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CONNECTIVE C*-ALGEBRAS

MARIUS DADARLAT AND ULRICH PENNIG

ABSTRACT. Connectivity is a homotopy invariant property of separable C^* -algebras which has three notable consequences: absence of nontrivial projections, quasidiagonality and a more geometric realization of KK-theory for nuclear C^* -algebras using asymptotic morphisms. The purpose of this paper is to further explore the class of connective C^* -algebras. We give new characterizations of connectivity for exact and for nuclear separable C^* -algebras and show that an extension of connective separable nuclear C^* -algebras is connective. We establish connectivity or lack of connectivity for C^* -algebras associated to certain classes of groups: virtually abelian groups, linear connected nilpotent Lie groups and linear connected semisimple Lie groups.

1. INTRODUCTION

Connectivity of separable C^* -algebras was introduced in our earlier paper [10] under different terminology, see Definition 2.1 below. The initial motivation for studying it stemmed from our search for homotopy-symmetric C^* -algebras. By a result of Loring and the first author [9], these are precisely the separable C^* -algebras for which one can unsuspend in the E-theory of Connes and Higson [6]. Using a result of Thomsen [34], we proved in [10] that connectivity is equivalent to homotopy-symmetry for all separable nuclear C^* -algebras. Moreover, we showed that connectivity has a number of important permanence properties. These facts allowed us to exhibit new classes of homotopy-symmetric C^* -algebras.

The purpose of this paper is to further explore the class of connective C^* -algebras. We are motivated by the following three properties they share:

(i) If A is a separable nuclear C^* -algebra, then KK(A, B) is isomorphic to the homotopy classes of completely positive and contractive (cpc) asymptotic morphisms from A to $B \otimes \mathcal{K}$ for any separable C^* -algebra B.

(ii) Connective C^* -algebras are quasidiagonal. In fact, if A is connective, then $A \otimes B$ is quasidiagonal for any C^* -algebra B. $(A \otimes B$ denotes the minimal tensor product.)

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(iii) Connective C^* -algebras do not have nonzero projections. In fact, if A is connective, then $A \otimes B$ does not have nonzero projections for any C^* -algebra B.

Connectivity is of particular interest in the case of group C^* -algebras. A countable discrete group G is called connective if the augmentation ideal I(G) defined as the kernel of the trivial representation $\iota: C^*(G) \to \mathbb{C}$ is a connective C^* -algebra. In view of properties (ii) and (iii) connectivity of G may be viewed as a stringent topological property that accounts simultaneously for the quasidiagonality of $C^*(G)$ and the verification of the Kadison-Kaplansky conjecture for certain classes of groups. Examples of nonabelian connective groups were exhibited in [10] and [11].

In this paper we give new characterizations of connectivity for exact and nuclear separable C^* -algebras, see Prop. 2.2, 2.3. We prove that connectivity of separable nuclear C^* -algebras is preserved under extensions, see Thm. 2.4. This is a key permanence property which does not hold for quasidiagonal C^* -algebras.

There is a close connection between the topology of the spectrum and connectivity, which we employ to reveal an obstruction to connectivity by using work of Blackadar and Cuntz [1] and Pasnicu and Rørdam [30]. In particular, we show that a countable discrete group G is not connective if the trivial representation ι is a shielded point of the unitary dual of G in the sense of Def. 2.9.

Motivated by this, we give a complete description of the neighborhood of ι in the spectrum of the Hantzsche-Wendt group G, which is a torsion free crystallographic group with holonomy $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, in Sec. 3.1. This allows us to prove that G is not connective in this case (Cor. 3.2). Moreover, we show that this group provides a counterexample to a conjecture from [8]. Specifically, we prove that the natural map $[[I(G), \mathcal{K}]] \to K^0(I(G))$ is not an isomorphism (Lem. 3.3). In contrast, we show that all torsion free crystallographic groups with cyclic holonomy are connective (Thm. 3.8).

Next, we investigate connectivity for C^* -algebras associated to Lie groups. We show that all noncompact linear connected nilpotent Lie groups have connective C^* -algebras (Thm. 4.3). Using classic results from representation theory in conjunction with permanence properties of connectivity, we show that if G is a linear connected complex semisimple Lie group, then $C_r^*(G)$ is connective if and only if G is not compact (Thm. 4.5). Moreover, if G is a linear connected real reductive Lie group, then $C_r^*(G)$ is connective if and only if G does not have a compact Cartan subgroup (Thm. 4.6).

