International Journal on Artificial Intelligence Tools
Vol. 6 , No. 4 (1997) 423-450

ORDER IN SPACE : A GENERAL FORMALISM FOR SPATIAL REASONING

BAHER A. EL-GERESY and ALIA I. ABDELMOTY
Dept. of Computer Studies, University of Glamorgan,
Treforest, Mid Glamorgan, Wales, CF37 1DL, U.K.
Email: bageresy,aiabdel@glam.ac.uk

Received (received date)
Revised (revised date)

In this paper we propose a general approach for reasoning in space. The approach is composed of a set of two general constraints to govern the spatial relationships between objects in space, and two rules to propagate relationships between those objects. The approach is based on a novel representation of the topology of the space as a connected set of components using a structure called adjacency matrix which can capture the topology of objects of different complexity in any space dimension. The relationships between objects are represented by the intersection of the space components. The approach is also shown to be applicable to reasoning in the temporal domain and is used to explain the conceptual neighbourhood phenomenon related to the reasoning process. The formalism is also used to explain composition resulting in indefinite and definite relations. A major advantage of the method is that reasoning between objects of any complexity can be achieved in a defined limited number of steps. Hence, the incorporation of spatial reasoning mechanisms in spatial information systems becomes possible.

Keywords: Spatial reasoning, knowledge representation, spatial relations, qualitative reasoning.

1. Introduction

Spatial reasoning is a field of AI research which studies formalisms for encoding qualitative spatial knowledge [1]. The ability to handle a certain level of indeterminacy makes techniques of spatial reasoning attractive to many application domains, such as computer vision, image processing, medical and geographic information systems (GIS). Precise information required in quantitative methods are sometimes neither available nor needed. Techniques for representing and reasoning over qualitative spatial knowledge are valuable in complementing traditional computational geometry in these domains. For example, in a GIS the fact that the river Thames is in Britain can be inferred directly if the facts that it passes through London and London is in Britain are known, without needing to execute a line-in-polygon geometric computation.

Qualitative treatment of the temporal knowledge is an established research area where different approaches exist for the representation of temporal entities and their relations (interval and point algebra) and reasoning over them (composition tables and constraint networks). A similar general treatment of spatial knowledge is still lacking. Composition tables (result of the reasoning process) need to be built for every new type of objects considered and techniques to derive them automatically presents a challenge to theorem provers [2]. Phenomena such as conceptual neighbourhood needs explanation [3].

Some research work [4] tried to exploit the well developed treatment of temporal knowledge [5] in handling the representation in the spatial domain. However, the multi-dimensionality and complexity of the topology of spatial entities, as opposed to the uni-dimensionality of temporal entities and their simpler topology prevented the generality of these approaches. General approaches which handles reasoning over objects of different types and random complexity are not yet achieved.

In this paper a general reasoning formalism for qualitative spatial relations is proposed. Section 2 describes the proposed approach by describing the underlying representation methodology and the reasoning formalism. Examples are given to show how the approach can be used to represent and reason over relationships between objects with random complexity. In section 3, analysis of the composition results is presented and a possible explanation on how the phenomenon of conceptual neighbourhood occurs is discussed and the application of the same approach to the representation and reasoning in the temporal domain is given. Section 4 gives a comparative description of related approaches and some conclusions and a view over future work are given in section 5.

2. The Formalism

The first part of the paper addresses the problem of qualitative representation of objects with random spatial complexity and their topological relationships. In the second part the reasoning formalism is presented, consisting of a) general constraints to govern the spatial relationships between objects in space, and b) general rules to propagate relationships between those objects. Both the constraints and the rules are based on a uniform representation of the topology of the objects, their embedding space and the representation of the relationships between them. The representation methodology is first described and examples are used to demonstrate how relationships between objects of random complexity can be represented.

2.1. *The General Representation*

Objects of interest and their embedding space are divided into components according to a required resolution. The connectivity of those components is explicitly represented. Spatial relations are represented by the intersection of object components [6] in a similar fashion to that described in [7] but with no restriction on object components to consist only of two parts (boundary and interior).

2.1.1. *The Underlying Representation of Object Topology*

Let S be the space in which the object is embedded. The object and its embedding space are assumed to be *dense* and *connected*. The embedding space is also assumed

	x_0	x_1	x_2	x_3
x_0	-	1	0	1
x_1	1	-	1	1
x_2	0	1	-	0
x_3	1	1	0	-

(b)

x_0			
1	x_1		
0	1	x_2	
1	1	0	x_3

(a) (c)

Figure 1: (a) Possible decomposition of a concave-shaped object and its embedding space. (b) Adjacency matrix of the shape in (a). (c) Half the symmetric adjacency matrix is sufficient to capture the object representation.

to be infinite. The object and its embedding space are decomposed into components which reflects the objects and space topology such that,

1. No overlap exists between any of the representative components.
2. The union of the components is equal to the embedding space.

The topology of the object and the embedding space can then be described by a matrix whose elements represent the connectivity relations between its components. This matrix shall be denoted *adjacency matrix.* In figure 1(a) a possible decomposition of a concave shaped object (for example an island with a bay) and its embedding space is shown and in 1(b) the adjacency matrix for its components is presented. The object is represented by two components a linear component x_1 (the shore line of the island) and an areal component x_2 and the rest of its embedding space is represented by a finite areal component x_3 (representing the bay of the island) and infinite areal component x_0 representing the surrounding area. The fact that two components are connected is represented by a (1) in the adjacency matrix and by a (0) otherwise. Since connectivity is a symmetric relation, the resulting matrix will be symmetric around the diagonal. Hence, only half the matrix is sufficient for the representation of the object's topology and the matrix can be collapsed to the structure in figure 1(c). In the decomposition strategy, the complement of the object in question shall be considered to be infinite. The suffix 0 (x_0) is used to represent this component.

