Generalised Frobenius numbers: geometry of upper bounds, Frobenius graphs and exact formulas for arithmetic sequences

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Dedication

This dissertation is expressly dedicated to the memory of my father, Haji mohammed Haji who left us with the most precious asset in life, knowledge. I know that he would be the happiest father in the world to know that his daughter has completed her PhD studies. I also dedicate my work to my lovely mother Adila Mousa for her support, encouragement, and constant love that have sustained me throughout my life.

I also dedicate this work and express my special thanks to all my family members, friends, and colleagues whose words of encouragement halped me to write this dissertation.

Summary

Given a positive integer vector $\boldsymbol{a} = (a_1, a_2 \ldots, a_k)^t$ with

$$1 < a_1 < \cdots < a_k \quad \text{and} \quad \gcd(a_1, \ldots, a_k) = 1\,.$$

The Frobenius number of the vector $\boldsymbol{a}$, $\mathrm{F}_k(\boldsymbol{a})$, is the largest positive integer that cannot be represented as $\sum_{i=1}^{k} a_i x_i$, where $x_1, \ldots, x_k$ are nonnegative integers. We also consider a generalised Frobenius number, known in the literature as the s-Frobenius number, $\mathrm{F}_s(a_1, a_2, \ldots, a_k)$, which is defined to be the largest integer that cannot be represented as $\sum_{i=1}^{k} a_i x_i$ in at least s distinct ways. The classical Frobenius number corresponds to the case $s = 1$.

The main result of the thesis is the new upper bound for the 2-Frobenius number,

$$\mathrm{F}_2(a_1, \ldots, a_k) \le \mathrm{F}_1(a_1, \ldots, a_k) + 2 \left(\frac{(k-1)!}{\binom{2(k-1)}{k-1}} \right)^{1/(k-1)} (a_1 \cdots a_k)^{1/(k-1)}\,, \tag{0.0.1}$$

that arises from studying the bounds for the quantity $\left(\mathrm{F}_s(\boldsymbol{a}) - \mathrm{F}_1(\boldsymbol{a})\right)(a_1 \cdots a_k)^{-1/(k-1)}$. The bound (0.0.1) is an improvement, for $s = 2$, on a bound given by Aliev, Fukshansky and Henk [2]. Our proofs rely on the geometry of numbers.

By using graph theoretic techniques, we also obtain an explicit formula for the 2-Frobenius number of the arithmetic progression $a, a+d, \ldots a+nd$ (i.e. the a_i's are in an arithmetic progression) with $\gcd(a, d) = 1$ and $1 \le d < a$.

$$\mathrm{F}_2(a, a+d, \ldots a+nd) = a \left\lfloor \frac{a}{n} \right\rfloor + d(a+1)\,, \quad n \in \{2, 3\}. \tag{0.0.2}$$

This result generalises Roperts's result [73] for the Frobenius number of general arithmetic sequences.

In the course of our investigations we derive a formula for the shortest path and the distance between any two vertices of a graph associated with the positive integers $a_1, \ldots, a_k$.

Based on our results, we observe a new pattern for the 2-Frobenius number of general arithmetic sequences $a, a+d, \ldots, a+nd$, $\gcd(a, d) = 1$, which we state as a conjecture.

Part of this work has appeared in [6].

Declaration

This work has not been submitted in substance for any other degree or award at this or any other university or place of learning, nor is being submitted concurrently in candidature for any degree or other award.

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Chapter 1

Introduction

1.1 A brief history of the Frobenius problem

The Frobenius problem can be formulated as follows: Given a positive integer k-dimensional vector $\boldsymbol{a} = (a_1, a_2, \ldots, a_k)^t \in \mathbb{Z}^k_{>0}$ with $\gcd(\boldsymbol{a}) := \gcd(a_1, a_2, \ldots, a_k) = 1$, find the largest integer $\mathrm{F}(\boldsymbol{a}) = \mathrm{F}(a_1, a_2, \ldots, a_k)$ that cannot be represented as a nonnegative integer linear combination of the entries of $\boldsymbol{a}$. We can write this as

$$\mathrm{F}(\boldsymbol{a}) = \max\{b \in \mathbb{Z} : b \neq \langle \boldsymbol{a}, \boldsymbol{z} \rangle \text{ for all } \boldsymbol{z} \in \mathbb{Z}^k_{\geq 0}\},$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^k$. The number $\mathrm{F}(\boldsymbol{a})$ is called the *Frobenius number* associated with the vector $\boldsymbol{a}$. The positive integers $a_1, a_2, \ldots, a_k$ are called the *basis* of the Frobenius number or the *Frobenius basis*. Historically this problem is often described in terms of coins of denominations $a_1, a_2, \ldots, a_k$, so that the Frobenius number is the largest amount of money which cannot be formed using these coins.

The Frobenius problem is an old problem that was originally considered by Ferdinand Georg Frobenius (1849-1917)[39]. According to Brauer [25], Frobenius occasionally raised the following question: "determine (or at least find non-trivial good bounds for) $\mathrm{F}(\boldsymbol{a})$" in his lectures in the early 1900s.

The Frobenius problem is known by other names in the literature, such as the money-changing problem (or the money-changing problem of Frobenius, or the coin-exchange problem of Frobenius) [95, 90, 20, 21, 17], the coin problem (or the Frobenius coin problem) [23, 85, 9, 65] and the Diophantine problem of Frobenius [81, 75, 18, 72].

The Frobenius problem is related to many other mathematical problems, and has applications in various fields including number theory, algebra, probability, graph theory, counting points in polytopes, and the geometry of numbers. There is a rich literature on the Frobenius problem and for a comprehensive survey on the history and different aspects of this problem we refer the reader to the book of Ramírez-Alfonsín [72].

In this present work we are not intending to survey all of the work related to the Frobenius problem. We aim to give an overview of the key results related to the scope of this thesis. For $k = 2$ it is well known (most probably at least to Sylvester [86]) that

$$\mathrm{F}(a_1, a_2) = a_1a_2 - (a_1 + a_2).$$

Sylvester also found that exactly half of the integers between 1 and $(a_1 - 1)(a_2 - 1)$ are representable (in terms a_1 and a_2). This result was posted as a mathematical problem in the Educational Times [86]. About half a century after Sylvester's result, I. Schur in his last lecture in Berlin in 1935 gave an upper bound for $\mathrm{F}(\boldsymbol{a})$ in the general case. This bound was published and later improved by Brauer [25, 26].

Remarkably, no closed formula exists for the Frobenius number with a Frobenius basis consisting of $k > 2$ elements, as shown by Curtis [31] in 1990. Johnson [54] was probably the first who developed an algorithm for computing the Frobenius number of three integers. Later Brauer and Shockley [27] found a simpler algorithm to compute the value of $\mathrm{F}(a_1, a_2, a_3)$. In 1978 Selmer and Beyer [82] developed a general method, based on a continued fractions algorithm, for determining the Frobenius number in the case $k = 3$. Their result was later simplified by Rödseth [75]. The fastest known algorithms for computing $\mathrm{F}(a_1, a_2, a_3)$ (according to the experiments in [19]) were discovered by Greenberg [43] in 1988 and Davison [32] in 1994.

For $k \geqslant 4$, formulas for $\mathrm{F}(a_1, \ldots, a_k)$ are known only in some special cases (for instance, where the a_i's are consecutive integers [25], or where the a_i's form an arithmetic progression [73, 13]. Computing the Frobenius number is NP-hard, as proved by Ramírez-Alfonsín [71] in 1996, who reduced it to the integer knapsack problem. On the other hand, in 1992 Kannan [56] established a polynomial time algorithm for computing the Frobenius number $\mathrm{F}(\boldsymbol{a})$ for any fixed k. However, Kannan's algorithm is known to be hard to implement, as it is based on a relation between the Frobenius number and the covering radius of a certain polytope. Barvinok and Woods [12] in 2003 proposed a polynomial time algorithm for computing the Frobenius number in fixed dimension, using the generating functions.

In 1962, Brauer & Shockley [27] suggested a method that allows us to determine the Frobenius number by computing a residue table of a_1 words. The method makes use of the following identity: (see also [71])

$$\mathrm{F}(\boldsymbol{a}) = \mathrm{F}(a_1, \ldots, a_k) = \max_{1 \le i \le a_1 - 1} \{w_i\} - a_1, \tag{1.1.1}$$

where w_i is the smallest positive integer such that $w_i \equiv i \pmod{a_1}$ that is representable as a nonnegative integer combination of $a_2, \ldots, a_k$. In other words

$$w_i = \min \left\{ \sum_{n=2}^{k} x_n a_n : x_n \in \mathbb{Z}_{\ge 0} \text{ for } n = 2, \ldots, k, \sum_{n=2}^{k} x_n a_n \equiv i \pmod{a_1} \right\}.$$

In 2007, Einstein, Lichtblau, Strzebonski and Wagon [36] presented an algorithm to compute the Frobenius number of a quadratic sequence of small length. For example, for $x \ge 2$,

$$\mathrm{F}(9x, 9x+1, 9x+4, 9x+9) = 9x^2 + 18x - 2\,.$$

There exists a number of useful relations between graph theory and the Frobenius numbers. For instance, Nijenhuis [66] developed an algorithm to determine the Frobenius number, constructing a corresponding graph with weighted edges and determining the path of minimum weight from one vertex to all the others. Then

$$\mathrm{F}(\boldsymbol{a}) = \mathrm{F}(a_1, \ldots, a_k) = \mathrm{diam}(G_w(\boldsymbol{a})) - a_1\,,$$

where $G_w(\boldsymbol{a})$ is a certain graph associated with a vector $\boldsymbol{a}$ and $\mathrm{diam}(\cdot)$ stands for the graph diameter. The correctness of Nijenhuis' algorithm follows from (1.1.1) (see also [72, p.20]). Nijenhuis' algorithm runs in time of order $O(k a_{\min} \log a_{\min})$ where $a_{\min} = \min\limits_{1 \le i \le k} \{a_i\}$. In this present work Nijhenius's formula will be applied to compute out the 2-Frobenius number of arithmetic progressions.

There is another algorithm constructed by Heap and Lynn [48] to compute $\mathrm{F}(a_1, \ldots, a_k)$ by finding the index of primitivity $\gamma(B)$ of a nonnegative matrix $B = (b_{i,j})$ (i.e. $b_{i,j} \ge 0$), $1 \le i, j \le k$ of order $(a_k + a_{k-1} - 1)$ via graph theory

$$\mathrm{F}(a_1, \ldots, a_k) = \gamma(B) - 2a_k + 1\,,$$

where $\gamma(B)$ is the smallest integer such that $B^{\gamma(B)} > 0$.

We note that other methods have been derived, but they will not be discussed here.

Historically, the problem of computing the Frobenius number for a given Frobenius basis has proved intractable, leading to considerable interest in obtaining bounds for $\mathrm{F}(\boldsymbol{a})$. For instance, there are various bounds on the Frobenius number given by Erdös and Graham [38], Selmer [81], Rödseth [75], Davison [32], Fukshansky and Robins [40], Aliev and Gruber [7], Aliev, Henk and Hinrichs [4] amongst others.

Beck and Robins [16] defined the *s-Frobenius number* as follows. Let s be a positive integer. The s-Frobenius number $\mathrm{F}_s(\boldsymbol{a}) = \mathrm{F}_s(a_1, \ldots, a_k)$ is the largest integer number that cannot be represented in at least s different ways as a nonnegative integer linear combination of $a_1, \ldots, a_k$. Beck and Robins [16] gave the formula for the case $k = 2$

$$\mathrm{F}_s(a_1, a_2) = sa_1a_2 - a_1 - a_2.$$

In particular, this identity generalises the well-known result in the setting of the (classical) Frobenius number $\mathrm{F}(\boldsymbol{a}) = \mathrm{F}_1(\boldsymbol{a})$ which corresponding to $s = 1$.

This natural generalisation of the classical Frobenius number $\mathrm{F}_1(\boldsymbol{a})$, has been studied recently by several authors. For instance, Aliev, Henk and Linke [5] obtained an optimal lower bound on the s-Frobenius number $\mathrm{F}_s(a_1, \ldots, a_k)$ for $k \geq 3$.

Aliev, Fukshansky and Henk [2] obtained an upper bound for the s-Frobenius number using the concept of s-covering radius. In this thesis we derive an upper bound for 2-Frobenius numbers, that improves on known results.

The next subsection summarise the main results of this thesis, which will be presented in the following chapters.

1.2 Organisation of the thesis

The present work is concerned with the generalised Frobenius number $\mathrm{F}_s(\boldsymbol{a})$ associated with a primitive vector $\boldsymbol{a} = (a_1, a_2, \ldots, a_k)^t \in \mathbb{Z}^k_{>0}$. In particular, we give an improved upper bound for the generalised Frobenius number $\mathrm{F}_s(\boldsymbol{a})$ with $s = 2$ and $k \geq 3$. Also we present a conjecture for computing the 2-Frobenius number $\mathrm{F}_2(\boldsymbol{a})$, when the entries a_i's are in arithmetic progressions.

To give structural overview of this thesis, in Chapter 1 we outline the existing results on the behaviour of the Frobenius numbers, accompanied by a brief history of the Frobenius problem, and also a literature review.

The concept of the generalised Frobenius number is then introduced in Chapter 2, where known results and ideas are discussed. In the end of the chapter, publications related to the discussed results are supplied for the interested reader.

In Chapter 3, we obtain a new upper bound on the s-Frobenius number when $s = 2$, using techniques from the geometry of numbers, which improves upon an upper bound given in [2] for $\mathrm{F}_s(\boldsymbol{a})$ where $s \geq 1$.

Basic graph-theoretic definitions are introduced in Chapter 4, as well as related concepts, lemmas and known results that we require for our proofs. The concept of directed circulant graphs is also introduced, where we note that such graphs are also referred to as Frobenius circulant graphs. Connection between graph theory and the Frobenius number is then discussed and new results derived. In particular, we present a new proof for the formula $\mathrm{F}_2(a_1, a_2) = 2a_1a_2 - a_1 - a_2$, using only graph theoretical methods.

In Chapter 5, we obtain an explicit formula for the shortest path and the minimum distance between any two vertices of a directed circulant graph $G_w(\boldsymbol{a})$ associated with a positive integer 3-dimensional primitive vector $(\boldsymbol{a}) = (a, a+d, a+2d)^t$. We also establish a relationship between representations of nonnegative integers and the shortest paths from one vertex to all other vertex in $G_w(\boldsymbol{a})$. This relationship is used to derive an explicit formula for computing the 2-Frobenius number of the arithmetic progression $a, a+d, a+2d$ with $\gcd(a, d) = 1$.

In Chapter 6, we extend the results of Chapter 5 to include the four term arithmetic progression (i.e. $a, a+d, a+2d, a+3d$). This yields an explicit formula for computing $\mathrm{F}_2(a, a+d, a+2d, a+3d)$. In particular, we propose a conjecture an explicit formula for the 2-Frobenius number of the general arithmetic sequences.

In the last chapter, we will summarize the main results in this thesis and future work.

Chapter 2

The Frobenius problem and its generalisations

In this chapter we give an overview of the Frobenius problem, introduce the generalised Frobenius number and define the s-covering radius, which plays an important role in subsequent chapters. In Sections 2.1 and 2.2 we introduce some definitions, accompanied by some examples of determining the Frobenius number for given Frobenius basis, $a_1, \ldots, a_k$. In Section 2.3 we discuss a known formula for the Frobenius number $\mathrm{F}(a_1, a_2)$. Some special cases for large values of k are presented, followed by results concerning the Frobenius number for general k. In Section 2.4 we examine a relationship between the Frobenius number of k positive integers and the covering radius of a certain simplex in $\mathbb{R}^{k-1}$. These results are generalised in Section 2.5, to encompass the relationship between the s-Frobenius number $\mathrm{F}_s(a_1, \ldots, a_k)$ and the s-covering radius.

2.1 Some preliminaries from number theory

We denote by $\mathbb{Z}_{>0}$ and $\mathbb{Z}_{\geq 0}$ the sets of all positive and nonnegative integer numbers, respectively. The *Minkowski sum* of two sets $A, B \subseteq \mathbb{R}^n$ is defined as the set $A + B = \{a + b : a \in A, b \in B\} \subseteq \mathbb{R}^n$ and $\lambda A = \{\lambda a : a \in A\}$ for $\lambda \in \mathbb{R}$. The cardinality of a set A is denoted $\#(A)$. For any real x, $\lfloor x \rfloor$ denotes the largest integer not exceeding x.

Let $a_1, \ldots, a_k$ be integers, not all zero. The *greatest common divisor* of $a_1, \ldots, a_k$ will be denoted by $\gcd(a_1, \ldots, a_k)$. If $\gcd(a_1, \ldots, a_k) = 1$ then these integers are said to be *relatively prime* (or *coprimes*).

We will need the following well-known result.

Theorem 2.1.1 (Theorem 5.15 p.172 in [88]). *Let a, b, c be integers with not both a and b equal to 0. Then the linear Diophantine equation*

$$ax + by = c \tag{2.1.1}$$

is solvable if and only if $\gcd(a, b)$ *divides c. Furthermore, if (x_0, y_0) is any particular solution to (2.1.1), then all integer solutions of (2.1.1) are given by*

$$\begin{aligned} x &= x_0 + tb/\gcd(a,b)\,, \\ y &= y_0 - ta/\gcd(a,b)\,, \end{aligned} \tag{2.1.2}$$

where t is an arbitrary integer.

Lattice

Let $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_k$ be linearly independent vectors in $\mathbb{R}^n$ and let $B = [\boldsymbol{b}_1, \ldots, \boldsymbol{b}_k] \in \mathbb{R}^{n \times k}$ be the matrix with columns $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_k$. The *lattice* L generated by $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_k$ (or, equivalently, by B) is the set

$$L = L(B) = \left\{ \sum_{i=1}^{k} x_i \boldsymbol{b}_i : x_i \in \mathbb{Z} \right\} = \left\{ B\boldsymbol{x} : \boldsymbol{x} \in \mathbb{Z}^k \right\} \tag{2.1.3}$$

of all integer linear combinations of the vectors $\boldsymbol{b}_i$'s.
The vectors $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_k$ (or, equivalently, B) are called a *basis for the lattice* (or *lattice basis*). The integers n and k are called the *dimension* and the *rank* of $L(B)$ respectively. When $k = n$ the lattice $L(B)$ is called a *full rank* or *full dimensional* lattice in $\mathbb{R}^n$.

The *fundamental parallelepiped* associated to $B = [\boldsymbol{b}_1, \ldots, \boldsymbol{b}_k] \in \mathbb{R}^{n \times k}$ is the set of points

$$\mathcal{P}(B) = \left\{ \sum_{i=0}^{k} \alpha_i \boldsymbol{b}_i : \alpha_i \in \mathbb{R}, 0 \leq \alpha_i < 1 \right\}.$$

The *determinant* $\det(L(B))$ of the lattice $L(B)$ is the k-dimensional volume of the fundamental parallelepiped $\mathcal{P}(B)$ associated to B

$$\det(L(B)) = \mathrm{vol}_k(\mathcal{P}(B)) = \sqrt{\det(B^t B)}\,,$$

where B^t is the transpose of B.

Remark 2.1.2. In this thesis we will mainly consider full rank lattices.

2.2 The Frobenius problem and representable integers

Let $k \geq 2$ be an integer and let $a_1, a_2, \ldots, a_k$ be positive relatively prime integers. We call an integer t *representable* by the vector $\boldsymbol{a} = (a_1, a_2, \ldots, a_k)^t$ if there exist nonnegative integers $x_1, x_2, \ldots, x_k$ such that

$$t = \sum_{i=1}^{k} x_i a_i \,, \tag{2.2.1}$$

and *nonrepresentable* otherwise.

We denote by $\mathrm{Sg}\,(\boldsymbol{a})$ the set of all representable integers in terms of $\boldsymbol{a}$. $\mathrm{Sg}\,(\boldsymbol{a})$ is a numerical semigroup generated by $a_1, a_2, \ldots, a_k$.

The Frobenius problem is an old problem named after the 19th century German mathematician Ferdinand Georg Frobenius who raised this problem in his lectures (according to Brauer [25]).

Given a positive integer k-dimensional primitive vector $\boldsymbol{a}$, i.e., $\boldsymbol{a} = (a_1, \ldots, a_k)^t \in \mathbb{Z}^k_{>0}$ with $\gcd(a_1, \ldots, a_k) = 1$, the Frobenius problem asks to find the *Frobenius number* $\mathrm{F}(\boldsymbol{a})$, that is the largest integer which is nonrepresentable in terms of $\boldsymbol{a}$. That is

$$\mathrm{F}(\boldsymbol{a}) = \mathrm{F}(a_1, \ldots, a_k) = \max\{b \in \mathbb{Z} : b \neq \langle \boldsymbol{a}, \boldsymbol{z} \rangle \text{ for all } \boldsymbol{z} \in \mathbb{Z}^k_{\geq 0}\}\,, \tag{2.2.2}$$

or, equivlently,

$$\mathrm{F}(\boldsymbol{a}) = \max\{x \in \mathbb{Z}_{\geq 0} : x \notin \mathrm{Sg}\,(\boldsymbol{a})\}\,. \tag{2.2.3}$$

The theorem below implies that $\mathrm{F}(\boldsymbol{a})$ exists.

Theorem 2.2.1 (Theorem 1.1.5 in [99])**.** *Let $\boldsymbol{a} = (a_1, a_2, \ldots, a_k)^t$ be a positive integer k-dimensional vector. There are only finitely many nonnegative integers that are not in* $\mathrm{Sg}\,(\boldsymbol{a})$ *if and only if* $\gcd(a_1, a_2, \ldots, a_k) = 1$.

Dozens of papers have been published since then, but no closed formula for Frobenius number $\mathrm{F}(\boldsymbol{a})$ is known up to now. The first published work on this problem is attributed to Sylvester [86] who determined that exactly half of the integers between 1 and $(a_1 - 1)(a_2 - 1)$ are representable in terms a_1 and a_2, when a_1 and a_2 are relatively prime. The modern study of the Frobenius problem began with the 1942 paper of Brauer [25].

Example 2.2.2. Let $\boldsymbol{a} = (3,8)^t$. Then

$$\mathrm{Sg}\,(\boldsymbol{a}) = \{3a + 8b : a, b \in \mathbb{Z}_{\geq 0}\} \tag{2.2.4}$$

and

$$\mathbb{Z}_{\geq 0} \setminus \mathrm{Sg}\,(\boldsymbol{a}) = \{1, 2, 4, 5, 7, 10, 13\}\,.$$

Hence the Frobenius number is $\mathrm{F}(\boldsymbol{a}) = 13$.

A special case of the Frobenius problem is the *McNuggets number problem*:

Problem 2.2.3. *(*Chicken McNuggets Problem*)[70, 83] At McDonald's, Chicken McNuggets are available in packs of either* $6, 9,$ *or* 20 *McNuggets. What is the largest number of McNuggets that one cannot purchase?*

Figure 2.1: McDonald's Chicken McNuggets in a box of 20

The answer is $\mathrm{F}(6, 9, 20) = 43$. To see that 43 is not representable, observe that we can choose either $0, 1,$ or 2 packs of 20. If we choose 0 or 1 or 2 packs, then we have to represent 43 or 23 or 3 as a nonnegative integer linear combination of 6 and 9, which is impossible.

To see that every larger number representable, note that

$$\begin{aligned}
44 &= 1 \cdot 20 + 0 \cdot 9 + 4 \cdot 6, \\
45 &= 0 \cdot 20 + 3 \cdot 9 + 3 \cdot 6, \\
46 &= 2 \cdot 20 + 0 \cdot 9 + 1 \cdot 6, \\
47 &= 1 \cdot 20 + 3 \cdot 9 + 0 \cdot 6, \\
48 &= 0 \cdot 20 + 0 \cdot 9 + 8 \cdot 6, \\
49 &= 2 \cdot 20 + 1 \cdot 9 + 0 \cdot 6\,.
\end{aligned}$$

Then all integers greater than 49 can be expressed in the form $6m + n$, where $m \in \mathbb{Z}_{>0}$ and $n \in \{44, 45, 46, 47, 48, 49\}$, so all the integers greater than or equal to 44 are in $\mathrm{Sg}\,(6, 9, 20)$. Therefore 43 is the largest integer that cannot be expressed in the form $6a + 9b + 20c$, with $a, b, c \in \mathbb{Z}_{\geq 0}$.

A geometric approach to the Frobenius problem is based on considering the so-called *knapsack polytope*

$$P(\boldsymbol{a}, b) = \{\boldsymbol{x} \in \mathbb{R}^k_{\geq 0} : \langle \boldsymbol{a}, \boldsymbol{x} \rangle = b\}\,.$$

$\mathrm{F}(\boldsymbol{a})$ is the largest integer b, such that the knapsack polytope $P(\boldsymbol{a}, b)$ does not contain an integer point. Figure 2.2 shows the geometry behind the knapsack polytope $P((3,5)^t, b)$ for the first few values of b. Note that the knapsack polytope corresponding to the Frobenius number $\mathrm{F}(3,5) = 7$ is a segment on the red line $3x + 5y = 7$.

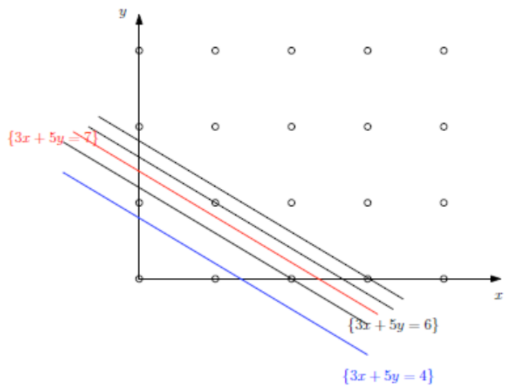

Figure 2.2: $3x + 5y = b\,, b = 1, 2, 3\ldots$

For given positive integers $a_1, a_2, \ldots, a_k$ with $\gcd(a_1, \ldots, a_k) = 1$, we also consider a function closely connected with $\mathrm{F}(a_1, \ldots, a_k)$, as observed by Brauer [25]

$$\mathrm{F}^{+}(a_1, \ldots, a_k) = \mathrm{F}(a_1, \ldots, a_k) + \sum_{i=1}^{k} a_i \,. \tag{2.2.5}$$

From the definition it follows that $\mathrm{F}^{+}(a_1, \ldots, a_k)$ is the largest integer which cannot be represented as a positive integer linear combination of a_i's. However in this present work we focus mainly on the property $\mathrm{F}(a_1, \ldots, a_k)$.

2.3 Frobenius number research directions

Broadly speaking, research work on the Frobenius problem can be divided into three different areas:

1. Explicit formulas for the Frobenius number in special cases.
2. Upper or lower bounds for the Frobenius number.
3. Algorithms for computing the Frobenius number.

2.3.1 Frobenius number formulas

There is a simple formula for the Frobenius number $\mathrm{F}(a_1, \ldots, a_k)$ when $k = 2$. But when $k = 3, 4$; formulae exist only for some special choices of $a_1, \ldots, a_k$. The explicit formula for the case $k = 2$ is given in the following theorem.

Theorem 2.3.1. *[86] Let a_1 and a_2 be positive relatively prime integers. Then*

$$\mathrm{F}(a_1, a_2) = (a_1 - 1)(a_2 - 1) - 1 = a_1 a_2 - (a_1 + a_2) \,. \tag{2.3.1}$$

The origin of this famous result is usually attributed to Sylvester [86] although some consider this to be a “Folklore result”.

In contrast to the case $k = 2$, it was shown in 1990 by Curtis [31] that closed form expression does not exist for the Frobenius number when $k \geq 3$. For the case $k = 3$ there are efficient algorithms to compute $\mathrm{F}(a_1, a_2, a_3)$, developed by Selmer and Beyer [82], Rödseth [75], Greenberg [43] and Davison [32].

In the following we will mention some results on the Frobenius number for special choices of a_1, a_2, a_3. In 1956, Roberts [74] showed that for any positive integers $a, z > 2$

$$\mathrm{F}(a, a+1, a+z) = \begin{cases} \left\lfloor \frac{a+1}{z} \right\rfloor a + (z-3)a, & \text{if } a \equiv -1 \pmod{z} \text{ and } a \geq z^2 - 5z + 3 \ , \\ \\ \left\lfloor \frac{a+1}{z} \right\rfloor (a+z) + (z-3)a, & \text{if } a \equiv -1 \pmod{z} \text{ and } a \geq z^2 - 4z + 2 \ . \end{cases}$$

In 1960 Johnson [54] show that if $a_3 \geq \mathrm{F}(\frac{a_1}{d}, \frac{a_2}{d})$ where $d = \gcd(a_1, a_2)$ then

$$\mathrm{F}(a_1, a_2, a_3) = d\big(a_1 a_2 - a_1 - a_2\big) + (d-1)a_3 \, .$$

In 1962, Brauer & Shockley [27] proved that if $a_1 | (a_2 + a_3)$, then

$$\mathrm{F}(a_1, a_2, a_3) = -a_1 + \max_{i=2,3} \left\{ a_i \left\lfloor \frac{a_1 a_{5-i}}{a_2 + a_3} \right\rfloor \right\} \, .$$

A sequence $a_1, \ldots, a_k$, is called *independent* if none of the basis elements can be represented as a nonnegative integer linear combination of the others.

In 1977, Selmer [81] showed that if a_1, a_2, a_3 are independent and $a_2 \geq t(q+1)$ then

$$\mathrm{F}(a_1, a_2, a_3) = \max\left\{(s-1)a_2 + (q-1)a_3, (r-1)a_2 + qa_3\right\} - a_1 \, ,$$

where s, t, q and r determined by

$$\begin{aligned} &a_3 \equiv s a_2 \pmod{a_1}, \ 1 < s < a_1, \\ &a_3 = s a_2 - t a_1, \ t > 0, \end{aligned}$$

and

$$a_1 = qs + r, \ 0 < r < s \, .$$

In 1987, Hujter [52] has proved for any integer $q > 2$,

$$\mathrm{F}(q^2, q^2+1, q^2+q) = 2q^3 - 2q^2 - 1 \, .$$

For the case $k = 4$, the Frobenius number is much more difficult to find then in the case $k = 3$. In 1964, Dulmage & Mendelsohn [34] found some interesting formulas for $\mathrm{F}(a, a+1, a+2, a+K)$, when $K = 4, 5, 6$, by using graphical methods. For instance when $K = 4$

$$\mathrm{F}(a, a+1, a+2, a+4) = (a+1)\left\lfloor \frac{a}{4} \right\rfloor + \left\lfloor \frac{a+1}{4} \right\rfloor + 2\left\lfloor \frac{a+2}{4} \right\rfloor - 1. \tag{2.3.2}$$

We will discuss the connection between the Frobenius numbers and graph theory in more detail in Chapter 4.

In the general case, Brauer & Shockley [27] found the following expression for the Frobenius number.

Theorem 2.3.2. *(Brauer and Shockley, 1962) Let* $\boldsymbol{a} = (a_1, \dots, a_k)^t$ *be a positive integer vector with* $\gcd(a_1, \dots, a_k) = 1$*. Then*

$$\mathrm{F}(a_1, \dots, a_k) = \max_{1 \le i \le a_1 - 1} \{w_i\} - a_1 \,, \tag{2.3.3}$$

where w_i *is the smallest positive integer with* $w_i \equiv i \pmod{a_1}$*, that can represented as a non-negative integer linear combination of* $a_2, \dots, a_k$*.*

In 1979, Nijenhuis [66] applied the above theorem to compute the Frobenius number $\mathrm{F}(\boldsymbol{a})$, using graph theoretical methods. The graph theory approach employs finding minimum paths in a certain graph associated with a vector $\boldsymbol{a}$. We will give more details of this method in Chapter 4.

2.3.2 Bounds on the Frobenius number

Computing Frobenius number is NP-hard as was shown by Ramírez-Alfonsín [71]. Hence it is important to obtain upper and lower bounds for $\mathrm{F}(\boldsymbol{a})$.

First we will mention several upper bounds. Suppose that $a_1 < \cdots < a_k$. In 1935, Schur proved in his last lecture (according to Brauer [25]) that

$$\mathrm{F}(a_1, a_2, \dots, a_k) \le (a_1 - 1)(a_k - 1) - 1 \,. \tag{2.3.4}$$

In 1942, Brauer [25] improved the bound (2.3.4) as follows:

$$\mathrm{F}(a_1, a_2, \dots, a_k) \le \sum_{i=1}^{k-1} a_{i+1} \frac{d_i}{d_{i+1}} - \sum_{i=1}^{k} a_i \,, \tag{2.3.5}$$

where $d_i = \gcd(a_1, \dots, a_i)$.

Brauer & Seelbinder [26](see also [67]) showed that the bound (2.3.5) is the best possible upper bound if and only if each of the integers $\frac{a_j}{d_j}$, for $j = 2, \ldots, k$, is representable in the form

$$\frac{a_j}{d_j} = \sum_{i=1}^{k-1} y_{ji} \left(\frac{a_i}{d_{j-1}} \right) \qquad \text{with } y_{ij} \geq 0.$$

In 1972, Erdös & Graham [38] showed that

$$\mathrm{F}(a_1, a_2, \ldots, a_k) \leq 2a_{k-1} \left\lfloor \frac{a_k}{k} \right\rfloor - a_k \,, \tag{2.3.6}$$

and in 1977 a similar bound was found by Selmer [81] for the case $a_1 \geq k$ (i.e. each element of the basis $a_1, a_2, \ldots, a_k$ is independent) as follows:

$$\mathrm{F}(a_1, a_2, \ldots, a_k) \leq 2a_k \left\lfloor \frac{a_1}{k} \right\rfloor - a_1 \,.$$

In 1975, Vitek [92] proved another bound for $k \geq 3$ (also see Lewin's work [58]) which says

$$\mathrm{F}(a_1, a_2, \ldots, a_k) \leq \left\lfloor \frac{(a_2 - 1)(a_k - 2)}{2} \right\rfloor - 1 \,.$$

In 1982, Rödseth [77] improved the bound (2.3.6) when k is odd to

$$\mathrm{F}(a_1, a_2, \ldots, a_k) \leq 2a_k \left\lfloor \frac{a_1 + 2}{k + 1} \right\rfloor - a_1.$$

In 2002, Beck, Diaz and Robins [14] showed that

$$\mathrm{F}(a_1, a_2 \ldots, a_k) \leq \frac{1}{2} \left(\sqrt{a_1 a_2 a_3 (a_1 + a_2 + a_3)} - a_1 - a_2 - a_3 \right).$$

There are also upper bounds for the small values of k. In 1975, Roberts [74] proved that for the integers a, b, m with $0 < a < b$, $\gcd(a, b) = 1$ and $m \geq 2$ we have

$$\mathrm{F}(m, m + a, m + b) \leq m \left(b - 2 + \left\lfloor \frac{m}{b} \right\rfloor \right) + (a - 1)(b - 1).$$

In 1976, Vitek [93] showed that if a_1, a_2, a_3 are independent (i.e. none of the a_i is representable by the other two) then

$$\mathrm{F}(a_1, a_2, a_3) \leq a_1 \left\lfloor \frac{a_3}{2} - 1 \right\rfloor \,.$$

A more recent upper bound for the Frobenius number was given by Fukshansky & Robins [40] and will be discussed in § 2.4.

There are also some results on lower bounds for the Frobenius number $\mathrm{F}(a_1, \ldots, a_k)$. Let $a_1, \ldots, a_k$ be positive integers with $\gcd(a_1, \ldots, a_k) = 1$. In 1994, Davison [32] established the following sharp lower bound for $k = 3$

$$\mathrm{F}(a_1, a_2, a_3) \geq \sqrt{3}\sqrt{a_1 a_2 a_3} - a_1 - a_2 - a_3 ,$$

where it is known that the constant $\sqrt{3}$ cannot be improved.

In 2000, Killingbergtrø's [57] proved in the general case that

$$\mathrm{F}(a_1, \ldots, a_k) \geq ((k-1)!\, a_1 \cdots a_k)^{1/(k-1)} - \sum_{i=1}^{k} a_i \,. \tag{2.3.7}$$

More recently, Aliev & Gruber [7] obtained an optimal lower bound for $\mathrm{F}(a_1, a_2, \ldots, a_k)$ in terms of the absolute inhomogeneous minimum of the standard simplex in $\mathbb{R}^{k-1}$. This is discussed further in § 2.4.

2.3.3 The Frobenius number for particular sequential bases

To date there are four main sequentially approaches to classifying the Frobenius basis $a_1, a_2, \ldots, a_k$. These consist of arithmetic sequences, almost arithmetic sequences, geometric sequences and arbitrary sequences.

1. **Arithmetic sequences**

   The sequence of positive integers $a_1, a_2, \ldots, a_k$ is called an *arithmetic sequence* if it satisfies the conditions.

   (i) $\gcd(a_1, a_2, \ldots, a_k) = 1$;

   (ii) $0 < a_1 < \cdots < a_k$ and $a_i = a_1 + (i-1)d$ for $i = 2, 3, \ldots, k$ and $d \geq 1$ (i.e., the integers are in an arithmetic progression with common difference d).

   When the a_i's are in arithmetic progressions, a formula for $\mathrm{F}(\boldsymbol{a})$ has been determined by several authors.

   Let a, d and n be positive integers with $a > n$ and $\gcd(a, d) = 1$. (Note that the condition $a > n$ guarantees that no term a_i is dependent on the other ones). Then in 1942, Brauer [25] (and indepentently, Schur) found the following formula for the Frobenius number of

n consecutive positive integers

$$F(a, a+1, \ldots, a+n-1) = a \left\lfloor \frac{a-2}{n-1} \right\rfloor + (a-1). \tag{2.3.8}$$

Roberts [73] generalised the formula (2.3.8) in 1965 (alse simpler proofs have later been given by Bateman [13] and other authors) for general arithmetic sequences such as

$$F(a, a+d, \ldots, a+nd) = a \left\lfloor \frac{a-2}{n} \right\rfloor + d(a-1). \tag{2.3.9}$$

In this thesis, we derive a formula for the 2-Frobenius number of the arithmetic Frobenius basis $a, a+d, \ldots, a+nd$ when $n \in \{2, 3\}$, using graph-theoretic techniques, which are discussed later in Chapters 5 and 6.

2. **Almost arithmetic sequences**

   The sequence of positive integers $a_1, a_2, \ldots, a_k$ is called an *almost arithmetic sequence* if some $k-1$ terms of $a_1, a_2, \ldots, a_k$ form an arithmetic sequence.

   Lewin [60, 59] was the first who studied the Frobenius number of almost arithmetic sequences. In 1977, Selmer [81] generalised Robert's results (2.3.9) for an almost arithmetic sequence (see also Rödseth's work [76]) as follows: Let $a, h, d, n \in \mathbb{Z}_{>0}$ with $\gcd(a, d) = 1$. Then,

$$F(a, ha+d, ha+2d, \ldots, ha+nd) = ha \left\lfloor \frac{a-2}{n} \right\rfloor + a(h-1) + d(a-1).$$

3. **Geometric sequences**

   A sequence of k terms of positive integers $a_1, a_2, \ldots, a_k$ is called a *geometric sequence* if and only if there is a constant r such that $a_i = ra_{i-1}$ for each $i = 2, 3, \ldots, k$. It follows that the *nth term* of a geometric sequence is given by $a_n = a_1 r^{n-1}$.
   In 2008, Ong & Ponomarenko [68] determined the Frobenius number for geometric sequences. Let x, y, n be integers with $\gcd(x, y) = 1$. Then,

$$F(x^n, x^{n-1}y, x^{n-2}y^2, \ldots, y^n) = y^{n-1}(xy - x - y) + \frac{(y-1)x^2(x^{n-1} - y^{n-1})}{(x-y)}.$$

4. **Mixed types of sequences**

   In 1966, Hofmeister [50] (see also [81]) considered the shifted geometric sequence defined for a, d, t are positive integers, $a, t > 1$ and $\gcd(a, d) = 1$. He obtained the following result

$$F(a, a+d, a+td, \ldots, a+t^{n-2}d) = a \left\lfloor \frac{a-2}{t^{n-2}} \right\rfloor + d(a-1),$$

which holds provided that d exceeds a certain (rather larger) bound.

In 1982, Hujter [51] considered the following sequence and showed for any arbitrary positive integer q, we have that

$$\mathrm{F}(q^{n-1}, q^{n-1}+1, q^{n-1}+q, \ldots, q^{n-1}+q^{n-2}) = (n-1)(q-1)q^{n-1} - 1 \, .$$

2.3.4 Algorithms for computing the Frobenius number

There are several known algorithms to compute $\mathrm{F}(a_1, a_2, \ldots, a_k)$ for a small fixed k. In 1960, Johnson [54] obtained an algorithm for computing the Frobenius number for the case $k = 3$. Later on, Brauer and Shockley [27] in 1962 provided a similar algorithm for finding the Frobenius number. In 1978, Selmer and Beyer [82] devised an algorithm for computating Frobenius number in the case $k = 3$ based on the continued fractions expansions of a ratio associated with a_1, a_2, a_3. Rödseth [75] simplified their result later on. Greenberg [43] and Davison [32] independently discovered a simple and fast algorithm to compute the Frobenius number for $k = 3$ in 1988 and 1994, respectively. This algorithm is the fastest algorithm in comparison with other algorithms in which the runtime is $O(\log a_1)$ and $O(\log a_2)$, respectively.

