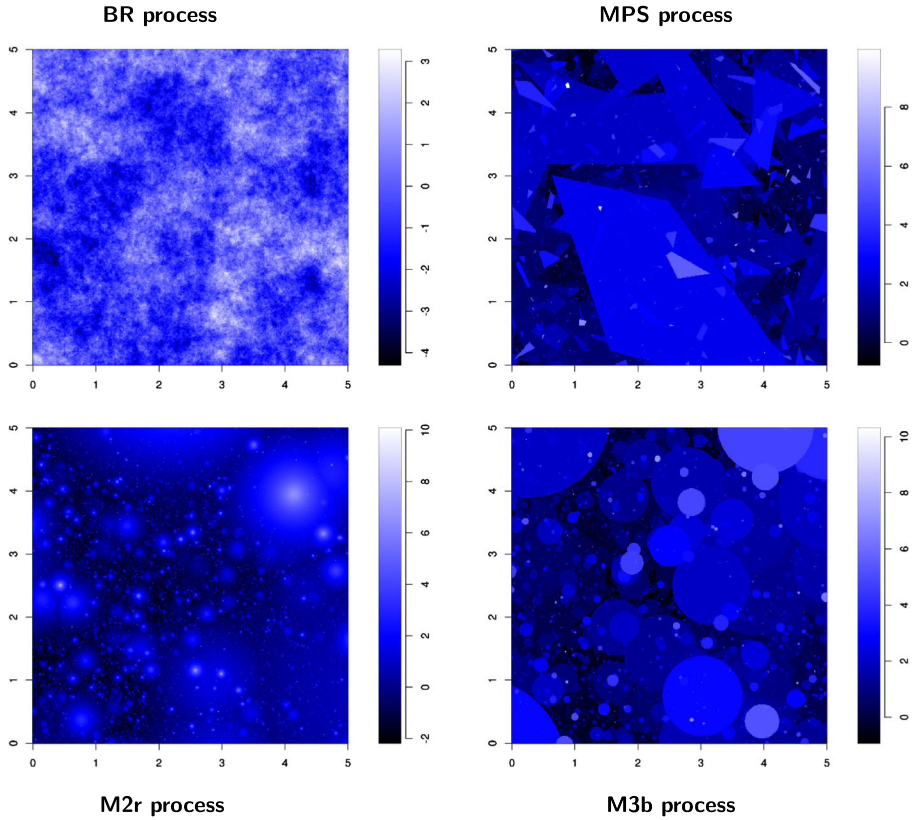

Fig. 2 Simulations of different mixing max-stable processes on $[0, 5]^2 \subset \mathbb{R}^2$ with identical tail correlation function $\chi(t) = \operatorname{erfc}(\sqrt{\|t\|})$ (see Proposition 17): Brown-Resnick process (BR), Mixed Poisson Storm process (MPS), two-dimensional section of an M2r process with deterministic shape (M2r), two-dimensional section of an M3b process of normalized ball indicator functions (M3b). The plots were transformed to standard Gumbel marginals

two-dimensional M2r and M3b processes. Figure 2 shows simulations of these processes in dimension $d = 2$ that were obtained using the R-package RandomFields V3.0 (Schlather et al. 2014).

Corollary 17 *The following four processes on $\mathbb{R}^2$ are stationary simple max-stable and share the same TCF $\chi(t) = \mathrm{erfc}\left(\sqrt{\|t\|}\right)$, $t \in \mathbb{R}^2$:*

(i) *the BR process on $\mathbb{R}^2$ associated to the variogram $\gamma(t) = 8\|t\|$ for $t \in \mathbb{R}^2$,*
(ii) *the MPS process on $\mathbb{R}^2$ with intensity mixing distribution*

$$G_\beta(s) = \begin{cases} 0 & \text{if } s \leq \pi/2, \\ 2\pi^{-1} \arctan\left(\sqrt{2\pi^{-1}s - 1}\right) & \text{if } s > \pi/2, \end{cases}$$

(iii) *the restriction of the M2r process on $\mathbb{R}^3$ to $\mathbb{R}^2 = \{(t_1, t_2, 0) : t \in \mathbb{R}^3\}$ that has the monotone shape function*

$$f(t) = \pi^{-3/2} (1 + \|4t\|)\|2t\|^{-5/2} e^{-\|2t\|}, \qquad t \in \mathbb{R}^3,$$

(iv) *the restriction of the M3b process on $\mathbb{R}^3$ to $\mathbb{R}^2 = \{(t_1, t_2, 0) : t \in \mathbb{R}^3\}$ with random radius R, where the density g_{2R} of $2R$ is given by*

$$g_{2R}(s) = 1/12\,(\pi s)^{-1/2}\left(4s^2 + 8s + 5\right) e^{-s}, \qquad s \in [0, \infty).$$

7 Concluding remarks

The present text puts particular emphasis on isotropic models, i.e., they are radially symmetric. Of course, all models can be combined with a linear or non-linear transformation of the space to account for observed anisotropies as commonly done in spatial applications.