A common denominator of our results concerning group C^* -algebras is that in all the cases we analyzed, $C^*(G)$ contains a large connective ideal.

2. Connective C^* -Algebras

2.1. Definitions and background. For a C^* -algebra A, the cone over A is defined as $CA = C_0[0, 1) \otimes A$ the suspension of A as $SA = C_0(0, 1) \otimes A$.

The first of the following two notions was introduced in [10, Def. 2.6 (i)], under a different terminology which we have abandoned. We use the abbreviation $cpc \ map$ for a completely positive and contractive map.

Definition 2.1. Let A be a C^* -algebra.

(a) A is connective if there is a *-monomorphism

$$\Phi \colon A \to \prod_n CL(\mathcal{H}) / \bigoplus_n CL(\mathcal{H})$$

which is liftable to a cpc map $\varphi \colon A \to \prod_n CL(\mathcal{H})$. (b) A is almost connective, if there is a (not necessarily liftable) *-monomorphism $\Phi \colon A \to \prod_n CL(\mathcal{H}) / \bigoplus_n CL(\mathcal{H})$.

For a discrete group G, we define I(G) to be the augmentation ideal, i.e. the kernel of the trivial representation $C^*(G) \to \mathbb{C}$. We will sometimes say that a discrete amenable group G is connective if the C^* -algebra I(G) is connective. Note that (almost) connective C^* -algebras do not have nonzero projections. Thus any connective C^* -algebra is nonunital. Our definition allows that the zero C^* -algebra $\{0\}$ is connective.

Let A and B be C^{*}-algebras. An asymptotic morphism is a family of maps $\{\varphi_t \colon A \to B\}_{t \in [0,\infty)}$ such that

a) for each $a \in A$ the map $t \mapsto \varphi_t(a)$ is norm-continuous and bounded,

b) for all $a, b \in A$ and $\lambda \in \mathbb{C}$, we have

ť

$$\lim_{t \to \infty} \|\varphi_t(a + \lambda b) - (\varphi_t(a) + \lambda \varphi_t(b))\| = 0$$
$$\lim_{t \to \infty} \|\varphi_t(ab) - \varphi_t(a) \varphi_t(b)\| = 0$$
$$\lim_{t \to \infty} \|\varphi_t(a^*) - \varphi_t(a)^*\| = 0$$

A discrete asymptotic morphism $(\varphi_n)_{n\in\mathbb{N}}$ between A and B is a family of maps $\varphi_n \colon A \to B$ that satisfies the analogous conditions as a) and b) above with the index set replaced by \mathbb{N} . A homotopy between two (discrete) asymptotic morphisms $(\varphi_t^0)_{t\in I}$ and $(\varphi_t^1)_{t\in I}$ is a (discrete) asymptotic morphism $H_t \colon A \to C[0, 1] \otimes B$, such that $\operatorname{ev}_i \circ H_t = \varphi_t^i$ for all $t \in I$, where I either denotes $[0, \infty)$ or \mathbb{N} . We will say that a (discrete) asymptotic morphism $(\varphi_t)_{t\in I}$ is completely positive and contractive (cpc) if each of the maps φ_t is cpc. The corresponding homotopy classes will be denoted as follows:

- [[A, B]] homotopy classes of asymptotic morphisms,
- $[[A, B]]_{\mathbb{N}}$ homotopy classes of discrete asymptotic morphisms,
- $[[A, B]]^{cp}_{\mathbb{N}}$ homotopy classes of discrete cpc asymp. morphisms

2.2. Characterizations of connectivity. In the following we give two more characterizations of connectivity for exact and respectively nuclear C^* -algebras.

Proposition 2.2. Let A be a separable exact C^* -algebra. Then A is connective if and only if there is an injective *-homomorphism $\pi: A \to \mathcal{O}_2$ which is null-homotopic as a discrete cpc asymptotic morphism. This means that $[[\pi]] = 0$ in the set $[[A, \mathcal{O}_2]]^{cp}_{\mathbb{N}}$.