In 2000, Killingbergtrø [57] developed an algorithm to compute the Frobenius number for $k = 3$. The algorithm works as follows: Let L_1 be the be the smallest integer such that $L_1 a_1$ can be represent as a nonnegative linear integer combination a_2 and a_3, i.e.

$$L_1 = \min \{L_1 : L_1 a_1 = a_2 \lambda_2 + a_3 \lambda_3 \text{ where } \lambda_2, \lambda_3 \in \mathbb{Z}_{\geq 0}\} \, ,$$

and similarly

$$\begin{aligned} L_2 &= \min \{L_2 : L_2 a_2 = a_1 \lambda_1 + a_3 \lambda_3 : \lambda_1, \lambda_3 \in \mathbb{Z}_{\geq 0}\} \, , \\ L_3 &= \min \{L_3 : L_3 a_3 = a_1 \lambda_1 + a_2 \lambda_2 : \lambda_1, \lambda_2 \in \mathbb{Z}_{\geq 0}\} \, . \end{aligned}$$

Suppose that $L_1 a_1$ can be written as inner product of (a_2, a_3) and (λ_2, λ_3) for some $\lambda_2, \lambda_3 \in \mathbb{Z}_{>0}$. Let $[x, y]$ denote the unit square with vertices at (x, y), $(x+1, y)$, $(x, y+1)$ and $(x+1, y+1)$. Consider the following sets

$$\begin{aligned} &C = \{[x, y] : x > 0 \text{ and } y > 0\}, \\ &C_1 = \{[x, y] : x > \lambda_2 \text{ and } y > \lambda_3\}, \\ &C_2 = \{[x, y] : x > L_2 \text{ and } y > 0\}, \\ &\text{and } C_3 = \{[x, y] : x > 0 \text{ and } y > L_3\} \, . \end{aligned}$$

Let the set $R[a_1, a_2, a_3] := C \setminus \{C_1 \cup C_2 \cup C_3\}$. Let $\mathcal{B}(R)$ denoted of all points $(c_1, c_2) \in R[a_1, a_2, a_3]$ such that the unit square $[c_1, c_2]$ is completely contained within $R[a_1, a_2, a_3]$, i.e.

$$\mathcal{B}(R) = \{(c_1, c_2) \in R[a_1, a_2, a_3] : [c_1, c_2] \subseteq R[a_1, a_2, a_3]\}.$$

Then

$$\mathrm{F}(\boldsymbol{a}) = \max\{c_1 a_2 + c_2 a_3 : (c_1, c_2) \in \mathcal{B}(R)\} - a_1 .$$

Killingbergtrø proposed that this method could be extended to all cases $k \geq 3$ but he has only demonstrated it for a very particular choice of numbers, namely $a_1 = 103$, $a_2 = 133$, $a_3 = 165$ and $a_4 = 228$.

There exist a variety of algorithms to compute $\mathrm{F}(a_1, a_2, \ldots, a_k)$ for any fixed k. The main ideas of these algorithms are based on notions from graph theory, mathematical programming, index of primitivity of nonnegative matrices, and geometry of numbers. In 1978, Wilf [95] developed a "circle of lights "algorithm to compute $\mathrm{F}(a_1, a_2, \ldots, a_k)$ where $1 < a_1 \cdots < a_k$, which employs a circle of a_k lights labelled by $l_0, l_1, \ldots, l_{k-1}$, moving in a clockwise direction. Suppose the light l_0 is on while all the others are off. Starting from l_0 and moving clockwise, consider the k lights that are at distance $a_1, \ldots, a_k$ away anti-clockwise. If any of them is on, then turn on the current light, if the light is already on then leave it on and move to the next light. The process halts until there are a_1 consecutive on lights. The Frobenius number is then given by

$$\mathrm{F}(a_1, a_2, \ldots, a_k) = r + (s(l_r) - 1)a_k\,, \tag{2.3.10}$$

where $s(l_r)$ is the number of times light visited l_r during the operation and let l_r be the last visited off light just before ending the process.

In 1980, Greenberg [42] developed an algorithm to compute $\mathrm{F}(\boldsymbol{a})$, by using mathematical programming. The correctness of both Wilf's and Greenberg's algorithms is based on Theorem 2.3.2 of Brauer and Shockley. In 1979, Nijenhuis [66] establish an algorithm to compute the Frobenius number $\mathrm{F}(\boldsymbol{a})$ by finding minimum paths in a directed circulant graph (Frobenius circulant graph). In 1989, Lovász [61] was probably the first who related the Frobenius number to study of maximal lattice point free convex bodies (i.e. interior does not contain any integral points). Lovász formulated a conjecture which he shows would imply a polynomial time algorithm for the Frobenius number for fixed k. In 1992, Kannan [56] established an algorithm that for any fixed k, computes the Frobenius number in polynomial time. His algorithm is based on the relation between the Frobenius number and the covering radius of a certain simplex. For variable k, the runtime of such algorithm has a double exponential dependency on k, and is not competitive for $k \geq 5$. Kannan's algorithm is very complicated and it's not easy to implement.

In 2005, Beihoffer et al. [19] developed a fast algorithm that can handle cases for $k = 10$ and $a_1 = 10^7$ to compute the Frobenius number. There is a rich literature on Frobenius numbers, and for an impressive survey on the history and the different aspects of the problem we refer to the book [72].

2.4 Frobenius numbers and the covering radius

In this section we will study the behaviour of $\mathrm{F}(\boldsymbol{a})$ by using techniques from the geometry of numbers.

2.4.1 The covering radius

A *convex body* is a convex subset K of $\mathbb{R}^k$ which is a compact (closed and bounded) and has nonempty interior. A convex body K is called *symmetric* if it is centrally symmetric with respect to the origin (i.e., $\boldsymbol{x} \in K$ if and only if $-\boldsymbol{x} \in K$). For this thesis will denote the family of all convex bodies and symmetric convex bodies in $\mathbb{R}^k$ as $\mathcal{K}^k$ and $\mathcal{K}_0^k$, respectively.

We denote by $\mathcal{L}^k$ the set of all k-dimensional lattices L in $\mathbb{R}^k$, and the lattice of all points with integer coordinates in $\mathbb{R}^k$ is denoted by $\mathbb{Z}^k$. The i th coordinate of a point $\boldsymbol{x} \in \mathbb{R}^k$ is denoted by x_i. Given a matrix $B \in \mathbb{R}^{k \times k}$ with $\det(B) \neq 0$ and a set $Q \subset \mathbb{R}^k$, let

$$BQ := \{B\boldsymbol{x} : \boldsymbol{x} \in Q\}$$

be the image of Q under linear map defined by B. Then we can write $\mathcal{L}^k$ as

$$\mathcal{L}^k = \{B\,\mathbb{Z}^k : B \in \mathbb{R}^{k \times k},\, \det(B) \neq 0\}\,.$$

For $L = B\,\mathbb{Z}^k \in \mathcal{L}^k$, $\det(L) = |\det(B)|$.

A lattice $L \in \mathcal{L}^k$ is called a *covering lattice* or *packing lattice* for a convex body K if $K + L$ covers $\mathbb{R}^k$ or if $\forall \boldsymbol{x}, \boldsymbol{y} \in L, \boldsymbol{x} \neq \boldsymbol{y}, (\boldsymbol{x} + K) \cap (\boldsymbol{y} + K) = \varnothing$, respectively. See [63, 24] or [69]).

The *covering radius* $\mu(K, L)$ (also known as the *inhomogeneous minimum*) of a convex body K with respect to the lattice L is defined as the smallest positive number t such that the dilated body tK covers $\mathbb{R}^k$ by translates of the lattice L. This can be formulated as

$$\mu(K, L) = \min\{t \in \mathbb{R}_{>0} : tK + L = \mathbb{R}^k\} . \tag{2.4.1}$$

Or equivalently, we can describe it as

$$\mu(K, L) = \min\{t \in \mathbb{R}_{>0} : \text{L is a covering lattice of } tK\} .$$

Further, for any arbitrary convex body K, the quantity $\vartheta_1(K)$ given by

$$\vartheta_1(K) = \min\{\mu(K, L) : \det(L) = 1\} \tag{2.4.2}$$

is called the *absolute inhomogeneous minimum* of K.

2.4.2 Kannan's formula

A number of results on the Frobenius numbers with an arbitrary number of variables have been found using the methods based in the geometry of numbers for which we refer to the books [46, 45]. In particular, Kannan [56] established a relation between the covering radius of simplex and the Frobenius number. More precisely, for a given primitive vector $\boldsymbol{a} = (a_1, a_2, \ldots, a_k) \in \mathbb{Z}^k_{>0}$, let

$$S_{\boldsymbol{a}} = \left\{ \boldsymbol{x} \in \mathbb{R}^{k-1}_{\geq 0} : \sum_{i=1}^{k-1} a_i x_i \leq 1 \right\} , \tag{2.4.3}$$

be the $(k-1)$-dimensional simplex with vertices $0, \frac{1}{a_i} e_i$ where e_i is the i th unit vector in $\mathbb{R}^{k-1}$, $1 \leq i \leq k-1$.

Define the lattice $\Lambda_{\boldsymbol{a}}$ in $\mathbb{R}^{k-1}$ by

$$\Lambda_{\boldsymbol{a}} = \left\{ \boldsymbol{x} \in \mathbb{Z}^{k-1} : \sum_{i=1}^{k-1} a_i z_i \equiv 0 \pmod{a_k} \right\} . \tag{2.4.4}$$

This simplex and lattice were introduced by Kannan in his studies of the Frobenius number [55, 56] where he proved the following relationship between the covering radius of $S_{\boldsymbol{a}}$ and the Frobenius number.

Theorem 2.4.1 (Theorem 2.5 in [55]). *We have*

$$\mu(S_{\boldsymbol{a}}, \Lambda_{\boldsymbol{a}}) = \mathrm{F}(a_1, a_2, \dots, a_k) + \sum_{i=1}^{k} a_i \,.$$

Then from Theorem 2.4.1 one could produce bounds on $\mathrm{F}(a_1, a_2, \dots, a_k)$ by bounding $\mu(S_{\boldsymbol{a}}, \Lambda_{\boldsymbol{a}})$. Standard techniques for bounding a covering radius only work in the case when the convex body is centrally symmetric with respect to the origin.

In 2006, Aliev and Gruber [7] found the following optimal lower bound for the Frobenius number in term of the absolute inhomogeneous minimum of the standard simplex S^{k-1}; $S^{k-1} = \left\{ \boldsymbol{x} \in \mathbb{R}^{k-1}_{\geq 0} : \sum_{i=1}^{k-1} x_i \leq 1 \right\}$. Indeed

$$\mathrm{F}(a_1, \dots, a_k) \geq \vartheta_1(S^{k-1})\,(a_1 \cdots a_k)^{1/(k-1)} - \sum_{i=1}^{k} a_i \,. \tag{2.4.5}$$

Here $\vartheta_1(S^{k-1})$ is the absolute inhomogeneous minimum of an $(k-1)$-dimensional standard simplex S^{k-1}. Since $\vartheta_1(S^{k-1}) > ((k-1)!)^{\frac{1}{k-1}}$, (see [46, Theorem 2, section 21] or [7, (7)]), this implies that

$$\mathrm{F}(a_1, \dots, a_k) > ((k-1)!\, a_1 \cdots a_k)^{1/(k-1)} - \sum_{i=1}^{k} a_i \,.$$

On the other hand, Fukshansky & Robins [40] in 2007 also used techniques from the geometry of numbers to obtain the following upper bound for $\mathrm{F}(\boldsymbol{a})$:

$$\mathrm{F}(a_1, \dots, a_k) \leq \left\lfloor \frac{(k-1)^2/\Gamma(\frac{k}{2}+1)}{\pi^{k/2}} \sum_{i=1}^{k} a_i \sqrt{(|\boldsymbol{a}|_2)^2 - a_i^2} + 1 \right\rfloor , \tag{2.4.6}$$

where $|\cdot|_2$ denotes the Euclidean norm. See [40] for details.

2.5 A generalisation of the Frobenius numbers

Beck and Robins [16] introduced and studied a generalised Frobenius number, sometimes called the *s-Frobenius number*. For a positive integer s, the s-Frobenius number $\mathrm{F}_s(a_1, \dots, a_k)$ associated with a vector $\boldsymbol{a}$ is defined to be the largest integer number that cannot be represented in at least s different ways as a nonnegative integer linear combination of the a_i's. That is

$$\mathrm{F}_s(\boldsymbol{a}) = \mathrm{F}_s(a_1, \dots, a_k) = \max\{b \in \mathbb{Z} : \#\{\boldsymbol{z} \in \mathbb{Z}^k_{\geq 0} : \langle \boldsymbol{a}, \boldsymbol{z} \rangle = b\} < s\} \,. \tag{2.5.1}$$

When $s = 1$ we have the (classical) Frobenius number $\mathrm{F}(\boldsymbol{a}) = \mathrm{F}_1(\boldsymbol{a})$.

Remark 2.5.1. To avoid any confusion with conflicting notation we remark that the term "s-Frobenius number", $\mathrm{F}^*_s = \mathrm{F}^*_s(\boldsymbol{a})$, is also used by some authors to denote the largest positive integer that has *exactly* s-representations in terms of a_i's. From herein we will adhere to the first definition, whereby the s-Frobenius number $\mathrm{F}_s(\boldsymbol{a})$ is the largest positive integer that has *less* than s-representations in terms of a_i's.

It has also been proven that $\mathrm{F}^*_s(a_1, a_2, a_3)$ is not necessarily increasing with s. For example, Brown et al. [28] indicated that $\mathrm{F}^*_{35}(4, 7, 19) = 181$ while $\mathrm{F}^*_{36}(4, 7, 19) = 180$. Furthermore Shallit and Stankewicz [84] proved that for any $s \geq 1$ and $k = 5$, the quantity $\mathrm{F}^*_1(\boldsymbol{a}) - \mathrm{F}^*_s(\boldsymbol{a})$ is unbounded. Furthermore, they provide examples with $\mathrm{F}^*_1(\boldsymbol{a}) > \mathrm{F}^*_s(\boldsymbol{a})$ for $k \geq 6$ and $\mathrm{F}^*_1(\boldsymbol{a}) > \mathrm{F}^*_2(\boldsymbol{a})$ for $k \geq 4$.

Remark 2.5.2. It should be noted that other generalisations of the Frobenius number have been investigated by different authors, including, but not limited, to Chapter 6 of [72], as well as more recent works in [87, 3].

Beck & Robins [16] showed that for $k = 2$

$$\mathrm{F}_s(a_1, a_2) = s a_1 a_2 - (a_1 + a_2)\,. \tag{2.5.2}$$

This identity generalises formula (2.3.1) that corresponds to $s = 1$. But for general k and s only bounds on the s-Frobenius number $\mathrm{F}_s(\boldsymbol{a})$ are available. It was recently shown by Aliev, De Loera and Louveaux [1] that $\mathrm{F}_s(\boldsymbol{a})$ can be computed in polynomial time for fixed dimension k and parameter s, extending well-known results of Kannan [56] and Barvinok and Woods [12] for the Frobenius number $\mathrm{F}_1(\boldsymbol{a})$.

In 2011, Beck & Curtis [15], presented an argument for computing $\mathrm{F}_s(a_1, \ldots, a_k)$, which generalises Theorem 2.3.2 of Brauer and Shockely. We state their result in the following lemma.

Lemma 2.5.3. *Let $\boldsymbol{a} = (a_1, \ldots, a_k)^t$ be a positive integer k-dimensional vector with* $\gcd(\boldsymbol{a}) = 1$ *and let $n_{j,s}$ be the smallest nonnegative integer with $n_{j,s} \equiv j \pmod{a_1}$, that has at least s-representations as a nonnegative integer linear combination of the given $a_1, a_2, \ldots, a_k$. Then*

$$\mathrm{F}_s(a_1, \ldots, a_k) = \max_{1 \leq j \leq a_1 - 1} \{n_{j,s}\} - a_1\,. \tag{2.5.3}$$

2.5.1 The s-covering radius

Let $s \in \mathbb{N}$, $K \in \mathcal{K}^k$ and $L \in \mathcal{L}^k$. Then the *s-covering radius* $\mu_s(K, L)$ (also known as the *s-inhomogeneous minimum*) of a convex body K with respect a lattice L is defined to be the smallest positive number μ such that any point $t \in \mathbb{R}^k$ is covered with multiplicity at least s by $\mu K + L$. This can be formulated as

$$\begin{aligned}\mu_s(K, L) = \min\{\mu > 0 : \text{for all } t \in \mathbb{R}^k \text{ there exist } b_1, \dots, b_s \in L \\ \text{such that } t \in b_i + \mu K, 1 \le i \le s\}\,.\end{aligned} \tag{2.5.4}$$

Alternatively, the s-covering radius $\mu_s(K, L)$ can be described equivalently as the smallest positive number μ such that any translate of μK contains at least s lattice points, i.e.,

$$\mu_s(K, L) = \min\{\mu > 0 : \#\{(t + \mu K) \cap L\} \ge s \text{ for all } t \in \mathbb{R}^k\}\,. \tag{2.5.5}$$

For $s = 1$ we get the well-known the covering radius, see e.g. Gruber [45] and Gruber and Lekkerkerker [46]. These books also serve as excellent sources for further information on lattices and convex bodies in the context of the geometry of numbers.

Gruber [44, (5)], defined for any convex body $K \subset \mathbb{R}^k$ the *absolute s-inhomogeneous minimum* $\vartheta_s(K)$ as follows:

$$\vartheta_s(K) = \inf \frac{\mu_s(K, L)}{\det(L)^{1/k}}\,, \tag{2.5.6}$$

where the infimum is taken over all k-dimensional lattices $L \in \mathbb{R}^k$. For s = 1 the formula reduces to the classical absolute inhomogeneous minimum used in (2.4.5).

In 2011, Aliev, Fukshansky and Henk [2], generalised Theorem 2.4.1 for the classical Frobenius number to $\mathrm{F}_s(\boldsymbol{a})$ as follows:

Theorem 2.5.4. *[2, Theorem 3.2] Let $k \ge 2$, $s \ge 1$ and let $a_1 < \cdots < a_k$. Then*

$$\mu_s(S_{\boldsymbol{a}}, \Lambda_{\boldsymbol{a}}) = \mathrm{F}_s(a_1, \dots, a_k) + \sum_{i=1}^{k} a_i\,.$$

For $s = 1$ the formula reduces to that of Kannan's Theorem 2.4.1.

2.5.2 Bounds on $\mathrm{F}_s(\boldsymbol{a})$ in terms of the s-covering radius

The successive minima of convex bodies with respect to lattice were first defined and investigated by Minkowski in the context of the geometry of numbers.

The i th *successive minimum* $\lambda_i = \lambda_i(K, L)$ of $K \in \mathcal{K}_0^k$ with respect to $L \in \mathcal{L}^k$ is the smallest positive real number λ such that λK contains at least i linearly independent lattice points of L (inside or on its boundary). That is

$$\lambda_i = \lambda_i(K, L) = \min\{\lambda \in \mathbb{R}_{>0} : \dim(\lambda K \cap L) \geq i\}, \ 1 \leq i \leq k\,. \tag{2.5.7}$$

Obviously, we have $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k$ and the first successive minimum $\lambda_1(K, L)$ is the smallest dilation factor such that $\lambda_1(K, L)K$ contains a nonzero lattice point of L. There exists a vast literature on successive minima (for example see [46, 30]).

Minkowski ([64]) proved two fundamental inequalities for the successive minima $\lambda_i(K, L)$ and the volume of $K \in \mathcal{K}_0^k$, which can be written in the following way:

Theorem 2.5.5. *(Minkowski, 1896):*

$$(\lambda_1(K, L))^k \ \mathrm{vol}\,(K) \leq 2^k \det(L)\,, \tag{2.5.8}$$

and

$$\frac{2^k}{k!} \det(L) \leq \prod_{i=1}^{k} \lambda_i(K, L)\, \mathrm{vol}\,(K) \leq 2^k \det(L)\,. \tag{2.5.9}$$

Note that the upper bound in the inequality (2.5.9) is a far reaching improvement of the inequality (2.5.8). The above are known as the *first and second Minkowski's theorems* on successive minima, respectively.

A relation between the covering radius and successive minima is given by Jarnik's inequalities [53].

Theorem 2.5.6. *Let K be a 0-symmetric convex body, with successive minima $\lambda_1, \lambda_2, \cdots, \lambda_k$ and covering radius $\mu(K, L)$. Then*

$$\frac{1}{2}\lambda_k \leq \mu(K, L) \leq \frac{1}{2}\sum_{i=1}^{k} \lambda_i.$$

In [2] bounds for the s-covering radius $\mu_s(K, L)$ of K are given, as described below.

Lemma 2.5.7. *[2, Lemma 2.2] Let $s \in \mathbb{N}$, $s \geq 1$, $K \in \mathcal{K}^k$ and $L \in \mathcal{L}^k$. Then*

$$s^{\frac{1}{k}} \left(\frac{\det(L)}{\mathrm{vol}\,(K)}\right)^{\frac{1}{k}} \leq \mu_s(K, L) \leq \mu_1(K, L) + (s-1)^{\frac{1}{k}} \left(\frac{\det(L)}{\mathrm{vol}\,(K)}\right)^{\frac{1}{k}}.$$

Aliev, Fukshansky and Henk [2] established an upper bound for the s-Frobenius number using Theorem 2.5.4 and Lemma 2.5.7 as follows:

Theorem 2.5.8. *Let $k \geq 2$, $s \geq 1$ and let $a_1 < \cdots < a_k$. Then*

$$\mathrm{F}_s(\boldsymbol{a}) \leq \mathrm{F}_1(\boldsymbol{a}) + ((s-1)(k-1)!)^{\frac{1}{k-1}} \left(\prod_{i=1}^{k} a_i\right)^{\frac{1}{k-1}} . \tag{2.5.10}$$

One of the main results of this thesis, is an improvement of the upper bound (2.5.10) in the case $s = 2$, where in Theorem 3.2.1 we show that

$$\mathrm{F}_2(\boldsymbol{a}) \leq \mathrm{F}_1(\boldsymbol{a}) + 2\left(\frac{(k-1)!}{\binom{2(k-1)}{k-1}}\right)^{\frac{1}{k-1}} \left(\prod_{i=1}^{k} a_i\right)^{\frac{1}{k-1}} .$$

We also note that Aliev, Henk and Linke [5] obtained an optimal lower bound for the s-Frobenius number by generalising the optimal lower bound (2.4.5) for classical Frobenius number as follows:

Theorem 2.5.9. *Let $k \geq 3$, $s \geq 1$. Then*

$$\mathrm{F}_s(a_1, \ldots, a_k) \geq \vartheta_s(S^{k-1})\,(a_1 \cdots a_k)^{\frac{1}{k-1}} - \sum_{i=1}^{k} a_i \,.$$

Here $\vartheta_s(S^{k-1})$ is the absolute s-inhomogeneous minimum of an $(k-1)$-dimensional standard simplex S^{k-1}.

Hence from (2.5.6), we have

$$\vartheta_s(S^{k-1}) \geq (s(k-1)!)^{\frac{1}{k-1}} .$$

This implies that

$$\mathrm{F}_s(a_1, \ldots, a_k) \geq s^{\frac{1}{k-1}}\big((k-1)!\, a_1 \cdots a_k\big)^{\frac{1}{k-1}} - \sum_{i=1}^{k} a_i \,. \tag{2.5.11}$$

For further information see [2].

Chapter 3

A new upper bound for the 2-Frobenius number

In this chapter we study the quantity

$$\big(\mathrm{F}_s(a_1,\ldots,a_k)-\mathrm{F}_1(a_1,\ldots,a_k)\big)\big(a_1\cdots a_k\big)^{-1/(k-1)},$$

for $k \geq 2$, deriving an improved upper bound on the 2-Frobenius number.

Let $\boldsymbol{a}=(a_1,\ldots,a_k)^t$, be an integer vector with

$$0 < a_1 < \cdots < a_k\,,\ \gcd(a_1,\ldots,a_k)=1\,. \tag{3.0.1}$$

In general setting, when dimension k is a part of input, computing $\mathrm{F}_s(\boldsymbol{a})$ is NP-hard already for $s=1$ due to a result of Ramírez-Alfonsín [72]. Thus the upper and lower bounds for $\mathrm{F}_s(\boldsymbol{a})$ are of special interest.

We have already mentioned in Subsection 2.5.2 that a sharp lower bound for $\mathrm{F}_s(\boldsymbol{a})$ was obtained in [5]. Upper bounds for the s-Frobenius number were established by Fukshansky and Schürmann [41] and Aliev, Fukshansky and Henk [2]. In particular, it was shown in [2] that

$$\mathrm{F}_s(\boldsymbol{a}) \leq \mathrm{F}_1(\boldsymbol{a}) + ((s-1)\,(k-1)!)^{\frac{1}{k-1}}\,\Pi(\boldsymbol{a})^{\frac{1}{k-1}}\,, \tag{3.0.2}$$

where $\Pi(\boldsymbol{a})=a_1\cdots a_k$. The inequality (3.0.2) allows us to use various upper bounds for the Frobenius number to estimate $\mathrm{F}_s(\boldsymbol{a})$.

In view of (3.0.2), to estimate $\mathrm{F}_s(\boldsymbol{a})$ from above it is natural to study the *(normalised) distance*

$$\tau_s(\boldsymbol{a}) = \frac{\mathrm{F}_s(\boldsymbol{a}) - \mathrm{F}_1(\boldsymbol{a})}{\Pi(\boldsymbol{a})^{\frac{1}{k-1}}}\,,$$

between $\mathrm{F}_s(\boldsymbol{a})$ and $\mathrm{F}_1(\boldsymbol{a})$ by considering the constant

$$c(k,s) = \sup_{\boldsymbol{a}} \tau_s(\boldsymbol{a})\,, \tag{3.0.3}$$

where the supremum in (3.0.3) is taken over all integer vectors satisfying (3.0.1). It follows that, (3.0.2) implies the bound

$$c(k,s) \leq ((s-1)\,(k-1)!)^{\frac{1}{k-1}}\,. \tag{3.0.4}$$

As the case $k=2$ is covered by (2.5.2), we now focus on the case $k \geq 3$.

3.1 A lower bound for $c(k,s)$

The first result below shows that, roughly speaking, cutting off special families of input vectors cannot make the order of magnitude of $\mathrm{F}_s(\boldsymbol{a}) - \mathrm{F}_1(\boldsymbol{a})$ smaller than $\Pi(\boldsymbol{a})^{\frac{1}{k-1}}$. This will imply a lower bound for $c(k,s)$.

Theorem 3.1.1. *Let $k \geq 3$ and $s \geq 2$. For any direction vector $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_{k-1})^t \in \mathbb{Q}^{k-1}$, with $0 < \alpha_1 < \cdots < \alpha_{k-1} < 1$, there exists an infinite sequence of distinct integer vectors $\boldsymbol{a}(t) = (a_1(t), \ldots, a_k(t))^t$, satisfying (3.0.1) such that*

(i) $\displaystyle\lim_{t\to\infty} \frac{a_i(t)}{a_k(t)} = \alpha_i$, $1 \leq i \leq k-1$,

(ii) $\displaystyle\lim_{t\to\infty} \tau_s(\boldsymbol{a}(t)) = p(k-1,s)$, *where* $p(d,s) = \min\{m \in \mathbb{Z}_{\geq 0} : \binom{m+d}{d} \geq s\}$.

It follows that $c(k,s) \geq p(k-1,s)$. Since for a fixed dimension $k \geq 3$ we have

$$p(k-1,s)((s-1)\,(k-1)!)^{-\frac{1}{k-1}} \to 1 \text{ as } s \to \infty,$$

Theorem 3.1.1 also implies that for large s the upper bound (3.0.4) (and hence (3.0.2)) cannot be significantly improved.

In order to prove Theorem 3.1.1 we require the following three lemmas.

Lemma 3.1.2. *Let $d \geq 2$, $s \geq 1$. Then*

$$\mu_s(S^d, \mathbb{Z}^d) = p(d,s) + d\,, \tag{3.1.1}$$

where S^d is the standard simplex in $\mathbb{R}^d$.

Proof. Let $F = [0,1)^d$ be the fundamental cell of the lattice $\mathbb{Z}^d$ with respect to the standard basis. It is straightforward to see that

$$\begin{aligned}\mu_s(S^d, \mathbb{Z}^d) = \min\{\mu > 0 : &\text{ there exist } \boldsymbol{b}_1, \ldots, \boldsymbol{b}_s \in \mathbb{Z}^d \\ &\text{such that } F \subset (\boldsymbol{b}_i + \mu S^d)\,,\ 1 \leq i \leq s\}\,.\end{aligned} \tag{3.1.2}$$

This implies, in particular, that $\mu_k(S^d, \mathbb{Z}^d)$ is a positive integer$\geq d$.

Suppose that F is covered by $\boldsymbol{u} + \bar{t}S^d$ with $\boldsymbol{u} \in \mathbb{Z}^d$. Then, $\boldsymbol{u} \in \mathbb{Z}^d_{\leq 0}$ and $\bar{t} \geq d$. We observe that

$$F \subset (\boldsymbol{u} + \bar{t}S^d) \iff \boldsymbol{0} \in (\boldsymbol{u} + (\bar{t} - d)S^d) \iff -\boldsymbol{u} \in (\bar{t} - d)S^d\,.$$

Hence, F is covered with multiplicity at least s by $(m+d)S^d + \mathbb{Z}^d$ if and only if mS^d contains at least s integer points. Therefore, by (3.1.2),

$$\mu_s(S^d, \mathbb{Z}^d) = \min\{m \in \mathbb{Z}_{\geq 0} : \#(mS^d \cap \mathbb{Z}^d) \geq s\} + d\,.$$

Noting that $\#(mS^d \cap \mathbb{Z}^d) = \binom{m+d}{d}$, we thus obtain (3.1.1).

□

Following Gruber [44], we say that a sequence S_t of convex bodies in $\mathbb{R}^d$ converges to a convex body S if the sequence of *distance functions* of S_t converges uniformly on the unit ball in $\mathbb{R}^d$ to the distance function of S. For the notion of convergence of a sequence of lattices to a given lattice we refer the reader to [46, p.178].

Lemma 3.1.3 (see Satz 1 in [44])**.** *Let S_t be a sequence of convex bodies in $\mathbb{R}^d$ which converges to a convex body S and let Λ_t be a sequence of lattices in $\mathbb{R}^d$ convergent to a lattice Λ. Then*

$$\lim_{t \to \infty} \mu_s(S_t, \Lambda_t) = \mu_s(S, \Lambda)\,.$$

The last ingredient required for the proof of Theorem 3.1.1 is the following result from [7] which is also implicit in Schinzel [80].

Lemma 3.1.4 (Theorem 1.2 in [7]). *For any lattice Λ with basis $\boldsymbol{b}_1, \dots, \boldsymbol{b}_d$, $\boldsymbol{b}_i \in \mathbb{Q}^d$, $i = 1, \dots, d$, and for all rationals $\alpha_1, \dots, \alpha_d$ with $0 < \alpha_1 < \dots < \alpha_d < 1$, there exists a sequence*

$$\boldsymbol{a}(t) = (a_1(t), \dots, a_d(t), a_{d+1}(t))^t \in \mathbb{Z}^{d+1}\,, t = 1, 2, \dots\,,$$

such that $\gcd(a_1(t), \dots, a_d(t), a_{d+1}(t)) = 1$ *and the lattice $\Lambda_{\boldsymbol{a}(t)}$ has a basis $\boldsymbol{b}_1(t), \dots, \boldsymbol{b}_d(t)$ with*

$$\frac{b_{ij}(t)}{n\,t} = b_{ij} + O\left(\frac{1}{t}\right)\,, \quad i, j = 1, \dots, d\,, \tag{3.1.3}$$

where $n \in \mathbb{N}$ is such that $n\, b_{ij}, n\, \alpha_j\, b_{ij} \in \mathbb{Z}$ for all $i, j = 1, \dots, d$. Moreover,

$$a_{d+1}(t) = \det(\Lambda) n^d t^d + O(t^{d-1})\,, \tag{3.1.4}$$

and

$$\alpha_i(t) := \frac{a_i(t)}{a_{d+1}(t)} = \alpha_i + O\left(\frac{1}{t}\right). \tag{3.1.5}$$

3.1.1 Proof of Theorem 3.1.1

Let $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_{k-1})^t$ be any rational vector in $\mathbb{Q}^{k-1}$ satisfying

$$0 < \alpha_1 < \ldots < \alpha_{k-1} < 1\,, \tag{3.1.6}$$

and let $D(\boldsymbol{\alpha}) = \operatorname{diag}(\alpha_1^{-1}, \dots, \alpha_{k-1}^{-1})$. Then $\Lambda(\boldsymbol{\alpha}) = D(\boldsymbol{\alpha})\mathbb{Z}^{k-1}$ is the lattice of determinant

$$\det(\Lambda(\boldsymbol{\alpha})) = |\det(D(\boldsymbol{\alpha}))| = (\Pi(\boldsymbol{\alpha}))^{-1},$$

and $S(\boldsymbol{\alpha}) = D(\boldsymbol{\alpha})S^{k-1}$ is the simplex of volume

$$\operatorname{vol}(S(\boldsymbol{\alpha})) = |\det(D(\boldsymbol{\alpha}))| \operatorname{vol}(S^{k-1}) = (\Pi(\boldsymbol{\alpha})(k-1)!)^{-1}.$$

Applying Lemma 3.1.4 to the lattice $\Lambda = \Lambda(\boldsymbol{\alpha})$ and the numbers $\alpha_1, \dots, \alpha_{k-1}$, we get a sequence $\boldsymbol{a}(t)$, satisfying (3.1.3), (3.1.4) and (3.1.5). Furthermore, by (3.1.6) and (3.1.5),

$$0 < a_1(t) < a_2(t) < \ldots < a_k(t)\,,$$

for sufficiently large t.

Now define the simplex S_t and the lattice Λ_t such that

$$S_t = a_k(t) S_{\boldsymbol{a}(t)} = \{(x_1, \dots, x_{k-1})^t \in \mathbb{R}^{k-1}_{\geq 0} : \sum_{i=1}^{k-1} \alpha_i(t) x_i \leq 1\}\,,$$

$$\Lambda_t = (\Pi(\boldsymbol{\alpha})a_k(t))^{-1/(k-1)}\Lambda_{\boldsymbol{a}(t)}\,.$$

Then, we have

$$\mu_s(S_{\boldsymbol{a}(t)},\Lambda_{\boldsymbol{a}(t)}) = \Pi(\boldsymbol{\alpha})^{1/(k-1)}a_k(t)^{k/(k-1)}\mu_s(S_t,\Lambda_t)\,, \tag{3.1.7}$$

and by (3.1.3) and (3.1.4), the sequence Λ_t converges to the lattice $\Lambda(\boldsymbol{\alpha})$. Next, the point $\boldsymbol{p} = (1/(2k),\ldots,1/(2k))$ is an inner point of the simplex $S(\boldsymbol{\alpha})$, and also for all the simplicies S_t for sufficiently large t. By (3.1.5) and Lemma 3.1.3, the sequence

$$\mu_s(S_t - \boldsymbol{p},\Lambda_t) \text{ converges to } \mu_s(S(\boldsymbol{\alpha}) - \boldsymbol{p},\Lambda(\boldsymbol{\alpha})).$$

Here we consider the sequence $\mu_s(S_t - \boldsymbol{p},\Lambda_t)$ instead of $\mu_s(S_t,\Lambda_t)$, because the distance functions of the family of convex bodies in Lemma 3.1.3 need to converge on the unit ball. Now, since s-covering radii are independent of translation, the sequence $\mu_s(S_t,\Lambda_t)$ converges to $\mu_s(S(\boldsymbol{\alpha}),\Lambda(\boldsymbol{\alpha}))$. It follows that,

$$\mu_s(S(\boldsymbol{\alpha}),\Lambda(\boldsymbol{\alpha})) = \mu_s(D(\boldsymbol{\alpha})^{-1}S(\boldsymbol{\alpha}),D(\boldsymbol{\alpha})^{-1}\Lambda(\boldsymbol{\alpha})) = \mu_s(S^{k-1},\mathbb{Z}^{k-1})\,.$$

Therefore, using Lemma 3.1.2, we have

$$\begin{aligned}\mu_s(S_t,\Lambda_t) - \mu_1(S_t,\Lambda_t) \to \mu_s(S^{k-1},\mathbb{Z}^{k-1}) &- \mu_1(S^{k-1},\mathbb{Z}^{k-1})\\ &= p(k-1,s)\,,\end{aligned}$$

as $t\to\infty$. Therefore, by Theorem 2.5.4, (3.1.7) and (3.1.5), we obtain

$$\begin{aligned}\tau_s(\boldsymbol{a}(t)) = \frac{F_s(\boldsymbol{a}(t)) - F_1(\boldsymbol{a}(t))}{\Pi(\boldsymbol{a}(t))^{\frac{1}{k-1}}} &= \frac{\mu_s(S_{\boldsymbol{a}(t)},\Lambda_{\boldsymbol{a}(t)}) - \mu_1(S_{\boldsymbol{a}(t)},\Lambda_{\boldsymbol{a}(t)})}{\Pi(\boldsymbol{a}(t))^{\frac{1}{k-1}}}\\ &= \frac{\Pi(\boldsymbol{\alpha})^{1/(k-1)}a_k(t)^{k/(k-1)}(\mu_s(S_t,\Lambda_t) - \mu_1(S_t,\Lambda_t))}{\Pi(\boldsymbol{a}(t))^{\frac{1}{k-1}}}\\ &= \frac{\Pi(\boldsymbol{\alpha})^{1/(k-1)}(\mu_s(S_t,\Lambda_t) - \mu_1(S_t,\Lambda_t))}{\left(\prod_{i=1}^{k-1}\alpha_i(t)\right)^{\frac{1}{k-1}}} \to p(k-1,s)\,,\end{aligned}$$

as $t\to\infty$. In conjunction with (3.1.5) this completes the proof of Theorem 3.1.1, and hence the result.

3.2 An upper bound for $c(k,s)$

The exact values of the constants $c(k,s)$ remain unknown apart of the case $c(2,s) = s-1$, which follows from (2.5.2). In this section we give a new upper bound for the case $s = 2$. The main result of this chapter is the following theorem.

Theorem 3.2.1. *Let $k \geq 3$. Then*

$$c(k,2) \leq 2\left(\frac{(k-1)!}{\binom{2(k-1)}{k-1}}\right)^{\frac{1}{k-1}}. \tag{3.2.1}$$

Theorem 3.2.1 improves (3.0.4) with the factor

$$f(k) = 2\binom{2(k-1)}{k-1}^{-\frac{1}{k-1}}.$$

The asymptotic behavior and bounds for $f(k)$ can be easily derived from results on extensively studied *Catalan numbers* $C_d = (d+1)^{-1}\binom{2d}{d}$, see for example [35].
In particular,

$$f(k) < \frac{1}{2}(4\pi(k-1)^2/(4(k-1)-1))^{1/(2(k-1))} < 0.82\,,$$

as illustrated in Figure 3.1.

Using Maple we obtain the asymptotic expansion of $f(k)$,

$$f(k) = \frac{1}{2} + \frac{\log k}{4k} + \frac{\log \pi}{4k} + o\left(\frac{1}{k}\right), \quad k \to \infty\,.$$

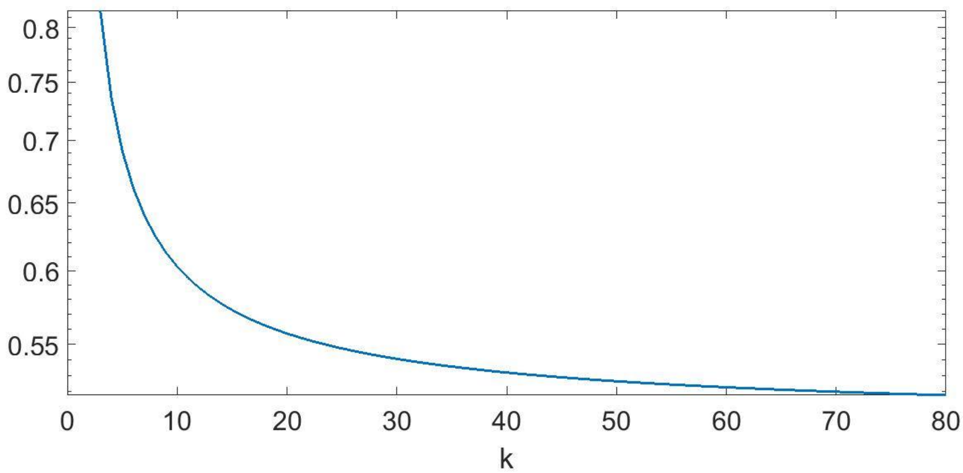

Figure 3.1: The function $f(k)$ for for $k = 3, \ldots, k$

The proof of Theorem 3.2.1 is based on the geometric approach used in [2], combined with results on the difference bodies dated back to works of Minkowski (see e.g. Gruber [45], Section

30.1) and Rogers and Shephard [79]. Let $K \in \mathcal{K}^k$. The *difference body* of K, denoted by D_K, is the origin-symmetric convex body defined as

$$D_K = K - K = K + (-K) = \{x - y : x \in K, y \in K\}.$$

It is well known that D_K can equivalently be described as follows,

$$D_K := \{x \in \mathbb{R}^k : K \cap (K + x) \neq \varnothing\}.$$

In 1957 Rogers and Shephard [79] inequality states that, for every k-dimensional convex body,

$$\mathrm{vol}\,(D_K) \leq \binom{2k}{k} \mathrm{vol}\,(K). \tag{3.2.2}$$

This inequality is sharp; indeed, it becomes an equality if and only if K is a simplex.