Note that different decomposition strategies for the objects and their embedding spaces can be used according to the precision of the relations required and the specific application considered. The higher the resolution used (or the finer the components of the space and the objects), the higher the precision of the resulting

x_0							
0	x_1						
0	1	x_2					
0	0	1	x_3				
1	0	0	1	x_4			
0	1	1	1	0	x_5		
0	0	0	1	0	1	x_6	
1	0	0	1	1	0	1	x_7

(a) (b)

Figure 2: Representation of a complex object of a map consisting of an island with a lake x_1 and x_2 and a river x_5, x_6 and x_7 by the adjacency matrix.

set of relations in the domain considered. For example, consider the objects in figure 2(a) which represents an island with a lake represented by x_1 and x_2 and a river represented by the components x_5, x_6 and x_7. The adjacency matrix for the map in (a) in given in (b). This example demonstrates the ability of the adjacency structure to represent complex objects such as a whole map. At a lower resolution the river object may be omitted by removing the rows and columns of components x_5, x_6 and x_7. This representation can also be used to represent virtual components as was seen in figure 1 which makes the method flexible for representation in any application domain.

2.1.2. *The Underlying Representation of Spatial Relations*

In this section, the representation of the topological relations through the intersection of their components [8,9] is adopted and generalized for objects of arbitrary complexity.

Distinction of topological relations is dependent on the strategy used in the decomposition of the objects and their related spaces. For example, in figure 3 different relationships between two objects representing a ship (x) and an island (y) are shown, where in 3(a) the ship is outside the bay and in 3(b) the ship is inside the bay. The concave region representing the island (y) is decomposed into two components y_1 and y_2 and the rest of the space associated with y is decomposed into two components (y_3 representing the bay and y_0 representing the rest of the ocean). Note that the component y_3 is a virtual component, i.e. with no physical boundary to delineate its spatial extension. It is the identification of this component that makes the distinction between the two relationships in the figure. The complete set of spatial relationships are represented by combinatorial

Figure 3: Different qualitative spatial relationships can be distinguished by identifying the appropriate components of the objects and the space.

intersection of the components of one space with those of the other space.

If $R(x, y)$ is a relation of interest between object x and object y, and X and Y are the spaces associated with the objects respectively such that m is the number of components in X and l is the number of components in Y, then a spatial relation $R(x, y)$ can be represented by one state of the following equation:

$$\begin{aligned} R(x, y) &= X \cap Y \\ &= \left(\bigcup_{i=1}^{m} x_i \right) \cap \left(\bigcup_{j=1}^{l} y_j \right) \\ &= (x_1 \cap y_1, \cdots, x_1 \cap y_l, x_2 \cap y_1, \cdots, x_m \cap y_l) \end{aligned}$$

The intersection $x_i \cap y_j$ can be an empty or a non-empty intersection. The above set of intersections shall be represented by an intersection matrix, as follows,

$$R(x, y) = \begin{array}{|c|c|c|c|c} \hline & y_0 & y_1 & y_2 & \cdots \\ \hline x_0 & & & & \\ \hline x_1 & & & & \\ \hline x_2 & & & & \\ \hline \vdots & & & & \end{array}$$

For example, the intersection matrices corresponding to the spatial relationships in figure 3 are shown in figure 4. The components x_1 and x_2 have a non-empty intersection with y_0 in 4(a) and with y_3 in 4(b).

Different combinations in the intersection matrix can represent different qualitative relations. The set of valid or sound spatial relationships between objects is dependent on the particular domain studied. For example, in considering relationships between two line objects in a network analysis application we might be interested in only those relationships where end points of lines are in contact. Also, properties of the objects would affect the set of possible spatial relationships that

	y_0	y_1	y_2	y_3
x_0	1	1	1	1
x_1	1	0	0	0
x_2	1	0	0	0

(a)

	y_0	y_1	y_2	y_3
x_0	1	1	1	1
x_1	0	0	0	1
x_2	0	0	0	1

(b)

Figure 4: The corresponding **intersection matrices** for the relationships in figure 3 respectively.

$$R_1 = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline 1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ \hline \end{array} \qquad R_2 = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ \hline \end{array}$$

$$R_3 = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline 1 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 \\ \hline \end{array} \qquad R_4 = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 \\ \hline \end{array}$$

$$R_5 = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 \\ \hline \end{array} \qquad R_6 = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 \\ \hline \end{array}$$

Figure 5: A set of 6 spatial relationships between two solid bodies. The decomposition of objects are as in figure 3.

can exist between them. For example, if one object is a solid object and the other is permeable, there cannot be any intersection of the inside of the solid object with any other component of the other object. Also, objects of different size or shape cannot be involved in certain spatial relations such as `equal` or `contain` between the smaller and the larger object.

The example in figure 5 demonstrates the six possible spatial relations that can exist between two *solid* objects, one having the shape of a convex region and the other a concave one along with their intersection matrices. The example can be used to represent many situations, for example, a solid object falling into a container full of liquid, a ball thrown into a net, or a ship entering a bay of an island, etc. Note that since object y is a solid object, the component y_2 will always have only one intersection relation with x_0.

2.2. *The General Reasoning Formalism*

The reasoning approach consists of: a) general constraints to govern the spatial relationships between objects in space, and b) general rules to propagate relationships between the objects.

2.2.1. *General Constraints*

The intersection matrix is in fact a set of constraints whose values identifies specific spatial relationships. For example, part of the constraints used to represent the relationship in figure 3(a) are $x_1 \cap y_1 = 0, x_1 \cap y_2 = 0, x_1 \cap y_3 = 0, x_1 \cap y_4 = 1, \cdots$

The process of spatial reasoning can be defined as the process of propagating the constraints of two spatial relations (for example, $R_1(A, B)$ and $R_2(B, C)$), to derive a new set of constraints between objects. The derived constraints can then be mapped to a specific spatial relation (i.e. the relation $R_3(A, C)$).