We showed that the TCF uniquely determines the distribution of some max-stable processes when certain subclasses of max-stable processes are considered. On the other hand different max-stable models may share the same TCF. This phenomenon arises not just from exotic coincidences, but happens systematically, even among well-known subclasses of max-stable processes, cf. Fig. 2. We conclude that the TCF should not be overrated as an informative dependence measure solely and that other criteria should be involved as well in the extreme value analysis of spatial data.

Our considerations exceed the max-stable setting. First, the TCF of a stochastic process in the domain of attraction of a max-stable process X coincides with the TCF of X. Second, the results also concern inverted max-stable processes (Wadsworth and Tawn 2012). Let $X^{\mathrm{inv}} := -\log(1 - \exp(-1/X))^{-1}$ be the inverted max-stable process associated to the max-stable process X with TCF χ and η the tail dependence function, where $\mathbb{P}\left(X_t^{\mathrm{inv}} > \tau\,,\, X_o^{\mathrm{inv}} > \tau\right) \sim \mathcal{L}(\tau)\,\tau^{-1/\eta(t)}$ for a slowly varying function $\mathcal{L}(\tau)$ as $\tau \to \infty$ (Ledford and Tawn 1996), Wadsworth and Tawn (2012) observe that

$$\eta(t) = (2 - \chi(t))^{-1} = \frac{1}{2}\left(1 - \frac{\chi(t)}{2}\right)^{-1} \in \left[\frac{1}{2}, 1\right], \qquad t \in \mathbb{R}^d.$$

In particular, η is also positive definite and not differentiable unless η is constant. The function η comes along with similar benefits and dangers in the regime of inverted max-stable processes as presented here for χ in the regime of max-stable processes.

Acknowledgements We thank two anonymous referees for many helpful comments that lead to significant improvements of the paper and Zakhar Kabluchko for a discussion on ergodicity and mixing of max-stable processes. Financial support by the German Research Foundation DFG for K. Strokorb who undertook this work as a member of the RTG 1023 and for M. Schlather by Volkswagen Stiftung within the 'WEX-MOP' project is gratefully acknowledged.

A Proofs

Lemma 18 *Let* $k_1 \leq k_2 \leq k_3$.

a) *The composition map* $\mathcal{V}_{k_1}\left(\mathbb{R}^{k_2}\right) \times \mathcal{V}_{k_2}\left(\mathbb{R}^{k_3}\right) \to \mathcal{V}_{k_1}\left(\mathbb{R}^{k_3}\right)$ *which maps* (A, B) *to* $B \circ A$ *is continuous.*

b) *If* $B \sim \sigma_{k_2}^{k_3}$ *is uniformly distributed on* $\mathcal{V}_{k_2}\left(\mathbb{R}^{k_3}\right)$ *and A is an independent (Borel-measurable) random variable with values in* $\mathcal{V}_{k_1}\left(\mathbb{R}^{k_2}\right)$*, then the composition* $B \circ A$ *will also be uniformly distributed* $B \circ A \sim \sigma_{k_1}^{k_3}$.

c) *The turning bands operator is compatible with compositions, i.e., we have* $TB_{k_2}^{k_3} \circ TB_{k_1}^{k_2} = TB_{k_1}^{k_3}$.

Proof The composition of matrices is continuous and here just restricted to a subspace. This shows a). Let f be a continuous function on $\mathcal{V}_{k_1}\left(\mathbb{R}^{k_3}\right)$, then (by dominated convergence) the function $g(b) := \mathbb{E}_A(f(b \circ A))$ will also be continuous on $\mathcal{V}_{k_2}\left(\mathbb{R}^{k_3}\right)$. Therefore, $\mathbb{E}_B\left(g\left(G^{-1}B\right)\right) = \mathbb{E}_B(g(B))$ for all $G \in \mathrm{O}(k_3)$, since $B \sim \sigma_{k_2}^{k_3}$. Thus, we also have for $G \in \mathrm{O}(k_3)$ that

$$\begin{aligned}\mathbb{E}f(G^{-1} \circ B \circ A) &= \mathbb{E}\left(\mathbb{E}\left(f\left(G^{-1} \circ B \circ A\right)|B\right)\right) = \mathbb{E}(g(G^{-1}B)) \\ &= \mathbb{E}(g(B)) = \mathbb{E}(\mathbb{E}(f(B \circ A)|B)) = \mathbb{E}f(B \circ A).\end{aligned}$$

Since $G \in \mathrm{O}(k_3)$ and f were arbitrary, the last relation implies that the distribution of $B \circ A$ is invariant to left actions of $\mathrm{O}(k_3)$. This fact and the uniqueness of the normalized Haar measure imply part b), which entails c). □