The proof of Theorem 3.2.1 is based on a link between lattice coverings with multiplicity at least two with usual lattice coverings and packings of convex bodies. Following the classical approach of Minkowski, we will use difference bodies and successive minima in our work with lattice packings.

Lemma 3.2.2. *Let $\Lambda \in \mathcal{L}^k$ and $K \in \mathcal{K}^k$. Then*

$$\mu_2(K, \Lambda) \leq \mu_1(K, \Lambda) + \lambda_1(D_K, \Lambda).$$

Proof. By (2.5.7) there exists a nonzero point $\boldsymbol{u} \in \Lambda$ in the set $\lambda_1 D_K$, where $\lambda_1 = \lambda_1(D_K, \Lambda)$. Then, by the definition of difference body, there exists a point $\boldsymbol{v} \in \mathbb{R}^k$ in the intersection $\lambda_1 K \cap (\boldsymbol{u} + \lambda_1 K)$. Indeed, $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$ with $\boldsymbol{u}_1, \boldsymbol{u}_2 \in \lambda_1 K$ and hence we can take

$$\boldsymbol{v} := \boldsymbol{u}_1 = \boldsymbol{u} + \boldsymbol{u}_2 \in \lambda_1 K \cap (\boldsymbol{u} + \lambda_1 K).$$

Next, given an arbitrary point $\boldsymbol{x} \in \mathbb{R}^k$ we know by the definition of the covering radius $\mu_1 = \mu_1(K, \Lambda)$ that there exists a point $\boldsymbol{z} \in \Lambda$ such that $\boldsymbol{x} - \boldsymbol{v} \in \boldsymbol{z} + \mu_1 K$.
Hence $\boldsymbol{x} \in \boldsymbol{v} + \boldsymbol{z} + \mu_1 K$, so that

$$\begin{aligned} &\boldsymbol{x} \in \boldsymbol{z} + (\mu_1 + \lambda_1)\, K \ \text{ and} \\ &\boldsymbol{x} \in \boldsymbol{z} + \boldsymbol{u} + (\mu_1 + \lambda_1)\, K, \end{aligned}$$

and we have that $\boldsymbol{x}$ is covered with multiplicity at least two by $(\mu_1 + \lambda_1)\, K + \Lambda$.
Therefore

$$\mu_2(K, \Lambda) \leq \mu_1 + \lambda_1\,,$$

as required. □

3.2.1 Proof of Theorem 3.2.1

Let $\boldsymbol{\alpha} = (1/a_1, \dots, 1/a_{k-1})$ and let $\Gamma_{\boldsymbol{a}} = D(\boldsymbol{\alpha})\Lambda_{\boldsymbol{a}}$, where in notation of Subsection 3.2.1 we set $D(\boldsymbol{\alpha}) = \mathrm{diag}(\alpha_1^{-1}, \dots, \alpha_{k-1}^{-1}) = \mathrm{diag}(a_1, \dots, a_{k-1})$. Then $\Gamma_{\boldsymbol{a}}$ is the lattice of determinant

$$\det(\Gamma_{\boldsymbol{a}}) = |\det(D(\boldsymbol{\alpha}))|\ \det(\Lambda_{\boldsymbol{a}}) = \Pi(\boldsymbol{\alpha})^{-1}(a_k) = \Pi(\boldsymbol{a})$$

and since $S^{k-1} = D(\boldsymbol{\alpha})S_{\boldsymbol{a}}$ is the standard simplex of volume

$$\mathrm{vol}\,(S^{k-1}) = |\det(D(\boldsymbol{\alpha}))|\ \mathrm{vol}\,(S_{\boldsymbol{a}}) = \Pi(\boldsymbol{\alpha})^{-1}\left((k-1)!\prod_{i=1}^{k-1} a_i\right)^{-1} = ((k-1)!)^{-1},$$

we have

$$\mu_s(S_{\boldsymbol{a}}, \Lambda_{\boldsymbol{a}}) = \mu_s(S^{k-1}, \Gamma_{\boldsymbol{a}})\,. \tag{3.2.3}$$

Combining Theorem 2.5.4 and Lemma 3.2.2, together with (3.2.3), with $s = 2$ we obtain

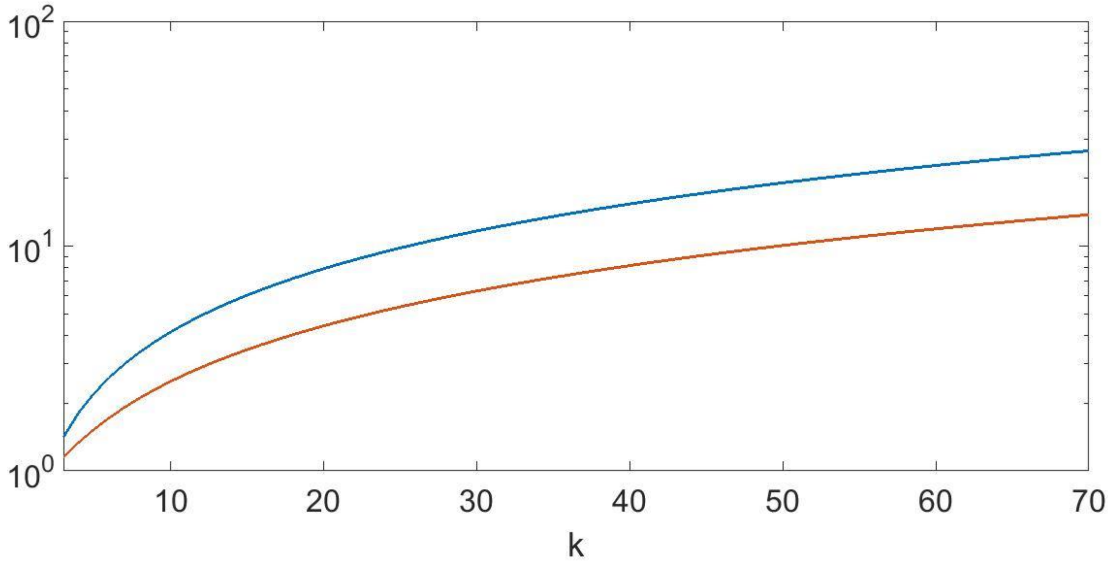

Figure 3.2: Comparison of the constants in the upper bound (3.2.7) (Orange) and in the upper bound (3.0.2) (Blue) with $s = 2$ for $k = 3, \dots, 70$

$$\frac{F_2(\boldsymbol{a}) - F_1(\boldsymbol{a})}{\Pi(\boldsymbol{a})^{\frac{1}{k-1}}} = \frac{\mu_2(S^{k-1}, \Gamma_{\boldsymbol{a}}) - \mu_1(S^{k-1}, \Gamma_{\boldsymbol{a}})}{\Pi(\boldsymbol{a})^{\frac{1}{k-1}}} \le \frac{\lambda_1(D_{S^{k-1}}, \Gamma_{\boldsymbol{a}})}{\Pi(\boldsymbol{a})^{\frac{1}{k-1}}}\,. \tag{3.2.4}$$

As was shown by Rogers and Shephard [79], the volume of a difference body $D_{S^{k-1}}$ is,

$$\mathrm{vol}\,(D_{S^{k-1}}) = \binom{2(k-1)}{k-1}\mathrm{vol}\,(S^{k-1}) = \binom{2(k-1)}{k-1}\Big/(k-1)!\,. \tag{3.2.5}$$

Hence, by Minkowski's second fundamental theorem (2.5.9), we deduce the inequality

$$\lambda_1(D_{S^{k-1}}, \Gamma_{\boldsymbol{a}}) \leq 2\left(\frac{\det(\Gamma_{\boldsymbol{a}})}{\mathrm{vol}\,(D_{S^{k-1}})}\right)^{\frac{1}{k-1}} = 2\left(\frac{(k-1)!}{\binom{2(k-1)}{k-1}}\right)^{\frac{1}{k-1}} \Pi(\boldsymbol{a})^{\frac{1}{k-1}}\,, \tag{3.2.6}$$

and combining (3.2.4), (3.2.5) and (3.2.6), we obtain the bound (3.2.1). See Figure 3.2. Therefore, we have

$$F_2(\boldsymbol{a}) - F_1(\boldsymbol{a}) \leq 2\left(\frac{(k-1)!}{\binom{2(k-1)}{k-1}}\right)^{\frac{1}{k-1}} \Pi(\boldsymbol{a})^{\frac{1}{k-1}}\,. \tag{3.2.7}$$

Remark 3.2.3. The results contained in this chapter, have been published the paper entitled "On the distance between Frobenius numbers", Moscow Journal of Combinatorics and Number Theory, **5** (2015), $No.4$, $3-12$.

Chapter 4

Frobenius numbers and graph theory

In the present chapter we provide an overview of the theory of graphs, and introduce some of the tools and concepts that will be employed throughout the latter part of the thesis. This includes the Nijenhuis's algorithm to determine the Frobenius number and known formula for the 2-Frobenius number of two coprime positive integers. In Section 4.1 we introduce some under planning notation and graph theoretic properties relevant to our work. In Section 4.2 we define the graph used in the Nijenhuis model, which we call a directed circulant graph and describe some of their properties, examining how they relate with the Frobenius numbers. In particular, we focus on the connectivity and the diameter. In Section 4.3 we apply graph theoretic techniques developed in order to construct a new proof for the formula of $\mathrm{F}_2(a_1, a_2)$ where $\gcd(a_1, a_2) = 1$.

4.1 Elements of graph theory

Let us begin by introducing some fundamental concepts and outlining the theory underpinning weighted directed graphs. The material presented here can be found in many introductory textbooks on graph theory (for example see [47, 10, 96, 94]).

Graphs

A *graph* is a pair $\mathcal{G} = (V, \mathcal{E})$, consisting of a nonempty finite set V of elements called *vertices* (or *points*) and a finite subset

$$\mathcal{E} \subseteq V \times V = \{\{u, v\} : u \, \text{and} \, v \in V, u \neq v\},$$

of unordered pairs of distinct vertices of V called *edges* (or *lines*). Graphs are so named since they can be viewed graphically, and this graphical representation helps us to understand and investigate many of their properties. An edge $\{u, v\}$ is said to *join* the vertices u and v, and is commonly abbreviated to uv or vu. The vertices u and v are called the *endvertices* of the edge uv. If $\varepsilon = uv \in \mathcal{E}(\mathcal{G})$, then u and v are said to be *adjacent* (or *neighbours*) vertices of $\mathcal{G}$ and the edge ε is said to be *incident* with the vertices u and v. Two edges are said to be *adjacent* if they have exactly one common endvertex. Graphs can have weights or other values associated with different properties of either the vertices or the edges, or both of these.

Directed graphs

A *directed graph* (sometimes referred to as *digraph*) is a pair $G = (V, E)$, consisting of a nonempty finite set V of vertices and a finite subset $\mathcal{E} \subseteq V \times V = \{(u, v) : u \, \text{and} \, v \in V, u \neq v\}$, of ordered pairs of distinct vertices of V called *arcs* (or *directed edges*). The *vertex set* of a digraph G is referred to as $V(G)$, its *arc set* as $E(G)$.

The *order* of G is defined to be the cardinality of its vertex set, $\#(V(G))$, whereas the *size* of G is defined to be the cardinality of its arc set, $\#(E(G))$.

We write $u \to v$, or (u, v), for the arc directed from u to v. Here u is the *initial vertex* and v is the *terminal vertex* of e. Moreover, u is said to be *adjacent to* v and v is said to be *adjacent from* u.

For a vertex $v \in V(G)$, the *out-neighbourhood* $N_G^+(v)$ of v is the set of out-neighbours of v in G; $N_G^+(v) = \{u \in V : (v, u) \in E\}$ and the *in-neighbourhood* $N_G^-(v)$ of v is the set of in-neighbours of v in G; $N_G^-(v) = \{u \in V : (u, v) \in E\}$. The *neighbourhood* $N_G(v)$ of a vertex v is given by

$$N_G(v) = N_G^+(v) \cup N_G^-(v).$$

The *out-degree* $\deg_G^+(v)$ and the *in-degree* $\deg_G^-(v)$ of a vertex $v \in V(G)$ are defined to be the cardinality of $N_G^+(v)$ and $N_G^-(v)$, respectively. The degree $\deg_G(v)$ of a vertex v is the cardinality of $N_G(v)$ and is given by

$$\deg_G(v) = \deg_G^+(v) + \deg_G^-(v).$$

A $u - v$ *directed path* in a directed G is a finite sequence

$$u = v_0, e_1, v_1, e_2, \dots, e_n, v_n = v,$$

of vertices and arcs, beginning with u and ending with v such that $e_i = (v_{i-1}, v_i) \in E(G)$ for $i = 1, 2, \dots, n$. The vertices u and v are called its endvertices. Note that a path may consist of a single vertex, in which case both endvertices are the same. The length of the path is the number arcs it contains, that is a $u - v$ path of length n. The path $v_0, e_1, v_1, e_2, \dots, e_n, v_n$ is said to be *simple* if there are no repeated vertices in the path, (except possibly that the initial vertex v_0 can be equal to the terminal vertex v_n).

A directed graph G is said to be *strongly connected* (resp. *connected*) if, for any two vertices v and w of G, there is a directed path (resp. path) from v to w. Consequently one finds that every strongly connected digraph is connected, but not all connected digraphs are strongly connected.

Weighted directed graphs

A *weighted directed graph* (or *weighted digraph*) $G_w = (V, E; w)$ is a directed graph (V, E) associated with a weight function $w : E \to \mathbb{R}^+$ that assigns a positive real value $w(e)$ with each arc $e \in E$, called its weight (or length). Weights can represent costs, times or capacities, etc., depending on the problem. Figure 4.1 shows an example of a weighted digraph.

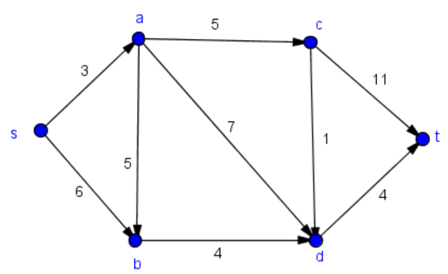

Figure 4.1: A weighted digraph with positive integer weights

The *length* (or *weight*) $w(p)$ of the $v_0 - v_k$ path $p = v_0, e_1, v_1, e_2, \cdots, e_k, v_k$ in G_w, is the sum of the weights on its arcs. That is

$$w(p) = \sum_{i=1}^{k} w(e_i)\,. \tag{4.1.1}$$

For any two vertices $u, v \in V$, the *shortest* (or *minimum*) $u-v$ path in G_w is a path whose weight is minimum among all $u-v$ paths. For example, Figure 4.2 shows the minimum (shortest) path from vertex s to vertex t.

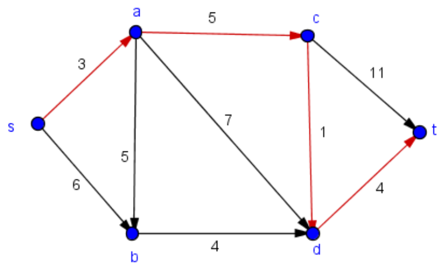

Figure 4.2: The shortest path from vertex s to vertex t

The *distance* (or *minimum distance*) $d_{G_w}(u, v)$ between two vertices u and v in a connected graph G_w is defined to be the weight of a shortest $u - v$ path. That is

$$d_{G_w}(u, v) = \begin{cases} \min\{w(p)\} & \text{if there is a } u - v \text{ path p,} \\ \infty & \text{otherwise.} \end{cases}$$

The *diameter* $\mathrm{diam}(G_w)$ of a connected graph G_w is defined to be the longest distance between any pair of vertices in G_w, so that

$$\mathrm{diam}(G_w) = \max_{i,j \in V(G_w)} d_{G_w}(i, j).$$

4.2 The Frobenius numbers and directed circulant graphs

In this section we consider properties of directed circulant graphs and we describe the relationship that exists between the Frobenius numbers and the diameters of directed circulant graphs.

The circulant graph is a natural generalisation of the *double-loop network*, which was first introduced by C.K. Wong and Don Coppersmith [97] in 1974, for organizing multimodule memory services. The term directed circulant graph was proposed by Elspas and Turner [37] with the weight function defined on the edges as described above. A directed circulant graph can be constructed as follows. Given a positive integer vector $\boldsymbol{a} = (a_1, \dots, a_k)^t$ with $1 < a_1 < \cdots < a_k$, the *directed circulant graph* (circulant digraph for short), $G_w(\boldsymbol{a})$, is defined to be a weighted directed graph with a_1 vertices labelled by $0, 1, \dots, a_1 - 1$ corresponding to the residue classes of integers modulo a_1, where for each vertex i, $(0 \le i \le a_1 - 1)$, there is an arc $i \to i + a_j \pmod{a_1}$ with weight $w_j = a_j$, for all $j = 2, \dots, k$. That is a directed circulant graph $G_w(\boldsymbol{a})$ is a graph with the vertex set

$$V(G_w(\boldsymbol{a})) = \mathbb{Z}_{a_1} = \{0, 1 \dots, a_1 - 1\},$$

and the arc set

$$E(G_w(\boldsymbol{a})) = \{(x, y) : \exists\, a_j, 2 \le j \le k \text{ such that } x + a_j \equiv y \pmod{a_1}\}.$$

Figure 4.3 shows two examples of the circulant digraphs $G_w(6, 8)$ and $G_w(11, 13, 14)$.

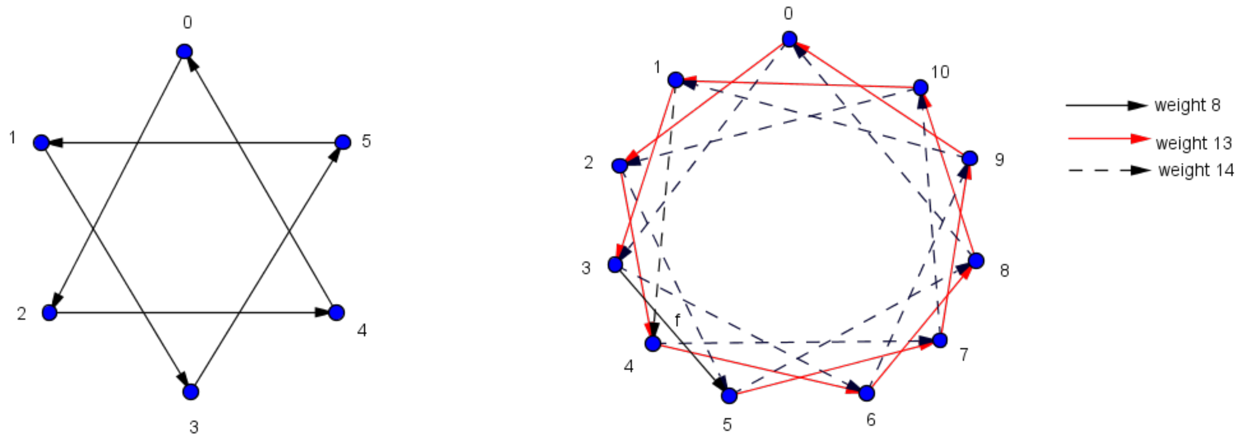

Figure 4.3: The circulant digraphs $G_w(6, 8)$ (left) and $G_w(11, 13, 14)$ (right)

The circulant digraphs $G_w(a_1, \dots, a_k)$ are the *Cayley digraphs* [89] over the cyclic group $\mathbb{Z}_{a_1}$ with respect to the generating set $\{a_2, \dots, a_k\}$. Circulant digraphs, also known as *Frobenius circulant graphs* in the literature [19].

In the literature [8, 33, 11] on circulant digraphs, the following definition and notation are also commonly used. Let $S = \{s_1, \dots, s_k\}$ be a set of integers such that $0 < s_1 < \cdots < s_k < n$. Then the circulant digraph $C_n(S)$ is defined to be the weighted digraph of order n with vertex set $V(C_n(S)) = \mathbb{Z}_n$ and edge set

$$E(C_n(S)) = \{(x, x + s_i \pmod{n}),\, x \in V(C_n(S)),\, 1 \le i \le k\}.$$

The set $\{s_1, s_2, \ldots, s_k\}$ is called a *connection set* of the graph $C_n(S)$.

The graph-theoretical properties of these graphs have been studied in several papers, e.g. in [8, 49, 62] and [91].

In 1974, Boesch and Tindell [22] obtained the following proposition, which gives a sufficient condition for circulant digraphs to be *strongly connected*.

Proposition 4.2.1. *If* $\gcd(a_1, \ldots, a_k) = d$ *then* $G_w(a_1, \ldots, a_k)$ *has* d *components. In particular,* $G_w(a_1, \ldots, a_k)$ *is strongly connected if and only if* $\gcd(a_1, \ldots, a_k) = 1$.

We refer to Boesch and Tindell [22] for further results concerning connectivity of circulant graphs. (See also [100, 98]).

Henceforth in this work we assume that our directed circulant graphs are strongly connected. Furthermore, it follows that every vertex of the circulant digraph $G_w(a_1, \ldots, a_k)$ has precisely $(k-1)$ *out-neighbours* and $(k-1)$ *in-neighbours*. Here the *neighbourhood* of any vertex i of $G_w(\boldsymbol{a})$ is given by

$$\{i \pm a_j \pmod{a_1} : \text{for } j = 2, 3, \ldots, k\}.$$

As we can observe from Figure 4.3, the neighbourhood of the vertex 4 of $G_w(11, 13, 14)$ is the set $\{1, 2, 6, 7\}$ of vertices.

An important concept employed is that given any two vertices r and s, an $r-s$ path in $G_w(\boldsymbol{a})$, can be associated with the integer vector $(\sigma_2, \sigma_3, \ldots, \sigma_k)^t \in \mathbb{Z}_{\geq 0}^{k-1}$, such that

$$\sum_{j=2}^{k} a_j \sigma_j \equiv s - r \pmod{a_1}, \tag{4.2.1}$$

(see for example [29, 19]).

In other words, σ_j is the number of arcs of weight a_j in a path from r to s. For each vertex v, the path that starts from vertex 0 to vertex v is called a *minimum path* (or *shortest path*) to vertex v if the weight of the path is minimum among all paths from 0 to v. This means that, from (4.2.1) we can determine the endvertex v for any path that starts at vertex 0 such that

$$\sum_{j=2}^{k} a_j \sigma_j \equiv v \pmod{a_1}. \tag{4.2.2}$$

It can be seen that, the total weight w of the path in $G_w(\boldsymbol{a})$ that starts at vertex 0 to vertex v

is given by

$$w = \sum_{j=2}^{k} a_j \sigma_j \equiv v \pmod{a_1}. \tag{4.2.3}$$

Let S_v be the *minimum weight of any path* (or *weight of any minimum path*) from vertex 0 to v in $G_w(\boldsymbol{a})$. Then (4.2.3) gives us

$$S_v = \sum_{j=2}^{k} a_j \sigma_j \equiv v \pmod{a_1}. \tag{4.2.4}$$

Nijenhuis [66] showed that there exists a relation between a solution $(x_1, \ldots, x_k)$ in nonnegative integers to (2.2.1) and a path in a circulant digraph $G_w^+(\boldsymbol{a})$ related to $G_w(\boldsymbol{a})$ from vertex 0 to any other vertex v in $G_w^+(\boldsymbol{a})$. From this, Nijenhuis [66] established an algorithm to compute $\mathrm{F}(\boldsymbol{a})$, by constructing for all vertices v in $G_w^+(\boldsymbol{a})$, a path from vertex 0 to v of minimum weight S_v. Indeed

$$\mathrm{F}(\boldsymbol{a}) = \max_{v \in V(G_w^+(\boldsymbol{a}))} \{S_v\} - a_1. \tag{4.2.5}$$

In 2005, Beihoffer at el [19] used the approach of Nijenhuis [66] on the circulant digraph $G_w(\boldsymbol{a})$ to established a link between the Frobenius number $\mathrm{F}(\boldsymbol{a})$ and the diameter of $G_w(\boldsymbol{a})$. The following lemma is implicit in [19], Section two.

Lemma 4.2.2. *For any vertex v of $G_w(\boldsymbol{a})$ there is a positive integer M such that*

$$M \equiv v \;(\mathit{mod}\; a_1).$$

Then M is representable in terms of $\boldsymbol{a} = (a_1, \ldots, a_k)^t$ if and only if $M \geq S_v$.

Proof. Suppose that $M \geq S_v$. We need to show that M can be representable in terms $a_1, \ldots, a_k$. Since S_v is the minimum weight of a path from vertex 0 to vertex v in $G_w(\boldsymbol{a})$, then (4.2.4) gives us

$$v \equiv S_v \pmod{a_1}.$$

Thus we have

$$M \equiv v \equiv S_v \pmod{a_1}, \quad \text{and } M \geq S_v.$$

It follows that there exist a nonnegative integer t such that

$$M = S_v + ta_1 .$$

Hence, M is representable in terms of $a_1, \ldots, a_k$.

Conversely, now let M is representable in terms of $a_1, \ldots, a_k$. Then there exist nonnegative integers $x_1, \ldots, x_k$ such that

$$M = \sum_{j=1}^{k} a_j x_j . \tag{4.2.6}$$

Hence

$$M \equiv \sum_{j=2}^{k} a_j x_j \pmod{a_1} . \tag{4.2.7}$$

Since $M \equiv v \pmod{a_1}$, from (4.2.7) we have

$$M \equiv \sum_{j=2}^{k} a_j x_j \equiv v \pmod{a_1} .$$

Then it follows from (4.2.3) that we have a path from 0 to v of weight $\sum\limits_{j=2}^{k} a_j x_j$. Thus

$$\sum_{j=2}^{k} a_j x_j \geq S_v .$$

From (4.2.6), $M = a_1x_1 + a_2x_2 + \ldots, a_kx_k$, we get

$$M \geq \sum_{j=2}^{k} a_j x_j \geq S_v ,$$

as required. □

Therefore, we have shown that the largest integer $M \equiv v \pmod{a_1}$, for any v of $G_w(\boldsymbol{a})$, that is nonrepresentable as a nonnegative integer linear combination of $a_1, \ldots, a_k$ is given by

$$M = S_v - a_1 .$$

We know that the diameter of the circulant digraphs $G_w(\boldsymbol{a})$ is given by

$$\mathrm{diam}(G_w(\boldsymbol{a})) = \max_{v \in V(G_w(\boldsymbol{a}))} \{S_v\}, \tag{4.2.8}$$

for example see [29] or [78].

From formula (4.2.8) and applying Lemma 4.2.2, we obtain the following result [19, 78].

Corollary 4.2.3. *We have*

$$\mathrm{F}(a_1, \ldots, a_k) = \mathrm{diam}(G_w(\boldsymbol{a})) - a_1. \tag{4.2.9}$$

In the next chapter we will use the same approach of Beihoffer at el [19] to establish a link between the 2-Frobenius number for the arithmetic progression $a, a+d, a+2d$ with $\gcd(a,d) = 1$ and shortest paths from vertex 0 to any other vertex v in the circulant digraph (Frobenius circulant graph) associated with the positive integers a, $a+d$,$a+2d$.

4.3 Diameters of 2-circulant digraphs and the 2-Frobenius numbers

In view of (4.2.9), one can ask does these exits a relationship between the generalised Frobenius number $\mathrm{F}_s(\boldsymbol{a})$ and diameters of certain graphs.

At the time of writing this thesis the existence of an analogue for (4.2.9) in the generalised setting is still an open question. In this chapter we explore a link between $\mathrm{F}_2(\boldsymbol{a})$ for $k = 2$ and diameters of special graphs, which we call 2-*circulant digraphs.*

Our starting point is the formula $\mathrm{F}_s(a_1, a_2) = sa_1a_2 - (a_1 + a_2)$. In the classical setting when $s = 1$, the circulant digraph has a_1 vertices, so it is natural to extend it to a circulant digraph with $2a_1$ vertices when $s = 2$.

We note that the ideas developed in the course of this work have been further utilised in Chapters 5 and 6, where new results on 2-Frobenius numbers of vectors with entries in arithmetic sequences are established.

4.3.1 2-circulant digraphs

Consider two positive integers a_1, a_2 such that $a_1 > 1$, and $a_2 \equiv 1 \pmod 2$. A 2-*circulant digraph*, denoted $Circ(a_1, a_2)$, is defined to be a weighted digraph with $2a_1$ vertices labelled by $0, 1, \ldots, 2a_1 - 1$, corresponding to the residue classes of integers modulo $2a_1$. For each vertex i, $(0 \leq i \leq 2a_1 - 1)$, there is an arc $i \to i + a_2 \pmod{2a_1}$ with weight a_2. That is a 2-circulant digraph $Circ(a_1, a_2)$ is a graph with the vertex set

$$V(Circ(a_1, a_2)) = \mathbb{Z}_{2a_1} = \{0, 1 \ldots, 2a_1 - 1\},$$

and the arc set

$$E(Circ(a_1, a_2)) = \Big\{(i, (i + a_2) \pmod{2a_1}) : i \in V(Circ(a_1, a_2))\Big\}.$$

Example 4.3.1. Figure 4.4 shows the 2-circulant digraphs $Circ(5, 3)$ and $Circ(5, 2)$.

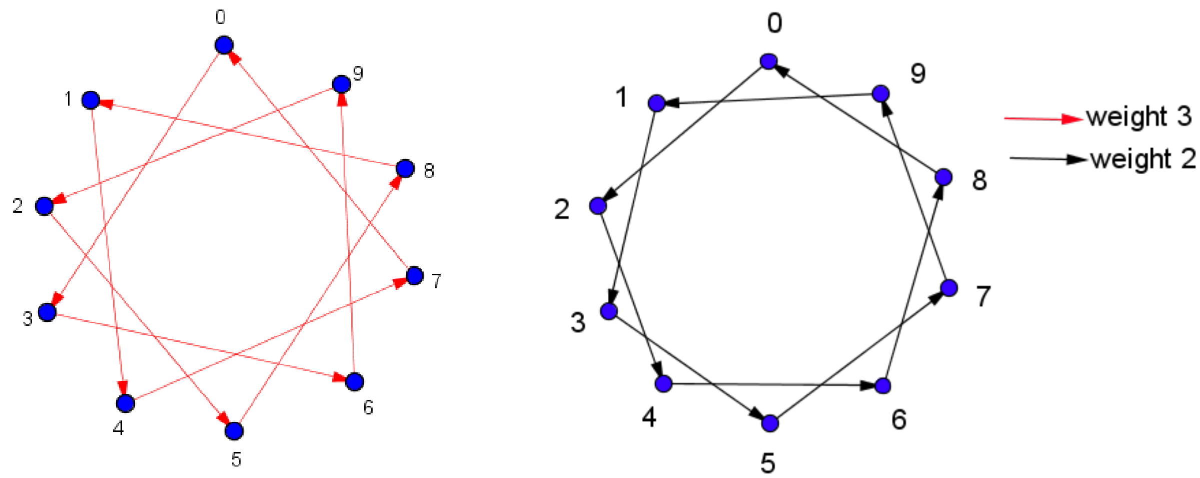

Figure 4.4: The 2-circulant digraphs Circ(5,3)(left) and Circ(5,2)(right) with arcs of weight 3 and 2, respectively

Moreover the *neighbourhood* for each vertex i in $Circ(a_1, a_2)$ is the set $\{i \pm a_2 \pmod{2a_1}\}$ of vertices.

Lemma 4.3.2. *A graph $Circ(a_1, a_2)$ is strongly connected if and only if* $\gcd(2a_1, a_2) = 1$.

Proof. The proof immediately follows from Proposition 4.2.1. $\square$

Given any two vertices r and s of $Circ(a_1, a_2)$, we denote by $y = y(p)$ the number of arcs in a $r - s$ path p of weight a_2, such that from (4.2.1), we have

$$a_2\, y(p) \equiv s - r \pmod{2a_1}.$$

Then by (4.1.1) and (4.2.3) it follows that the weight w of a $r - s$ path p is given by

$$w = a_2\, y(p) \equiv s - r \pmod{2a_1}. \tag{4.3.1}$$

Then in particular, one can determine the endvertex v for any path p that starts at vertex 0,

$$w = a_2\, y(p) \equiv v \pmod{2a_1}. \tag{4.3.2}$$

Let us assume that $\gcd(2a_1, a_2) = 1$, so that the graph $Circ(a_1, a_2)$ is connected. This condition ensures that a_2 is odd. For example as shown in Figure 4.4, since $a_1 = 5$ and $a_2 = 2$ such that $\gcd(10, 2) \neq 1$, then the graph $Circ(5, 2)$ will be disconnected. In such cases we will consider the graph with a_1 and a_2 swapped, namely $Circ(2, 5)$ as shown in Figure 4.5, thus covering this all possible cases for $\mathrm{F}_2(a_1, a_2)$. We can do this because the ordering of the positive integers in the Frobenius basis does not effect the s-Frobenius number $\mathrm{F}_s(a_1, \ldots, a_k)$ in general.

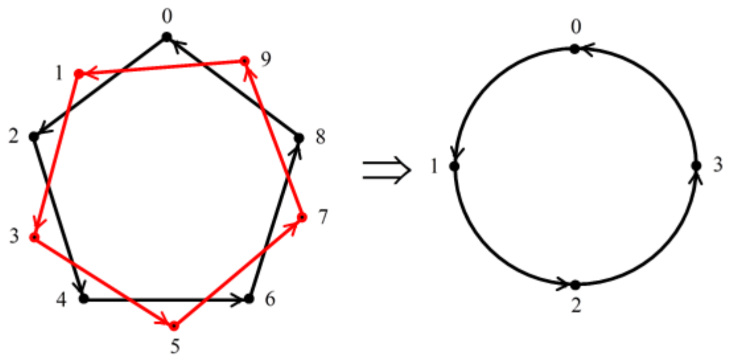

Figure 4.5: A swapped Frobenius basis for the two 2-circulant digraphs Circ(5,2) (left) and Circ(2,5) (right)

The connectedness property enables us to order the vertices of the 2-circulant digraph $Circ(a_1, a_2)$ in the order $v_0, v_1, \ldots, v_{2a_1-1}$, moving in an anti-clockwise direction around the graph $Circ(a_1, a_2)$, as shown in Figure 4.6. Here we have

$$v_j \equiv ja_2 \pmod{2a_1}, \qquad \text{for } 0 \leq j \leq 2a_1 - 1. \tag{4.3.3}$$

And, the minimum weight S_{v_j} of any path from 0 to v_j in $Circ(a_1, a_2)$ with $0 \leq j \leq 2a_1 - 1$ is defined by

$$S_{v_j} = ja_2\,. \tag{4.3.4}$$

Hence from (4.3.2) and (4.3.3), we find that

$$S_{v_j} \equiv v_j \pmod{a_1}. \tag{4.3.5}$$

It can be seen that by (4.2.8) and (4.3.4), the diameter of $Circ(a_1, a_2)$ is given by

$$\begin{aligned} \operatorname{diam}(Circ(a_1, a_2)) = \max_{0 \le j \le 2a_1 - 1} S_{v_j} &= S_{v_{2a_1-1}} \\ &= (2a_1 - 1)a_2 \,. \end{aligned} \tag{4.3.6}$$

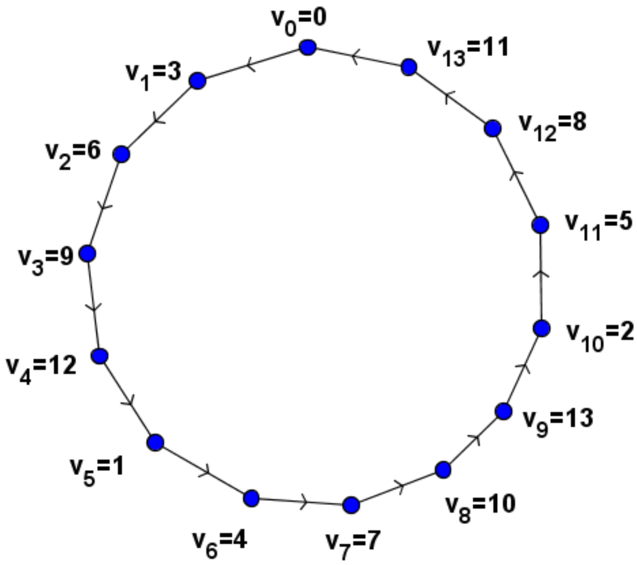

Figure 4.6: Circ(7,3) with 14 arcs of weight 3

The condition $\gcd(a_1, a_2) = 1$, imply that the minimum weight S_{v_j} of a path from 0 to v_j given by (4.3.4), can be represented exactly one way as a nonnegative integer linear combination of a_1, a_2 when $0 \le j \le a_1 - 1$.

With regard the remaining vertices v_j with $a_1 \le j \le 2a_1 - 1$, we consider the vertex v_{a_1+h} with $0 \le h \le a_1 - 1$. In this instance, the minimum weight $S_{v_{a_1+h}}$ of a path from 0 to v_j, can be represented in exactly two distinct ways as a nonnegative integer linear combination of a_1, a_2 such that

$$S_{v_{a_1+h}} = (a_1 + h)a_2 = a_2 a_1 + h a_2 \,. \tag{4.3.7}$$

4.3.2 An expression for 2-Frobenius numbers

Here, we obtain a formula for the 2-Frobenius number by using the diameter of $(Circ(a_1, a_2))$. We note that a general formula for the 2-Frobenius number $\mathrm{F}_2(a_1, a_2)$ is well known. The main challenge in this part of our work is to understand the relationship that exists between representations of nonnegative integer in terms a_1, a_2 and the shortest path in $Circ(a_1, a_2)$. From this we establish the formula $\mathrm{F}_2(a_1, a_2) = 2a_1a_2 - a_1 - a_2$, using only the properties of the graph $Circ(a_1, a_2)$.

Theorem 4.3.3. *Let a_1, a_2 be positive integers with $a_2 \equiv 1 \ (mod\ 2)$ and $\gcd(2a_1, a_2) = 1$. Then*

$$\mathrm{F}_2(a_1, a_2) = \mathrm{diam}(Circ(a_1, a_2)) - a_1. \tag{4.3.8}$$

Proof. Let v_j be any vertex of $Circ(a_1, a_2)$ with $0 \le j \le 2a_1 - 1$ and let M be a positive integer, such that

$$M \equiv v_j - a_1 \pmod{2a_1}. \tag{4.3.9}$$

To prove Theorem 4.3.3 we need the following two lemmas.

Lemma 4.3.4. *Let $0 \le j \le a_1 - 1$. Then the positive integer $M \equiv v_j - a_1 \ (mod\ 2a_1)$ is representable in at least two distinct ways as a nonnegative integer linear combination of a_1 and a_2 if and only if $M \ge S_{v_{j+a_1}}$.*

Proof. Suppose that $M \ge S_{v_{j+a_1}}$. We have to show that M is represented in at least two distinct ways.

First, we will show that

$$M \equiv S_{v_{j+a_1}} \pmod{2a_1}.$$

By (4.3.5) we have $v_j \equiv S_{v_j} \pmod{2a_1}$ so that $v_j - 2a_1 \equiv S_{v_j} \pmod{2a_1}$. Hence, there is a nonnegative integer t such that

$$v_j - a_1 = ja_2 + a_1 + t(2a_1)$$

and adding $0 = a_1a_2 - a_1a_2$ to the right hand side of the above equation, gives us

$$v_j - a_1 = a_2(j - a_1) + a_1(a_2 + 1) + t(2a_1).$$

Since a_2 is odd, we can write $a_2 + 1 = 2b$ for some positive integer b. Hence

$$v_j - a_1 = a_2(j - a_1) + a_1(2b) + t(2a_1)\,,$$

and so

$$v_j - a_1 \equiv a_2(j - a_1) \equiv a_2(j + a_1) \equiv S_{v_{j+a_1}} \pmod{2a_1}. \tag{4.3.10}$$

Thus, we have

$$M \equiv v_j - a_1 \equiv S_{v_{j+a_1}} \pmod{2a_1} \qquad \text{and} \qquad M \geq S_{v_{j+a_1}}.$$

Consequently, there exists a nonnegative integer t such that

$$M = S_{v_{j+a_1}} + t(2a_1) = (j + a_1)a_2 + t(2a_1)\,.$$

By (4.3.7), we deduce that M is represented in at least two distinct ways as a nonnegative integer linear combination of a_1 and a_2.