A subset of the set of constraints defining all spatial relations are general and are applicable to any relationship between any objects. These general constraints are a consequence of the initial assumptions used in the definition of the object and space topology. The identification of these constraints complements the reasoning rules and shall be used later in the paper to give some insight in the propagation of spatial relations.

The two general constraints are:

1. Every unbounded (infinite) component of one space must intersect with at least one unbounded (infinite) component of the other space.

   Intuitively this rule says that it is impossible for an infinite component in the space to only have an intersection with finite component(s). In this case the infinite component becomes a subset of the finite component(s) which is not possible. In figure 5, x_0 and y_0 always have a non-empty intersection.

2. Every component from one space must intersect with at least one component from the other space.

   If one component of one space does not intersect with any component of the other space, either the two spaces are not equal or the spaces are not dense or *connected*. Both conditions are excluded by the initial assumptions. This implies that there cannot exist a row or a column in the intersection matrix whose elements are all empty intersections, hence the combinatorial cases in the matrix where this case exists can be ignored.

2.2.2. *General Reasoning Rules*

Composition of spatial relations is the process through which the possible relationship(s) between two object x and z is derived given two relationships: R_1 between x and y and R_2 between y and z. Two general reasoning rules for the propagation of intersection constraints are presented. The rules are characterized by the ability to reason over spatial relationships between objects of arbitrary complexity in any space dimension. These rules allow for the automatic derivation of the composition (transitivity) tables between any spatial shapes [10,2].

Reasoning Rules

Composition of spatial relations using the *intersection* representation approach is based on the transitive property of the subset relations. In what follows the following subset notation is used. If x' is a set of components (set of point-sets)

$\{x_1, \cdots, x_{m'}\}$ in a space X, and y_j is a component in space Y, then $\sqsubseteq$ denotes the following subset relationship.

- $y_j \sqsubseteq x'$ denotes the subset relationship such that: $\forall x_i \in x'(y_j \cap x_i \neq \phi) \wedge y_j \cap (X - x_1 - x_2 \cdots - x_m) = \phi \; where \; i = 1, \cdots m'$. Intuitively, this symbol indicates that the component y_j intersects with every set in the collection x' and does not intersect with any set outside of x'.

If x_i, y_j and z_k are components of objects x, y and z respectively, then if there is a non-empty intersection between x_i and y_j, and y_j is a subset of z_k, then it can be concluded that there is also a non-empty intersection between x_i and z_k.

$$(x_i \cap y_j \neq \phi) \;\wedge\; (y_j \subseteq z_k) \rightarrow (x_i \cap z_k \neq \phi)$$

This relation can be generalized in the following two rules. The rules describe the propagation of intersections between the components of objects and their related spaces involved in the spatial composition.

Rule 1: Propagation of Non-Empty Intersections

Let $x' = \{x_1, x_2, \cdots, x_{m'}\}$ be a subset of the set of components of space X whose total number of components is m and $m' \leq m$; $x' \subseteq X$. Let $z' = \{z_1, z_2, \cdots, z_{n'}\}$ be a subset of the set of components of space Z whose total number of components is n and $n' \leq n$; $z' \subseteq Z$. If y_j is a component of space Y, the following is a governing rule of interaction for the three spaces X, Y and Z.

$$\begin{array}{rcl}
(x' \sqsupseteq y_j) & \wedge & (y_j \sqsubseteq z') \\
 & \rightarrow & (x' \cap z' \neq \phi) \\
 & \equiv & (x_1 \cap z_1 \neq \phi \vee \cdots \vee x_1 \cap z_{n'} \neq \phi) \\
 & & \wedge (x_2 \cap z_1 \neq \phi \vee \cdots \vee x_2 \cap z_{n'} \neq \phi) \\
 & & \wedge \cdots \\
 & & \wedge (x_{m'} \cap z_1 \neq \phi \vee \cdots \vee x_{m'} \cap z_{n'} \neq \phi)
\end{array}$$

The above rule states that if the component y_j in space Y has a positive intersection with every component from the sets x' and z', then each component of the set x' must intersect with at least one component of the set z' and vice versa.

The constraint $x_i \cap z_1 \neq \phi \;\vee\; x_i \cap z_2 \neq \phi \cdots \;\vee\; x_i \cap z_{n'} \neq \phi$ can be expressed in the intersection matrix by a label, for example the label a_r ($r = 1$ or 2) in the following matrix indicates $x_1 \cap (z_2 \cup z_4) \neq \phi$ (x_1 has a positive intersection with z_2, or with z_4 or with both). A $-$ in the matrix indicates that the intersection is either positive or negative.

	z_1	z_2	z_3	z_4	$\cdots$	z_n
x_1	$-$	a_1	$-$	a_2	$-$	$-$

Rule 1 represents the propagation of non-empty intersections of components in space. A different version of the rule for the propagation of empty intersections can be stated as follows.

Rule 2: Propagation of Empty Intersections

Let $z' = \{z_1, z_2, \cdots, z_{n'}\}$ be a subset of the set of components of space Z whose total number of components is n and $n' < n$; $z' \subset Z$. Let $y' = \{y_1, y_2, \cdots, y_{l'}\}$ be a subset of the set of components of space Y whose total number of components is l and $l' < l$; $y' \subset Y$. Let x_i be a component of the space X. Then the following is a governing rule for the spaces X, Y and Z.

$$\begin{aligned} (x_i \sqsubseteq y') \quad &\wedge \quad (y' \sqsubseteq z') \\ &\rightarrow \quad (x_i \cap (Z - z_1 - z_2 \cdots - z_{n'}) = \phi) \end{aligned}$$

Remark: if $n' = n$, i.e. x_i may intersect with every element in Z, then no empty intersections can be propagated. Rules 1 and 2 are the two general rules for propagating empty and non-empty intersections of components of spaces.