Proof. (\Rightarrow) By assumption, there is a cpc discrete asymptotic morphism $\{\varphi_n : A \to C[0,1] \otimes \mathcal{O}_2\}_n$ such that $\varphi_n^{(0)} = \pi$ is an injective *-homomorphism and $\varphi_n^{(1)} = 0$. Thus, we can view $\{\varphi_n\}_n$ as an injective discrete asymptotic morphism $\{\varphi_n : A \to C_0[0,1) \otimes \mathcal{O}_2 \subset CL(\mathcal{H})\}_n$ and hence A is connective.

(\Leftarrow) Suppose that A is a separable exact connective C^* -algebra. By [10, Prop. 2.11] it follows that $[[A, \mathcal{O}_2 \otimes \mathcal{K}]]^{cp}_{\mathbb{N}}$ is an abelian group. By Kirchberg's embedding theorem, there is an injective *-homomorphism $\pi : A \to \mathcal{O}_2 \otimes \mathcal{K}$. Moreover $\pi \oplus \pi : A \to \mathcal{O}_2 \otimes \mathcal{K}$ is unitarily homotopy equivalent to π . It follows that $[[\pi]] \oplus [[\pi]] = [[\pi]]$ in the group $[[A, \mathcal{O}_2 \otimes \mathcal{K}]]^{cp}_{\mathbb{N}}$ and hence $[[\pi]] = 0$. After embedding $\mathcal{O}_2 \otimes \mathcal{K}$ into \mathcal{O}_2 we obtain the desired conclusion.

Proposition 2.3. Let A be a separable nuclear C^* -algebra. The following properties are equivalent.

- (i) A is connective.
- (ii) $A \otimes B$ is connective for some C^* -algebra B that contains a nonzero projection.
- (iii) $A \otimes B$ is connective for all C^* -algebras B
- (iv) $[[A, \mathcal{O}_2 \otimes \mathcal{K}]] = 0.$
- $(v) [[A, L(\mathcal{H}) \otimes \mathcal{K}]] = 0.$

Proof. The equivalences $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$ were established in [10]. (For $(ii) \Rightarrow (i)$ observe that A is a subalgebra of $A \otimes B$ if B contains a nonzero projection.)

 $(i) \Rightarrow (iv)$ and $(i) \Rightarrow (v)$. Let *B* be a σ -unital *C*^{*}-algebra such that KK(A, B) = 0, for instance $B = \mathcal{O}_2$ or $B = L(\mathcal{H})$. If *A* is connective, then *A* is homotopy symmetric and hence $[[A, B \otimes \mathcal{K}]] \cong KK(A, B) = 0$ by [10, Thm. 3.1]. Note that even though [10, Thm. 3.1] was stated for separable *C*^{*}-algebras *B* it is routine to extend the result to general *C*^{*}-algebras using the separability of *A*.

 $(iv) \Rightarrow (i)$ and $(v) \Rightarrow (i)$. Fix an embedding $\pi: A \to \mathcal{O}_2 \subset L(\mathcal{H}) \otimes \mathcal{K}$ and regard it as a constant asymptotic morphism $\{\pi_t: A \to L(\mathcal{H}) \otimes \mathcal{K}\}_t$. By assumption, $[[\pi_t]] = 0$ in $[[A, L(\mathcal{H}) \otimes \mathcal{K}]]$ and hence by restriction, $[[\pi_n]] = 0$ in $[[A, L(\mathcal{H}) \otimes \mathcal{K}]]_{\mathbb{N}}$. We shall view $L(\mathcal{H}) \otimes \mathcal{K}$ as a subalgebra of $L(\mathcal{H})$. The corresponding homotopy from the constant discrete morphism $\{\pi_n\}_n$ to zero will induce an embedding $\Phi: A \to \prod_n CL(\mathcal{H}) / \bigoplus_n CL(\mathcal{H})$ which is liftable to a cpc map $\varphi: A \to \prod_n CL(\mathcal{H})$ by the nuclearity of A.

2.3. Extensions of connective C^* -algebras. Connectivity of C^* -algebras has a plethora of permanence properties as proven in [10, Thm. 3.3]. In particular, it is inherited by split extensions [10, Thm. 3.3 (d)]. In the following theorem this result is extended to non-split extensions as well.

Theorem 2.4. Let $0 \to J \to A \to B \to 0$ be an exact sequence of separable nuclear C^* -algebras. If J and B are connective, then A is connective.