Conversely, now suppose that M has at least two distinct representations in terms of a_1, a_2. Then there exists nonnegative integers x_1, y_1, x_2, y_2 with $x_1 \neq x_2, y_1 \neq y_2$ such that

$$M = a_1x_1 + a_2y_1 = a_1x_2 + a_2y_2\,. \tag{4.3.11}$$

Now we consider the cases when x_1 and x_2 both odd, both even and when x_1 and x_2 are of opposite parity.

If x_1 **and** x_2 **are both odd**, then we may write $x_1 = 2X_1 + 1$ and $x_2 = 2X_2 + 1$, to obtain

$$M = a_1(2X_1 + 1) + a_2y_1 = a_1(2X_2 + 1) + a_2y_2\,,$$

for some nonnegative integers X_1 and X_2. We have

$$M \equiv a_1 + a_2y_1 \equiv a_1 + a_2y_2 \pmod{2a_1},$$

which implies

$$y_1 \equiv y_2 \pmod{2a_1}\,.$$

Without loss of generality, we may assume that $y_2 > y_1$. Then there exists $t \in \mathbb{Z}_{>0}$ such that $y_2 = y_1 + t(2a_1)$ and thus $y_2 \geq 2a_1$. By (4.3.11), $M = a_1x_2 + a_2y_2$, we have

$$M \geq a_1 + 2a_1a_2 > S_{v_{j+a_1}}\,,$$

as required.

Alternatively, **if x_1 and x_2 are both even**, then we may write $x_1 = 2X_1$ and $x_2 = 2X_2$, to obtain

$$M = a_1(2X_1) + a_2y_1 = a_1(2X_2) + a_2y_2\,,$$

for some nonnegative integers X_1 and X_2. We have

$$M \equiv a_2y_1 \equiv a_2y_2 \pmod{2a_1}\,,$$

and hence

$$y_1 \equiv y_2 \pmod{2a_1}\,.$$

Without loss of generality, we may therefore assume that $y_2 > y_1$, so that $y_2 = y_1 + t(2a_1)$, for some $t \in \mathbb{Z}_{>0}$. Thus, similar to the previous case, $y_2 \geq 2a_1$ and

$$M \geq 2a_1a_2 > S_{v_{j+a_1}}\,,$$

as required.

If **x_1 is even and x_2 is odd**, then we can write $x_1 = 2X_1$ and $x_2 = 2X_2 + 1$, to obtain

$$M = a_1(2X_1) + a_2y_1 = a_1(2X_2 + 1) + a_2y_2\,,$$

for some nonnegative integers X_1 and X_2. So that by (4.3.9),

$$M \equiv a_2y_1 \equiv a_1 + a_2y_2 \equiv v_j - a_1 \equiv ja_2 - a_1 \pmod{2a_1}\,. \tag{4.3.12}$$

Since a_2 is odd, we find that

$$y_1 \equiv j - a_1 \pmod{2a_1} \tag{4.3.13}$$

is the solution to (4.3.12).

Since $0 \leq j \leq a - 1$ and $y_1 \in \mathbb{Z}_{\geq 0}$, we must have $y_1 > j - a_1$ in (4.3.13). Therefore there exist a positive integer k such that

$$y_1 = j - a_1 + k(2a_1)\,.$$

and, consequently

$$y_1 \geq j + a_1\,.$$

Since $M = a_1x_1 + a_2y_1$, we have

$$M \geq a_2y_1 \geq a_2(j + a_1) = S_{v_{j+a_1}}\,.$$

The case when **x_1 is odd and x_2 is even** then follows by symmetry. □

Lemma 4.3.4 implies that the largest integer $M \equiv v_j - a_1 \pmod{2a_1}$ with $0 \leq j \leq a_1 - 1$, that is nonrepresentable in at least two distinct ways as a nonnegative integer linear combination of a_1 and a_2 is given by

$$M = S_{v_{j+a_1}} - 2a_1 = (j + a_1)a_2 - 2a_1 \,.$$

Since in this case, $j_{\max} = a_1 - 1$, we find that

$$(j + a_1)a_2 - 2a_1 \leq (j_{\max} + a_1)a_2 - 2a_1 = (2a_1 - 1)a_2 - 2a_1 \,.$$

Then using formula (4.3.6), we get

$$S_{v_{j+a_1}} - 2a_1 \leq \operatorname{diam}(Circ(a_1, a_2)) - 2a_1 \,.$$

Lemma 4.3.5. *Let $a_1 \leq j \leq 2a_1 - 1$. Then the positive integer $M \equiv v_j - a_1 \pmod{2a_1}$ is representable in at least two distinct ways as a nonnegative integer linear combination of a_1 and a_2 if and only if $M \geq S_{v_{j-a_1}} + a_1(a_2 + 1)$.*

Proof. Suppose that $M \geq S_{v_{j-a_1}} + a_1(a_2 + 1)$. We need to show that M can be represented in at least two distinct ways. Recall that

$$v_j \equiv S_{v_j} \pmod{2a_1} \,.$$

Then using (4.3.10), we have

$$v_j - a_1 \equiv S_{v_{j-a_1}} \pmod{2a_1} \,.$$

Since $a_2 + 1$ is always even, we can write the above congruence as follows:

$$v_j - a_1 \equiv S_{v_{j-a_1}} \equiv S_{v_{j-a_1}} + a_1(a_2 + 1) \pmod{2a_1} \,.$$

Thus we have

$$M \equiv v_j - a_1 \equiv S_{v_{j-a_1}} + a_1(a_2 + 1) \pmod{2a_1} \qquad \text{and} \qquad M \geq S_{v_{j-a_1}} + a_1(a_2 + 1) \,.$$

It follow that there exist a nonnegative integer t such that

$$M = S_{v_{j-a_1}} + a_1(a_2 + 1) + t(2a_1) = ja_2 + a_1 + t(2a_1) \,.$$

Since $a_1 \leq j \leq 2a_1 - 1$, then it follows that M is represented in at least two distinct ways as a nonnegative integer linear combination of a_1 and a_2.

Conversely, now suppose that M has at least two distinct representations in terms of a_1 and a_2. Then there exists nonnegative integers x_1, y_1, x_2, y_2 with $x_1 \neq x_2, y_1 \neq y_2$ such that

$$M = a_1x_1 + a_2y_1 = a_1x_2 + a_2y_2\,. \tag{4.3.14}$$

We again consider the cases when x_1 and x_2 both odd, both even and when x_1 and x_2 are of opposite parity.

If x_1 **and** x_2 **both are odd**, then we may write $x_1 = 2X_1 + 1$ and $x_2 = 2X_2 + 1$, to obtain

$$M = a_1(2X_1 + 1) + a_2y_1 = a_1(2X_2 + 1) + a_2y_2\,,$$

for some nonnegative integers X_1 and X_2. Then

$$M \equiv a_1 + a_2y_1 \equiv a_1 + a_2y_2 \pmod{2a_1}\,,$$

and hence

$$y_1 \equiv y_2 \pmod{2a_1}\,.$$

Without loss of generality, we can assume that $y_1 > y_2$, so that there is a positive integer k such that

$$y_1 = y_2 + k(2a_1).$$

This implies that $y_1 \geq 2a_1$. Since x_1 and x_2 are odd, from (4.3.14), $M = a_1x_1 + a_2y_1$, we have

$$M \geq a_1 + 2a_1a_2 > S_{v_{j-a_1}} + a_1(a_2 + 1) \qquad \text{for } a_1 \leq j \leq 2a_1 - 1,$$

as required.

If x_1 **and** x_2 **both are even**, then we may write $x_1 = 2X_1$ and $x_2 = 2X_2$, to obtain

$$M = a_1(2X_1) + a_2y_1 = a_1(2X_2) + a_2y_2\,,$$

for some nonnegative integers X_1 and X_2. Then (4.3.9) gives

$$M \equiv a_2y_1 \equiv a_2y_2 \equiv ja_2 - a_1 \pmod{2a_1}\,.$$

Since a_2 is odd, we have

$$y_1 \equiv y_2 \equiv j - a_1 \pmod{2a_1}\,. \tag{4.3.15}$$

Assume without loss of generality that $y_1 > y_2$. Since $a_1 \leq j \leq 2a_1 - 1$ and $y_1, y_1 \in \mathbb{Z}_{\geq 0}$, then by a simple argument it can be seen that $y_1 > j - a_1$ in (4.3.15). Therefore there exits a positive integer t such that

$$y_1 = j - a_1 + t(2a_1)\,.$$

Thus, similarly to the previous case, we find that $y_1 \geq j + a_1$ and

$$M \geq (j + a_1)a_2 \geq S_{v_{j-a_1}} + a_1(a_2 + 1). \tag{4.3.16}$$

If x_1 **is odd and** x_2 **is even**, then by (4.3.9) we have

$$M \equiv a_1 + a_2 y_1 \equiv a_2 y_2 \equiv a_2(j - a_1) \pmod{2a_1}\,.$$

Hence

$$y_2 \equiv j - a_1 \pmod{2a_1}\,.$$

We will first consider the case $y_2 = j - a_1$, where we observe that since $a_1(x_1 - x_2) = a_2(y_2 - y_1)$ $\gcd(a_1, a_2) = 1$, we have

$$\begin{aligned} x_1 - x_2 &= t a_2\,, \\ y_2 - y_1 &= t a_1\,, \end{aligned} \tag{4.3.17}$$

with $t \in \mathbb{Z}$, $t \neq 0$.
Therefore, according to our assumption we have $j - a_1 - y_1 = t a_1$ and then $y_1 = j - a_1(1 + t)$. This implies that, $t \in \mathbb{Z}_{<0}$, so that

$$t = -q; \qquad q \in \mathbb{Z}_{>0}\,.$$

Consequently, by (4.3.17),

$$x_2 \geq 1 + q a_2 \geq 1 + a_2\,.$$

From (4.3.14), $M = a_1 x_2 + a_2 y_2$, we obtain

$$M \geq a_1(1 + a_2) + a_2(j - a_1) = S_{v_{j-a_1}} + a_1(a_2 + 1)\,.$$

Next, if $y_2 > j - a_1$, then there exist a positive integer s such that

$$y_2 = j - a_1 + s(2a_1)\,,$$

which implies

$$y_1 \geq j + a_1\,.$$

By (4.3.16), we deduce that

$$M > S_{v_{j-a_1}} + a_1(a_2+1)\,.$$

Finally, the case when x_1 **is even and** x_2 **is odd** follows by symmetry. $\square$

Lemma 4.3.5 implies that the largest integer $M \equiv v_j - a_1 \pmod{2a_1}$ with $a_1 \leq j \leq 2a_1 - 1$, that is nonrepresentable in at least two distinct ways as a nonnegative integer linear combination of a_1 and a_2 is given by

$$M = \Big(S_{v_{j-a_1}} + a_1(a_2+1)\Big) - 2a_1 = S_{v_{j-a_1}} + a_1(a_2-1)\,.$$

Since in this case, $j_{\max} = 2a_1 - 1$, we have

$$S_{v_{j-a_1}} + a_1(a_2-1) = ja_2 - a_1 \leq j_{\max}\, a_2 - a_1 = (2a_1-1)a_2 - a_1\,.$$

Hence from formula (4.3.6), we get

$$S_{v_{j-a_1}} + a_1(a_2-1) \leq \mathrm{diam}(Circ(a_1,a_2)) - a_1\,.$$

$\square$

Proof of Theorem 4.3.3. Combining Lemmas 4.3.4 with 4.3.5, we conclude that the largest integer $M \equiv v_j - a_1 \pmod{2a_1}$ with $0 \leq j \leq 2a_1 - 1$, that is nonrepresentable in at least two distinct ways as a nonnegative integer linear combination of a_1 and a_2 is given by

$$M = \max\Bigg(\mathrm{diam}(Circ(a_1,a_2)) - 2a_1, \mathrm{diam}(Circ(a_1,a_2)) - a_1\Bigg) = \mathrm{diam}(Circ(a_1,a_2)) - a_1\,.$$

Thus the 2-Frobenius number of the positive integers a_1 and a_1, is given by

$$\mathrm{F}_2(a_1,a_2) = diam(Circ(a_1,a_2)) - a_1\,.$$

This completes the proof of Theorem 4.3.3. $\square$

Remark: Lemma 4.3.5 shows that the largest $M \equiv v_j - a_1 \pmod{2a_1}$ with $0 \leq j \leq 2a_1 - 1$, that is nonrepresentable in at least two distinct ways corresponds to the vertex v_{2a_1-1} in $Circ(a_1,a_2)$ (i.e., $j = 2a_1 - 1$).

Chapter 5

The 2-Frobenius numbers of $\boldsymbol{a} = (a, a+d, a+2d)^t$

Its was shown by Roberts [73] in 1956 that the Frobenius number for the general arithmetic sequence $a, a+d, \ldots, a+nd$, with $\gcd(a,d) = 1$, is given by

$$\mathrm{F}(a, a+d, \ldots, a+nd) = a \left\lfloor \frac{a-2}{n} \right\rfloor + d(a-1)\,.$$

In this chapter, we extend Roberts's result to encompass the 2-Frobenius number $\mathrm{F}_2(a, a+d, a+2d)$ for three integers in an arithmetic progression. Our main result here says that

$$\mathrm{F}_2(a, a+d, a+2d) = a \left\lfloor \frac{a}{2} \right\rfloor + d(a+1)\,.$$

In order to prove this relation we first need to set up some notation.

Remark 5.0.6. Note that the notation in this chapter is quite different from the previous chapters. For instance, K here means number of shifts and L is the number of jumps, respectively, which will be introduced shortly.

Let $G_w(\boldsymbol{a})$ be the circulant digraph of the positive integer vector $\boldsymbol{a} = (a, a+d, a+2d)^t$ with $1 \le d < a$ and $\gcd(a,d) = 1$. We will establish a relation between the minimum weight S_{v_j} of paths from the initial vertex v_0 to the terminal vertex v_j in $G_w(a, a+d, a+2d)$, where $v_j \equiv jd \pmod{a}$, and representations of nonnegative integers in terms of a, $a+d$ and $a+2d$, (or the solutions of (2.2.1) in nonnegative integers).

Any arc on the graph $G_w(\boldsymbol{a})$ of weight $a+2d$ will be called a *jump step*, or *jump*. Any arc on the graph $G_w(\boldsymbol{a})$ of weight $a+d$ will be called a *shift step* or *shift*. We will say that any path $\mathcal{T}$ in $G_w(\boldsymbol{a})$ that consists of L jumps and K shifts has the form

$$\mathcal{T} = L\mathcal{J} + K\mathcal{S}\,,$$

where $\mathcal{J}$ and $\mathcal{S}$ stand for jumps and shifts, respectively.
Furthermore, since $\deg^+_{G_w(\boldsymbol{a})}(v_j) = 2$, for $0 \leq j \leq a-1$, we have one shift $\mathcal{S}$ (i.e. an arc of weight $a+d$), namely

$$v_j + \mathcal{S} \equiv v_{j+1} \pmod{a}\,.$$

An one jump $\mathcal{J}$ (i.e. an arc of weight $a+2d$), namely

$$v_j + \mathcal{J} \equiv v_j + 2\,\mathcal{S} \equiv v_{j+2} \pmod{a}\,. \tag{5.0.1}$$

It follows from (5.0.1) that $\mathcal{J} \equiv 2\,\mathcal{S}$, (see Figure 5.1).
Thus, one can easily see that any path from v_j to v_{j+2} in $G_w(\boldsymbol{a})$ contains either one jump or

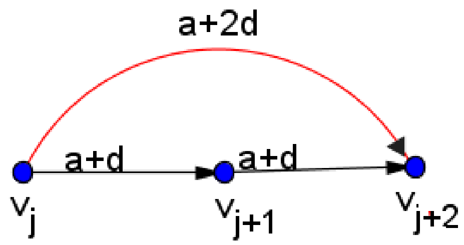

Figure 5.1: Two paths from vertex v_j to vertex v_{j+2}

two shifts and since $a+2d < 2(a+d)$. Hence, the minimum weight of any path from v_j to v_{j+2} is given by $a+2d$, (as illustrate in Figure 5.1).

Example 5.0.7. Let $a = 10$, $d = 3$ so that $\boldsymbol{a} = (10, 13, 16)^t$. Figure 5.5 shows the circulant digraph of $\boldsymbol{a}$.

As we can observe from Figure 5.2, that $\gcd(10, 13) = 1$ and the arcs of weight 13 connect all the vertices of $G_w(10, 13, 16)$ together. On other hand, as $\gcd(10, 16) = 2$ then the arcs of weight 16 partition $G_w(10, 13, 16)$ into two complements with vertex set $\{0, 2, 4, 6, 8\}$ and $\{1, 3, 5, 7, 9\}$, which can be connected by arcs of weight 13.

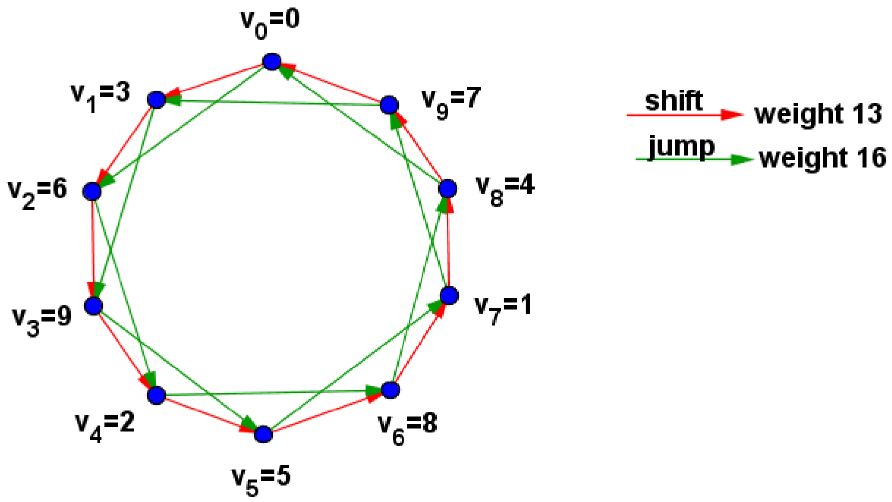

Figure 5.2: The circulant digraphs for the vector $(10, 13, 16)^t$. There are 10 red arcs of weight 13 and 10 green arcs of weight 16

Figure 5.2 shows the minimum path $v_2 \longrightarrow v_4 \longrightarrow v_6 \longrightarrow v_7$ from vertex v_2 to vertex v_7 contains two jumps and one shift. That is the minimum $v_2 - v_7$ path can be written in form

$$2\mathcal{J} + \mathcal{S}\,.$$

There are other equivalent minimum $v_2 - v_7$ paths (for example $v_2 \longrightarrow v_3 \longrightarrow v_5 \longrightarrow v_7$), consisting of the same edge weights but in a different order.

In the next section we presented an explicit formula for minimum weight S_{v_j} of a path from v_0 to v_j in the circulant digraph $G_w(\boldsymbol{a})$, for $0 \leq j \leq a-1$, (defined in Section 4.3).

5.1 The shortest path method

In the following theorem we give a formula for the *shortest path* and the *distance* between any two vertices of $G_w(\boldsymbol{a})$, moving in an anti-clockwise direction around the graph.

Theorem 5.1.1 (Minimum Path Theorem)**.** *The minimum path from vertex v_i to vertex v_j in $G_w(\boldsymbol{a})$, with $0 \leq i < j \leq a-1$, consists of exactly $\left(\frac{j-i-\delta}{2}\right)$ jumps and δ shifts, where $\delta \equiv j - i \pmod 2$, with $\delta \in \{0, 1\}$.*

The proof of Theorem 5.1.1, follow immediately from the next two lemmas.

Lemma 5.1.2. *Let $a \equiv 0 \pmod 2$. Then The minimum path from vertex v_i to vertex v_j in $G_w(\boldsymbol{a})$, with $0 \le i < j \le a-1$, consists of exactly $\left(\frac{j-i-\delta}{2}\right)$ jumps and δ shifts, where $\delta \equiv j-i \pmod 2$, with $\delta \in \{0,1\}$.*

Proof. Let v_i and v_j be any two distinct vertices in the circulant digraph $G_w(\boldsymbol{a})$. To find the minimum $v_i - v_j$ path, we have to consider two cases:

Case 1: Let us suppose that $j-i \equiv 1 \pmod 2$, (i.e. $\delta = 1$), and let N be the maximum number of jumps in a path from vertex v_i to vertex v_j that does not contains v_j as an intermediate vertex and where no arc is repeated.
Then any path from v_i to v_j can be written as

$$(N-M)\,\mathcal{J} + K\,\mathcal{S}\,, \tag{5.1.1}$$

where $N = \frac{a+j-i-1}{2}$, $0 \le M \le N$, $K = 2M+1 \pmod a$.
Substituting the weight for the jump steps and shift steps into (5.1.1) gives us

$$(N-M)(a+2d) + K(a+d)\,. \tag{5.1.2}$$

Since $2M+1$ can take the values $1, 3, \ldots, a-1, a+1, \ldots, 2N+1$. So we will consider two possibilities:

$$2M+1 < a \quad \text{and} \quad 2M+1 > a\,.$$

1. Suppose that $1 \le 2M+1 \le a-1$. Since $2M+1 < a$, we have $K = 2M+1$. Hence expression (5.1.2) becomes

   $$(N-M)(a+2d) + (2M+1)(a+d) = N(a+2d) + (a+d) + Ma\,.$$

   Now let $c(M)$ be a weight function in terms of M defined by

   $$c(M) = N(a+2d) + (a+d) + Ma$$

   for

   $$0 \le M \le \frac{a-2}{2}\,.$$

   Since N, a and d all positive, the minimum weight occurs when $M = 0$. Therefore the weight of the minimum path (distance) from v_i to v_j in $G_w(\boldsymbol{a})$, is given by

   $$\min_{0 \le M \le (a-2)/2} c(M) = c(0) = N(a+2d) + (a+d)\,. \tag{5.1.3}$$

2. Suppose that $a+1 \le 2M+1 \le 2N+1 < 2a$. Since $2M+1 > a$, so

$$K = 2M+1 \pmod{a} = 2M+1-a.$$

Hence expression (5.1.2) becomes

$$(N-M)(a+2d) + (2M+1-a)(a+d) = N(a+2d) + (1-a)(a+d) + Ma\,.$$

Now let

$$c(M) = N(a+2d) + (1-a)(a+d) + Ma$$

for

$$\frac{a}{2} \le M \le N\,.$$

As we know that N, a and d are positive integers. Then the minimum weight occurs when $M = \frac{a}{2}$. Therefore the weight of the minimum path from v_i to v_j in $G_w(\boldsymbol{a})$, is given by

$$\min_{a/2 \le M \le N} c(M) = c(a/2) = N(a+2d) + (a+d) - \frac{a}{2}(a+2d)\,. \tag{5.1.4}$$

From (5.1.3) and (5.1.4), we deduce that the weight of the minimum $v_i - v_j$ path with $0 \le M \le N$ corresponds to the choice $M = \frac{a}{2}$. So we have

$$\min_{0 \le M \le N} c(M) = c(a/2) = N(a+2d) + (a+d) - \frac{a}{2}(a+2d)\,. \tag{5.1.5}$$

Substituting N into (5.1.5) gives

$$\min_{0 \le M \le N} c(M) = c(a/2) = \frac{j-i-1}{2}(a+2d) + (a+d)\,.$$

It follows that, the distance from vertex v_i to vertex v_j in $G_w(\boldsymbol{a})$ with $0 \le i < j \le a-1$ and $j - i \equiv 1 \pmod{2}$, is given by

$$\frac{j-i-1}{2}(a+2d) + (a+d)\,. \tag{5.1.6}$$

Thus, the minimum path Q from v_i to v_j in $G_w(\boldsymbol{a})$ when $j - i \equiv 1 \pmod{2}$, consists of exactly $\frac{j-i-1}{2}$ jump steps and one shift step. That is

$$Q = \frac{j-i-1}{2}\,\mathcal{J} + \mathcal{S}\,.$$

Case 2: Let us suppose that $j - i \equiv 0 \pmod{2}$, (i.e. $\delta = 0$). Then any path from vertex v_i to vertex v_j in $G_w(\boldsymbol{a})$ can be written as

$$(N-M)\,\mathcal{J} + K\,\mathcal{S}\,, \tag{5.1.7}$$

where $N = \frac{a+j-i-2}{2}$, $0 \le M \le N$ and $K = 2M + 2 \pmod{a}$.
Substituting the weight for the jumps and shifts into (5.1.7) gives us

$$(N - M)(a + 2d) + K(a + d)\,. \tag{5.1.8}$$

Since $2M+2$ can take the values $2, 4, \dots, a-2, a, \dots, 2N+2$. So we will consider two possibilities:

$$2M + 2 < a \quad \text{and} \quad 2M + 2 \ge a\,.$$

1. Let $2 \le 2M + 2 \le a - 2$. Since $2M + 2 < a$, we have $K = 2M + 2$. Hence (5.1.8) becomes

   $$(N - M)(a + 2d) + (2M + 2)(a + d) = N(a + 2d) + 2(a + d) + Ma\,.$$

   Now let $c(M)$ be the weight function in term of M defined by

   $$c(M) = N(a + 2d) + 2(a + d) + Ma$$

   for

   $$0 \le M \le \frac{a-4}{2}\,.$$

   Since N, a and d are all positive, the minimum weight occurs when $M = 0$. Then the weight of the minimum $v_i - v_j$ path, is given by

   $$\min_{0 \le M \le (a-4)/2} c(M) = c(0) = N(a + 2d) + 2(a + d)\,. \tag{5.1.9}$$

2. Let $a \le 2M + 2 \le 2N + 2 < 2a$. Then $K = 2M + 2 - a$ and (5.1.8) becomes

   $$(N - M)(a + 2d) + (2M + 2 - a)(a + d) = N(a + 2d) + (2 - a)(a + d) + Ma\,.$$

   Now let

   $$c(M) = N(a + 2d) + (2 - a)(a + d) + Ma$$

   for

   $$\frac{a-2}{2} \le M \le N\,.$$

   According to N, a and d are all positive integers, the minimum weight occurs when $M = \frac{a-2}{2}$. Thus the weight of the minimum path from v_i to v_j is given by

   $$\min_{(a-2)/2 \le M \le N} c(M) = c((a-2)/2) = N(a + 2d) + 2(a + d) - a\left(\frac{a+2}{2} + d\right)\,. \tag{5.1.10}$$

From (5.1.9) and (5.1.10), we can see the weight of the minimum path from v_i to v_j in $G_w(\boldsymbol{a})$ with $0 \leq M \leq N$ corresponds to the choice $M = \frac{a-2}{2}$. Therefore we have

$$\min_{0 \leq M \leq N} c(M) = c((a-2)/2) = N(a+2d) + 2(a+d) - a\left(\frac{a+2}{2} + d\right). \tag{5.1.11}$$

Substituting N into (5.1.11) gives

$$\min_{0 \leq M \leq N} c(M) = c((a-2)/2) = \frac{j-i}{2}(a+2d).$$

This implies that, the distance from v_i to v_j in $G_w(\boldsymbol{a})$ with $0 \leq i < j \leq a-1$ and $j - i \equiv 0 \pmod 2$, is

$$\frac{j-i}{2}(a+2d). \tag{5.1.12}$$

Hence, the minimum path Q from v_i to v_j in $G_w(\boldsymbol{a})$ when $j - i \equiv 0 \pmod 2$, consists of exactly $\frac{j-i}{2}$ jump steps. That is

$$Q = \frac{j-i}{2}\,\mathcal{J}.$$

Combining the above cases, we deduce that the minimum $v_i - v_j$ path Q in $G_w(a, a+d, a+2d)$ with $0 \leq i < j \leq a-1$ and $a \equiv 0 \pmod 2$, consists of exactly $\left(\frac{j-i-\delta}{2}\right)$ jumps and δ shifts. That is

$$Q = \left(\frac{j-i-\delta}{2}\right)\mathcal{J} + \delta\,\mathcal{S},$$

where $\delta \equiv j - i \pmod 2$, with $\delta \in \{0, 1\}$. $\square$

We now consider the case where a is odd.

Lemma 5.1.3. *Let $a \equiv 1 \pmod 2$. Then the minimum path from vertex v_i to vertex v_j in $G_w(\boldsymbol{a})$, with $0 \leq i < j \leq a-1$, consists of exactly $\left(\frac{j-i-\delta}{2}\right)$ jumps and δ shifts, where $\delta \equiv j - i \pmod 2$, with $\delta \in \{0, 1\}$.*

The proof will follow the same strategy as in the proof of Lemma 5.1.2.

Proof. Let v_i and v_j be any two distinct vertices of $G_w(\boldsymbol{a})$. To find the minimum $v_i - v_j$ path. Again we need to consider two cases:

Case 1: Assume $j - i \equiv 1 \pmod 2$, (i.e. $\delta = 1$). Let N be the maximum number of jumps in a path from vertex v_i to vertex v_j that does not contains v_j as an intermediate vertex and where no arc is repeated. Then any path from v_i to v_j can be written as

$$(N - M)\,\mathcal{J} + K\,\mathcal{S}, \tag{5.1.13}$$

where $N = \frac{a+j-i}{2}$, $0 \le M \le N$ and $K = 2M \pmod a$.
Substituting the weight for the jump steps and shift steps into expression (5.1.13) gives us

$$(N - M)(a + 2d) + K(a + d)\,. \tag{5.1.14}$$

Since $2M$ can take the values $0, 2, \ldots, a-1, a+1, \ldots, 2N$. We have to consider two possibilities according to whether

$$2M < a \quad \text{or} \quad 2M > a\,.$$

1. Let $0 \le 2M \le a - 1$. Since $2M \le a - 1$, we have $K = 2M$. Hence expression (5.1.14) becomes

   $$(N - M)(a + 2d) + 2M(a + d) = N(a + 2d) + Ma\,.$$

   Now let $c(M)$ be the weight function in terms of M defined by

   $$c(M) = N(a + 2d) + Ma$$

   for

   $$0 \le M \le \frac{a-1}{2}\,.$$

   Since N, a and d are all positive, the minimum weight occurs when $M = 0$. So that the weight of the minimum path (distance) from v_i to v_j, is given by

   $$\min_{0 \le M \le (a-1)/2} c(M) = c(0) = N(a + 2d)\,. \tag{5.1.15}$$

2. Let $a + 1 \le 2M \le 2N < 2a$. Then $K = 2M - a$ and expression (5.1.14) gives us

   $$(N - M)(a + 2d) + (2M - a)(a + d) = N(a + 2d) - a(a + d) + Ma\,.$$

   Now let

   $$c(M) = N(a + 2d) - a(a + d) + Ma$$

for

$$\frac{a+1}{2} \leq M \leq N\,.$$

Since N, a and d are all positive, the minimum weight occurs when $M = \frac{a+1}{2}$. Therefore the weight of the minimum path from v_i to v_j in $G_w(\boldsymbol{a})$, is given by

$$\min_{(a+1)/2 \leq M \leq N} c(M) = c\left((a+1)/2\right) = N(a+2d) + (a+d) - \frac{a+1}{2}(a+2d)\,. \quad (5.1.16)$$

From (5.1.15) and (5.1.16), we can see that the weight of the minimum $v_i - v_j$ path in $G_w(\boldsymbol{a})$ with $0 \leq M \leq N$ corresponds to the choice $M = \frac{a+1}{2}$. Thus

$$\min_{0 \leq M \leq N} c(M) = c\left(\frac{a+1}{2}\right) = N(a+2d) + (a+d) - \frac{a+1}{2}(a+2d)\,. \quad (5.1.17)$$

Substituting N into (5.1.17), we get

$$\min_{0 \leq M \leq N} c(M) = c((a+1)/2) = \frac{j-i-1}{2}(a+2d) + (a+d)\,.$$

This means that, the distance from v_i to v_j in $G_w(\boldsymbol{a})$ with $0 \leq i < j \leq a-1$ and $j - i \equiv 1 \pmod 2$, is

$$\frac{j-i-1}{2}(a+2d) + (a+d)\,. \quad (5.1.18)$$

Therefore, the minimum path Q from v_i to v_j in $G_w(\boldsymbol{a})$ when $j - i \equiv 0 \pmod 2$, consists of exactly $\frac{j-i-1}{2}$ jump steps and one shift step. That is

$$Q = \frac{j-i-1}{2}\,\mathcal{J} + \mathcal{S}\,.$$

Case 2: Here assume $j - i \equiv 0 \pmod 2$, (i.e. $\delta = 0$). Then any path from v_i to v_j can be written as

$$(N-M)\,\mathcal{J} + K\,\mathcal{S}\,, \quad (5.1.19)$$

where $N = \frac{a+j-i-1}{2}$, $0 \leq M \leq N$ and $K = (2M+1) \pmod a$.
Substituting the weight for the jump steps and shift steps into (5.1.19) gives us

$$(N-M)(a+2d) + K(a+d)\,. \quad (5.1.20)$$

Since $2M+1$ can take the values $1, 3, \ldots, a-2, a, \ldots, 2N+1$. Now let us consider two possibilities:

$$2M+1 < a \quad \text{and} \quad 2M+1 \geq a\,.$$

1. Let $1 \le 2M + 1 \le a - 2$. Then $K = 2M + 1$, so that (5.1.20) becomes

$$(N - M)(a + 2d) + (2M + 1)(a + d) = N(a + 2d) + (a + d) + Ma\,.$$

Now let

$$c(M) = N(a + 2d) + (a + d) + Ma$$

for

$$0 \le M \le \frac{a-3}{2}\,.$$

Since N, a and d are all positive, the minimum weight occurs when $M = 0$. Hence the weight of the minimum path from v_i to v_j, is given by

$$\min_{0 \le M \le (a-3)/2} c(M) = c(0) = N(a + 2d) + (a + d)\,. \tag{5.1.21}$$

2. Let $a \le 2M + 1 \le 2N + 1 < 2a$. Then $K = 2M + 1 - a$. Thus (5.1.20) becomes

$$(N - M)(a + 2d) + (2M + 1 - a)(a + d) = N(a + 2d) + (1 - a)(a + d) + Ma\,.$$

Now let

$$c(M) = N(a + 2d) + (1 - a)(a + d) + Ma$$

for

$$\frac{a-1}{2} \le M \le N\,.$$

Since N, a and d are all positive, the minimum weight occurs when $M = \frac{a-1}{2}$. So the weight of the minimum path from v_i to v_j is

$$\min_{(a-1)/2 \le M \le N} c(M) = c\left((a-1)/2\right) = N(a + 2d) - \frac{a-1}{2}(a + 2d)\,. \tag{5.1.22}$$

Then from (5.1.21) and (5.1.22), we deduce that the weight of the minimum path from v_i to v_j with $0 \le M \le N$ occurs when $M = \frac{a-1}{2}$. Then

$$\min_{0 \le M \le N} c(M) = c\left((a-1)/2\right) = N(a + 2d) - \frac{a-1}{2}(a + 2d)\,. \tag{5.1.23}$$

Consequently, the distance from v_i to v_j in $G_w(\boldsymbol{a})$ with $0 \le i < j \le a-1$ and $j - i \equiv 0 \pmod 2$, is

$$\frac{j-i}{2}(a + 2d)\,. \tag{5.1.24}$$

Thus, the minimum path Q from v_i to v_j in $G_w(\boldsymbol{a})$ when $j - i \equiv 0 \pmod 2$, consists of exactly $\frac{j-i}{2}$ jump steps. That is

$$Q = \frac{j-i}{2}\mathcal{J}\,.$$

By considering the above cases, we have shown that the minimum path Q from v_i to v_j in $G_w(a, a+d, a+2d)$ with $0 \leq i < j \leq a-1$ and $a \equiv 1 \pmod 2$ consists of exactly $\left(\frac{j-i-\delta}{2}\right)$ jumps and δ shifts. That is

$$Q = \left(\frac{j-i-\delta}{2}\right)\mathcal{J} + \delta\,\mathcal{S}\,,$$

where $\delta \equiv j - i \pmod 2$, with $\delta \in \{0, 1\}$. ☐

Proof of Theorem 5.1.1. , Combining Lemmas 5.1.2 and 5.1.3 we deduce Theorem 5.1.1. ☐

We now give an example to illustrate Theorem 5.1.1 as follows:

Example 5.1.4. Let $a = 9$ and $d = 2$, then $\boldsymbol{a} = (9, 11, 13)^t$. To find the shortest $v_2 - v_7$ path in $G_w(9, 11, 13)$, we need to find all possible paths from v_2 to v_7 with different weights.

Then we will use the notation $\xmapsto{a+d}$ for the arc of weight $a + d$ and $\xmapsto{a+2d}$ for the arc of weight $a + 2d$ in $G_w(9, 11, 13)$. We have the following possibilities:

1. A $v_2 - v_7$ path $\mathcal{T}_1$ of weight $7(a + 2d)$, has the form

$$\mathcal{T}_1 = v_2 \xmapsto{a+2d} v_4 \xmapsto{a+2d} v_6 \xmapsto{a+2d} v_8 \xmapsto{a+2d} v_1 \xmapsto{a+2d} v_3 \xmapsto{a+2d} v_5 \xmapsto{a+2d} v_7\,.$$

2. A $v_2 - v_7$ path $\mathcal{T}_2$ of weight $6(a + 2d) + 2(a + d)$, has the form

$$\mathcal{T}_2 = v_2 \xmapsto{a+2d} v_4 \xmapsto{a+2d} v_6 \xmapsto{a+2d} v_8 \xmapsto{a+2d} v_1 \xmapsto{a+2d} v_3 \xmapsto{a+2d} v_5 \xmapsto{a+d} v_6 \xmapsto{a+d} v_7\,.$$

3. A $v_2 - v_7$ path $\mathcal{T}_3$ of weight $5(a + 2d) + 4(a + d)$, has the form

$$\mathcal{T}_3 = v_2 \xmapsto{a+2d} v_4 \xmapsto{a+2d} v_6 \xmapsto{a+2d} v_8 \xmapsto{a+2d} v_1 \xmapsto{a+2d} v_3 \xmapsto{a+d} v_4 \xmapsto{a+d} v_5 \xmapsto{a+d} v_6 \xmapsto{a+d} v_7\,.$$

4. A $v_2 - v_7$ path $\mathcal{T}_4$ of weight $4(a + 2d) + 6(a + d)$, has the form

$$\mathcal{T}_4 = v_2 \xmapsto{a+2d} v_4 \xmapsto{a+2d} v_6 \xmapsto{a+2d} v_8 \xmapsto{a+2d} v_1 \xmapsto{a+d} v_2 \xmapsto{a+d} v_3 \xmapsto{a+d} v_4 \xmapsto{a+d} v_5 \xmapsto{a+d} v_6 \xmapsto{a+d} v_7\,.$$

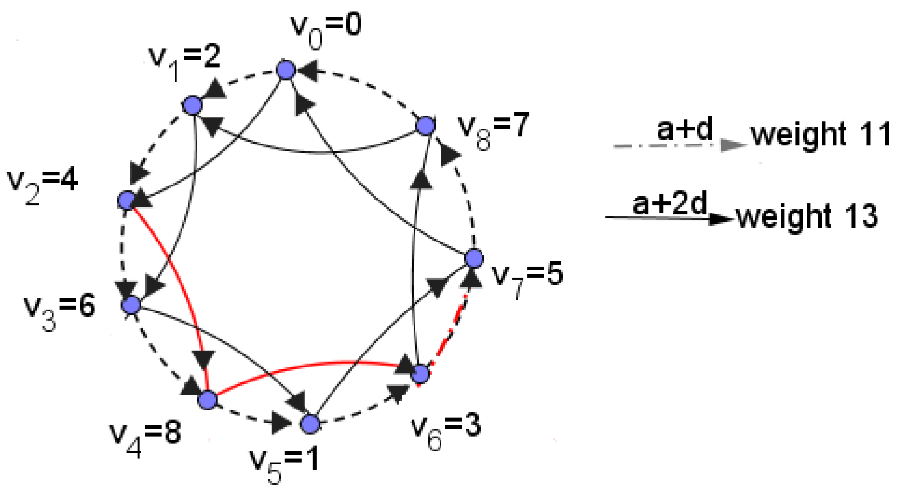

Figure 5.3: The shortest $v_2 - v_7$ path in $G_w(9, 11, 13)$

5. A $v_2 - v_7$ path $\mathcal{T}_5$ of weight $3(a+2d) + 8(a+d)$, has the form

$$\mathcal{T}_5 = v_2 \xmapsto{a+2d} v_4 \xmapsto{a+2d} v_6 \xmapsto{a+2d} v_8 \xmapsto{a+d} v_0 \xmapsto{a+d} v_1 \xmapsto{a+d} v_2 \xmapsto{a+d} v_3 \xmapsto{a+d} v_4 \xmapsto{a+d} v_5 \xmapsto{a+d} v_6 \xmapsto{a+d} v_7 .$$

6. A $v_2 - v_7$ path $\mathcal{T}_6$ of weight $2(a+2d) + (a+d)$, has the form

$$\mathcal{T}_6 = v_2 \xmapsto{a+2d} v_4 \xmapsto{a+2d} v_6 \xmapsto{a+d} v_7 .$$

7. A $v_2 - v_7$ path $\mathcal{T}_7$ of weight $(a+2d) + 3(a+d)$, has the form

$$\mathcal{T}_7 = v_2 \xmapsto{a+2d} v_4 \xmapsto{a+d} v_5 \xmapsto{a+d} v_6 \xmapsto{a+d} v_7 .$$

8. A $v_2 - v_7$ path $\mathcal{T}_8$ of weight $5(a+d)$, has the form

$$\mathcal{T}_8 = v_2 \xmapsto{a+d} v_3 \xmapsto{a+d} v_4 \xmapsto{a+d} v_5 \xmapsto{a+d} v_6 \xmapsto{a+d} v_7 .$$

We can clearly see that, path $\mathcal{T}_6$ is the shortest $v_2 - v_7$ path, that consists of exactly

$$2\mathcal{J} + 1\mathcal{S} .$$

Consequently, the distance from v_2 to v_7 in $G_w(9, 11, 13)$ will be

$$2(13) + 11 = 37 .$$

In the following theorem we consider the alternative case when $0 \leq j < i \leq a-1$.