Proof (of Lemma 2) Let M be a non-empty finite subset of $\mathbb{R}^d$ and $x \in (0, \infty)^M$. The finite-dimensional distributions of Y are determined by

$$-\log \mathbb{P}\left(Y_t \leq x_t,\ t \in M\right) = \int_{\mathcal{V}_k(\mathbb{R}^d)} \int_E \bigvee_{t \in M} x_t^{-1} V_{A^{\mathrm{T}}t}(e)\, \mu(\mathrm{d}e)\, \sigma_k^d(\mathrm{d}A).$$

If X is stationary, then we have for all $h \in \mathbb{R}^d$ and all $A \in \mathcal{V}_k\left(\mathbb{R}^d\right)$ that $\int_E \bigvee_{t \in M} x_t^{-1} V_{A^{\mathrm{T}}(t+h)}(e)\, \mu(\mathrm{d}e) = \int_E \bigvee_{t \in M} x_t^{-1} V_{A^{\mathrm{T}}t}(e)\, \mu(\mathrm{d}e)$, since A is linear, which implies the assertion a). Subsequently, part b) follows since σ_k^d is O(d)-invariant. Part c) follows with Eq.3. □

Proof (of Proposition 3) In view of Lemma 2 we need to show that continuous TCFs on $\mathbb{R}^k$ coincide with the TCFs of stochastically continuous processes on $\mathbb{R}^k$. Let χ be a continuous TCF on $\mathbb{R}^k$ and let X be a corresponding stationary max-stable process. Let θ be the extremal coefficient function (ECF) of X as in Strokorb and Schlather (2013) and let X^* be the associated Tawn-Molchanov process as in Theorem 8 therein. Note that $\chi(h) = 2 - \theta(\{h, o\})$. By construction, X^* is also stationary and has TCF χ. Additionally, X^* is stochastically continuous due to Theorem 25 therein. □

Proof (of Lemma 6) Let M be a non-empty finite subset of $\mathbb{R}^d$ and $x \in (0,\infty)^M$. The finite-dimensional distributions of Y are determined by

$$-\log \mathbb{P}\left(Y_t \leq x_t,\ t \in M\right) = \mathbb{E}_B \int_{\mathbb{R}^d} \int_E \bigvee_{t \in M} c_B^{-1} x_t^{-1} B(t-z) V_t(e)\, \mu(\,\mathrm{d}e)\ \mathrm{d}z.$$

a) If X is stationary, we have for all $h \in \mathbb{R}^d$, $z \in \mathbb{R}^d$ and $B \in \{0,1\}^{\mathbb{R}^d}$ that $\int_E \bigvee_{t\in M} \frac{B(t-z)V_{t+h}(e)}{x_t}\, \mu(\,\mathrm{d}e) = \int_E \bigvee_{t\in M} B(t-z)V_t(e)x_t\, \mu(\,\mathrm{d}e)$, which entails for all $h \in \mathbb{R}^d$ and all integrable functions $B \in \{0,1\}^{\mathbb{R}^d}$ that $\int_{\mathbb{R}^d}\int_E \bigvee_{t\in M} \frac{B((t+h)-z)V_{t+h}(e)}{x_t}\, \mu(\,\mathrm{d}e)\ \mathrm{d}z = \int_{\mathbb{R}^d}\int_E \bigvee_{t\in M} \frac{B(t-z)V_t(e)}{x_t}\, \mu(\,\mathrm{d}e)\ \mathrm{d}z$.

b) The assertion follows from Eq.3 and the fact that $b_1 v_1 \wedge b_2 v_2 = b_1 b_2 (v_1 \wedge v_2)$ for non-negative real numbers b_1, b_2, v_1, v_2 with $b_i \in \{0,1\}$ for $i = 1, 2$. □

It is shown already in Gneiting (1999c) that H_d and the *Mittal-Berman class* V_d coincide (for $d \geq 2$; cf. Gneiting (1999c, Eq. 40) and Mittal (1976)). Here, V_d is the class of functions φ on $[0,\infty)$ of the form

$$\varphi(t) = 2\int_{t/2}^{\infty} S_{d,u,\theta(t,u)}\, S_{d,u,\pi}^{-1}\, p(u)\ \mathrm{d}u, \tag{27}$$

where p is a probability density function on $(0,\infty)$, such that $p(u)/u^{d-1}$ is non-increasing, and $S_{d,u,\theta}$ is the surface area of the sphere $\{x : \|x\| = u\} \subset \mathbb{R}^d$ intersected by the cone of angle $\theta(t,u) = \arccos(t/(2u))$ (with apex the origin). In what follows, we show that we have

$$H_d = T^d_{\mathrm{M3r}} = T^d_{\mathrm{M2r}} = T^d_{\mathrm{M3b}}\ (= V_d) \qquad \text{for } d \geq 1\ (d \geq 2). \tag{28}$$

Proof (of Proposition 8 and Proposition 14c)
We divide the proof into five steps:

1st step $H_d = T^d_{\mathrm{M3b}}$ for $d \geq 1$.