Note that in both rules the intermediate object (y) and its space components plays the main role in the propagation of intersections. Indeed, it shall be shown in the next example how the first rule is applied a number of times equal to the number of components of the space of the intermediate object. Hence, the composition of spatial relations using this method becomes a tractable problem which can be performed in a defined limited number of steps.

Soundness and Completeness of the Reasoning Rules

Applying the reasoning rules over relations between objects x and y and y and z results in an intersection matrix between objects x and z. Values of elements in the result matrix will be either 0 or 1 or $-$ (indicating an indefinite intersection of $0; 1$). These values are the result of the following conditions:

a. $x_i \cap z_k = 1$ if $x_i \cap y_j = 1 \wedge y_j \cap z_k = 1 \wedge (x_i \supseteq y_j \vee y_j \subseteq z_k)$.

b. $x_i \cap z_k = 0$ if $(x_i \cap y_j = 0 \wedge y_j \supseteq z_k) \vee (x_i \subseteq y_j \wedge y_j \cap z_k = 0)$.

c. $x_i \cap z_k = -$ if $x_i \cap y_j = 1 \wedge y_j \cap z_k = 1 \wedge (y_j \not\subseteq x_i \wedge y_j \not\subseteq z_k)$.

I. If the formalism is not sound then one or more of the values in the derived matrix will be incorrect. A value of 1 or 0 or $-$ will be driven instead of 0 or 1 or (either 0 or 1) respectively. From the above three conditions, this is true if:

- the entries of the original matrices are wrong, for example, $x_i \cap y_j = 1$ instead of 0 and vice versa.
- the intermediate space Y is not dense or the three space X, Y and Z are not equal, for example space Y is smaller than spaces X or Z, (i.e. $x_i \subseteq y_j$ is wrongly interpreted).

The latter problem contradicts the original assumptions while the former one can result from an initial error in the initial intersection matrix.

II. If the formalism is not complete then one or more values in the derived matrix will have a definite value of 0 or 1 instead of an indefinite value of $-$. Again this is possible only if,

- the entries of the original matrices are wrong, for example $x_i \cap y_j = 0$ instead of 1.

	y_0	y_1	y_2	y_3	y_4
x_0	1	1	1	0	0
x_1	1	0	0	0	0
x_2	1	0	0	0	0
x_3	1	1	1	1	1

(c)

	z_0	z_1	z_2
y_0	1	0	0
y_1	1	0	0
y_2	1	1	1
y_3	1	1	0
y_4	1	0	0

(d)

Figure 6: (a) and (b) Spatial relationships between non-simple objects x, y and z. (c) and (d) Corresponding intersection matrices.

- The intermediate space Y is not dense or spaces X, Y and Z are not equal.

Similar to the above this problem contradicts the original assumptions or an indication of initial errors in the initial intersection matrix.

2.3. *Example of Spatial Reasoning with Complex Objects*

The example in figure 6 is used for demonstrating the composition of relations using non-simple spatial objects. Figure 6(a) shows the relationship between a concave region x and a region with a hole y and 6(b) shows the relationship between object y and a simple convex region z where z touches the the hole in y. The intersection matrices corresponding to the two relationships are also shown.

Given that the possible set of relationships that can occur between x and z in a certain domain are as shown in figure 5, it is required to derive the possible relationships between these two objects given the situation in figure 6.

The reasoning rules are used to propagate the intersections between the components of objects x and z as follows. From rule 1 we have,

- y_0 intersections:

$$\begin{aligned} \{x_0, x_1, x_2, x_3\} \sqsupseteq y_0 \quad & \wedge \quad y_0 \sqsubseteq \{z_0\} \\ & \rightarrow \quad x_0 \cap z_0 \neq \phi \wedge x_1 \cap z_0 \neq \phi \\ & \wedge \quad x_2 \cap z_0 \neq \phi \wedge x_3 \cap z_0 \neq \phi \end{aligned}$$

- y_1 intersections:

$$\{x_0, x_3\} \sqsupseteq y_1 \wedge y_1 \sqsubseteq \{z_0\} \rightarrow x_1 \cap z_0 \neq \phi \wedge x_3 \cap z_0 \neq \phi$$

- y_2 intersections:

$$\begin{aligned} \{x_0, x_3\} \sqsupseteq y_2 \quad & \wedge \quad y_2 \sqsubseteq \{z_0, z_1, z_2\} \\ & \rightarrow \quad x_0 \cap (z_0 \cup z_1 \cup z_2) \neq \phi \\ & \wedge \quad x_3 \cap (z_0 \cup z_1 \cup z_2) \neq \phi \end{aligned}$$

- y_3 intersections:

$$\begin{aligned} \{x_3\} \sqsupseteq y_3 \quad & \wedge \quad y_3 \sqsubseteq \{z_0, z_1\} \\ & \rightarrow \quad x_3 \cap z_0 \neq \phi \wedge x_3 \cap z_1 \neq \phi \end{aligned}$$

- y_4 intersections:

$$\{x_3\} \sqsupseteq y_4 \ \wedge \ y_4 \sqsubseteq \{z_0\} \quad \rightarrow \quad x_3 \cap z_0 \neq \phi$$

Applying rule 2 we get the following,

- $x_0 \sqsubseteq \{y_0, y_1, y_2\} \ \wedge \ \{y_0, y_1, y_2\} \sqsubseteq \{z_0, z_1, z_2\}$
  x_0 has no empty intersections with components in Z.
- $x_1 \sqsubseteq y_0 \ \wedge \ y_0 \sqsubseteq \{z_0\} \rightarrow x_1 \cap z_1 = \phi \wedge x_1 \cap z_2 = \phi$
- $x_2 \sqsubseteq y_0 \ \wedge \ y_0 \sqsubseteq \{z_0\} \rightarrow x_2 \cap z_1 = \phi \wedge x_2 \cap z_2 = \phi$
- $x_3 \sqsubseteq \{y_0, y_1, y_2, y_3, y_4\} \ \wedge \ \{y_0, y_1, y_2, y_3, y_4\} \sqsubseteq \{z_0, z_1, z_2\}$
  x_3 has no empty intersections with components in Z.