Theorem 5.1.5. *The minimum path from v_i to v_j, with $0 \le j < i \le a-1$, in $G_w(a, a+d, a+2d)$ consists of exactly $\left(\frac{a+j-i-\delta}{2}\right)$ jump steps and δ shift steps, where $\delta \equiv (a+j-i) \pmod 2$, $\delta \in \{0,1\}$.*

Proof. The graph $G_w(\boldsymbol{a})$ is a symmetric graph. Let R be the function that maps vertex v_i to vertex $v_0 = 0$ for all $1 \le i \le a-1$, so that $R(v_i) = v_0$, and $R(v_j) = v_{j+(a-i)}$ (from the geometric viewpoint we rotates v_i anti-clockwise by $(\frac{a-i}{a})2\pi$ on the graph). Setting $j' = j + (a-i)$ gives $R(v_j) = v_{j'}$ and $R(v_i) = 0$. By applying Theorem 5.1.1, we obtain the defined result. □

From Theorems 5.1.1 and 5.1.5 we immediately obtain the following corollary.

Theorem 5.1.6. *Let $a^{'} \equiv a \pmod 2$, with $a^{'} \in \{0,1\}$. For $0 \le j \le a-1$ the minimum (nontrivial) path T from vertex v_j back to itself in $G_w(\boldsymbol{a})$, consists of exactly $\frac{a-a^{'}}{2}$ jump steps and $a^{'}$ shift steps. That is*

$$T = \frac{a-a^{'}}{2}\,\mathcal{J} + a^{'}\,\mathcal{S}\,. \tag{5.1.25}$$

Proof. Let v_j be any vertex of the circulant digraph $G_w(\boldsymbol{a})$. We need to show that the minimum weight of a (nontrivial) $v_j - v_j$ path T is

$$\frac{a-a^{'}}{2}(a+2d) + a^{'}(a+d)\,,$$

where $a^{'} \equiv a \pmod 2$. Observe that $\deg^{-}_{G_w}(v_j) = 2$, and

$$\begin{aligned} v_{j-1} + \mathcal{S} &\equiv jd \equiv v_j \pmod a\,, \quad \text{and} \\ v_{j-2} + \mathcal{J} &\equiv jd \equiv v_j \pmod a. \end{aligned}$$

where $\mathcal{S}$ and $\mathcal{J}$ are arcs of weight $a+d$ and $a+2d$, respectively.
Then, in order to take any (nontrivial) path from v_j back to v_j in $G_w(\boldsymbol{a})$. We have consider two possibilities, according to the in-neighborhood $N_{G_w}(v_j)$ of the vertex v_j.

1. A $v_j - v_j$ path W has the form

$$W = P \cup \mathcal{S}\,,$$

   where P is any $v_j - v_{j-1}$ path and $\mathcal{S}$ is an arc from v_{j-1} to v_j of weight $a+d$. Therefore, using Theorems 5.1.5 and 5.1.1, the minimum weight v of the path W is given by

$$v = \begin{cases} \left(\frac{a-2}{2}(a+2d) + (a+d)\right) + (a+d), & \text{if } a \equiv 0 \pmod 2, \\ \\ \left(\frac{a-1}{2}(a+2d)\right) + (a+d), & \text{if } a \equiv 1 \pmod 2. \end{cases} \tag{5.1.26}$$

2. A $v_j - v_j$ path U has the form

$$U = Q \cup \mathcal{J},$$

where Q is any $v_j - v_{j-2}$ path and $\mathcal{J}$ is an arc from v_{j-2} to v_j of weight $a + 2d$. By Theorems 5.1.5 and 5.1.1, the minimum weight y of the path U is given by

$$y = \begin{cases} \left(\frac{a-2}{2}(a+2d)\right) + (a+2d), & \text{if } a \equiv 0 \pmod 2, \\ \\ \left(\frac{a-3}{2}(a+2d) + (a+d)\right) + (a+2d), & \text{if } a \equiv 1 \pmod 2. \end{cases} \tag{5.1.27}$$

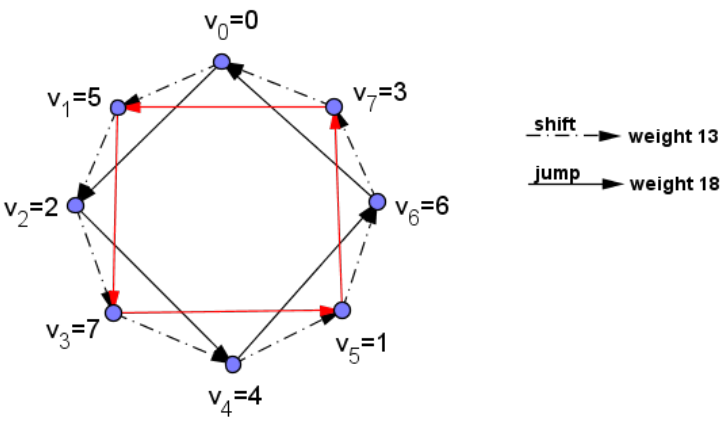

Figure 5.4: The shortest (nontrivial) path from v_3 back to v_3 in $G_w(8, 13, 18)$ consisting of exactly 4 jumps

From (5.1.26) and (5.1.27), it can be argued that the weight y is less than or equal to the weight v, since

$$v = (1 - a')a + y.$$

where $a' \equiv a \pmod 2$, $a' \in \{0,1\}$.
Thus we can write the weight y as

$$y = \frac{a-a'}{2}(a+2d) + a'(a+d)\,.$$

Consequently, the minimum weight of a (nontrivial) path T (distance) from v_j back to v_j will be

$$\frac{a-a'}{2}(a+2d) + a'(a+d)\,.$$

Hence, the minimum path T from v_j back to v_j in $G_w(\boldsymbol{a})$, consisting of exactly $\frac{a-a'}{2}$ jumps and a' shifts. That is

$$T = \frac{a-a'}{2}\,\mathcal{J} + a'\,\mathcal{S}\,.$$

The theorem is proved. □

This corollary is important for establishing our main result of this chapter.

Corollary 5.1.7 (To Theorems 5.1.1 and 5.1.5)**.** *For any $0 \le j \le a-1$, let S_{v_j} be the minimum weight of the (nontrivial) path from $v_0 = 0$ to v_j in $G_w(\boldsymbol{a})$. Then*

$$S_{v_j} = \begin{cases} \frac{a-a'}{2}(a+2d) + a'(a+d)\,, & \textit{if } j = 0\,, \\ \\ \frac{j-1}{2}(a+2d) + (a+d)\,, & \textit{if } j \equiv 1 \ (\textit{mod } 2)\,, \\ \\ \frac{j}{2}(a+2d)\,, & \textit{if } j \equiv 0 \ (\textit{mod } 2),\ j \neq 0\,, \end{cases}$$

where $a' \equiv a \ (mod\ 2)$, $a' \in \{0,1\}$.

Proof. The proof follows from Theorems 5.1.1 and 5.1.5. □

Corollary 5.1.8. *The minimum weight S_{v_j} of a (nontrivial) path from $v_0 = 0$ to v_j, given in Corollary 5.1.7, has two distinct representations in terms of a, $a+d$ and $a+2d$ when $j = 0$.*

Proof. Let S_{v_0} be the minimum weight of a (nontrivial) $v_0 - v_0$ path in $G_w(\boldsymbol{a})$. We have to show that S_{v_0} can be presented in two distinct ways.
From Corollary 5.1.7

$$S_{v_0} = \frac{a-a'}{2}(a+2d) + a'(a+d)\,,$$

where $a^{'} \equiv a \pmod 2$, $a^{'} \in \{0, 1\}$.
Since $\gcd(a, d) = 1$, we can write S_{v_0} as

$$\begin{aligned} S_{v_0} &= \frac{a - a^{'}}{2}(a + 2d) + a^{'}(a + d)\,, \quad \text{and} \\ S_{v_0} &= a\,(\frac{a + a^{'}}{2} + d)\,. \end{aligned}$$

This implies that, S_{v_0} has two distinct representations in terms of a, $a + d$ and $a + 2d$.

The corollary is proved. □

The following is a fundamental step in the proof of the main result in this chapter.

Theorem 5.1.9 (Unique Representation of S_{v_j}). *With $1 \le j \le a-1$, the minimum weight S_{v_j} of a path from $v_0 = 0$ to v_j, given in Corollary 5.1.7, has exactly one representation in terms of a, $a + d$ and $a + 2d$.*

Proof. Suppose, on the contrary, that S_{v_j} for $1 \le j \le a - 1$, can be represented in at least two distinct ways. There exists nonnegative integers $x_1, x_2, x_3, y_1, y_2, y_3$ with $x_j \neq y_j$ such that

$$\begin{aligned} S_{v_j} &= ax_1 + (a + d)x_2 + (a + 2d)x_3, \\ S_{v_j} &= ay_1 + (a + d)y_2 + (a + 2d)y_3\,. \end{aligned}$$

We will consider two cases: $j \equiv 0 \pmod 2$, $j \neq 0$ and $j \equiv 1 \pmod 2$.

Case 1: Suppose that $j \equiv 0 \pmod 2$, $j \neq 0$. Then by Corollary 5.1.7

$$S_{v_j} = \frac{j}{2}(a + 2d).$$

Thus by assumption S_{v_j} can be represented in at least two distinct ways, as

$$S_{v_j} = \frac{j}{2}(a + 2d) = ay_1 + (a + d)y_2 + (a + 2d)y_3\,. \tag{5.1.28}$$

If $y_3 = j/2$ then as $\gcd(a, d) = 1$ and $y_1, y_2 \ge 0$ we must have $y_1 = y_2 = 0$. Consequently if there exists a second representation then we must have $y_3 < j/2$ and at least one of y_1, y_2 has to be nonzero. Set $k = j/2 - y_3 \in \mathbb{Z}_{>0}$. Then (5.1.28) gives

$$(k - y_1 - y_2)a = (y_2 - 2k)d\,.$$

Now as $\gcd(a,d)=1$ we must have

$$k-y_1-y_2=dt\,, \tag{5.1.29a}$$

$$y_2-2k=at\,, \tag{5.1.29b}$$

with $t\in\mathbb{Z}$. We now have three choices for t. If $t=0$, then (5.1.29b) and (5.1.29a) gives us

$$y_1=-k.$$

Which is a contradiction as y_1 and k are both nonnegative integers. If $t>0$, then from (5.1.29b) we obtain $y_2=at+2k$. Substituting y_2 in (5.1.29a) gives

$$dt=-(k+y_1+at),$$

which also contradicts the fact that $d>0$. Finally, if $t<0$, then $t=-h$, where h is a positive integer. From (5.1.29b) we have $y_2+ah=2k$, implying

$$2k\geq ah\,. \tag{5.1.30}$$

However, we know that $j=2y_3+2k$, and hence $2k<a$ (that contradicts (5.1.30)) (as $1\leq j<a$). Thus, we conclude that S_{v_j} can be represented in exactly one way in terms of a, $a+d$ and $a+2d$ when $j\equiv 0 \pmod 2$, $j\neq 0$.

Case 2: Suppose that $j\equiv 1 \pmod 2$. Then by Corollary 5.1.7

$$S_{v_j}=(a+d)+\frac{j-1}{2}(a+2d).$$

Since S_{v_j} can be represented in at least two distinct ways, we have

$$S_{v_j}=(a+d)+\frac{j-1}{2}\big(a+2d\big)=ay_1+(a+d)y_2+(a+2d)y_3\,. \tag{5.1.31}$$

Therefore,

$$\left(\frac{j+1}{2}-y_1-y_2-y_3\right)a=(y_2+2y_3-j)d\,.$$

Now as $\gcd(a,d)=1$ we must have

$$(j+1)-2(y_1+y_2+y_3)=2dt\,, \tag{5.1.32a}$$

$$y_2+2y_3-j=at\,, \tag{5.1.32b}$$

with $t\in\mathbb{Z}$. Again there are three choices for t. If $t=0$, then from (5.1.32b) and (5.1.32a) we have

$$2y_1+y_2=1\,,$$

it follows that $y_1 = 0$ and $y_2 = 1$. Then from (5.1.31) implies that $y_3 = \frac{j-1}{2}$, and so the representations of S_{v_j} in (5.1.31) are the same. If $t > 0$, then (5.1.32a) and (5.1.32b) gives us

$$2dt + at + y_2 + 2y_1 = 1 \,.$$

Which contradicts the fact that $a > 1$ and $d \geq 1$. Finally, if $t < 0$, then $t = -h$, where $h \in \mathbb{Z}_{>0}$. Hence by (5.1.32b) we deduce that

$$y_2 + 2y_3 + ah = j \,.$$

This implies $j \geq ah$, which is a contradiction to our strategy that $1 \leq j \leq a-1$.
Thus, S_{v_j} can be represented in exactly one way in terms of a, $a+d$ and $a+2d$ when $j \equiv 1 \pmod 2$.

Combining the above arguments, we get the minimum weight S_{v_j} of a path from v_0 to v_j in $G_w(\boldsymbol{a})$, for $1 \leq j \leq a-1$, has exactly one representation in terms of a, $a+d$ and $a+2d$. □

5.2 The 2-Frobenius number of $\boldsymbol{a} = (a, a+d, a+2d)^t$ when a is even

In this section we obtain a formula for determining the 2-Frobenius number of three integers a, $a+d$ and $a+2d$ with $a \equiv 0 \pmod 2$ and $\gcd(a, d) = 1$ as follows:

Proposition 5.2.1. *Let $\boldsymbol{a} = (a, a+d, a+2d)^t$ be a positive integer vector with $a \equiv 0$ (mod 2), $1 \leq d < a$ and $\gcd(a, d) = 1$. Then*

$$\mathrm{F}_2(a, a+d, a+2d) = a\left(\frac{a}{2}\right) + d(a+1)\,. \tag{5.2.1}$$

Proof. Let v_j be any vertex of $G_w(\boldsymbol{a})$ with $0 \leq j \leq a-1$ and M be a positive integer. Then

$$M \equiv v_j \pmod a. \tag{5.2.2}$$

To prove Proposition 5.2.1, we need the following four lemmas.

Lemma 5.2.2. *Let $2 \leq j \leq a-2$ and $j \equiv 0$ (mod 2). Then the positive integer $M \equiv v_j$ (mod a) is representable in at least two distinct ways as a nonnegative integer linear combination of a, $a+d$ and $a+2d$ if and only if $M \geq S_{v_j} + a$.*

Proof. Let $M \geq S_{v_j} + a$. We need to show that M can be represented in at least two distinct ways. By (4.2.4), $v_j \equiv S_{v_j} \pmod{a}$ so that $v_j \equiv (S_{v_j} + a) \pmod{a}$. Thus we have

$$M \equiv (S_{v_j} + a) \pmod{a} \quad \textit{and} \quad M \geq S_{v_j} + a\,.$$

It follows that there is a nonnegative integer t such that

$$M = (S_{v_j} + a) + ta\,.$$

By Corollary 5.1.7

$$S_{v_j} = \frac{j}{2}(a + 2d)\,.$$

Therefore we can write M as

$$\begin{aligned} M &= (t+1)a + \frac{j}{2}(a+2d)\,, \\ \text{and} \qquad M &= ta + 2(a+d) + \left(\frac{j-2}{2}\right)(a+2d)\,. \end{aligned}$$

Hence, M is represented in at least two distinct ways as a nonnegative integer linear combination of a, $a+d$ and $a+2d$.

Conversely, let us assume that M has at least two distinct representations, then there exist nonnegative integers $x_1, y_1, z_1, x_2, y_2, z_2$ such that

$$M = ax_1 + (a+d)y_1 + (a+2d)z_1 = ax_2 + (a+d)y_2 + (a+2d)z_2\,. \tag{5.2.3}$$

We are required to prove that

$$M \geq S_{v_j} + a\,.$$

Since $M \equiv v_j \pmod{a}$, (5.2.3) gives us

$$M \equiv (a+d)y_1 + (a+2d)z_1 \equiv (a+d)y_2 + (a+2d)z_2 \equiv v_j \equiv S_{v_j} \pmod{a}\,. \tag{5.2.4}$$

Since $j \equiv 0 \pmod{2}$, both y_1 and y_2 are even numbers. We observe that S_{v_j} is maximum when $j = j_{\max} = a - 2$. Then

$$S_{v_{j_{\max}}} = \frac{a-2}{2}(a+2d)\,.$$

We now consider four cases according to the value of y_i and z_i, for $i = 1, 2$.

Case 1: Suppose that $y_1 = y_2 = 2t$, where $t \in \mathbb{Z}_{\geq 0}$. Then $z_1 \neq z_2$, and we may assume w.l.o.g. that $z_1 > z_2$ (as we may swap z_1 with z_2). This implies that

$$(a+d)y_1 + (a+2d)z_1 > (a+d)y_2 + (a+2d)z_2 \,.$$

Next, (5.2.3) gives

$$((x_2 - x_1) + (z_2 - z_1))\, a = 2(z_1 - z_2)d \,.$$

This means that either $\gcd(a, d) \neq 1$, which contradicts our assumptions, or

$$\begin{aligned} (x_2 - x_1) + (z_2 - z_1) &= dk \,, \\ 2(z_1 - z_2) &= ak \,, \end{aligned} \tag{5.2.5}$$

where $k \in \mathbb{Z}_{>0}$. Then, from (5.2.5) we get

$$z_1 \geq \frac{ak}{2} \,.$$

By (5.2.3), $M = ax_1 + (a+d)y_1 + (a+2d)z_1$, which gives us

$$M \geq (a+d)y_1 + (a+2d)z_1 \geq (a+d)y_1 + (a+2d)\frac{ak}{2} \geq (a+2d)\frac{ak}{2}.$$

Thus

$$M \geq \frac{a}{2}(a+2d) > S_{v_{j\max}} + a \geq S_{v_j} + a \,, \tag{5.2.6}$$

as required.

Case 2: Suppose that $z_1 = z_2 = t$, where $t \in \mathbb{Z}_{\geq 0}$. Then $y_1 \neq y_2$, and we may assume w. l. o. g. that $y_1 > y_2$ (as we may swap y_1 with y_2), hence

$$(a+d)y_1 + (a+2d)z_1 > (a+d)y_2 + (a+2d)z_2 \,.$$

By (5.2.3), we have

$$((x_2 - x_1) + (y_2 - y_1))\, a = (y_1 - y_2)d \,.$$

Now as $\gcd(a, d) = 1$ we must have

$$\begin{aligned} (x_2 - x_1) + (y_2 - y_1) &= dk \,, \\ y_1 - y_2 &= ak \,, \end{aligned} \tag{5.2.7}$$

where $k \in \mathbb{Z}_{>0}$. Therefore (5.2.7) gives

$$y_1 \geq ak\,.$$

Since $M = ax_1 + (a+d)y_1 + (a+2d)z_1$, we get

$$M \geq (a+d)y_1 + (a+2d)z_1 \geq (a+d)ak + (a+2d)z_1 \geq (a+d)ak$$

and, consequently

$$M \geq a(a+d) > S_{v_{j\max}} + a \geq S_{v_j} + a\,. \tag{5.2.8}$$

Case 3: Suppose that $y_1 > y_2$ and $z_1 > z_2$. Then

$$(a+d)y_1 + (a+2d)z_1 > (a+d)y_2 + (a+2d)z_2\,. \tag{5.2.9}$$

By (5.2.4) and (4.2.3), both the left and right hand sides of (5.2.9) represent two different paths from $v_0 = 0$ to v_j in $G_w(\boldsymbol{a})$ of weights $(a+d)y_1 + (a+2d)z_1$ and $(a+d)y_1 + (a+2d)z_1$. The weight $(a+d)y_2 + (a+2d)z_2$ has to be at least minimum weight S_{v_j} of the path from 0 to v_j in $G_w(\boldsymbol{a})$. Then by (5.2.4), there exists a positive integer h such that

$$\begin{aligned}(a+d)y_1 + (a+2d)z_1 = (a+d)y_2 + (a+2d)z_2 + ha &\geq S_{v_j} + ha\\ &\geq S_{v_j} + a\,.\end{aligned}$$

By (5.2.3), it follows that

$$M \geq (a+d)y_1 + (a+2d)z_1 \geq S_{v_j} + a\,. \tag{5.2.10}$$

Case 4: Suppose that $y_1 > y_2$ and $z_1 < z_2$. Then (5.2.3) gives

$$((x_1 - x_2) + (y_1 - y_2) - (z_2 - z_1))\,a = (2(z_2 - z_1) - (y_1 - y_2))\,d\,.$$

Now as $\gcd(a, d) = 1$ we must have

$$\begin{aligned}(x_1 - x_2) + (y_1 - y_2) - (z_2 - z_1) &= dk\,,\\ 2(z_2 - z_1) - (y_1 - y_2) &= ak\,,\end{aligned} \tag{5.2.11}$$

where $k \in \mathbb{Z}$. To solve (5.2.11), we will consider two possibilities:

$$z_2 - z_1 \geq y_1 - y_2 \quad \text{or} \quad z_2 - z_1 < y_1 - y_2\,.$$

1: If $z_2 - z_1 \geq y_1 - y_2$, then from (5.2.11), $k \in \mathbb{Z}_{>0}$. Hence

$$x_1 > x_2 \quad \text{and} \qquad z_2 - z_1 > \frac{ak}{2}\,.$$

The latter implies $z_2 > \frac{ak}{2}$. By (5.2.6) we have

$$M > S_{v_{j\max}} + a \geq S_{v_j} + a\,.$$

2: If $z_2 - z_1 < y_1 - y_2$. We again consider two subcases:

Firstly, let us assume $x_1 = x_2$. Then from (5.2.11) we have

$$\begin{cases} y_1 - y_2 = (a+2d)k, & \text{and} \\ z_2 - z_1 = (a+d)k, \end{cases}$$

where $k \in \mathbb{Z}_{>0}$. This implies that,

$$y_1 \geq (a+2d)k > ak \quad \text{and} \quad z_2 \geq (a+d)k > \frac{ak}{2}\,.$$

Then from (5.2.8) or (5.2.6), we obtain

$$M > S_{v_{j\max}} + a \geq S_{v_j} + a\,,$$

as required.

Secondly, let us assume $x_1 \neq x_2$. In this subcase, we have three options for k.

(i) Let $y_1 - y_2 > 2(z_2 - z_1)$. Then from (5.2.11), $k \in \mathbb{Z}_{<0}$, so that

$$k = -q\,, \quad \text{where } q \in \mathbb{Z}_{>0}\,.$$

Thus

$$y_1 - y_2 = 2(z_2 - z_1) + aq,$$

and, consequently

$$y_1 > aq\,.$$

So (5.2.8) gives

$$M > S_{v_{j\max}} + a \geq S_{v_j} + a\,.$$

(ii) Now, let $y_1 - y_2 < 2(z_2 - z_1)$. Then by (5.2.11), $k \in \mathbb{Z}_{>0}$, and we have

$$2(z_2 - z_1) = ak + y_1 - y_2\,,$$

which implies

$$z_2 - z_1 > \frac{ak}{2}\,,$$

and hence

$$z_2 > \frac{ak}{2}\,.$$

Using (5.2.6) we deduce that

$$M > S_{v_{j\max}} + a \geq S_{v_j} + a\,.$$

(iii) Finally, let $y_1 - y_2 = 2(z_2 - z_1)$. By (5.2.11), $k = 0$ and w.l.o.g. we may assume

$$(a+d)y_1 + (a+2d)z_1 > (a+d)y_2 + (a+2d)z_2. \tag{5.2.12}$$

Then from (5.2.10) we obtain

$$M \geq S_{v_j} + a\,,$$

as required.

□

Collectively all the above cases show that the largest integer $M \equiv v_j \pmod{a}$, with $2 \leq j \leq a-2$ and $j \equiv 0 \pmod{2}$, that is nonrepresentable in at least two distinct ways as a nonnegative integer combination of a, $a+d$ and $a+2d$ is given by

$$M = (S_{v_j} + a) - a = S_{v_j}\,.$$

Lemma 5.2.3. *Let $3 \leq j \leq a-1$ and $j \equiv 1 \pmod{2}$. Then the positive integer $M \equiv v_j \pmod{a}$ is representable in at least two distinct ways as a nonnegative integer linear combination of a, $a+d$ and $a+2d$ if and only if $M \geq S_{v_j} + a$.*

Proof. Suppose $M \geq S_{v_j} + a$. We have to prove that M can be represented in at least two distinct ways. By (4.2.4), $v_j \equiv S_{v_j} \pmod{a}$ so that $v_j \equiv (S_{v_j} + a) \pmod{a}$. Thus

$$M \equiv (S_{v_j} + a) \pmod{a} \quad \text{and} \quad M \geq S_{v_j} + a\,.$$

It follows that there is a nonnegative integer t such that

$$M = (S_{v_j} + a) + ta \,.$$

By Corollary 5.1.7

$$S_{v_j} = (a+d) + \left(\frac{j-1}{2}\right)(a+2d) \,.$$

Therefore we can write M as

$$\begin{array}{rrcl} & M & = & (t+1)a + (a+d) + \left(\frac{j-1}{2}\right)(a+2d) \,, \\ \text{and} & M & = & ta + 3(a+d) + \left(\frac{j-3}{2}\right)(a+2d) \,. \end{array}$$

Consequently, M is represented in at least two distinct ways as a nonnegative integer linear combination of a, $a+d$ and $a+2d$.

Conversely, now assume that M has at least two distinct representations, then by (5.2.3)

$$M = ax_1 + (a+d)y_1 + (a+2d)z_1 = ax_2 + (a+d)y_2 + (a+2d)z_2 \,.$$

We have to show that

$$M \geq S_{v_j} + a \,.$$

Since $M \equiv v_j \pmod{a}$, then (5.2.4) gives

$$M \equiv (a+d)y_1 + (a+2d)z_1 \equiv (a+d)y_2 + (a+2d)z_2 \equiv v_j \pmod{a} \,.$$

In view of $j \equiv 1 \pmod{2}$ and $\gcd(a, d) = 1$, both y_1 and y_2 are odd numbers. We observe that S_{v_j} is maximum when $j = j_{\max} = a - 1$. Then

$$S_{v_{j_{\max}}} = (a+d) + \frac{a-2}{2}(a+2d) \,.$$

As with Lemma 5.2.2 we will consider four cases:

Case 1: Suppose that $y_1 = y_2 = 2t+1$, where $t \in \mathbb{Z}_{\geq 0}$. Then $z_1 \neq z_2$, and we may assume w.l.o.g. that $z_1 > z_2$ (as we may swap z_1 with z_2), implying

$$(a+d)y_1 + (a+2d)z_1 > (a+d)y_2 + (a+2d)z_2 \,.$$

Next, (5.2.3) gives

$$\big((x_2 - x_1) + (z_2 - z_1)\big)a = 2(z_1 - z_2)d\,.$$

This means that either $\gcd(a, d) \neq 1$, which contradicts our assumptions, or

$$\begin{aligned}(x_2 - x_1) + (z_2 - z_1) &= dk\,,\\ 2(z_1 - z_2) &= ak\,,\end{aligned} \tag{5.2.13}$$

where $k \in \mathbb{Z}_{>0}$. By (5.2.13),

$$z_1 \geq \frac{ak}{2}\,.$$

Then (5.2.3), $M = ax_1 + (a+d)y_1 + (a+2d)z_1$, which gives us

$$M \geq (a+d)y_1 + (a+2d)z_1 \geq (a+d) + (a+2d)\frac{ak}{2}\,,$$

thus

$$M \geq (a+d) + \frac{a}{2}(a+2d) > S_{v_{j_{\max}}} + a \geq S_{v_j} + a\,. \tag{5.2.14}$$

Case 2: Suppose that $z_1 = z_2 = t \in \mathbb{Z}_{\geq 0}$. Then $y_1 \neq y_2$, and we may assume w. l. o. g. that $y_1 > y_2$ (as we may swap y_1 with y_2) and hence

$$(a+d)y_1 + (a+2d)z_1 > (a+d)y_2 + (a+2d)z_2\,.$$

Next, (5.2.3) gives

$$\big((x_2 - x_1) + (y_2 - y_1)\big)a = (y_1 - y_2)d\,.$$

Now as $\gcd(a, d) = 1$ we must have

$$\begin{aligned}(x_2 - x_1) + (y_2 - y_1) &= dk\,,\\ y_1 - y_2 &= ak\,,\end{aligned} \tag{5.2.15}$$

where $k \in \mathbb{Z}_{>0}$. Since y_1 and y_2 are both odd numbers and a is an even number, from (5.2.15) we obtain

$$y_1 \geq ak + 1\,.$$

Using (5.2.3), we deduce that

$$M \geq (a+d)y_1 + (a+2d)z_1 \geq (a+d)(ak+1) + (a+2d)z_1 \geq (a+d)(a+1)\,.$$

Therefore,

$$M > (a+d) + \frac{a}{2}(a+2d) > S_{v_{j\max}} + a \geq S_{v_j} + a\,. \tag{5.2.16}$$

Case 3: Suppose that $y_1 > y_2$ and $z_1 > z_2$. Then we have

$$(a+d)y_1 + (a+2d)z_1 > (a+d)y_2 + (a+2d)z_2\,.$$

Hence, by (5.2.4) and (4.2.3), we have two different paths from v_0 to v_j in $G_w(\boldsymbol{a})$ of weights $(a+d)y_1 + (a+2d)z_1$ and $(a+d)y_2 + (a+2d)z_2$. The weight $(a+d)y_2 + (a+2d)z_2$ has to be at least minimum weight S_{v_j} of the path from v_0 to v_j in $G_w(\boldsymbol{a})$. Therefore by (5.2.4),

$$(a+d)y_1 + (a+2d)z_1 \equiv (a+d)y_2 + (a+2d)z_2 \pmod{a}\,,$$

and there exists a positive integer h such that

$$\begin{aligned}(a+d)y_1 + (a+2d)z_1 = (a+d)y_2 + (a+2d)z_2 + ha &\geq S_{v_j} + ha\\ &\geq S_{v_j} + a\,.\end{aligned}$$

From (5.2.3), it follows that

$$M \geq (a+d)y_1 + (a+2d)z_1 \geq S_{v_j} + a\,. \tag{5.2.17}$$

Case 4: Suppose that $y_1 > y_2$ and $z_1 < z_2$. Then from (5.2.3), we have

$$((x_1 - x_2) + (y_1 - y_2) - (z_2 - z_1))\,a = (2(z_2 - z_1) - (y_1 - y_2))\,d\,.$$

Now as $\gcd(a, d) = 1$ we must have

$$\begin{aligned}(x_1 - x_2) + (y_1 - y_2) - (z_2 - z_1) &= dk\,,\\ 2(z_2 - z_1) - (y_1 - y_2) &= ak\,,\end{aligned} \tag{5.2.18}$$

where $k \in \mathbb{Z}$. To solve (5.2.18) we have to consider two possibilities:

$$z_2 - z_1 \geq y_1 - y_2 \quad \text{or} \quad z_2 - z_1 < y_1 - y_2\,.$$

1: If $z_2 - z_1 \geq y_1 - y_2$, then by (5.2.18), $k \in \mathbb{Z}_{>0}$. Thus

$$x_1 > x_2 \quad \text{and} \quad z_2 - z_1 > \frac{ak}{2}\,,$$

This implies $z_2 > \frac{ak}{2}$. Hence (5.2.14) gives

$$M > S_{v_{j\max}} + a \geq S_{v_j} + a\,.$$

2: If $z_2 - z_1 < y_1 - y_2$. Here we will consider two subcases:

Firstly, let $x_1 = x_2$. Then from (5.2.18) we find that

$$\begin{cases} y_1 - y_2 = (a+2d)k, & \text{and} \\ z_2 - z_1 = (a+d)k, \end{cases}$$

where $k \in \mathbb{Z}_{>0}$. This implies that

$$y_1 > (a+2d)k \quad \text{and} \quad z_2 \geq (a+d)k\,,$$

and using (5.2.3), $M = ax_1 + (a+d)y_1 + (a+2d)z_1$, we have

$$M \geq (a+d)y_1 + (a+2d)z_1 > (a+d)(a+2d) + (a+2d)z_1 > (a+d)(a+2)\,.$$

Thus

$$M > 2(a+d) + \frac{a}{2}(a+2d) > S_{v_{j\max}} + a \geq S_{v_j} + a\,. \tag{5.2.19}$$

Secondly, let $x_1 \neq x_2$. In this subcase we have three choices for k.

(i) Assume now that $y_1 - y_2 > 2(z_2 - z_1)$. Therefore by (5.2.18), $k \in \mathbb{Z}_{<0}$, so that $k = -q$, where $q \in \mathbb{Z}_{>0}$. Then

$$y_1 - y_2 = 2(z_2 - z_1) + aq\,,$$

which implies

$$y_1 > aq\,,$$

so that by (5.2.16),

$$M > S_{v_{j\max}} + a \geq S_{v_j} + a\,.$$

(ii) Here we assume that $y_1 - y_2 < 2(z_2 - z_1)$. By (5.2.18), $k \in \mathbb{Z}_{>0}$ and we have

$$2(z_2 - z_1) = ak + (y_1 - y_2)\,,$$

yields

$$z_2 - z_1 > \frac{ak}{2}.$$

Thus

$$z_2 > \frac{ak}{2}\,.$$

From (5.2.14) we get

$$M > S_{v_{j\max}} + a \geq S_{v_j} + a\,.$$

(iii) Finally, let $y_1 - y_2 = 2(z_2 - z_1)$. Then from (5.2.18) $k = 0$, so w.l.o.g. we may assume that

$$(a+d)y_1 + (a+2d)z_1 > (a+d)y_2 + (a+2d)z_2. \tag{5.2.20}$$

Therefore, from (5.2.17) we deduce that

$$M \geq S_{v_j} + a\,.$$

□

Collectively, the above cases imply that the largest integer $M \equiv v_j \pmod{a}$, with $3 \leq j \leq a-1$ and $j \equiv 1 \pmod{2}$, that is nonrepresentable in at least two distinct ways as a nonnegative integer combination of a, $a+d$ and $a+2d$, is

$$M = (S_{v_j} + a) - a = S_{v_j}\,.$$

Lemma 5.2.4. *For $j = 1$, the positive integer $M \equiv v_j \pmod{a}$ is representable in at least two distinct ways as a nonnegative integer linear combination of a, $a+d$ and $a+2d$ if and only if $M \geq S_{v_1} + a\left(\frac{a}{2} + d\right)$.*

Proof. Assume that $M \geq S_{v_1} + a(\frac{a}{2} + d)$. We need to show that M can be represented in at least two distinct ways. By the definition of v_j in $G_w(\boldsymbol{a})$ we have $v_1 \equiv S_{v_1} \pmod{a}$ and then $v_1 \equiv S_{v_1} + a(\frac{a}{2} + d) \pmod{a}$. Thus we have

$$M \equiv v_1 \equiv S_{v_1} + a\left(\frac{a}{2} + d\right) \pmod{a} \quad \text{and} \quad M \geq S_{v_1} + a\left(\frac{a}{2} + d\right).$$

It follows that there is a nonnegative integer t such that

$$M = \left(S_{v_1} + a(\frac{a}{2} + d)\right) + ta.$$

By Corollary 5.1.7

$$S_{v_1} = a + d\,.$$

Hence

$$\begin{aligned} M &= a\left(\frac{a}{2} + d + t\right) + (a+d)\,, \\ \text{and} \qquad M &= at + (a+d) + \frac{a}{2}(a+2d). \end{aligned}$$

Consequently, M is represented in at least two distinct ways as a nonnegative integer linear combination of a, $a+d$ and $a+2d$.

Conversely, suppose M has at least two distinct representations. Then by (5.2.3),

$$M = ax_1 + (a+d)y_1 + (a+2d)z_1 = ax_2 + (a+d)y_2 + (a+2d)z_2\,.$$

Since $M \equiv v_1 \pmod{a}$, (5.2.4) gives us

$$M \equiv (a+d)y_1 + (a+2d)z_1 \equiv (a+d)y_2 + (a+2d)z_2 \equiv v_1 \pmod{a}\,. \tag{5.2.21}$$

We are required to prove

$$M \geq S_{v_1} + a\left(\frac{a}{2} + d\right)\,.$$

In view of $j = 1$ and $a \equiv 0 \pmod 2$, both positive integers y_1, y_2 are odd numbers.
Again we examine four cases:

Case 1: Suppose that $y_1 = y_2 = 2t+1$, where $t \in \mathbb{Z}_{\geq 0}$. Then $z_1 \neq z_2$, and we may assume w. l. o. g. that $z_1 > z_2$ (as we may swap z_1 with z_2), hence

$$(a+d)y_1 + (a+2d)z_1 > (a+d)y_2 + (a+2d)z_2\,.$$

Next, (5.2.3) gives

$$\big((x_2 - x_1) + (z_2 - z_1)\big)a = 2(z_1 - z_2)d\,.$$

This means that either $\gcd(a,d) \neq 1$, which contradicts our assumptions, or

$$\begin{aligned} (x_2 - x_1) + (z_2 - z_1) &= dk\,,\\ 2(z_1 - z_2) &= ak\,, \end{aligned} \tag{5.2.22}$$

where $k \in \mathbb{Z}_{>0}$. By (5.2.22),

$$z_1 \geq \frac{ak}{2}.$$

From (5.2.3), $M = ax_1(a+d)y_1 + (a+2d)z_1$, it follows that

$$M \geq (a+d)y_1 + (a+2d)z_1 \geq (a+d) + (a+2d)\frac{ak}{2}\,.$$

Therefore,

$$M \geq (a+d) + \frac{a}{2}(a+2d) = (a+d) + a\left(\frac{a}{2} + d\right) = S_{v_1} + a\left(\frac{a}{2} + d\right). \tag{5.2.23}$$

Case 2: Suppose that $z_1 = z_2 = t \in \mathbb{Z}_{\geq 0}$. Then $y_1 \neq y_2$, and we may assume w. l. o. g. that $y_1 > y_2$ (as we may swap y_1 with y_2). This implies

$$(a+d)y_1 + (a+2d)z_1 > (a+d)y_2 + (a+2d)z_2 \,.$$

Next, (5.2.3) gives

$$\big((x_2 - x_1) + (y_2 - y_1)\big)a = (y_1 - y_2)d \,.$$

Now as $\gcd(a, d) = 1$ we must have

$$\begin{aligned} (x_2 - x_1) + (y_2 - y_1) &= dk \,, \\ y_1 - y_2 &= ak \,, \end{aligned} \tag{5.2.24}$$

where $k \in \mathbb{Z}_{>0}$. Since y_1, y_2 are both odd numbers and a is an even number, from (5.2.24) we have

$$y_1 \geq ak + 1.$$

Using (5.2.3) we find that

$$M \geq (a+d)y_1 + (a+2d)z_1 \geq (a+d)(ak+1) + (a+2d)z_1 \geq (a+d)(a+1) \,,$$

thus

$$M > (a+d) + a(\frac{a}{2} + d) = S_{v_1} + a\left(\frac{a}{2} + d\right) \,, \tag{5.2.25}$$

as required.

Case 3: Suppose that $y_1 > y_2$ and $z_1 > z_2$, then we have

$$(a+d)y_1 + (a+2d)z_1 > (a+d)y_2 + (a+2d)z_2 \,. \tag{5.2.26}$$

By (5.2.21) and (4.2.3), we have two different paths from v_0 to v_1 in $G_w(\boldsymbol{a})$ of weights $(a+d)y_1 + (a+2d)z_1$ and $(a+d)y_2 + (a+2d)z_2$. The weight $(a+d)y_2 + (a+2d)z_2$ has to be at least minimum weight S_{v_1} of the path from v_0 to v_1 in $G_w(\boldsymbol{a})$. Therefore from (5.2.21),

$$(a+d)y_1 + (a+2d)z_1 \equiv (a+d)y_2 + (a+2d)z_2 \pmod{a},$$

and there is a positive integer k such that

$$(a+d)y_1 + (a+2d)z_1 = (a+d)y_2 + (a+2d)z_2 + ak \geq S_{v_1} + ak \,. \tag{5.2.27}$$

Since $M = ax_1 + (a+d)y_1 + (a+2d)z_1$, we have

$$M \geq (a+d)y_1 + (a+2d)z_1 \geq S_{v_1} + ak\,. \tag{5.2.28}$$

So to prove $M \geq S_{v_1} + a\left(\frac{a}{2} + d\right)$, we need only to show that

$$k \geq \frac{a}{2} + d\,.$$

Since $\deg^-_{G_w(\boldsymbol{a})}(v_1) = 2$ then in order to take any path from v_0 to v_1 in $G_w(\boldsymbol{a})$, we have to consider four possibilities:

(i) $v_0 - v_1$ path P of weight $a+d$.