The assertion follows immediately from Eq. 15.

2nd step $T^d_{\mathrm{M2r}} = V_d = H_d$ for $d \geq 2$ and Eq. 25 holds for $d \geq 2$.

Members of T^d_{M2r} depend on a shape function $f \geq 0$ with $\int_{\mathbb{R}^d} f(\|t\|)\,\mathrm{d}t = 1$, which is non-increasing as the radius grows, whereas members of V_d depend on a probability density function p on $(0,\infty)$ with $p(u)/u^{d-1}$ non-increasing in $u > 0$. Integration along the radius shows that both functions are in one-to-one correspondence via

$$f(\|t\|) = S^{-1}_{d,\|t\|,\pi}\, p(\|t\|).$$

Moreover, since f is non-increasing, this correspondence is compatible with the integration in Eq. 27 and the TCF for M2r processes in Table 1, that is

$$\begin{aligned}\int_{\mathbb{R}^d} f(\|z\|) \wedge f(\|z-t\|)\ \mathrm{d}z &= 2\int_{\|t\|/2}^{\infty} S_{d,u,\theta(\|t\|,u)}\, f(u)\ \mathrm{d}u \\ &= 2\int_{\|t\|/2}^{\infty} S_{d,u,\theta(\|t\|,u)}\, S^{-1}_{d,u,\pi}\, p(u)\ \mathrm{d}u. \qquad (29)\end{aligned}$$

Hence $T^d_{\mathrm{M2r}} = V_d$ for $d \geq 2$. From Gneiting (1999c) we already know that $H_d = V_d$. In particular, f and $G_{1/(2R)} = G$ from Proposition 14 can be recovered from each other by (44) and (45) in Gneiting (1999c) with $n \geq 2$ (Theorem 5.2 therein), or, equivalently, f and $G_{1/(2R)}$ can be recovered from each other by (25) with $d \geq 2$ here. Note that our f corresponds to g in Gneiting (1999c).

3rd step $T^1_{\mathrm{M2r}} = H_1$ and Eq. 25 holds for $d = 1$.

If $d = 1$, it is straightforward to check that, for $\chi \in T^1_{\mathrm{M2r}}$ depending on a single shape function f, we have

$$\chi(t) = \int_{\mathbb{R}} f(z) \wedge f(z-t)\ \mathrm{d}z = 2\int_{t/2}^{\infty} f(u)\ \mathrm{d}u \qquad (30)$$

(similarly to the integration along the radius in Eq. 29). Now, precisely the same proof as the proof of Theorem 5.2. in Gneiting (1999c) applies here when we set $n = 1$, $g = f$, $\varphi = \chi$ and omit the term $S_{n,u,\theta}$ in (48) and (49) therein, showing that $T^1_{\mathrm{M2r}} = H_1$. In particular, f and $G_{1/(2R)} = G$ from Proposition 14 can also be recovered from each other by (44) and (45) in Gneiting (1999c) with $n = 1$ or, equivalently, f and $G_{1/(2R)}$ can be recovered from each other by Eq. 25 with $d = 1$ here (where our f corresponds to g therein).

4th step $T^d_{\mathrm{M3r}} \subset H_d$ for $d \geq 1$.

From the 2nd and 3rd step we know that $T^d_{\mathrm{M2r}} = H_d$ for $d \geq 1$. That means for each (single deterministic) radially symmetric non-increasing shape function $f \geq 0$ on $[0,\infty)$ with $0 < \|f\|_{\mathrm{L}^1(\mathbb{R}^d)} < \infty$ we may define a unique distribution function $G_{f/\|f\|_{\mathrm{L}^1(\mathbb{R}^d)}}$ via Eq. 25. We set

$$A(f)_s = \|f\|_{\mathrm{L}^1(\mathbb{R}^d)}\ G_{f/\|f\|_{\mathrm{L}^1(\mathbb{R}^d)}}(s), \qquad s > 0,$$

such that $A(f)$ is non-decreasing on $(0,\infty)$ with $A(f)_{0+} = 0$, $A(f)$ is right-continuous and $A(f)$ has total variation $\|f\|_{\mathrm{L}^1(\mathbb{R}^d)}$. It is coherent to set $A_0 \equiv 0$. Now, consider a member χ of T^d_{M3r} and its corresponding measurable map $f : \mathbb{R}^d \times \Omega \to [0, \infty]$, which satisfies

$$\int_\Omega \| f(\cdot, \omega) \|_{\mathrm{L}^1(\mathbb{R}^d)} \, \nu(\,\mathrm{d}\omega) = 1.$$