Proof. Since connectivity passes to nuclear subalgebras we may replace the given extension by

$$0 \to J \otimes \mathcal{O}_2 \otimes \mathcal{K} \to A \otimes \mathcal{O}_2 \otimes \mathcal{K} \to B \otimes \mathcal{O}_2 \otimes \mathcal{K} \to 0.$$

Adding to this extension a trivial absorbing extension, using the addition in Ext-theory, we obtain an absorbing extension

(1)
$$0 \to J \otimes \mathcal{O}_2 \otimes \mathcal{K} \to E \to B \otimes \mathcal{O}_2 \otimes \mathcal{K} \to 0,$$

which by construction has the property that $A \subset A \otimes \mathcal{O}_2 \otimes \mathcal{K} \subset E$, see for instance [4, Lemma 2.2]. Since $Ext(B \otimes \mathcal{O}_2, J \otimes \mathcal{O}_2) = 0$ as \mathcal{O}_2 is KKcontractible and we are dealing with an absorbing extension, it follows that the extension (1) splits by [24, Sec. 7] and so E is connective by [10, Thm. 3.3]. We conclude that $A \subset E$ is connective.

In the sequel we will need to use the following result from [10], which is based on [2].

Theorem 2.5 ([10], Cor. 3.4.). Let A be a separable continuous field of nuclear C^* -algebras over a compact connected metrizable space X. If one of the fibers of A is connective, then A is connective.

Corollary 2.6. Let A be a separable continuous field of nuclear C^* -algebras over a locally compact metrizable space X that has no compact open subsets. Then A is connective.

Proof. Let $Y = X \cup \{y_0\}$ be the one-point compactification of X. Then Y is a compact metrizable space which must be connected. Indeed, arguing by contradiction, say that $Y = U \cup V$ with U, V open and nonempty with $y_0 \in U$ and $U \cap V = \emptyset$. Then $V = V \cap X$ is both an open and compact subset of X.

We can view A as a continuous field over Y (see the remark on page 145 of [2]) and the fiber over y_0 satisfies $A(y_0) = \{0\}$. It follows that A is connective by Thm. 2.5.

2.4. Obstructions to connectivity. If A is a C^* -algebra, we denote by \widehat{A} the spectrum of A, which consists of all unitary equivalence classes of irreducible representations and by $\operatorname{Prim}(A)$ the primitive spectrum of A consisting of kernels of irreducible representations. The unitary dual \widehat{G} of a group G identifies with $\widehat{C^*(G)}$. Recall that \widehat{A} is topologized by pulling-back the Jacobson topology of $\operatorname{Prim}(A)$ under the natural map $\widehat{A} \to \operatorname{Prim}(A)$, $\pi \mapsto \ker \pi$, [14]. Let π and $(\pi_n)_n$ be irreducible representations of A acting on the same separable Hilbert space H. Suppose that $\|\pi_n(a)\xi - \pi(a)\xi\| \to 0$ for all $a \in A$ and $\xi \in H$. Then the sequence $(\pi_n)_n$ converges to π in the topology of \widehat{A} , see [14, Sec. 3.5].

Proposition 2.7. Let A be a separable C^* -algebra.

- (i) If Prim(A) contains a non-empty compact open subset, then A is not connective.
- (ii) If A is nuclear and Prim(A) is Hausdorff, then A is connective if and only if Prim(A) does not contain a non-empty compact open subset.

Proof. (i) Set X = Prim(A). If X has a non-empty compact open subset, then $A \otimes \mathcal{O}_2$ contains a nonzero projection by [30, Prop. 2.7] and hence A cannot be connective.

(ii) One implication follows from (i). For the other implication suppose that X = Prim(A) does not contain a non-empty compact open subset. Since X is Hausdorff by assumption, A is a nuclear separable continuous field over the locally compact space X, [16]. This is explained in detail in [3, Sec. 2.2.2]. Now we apply Cor. 2.6.

We would like thank Gabor Szabo for pointing out the following invariance property of connectivity.

Proposition 2.8. Let A and B be separable nuclear C^* -algebra with homeomorphic primitive spectra. Then A is connective if and only if B is connective.

Proof. Kirchberg's classification theorem [26] implies that if A and B are as in the statement, then $A \otimes \mathcal{O}_2 \otimes \mathcal{K} \cong B \otimes \mathcal{O}_2 \otimes \mathcal{K}$. The desired conclusion follows now from Proposition 2.3.