Refining the above constraints, we get the following intersection matrix.

	z_0	z_1	z_2
x_0	1	-	a_1
x_1	1	0	0
x_2	1	0	0
x_3	1	1	a_2

Comparing the resulting matrix above with the matrices in figure 5, it can be seen that the result matrix corresponds to two possible relationships between objects x and z, namely the relationships R_3 and R_5.

A different conclusion is obtained if the relationship between objects y and z is as shown in figure 7(a). The composition of the relationships between x, y and z in this case will result in the definite matrix in figure 7(b) which corresponds to R_5 in figure 5.

2.4. *Reasoning between Object with Different Dimensions*

Spatial reasoning is needed between spatial objects of different dimension and not only between objects with similar dimension. The set of valid relations between

(a)

	z_0	z_1	z_2
x_0	1	0	0
x_1	1	0	0
x_2	1	0	0
x_3	1	1	1

	z_0	z_1	z_2
y_0	1	0	0
y_1	1	0	0
y_2	1	0	0
y_3	1	0	0
y_4	1	1	1

(b)

Figure 7: Given the relationship between objects x and y as in figure 6(a) and (c) and the relation between the objects y and z as defined in (a) in this figure the composition shall result in the definite intersection matrix between x and z shown in (b).

Figure 8: Relationships between object with different dimension.

regions and between lines and regions have been identified [11]. As an example of reasoning between regions and lines is shown in figure 8. The matrices for the relations in the figure are as follows.

$$R(x,z) = \begin{array}{c|c|c|c|} & y_0 & y_1 & y_2 \\ \hline x_0 & 1 & 1 & 1 \\ \hline x_1 & 0 & 0 & 1 \\ \hline x_2 & 0 & 0 & 1 \\ \hline \end{array} \circ \begin{array}{c|c|c|c|c|} & z_0 & z_1 & z_2 & z_3 \\ \hline y_0 & 0 & 0 & 0 & 1 \\ \hline y_1 & 1 & 1 & 1 & 1 \\ \hline y_2 & 1 & 0 & 0 & 1 \\ \hline \end{array}$$

From rule 1 we have,

- y_0 intersections:

$$\begin{aligned} \{x_0\} \sqsupseteq y_0 \quad & \wedge \quad y_0 \sqsubseteq \{z_3\} \\ & \rightarrow \quad x_0 \cap z_3 \neq \phi \end{aligned}$$

- y_1 intersections:

$$\begin{aligned} \{x_0\} \sqsupseteq y_1 \quad & \wedge \quad y_1 \sqsubseteq \{z_0, z_1, z_2, z_3\} \\ & \rightarrow \quad x_0 \cap z_0 \neq \phi \wedge x_0 \cap z_1 \neq \phi \\ & \wedge \quad x_0 \cap z_2 \neq \phi x_0 \cap z_3 \neq \phi \end{aligned}$$

- y_2 intersections:

$$\begin{aligned} \{x_0, x_1, x_2\} \sqsupseteq y_2 \quad & \wedge \quad y_2 \sqsubseteq \{z_0, z_3\} \\ & \rightarrow \quad x_0 \cap (z_0 \cup z_3) \neq \phi \\ & \wedge \quad x_1 \cap (z_0 \cup z_3) \neq \phi \\ & \wedge \quad x_2 \cap (z_0 \cup z_3) \neq \phi \end{aligned}$$

Applying rule 2 we get the following,

- $x_1 \sqsubseteq \{y_2\} \ \wedge \ \{y_2\} \sqsubseteq \{z_0, z_3\} \rightarrow x_1 \cap \{z_1 \cap z_2\} = \phi$
- $x_2 \sqsubseteq y_2 \ \wedge \ y_2 \sqsubseteq \{z_0, z_3\} \rightarrow x_2 \cap \{z_1 \cap z_2\} = \phi$

Refining the above constraints, we get the following intersection matrix.

	z_0	z_1	z_2	z_3
x_0	1	1	1	1
x_1	a_1	0	0	a_2
x_2	b_1	0	0	b_2

The matrix represents the possible relations in figure 9. The formalism was used to derive the full composition table between two regions and a region and a non-directed line. The full table is given in the appendix. The table shows the conceptual neighbourhood phenomenon observed by Freksa [12], namely that in the case of indefinite composition the disjunctive set of relations are conceptual neighbours.

Note that in the above example a directed line is used, with two different end points in this case, the result of the reasoning are the same if a non-directed line

Figure 9: Possible relations resulting from the composition in figure 8.

was used. This is due to that the components z_1 and z_2 always have a non-empty intersection with x_0 only.

3. Applications

In this section the generality of the formalism is demonstrated by using it to offer explanations to some aspects in qualitative reasoning and by applying it to the order domain.

3.1. *Definite and Indefinite Compositions*

As seen from the previous example and from the composition table in the appendix, indefinite compositions are those where the the result of the spatial reasoning problem is a set of disjunctive spatial relations as opposed to one definite relation. On the other hand definite compositions result in only one relation. In fact in composition tables, which hold the results of reasoning between all the possible set of relations between the concerned objects, many of the entries are disjunctive sets of spatial relations [7,13]. If m' and n' are the number of components of the sets x' and z' respectively and m and n are the total number of components of the spaces X and Z respectively and $x' \sqsubseteq X$ and $z' \sqsubseteq Z$. Using Rule 1 $((x' \sqsupseteq y_j) \ \wedge \ (y_j \sqsubseteq z') \rightarrow (x' \cap z' \neq \phi))$, the composition of relations can be classified into the following.