Then the assignment $\omega \mapsto (A(f(\cdot, \omega))_s)_{s>0}$ defines a non-decreasing, right-continuous process A on $(0, \infty)$, such that $\mathbb{E}\,(A_\infty) = 1$ and $A_{0+} = 0$. Moreover, note that (by the correspondence $T^d_{\mathrm{M2r}} = H_d$)

$$\chi(t) = \mathbb{E} \int_0^\infty h_d(st)\ \mathrm{d}A_s\,,$$

where the expectation is taken with respect to the process A (the expectation comes from the measure μ). Set $G(s) = \mathbb{E}A_s$. Then G is also non-decreasing, right-continuous with total variation 1 and with $G(0+) = 0$ (by dominated convergence). Finally, we obtain (again by dominated convergence) that

$$\chi(t) = \int_0^\infty h_d(st)\ \mathrm{d}\mathbb{E}A_s = \int_0^\infty h_d(st)\ \mathrm{d}G(s)$$

as desired. Hence $T^d_{\mathrm{M3r}} \subset H_d$.

5th step (Summary) From the previous steps we know that $T^d_{\mathrm{M3r}} \subset H_d = T^d_{\mathrm{M3b}} = T^d_{\mathrm{M2r}}$ for $d \geq 1$. Clearly, $T^d_{\mathrm{M3b}} \subset T^d_{\mathrm{M3r}}$ by definition. □

Proof (of Proposition 9) a) If φ is completely monotone, then also $-\varphi'$ and $-\varphi'(\sqrt{\cdot})$ will be completely monotone, since $\sqrt{\cdot}$ is a Bernstein function, cf. Berg et al (1984, p. 141/142) and Bochner (1955, p. 83) (where Bernstein functions are confusingly also called "completely monotone"). This shows $T^d_{\mathrm{MPS}} \subset H_\infty$. Clearly, $H_\infty \subset H_d$ and the other equalities are restated from Proposition 9.

b) Clearly, $T^d_{\mathrm{BR}} \subset T^d_{\mathrm{VBR}}$, since BR processes form a proper subclass of VBR processes. The inclusion $H_\infty \subset T^d_{\mathrm{VBR}}$ follows from Eq. 20, since $\gamma(t) = 8\|t\|^2$ is a valid variogram in each dimension.

c) The variogram $\gamma(t) = 8\|t\|^{2\alpha}$ is valid in each dimension for $\alpha \in (0, 1]$ (corresponding to fractional Brownian motion). Hence $\mathrm{erfc}(t^\alpha)$ is a valid TCF of a BR process for $\alpha \in (0, 1]$. Moreover, the function $\mathrm{erfc}(t^\alpha)$ is completely monotone ($\Leftrightarrow$ it belongs to T^d_{MPS}) if and only if $\alpha \leq 0.5$.

d) The class $T^d_{\mathrm{M3r}} = H_d$ naturally contains functions with compact support, e.g. the function h_d, see Eq. 14, whereas T^d_{VBR} cannot contain such functions. To see this, recall (19) and observe that members of H_∞ are scale mixtures of erfc that cannot have compact support. Thus, the involved variogram in Eq. 20 would have to take the value ∞ outside a compact region.

e) Consider the simple erfc-mixture

$$\chi(\|t\|) = 0.25 \cdot \mathrm{erfc}(\|t\|) + 0.75 \cdot \mathrm{erfc}(5\|t\|), \qquad t \in \mathbb{R}^d.$$

Surely, χ is a member of H_∞, cf. (19). Suppose that there is a BR process on $\mathbb{R}^d$ corresponding to a variogram $\tilde{\gamma}$ such that its TCF $\tilde{\chi}$ coincides with χ. We will show now that this cannot be true for any dimension d. Otherwise,

$$\tilde{\gamma}(\|t\|) = 8\left[\operatorname{erfc}^{-1}(0.25 \cdot \operatorname{erfc}(\|t\|) + 0.75 \cdot \operatorname{erfc}(5\|t\|))\right]^2, \qquad t \in \mathbb{R}^d$$

is a variogram for any dimension d. In particular, $\tilde{\gamma}(\|\cdot\|)$ is for any dimension d a continuous negative definite function on $\mathbb{R}^d$. By Berg et al (1984, 5.1.8) it follows that the function

$$\psi(r) = \left[\operatorname{erfc}^{-1}\left(0.25 \cdot \operatorname{erfc}\left(\sqrt{r}\right) + 0.75 \cdot \operatorname{erfc}\left(5\sqrt{r}\right)\right)\right]^2, \qquad r \in [0, \infty)$$

is a (continuous) negative definite function on $[0, \infty)$ in the semigroup sense and obviously $\psi(r) \geq 0$. Hence $\psi(r)$ is a Bernstein function, cf. Berg et al (1984, 4.4.3). However, the second derivative of $\psi(r)$ has a local minimum. So, the assertion fails and our assumption must be wrong. That means there is a dimension d_0 such that the above $\chi \in H_\infty$ cannot be realized as a TCF of a BR process for any dimension $d \geq d_0$.

□

Lemma 19 *For all* $1 \leq k \leq d$ *the turning bands operator* TB_k^d *transfers members of the class* H_1 *into members of* H_1.