(ii) A $v_0 - v_1$ path W has the form

$$W = R \cup D\,,$$

where R is a (nontrivial) $v_0 - v_0$ path (or full cycle) in $G_w(\boldsymbol{a})$ and D is an arc from v_0 to v_1 of weight $a+d$. From Theorem 5.1.6, the minimum weight m of the path W is given by

$$m = \frac{a}{2}(a+2d) + (a+d) = a\left(\frac{a}{2} + d\right) + (a+d)\,, \tag{5.2.29}$$

as the weight of R is $\frac{a}{2}(a+2d)$ and the weight of D is $a+d$.

(iii) A $v_0 - v_1$ path V has the form

$$V = S \cup N \cup D\,,$$

where S is a $v_0 - v_{a-1}$ path in $G_w(\boldsymbol{a})$ and N is an arc from v_{a-1} to $v_0 = 0$ of weight $a+d$. So, from Theorem 5.1.1, the minimum weight n of the path V will be

$$n = \left((a+d) + \frac{a-2}{2}(a+2d)\right) + 2(a+d)\,, \tag{5.2.30}$$

as the weight of S is $\left((a+d) + \frac{a-2}{2}(a+2d)\right)$, the weight of N is $a+d$ and the weight of D is $a+d$.

(iv) A $v_0 - v_1$ path U has the form

$$U = S \cup J\,,$$

where S is a $v_0 - v_{a-1}$ path in $G_w(\boldsymbol{a})$ and J is an arc from v_{a-1} to v_1 of weight $a+2d$. Hence, by Theorem 5.1.1, the minimum weight z of the path U is

$$z = \left((a+d) + \frac{a-2}{2}(a+2d)\right) + (a+2d) = (a+d) + \frac{a}{2}(a+2d)\,, \tag{5.2.31}$$

which agrees with the given in (5.2.29).

Comparing (5.2.29) and (5.2.30), we see that the minimum weight m is less than the minimum weight n, where

$$m = n - a\,.$$

Hence, the minimum weight of the path from v_0 to v_1 in $G_w(\boldsymbol{a})$ around the full cycle will be

$$(a+d) + a\left(\frac{a}{2} + d\right) = S_{v_1} + a\left(\frac{a}{2} + d\right)\,.$$

Consequently, the value of ak in (5.2.27) has to be at least $a(\frac{a}{2} + d)$, which implies

$$k \geq \frac{a}{2} + d\,.$$

Then (5.2.28) gives us

$$M \geq S_{v_1} + ak \geq S_{v_1} + a\left(\frac{a}{2} + d\right)\,, \tag{5.2.32}$$

as required.

Case 4: Suppose that $y_1 > y_2$ and $z_1 < z_2$. Then from (5.2.3), we have

$$\big((x_1 - x_2) + (y_1 - y_2) - (z_2 - z_1)\big)a = \big(2(z_2 - z_1) - (y_1 - y_2)\big)d\,.$$

This means that either $\gcd(a, d) \neq 1$, which is contradicts our assumptions, or

$$\begin{aligned} (x_1 - x_2) + (y_1 - y_2) - (z_2 - z_1) &= dk\,, \\ 2(z_2 - z_1) - (y_1 - y_2) &= ak\,, \end{aligned} \tag{5.2.33}$$

where $k \in \mathbb{Z}$. To solve (5.2.33) we have to consider two possibilities:

$$z_2 - z_1 \geq y_1 - y_2 \quad \text{or} \quad z_2 - z_1 < y_1 - y_2\,.$$

1: Let $z_2 - z_1 \geq y_1 - y_2$. Then by (5.2.33), $k \in \mathbb{Z}_{>0}$. Hence

$$x_1 > x_2 \quad \text{and} \quad z_2 - z_1 > \frac{ak}{2}\,,$$

which implies

$$z_2 > \frac{ak}{2}\,.$$

From (5.2.23) we get

$$M \geq S_{v_1} + a\left(\frac{a}{2} + d\right)\,.$$

2: Let $z_2 - z_1 < y_1 - y_2$. We again consider two subcases:
Firstly, if $x_1 = x_2$ then (5.2.33) gives

$$\begin{cases} y_1 - y_2 = (a+2d)k, & \text{and} \\ z_2 - z_1 = (a+d)k, \end{cases}$$

where $k \in \mathbb{Z}_{>0}$. This implies that

$$y_1 > (a+2d)k \quad \text{and} \quad z_2 \geq (a+d)k\,.$$

Then by (5.2.3), $M = ax_1 + (a+d)y_1 + (a+2d)z_1$, we find that

$$M \geq (a+d)y_1 + (a+2d)z_1 > (a+d)(a+2d)k + (a+2d)z_1 > (a+d)(a+2d)\,.$$

Thus

$$M > 2(a+d) + \frac{a}{2}(a+2d) > S_{v_1} + a\left(\frac{a}{2} + d\right)\,, \tag{5.2.34}$$

as required.

Secondly, if $x_1 \neq x_2$, so in this subcase we have three options for k.

(i) Let $y_1 - y_2 > 2(z_2 - z_1)$. Then $k \in \mathbb{Z}_{<0}$ in (5.2.33), and so

$$k = -q; \quad q \in \mathbb{Z}_{>0}\,.$$

Therefore, we get

$$y_1 - y_2 = 2(z_2 - z_1) + aq\,,$$

which implies

$$y_1 \geq aq + 2\,.$$

Hence from (5.2.34) we have

$$M > S_{v_1} + a\left(\frac{a}{2} + d\right)\,.$$

(ii) Let $y_1 - y_2 < 2(z_2 - z_1)$. Then $k \in \mathbb{Z}_{>0}$ in (5.2.33), thus

$$2(z_2 - z_1) = ak + (y_1 - y_2)\,,$$

and consequently

$$z_2 > \frac{ak}{2}\,.$$

By (5.2.23), we therefore have

$$M \geq S_{v_1} + a\left(\frac{a}{2} + d\right)\,.$$

(iii) Finally, let $y_1 - y_2 = 2(z_2 - z_1)$. Then by (5.2.33), $k = 0$ and w.l.o.g. we can assume that

$$(a+d)y_1 + (a+2d)z_1 > (a+d)y_2 + (a+2d)z_2\,. \tag{5.2.35}$$

Therefore, (5.2.32) gives us

$$M \geq S_{v_1} + a\left(\frac{a}{2} + d\right)\,.$$

□

Collectively considering the above cases, we have shown that the largest integer $M \equiv v_1 \pmod{a}$, that is nonrepresentable in at least two distinct ways as a nonnegative integer combination of a, $a+d$ and $a+2d$ is given by

$$M = \left(S_{v_1} + a(\frac{a}{2} + d)\right) - a = S_{v_1} + a\left(\frac{a}{2} + d - 1\right)\,.$$

Lemma 5.2.5. *For $j = 0$, the number $M \equiv v_0$ (mod a) is representable in at least two distinct ways as a nonnegative integer linear combination of a, $a+d$ and $a+2d$ if and only if $M \geq S_{v_0}$.*

Proof. Using the same techniques as in Lemmas 5.2.2 and 5.2.3, we immediately obtain the proof of Lemma 5.2.5. □

Combining Lemmas 5.2.2, 5.2.3, 5.2.4 and 5.2.5, we conclude that the largest integer $M \equiv v_j \pmod{a}$ with $0 \leq j \leq a-1$, that is nonrepresentable in at least two distinct ways as a nonnegative integer combination of a , $a+d$ and $a+2d$ is equal to

$$\begin{aligned} S_{v_1} + a\left(\frac{a}{2} + d - 1\right) &= (a+d) + a\left(\frac{a}{2} + d - 1\right) \\ &= a\left(\frac{a}{2}\right) + d(a+1)\,. \end{aligned}$$

Thus, the 2-Frobenius number of the Frobenius basis $a, a+d, a+2d$ when $a \equiv 0 \pmod{2}$, $1 \leq d < a$ and $\gcd(a,d) = 1$, is given by

$$\mathrm{F}_2(a, a+d, a+2d) = a\left(\frac{a}{2}\right) + d(a+1)\,,$$

and hence Proposition 5.2.1.

□

Remark: Lemma 5.2.4 shows that the largest integer number $M \equiv v_j \pmod{a}$ with $0 \leq j \leq a-1$, that is nonrepresented in at least two distinct ways always corresponds to the vertex v_1 in $G_w(\boldsymbol{a})$ (i.e. $j = 1$).

We now illustrate Proposition 5.2.1 on the following example.

Example 5.2.6. To determine the 2-Frobenius number of the arithmetic progression $10, 13, 16$, we begin by finding the largest positive integer number

$$M_j \equiv v_j \equiv jd \pmod{10}, \qquad \text{for } 0 \leq j \leq 9.$$

for all vertices in the circulant digraph $G_w(10, 13, 16)$ (see Figure 5.5), that cannot be represented in least two distinct ways. This means that for each vertex v_j we can associate a corresponding positive integer M_j which cannot be represented in least two distinct ways as a nonnegative integer linear combination of the Frobenius basis $10, 13, 16$.

We give the calculations for the three cases, when $j \in \{0, 1, 2\}$, as follows:

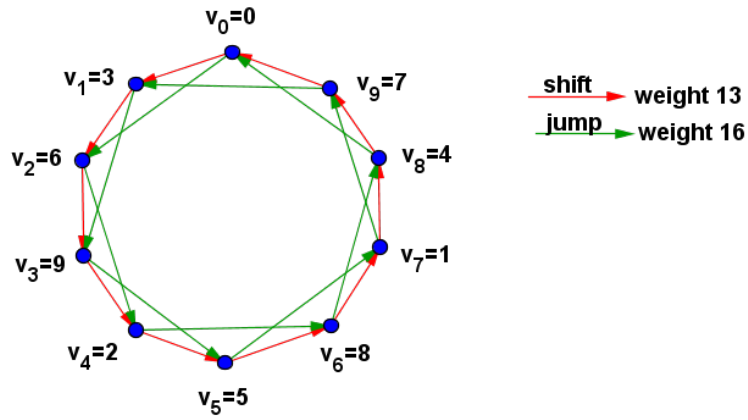

Figure 5.5: The circulant digraph for the arithmetic progression $10, 13, 16$

Let $j = 0$, we have to find a largest integer number

$$M_0 \equiv v_0 \equiv 0 \pmod{10},$$

that cannot represented in at least two distinct ways as a nonnegative integer linear combination of $10, 13, 16$. Therefore by Lemma 5.2.5 and Corollary 5.1.7,

$$\begin{aligned} M_0 = S_{v_0} - 10 &= 5(16) - 10 \\ &= 70. \end{aligned}$$

Then from Lemma 5.2.5, it follows that, any positive integer $M_0 > 70$ is represented in at least two distinct ways in terms of $10, 13$ and 16.
As, $80 \equiv 0 \pmod{10}$ and 80 has at least two distinct representations in terms of $10, 13$ and 16, as follows:

$$80 = 10(8) = 16(5)\,.$$

Let $j = 1$, a largest integer number

$$M_1 \equiv v_1 \equiv 3 \pmod{10}\,,$$

that cannot represented in at least two distinct ways as a nonnegative integer linear combination of $10, 13, 16$ is given by Lemma 5.2.4 and Corollary 5.1.7, as follows

$$\begin{aligned} M_1 &= S_{v_1} + 10(5+3-1) \\ &13 + 70 = 83\,. \end{aligned}$$

Thus Lemma 5.2.4, gives us any positive integer $M_1 > 83$ is represented in at least two distinct ways in terms of $10, 13$ and 16.
As, $93 \equiv 3 \pmod{10}$ and 93 has at least two distinct representations in terms of $10, 13$ and 16, as follows:

$$93 = 13 + 10(8) = 13 + 16(5)\,.$$

Let $j = 2$. Therefore by Lemma 5.2.2 and Corollary 5.1.7, a largest integer number

$$M_2 \equiv v_2 \equiv 6 \pmod{10}\,,$$

will be

$$M_2 = S_{v_2} = 16\,.$$

Hence Lemma 5.2.2, yields any positive integer $M_2 > 16$ is represented in at least two distinct ways in terms of $10, 13, 16$.
As we observe that $36 \equiv 6 \pmod{10}$ and 36 has at least two distinct representations in terms of $10, 13, 16$ as follows:

$$36 = 10(2) + 16 = 10 + 13(2)\,.$$

Thus, by the same way we can find the others M_j, $j = 3, 4, \ldots, 9$, as shown in the Table 5.1.

Table 5.1: A largest number $M_j \equiv v_j \pmod{10}$ with $0 \leq j \leq 9$, that cannot represented in at least two distinct ways as a nonnegative integer linear combination of $10, 13, 16$.

vertices of $G_w(10, 13, 16)$										
j	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9
v_j	0	3	6	9	2	5	8	1	4	7
M_j	70	83	16	29	32	45	48	61	64	77

Therefore Proposition (5.2.1) implies

$$\mathrm{F}_2(10, 13, 16) = \max_{0 \leq j \leq 9} \{M_j\} = \max\{70, 83, 16, 29, 32, 45, 48, 61, 64, 77\} = 83\,.$$

Note that by (5.2.1), $\mathrm{F}_2(10, 13, 16) = 10\left(\frac{10}{2}\right) + 3(10 + 1) = 83\,.$

We will now present two additional examples to compute the formula for $\mathrm{F}_2(a, a + d, a + 2d)$ when $a \equiv 0 \pmod 2$, using the MATLAB programming software package.

Example 5.2.7. Let $\boldsymbol{a} = (200, 207, 214)^t$, the largest integer number which connot represented in at least two distinct ways in terms of $\boldsymbol{a}$, is

$$21407 = 200(106) + 207\,.$$

Thus $\mathrm{F}_2(200, 204, 214) = 21407\,.$
Note that by Proposition 5.2.1,

$$\mathrm{F}_2(200, 204, 214) = 200\left(\frac{200}{2}\right) + 7(200 + 1) = 21407\,.$$

Example 5.2.8. Let $\boldsymbol{a} = (350, 359, 368)^t$, then the largest integer number which connot represented in at least two distinct ways in terms of $350, 359, 368$ is

$$64409 = 350(183) + 359\,,$$

which implies $\mathrm{F}_2(350, 359, 368) = 64409\,.$
Note by Proposition 5.2.1,

$$\mathrm{F}_2(350, 359, 368) = 350\left(\frac{350}{2}\right) + 9(350 + 1) = 64409\,.$$

5.3 The 2-Frobenius number of $\boldsymbol{a} = (a, a+d, a+2d)^t$ when a is odd

In this section we also obtain a formula for determining $\mathrm{F}_2(a, a+d, a+2d)$ for three integers in an arithmetic sequence with $a \equiv 1 \pmod 2$ and $\gcd(a, d) = 1$ as follows:

Proposition 5.3.1. *Let $\boldsymbol{a} = (a, a+d, a+2d)^t$ be a positive integer vector with $a \equiv 1$ (mod 2), $1 \le d < a$ and $\gcd(a, d) = 1$. Then*

$$\mathrm{F}_2(a, a+d, a+2d) = a\left(\frac{a-1}{2}\right) + d(a+1)\,. \tag{5.3.1}$$

We will follow the same strategy as in the proof of Proposition 5.2.1.

Proof. Let v_j be any vertex of $G_w(\boldsymbol{a})$ with $0 \le j \le a-1$ and let M be a positive integer. Then

$$M \equiv v_j \pmod a. \tag{5.3.2}$$

To prove Proposition 5.3.1, we need the following three lemmas.

Lemma 5.3.2. *For $2 \le j \le a-1$, $j \equiv 0, 1$ (mod 2), $j \neq 0$, the positive integer number $M \equiv v_j$ (mod a) is representable in at least two distinct ways as a nonnegative integer linear combination of a, $a+d$ and $a+2d$ if and only if $M \ge S_{v_j} + a$.*

Proof. Let $M \ge S_{v_j} + a$. We need to prove that M can be represented in at least two distinct ways. By (4.2.4), $v_j \equiv S_{v_j} \pmod a$ and then $v_j \equiv (S_{v_j} + a) \pmod a$. Thus

$$M \equiv (S_{v_j} + a) \pmod a \quad \text{and} \quad M \ge S_{v_j} + a\,.$$

It follows that there is a nonnegative integer t such that

$$M = (S_{v_j} + a) + ta\,.$$

By Corollary 5.1.7

$$S_{v_j} = \begin{cases} \frac{j}{2}(a+2d)\,, & \text{if } j \equiv 0 \pmod 2,\ j \neq 0\,, \\ \\ \frac{j-1}{2}(a+2d) + (a+d)\,, & \text{if } j \equiv 1 \pmod 2)\,. \end{cases}$$

Therefore, for $j \equiv 0 \pmod 2$, we can write M as

$$\begin{aligned} M &= a(t+1) + \frac{j}{2}(a+2d)\,, \quad \text{and} \\ M &= at + 2(a+d) + \left(\frac{j-2}{2}\right)(a+2d)\,. \end{aligned}$$

For $j \equiv 1 \pmod 2$, we can write M as

$$\begin{aligned} M &= a(t+1) + (a+d) + \left(\frac{j-1}{2}\right)(a+2d)\,, \quad \text{and} \\ M &= at + 3(a+d) + \left(\frac{j-3}{2}\right)(a+2d)\,. \end{aligned}$$

Consequently, M is represented in at least two distinct ways as a nonnegative integer linear combination of a, $a+d$ and $a+2d$ when $j \equiv 0, 1 \pmod 2\,, j \neq 0$.

Conversely, now let us assume that M has at least two distinct representations, then there exist nonnegative integers $x_1, y_1, z_1, x_2, y_2, z_2$ such that

$$M = ax_1 + (a+d)y_1 + (a+2d)z_1 = ax_2 + (a+d)y_2 + (a+2d)z_2\,. \tag{5.3.3}$$

We need to show that

$$M \geq S_{v_j} + a\,.$$

Since $M \equiv v_j \pmod a$, then (5.3.3) gives

$$M \equiv (a+d)y_1 + (a+2d)z_1 \equiv (a+d)y_2 + (a+2d)z_2 \equiv v_j \equiv S_{v_j} \pmod a\,. \tag{5.3.4}$$

We observe that S_{v_j} has maximum weight when $j = j_{\max}$. Then

$$S_{v_{j_{\max}}} = \begin{cases} \frac{a-1}{2}(a+2d)\,, & \text{if } j \equiv 0 \pmod 2,\ j \neq 0\,, \\ \\ \frac{a-3}{2}(a+2d) + (a+d)\,, & \text{if } j \equiv 1 \pmod 2\,. \end{cases}$$

We now consider four cases:

Case 1: Suppose that $y_1 = y_2 = t \in \mathbb{Z}_{\geq 0}$. Then $z_1 \neq z_2$, and we may assume w. l. o. g. that $z_1 > z_2$ (as we may swap z_1 with z_2). Thus

$$(a+d)y_1 + (a+2d)z_1 > (a+d)y_2 + (a+2d)z_2\,.$$

Next, (5.3.3) gives

$$((x_2 - x_1) + (z_2 - z_1))\, a = 2(z_1 - z_2)d\,.$$

This means that either $\gcd(a, d) \neq 1$, which contradicts our assumptions, or

$$\begin{aligned} (x_2 - x_1) + (z_2 - z_1) &= 2dk\,, \\ z_1 - z_2 &= ak\,, \end{aligned} \tag{5.3.5}$$

where $k \in \mathbb{Z}_{>0}$. So by (5.3.5) we obtain

$$z_1 \geq ak.$$

From (5.3.3), $M = ax_1 + (a+d)y_1 + (a+2d)z_1$, we find that

$$M \geq (a+d)y_1 + (a+2d)z_1 \geq (a+d)y_1 + (a+2d)ak \geq (a+2d)ak\,.$$

Hence

$$M \geq a(a+2d) > S_{v_{j_{\max}}} + a \geq S_{v_j} + a\,. \tag{5.3.6}$$

Case 2: Suppose that $z_1 = z_2 = t \in \mathbb{Z}_{\geq 0}$. Then $y_1 \neq y_2$, and we may assume w. l. o. g. that $y_1 > y_2$ (as we may swap y_1 with y_2), yields

$$(a+d)y_1 + (a+2d)z_1 > (a+d)y_2 + (a+2d)z_2\,.$$

Using (5.3.3) we get

$$((x_2 - x_1) + (y_2 - y_1))\, a = (y_1 - y_2)d\,.$$

Now as $\gcd(a, d) = 1$ we must have

$$\begin{aligned} (x_2 - x_1) + (y_2 - y_1) &= dk\,, \\ y_1 - y_2 &= ak\,, \end{aligned} \tag{5.3.7}$$

where $k \in \mathbb{Z}_{>0}$. By (5.3.7),

$$y_1 \geq ak\,.$$

Since $M = ax_1 + (a+d)y_1 + (a+2d)z_1$, we have

$$M \geq (a+d)y_1 + (a+2d)z_1 \geq (a+d)ak + (a+2d)z_1 \geq (a+d)ak\,,$$

which implies

$$M \geq a(a+d) > S_{v_{j\max}} + a \geq S_{v_j} + a\,, \tag{5.3.8}$$

as required.

Case 3: Suppose that $y_1 > y_2$ and $z_1 > z_2$. Then we have

$$(a+d)y_1 + (a+2d)z_1 > (a+d)y_2 + (a+2d)z_2\,. \tag{5.3.9}$$

By (5.3.4) and (4.2.3), both the left and right hand sides of (5.3.9) represent two different paths from v_0 to v_j in $G_w(\boldsymbol{a})$ of weights $(a+d)y_1 + (a+2d)z_1$ and $(a+d)y_2 + (a+2d)z_2$. The weight $(a+d)y_2 + (a+2d)z_2$ has to be at least minimum weight S_{v_j} of the path from v_0 to v_j in $G_w(\boldsymbol{a})$. Then by (5.3.4),

$$(a+d)y_1 + (a+2d)z_1 \equiv (a+d)y_2 + (a+2d)z_2,$$

and there exists a positive integer h such that

$$\begin{aligned}(a+d)y_1 + (a+2d)z_1 = (a+d)y_2 + (a+2d)z_2 + ha &\geq S_{v_j} + ha\\ &\geq S_{v_j} + a\,.\end{aligned}$$

Hence from (5.3.3), we get

$$M \geq (a+d)y_1 + (a+2d)z_1 \geq S_{v_j} + a\,. \tag{5.3.10}$$

Case 4: Suppose that $y_1 > y_2$ and $z_1 < z_2$. Then from (5.3.3), we have

$$((x_1 - x_2) + (y_1 - y_2) - (z_2 - z_1))\, a = (2(z_2 - z_1) - (y_1 - y_2))\, d\,.$$

Now as $\gcd(a, d) \neq 1$ we must have

$$\begin{aligned}(x_1 - x_2) + (y_1 - y_2) - (z_2 - z_1) &= dk\,,\\ 2(z_2 - z_1) - (y_1 - y_2) &= ak\,,\end{aligned} \tag{5.3.11}$$

where $k \in \mathbb{Z}$. To solve (5.3.11), we will consider two possibilities:

$$z_2 - z_1 \geq y_1 - y_2 \quad \text{or} \quad z_2 - z_1 < y_1 - y_2.$$

1: If $z_2 - z_1 \geq y_1 - y_2$. Then from (5.3.11), $k \in \mathbb{Z}_{>0}$. Thus

$$x_1 > x_2 \quad \text{and} \quad z_2 - z_1 \geq \frac{ak+1}{2}\,,$$

which implies

$$z_2 \geq \frac{ak+1}{2}.$$

Therefore by (5.3.3),

$$M \geq (a+d)y_2 + (a+2d)z_2 \geq (a+d)y_2 + (a+2d)\frac{ak+1}{2}\,,$$

and hence

$$M \geq \left(\frac{a+1}{2}\right)(a+2d) > S_{v_{j\max}} + a \geq S_{v_j} + a\,. \tag{5.3.12}$$

2: If $z_2 - z_1 < y_1 - y_2$. Here again we consider two subcases:

Firstly, let $x_1 = x_2$. Then (5.3.11) gives us

$$\begin{cases} y_1 - y_2 = (a+2d)k, & \text{and} \\ z_2 - z_1 = (a+d)k, \end{cases}$$

where $k \in \mathbb{Z}_{>0}$. This implies that

$$y_1 \geq (a+2d)k \quad \text{and} \quad z_2 \geq (a+d)k.$$

Then by (5.3.8) or by (5.3.6) we get

$$M > S_{v_{j\max}} + a \geq S_{v_j} + a\,.$$

Secondly, let $x_1 \neq x_2$. In this subcase we have three options for k.

(i) Let $y_1 - y_2 > 2(z_2 - z_1)$. Then $k \in \mathbb{Z}_{<0}$ in (5.3.11), and it follows that

$$k = -q; \quad q \in \mathbb{Z}_{>0}.$$

Thus

$$y_1 - y_2 = 2(z_2 - z_1) + aq$$

and, consequently,

$$y_1 > aq\,.$$

Therefore, by (5.3.8),

$$M > S_{v_{j\max}} + a \geq S_{v_j} + a\,.$$

(ii) Let $y_1 - y_2 < 2(z_2 - z_1)$. Then $k \in \mathbb{Z}_{>0}$ in (5.3.11), implies

$$2(z_2 - z_1) = ak + (y_1 - y_2)\,.$$

Hence

$$z_2 \geq \frac{ak+1}{2}\,.$$

From (5.3.12) we deduce that

$$M > S_{v_{j\max}} + a \geq S_{v_j} + a\,.$$

(iii) Finally, let $y_1 - y_2 = 2(z_2 - z_1)$. Then $k = 0$ in (5.3.11), so w.l.o.g. we may assume that

$$(a+d)y_1 + (a+2d)z_1 > (a+d)y_2 + (a+2d)z_2. \tag{5.3.13}$$

Therefore, from (5.3.10) we get

$$M \geq S_{v_j} + a\,.$$

□

As a result, we have shown that the largest integer $M \equiv v_j \pmod{a}$, with $2 \leq j \leq a-1$, and $j \equiv 0 \pmod 2$ or $j \equiv 1 \pmod 2$, that is nonrepresentable in at least two distinct ways as a nonnegative integer combination of a, $a+d$ and $a+2d$ is given by

$$M = (S_{v_j} + a) - a = S_{v_j}\,.$$

Lemma 5.3.3. *For $j = 1$, the number $M \equiv v_j \pmod{a}$ is representable in at least two distinct ways as a nonnegative integer linear combination of a, $a+d$ and $a+2d$ if and only if $M \geq S_{v_1} + a\left(\frac{a-1}{2} + d\right)$.*

Proof. Assume $M \geq S_{v_1} + a(\frac{a-1}{2} + d)$. We have to show that M can be represented in at least two distinct ways. By (4.2.4), $v_1 \equiv S_{v_1} \pmod{a}$ so that $v_1 \equiv S_{v_1} + a(\frac{a-1}{2} + d) \pmod{a}$. Thus

$$M \equiv v_1 \equiv S_{v_1} + a\left(\frac{a-1}{2} + d\right) \pmod{a} \quad \text{and} \quad M \geq S_{v_1} + a\left(\frac{a-1}{2} + d\right)\,.$$

It follows that there is a nonnegative integer t such that

$$M = S_{v_1} + a\left(\frac{a-1}{2} + d\right) + ta\,.$$

By Corollary 5.1.7

$$S_{v_1} = a + d\,.$$

Therefore,

$$\begin{aligned} M &= a\left(\frac{a-1}{2} + d + t\right) + (a+d)\,, \\ \text{and} \qquad M &= at + \left(\frac{a+1}{2}\right)(a+2d). \end{aligned}$$

Thus, M is represented in at least two distinct ways as a nonnegative integer linear combination of a, $a+d$ and $a+2d$.

Conversely, let us assume M has at least two distinct representations, then (5.3.3) gives

$$M = ax_1 + (a+d)y_1 + (a+2d)z_1 = ax_2 + (a+d)y_2 + (a+2d)z_2\,.$$

Since $M \equiv v_1 \pmod a$, hence from (5.3.3), we have

$$M \equiv (a+d)y_1 + (a+2d)z_1 \equiv (a+d)y_2 + (a+2d)z_2 \equiv v_1 \equiv d \pmod a\,. \tag{5.3.14}$$

We are required to show that

$$M \geq S_{v_1} + a\left(\frac{a-1}{2} + d\right)\,.$$

Again, we have to consider four cases here:

Case 1: Suppose that $y_1 = y_2 = t \in \mathbb{Z}_{>0}$. Then $z_1 \neq z_2$, and we may assume w.l.o.g. that $z_1 > z_2$ (as we may swap z_1 with z_2) and hence

$$(a+d)y_1 + (a+2d)z_1 > (a+d)y_2 + (a+2d)z_2\,.$$

Next (5.3.3) gives

$$\big((x_2 - x_1) + (z_2 - z_1)\big)a = 2(z_1 - z_2)d\,.$$

Now as $\gcd(a, d) = 1$ we must have

$$\begin{aligned} (x_2 - x_1) + (z_2 - z_1) &= 2dk\,, \\ z_1 - z_2 &= ak\,, \end{aligned} \tag{5.3.15}$$

where $k \in \mathbb{Z}_{>0}$. By (5.3.15),

$$z_1 \geq ak.$$

Then from expression (5.3.3), $M = ax_1 + (a+d)y_1 + (a+2d)z_1$, we have

$$M \geq (a+d)y_1 + (a+2d)z_1 \geq (a+d)y_1 + (a+2d)ak\,.$$

Therefore,

$$M \geq (a+2d)a > S_{v_1} + a\left(\frac{a-1}{2} + d\right)\,. \tag{5.3.16}$$

Case 2: Suppose that $z_1 = z_2 = t \in \mathbb{Z}_{\geq 0}$. Then $y_1 \neq y_2$, we and may assume w. l. o. g. that $y_1 > y_2$ (as we may else swap y_1 with y_2), implying that

$$(a+d)y_1 + (a+2d)z_1 > (a+d)y_2 + (a+2d)z_2\,.$$

Next (5.3.3) gives

$$\big((x_2 - x_1) + (y_2 - y_1)\big)a = (y_1 - y_2)d\,.$$

This means that either $\gcd(a, d) \neq 1$, which contradicts our assumptions, or

$$\begin{aligned}(x_2 - x_1) + (y_2 - y_1) &= dk\,,\\ y_1 - y_2 &= ak\,,\end{aligned} \tag{5.3.17}$$

where $k \in \mathbb{Z}_{>0}$. Then from (5.3.17) it follows that,

$$y_1 \geq a + y_2\,. \tag{5.3.18}$$

Hence by (5.3.14) and (5.3.18) we have

$$(a+d)(a+y_2) + (a+2d)t \equiv (a+d)y_2 + (a+2d)t \equiv v_1 \equiv d \pmod{a}.$$

This implies that

$$y_2 = (sa+1) - 2t, \qquad s \in \mathbb{Z}_{\geq 0}\,. \tag{5.3.19}$$

In particular,

$$y_2 = 0 \text{ if } t = \frac{sa+1}{2}.$$

Therefore, from (5.3.3), $M = ax_1 + (a+d)y_1 + (a+2d)z_1$, (5.3.19) and (5.3.18) we find that

$$M \geq (a+d)y_1 + (a+2d)z_1 \geq (a+d)(ak+1) \geq (a+d)(a+1)\,.$$

Thus

$$M > \frac{a+1}{2}(a+2d) = S_{v_1} + a\left(\frac{a-1}{2} + d\right), \tag{5.3.20}$$

as required.

Case 3: Suppose that $y_1 > y_2$ and $z_1 > z_2$. Then we have

$$(a+d)y_1 + (a+2d)z_1 > (a+d)y_2 + (a+2d)z_2\,. \tag{5.3.21}$$

Hence by (5.3.14) and (4.2.3), we have two different paths from v_0 to v_1 in $G_w(\boldsymbol{a})$ of weights $(a+d)y_1 + (a+2d)z_1$ and $(a+d)y_2 + (a+2d)z_2$. The weight $(a+d)y_2 + (a+2d)z_2$ has to be at least minimum weight S_{v_1} of the path from v_0 to v_1 in $G_w(\boldsymbol{a})$. Therefore by (5.3.14),

$$(a+d)y_1 + (a+2d)z_1 \equiv (a+d)y_2 + (a+2d)z_2 \pmod{a},$$

and there is a positive integer k such that

$$(a+d)y_1 + (a+2d)z_1 = (a+d)y_2 + (a+2d)z_2 + ak \geq S_{v_1} + ak\,. \tag{5.3.22}$$

Since $M = ax_1 + (a+d)y_1 + (a+2d)z_1$, we have

$$M \geq (a+d)y_1 + (a+2d)z_1 \geq S_{v_1} + ak\,. \tag{5.3.23}$$

In order to prove $M \geq S_{v_1} + a\left(\frac{a-1}{2} + d\right)$, we therefore only need to show that

$$k \geq \frac{a-1}{2} + d\,.$$

In order to take any path from v_0 to v_1 in $G_w(a, a+d, a+2d)$ with $a \equiv 1 \pmod{2}$, we have to consider four possibilities:

1. A v_0 to v_1 path P of weight $a+d$.

2. A $v_0 - v_1$ path W has the form

   $$W = R \cup D\,,$$

   where R is a (nontrivial) $v_0 - v_0$ path (or a full cycle) in $G_w(\boldsymbol{a})$ and D is an arc from v_0 to v_1 of weight $a+d$. Therefore, by Theorem 5.1.6, the minimum weight m of the path W is

   $$m = \frac{a-1}{2}(a+2d) + 2(a+d)\,, \tag{5.3.24}$$

   as the weight of R is $\left(\frac{a-1}{2}(a+2d) + (a+d)\right)$ and the weight of D is $a+d$.

3. A $v_0 - v_1$ path U has the form

$$U = S \cup J\,,$$

where S is a $v_0 - v_{a-1}$ path in $G_w(\boldsymbol{a})$ and J is an arc from v_{a-1} to v_1 of weight $a + 2d$. Hence, by Theorem 5.1.1, the minimum weight n of the path U is

$$n = \frac{a-1}{2}(a+2d) + (a+2d) = \frac{a+1}{2}(a+2d)\,, \tag{5.3.25}$$

as the weight of S is $\frac{a-1}{2}(a+2d)$ and the weight of J is $a + 2d$.

4. A $v_0 - v_1$ path V has the form

$$V = S \cup N \cup D\,,$$

where S is a $v_0 - v_{a-1}$ path in $G_w(\boldsymbol{a})$ and N is an arc from v_{a-1} to $v_0 = 0$ of weight $a+d$. So, from Theorem 5.1.1, the minimum weight z of the path V is

$$z = \left(\frac{a-1}{2}(a+2d)\right) + 2(a+d)\,, \tag{5.3.26}$$

which agrees with weight given in (5.3.24).

By comparing (5.3.24) with (5.3.25), we can easily find that the minimum weight n is less than the minimum weight m, where

$$n = m - a\,.$$

We can rewrite the weight n as follows

$$\begin{aligned} n &= (a+d) + a\left(\frac{a-1}{2} + d\right) \\ &= S_{v_1} + a\left(\frac{a-1}{2} + d\right)\,. \end{aligned}$$

Thus, the minimum weight of a $v_0 - v_1$ path in $G_w(\boldsymbol{a})$ around the full cycle will be

$$S_{v_1} + a\left(\frac{a-1}{2} + d\right)\,.$$

This implies that, the value of a positive integer ak in (5.3.22) has to be at least $a\left(\frac{a-1}{2} + d\right)$ and consequently

$$k \geq \frac{a-1}{2} + d\,.$$

Therefore, from (5.3.23) we deduce that

$$M \geq S_{v_1} + ak \geq S_{v_1} + a\left(\frac{a-1}{2} + d\right), \tag{5.3.27}$$

as required.

Case 4: Suppose that $y_1 > y_2$ and $z_1 < z_2$. Then from (5.3.3), we have

$$((x_1 - x_2) + (y_1 - y_2) - (z_2 - z_1))\, a = (2(z_2 - z_1) - (y_1 - y_2))\, d\,.$$

Now as know $\gcd(a, d) = 1$ we must have

$$\begin{aligned} (x_1 - x_2) + (y_1 - y_2) - (z_2 - z_1) &= dk\,, \\ 2(z_2 - z_1) - (y_1 - y_2) &= ak\,, \end{aligned} \tag{5.3.28}$$

where $k \in \mathbb{Z}$. To solve (5.3.28) we will consider two possibilities:

$$z_2 - z_1 \geq y_1 - y_2 \quad \text{or} \quad z_2 - z_1 < y_1 - y_2.$$

1: Let $z_2 - z_1 \geq y_1 - y_2$. Then from (5.3.28), $k \in \mathbb{Z}_{>0}$ and hence

$$z_2 - z_1 \geq \frac{ak+1}{2}$$

which implies that

$$z_2 \geq \frac{ak+1}{2}.$$

From (5.3.3), $M = ax_1 + (a+d)y_1 + (a+2d)z_1$, we get

$$M \geq (a+d)y_1 + (a+2d)z_1 \geq (a+d)y_1 + (a+2d)\frac{ak+1}{2}\,.$$

Therefore,

$$M \geq \frac{a+1}{2}(a+2d) = S_{v_1} + a\left(\frac{a-1}{2} + d\right). \tag{5.3.29}$$

2: Let $z_2 - z_1 < y_1 - y_2$. Again we consider two subcases:

Firstly, if $x_1 = x_2$. Then from (5.3.28) we find that

$$\begin{cases} y_1 - y_2 = (a+2d)k, & \text{and} \\ z_2 - z_1 = (a+d)k, \end{cases}$$

where $k \in \mathbb{Z}_{>0}$. Thus we have

$$y_1 \geq (a+2d)k \quad \text{and} \quad z_2 \geq (a+d)k > \frac{ak+1}{2}\,.$$

Therefore by (5.3.29),

$$M \geq S_{v_1} + a\left(\frac{a-1}{2} + d\right).$$

Secondly, if $x_1 \neq x_2$. In this subcase we have three options for k.

(i) Let $y_1 - y_2 > 2(z_2 - z_1)$. Then $k \in \mathbb{Z}_{<0}$ in (5.3.28) and so

$$k = -q;\ q \in \mathbb{Z}_{>0}.$$

Therefore we have

$$y_1 - y_2 = 2(z_2 - z_1) + aq\,,$$

which implies

$$y_1 \geq aq + 2\,.$$

Hence by (5.3.20),

$$M > S_{v_1} + a\left(\frac{a-1}{2} + d\right).$$

(ii) Let $y_1 - y_2 < 2(z_2 - z_1)$. Then $k \in \mathbb{Z}_{>0}$ in (5.3.28), thus

$$2(z_2 - z_1) = ak + (y_1 - y_2)$$

implies

$$z_2 - z_1 > \frac{ak+1}{2}\,,$$

and hence

$$z_2 > \frac{ak+1}{2}\,.$$

Using (5.3.29), we get

$$M \geq S_{v_1} + a\left(\frac{a-1}{2} + d\right).$$

(iii) Finally, let $y_1 - y_2 = 2(z_2 - z_1)$. Then by (5.3.28), $k = 0$ and we can assume w.l.o.g., that

$$(a+d)y_1 + (a+2d)z_1 > (a+d)y_2 + (a+2d)z_2\,. \tag{5.3.30}$$

Hence (5.3.27), yields

$$M \geq S_{v_1} + a\left(\frac{a-1}{2} + d\right).$$

□

Therefore by considering all above cases, we have proved that the largest integer $M \equiv v_1 \pmod{a}$, that is nonrepresentable in at least two distinct ways as a nonnegative integer combination of a, $a+d$ and $a+2d$ is given by

$$M = \left(S_{v_1} + a\left(\frac{a-1}{2} + d\right)\right) - a = S_{v_1} + a\left(\frac{a-1}{2} + d - 1\right).$$

Lemma 5.3.4. *For $j = 0$, the number $M \equiv v_j \pmod{a}$ is representable in at least two distinct ways as a nonnegative integer linear combination of a, $a+d$ and $a+2d$ if and only if $M \geq S_{v_0}$.*

Proof. Using the same techniques as in Lemma 5.3.2 and 5.3.3, we immediately get the proof of Lemma 5.3.4. □

By combining Lemmas 5.3.2, 5.3.3, and 5.3.4, we conclude that the largest integer $M \equiv v_j \pmod{a}$, with $0 \leq j \leq a-1$, that is nonrepresentable in at least two distinct ways as a nonnegative integer combination of a, $a+d$ and $a+2d$ is equal to

$$\begin{aligned} S_{v_1} + a\left(\frac{a-1}{2} + d - 1\right) &= (a+d) + a\left(\frac{a-1}{2} + d - 1\right) \\ &= a\left(\frac{a-1}{2}\right) + d(a+1). \end{aligned}$$

Thus, the 2-Frobenius number of the Frobenuis basis $a, a+d, a+2d$ when $a \equiv 1 \pmod{2}$, $1 \leq d < a$ and $\gcd(a, d) = 1$ will be

$$\mathrm{F}_2(a, a+d, a+2d) = a\left(\frac{a-1}{2}\right) + d(a+1).$$

This completes the proof of Proposition 5.3.1. □

Furthermore, Lemma 5.3.3 shows that the largest integer number $M \equiv v_j \pmod{a}$, with $0 \leq j \leq a-1$, that is nonrepresented in at least two distinct ways always corresponds to the vertex v_1 in $G_w(\boldsymbol{a})$ (i.e. $j = 1$).