I. If $(m' = 1 \ \vee \ n' = 1)$, then the rule shall propagate a definite set of intersections. For example, if y_j intersects the only element of x', then this element of x' must have a non-empty intersection with every element from the set z'. Also, if y_j intersects with the only element of z', then this element of z' must have a non-empty intersection with every element from the set x'. If this property holds for every component of the intermediate space Y then the composition must result in a definite relation. An example of this case is the composition of the inside relationship between two simple convex polygons : $inside(A, B) \wedge inside(B, C) \rightarrow inside(A, C)$

II. If $(m' > 1 \wedge n' > 1)$, for at least one y_j of the space Y no definite intersections are propagated (i.e. $x' \cap z' \neq \phi$). If after the application of the reasoning rules this result still holds, then the composition shall produce a non-definite set of disjunctive relations.

III. If $(m' = m \wedge n' = n)$, i.e. $(X \sqsupseteq y_j) \wedge (y_j \sqsubseteq Z)$, no distinguishing constraints can be propagated from the component y_j, as this case is an expression of the first general constraint in section . Also since the implication of such constraint is that every component of one space may intersect with all the components of the other space no empty intersection will be propagated (using rule 2) for any component.

IV. If $(m' = 1 \wedge n' = 1 \wedge x' = \{x_0\} \wedge z' = \{z_0\})$, i.e. x' is the infinite component and z' is the infinite component, then the rule becomes an expression of the second general constraint in section , i.e. no distinguishing constraint will be propagated.

V. If all the propagated intersections for the set of components of the intermediate space are either of type 3 or 4 above then the composition results in the universal relation (disjunction of set of all possible relationships) - since the only constraints propagated are the general ones, i.e no specific constraint propagated. An example is the compositions: $overlap(A, B) \wedge overlap(B, C)$ and $disjoint(A, B) \wedge disjoint(B, C)$ for two simple convex polygons.

3.2. *Conceptual Neighbourhood*

An observation made by Freksa [14] on the temporal composition table derived by Allen [5] is that the table entries which are a disjunctive set of relations are always sets of relations which are conceptual neighbors. "Two relations between pairs of events are conceptual neighbors if they can be directly transformed into one another by continuous deformation (i.e., shortening or lengthening) of the events [12]". The same observation was made for the composition tables derived in the spatial domain [12] and this property was utilized in making the reasoning process more efficient. However, there was no explanation on why this phenomenon occurs [3].

In this section the reasoning formalism developed shall be used to give an explanation on phenomenon of the conceptual neighbourhood. The main condition for conceptual neighbourhood relations can be defined in terms of component intersection. Conceptual neighbour relations are created by continuous deformation of one object (shortening or lengthening) [12]. When an object is deformed while in contact with another object one or more of its components moves into neighbouring (adjacent) components of the other object. In a dense connected space if a component x_1 of object x has a non-empty intersection with a component y_1 from object y, when object x is deformed (or moved) and if such deformation involves the component x_1, then x_1 has to intersect with other components of y, for example y_2, such that y_2 is connected to y_1. Thus, conceptual neighbourhood are characterized as follows: the union of the components that a specific component intersect with in both relations from the same object will be connected as defined by the adjacency matrix or else the continuity of the deformation is lost (i.e. non conceptual neighbours). I.e. if $R_1(x, y)$ and $R_2(x, y)$ are conceptual neighbours then if in R_1, $x_1 \cap y_1 = 1$ and in R_2, $x_1 \cap y_2 = 1$ then y_1 and y_2 must be connected in the adjacency matrix.

	y_0	y_1	y_2
x_0	1	0	0
x_1	1	1	0
x_2	1	1	1

	z_0	z_1	z_2
y_0	1	0	0
y_1	1	1	0
y_2	1	1	1

	z_0	z_1	z_2
x_0	1	0	0
x_1	1	-	0
x_2	1	1	1

Figure 10: (a) x covers y and y covers z, (b) Result of the composition.

The initial assumptions of our formalism states that all the components of the objects and the space are dense and connected. From rule 1, if $y_j \sqsubseteq x'$ and $x' = (x_1 \cup x_2)$ then $x'_1 \sqsubseteq x_1$ and $x'_2 \sqsubseteq x_2$ are the two sets of points that intersect with the point set y_j, i.e. $y_j = (x'_1 \cup x'_2)$.

In [15] connectedness is defined as follows: a topological space is separated if it is the union of two disjoint non-empty open sets and a space is connected it is is not separated. From the above if x'_1 does not connect to x'_2 then $x'_1 \cup x'_2$ is separated, i.e. y_j is separated. Since y_j is connected (by assumption) then x'_1 and x'_2 are connected, and hence x_1 and x_2 are also connected. The same applies to $y_j \sqsubseteq z'$, and hence the elements of z' are connected. Since $x' \sqsupseteq y_j \wedge y_j \sqsubseteq z' \rightarrow x' \cap z' \neq \phi$, Then any element of x' can intersect with connected elements of z' and vice versa. For example, if $x' = x_1 \cup x_2$ and $z' = z_1 \cup z_2$ then $(x_1 \cap z_1 \neq \phi \vee x_1 \cap z_2 \neq \phi)$, i.e. x_1 may intersect with connected components, which will result in a disjunctive set of conceptual neighbourhood relations.

The following example illustrates the above argument. Consider the composition of the relationships between simple convex regions in figure 10. By applying the reasoning rules we have that $x_1 \cap (z_1 \cup z_2) \neq \phi$ z_1 is connected to z_2 and thus the possible relationships from this composition are conceptual neighbors as shown in the figure.