Proof The class H_1 is the class of continuous functions h on $[0, \infty)$ that are convex and satisfy $h(0) = 1$ and $\lim_{t\to\infty} h(t) = 0$. All properties are preserved under TB_k^d. For continuity and $\lim_{t\to\infty} h(t) = 0$ use the dominated convergence theorem. Preservation of convexity follows from $\mathrm{TB}_k^d(h)(r) = \mathbb{E}_A(h(rc(A)))$ for $r \geq 0$ with $A \sim \sigma_k^d$ and $c(A) = \left\|A^{\mathrm{T}}(1, 0, \ldots, 0)^{\mathrm{T}}\right\|$. □

Proof (of Proposition 10) A priori it is clear that $\varphi_1 = h_1$ does not belong to H_k for $k \geq 2$ (Gneiting 1999c).

a) Because of Proposition 3 the function φ_d is a radial TCF on $\mathbb{R}^d$. Lemma 19 shows that $\varphi_d = \mathrm{TB}_1^d(h_1)$ belongs to H_1.
b) By Eq. 9 φ_d can be expressed as

$$\varphi_d(t) = 2\pi^{-1/2}\,\Gamma(d/2)\,\Gamma((d-1)/2)^{-1}\int_0^1 h_1(tw)\left(1 - w^2\right)^{(d-3)/2}\,\mathrm{d}w. \tag{31}$$

Thus, we have for $d \geq 2$ that

$$-\varphi_d'\left(\sqrt{t}\right) = \beta_d \begin{cases} 1 & t \leq 1 \\ 1 - (1 - 1/t)^{(d-1)/2} & t > 1 \end{cases}, \tag{32}$$

where β_d is the constant from Eq. 23. Clearly, $-\varphi_d'\left(\sqrt{t}\right)$ is not convex. From Eq. 17 (which is an equality for $d = 3$) we see that φ_d cannot belong to H_3.
c) We verify that one of the conditions of Theorem 3.3 in Gneiting (1999c) (that is necessary to belong to the class H_2) is not fulfilled: Namely, we show that for all

$d \geq 6$ the function

$$c(t) := \int_0^t \sqrt{v^{-1}(t-v)} \left(-\varphi_d'\left(1/\sqrt{v}\right)\right) \, \mathrm{d}v \tag{33}$$

is not convex. From Eq. 32 we see that

$$-\varphi_d'\left(1/\sqrt{v}\right) = \beta_d \begin{cases} 1-(1-v)^{(d-1)/2} & v < 1 \\ 1 & v \geq 1 \end{cases}$$

Since $d \geq 6$ we can compute the second derivative of c at 1:

$$\begin{aligned} c''(1) &= \int_0^1 \sqrt{w^{-1}(1-w)} \cdot \left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=1} \left(-\varphi_d'\left(1/\sqrt{tw}\right)\cdot t\right) \, \mathrm{d}w \\ &= -\beta_d(d-1)\left(3/16\sqrt{\pi}\,\Gamma(d/2-2)\Gamma((d+1)/2)^{-1}\right) < 0 \end{aligned}$$

Since $c''(1)$ is negative, the function c cannot be convex.

Lemma 20 *If $f, g \in H_1$ then $fg \in H_1$ as well.* □

Proof This is an immediate consequence of Gneiting (1999c, Lemma 4.7) or Williamson (1956, Lemma 2), which states that if f and g are non-negative, non-increasing and convex on an interval, then the product fg is also non-negative, non-increasing and convex there. □

Proof (of Proposition 12) a) From Proposition 10 we know that $\varphi_d(2t)$ is a radial TCF on $\mathbb{R}^d$ that belongs to H_1. Since $h_d(t)$ belongs to H_d it follows from Example 7 that the product $\chi_d(t) = \varphi_d(2t)h_d(t)$ is a radial TCF on $\mathbb{R}^d$. Moreover $h_d(t)$ also belongs to $H_d \subset H_1$ and therefore $\chi_d \in H_1$ due to Lemma 20. However, $\chi_d \notin T^d_{\mathrm{VBR}}$ because of its compact support (cf. Propostion 9 d).

b) It suffices to show that the function

$$f(t) = -\chi_3'\left(\sqrt{t}\right) = -2\varphi_3'\left(\sqrt{4t}\right)h_3\left(\sqrt{t}\right) + \varphi_3\left(\sqrt{4t}\right)\left(-h_3'\left(\sqrt{t}\right)\right)$$

is not convex, because then one of the conditions of Eq. 17 (which is an equality for $d = 3$) is not fulfilled. From Eq. 14), Eq. 31 and Eq. 32 we see that for $t \in [0, 1]$

$$\begin{aligned} h_3\left(\sqrt{t}\right) &= \frac{1}{2}\left(2-3t^{1/2}+t^{3/2}\right), & -h_3'\left(\sqrt{t}\right) &= \frac{3}{2}(1-t), \\ \varphi_3\left(\sqrt{4t}\right) &= \begin{cases} 1-\sqrt{t} & t \leq 1/4 \\ 1/\left(4\sqrt{t}\right) & t \geq 1/4 \end{cases}, & -2\varphi_3'\left(\sqrt{4t}\right) &= \begin{cases} 1 & t \leq 1/4 \\ 1/(4t) & t \geq 1/4 \end{cases}. \end{aligned}$$