We now illustrate Proposition 5.3.1 by the following example.

Example 5.3.5. To compute the 2-Frobenius number of the arithmetic sequence $9, 13, 17$, we begin by finding the largest integer number

$$M_j \equiv v_j \pmod 9 \qquad 0 \le j \le 8\,.$$

that cannot be represented in at least two distinct ways. This means that, for each vertex v_j of $G_w(9, 13, 17)$ (as shown in Figure 5.6) we can associate a corresponding positive integer M_j which cannot be represented in least two distinct ways as a nonnegative integer linear combination of $9, 13$ and 17.

We give the calculations for the three cases, when $j \in \{0, 3, 8\}$, as follows:

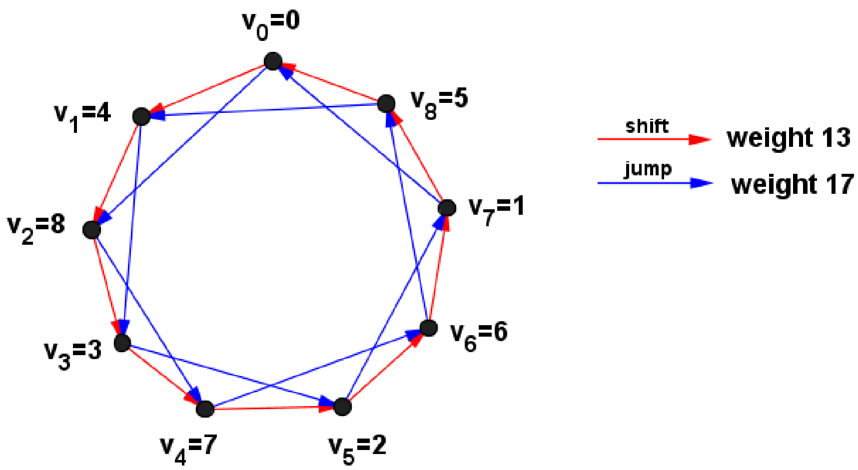

Figure 5.6: The circulant digraph of the arithmetic progression $9, 13, 17$

Let $j = 0$, we have to find the largest integer number

$$M_0 \equiv v_0 \equiv 0 \pmod 9\,,$$

that cannot represented in at least two distinct ways as a nonnegative integer linear combination of $9, 13, 17$. Therefore by Lemma 5.3.4 and Corollary 5.1.7,

$$\begin{aligned} M_0 = S_{v_0} - 9 &= (4(17) + 13) - 9 \\ &= 72\,. \end{aligned}$$

From Lemma 5.3.4, it follows that any positive integer $M_0 > 72$ is represented in at least two distinct ways in terms of $9, 13, 17$.

As, $81 \equiv 0 \pmod 9$ and 81 has at least two distinct representations in terms of $9, 13, 17$ as follows:

$$81 = 13 + 17(4) = 9(9)\,.$$

Let $j = 1$. Then by Lemma 5.3.3 and Corollary 5.1.7, we deduce that largest integer number

$$M_1 \equiv v_1 \equiv 4 \pmod 9\,,$$

will be

$$\begin{aligned} M_1 &= \left(S_{v_1} + 9(\frac{8}{2} + 4)\right) - 9 \\ &= 13 + 63 = 76\,. \end{aligned}$$

Using Lemma 5.3.3 we obtain that, any positive integer $M_1 > 76$ is represented in at least two distinct ways in terms of $9, 13, 17$.
As, $85 \equiv 4 \pmod 9$ and 85 has at least two distinct representations in terms of $9, 13, 17$, as follows:

$$85 = 9(8) + 13 = 17(5)\,.$$

Let $j = 8$. Then from Lemma 5.3.2 and Corollary 5.1.7 we get, the largest integer number

$$M_8 \equiv v_8 \equiv 5 \pmod 9\,,$$

is given by

$$M_8 = S_{v_8} = 17(4) = 68\,.$$

Hence Lemma 5.2.2 gives us, any positive integer $M_8 > 68$ is represented in at least two distinct ways in terms of $9, 13, 17$.
As see $95 \equiv 5 \pmod 9$ and 95 has at least two distinct representations terms of $9, 13, 17$, as follows:

$$95 = 9 + 13(4) + 17 = 13(6) + 17 = 9(3) + 17(4)\,.$$

Then by the same way we can find the others M_j, $j = 2, 3, \ldots, 7$ as shown in the Table 5.2. Hence from Proposition 5.3.1, 2-Frobenius number of the arithmetic progression $9, 13, 17$ is therefore

$$\mathrm{F}_2(9, 13, 17) = \max_{0 \le j \le 8} \{M_j\} = \max\{72, 76, 17, 30, 34, 47, 51, 64, 68\} = 76\,.$$

Table 5.2: The largest number $M_j \equiv v_j \pmod 9$ with $0 \le j \le 8$, that cannot represented in at least two distinct ways as a nonnegative integer linear combination of $9, 13$ and 17

vertices of $G_w(9,13,17)$									
j	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
v_j	0	4	8	3	7	2	6	1	5
M_j	72	76	17	30	34	47	51	64	68

Note that by (5.3.1),

$$\mathrm{F}_2(9,13,17) = 9\left(\frac{8}{2}\right) + 4(9+1) = 76\,.$$

In addition, let us present another two examples to compute the formula for $\mathrm{F}_2(\boldsymbol{a})$, using MATLAB.

Example 5.3.6. Let $\boldsymbol{a} = (357, 362, 367)^t$. The largest integer number which connot represented in at least two distinct ways in terms of $\boldsymbol{a}$ is

$$65336 = 182(357) + 362\,.$$

Hence the 2-Frobenius number will be

$$\mathrm{F}_2(357, 362, 367) = 65336\,.$$

Note that by Proposition 5.3.1

$$\mathrm{F}_2(357, 362, 367) = 357\left(\frac{357-1}{2}\right) + 5(357+1) = 65336\,.$$

Example 5.3.7. For $\boldsymbol{a} = (215, 221, 227)^t$, the largest integer number which connot represented in at least two distinct ways in terms of $215, 221, 227$ is

$$24301 = (112)215 + 221\,.$$

This implies that

$$\mathrm{F}_2(215, 221, 227) = 24301\,.$$

Note that by Proposition 5.3.1

$$\mathrm{F}_2(215, 221, 227) = 215\left(\frac{214}{2}\right) + 6(215+1) = 24301\,.$$

Now we are in a position to present the main result of this chapter.

Theorem 5.3.8. *Let* a *and* d *be coprime positive integers such that* $1 \leq d < a$*. Then*

$$\mathrm{F}_2(a, a+d, a+2d) = a \left\lfloor \frac{a}{2} \right\rfloor + d(a+1)\,. \tag{5.3.31}$$

Proof. The proof its follows immediately from Propositions 5.2.1 and 5.3.1. □

5.3.1 Conclusion for $\mathrm{F}_2(a, a+d, a+2d)$

Let $a, a+d, a+2d$ be positive integers with $1 \leq d < a$ and $\gcd(a, d) = 1$. Then we have

$$\mathrm{F}_2(a, a+d, a+2d) = \mathrm{F}_1(a, a+d, a+2d) + (a+2d)\,. \tag{5.3.32}$$

Chapter 6

The 2-Frobenius numbers of $\boldsymbol{a} = (a, a+d, a+2d, a+3d)^t$

In this chapter we extend the results of Chapter 5 by introducing the positive integer $a+3d$ to the arithmetic sequence $a, a+d, a+2d$ which used in Chapter 5 to be the 4 th term of it. This yields an explicit formula for computing the 2-Frobenius number $\mathrm{F}_2(a, a+d, a+2d, a+3d)$ for four integers in an arithmetic sequence.

We give a sketch of the proof of this formula omitting some technical details due to the size limitation. The method of proof employed here slightly different compared with that used in Chapter 5.

In order to simplify the argument, we first need to set up some notation. Let $G_w(\boldsymbol{a})$ be the circulant digraph associated with a positive integer vector $\boldsymbol{a} = (a, a+d, a+2d, a+3d)^t$ with $1 \le d < a$ and $\gcd(a, d) = 1$.

Recall that any arc on the graph $G_w(\boldsymbol{a})$ of weight $a+2d$ is the jump step, or jump and any arc of weight $a+d$ on the graph $G_w(\boldsymbol{a})$ is shift step or shift. Moreover, any arc on the graph $G_w(\boldsymbol{a})$ of weight $a+3d$ will be called a *long jump step*, or *long jump*. Then any path $\mathcal{T}$ in $G_w(\boldsymbol{a})$ that consists of K long jumps, L jumps and N shifts has the form

$$\mathcal{T} = K\mathcal{J}_l + L\mathcal{J} + N\mathcal{S}\,,$$

where $\mathcal{J}_l$, $\mathcal{J}$ and $\mathcal{S}$ stand for long jumps, jumps and shifts, respectively.

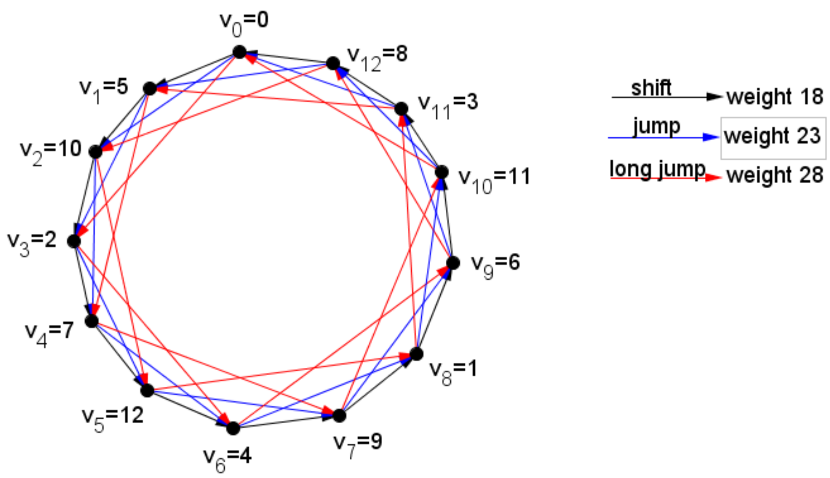

Figure 6.1: The Frobenius circulant graph of the arithmetic progression $13, 18, 23, 28$

For example Figure 6.1 shows the circulant digraph of the arithmetic progression $13, 18, 23, 28$.

Furthermore, $\deg^+_{G_w(\boldsymbol{a})}(v_j) = 3$, for $0 \le j \le a-1$, we have one shift $\mathcal{S}$ (i.e. an arc of weight $a+d$), namely

$$v_j + \mathcal{S} \equiv v_{j+1} \pmod a\,.$$

An one jump $\mathcal{J}$ (i.e. an arc of weight $a+2d$), namely

$$v_j + \mathcal{J} \equiv v_j + 2\,\mathcal{S} \equiv v_{j+2} \pmod a\,.$$

This implies that $\mathcal{J} \equiv 2\,\mathcal{S}$.
An one long jump $\mathcal{J}_l$ (i.e. an arc of weight $a+3d$), namely

$$v_j + \mathcal{J}_l \equiv v_j + \mathcal{J} + \mathcal{S} \equiv v_j + 3\,\mathcal{S} \equiv v_{j+3} \pmod a\,. \tag{6.0.1}$$

Hence, $\mathcal{J}_l \equiv \mathcal{J} + \mathcal{S} \equiv 3\,\mathcal{S}$, (see Figure 6.1).
Form (6.0.1), it can be seen that any path from v_j to v_{j+3} in $G_w(a, a+d, a+2d, a+3d)$ contains either a long jump or one jump and one shift or three shifts and since

$$a + 3d < (a+2d) + (a+d) < 3(a+d)\,.$$

Consequently, minimum weight of a path from v_j to v_{j+3} in $G_w(a, a+d, a+2d, a+3d)$ will be $a+3d$.

6.1 The shortest path method

The following theorem gives an explicit formula for the *shortest path* and the *distance between* any two distinct vertices of $G_w(a, a+d, a+2d, a+3d)$.

Theorem 6.1.1 (Minimum Path Theorem)**.** *The minimum path from vertex v_i to vertex v_j in $G_w(\boldsymbol{a})$, with $0 \le i < j \le a-1$, consists of exactly $\left(\frac{j-i-\delta}{3}\right)$ long jump steps, $\delta(2-\delta)$ shift steps and $\frac{\delta(\delta-1)}{2}$ jump steps. That is the minimum path from vertex v_i to vertex v_j is given by*

$$\left(\frac{j-i-\delta}{3}\right) \mathcal{J}_l + \delta(2-\delta)\,\mathcal{S} + \frac{\delta(\delta-1)}{2}\,\mathcal{J}\,,$$

where $\delta \equiv j-i \pmod 3$, with $\delta \in \{0,1,2\}$.

Proof. Let v_i and v_j be any two distinct vertices of $G_w(a, a+d, a+2d, a+3d)$. To find the minimum $v_i - v_j$ path, we have to consider three cases:

Case 1: Let us suppose that $j - i \equiv 0 \pmod 3$, (*i.e.* $\delta = 0$) and let K be the maximum number of long jumps in a path from v_i to v_j that does not pass the vertex v_j and where no vertex and no arc is repeated (i.e. $v_i + K\,\mathcal{J}_l \equiv v_j \pmod a$). Then any path from v_i to v_j can be written as

$$(K-M)\,\mathcal{J}_l + (3M-2N)\,\mathcal{S} + N\,\mathcal{J}\,. \tag{6.1.1}$$

where $K = \frac{j-i}{3}$. Since M and N must be positive integers then from (6.1.1) we get

$$0 \le M \le K \quad \text{and} \quad 0 \le N \le \left\lfloor \frac{3}{2} M \right\rfloor\,.$$

Substituting the weight for the long jump steps, shift steps and jump steps into (6.1.1), gives us

$$\begin{aligned}(K-M)\,(a+3d) + (3M-2N)\,(a+d) + N\,(a+2d)& \\ = K(a+3d) + 2Ma - Na\,.&\end{aligned}$$

Now let $c(M,N)$ be the weight function in terms of M and N defined by

$$c(M,N) = K(a+3d) + 2Ma - Na\,.$$

Since K, M, N, a, d are all positive integers and $N \leq 2M$, the minimum weight occurs when $N = 2M$. In particular, $N = 2M$ when $M = 0$ such that $N = 0$. Therefore we have

$$\min_{0 \leq M \leq K,\ 0 \leq N \leq \lfloor \frac{3M}{2} \rfloor} c(M, N) = c(0, 0) = K(a + 3d)\,. \qquad (6.1.2)$$

Substituting K into (6.1.2) we find that, the minimum weight of the path (distance) from v_i to v_j in $G_w(\boldsymbol{a})$, with $0 \leq i < j \leq a-1$ and $j - i \equiv 0 \pmod 3$, is given by

$$\frac{j-i}{3}(a + 3d)\,.$$

Consequently, the shortest path Q from v_i to v_j in $G_w(\boldsymbol{a})$, when $j - i \equiv 0 \pmod 3$, consists of exactly $\frac{j-i}{3}$ long jump steps. That is

$$Q = \frac{j-i}{3}\,\mathcal{J}_l\,.$$

Case 2: Let us suppose that $j - i \equiv 1 \pmod 3$, (*i.e.* $\delta = 1$) and let K be the maximum number of long jumps in a path from v_i to v_{j-1} that does not pass the vertex v_j and where no vertex and no arc is repeated (i.e. $v_i + K\,\mathcal{J}_l + \mathcal{S} \equiv v_j \pmod a$). Then any path from vertex v_i to vertex v_j in $G_w(\boldsymbol{a})$ can be written as

$$(K - M)\,\mathcal{J}_l + (3M - 2N + 1)\,\mathcal{S} + N\,\mathcal{J}\,, \qquad (6.1.3)$$

where $K = \frac{j-i-1}{3}$. Since M and N must be positive integers then from (6.1.3) we find that

$$0 \leq M \leq K \quad \text{and} \quad 0 \leq N \leq \left\lfloor \frac{3M+1}{2} \right\rfloor\,.$$

Substituting the weight for the long jump steps, shift steps and jump steps into (6.1.3) gives us

$$\begin{aligned}(K - M)\,(a + 3d) + (3M - 2N + 1)\,(a + d) + N\,(a + 2d)& \\ = K(a + 3d) + (a + d) + 2Ma - Na\,.&\end{aligned}$$

Now let

$$c(M, N) = K(a + 3d) + (a + d) + 2Ma - Na\,.$$

Since K, M, N, a, d are all positive integers and $0 \leq N \leq \lfloor \frac{3M+1}{2} \rfloor \leq 2M$, the minimum weight occurs when $N = 2M$. In particular $N = 2M$ if either $M = 0$ such that $N = 0$ or $M = 1$ such that $N = 2$. Thus

$$\min_{0 \leq M \leq K,\ 0 \leq N \leq \lfloor \frac{3M+1}{2} \rfloor} c(M, N) = c(0, 0) = c(1, 2) = K(a + 3d) + (a + d)\,. \qquad (6.1.4)$$

Substituting K into (6.1.4) yields the distance from v_i to v_j in $G_w(\boldsymbol{a})$, with $0 \leq i < j \leq a-1$ and $j-i \equiv 1 \pmod 3$, is

$$\frac{j-i-1}{3}(a+3d)+(a+d)\,.$$

Then, the minimum path Q from v_i to v_j in $G_w(\boldsymbol{a})$, when $j-i \equiv 1 \pmod 3$, consists of exactly $\frac{j-i-1}{3}$ long jump steps and one shift step. That is

$$Q=\frac{j-i-1}{3}\mathcal{J}_l+\mathcal{S}\,.$$

Case 3: Let us suppose that $j-i \equiv 2 \pmod 3$, ($i.e. \, \delta = 2$) and let K be the maximum number of long jumps in a path from v_i to v_{j-2} that does not pass the vertex v_j and where no vertex and no arc is repeated (i.e. $v_i + K\,\mathcal{J}_l + 2\,\mathcal{S} \equiv v_i + K\,\mathcal{J}_l + \mathcal{J} \equiv v_j \pmod a$). Then any path from v_i to v_j in $G_w(\boldsymbol{a})$ can be written as

$$(K-M)\,\mathcal{J}_l+(3M-2N+2)\,\mathcal{S}+N\,\mathcal{J}\,, \tag{6.1.5}$$

where $K=\frac{j-i-2}{3}$, $0 \leq M \leq K$ and $0 \leq N \leq \left\lfloor \frac{3M+2}{2} \right\rfloor$.

Substituting the weight for the long jump steps, shift steps and jump steps into expression (6.1.5), gives us

$$\begin{aligned}(K-M)\,(a+3d)+(3M-2N+2)\,(a+d)+N\,(a+2d)\\ =K(a+3d)+2(a+d)+2Ma-Na\,.\end{aligned}$$

Now let

$$c(M,N)=K(a+3d)+2(a+d)+2Ma-Na\,.$$

As we know that K, M, N, a, d are all positive integers and $0 \leq N \leq \left\lfloor \frac{3M+2}{2} \right\rfloor$, the minimum weight occurs when $N > 2M$. In particular $N > 2M$ when $M = 0$ and consequently $N = 1$. Hence we have

$$\min_{0\leq M\leq K,\; 0\leq N\leq \left\lfloor \frac{3M+2}{2} \right\rfloor} c(M,N)=c(0,1)=K(a+3d)+(a+2d)\,. \tag{6.1.6}$$

Substituting K into (6.1.6), we get the distance from v_i to v_j in $G_w(\boldsymbol{a})$, with $0 \leq i < j \leq a-1$ and $j-i \equiv 2 \pmod 3$ is

$$\frac{j-i-2}{3}(a+3d)+(a+2d)\,.$$

Therefore, the minimum path Q from v_i to v_j when $j-i \equiv 2 \pmod 3$, consists of exactly $\frac{j-i-2}{3}$ long jump steps and one jump step. That is

$$Q=\frac{j-i-2}{3}\mathcal{J}_l+\mathcal{J}\,.$$

Combining the results in the three cases given above, we see that the weight of any path from v_i to v_j in $G_w(\boldsymbol{a})$, for $0 \leq i < j \leq a-1$, can be written as

$$\begin{aligned}(K-M)\,(a+3d) + (3M-2N+\delta)\,(a+d) + N\,(a+2d) \\ = K\,(a+3d) + \delta(a+d) + 2Ma - Na\,.\end{aligned}$$

where

$\delta \equiv (j-i) \pmod 3, \text{with } \delta \in \{0,1,2\},$
$K = \dfrac{j-i-\delta}{3}\,;$
(i.e. K be the maximum number of long jumps in a path from v_i to $v_{j-\delta}$ that does not pass the vertex v_j and where no vertex and no arc is repeated) ,

$0 \leq M \leq K\,, \quad$ and
$0 \leq N \leq \left\lfloor \dfrac{3M+\delta}{2} \right\rfloor .$

Now let

$$c(M,N) = K\,(a+3d) + \delta(a+d) + 2Ma - Na\,.$$

Since the path needs to be minimum, then the value of M has to be minimum (i.e. $M = 0$), and the value of N has to be maximum (i.e. $N = \lfloor \frac{\delta}{2} \rfloor$). So for our purpose it can easily shown that

$$N = \left\lfloor \frac{\delta}{2} \right\rfloor = \frac{\delta(\delta-1)}{2},$$

since δ can only take the values $0,1,2$. Then the minimum weight of the path (distance) from v_i to v_j occurs when $M = 0$, and consequently $N = \frac{\delta(\delta-1)}{2}$. That is

$$
\begin{aligned}
\min_{0\le M\le K,\,0\le N\le\lfloor\frac{3M+\delta}{2}\rfloor} c(M,N) &= K(a+3d)+\delta(a+d)-\left\lfloor\frac{\delta}{2}\right\rfloor a\\
&= K(a+3d)+\delta(a+d)-\frac{\delta(\delta-1)}{2}a\\
&= K(a+3d)+2\delta(a+d)-\delta(a+d)-\frac{\delta(\delta-1)}{2}a\\
&= K(a+3d)+2\delta(a+d)-\frac{1}{2}\delta a-\delta d-\frac{1}{2}\delta^2 a\\
&= K(a+3d)+2\delta(a+d)-\frac{1}{2}\delta(a+2d)-\frac{1}{2}\delta^2 a\\
&= K(a+3d)+2\delta(a+d)-\delta^2(a+d)+\delta(\delta-1)d+\frac{\delta(\delta-1)}{2}a\\
&= K(a+3d)+\delta(2-\delta)(a+d)+\frac{\delta(\delta-1)}{2}(a+2d).
\end{aligned}
\tag{6.1.7}
$$

Substituting the value of K into (6.1.7) we obtain the distance from v_i to v_j in $G_w(\boldsymbol{a})$, with $0 \le i < j \le a-1$, is given by

$$\left(\frac{j-i-\delta}{3}\right)(a+3d)+\delta(2-\delta)(a+d)+\frac{\delta(\delta-1)}{2}(a+2d)\,.$$

This implies that, the minimum path from v_i to v_j in $G_w(\boldsymbol{a})$ with $0 \le i < j \le a-1$, consists of exactly $\left(\frac{j-i-\delta}{3}\right)$ long jump steps, $\delta(2-\delta)$ shift steps and $\frac{\delta(\delta-1)}{2}$ jump steps.

This completes the proof of Theorem 6.1.1. □

In the next theorem we give also a formula of the shortest path between any two vertices v_i and v_j in $G_w(\boldsymbol{a})$, that has opposite direction of the shortest path, that defined in Theorem 6.1.1 (i.e., in this case $i > j$).

Theorem 6.1.2. *The minimum path T from vertex v_i to vertex v_j in $G_w(\boldsymbol{a})$, with $0 \le j < i \le a-1$, consists of exactly $\left(\frac{a+j-i-\delta}{3}\right)$ long jump steps, $\delta(2-\delta)$ shift steps and $\frac{\delta(\delta-1)}{2}$ jump steps. That is*

$$T=\left(\frac{a+j-i-\delta}{3}\right)\mathcal{J}_l+\delta(2-\delta)\,\mathcal{S}+\frac{\delta(\delta-1)}{2}\,\mathcal{J}\,,$$

where $\delta \equiv a+j-i \ (mod\ 3)$, $\delta \in \{0,1,2\}$.

Proof. The graph $G_w(\boldsymbol{a})$ is a symmetric. Let R be the function that maps vertex v_i to vertex $v_0 = 0$ for all $1 \le i \le a-1$, so that $R(v_i) = v_0$ and $R(v_j) = v_{j+(a-i)}$ (from the geometry viewpoint we rotates v_i anti-clockwise by $\frac{a-i}{a}2\pi$ on the graph). Setting $j' = j + (a-i)$ gives $R(v_j) = v_{j'}$ and $R(v_i) = v_0$. Now we can apply Theorem 6.1.1 to deduce the result. $\square$

Combining Theorems 6.1.1 and 6.1.2, we immediately obtain the following theorem.

Theorem 6.1.3. *Let* $a' \equiv a \pmod 3$, *with* $a' \in \{0,1,2\}$. *For* $0 \le j \le a-1$ *the minimum (nontrivial) path* Q *from vertex* v_j *back to itself in* $G_w(\boldsymbol{a})$, *consists of exactly* $\frac{a-a'}{3}$ *long jump steps,* $a'(2-a')$ *shift steps and* $\frac{a'(a'-1)}{2}$ *jump steps. That is the minimum (nontrivial)* $v_j - v_j$ *path* Q *is given by*

$$Q = \frac{a-a'}{3}\,\mathcal{J}_l + a'(2-a')\,\mathcal{S} + \frac{a'(a'-1)}{2}\,\mathcal{J}\,.$$

Proof. Let v_j be any vertex of $G_w(a, a+d, a+2d, a+3d)$. We need to show that the minimum weight of a (nontrivial) path Q (or distance) from v_j back to v_j in $G_w(\boldsymbol{a})$, is

$$\frac{a-a'}{3}(a+3d) + a'(2-a')(a+d) + \frac{a'(a'-1)}{2}(a+2d)\,,$$

where $a' \equiv a \pmod 3$, $a' \in \{0,1,2\}$.
Since $\deg^-_{G_w}(v_j) = 3$, we have

$$\begin{aligned} v_{j-1} + \mathcal{S} &\equiv v_j \pmod a\,, \\ v_{j-2} + \mathcal{J} &\equiv v_j \pmod a\,, \quad \text{and} \\ v_{j-3} + \mathcal{J}_l &\equiv v_j \pmod a\,, \end{aligned}$$

where $\mathcal{S}$, $\mathcal{J}$ and $\mathcal{J}_l$ are arcs in $G_w(a, a+d, a+2d, a+3d)$ of weight $a+d$, $a+2d$ and $a+3d$, respectively.
Then, in order to take any (nontrivial) path from v_j back to v_j in $G_w(\boldsymbol{a})$. We will consider three possibilities according to the in-neighborhood $N_{G_w}(v_j)$ of the vertex v_j, (as illustrate in Figure 6.2).

1. A $v_j - v_j$ path P_1 has the form

$$P_1 = R \cup \mathcal{S}\,,$$

where R is any $v_j - v_{j-1}$ path and $\mathcal{S}$ is an arc from v_{j-1} to v_j of weight $a+d$. By using Theorems 6.1.2 and 6.1.1, the minimum weight x of the path P_1 is given by

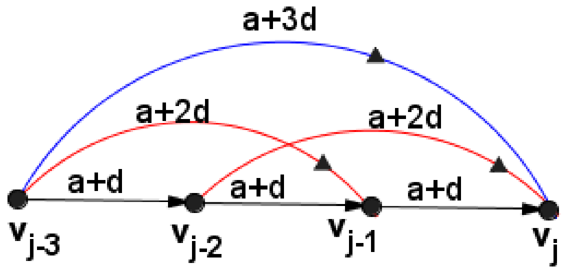

Figure 6.2: Three paths from vertex v_{j-3} to vertex v_j

$$x = \begin{cases} \left(\frac{a-3}{3}(a+3d) + (a+2d)\right) + (a+d), & \text{if } a \equiv 0 \pmod{3}\,, \\ \left(\frac{a-1}{3}(a+3d)\right) + (a+d), & \text{if } a \equiv 1 \pmod{3}\,, \\ \left(\frac{a-2}{3}(a+3d) + (a+d)\right) + (a+d), & \text{if } a \equiv 2 \pmod{3}\,. \end{cases} \tag{6.1.8}$$

2. A $v_j - v_j$ path P_2 has the form

$$P_2 = U \cup \mathcal{J}\,,$$

where U is any $v_j - v_{j-2}$ path and $\mathcal{J}$ is an arc from v_{j-2} to v_j of weight $a+2d$. Therefore from Theorems 6.1.2 and 6.1.1, the minimum weight y of the path P_2 will be

$$y = \begin{cases} \left(\frac{a-3}{3}(a+3d) + (a+d)\right) + (a+2d), & \text{if } a \equiv 0 \pmod{3}\,, \\ \left(\frac{a-4}{3}(a+3d) + (a+2d)\right) + (a+2d), & \text{if } a \equiv 1 \pmod{3}\,, \\ \left(\frac{a-2}{3}(a+3d)\right) + (a+2d), & \text{if } a \equiv 2 \pmod{3}\,. \end{cases} \tag{6.1.9}$$

3. A $v_j - v_j$ path P_3 has form

$$P_3 = V \cup \mathcal{J}_l\,,$$

where V is any $v_j - v_{j-3}$ path and $\mathcal{J}_l$ is an arc from v_{j-3} to v_j of weight $a+3d$. Then again from Theorems 6.1.2 and 6.1.1, the minimum weight z of the path P_3 is given by

$$z = \begin{cases} \left(\frac{a-3}{3}(a+3d)\right) + (a+3d), & \text{if } a \equiv 0 \pmod 3\,, \\ \left(\frac{a-4}{3}(a+3d) + (a+d)\right) + (a+3d), & \text{if } a \equiv 1 \pmod 3, \\ \left(\frac{a-5}{3}(a+3d) + (a+2d)\right) + (a+3d), & \text{if } a \equiv 2 \pmod 3\,. \end{cases} \tag{6.1.10}$$

For example, Figure 6.3 shows the shortest path from v_2 back to itself in $G_w(11, 15, 19, 23)$.

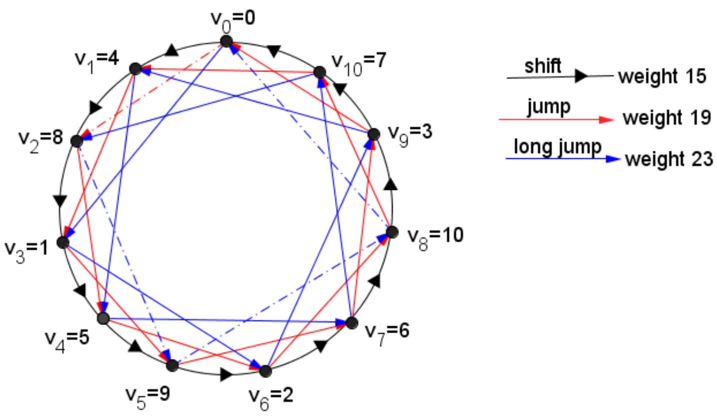

Figure 6.3: The shortest (nontrivial) path from v_2 back to v_2 in $G_w(11, 15, 19, 23)$ consists of exactly 3 long jumps and one jump

Therefore, by comparing (6.1.8), (6.1.9) and (6.1.10), we conclude that the minimum weight of a (nontrivial) path from v_j back to v_j, will be the weight z. We can rewrite the weight z as follows

$$z = \frac{a - a'}{3}(a+3d) + a'(2-a')(a+d) + \frac{a'(a'-1)}{2}(a+2d)\,,$$

where $a' \equiv a \pmod 3$, $a' \in \{0, 1, 2\}$.

Consequently, the minimum weight of a (nontrivial) path (distance) from v_j back to itself, is given by

$$\frac{a - a'}{3}(a+3d) + a'(2-a')(a+d) + \frac{a'(a'-1)}{2}(a+2d)\,.$$

Thus, the minimum (nontrivial) path Q from v_j back to itself, consists of exactly $\frac{a-a'}{3}$ long jumps, $a'(2-a')$ shifts and $\frac{a'(a'-1)}{2}$ jumps. That is

$$Q = \frac{a-a'}{3}\,\mathcal{J}_l + a'(2-a')\,\mathcal{S} + \frac{a'(a'-1)}{2}\,\mathcal{J}\,.$$

The theorem is proved. □

Corollary 6.1.4 (To Theorems 6.1.1 and 6.1.2)**.** *For any $0 \le j \le a-1$, let S_{v_j} be the minimum weight of the path from 0 to v_j in $G_w(a, a+d, a+2d, a+3d)$. Then*

$$S_{v_j} = \begin{cases} \frac{a-a'}{3}(a+3d) + \frac{a'(a'-1)}{2}(a+2d) + a'(2-a')(a+d)\,, & \text{if } j = 0\,, \\ \\ \frac{j-1}{3}(a+3d) + (a+d)\,, & \text{if } j \equiv 1 \ (mod\ 3)\,, \\ \\ \frac{j-2}{3}(a+3d) + (a+2d)\,, & \text{if } j \equiv 2 \ (mod\ 3)\,, \\ \\ \frac{j}{3}(a+3d)\,, & \text{if } j \equiv 0 \ (mod\ 3), j \neq 0\,, \end{cases}$$

where $a' \equiv a \ (mod\ 3)$, $a' \in \{0,1,2\}$.

Proof. The proof follows directly from Theorems 6.1.1 and 6.1.2. □

More generally, according to Corollaries 5.1.7 and 6.1.4, we propose the following conjecture.

Conjecture 1. *For any $0 < j \le a-1$, let S_{v_j} be the minimum weight of the path from $v_0 = 0$ to v_j in $G_w(a, a+d, a+2d, \ldots, a+nd)$. Then*

$$S_{v_j} = \begin{cases} \frac{j-t}{n}(a+nd) + (a+td)\,, & \text{if } j \equiv t \ (mod\ n)\,, 1 \le t < n \\ \\ \frac{j}{n}(a+nd)\,, & \text{if } j \equiv 0 \ (mod\ n), j \neq 0\,. \end{cases}$$

An important step in the proof of the main result in this chapter is the following theorem.

Theorem 6.1.5 (Unique Representation of S_{v_j})**.** *With $1 \le j \le a-1$, the minimum weight S_{v_j} of the path from 0 to v_j, given in Corollary 6.1.4, has exactly one representation in terms of $a, a+d, a+2d$ and $a+3d$ when $j \equiv 0 \ (mod\ 3)$, $j \neq 0$ or $j \equiv 2 \ (mod\ 3)$ or $j = 1$.*

Proof. Assume, to the contrary, that S_{v_j} for $1 < j \leq a-1$, can be represented in at least two distinct ways. There exits nonnegative integers $x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4$ with $x_j \neq y_j$ such that

$$\begin{aligned} S_{v_j} &= ax_1 + (a+d)x_2 + (a+2d)x_3 + (a+3d)x_4\,, \qquad \text{and} \\ S_{v_j} &= ay_1 + (a+d)y_2 + (a+2d)y_3 + (a+3d)y_4\,. \end{aligned}$$

we will consider three cases, according as $j \equiv 0 \pmod 3$, $j \neq 0$ or $j \equiv 2 \pmod 3$ or $j = 1$.

Case 1: Suppose that $j \equiv 0 \pmod 3$, $j \neq 0$. Then by Corollary 6.1.4

$$S_{v_j} = \frac{j}{3}(a+3d)\,.$$

By assumption, S_{v_j} can be represented in at least two distinct ways, as

$$S_{v_j} = \frac{j}{3}(a+3d) = ay_1 + (a+d)y_2 + (a+2d)y_3 + (a+3d)y_4\,. \tag{6.1.11}$$

Hence

$$\left(\frac{j}{3} - y_1 - y_2 - y_3 - y_4\right)a = \left(y_2 + 2y_3 + 3y_4 - j\right)d\,.$$

This means that either $\gcd(a,d) \neq 1$, which contradicts our assumption, or

$$j - 3(y_1 + y_2 + y_3 + y_4) = 3dt\,, \tag{6.1.12a}$$

$$y_2 + 2y_3 + 3y_4 - j = at\,, \tag{6.1.12b}$$

with $t \in \mathbb{Z}$. We now have three options for t. If $t = 0$, then from (6.1.12a) and (6.1.12b) we have

$$3y_1 + 2y_2 + y_3 = 0.$$

It follows that $y_3 = y_1 = y_2 = 0$ and from (6.1.11), implying

$$y_4 = \frac{j}{3}.$$

Thus, the representations of S_{v_j} in (6.1.11) are the same. If $t > 0$, therefore (6.1.12a) and (6.1.12b) gives us

$$3dt + at + 3y_1 + 2y_2 + y_3 = 0,$$

contradicting the fact that $a > 1$ and $d \geq 1$.

Finally, if $t < 0$, then $t = -h$, where h is a positive integer number. Substituting t into (6.1.12b) we obtain

$$y_2 + 2y_3 + 3y_4 + ah = j.$$

This implies that $j \geq ah$, which again contradicts our assumption, that $j \leq a-1$.

Therefore, we have shown that S_{v_j}, is represented in exactly one way in terms of a, $a+d$, $a+2d$ and $a+3d$, when $j \equiv 0 \pmod 3$, $j \neq 0$.

Case 2: Suppose that $j \equiv 2 \pmod 3$. Then by Corollary 6.1.4

$$S_{v_j} = \frac{j-2}{3}(a+3d) + (a+2d)\,.$$

Since S_{v_j} is represented in at least two distinct ways, we have

$$S_{v_j} = \frac{j-2}{3}(a+3d) + (a+2d) = ay_1 + (a+d)y_2 + (a+2d)y_3 + (a+3d)y_4 \tag{6.1.13}$$

and, consequently

$$\left(\frac{j+1}{3} - y_1 - y_2 - y_3 - y_4\right) a = \left(-j + y_2 + 2y_3 + 3y_4\right) d\,.$$

Now as $\gcd(a,d) = 1$ we must have

$$(j+1) - 3(y_1 + y_2 + y_3 + y_4) = 3dt\,, \tag{6.1.14a}$$

$$y_2 + 2y_3 + 3y_4 - j = at\,, \tag{6.1.14b}$$

with $t \in \mathbb{Z}$. Again there are three options for t. If $t = 0$, then we deduce from (6.1.14a) and (6.1.14b) that

$$3y_1 + 2y_2 + y_3 = 1.$$

This implies $y_3 = 1$ and $y_1 = y_2 = 0$. Hence by (6.1.13),

$$y_4 = \frac{j-2}{3}.$$

Hence, the two representations of S_{v_j} in (6.1.13) are the same. If $t > 0$ then by (6.1.14a) and (6.1.14b) we obtain

$$3dt + at + 3y_1 + 2y_2 + y_3 = 1.$$

Which contradicts the fact that $a > 1$ and $d \geq 1$. Finally, if $t < 0$, then $t = -h$, where h is a positive integer. Substituting t into (6.1.14b), yields

$$y_2 + 2y_3 + 3y_4 + ah = j.$$

It follows that, $j \geq ah$, which contradicts $0 \leq j \leq a-1$.

Thus, we have proved that S_{v_j}, is represented in exactly one way in terms of a, $a+d$, $a+2d$ and $a+3d$ when $j \equiv 2 \pmod 3$.

Case 3: Suppose that $j = 1$. Then by Corollary 6.1.4

$$S_{v_1} = a + d\,.$$

By combining the above three cases, we deduce that minimum weight S_{v_j} of the path from v_0 to v_j, in $G_w(a, a+d, a+2d, a+3d)$, for $1 < j \le a-1$, has exactly one representation in terms of a, $a+d$, $a+2d$ and $a+3d$ when $j \equiv 0 \pmod 3$, $j \neq 0$ or $j \equiv 2 \pmod 3$ or $j = 1$. ☐

Corollary 6.1.6. *For $0 \le j \le a-1$, the minimum weight S_{v_j} of the (nontrivial) path given in Corollary 6.1.4, has two distinct representations in terms of a, $a+d$, $a+2d$ and $a+3d$ when $j \equiv 1$ (mod 3), $j \neq 1$ or $j = 0$.*

Proof. Let S_{v_j} be the minimum weight of the (nontrivial) v_0 to v_j path in $G_w(a, a+d, a+2d, a+3d)$. We need to show that S_{v_j} can be represented in at least two distinct ways as a nonnegative integer linear combination of a, $a+d$, $a+2d$ and $a+3d$, when $j \equiv 1 \pmod 3$, $j \neq 1$ or $j = 0$.