Thus if any component of the intermediate space (y_j) is not connected the composition does not guarantee that the resulting relations from the composition are conceptual neighbours. Bennett [3] gave an example for a discontinuous composition

Figure 11: Discontinuous composition due to disconnected components [Bennet-94].

f_0				
1	f			
0	1	t		
0	0	1	s	
0	0	0	1	p_0

Figure 12: a) An event, b) representation of the event by adjacency matrix.

which falls into the last category, see figure 11. In the figure, the two concave cavities in the intermediate space B are not connected, yet are treated implicitly as one component. If we consider the two cavities as two components, then the relation between C and B $(c_1 \vee c_2)$ will then become a disjunctive set of disjoint relations. This is the only other possibility for discontinuous composition. As can be easily recognized, the two possibilities can be mapped into one another by changing object composition. Note that if the three objects A, B and C were three dimensional objects with the two cavities now forming one connected path (for example, object B represents two wheels connected by a shaft) it can be intuitively seen that the discontinuity in the resulting composition no longer exist since no separated component is considered.

3.3. *Applying the Reasoning Formalism to Temporal Relations*

The reasoning formalism can be applied to order relations by considering a 1D space where the object (or value) divides that space into two semi-infinite lines, one representing all objects (values) with the relation $<$ and the other for values $>$. In the temporal domain (an order domain), consider an event e in an event space E as shown in figure 12. e can be decomposed into the following components: s: its start, f: its finish, t: its duration. The event space E is composed of e and p_0: a semi-infinite line representing the past of e and f_0: a semi infinite line representing the future of e. The connectivity matrix for E is as shown in figure 12(b).

Relationship between two events can be represented by an intersection matrix. For example the overlap relationship in figure 13 can be represented by the matrix in the same figure. Both the general space constraints in section are also applicable in the temporal domain. In the above example, $f_{01} \cap f_{02} \neq \phi$ and $p_{01} \cap p_{02} \neq \phi$, i.e.

(a)

	f_{01}	f_1	t_1	s_1	p_{01}
f_{02}	1	0	0	0	0
f_2	1	0	0	0	0
t_2	1	1	1	0	0
s_2	0	0	1	0	0
p_{02}	0	0	1	1	1
	f_{02}	f_2	t_2	s_2	p_{02}
f_{03}	1	0	0	0	0
f_3	1	0	0	0	0
t_3	1	1	1	0	0
s_3	0	0	1	0	0
p_{03}	0	0	1	1	1

(b)

Figure 13: (a) An overlap relationship between two events. (b) adjacency matrix corresponding to the relationship in (a).

the future as well as the past of any two events must intersect.

The analysis of indefinite and definite intersections given earlier is also applicable here. For example, if $during(A, B)$ and $during(C, B)$, then all the components of B either intersect only with the futures or pasts of both A and C or with every component in A and C, i.e. propagates only the two general constraints and hence result in the universal relation. The explanation of conceptual neighbourhood is also applicable in the temporal domain. The two reasoning rules proposed are also applicable in the temporal domain. For example, consider the composition of the two relationships: $overlap(e1, e2)$ and $overlap(e2, e3)$ as shown in figure 13. Applying the two reasoning rules over the above matrices as in section , we get the result matrix in figure 14 which can corresponds to one of the three relations shown in the figure.

4. Approaches to Spatial Reasoning

The formalism proposed in this paper was shown to be applicable to reasoning over topological as well as order relations between complex objects. Approaches to spatial reasoning in the literature can generally be classified into a) using *transitive propagation* and b) using *theorem proving*.

- Transitive propagation: In this approach the transitive property of some spatial relations is utilized to carry out the required reasoning. This applies to the *order relations*, such as `before`, `after` and $(<, =, >)$ (for example, $a < b \wedge b < c \rightarrow a < c$), and to the subset relations such as `contain` and `inside` (for example, $inside(A, B) \wedge inside(B, C) \rightarrow inside(A, C)$, $east(A, B) \wedge east(B, C) \rightarrow east(A, C)$).

  Transitive property of the subset relations was employed by Egenhofer [7] for reasoning over topological relationships. Transitive property of the order rela-

	f_{01}	f_1	t_1	s_1	p_{01}
f_{03}	1	0	0	0	0
f_3	1	0	0	0	0
t_3	1	-	-	0	0
s_3	-	-	-	0	0
p_{03}	-	-	1	1	1

Figure 14: Result of the composition in figure 13 is a set of disjunctive relations $before(e_1, e_3) \vee meet(e_1, e_3) \vee overlap(e_1, e_3)$.

tions has been utilized by Mukerjee & Joe [17], Guesgen [4], Chang & Lu [18], Lee & Hsu [19] and Papadias & Sellis [20]. Although order relations can be utilized in reasoning over point-shaped objects, they cannot be directly applied when the actual shapes and proximity of objects are considered.

- Theorem proving (elimination): where reasoning can be carried out by checking every relation in the full set of *sound* relations in the domain to see whether it is a valid consequence of the composition considered (theorems to be proved) and eliminating the ones which are not consistent with the composition [21].

  Bennett [16] have proposed a propositional calculus for the derivation of the composition of topological relations between simple regions using this method. However, checking each relation in the composition table to prove or eliminate is not possible in general cases and is considered a challenge for theorem provers [2].

In general the limitation of all the methods in the above two approaches are as follows:

- Spatial reasoning is studied only between objects of similar types, e.g. between two lines or two simple areas. Spatial relations exist between objects of any type and it is limiting to consider the composition of only specific object shapes.
- Spatial reasoning was carried out only between objects with the same dimension as the space they are embedded in, e.g. between two lines in 1D, between

two regions in 2D, etc.

- Spatial reasoning is studied mainly between simple object shapes or objects with controlled complexity, for example, regions with holes treated as concentric simple regions. No method has yet been presented for spatial reasoning between objects with arbitrary complexity.