Thus, $f(t)$ is a decreasing function on $[0, 1]$ with the following left-hand and right-hand derivative at $1/4$

$$\lim_{t \uparrow 1/4} f'(t) = -3 \qquad \text{and} \qquad \lim_{t \downarrow 1/4} f'(t) = -17/4.$$

Hence, f cannot be convex in a neighbourhood of $1/4$.

□

Proof (of Proposition 14) c) The recovery of f and $G_{1/(2R)}$ has been proved already alongside the Proof of Proposition 8.

b) The recovery of $G_{1/(2R)}$ is stated in Gneiting (1999c), Eq. 18 (case $d = 1$), Theorem 3.2 (case $d \geq 3$ odd), Theorem 3.4 (case $d \geq 2$ odd) with $G_{1/(2R)} = G$ therein.

a) The recovery of f is obtained from part b) when Eq. 25 from part c) is applied. In case d is odd, we can simplify the result as follows

$$f(u) = \frac{1}{\kappa_d} \int_0^{1/(2u)} (2s)^d \, \mathrm{d}G(s) = \frac{2^d \sqrt{\pi}}{\kappa_d \, d \, \Gamma(d/2)} \int_0^{1/(2u)} \mathrm{d}\lambda \left(\frac{1}{s^2}\right)$$
$$= \left(\frac{2}{\sqrt{\pi}}\right)^{d-1} \left(\lambda\left(4u^2\right) - \lim_{x\to\infty} \lambda(x)\right)$$

But $\lim_{x\to\infty} \lambda(x)$ necessarily vanishes, since $\lambda(t) = -a'(t)$ for a non-negative (i.e. bounded from below), non-increasing and convex function $a(t)$ due to (Gneiting 1999c, Eq. 22).

□

Proof (of Table 3) If the density $g_{1/(2R)}$ of the cdf $G_{1/(2R)}$ of $1/(2R)$ exists, then the density g_{2R} of $2R$ is given by $g_{2R}(s) = g_{1/(2R)}(1/s)/s^2$. The cases $d = 1$ and $d = 3$ follow directly from Proposition 14. The case $d = 2$ has been derived in a tedious calculation that can be found in Strokorb (2013, p. 100, Proof of Table 4.2.) under the additional assumption that $\chi \in H_5$. □

Proof (of Lemma 15 analogously to Gneiting (1999c), p. 104) Replacing t by t^2 and r by s^2 yields

$$-\varphi'(t) = \int_0^\infty 2se^{-s^2t^2} f\left(s^2\right) \, \mathrm{d}s = \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t} [-\operatorname{erfc}(st)] \, g(s) \, \mathrm{d}s.$$

Applying Fubini's theorem when integrating w.r.t. t gives

$$\varphi(0) - \varphi(t) = \int_0^\infty [\operatorname{erfc}(0) - \operatorname{erfc}(st)] \, g(s) \, \mathrm{d}s,$$

which entails the claim, since g is a density on $(0, \infty)$ and $\varphi(0) = 1$. □

Proof (of Table 4) We apply Lemma 15 and derive this table from known Laplace transforms in Polyanin and Manzhirov (2008) using (in this order) Eq.'s [p. 964 5.3 (11)], [p. 964 5.3 (12), p. 963 5.2 (12) and p. 962 5.1 (26)], [p. 963 5.3 (1)] and [p. 963 5.3. (3) with $\nu = 1.5$] therein. □

Proof (of Corollary 17) The variogram $\gamma(t) = 8\|t\|$ corresponds to Brownian motion and is a valid variogram that determines a BR process with TCF

$$\chi(t) = \operatorname{erfc}\left(\sqrt{\gamma(t)/8}\right) = \operatorname{erfc}(\sqrt{\|t\|}).$$

Proposition 9 ensures the existence an intensity mixing distribution G_β of an MPS process (Part c)), of a monotone shape function f of an M2r process (Part a)) and a random radius R of an M3b process (Part a)), such that all involved processes share the same TCF χ as above. In fact, f, R and G_β are uniquely determined by χ, cf. Section 5. The quantities f and R (that is the density g_{2R} of $2R$) are recovered from χ as in Table 3 through

$$\begin{aligned} f(u) &= \chi''(2u)/(\pi u) = \pi^{-3/2}(1+4u)(2u)^{-5/2}e^{-2u}, \\ g_{2R}(s) &= \frac{s}{3}\left(\chi''(s) - s\chi'''(s)\right) = 1/12(\pi s)^{-1/2}\left(4s^2+8s+5\right)e^{-s}. \end{aligned}$$