Case 1: Let $j \equiv 1 \pmod 3$, $j \neq 1$. Then by Corollary 6.1.4

$$S_{v_j} = \frac{j-1}{3}(a+3d) + (a+d).$$

Since $\gcd(a, d) = 1$, we can write S_{v_j} as

$$\begin{aligned} S_{v_j} &= \frac{j-1}{3}(a+3d) + (a+d)\,, \\ \text{and} \qquad S_{v_j} &= \frac{j-4}{3}(a+3d) + 2(a+2d)\,. \end{aligned}$$

Hence, S_{v_j} for $0 \le j \le a-1$ can be represented in two distinct ways in terms of a, $a+d$, $a+2d$ and $a+3d$, when $j \equiv 1 \pmod 3$, $j \neq 1$.

Case 2: Let $j = 0$. Then by Corollary 6.1.4

$$S_{v_0} = \frac{a - a'}{3}(a+3d) + \frac{a'(a'-1)}{2}(a+2d) + a'(2-a')(a+d)\,,$$

where $a' \equiv a \pmod 3$, $a' \in \{0, 1, 2\}$.

Therefore, we can write S_{v_0} as

$$\begin{aligned}
S_{v_0} &= \frac{a-a^{'}}{3}(a+3d) + \frac{a^{'}(a^{'}-1)}{2}(a+2d) + a^{'}(2-a^{'})(a+d)\,,\\
S_{v_0} &= \left(\frac{a^{'}(2-a^{'})(a-a^{'}-3)}{3}\right)(a+3d) + \left(2a^{'}(2-a^{'})\right)(a+2d)\,,\\
\text{and} \qquad S_{v_0} &= \left(\frac{a-a^{'}}{3} + \frac{a^{'}(3-a^{'})}{2} + d\right)a\,.
\end{aligned}$$

This implies that, S_{v_0} can be represented in at least two distinct ways in terms of a, $a+d$, $a+2d$ and $a+3d$.

By combining results of the above two cases, we complete the proof. $\square$

We now state a conjecture, based on the result and other empirical result observed.

Conjecture 2. *Let $0 < j \leq a-1$. Then the minimum weight S_{v_j} of the path from v_0 to v_j, given in Conjecture 1, has exactly one representation in terms of $a, a+d, \ldots, a+nd$, when $j \equiv 0 \ (mod\ n), j \neq 0$ or $j \equiv n-1 \ (mod\ n)$ or $j = 1$, and otherwise S_{v_j} has at least two distinct representations in terms of $a, a+d, \ldots, a+nd$.*

6.2 The 2-Frobenius number of $\boldsymbol{a} = (a, a+d, a+2d, a+3d)^t$

In this section we give an explicit formula for $\mathrm{F}_2(a, a+d, a+2d, a+3d)$ of a 4-terms arithmetic progression with $\gcd(a, d) = 1$.

We are now ready to sketch a proof of the main theorem of this chapter.

Theorem 6.2.1 (Main Theorem). *Let $\boldsymbol{a} = (a, a+d, a+2d, a+3d)^t$ be a positive integer vector with $1 \leq d < a$ and $\gcd(a, d) = 1$. Then*

$$\mathrm{F}_2(a, a+d, a+2d, a+3d) = \left\lfloor \frac{a}{3} \right\rfloor + d(a+1)\,. \tag{6.2.1}$$

Proof. Let v_j by any vertex of $G_w(\boldsymbol{a})$ with $0 \leq j \leq a-1$ and let $M \in \mathbb{Z}_{>0}$. Then

$$M \equiv v_j \pmod{a}. \tag{6.2.2}$$

In order to prove Theorem 6.2.1, we need three lemmas.

Lemma 6.2.2. *For* $1 \le j \le a-1$, $j \equiv 0, 2$ *(mod 3) the positive integer* $M \equiv v_j$ *(mod* a*) is representable in at least two distinct ways as a nonnegative integer linear combination of* $a, a+d, a+2d$ *and* $a+3d$ *if and only if* $M \ge S_{v_j} + a$*.*

Proof. First, we assume that $M \ge S_{v_j} + a$. We need to show that M can be represented in at least two distinct ways.

By (4.2.4), $v_j \equiv S_{v_j} \pmod a$ so that $v_j \equiv (S_{v_j} + a) \pmod a$. Thus we have

$$M \equiv (S_{v_j} + a) \pmod a \quad \textit{and} \quad M \ge S_{v_j} + a\,.$$

It follows that there is a nonnegative integer t such that

$$M = (S_{v_j} + a) + ta\,.$$

By Corollary 6.1.4

$$S_{v_j} = \begin{cases} \frac{j}{3}(a+3d)\,, & \text{if } j \equiv 0 \pmod 3,\ j \neq 0, \\ \\ \frac{j-2}{3}(a+3d) + (a+2d)\,, & \text{if } j \equiv 2 \pmod 3)\,. \end{cases}$$

Therefore, for $j \equiv 0 \pmod 3$, we can write M as

$$\begin{aligned} M &= a(t+1) + \frac{j}{3}(a+3d)\,, \quad \text{and} \\ M &= at + (a+d) + (a+2d) + \left(\frac{j-3}{3}\right)(a+3d)\,. \end{aligned}$$

For $j \equiv 2 \pmod 3$, we can write M as

$$\begin{aligned} M &= a(t+1) + (a+2d) + \left(\frac{j-2}{3}\right)(a+3d)\,, \quad \text{and} \\ M &= at + 2(a+d) + \left(\frac{j-2}{3}\right)(a+3d)\,. \end{aligned}$$

Consequently, M is represented in at least two distinct ways as a nonnegative integer linear combination of $a, a+d, a+2d$ and $a+3d$, when $j \equiv 0, 2 \pmod 3$.

Conversely, assume that M has at least two distinct representations, so that there exist nonnegative integers $x_1, y_1, z_1, w_1, x_2, y_2, z_2, w_2$ such that

$$M = ax_1 + (a+d)y_1 + (a+2d)z_1 + (a+3d)w_1 = ax_2 + (a+d)y_2 + (a+2d)z_2 + (a+3d)w_2\,. \tag{6.2.3}$$

We have to prove that

$$M \geq S_{v_j} + a\,.$$

Since $M \equiv v_j \pmod{a}$, (6.2.3) gives us

$$\begin{aligned} M \equiv (a+d)y_1 + (a+2d)z_1 + (a+3d)w_1 &\equiv (a+d)y_2 + (a+2d)z_2 + (a+3d)w_2 \\ &\equiv v_j \equiv S_{v_j} \pmod{a}\,. \end{aligned} \tag{6.2.4}$$

Therefore, it follows from (4.2.3) that we have two paths from vertex $v_0 = 0$ to v_j in $G_w(\boldsymbol{a})$ of weights

$$(a+d)y_1 + (a+2d)z_1 + (a+3d)w_1 \quad \text{and} \quad (a+d)y_2 + (a+2d)z_2 + (a+3d)w_2\,.$$

Hence, we have three possibilities for the minimum weight S_{v_j} of a path from vertex 0 to v_j:

1. Assume that

   $$(a+d)y_1 + (a+2d)z_1 + (a+3d)w_1 = (a+d)y_2 + (a+2d)z_2 + (a+3d)w_2 = S_{v_j}\,.$$

   This implies that, the minimum weight S_{v_j} of a path vertex 0 to v_j in $G_w(\boldsymbol{a})$, can be represented in two distinct ways as a nonnegative integer linear combination of $a+d, a+2d$ and $a+3d$, when $j \equiv 0 \pmod 3$, $j \neq 0$ or $j \equiv 2 \pmod 3$. This contradicts Theorem 6.1.5.

2. Assume that

   $$(a+d)y_1 + (a+2d)z_1 + (a+3d)w_1 = (a+d)y_2 + (a+2d)z_2 + (a+3d)w_2 > S_{v_j}\,.$$

   Then from (6.2.4) there exist a positive integer h such that

   $$\begin{aligned} (a+d)y_1 + (a+2d)z_1 + (a+3d)w_1 &= (a+d)y_2 + (a+2d)z_2 + (a+3d)w_2 \\ &= S_{v_j} + ah \geq S_{v_j} + a\,. \end{aligned}$$

   Using (6.2.3), $M = ax_1 + (a+d)y_1 + (a+2d)z_1 + (a+3d)w_1$, we deduce that

   $$M \geq (a+d)y_1 + (a+2d)z_1 + (a+3d)w_1 \geq S_{v_j} + a\,.$$

3. W.l.o.g. we may assume that

   $$(a+d)y_1 + (a+2d)z_1 + (a+3d)w_1 > (a+d)y_2 + (a+2d)z_2 + (a+3d)w_2 \geq S_{v_j}\,.$$

Thus, the weight $(a+d)y_2 + (a+2d)z_2 + (a+3d)w_2$, has to be at least minimum weight S_{v_j} of a path from vertex 0 to v_j in $G_w(\boldsymbol{a})$. From (6.2.4), there exist a positive integer h such that

$$\begin{aligned}(a+d)y_1 + (a+2d)z_1 + (a+3d)w_1 &= (a+d)y_2 + (a+2d)z_2 + (a+3d)w_2 + ah \\ &\geq S_{v_j} + ah \geq S_{v_j} + a\,.\end{aligned}$$

Since $M = ax_1 + (a+d)y_1 + (a+2d)z_1 + (a+3d)w_1$, we find that that

$$M \geq (a+d)y_1 + (a+2d)z_1 + (a+3d)w_1 \geq S_{v_j} + a\,,$$

as required. □

Collectively considering the above cases, we have shown that the largest integer $M \equiv v_j \pmod a$ with $1 \leq j \leq a-1$ and $j \equiv 0 \pmod 3$, $j \neq 0$ or $j \equiv 2 \pmod 3$, that is nonrepresentable in at least two distinct ways as a nonnegative integer linear combination of a, $a+d$, $a+2d$ and $a+3d$, is given by

$$M = \big(S_{v_j} + a\big) - a = S_{v_j}\,.$$

Lemma 6.2.3. *Let* $0 \leq j \leq a-1$, $j \equiv 1, 0 \pmod 3$, $j \neq 1$ *the positive integer* $M \equiv v_j \pmod a$ *is representable in at least two distinct ways as a nonnegative integer linear combination of* $a, a+d, a+2d$ *and* $a+3d$ *if and only if* $M \geq S_{v_j}$.

Proof. Let us assume that $M \geq S_{v_j}$. We have to show that M can be represented in at least two distinct ways. (4.2.4), $v_j \equiv S_{v_j} \pmod a$. Thus we have

$$M \equiv S_{v_j} \pmod a \quad \textit{and} \quad M \geq S_{v_j}\,.$$

It follows that there is a nonnegative integer t such that

$$M = S_{v_j} + ta\,.$$

From Corollary 5.1.7

$$S_{v_j} = \begin{cases} \frac{j-1}{3}(a+3d) + (a+d)\,, & \text{if } j \equiv 1 \pmod 3\,, \\ \\ \frac{a-a'}{3}(a+3d) + a'(2-a')(a+d) + \frac{a'(a'-1)}{2}(a+2d)\,, & \text{if } j = 0\,, \end{cases}$$

where $a^{'} = a \pmod 3$, with $a^{'} \in \{0, 1, 2\}$.

Then, for $j \equiv 1 \pmod 3$, we can write M as

$$\begin{aligned} M &= at + (a+d) + \left(\frac{j-1}{3}\right)(a+3d)\,, \quad \text{and} \\ M &= at + 2(a+2d) + \left(\frac{j-4}{3}\right)(a+3d)\,. \end{aligned}$$

For $j = 0$, we can write M as

$$\begin{aligned} M &= at + (\frac{a-a^{'}}{3})(a+3d) + a^{'}(2-a^{'})(a+d) + \frac{a^{'}(a^{'}-1)}{2}(a+2d)\,, \\ M &= at + \left(\frac{a^{'}(2-a^{'})(a-a^{'}-3)}{3}\right)(a+3d) + \left(2a^{'}(2-a^{'})\right)(a+2d)+\,, \quad \text{and} \\ M &= a\left(t + \frac{a-a^{'}}{3} + \frac{a^{'}(3-a^{'})}{2} + d\right)\,. \end{aligned}$$

Hence, M is represented in at least two distinct ways as a nonnegative integer linear combination of $a, a+d, a+2d$ and $a+3d$, when $j \equiv 1, 0 \pmod 3$, $j \neq 1$.

Conversely, now let M has at least two different representations, so that there exist nonnegative integers $x_1, y_1, z_1, w_1, x_2, y_2, z_2, w_2$ such that

$$M = ax_1 + (a+d)y_1 + (a+2d)z_1 + (a+3d)w_1 = ax_2 + (a+d)y_2 + (a+2d)z_2 + (a+3d)w_2\,. \tag{6.2.5}$$

We have to prove that

$$M \geq S_{v_j}\,.$$

Since $M \equiv v_j \pmod a$, then (6.2.5) gives us

$$M \equiv (a+d)y_1 + (a+2d)z_1 + (a+3d)w_1 \equiv (a+d)y_2 + (a+2d)z_2 + (a+3d)w_2 \equiv v_j \pmod a\,. \tag{6.2.6}$$

Thus from (6.2.6) and (4.2.3) implies we have two paths from v_0 to v_j in $G_w(\boldsymbol{a})$ of weights

$$(a+d)y_1 + (a+2d)z_1 + (a+3d)w_1 \quad \text{and} \quad (a+d)y_2 + (a+2d)z_2 + (a+3d)w_2\,.$$

Thus w.l .o.g. we can assume

$$(a+d)y_1 + (a+2d)z_1 + (a+3d)w_1 \geq (a+d)y_2 + (a+2d)z_2 + (a+3d)w_2\,.$$

So by applying Corollary 6.1.6, the minimum weight S_{v_j} of a path from v_0 to v_j can be represented in two distinct ways when $j \equiv 1 \pmod 3$, $j \neq 1$ or $j = 0$, we deduce that

$$(a+d)y_1 + (a+2d)z_1 + (a+3d)w_1 \geq (a+d)y_2 + (a+2d)z_2 + (a+3d)w_2 \geq S_{v_j}\,.$$

Thus from (6.2.6), there exist a nonnegative integer h such that

$$\begin{aligned}(a+d)y_1 + (a+2d)z_1 + (a+3d)w_1 &= (a+d)y_2 + (a+2d)z_2 + (a+3d)w_2 + ah \\ &\geq S_{v_j} + ah \geq S_{v_j}\,.\end{aligned} \tag{6.2.7}$$

Since, $M = ax_1 + (a+d)y_1 + (a+2d)z_1 + (a+3d)w_1$, we get

$$M \geq (a+d)y_1 + (a+2d)z_1 + (a+3d)w_1 \geq S_{v_j}\,,$$

as required. □

Therefore we have proved that the largest integer $M \equiv v_j \pmod a$ with $0 \leq j \leq a-1$ and $j \equiv 1 \pmod 3$, $j \neq 1$ or $j = 0$, that is nonrepresentable in at least two distinct ways as a nonnegative integer linear combination of a, $a+d$, $a+2d$ and $a+3d$ is

$$M = S_{v_j} - a\,.$$

Lemma 6.2.4. *For $j = 1$, the number $M \equiv v_j \pmod a$ is representable in at least two distinct ways as a nonnegative integer linear combination of $a, a+d, a+2d$ and $a+3d$ if and only if $M \geq S_{v_1} + a\left(\left\lfloor \frac{a}{3} \right\rfloor + d\right)$.*

Proof. Let $M \geq S_{v_1} + a\left(\left\lfloor \frac{a}{3} \right\rfloor + d\right)$. We need to show that M can be represented in at least two distinct ways. By (4.2.4), $v_1 \equiv S_{v_1} \pmod a$ so that $v_1 \equiv S_{v_1} + a\left(\left\lfloor \frac{a}{3} \right\rfloor + d\right) \pmod a$. Thus, we have

$$M \equiv S_{v_1} + a\left(\left\lfloor \frac{a}{3} \right\rfloor + d\right) \pmod a \quad \textit{and} \quad M \geq S_{v_1} + a\left(\left\lfloor \frac{a}{3} \right\rfloor + d\right).$$

Consequently, there is a nonnegative integer t such that

$$M = S_{v_1} + a\left(\left\lfloor \frac{a}{3} \right\rfloor + d\right) + ta\,,$$

by Corollary 5.1.7

$$S_{v_1} = a + d\,.$$

Observe that

$$\left\lfloor \frac{a}{3} \right\rfloor = \begin{cases} \frac{a}{3}, & \text{if } a \equiv 0 \pmod 3, \\ \frac{a-1}{3}, & \text{if } a \equiv 1 \pmod 3, \\ \frac{a-2}{3}, & \text{if } a \equiv 2 \pmod 3. \end{cases}$$

Therefore, for $a \equiv 0 \pmod 3$, we can write M as

$$\begin{aligned} M &= a\left(\frac{a}{3} + d + t\right) + (a+d), \\ M &= at + (a+d) + \frac{a}{3}(a+3d), \quad \text{and} \\ M &= at + 2(a+2d) + \left(\frac{a-3}{3}\right)(a+3d). \end{aligned}$$

For $a \equiv 1 \pmod 3$, we can write M as

$$\begin{aligned} M &= a\left(\frac{a-1}{3} + d + t\right) + (a+d), \quad \text{and} \\ M &= at + (a+2d) + \left(\frac{a-1}{3}\right)(a+3d). \end{aligned}$$

For $a \equiv 2 \pmod 3$,

$$\begin{aligned} M &= a\left(\frac{a-2}{3} + d + t\right) + (a+d), \quad \text{and} \\ M &= at + \left(\frac{a+1}{3}\right)(a+3d). \end{aligned}$$

Hence, M is represented in at least two distinct ways as a nonnegative integer linear combination of $a, a+d, a+2d$ and $a+3d$ when $j = 1$.

Conversely, let us assume that M has at least two distinct representations, so that there exist nonnegative integers $x_1, y_1, z_1, w_1, x_2, y_2, z_2, w_2$ such that

$$M = ax_1 + (a+d)y_1 + (a+2d)z_1 + (a+3d)w_1 = ax_2 + (a+d)y_2 + (a+2d)z_2 + (a+3d)w_2. \quad (6.2.8)$$

We are required to prove that

$$M \geq S_{v_1} + a\left(\left\lfloor \frac{a}{3} \right\rfloor + d\right) = S_{v_1} + a\left(\frac{a-a'}{3} + d\right),$$

where $a^{'} \equiv a \pmod 3$, $a^{'} \in \{0, 1, 2\}$.
Using (6.2.2), $M \equiv v_1 \pmod a$, and (6.2.8), we get

$$\begin{aligned} M \equiv (a+d)y_1 + (a+2d)z_1 + (a+3d)w_1 &\equiv (a+d)y_2 + (a+2d)z_2 + (a+3d)w_2 \\ &\equiv v_1 \equiv S_{v_1} \pmod a . \end{aligned} \tag{6.2.9}$$

Then from (4.2.3) implies, we have two paths from v_0 to v_1 in $G_w(\boldsymbol{a})$ of weights

$$(a+d)y_1 + (a+2d)z_1 + (a+3d)w_1 \quad \text{and} \quad (a+d)y_2 + (a+2d)z_2 + (a+3d)w_2 .$$

So there are three possibilities to consider according to minimum weight S_{v_1} of a $v_0 - v_1$ path.

1. Assume that

   $$(a+d)y_1 + (a+2d)z_1 + (a+3d)w_1 = (a+d)y_2 + (a+2d)z_2 + (a+3d)w_2 = S_{v_1} .$$

   This implies that, S_{v_1} has two distinct representations in terms of $a, a+d, a+2d, a+3d$. This contradicts that $S_{v_1} = a+d$ is represented in exactly one way.

2. Assume that

   $$(a+d)y_1 + (a+2d)z_1 + (a+3d)w_1 = (a+d)y_2 + (a+2d)z_2 + (a+3d)w_2 > S_{v_1} .$$

   Hence by (6.2.9), there exist a positive integer h such that

   $$\begin{aligned} (a+d)y_1 + (a+2d)z_1 + (a+3d)w_1 &= (a+d)y_2 + (a+2d)z_2 + (a+3d)w_2 \\ &= S_{v_1} + ah . \end{aligned} \tag{6.2.10}$$

3. W.l.o.g. we may assume that

   $$(a+d)y_1 + (a+2d)z_1 + (a+3d)w_1 > (a+d)y_2 + (a+2d)z_2 + (a+3d)w_2 \geq S_{v_1} .$$

   This implies that, the weight $(a+d)y_2 + (a+2d)z_2 + (a+3d)w_2$ has to be at least minimum weight S_{v_1} of a $v_0 - v_1$ path in $G_w(\boldsymbol{a})$. Then by (6.2.9), there exist a positive integer h such that

   $$\begin{aligned} (a+d)y_1 + (a+2d)z_1 + (a+3d)w_1 &= (a+d)y_2 + (a+2d)z_2 + (a+3d)w_2 + ah \\ &\geq S_{v_1} + ah . \end{aligned} \tag{6.2.11}$$

Then from (6.2.8), $M = ax_1 + (a+d)y_1 + (a+2d)z_1 + (a+3d)w_1$, (6.2.10) and (6.2.11), we have

$$M \geq (a+d)y_1 + (a+2d)z_1 + (a+3d)w_1 \geq S_{v_1} + ah\,. \tag{6.2.12}$$

In order to prove $M \geq S_{v_1} + a\left(\frac{a-a'}{3} + d\right)$, we only need to show that

$$h \geq \frac{a-a'}{3} + d$$

Since $\deg^-_{G_w(\boldsymbol{a})}(v) = 3$, for all vertex v in $G_w(\boldsymbol{a})$, then in order to take any $v_0 - v_1$ path in $G_w(\boldsymbol{a})$, we have to consider five options, shown as in Figure 6.4.

1. A $v_0 - v_1$ path P_0 of weight $a+d$.
2. A $v_0 - v_1$ path P_1 has the form

$$P_1 = R \cup D\,,$$

where R is a (nontrivial) $v_0 - v_0$ path in $G_w(\boldsymbol{a})$ (or full cycle) and D is an arc from v_0 to v_1 of weight $a+d$. Hence by Theorem 6.1.3, the minimum weight w_1 of the path P_1, is

$$w_1 = \begin{cases} \left(\frac{a}{3}(a+3d)\right) + (a+d)\,, & \text{if } a \equiv 0 \pmod 3\,, \\ \\ \left(\frac{a-1}{3}(a+3d) + (a+d)\right) + (a+d)\,, & \text{if } a \equiv 1 \pmod 3\,, \\ \\ \left(\frac{a-2}{3}(a+3d) + (a+2d)\right) + (a+d)\,, & \text{if } a \equiv 2 \pmod 3\,. \end{cases} \tag{6.2.13}$$

3. A $v_0 - v_1$ path P_2 has the form

$$P_2 = S \cup N \cup D\,,$$

where S is a $v_0 - v_{a-1}$ path in $G_w(\boldsymbol{a})$ and N is an arc from v_{a-1} to v_0 of weight $a+d$. From Theorem 6.1.1, the minimum weight w_2 of the path P_2, is given by

$$w_2 = \begin{cases} \left(\frac{a-3}{3}(a+3d) + (a+2d)\right) + 2(a+d)\,, & \text{if } a \equiv 0 \pmod 3\,, \\ \\ \left(\frac{a-1}{3}(a+3d)\right) + 2(a+d)\,, & \text{if } a \equiv 1 \pmod 3\,, \\ \\ \left(\frac{a-2}{3}(a+3d) + (a+d)\right) + 2(a+d)\,, & \text{if } a \equiv 2 \pmod 3\,. \end{cases} \tag{6.2.14}$$

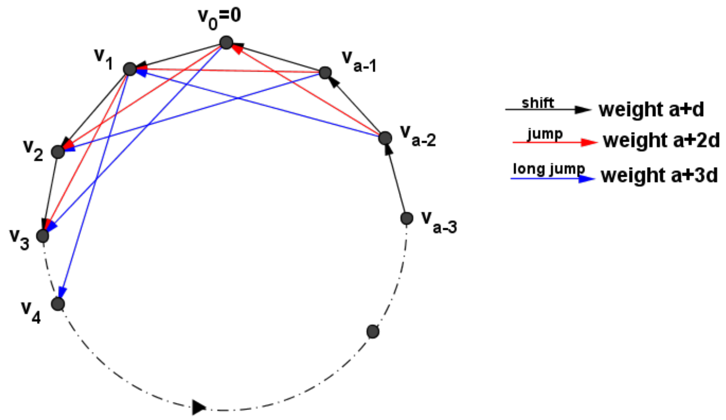

Figure 6.4: Number of paths from v_0 to v_1 in $G_w(\boldsymbol{a})$ around the full cycle

4. A $v_0 - v_1$ path P_3 has the form

$$P_3 = S \cup J\,,$$

where S is a $v_0 - v_{a-1}$ path in $G_w(\boldsymbol{a})$ and J is an arc from v_{a-1} to v_1 of weight $a + 2d$. Similar by Theorem 6.1.1, the minimum weight w_3 of the path P_3 is

$$w_3 = \begin{cases} \left(\frac{a-3}{3}(a+3d) + (a+2d)\right) + (a+2d)\,, & \text{if } a \equiv 0 \pmod 3\,, \\ \left(\frac{a-1}{3}(a+3d)\right) + (a+2d)\,, & \text{if } a \equiv 1 \pmod 3\,, \\ \left(\frac{a-2}{3}(a+3d) + (a+d)\right) + (a+2d)\,, & \text{if } a \equiv 2 \pmod 3\,. \end{cases} \tag{6.2.15}$$

5. A $v_0 - v_1$ path P_4 has the form

$$P_4 = V \cup \mathcal{T}\,,$$

where V is a path from $v_0 - v_{a-2}$ and $\mathcal{T}$ is an arc from v_{a-2} to v_1 of weight $a + 3d$ in $G_w(\boldsymbol{a})$. Similar by using Theorem 6.1.1, the minimum weight w_4 of the path P_4 is given by

$$w_4 = \begin{cases} \left(\frac{a-3}{3}(a+3d) + (a+d)\right) + (a+3d)\,, & \text{if } a \equiv 0 \pmod 3\,, \\ \\ \left(\frac{a-4}{3}(a+3d) + (a+2d)\right) + (a+3d)\,, & \text{if } a \equiv 1 \pmod 3\,, \\ \\ \left(\frac{a-2}{3}(a+3d)\right) + (a+3d)\,, & \text{if } a \equiv 2 \pmod 3\,. \end{cases} \tag{6.2.16}$$

By comparing (6.2.13), (6.2.14), (6.2.15) and (6.2.16) we conclude that

$$w_4 \leq w_3 \leq w_1 \leq w_2\,.$$

Then it immediately follows that the minimum weight of a $v_0 - v_1$ path around the full cycle will be the weight w_4. We can the minimum weight w_4 as

$$\begin{aligned} w_4 &= (a+d) + a\left(\frac{a-a^{'}}{3} + d\right) \\ &= S_{v_1} + a\left(\frac{a-a^{'}}{3} + d\right)\,, \end{aligned}$$

where $a^{'} \equiv a \pmod 3$, with $a^{'} \in \{0, 1, 2\}$.

Consequently, minimum value of a positive integer ah in (6.2.10) and (6.2.11) has to be at least $a(\frac{a-a^{'}}{3} + d)$. This implies that

$$h \geq \frac{a-a^{'}}{3} + d.$$

Using (6.2.12) we get

$$M \geq S_{v_1} + a\left(\frac{a-a^{'}}{3} + d\right) = S_{v_1} + a\left(\left\lfloor\frac{a}{3}\right\rfloor + d\right)\,.$$

as required. $\square$

Therefore, we have shown that the largest integer $M \equiv v_j \pmod a$ with $j = 1$, that is nonrepresentable in at least two distinct ways as a nonnegative integer linear combination of a, $a+d$, $a+2d$ and $a+3d$, is given by

$$M = \left(S_{v_1} + a\left(\left\lfloor\frac{a}{3}\right\rfloor + d\right)\right) - a = S_{v_1} + a\left(\left\lfloor\frac{a}{3}\right\rfloor + d - 1\right)\,.$$

Combining Lemmas 6.2.2, 6.2.3 and 6.2.4, we conclude that the largest integer $M \equiv v_j \pmod a$ with $0 \le j \le a-1$, nonrepresentable in at least two distinct ways as a nonnegative integer combination of $a, a+d, a+2d$ and $a+3d$, is given by

$$\begin{aligned} S_{v_1} + a\left(\left\lfloor \frac{a}{3} \right\rfloor + d\right) - a &= (a+d) + a\left(\left\lfloor \frac{a}{3} \right\rfloor + d - 1\right) \\ &= a\left\lfloor \frac{a}{3} \right\rfloor + d(a+1)\,. \end{aligned}$$

Thus, the 2-Frobenius number of the arithmetic progression $a, a+d, a+2d, a+3d$ with $1 \le d < a$ will be

$$\mathrm{F}_2(a, a+d, a+2d, a+3d) = a\left\lfloor \frac{a}{3} \right\rfloor + d(a+1)\,.$$

This completes the proof of Theorem 6.2.1. □

Furthermore, Lemma 6.2.4 shows that the largest integer $M \equiv v_j \pmod a$ with $0 \le j \le a-1$, that is nonrepresented in at least two distinct ways always corresponds to the vertex v_1 in $G_w(\boldsymbol{a})$ (i.e. $j = 1$).

In the following example we apply Theorem 6.2.1, to determine $\mathrm{F}_2(13, 18, 23, 28)$.

Example 6.2.5. To compute $\mathrm{F}_2(13, 18, 23, 28)$ of a 4 terms arithmetic progression, all we need to find the largest positive integer

$$M_j \equiv v_j \equiv jd \pmod{13}\,, \qquad 0 \le j \le 12\,,$$

for all vertex v_j in $G_w(13, 18, 23, 28)$ (as shown in Figure 6.5), that cannot be represented in least two distinct ways as a nonnegative integer linear combination of $13, 18, 23$ and 28. We give the calculations for the three cases when $j \in \{0, 7, 11\}$.

Let $j = 0$, we have to find the largest integer number

$$M_0 \equiv v_0 \equiv 0 \pmod{13}\,,$$

that cannot represented in at least two distinct ways as a nonnegative integer linear combination of $13, 18, 23$ and 28. Therefore by Lemma 6.2.3 and Corollary 6.1.4, we get

$$M_0 = S_{v_0} - 13 = (4(28) + 18) - 13 = 117\,.$$

And from Lemma 6.2.3, we deduce that any positive integer $M_0 > 117$ is represented in terms of $13, 18, 23, 28$.

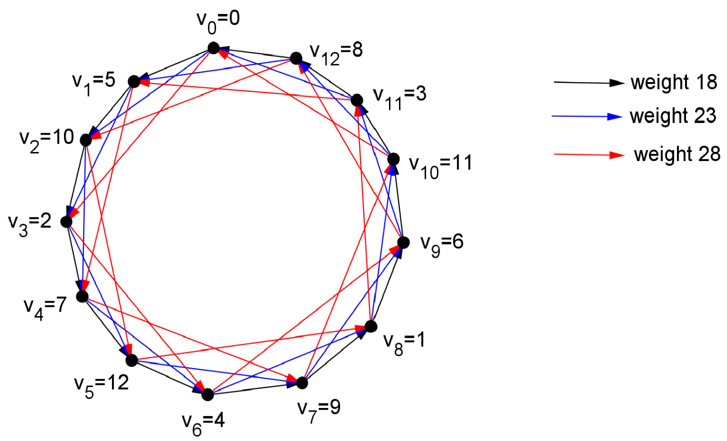

Figure 6.5: The Frobenius circulant graph of the arthmetic progression $13, 18, 23, 28$

Observe that, $130 \equiv 0 \pmod{13}$ and 130 has at least two distinct representations in terms of $13, 18, 23, 28$, as follows:

$$130 = 18 + 4 \cdot 28 = 2(23) + 3(28) = 10(13)\,.$$

Let $j = 7$. Then by Lemma 6.2.3 and Corollary 6.1.4, we deduce the largest integer

$$M_7 \equiv v_7 \equiv 9 \pmod{13}\,.$$

that cannot represented in at least two distinct ways as a nonnegative integer linear combination of $13, 18, 23$ and 28 is given by

$$M_7 = S_{v_7} - 13 = (2(28) + 18) - 13 = 61\,.$$

Hence from Lemma 6.2.3, we find that any positive integer $M_7 > 61$ is represented in terms of $13, 18, 23, 28$.

As we observe that $74 \equiv 9 \pmod{13}$ and 74 has at least two distinct representations in terms of $13, 18, 23, 28$ as follows:

$$74 = 18 + 2(28) = 2(23) + 28\,.$$

Let $j = 11$, then we have

$$M_{11} \equiv v_{11} \equiv 3 \pmod{13}\,.$$

Therefore by Lemma 6.2.2 and Corollary 6.1.4,

$$M_{11} = S_{v_{11}} = 3(28) + 23 = 107\,.$$

Using Lemma 6.2.2 yields any positive integer $M_{11} > 107$ can be represented in terms of $13, 18, 23, 28$.
observe that $120 \equiv 3 \pmod{13}$ and 120 has at least two distinct representations in terms of $13, 18, 23, 28$ as follows:

$$120 = 4(23) + 28 = 18 + 2(23) + 2(28) = 2(18) + 3(28) = 13 + 23 + 3(28)\,.$$

In exactly the same way, we can determine the others M_j, as shown in the table 6.1.

Table 6.1: A largest number $M_j \equiv v_j \pmod{13}$ with $0 \le j \le 12$, that cannot represented in at least two distinct ways as a nonnegative integer linear combination of $13, 18, 23, 28$.

vertices of $G_w(13, 18, 23, 28)$													
j	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}
v_j	0	5	10	2	7	12	4	9	1	6	11	3	8
M_j	117	122	23	28	33	51	56	61	79	84	89	107	112

Therefore by Theorem 6.2.1, the 2-Frobenius number of the arithmetic progression $13, 18, 23, 28$, is

$$\mathrm{F}_2(13, 18, 23, 28) = \max_{0 \le j \le 12}\{M_j\} = \max\{117, 122, 23, 28, 33, 51, 56, 61, 79, 84, 89, 107, 112\} = 122\,.$$

Note that by (6.2.1),

$$\mathrm{F}_2(13, 18, 23, 28) = 13\left(\frac{12}{3}\right) + 5(13+1) = 122\,.$$

6.2.1 Conclusion for $\mathrm{F}_2(a, a+d, a+2d, a+3d)$

Let $a, a+d, a+2d, a+3d$ be positive integers with $1 \le d < a$ and $\gcd(a, d) = 1$. Then we have

$$\mathrm{F}_2(a, a+d, a+2d, a+3d) = \begin{cases} \mathrm{F}_1(a, a+d, a+2d, a+3d) + 2d\,, & \text{if } a \equiv 2 \pmod{3}\,, \\ \\ \mathrm{F}_1(a, a+d, a+2d, a+3d) + (a+2d)\,, & \text{otherwise}\,. \end{cases} \tag{6.2.17}$$

We propose the following conjecture as a generalisation of Theorems 5.3.8 and 6.2.1.

Conjecture 3. *Let $n \geq 2$ be an integer and let a and d be coprime positive integers such that $1 \leq d < a$. Then*

$$\mathrm{F}_2(a, a+d, \ldots, a+nd) = a\left\lfloor \frac{a}{n} \right\rfloor + d(a+1)\,.$$

We checked numerical examples to verify Conjecture 3 up to $a = 10^5$, $d < a = a-1$ and $n = 15$, using MATLAB. For instance,

$$\mathrm{F}_2(24, 29, 34, 39, 44, 49, 54, 59, 64) = 24\left\lfloor \frac{24}{8} \right\rfloor + 5(25) = 197\,.$$

$$\mathrm{F}_2(34, 41, 48, 55, 62, 69) = 34\left\lfloor \frac{34}{5} \right\rfloor + 7(35) = 449\,.$$

$$\mathrm{F}_2(500, 990, 1480, 1970, 2460, 2950, 3440) = 500\left\lfloor \frac{500}{6} \right\rfloor + 490(501) = 286990\,.$$

From (5.3.32) and (6.2.17) we propose the following conjecture.

Conjecture 4. *Let $n \geq 2$ be an integer and let a and d be coprime positive integers such that $1 \leq d < a$. Then*

$$\mathrm{F}_2(a, a+d, \ldots, a+nd) = \begin{cases} \mathrm{F}_1(a, a+d, \ldots, a+nd) + 2d\,, & \textit{if } a \equiv t \ (\textit{mod } n)\,; 1 < t \leq (n-1) \\ \mathrm{F}_1(a, a+d, \ldots, a+nd) + (a+2d)\,, & \textit{otherwise}\,. \end{cases} \tag{6.2.18}$$

Let us present two numerical examples using MATLAB to explain Conjecture 4.

- To determine $\mathrm{F}_2(27, 31, 35, 39, 43, 47, 51, 55, 59, 63, 67, 71, 75, 79)$, we begin by finding t such that

  $$27 \equiv t \pmod{13}, \qquad t \in \mathbb{Z}_{\geq 0}\,.$$

  As we observe $27 \equiv 1 \pmod{13}$, we have

  $$\begin{aligned} &\mathrm{F}_2(27, 31, 35, 39, 43, 47, 51, 55, 59, 63, 67, 71, 75, 79) \\ &= \mathrm{F}_1(27, 31, 35, 39, 43, 47, 51, 55, 59, 63, 67, 71, 75, 79) + 35 \\ &= 131 + 35 = 166\,. \end{aligned}$$

- To find $\mathrm{F}_2(24, 29, 34, 39, 44, 49, 54, 59, 64, 69)$ we notice that

$$24 \equiv 6 \pmod 9\,.$$

Then

$$\begin{aligned}\mathrm{F}_2(24, 29, 34, 39, 44, 49, 54, 59, 64, 69) &= \mathrm{F}_1(24, 29, 34, 39, 44, 49, 54, 59, 64, 69) + 10\\ &= 163 + 10 = 164\,.\end{aligned}$$

Chapter 7

Conclusion and future work

In this final chapter, we conclude the results of this dissertation, and discuss the future work.

7.1 Conclusion

In Chapter 3 we studied the (normalised) distance between generalised Frobenius number $\mathrm{F}_s(\boldsymbol{a})$ and $\mathrm{F}_1(\boldsymbol{a})$ and covering radius of a difference body. We obtained a new upper bound for the generalised Frobenius number $\mathrm{F}_s(\boldsymbol{a})$, associated with a primitive vector $\boldsymbol{a} = (a_1, \ldots, a_k)^t \in \mathbb{Z}^k_{>0}$ when $s = 2$ and $k \geq 3$. This research is based on several results from geometry of numbers. We obtain an improvement on a result given by Aliev, Fukshansky, and Henk [2]. This part of the thesis has been published in [6].

In Chapter 4 we presented a special graph ($Circ(a_1, a_2)$), which we call 2-circulant graph and we given a new proof for the formula $\mathrm{F}_2(a_1, a_2) = 2a_1a_2 - a_1 - a_2$ by using only graph theoretical methods.

In Chapters 5 and 6, we considered a directed circulant graph (sometimes referred to as Frobenius circulant graph) $G_w(\boldsymbol{a})$ associated with a positive integer primitive vector $\boldsymbol{a} = (a_1, \ldots, a_k)^t$ in dimensions $k = 3$ and 4, respectively. Here a_i's are in the arithmetic progression $a, a + d, \ldots, a + nd$. We presented an explicit formula for the shortest path and the minimum distance between any two vertices of $G_w(\boldsymbol{a})$. Then we used Nijenhuis [66] approach to derive a relationship between the minimum weight S_{v_j} of paths from the initial vertex v_0 to the terminal vertex v_j in $G_w(\boldsymbol{a})$ and the representations of nonnegative integers in terms of $\boldsymbol{a}$. From this we obtained an explicit formula for computing the 2-Frobenius number $\mathrm{F}_2(a, a+d, \ldots, a+nd)$ for three or four integers in an arithmetic sequence ($n \in \{2,3\}$) with $1 \le d < a$ and $\gcd(a,d) = 1$,

$$\mathrm{F}_2(a, a+d, \ldots, a+nd) = a \left\lfloor \frac{a}{n} \right\rfloor + d(a+1)\,,$$

which is a generalisation on a result given by Roberts [73]. Based on these results, we state a conjecture on the behaviour of the 2-Frobenius numbers of a general arithmetic sequence $a, a+d, \ldots, a+nd$.

We also obtained a relationship between the 2-Frobenius number and the (classical) Frobenius number of the arithmetic sequence $a, a+d, \ldots, a+nd$, when $n \in \{2,3\}$.

7.2 Future work

In future work, we plan to prove (or disprove) Conjecture 3 and obtain a formula for the s-Frobenius number for arithmetic sequences of a given length. Using graph theoretic techniques, we expect to obtain an explicit formula for the minimum weight S_{v_j} of the path from an initial vertex v_0 to a terminal vertex v_j in $G_w(a, a+d, \ldots, a+nd)$, $n > 3$ and to analyse the relationship between S_{v_j} and the s-representations of a nonnegative integer in terms of $a, a+d, \ldots, a+nd$. We are aiming to use the same strategies as for the proof of Theorems 5.3.8 and 6.2.1, applying the approach of Nijenhuis [66]. We are also aiming to prove (or disprove) Conjecture 4.

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