The method proposed here is simple and general - only two rules are used to derive composition between objects of random complexity and is applicable to different types of spatial relations (topological and order).

5. Conclusions

A general approach for spatial reasoning is proposed. The approach consists of a set of two general constraints to govern the spatial relationships between objects in space, and two general rules to propagate relationships between objects in space. The following conclusions may be drawn:

- The reasoning process is general and can be applied on any types of objects with random complexity.
- The approach is simple and is based on the application of two rules for the propagation of empty and non-empty intersections between object components.
- The approach is based on a uniform representation of the topology of the space as a connected set of components. A structure called adjacency matrix is proposed to capture the topology of objects of different complexity in any space dimension.

A classification is given of the conditions where definite and indefinite compositions result. The reasoning method was used to explain the phenomenon of the conceptual neighbourhood. The approach was shown to be applicable to the representation and reasoning over events in the temporal domain. Finally, the method is applied in a finite known number of steps (equal to the number of components of the intermediate objects) which allows its implementation in spatial information systems.

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	$disjoint(x,y)$	$meet(x,y)$	$inside(x,y)$	$coverdBy(x,y)$	$contain(x,y)$	$cover(x,y)$	$overlap(x,y)$
$R_1(y,z)$	*all*	*all*	1	1	*all*	*all*	*all*
$R_2(y,z)$	1, 2, 4, 16, 17, 18, 19	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 14, 15, 16, 17, 18, 19	1	1, 2	9, 10, 11, 12, 13, 14, 17,	2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17	*all*
$R_3(y,z)$	1, 18, 19	1, 2, 3, 4, 5, 6, 7, 8, 15, 16, 18, 19	1	1, 2, 3,	11, 12, 13	3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15	*all*
$R_4(y,z)$	1, 2, 4, 16, 17, 18, 19	1, 2, 3, 4, 5, 6, 7, 9, 14, 15, 16, 17, 18, 19	1	1, 2, 4, 19, ,	9, 10, 11, 12, 13, 14, 17	4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17	*all*
$R_5(y,z)$	1, 18, 19,	1, 2, 3, 4, 5, 6, 7, 15, 16, 18, 19	1	1, 2, 3, 4, 5, 19	11, 12, 13	5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15	*all*
$R_6(y,z)$	1	1, 2, 3, 4, 5, 6, 19	1	1, 2, 3, 4, 5, 6, 19	11	6, 7, 8, 9, 10, 11, 12	*all*
$R_7(y,z)$	1	1, 2, 3, 4, 5, 19	1, 18, 19	1, 2, 3, 4, 5, 6, 7, 15, 16, 18, 19	11	7, 8, 9, 10, 11, 12	*all*
$R_8(y,z)$	1	1, 2, 3	1, 18, 19	1, 2, 3, 4, 5, 6, 7, 8, 15, 16, 18, 19	11	8, 10, 11	*all*
$R_9(y,z)$	1	1, 2, 4, 19	1, 2, 4, 16, 17, 18, 19	1, 2, 3, 4, 5, 6, 7, 9, 14, 15, 16, 17, 18, 19	11	9, 10, 11, 12	*all*
$R_{10}(y,z)$	1	1, 2	1, 2, 4, 16, 17, 18, 19	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 14, 15, 16, 17, 18, 19	11	10, 11	*all*
$R_{11}(y,z)$	1	1	*all*	*all*	11	11	*all*
$R_{12}(y,z)$	1	1, 19	1, 2, 3, 4, 5, 13, 14, 15, 16, 17, 18, 19	1, 2, 3, 4, 5, 6, 7, 9, 12, 13, 14, 15, 16, 17, 18, 19	11	11, 12	*all*
$R_{13}(y,z)$	1, 18, 19	1, 18, 19	1, 2, 3, 4, 5, 13, 14, 15, 16, 17, 18, 19	1, 2, 3, 4, 5, 13, 14, 15, 16, 17, 18, 19	11, 12, 13	11, 12, 13	*all*
$R_{14}(y,z)$	1, 18, 19	1, 2, 4, 16, 18, 19	1, 2, 4, 16, 17, 18, 19	1, 2, 3, 4, 5, 14, 15, 16, 17, 18, 19	11, 12, 13	9, 10, 11, 12, 13, 14	*all*
$R_{15}(y,z)$	1, 18, 19	1, 2, 3, 4, 5, 15, 16, 18, 19	1, 18, 19	1, 2, 3, 4, 5, 15, 16, 18, 19	11, 12, 13,	7, 8, 9, 10, 11, 12, 13, 14, 15	*all*
$R_{16}(y,z)$	1, 2, 4, 16, 17, 18, 19	1, 2, 3, 4, 5, 14, 15, 16, 17, 18, 19	1, 18, 19	1, 2, 4, 16, 18, 19,	9, 10, 11, 12, 13, 14, 17	7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17	*all*
$R_{17}(y,z)$	1, 2, 4, 16, 17, 18, 19	1, 2, 4, 16, 17, 18, 19	1, 2, 4, 16, 17, 18, 19	1, 2, 4, 16, 17, 18, 19	9, 10, 11, 12, 13, 14, 17	9, 10, 11, 12, 13, 14, 17	*all*
$R_{18}(y,z)$	1, 2, 3, 4, 5, 13, 14, 15, 16, 17, 18, 19	1, 2, 3, 4, 5, 13, 14, 15, 16, 17, 18, 19	1, 18, 19	1, 18, 19	7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18	7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18	*all*
$R_{19}(y,z)$	1, 2, 3, 4, 5, 13, 14, 15, 16, 17, 18, 19	1, 2, 3, 4, 5, 6, 7, 9, 12, 13, 14, 15, 16, 17, 18, 19	1	1, 19	7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18	4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16 17, 18, 19	*all*

Table 1: The composition table between two regions and a region and a line. The numbers in the table correspond to relations R_1 to R_{19} between a region and a line.