Derivatives are taken with respect to $\|t\|$ (not $t \in \mathbb{R}^d$) and the monotone shape function f depends on $u = \|t\|$ only. Furthermore, it follows from Gradshteyn and Ryzhik (2007, p. 1100, 17.13.5) that the Laplace transform of

$$G(s) = \begin{cases} 0 & \text{if } s \leq a, \\ 2\pi^{-1}\arctan\left(\sqrt{a^{-1}s-1}\right) & \text{if } s > a, \end{cases}$$

which admits the density $\sqrt{a}\mathbf{1}_{x\geq a}/(\pi s(s-a))$, is given by $\operatorname{erfc}(\sqrt{ax})$ for $|a| < \pi$, that is $\mathcal{L}(G)(x) = \operatorname{erfc}(\sqrt{ax})$. From the MPS entry of Table 1 we know that $\chi(x/c_d) = \mathcal{L}(G_\beta)(x)$ holds for the TCF χ of an MPS process where $c_d = 2\kappa_{d-1}/(d\kappa_d)$ is a dimension specific constant. With $d = 2$, we obtain $c_d = 2/\pi$. Therefore, choosing $a = 1/c_d = \pi/2$ for G yields the desired intensity mixing distribution G_β of an MPS process with TCF $\operatorname{erfc}(\sqrt{\|t\|})$. □

B Monotonicity properties of continuous functions

Let (a, b) be an open interval and $n \in \mathbb{N}$. A real-valued function f on (a, b) is called n-times monotone, where $n \geq 2$, if it is differentiable up to order $n-2$ and $(-1)^k f^{(k)}$ is non-negative, non-increasing and convex on (a, b) for $k = 0, 1, \ldots, n-2$. If $n = 1$, we simply require f to be non-negative and non-increasing (McNeil and Nešlehová 2009; Williamson 1956). The function f is called *completely monotone* if it is n-times monotone of any order n, which is equivalent to require that it has derivatives of all orders and that $(-1)^k f^{(k)}(x) \geq 0$ for all $x \in (a, b)$ and $k \in \mathbb{N} \cup \{0\}$ (Widder 1946, Chapter IV). If I is a closed or half-open interval, a function f on I is called *n-times monotone (resp. completely monotone)* if f has this property when restricted to the interior $\mathring{I}$ and if f is continuous at the boundary points of I. In the literature, the focus often lies on the intervals $I = (0, \infty)$ or $I = [0, \infty)$, since completely monotone functions on $[0, \infty)$ are precisely the functions f, such that $f(\|\cdot\|^2)$ is positive definite on $\mathbb{R}^d$ for all dimensions d (cf. e.g. Berg et al (1984, 5.1.5 and 5.1.6)). Such functions are characterized as *Laplace transforms* of non-decreasing functions, or, equivalently, positive measures (Widder 1946, Chapter IV, Theorem 12).

Theorem 21 (**Bernstein**) *A function $f : (0, \infty) \to \mathbb{R}$ is completely monotone on $(0, \infty)$ if and only if it has an integral representation of the form*

$$f(x) = \int_{[0,\infty)} \exp(-tx)\; dF(t) \tag{34}$$

for some non-decreasing function $F : [0, \infty) \to \mathbb{R}$*, such that the integral converges for* $x \in (0, \infty)$*. Furthermore, the function* f *can be extended continuously to* $[0, \infty)$ *– and, thus, is completely monotone on* $[0, \infty)$ *– if and only if* F *is bounded. In this case* $f(0) = F(\infty) - F(0)$.

An analogous integral representation with Bernstein's theorem as the limiting case holds for n-times monotone functions (Williamson 1956). It presents n-times monotone functions as scale mixtures of Askey's function.

Theorem 22 (**Williamson**) *A function* $f : (0, \infty) \to \mathbb{R}$ *is* n*-times monotone on* $(0, \infty)$ *if and only if it has an integral representation of the form*

$$f(x) = \int_{[0,\infty)} (1 - tx)_+^{n-1} \, dF(t) \tag{35}$$

for some non-decreasing function $F : [0, \infty) \to \mathbb{R}$ *bounded from below. This representation is unique in the sense that when* F *is normalized to* $F(0) = 0$*, the value* $F(t)$ *is determined at continuity points* $t > 0$ *of* F.

Finally, this motivates the definition of α-times monotone functions for real $\alpha \geq 1$ according to Williamson (1956). A real-valued function f on $(0, \infty)$ (resp. $[0, \infty)$) is called α*-times monotone* if it has an integral representation of the form Eq. 35 with $n = \alpha$ for some non-decreasing function $F : [0, \infty) \to \mathbb{R}$ with $F(0) = 0$ (resp. additionally $f(0+) = f(0)$).

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