Definition 2.9. Let A be a separable C^* -algebra. A point $\pi \in \widehat{A}$ is called *shielded*, if $\widehat{A} \setminus \{\pi\} \neq \emptyset$ and any sequence $(\pi_n)_n$ in $\widehat{A} \setminus \{\pi\}$ which converges to π also converges to another point $\eta \in \widehat{A} \setminus \{\pi\}$.

Lemma 2.10. Let A be a unital separable C^* -algebra. If a point $\pi \in \widehat{A}$ is closed and shielded, then $I = \ker \pi$ is not connective.

Proof. Observe that $I \neq \{0\}$, since $\widehat{A} \setminus \{\pi\} \neq \emptyset$ and $\{\pi\}$ is closed. By Proposition 2.7 it suffices to show that $\operatorname{Prim}(I) = \operatorname{Prim}(A) \setminus \{I\}$ is a nonempty compact-open subset of $\operatorname{Prim}(A)$. Since $\{\pi\}$ is closed, it follows that $q^{-1}(I) = \{\pi\}$ and hence $q(\widehat{A} \setminus \{\pi\}) = \operatorname{Prim}(A) \setminus \{I\}$. The quotient map $q: \widehat{A} \to \operatorname{Prim}(A)$ is continuous and open, since the topology of \widehat{A} is defined as the preimage of the topology of $\operatorname{Prim}(A)$. Therefore, the lemma follows if we show that $\widehat{A} \setminus \{\pi\}$ is compact and open. $\widehat{A} \setminus \{\pi\}$ is open because $\{\pi\}$ is closed. Since \widehat{A} is compact and satisfies the second axiom of countability [14], it suffices to show that $\widehat{A} \setminus \{\pi\}$ is sequentially compact, [25, p. 138]. Let $(\pi_n)_n$ be a sequence in $\widehat{A} \setminus \{\pi\}$. By compactness of \widehat{A} it contains a subsequence $(\pi_{n_k})_k$ converging in \widehat{A} . If it converges to $\pi \in \widehat{A}$, then it also converges to some other point $\eta \in \widehat{A} \setminus \{\pi\}$, because π is shielded. Hence $\widehat{A} \setminus \{\pi\}$ is also compact.

Corollary 2.11. Let G be a countable discrete group. If the trivial representation $\iota \in \widehat{G}$ is shielded, then I(G) is not connective.

Proof. Since ι is a one-dimensional representation, it follows that $\{\iota\}$ is closed in \widehat{G} . Thus, the statement follows from Lemma 2.10.

3. Connectivity of crystallographic groups

It is known that there are precisely 10 closed flat 3-dimensional manifolds. Conway and Rossetti [7] call these manifolds platycosms ("flat universes"). The Hantzsche-Wendt manifold [17], or the didicosm in the terminology of [7], is the only platycosm with finite homology. Its fundamental group G, known as the Hantzsche-Wendt group, is generated by two elements x and y subject to two relations:

$$x^2yx^2 = y, \quad y^2xy^2 = x.$$

The group G is one of the classic torsion free 3-dimensional crystallographic groups, [17, 7]. It is useful to introduce the notation $z = (xy)^{-1}$.

A concrete realization of G as rigid motions of \mathbb{R}^3 is given by the following transformations X, Y, Z that correspond to the group elements x, y and z.

$$X(\xi) = A\xi + a, \quad Y(\xi) = B\xi + b, \quad Z(\xi) = C\xi + c, \quad \xi \in \mathbb{R}^3,$$

where

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and

$$a = \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 1/2 \\ 1/2 \end{pmatrix}, \quad c = \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \end{pmatrix}.$$

The transformations X^2 , Y^2 and Z^2 are just translations by unit vectors in the positive directions of the coordinate axes.

One shows (independently of the previous concrete realization) that the elements x^2 , y^2 and z^2 commute. Moreover one has the following relations in G:

$$\begin{array}{ll} xx^2x^{-1}=x^2, & xy^2x^{-1}=y^{-2}, & xz^2x^{-1}=z^{-2}\\ yx^2y^{-1}=x^{-2}, & yy^2y^{-1}=y^2, & yz^2y^{-1}=z^{-2}\\ zx^2z^{-1}=x^{-2}, & zy^2z^{-1}=y^{-2}, & zz^2z^{-1}=z^2 \end{array}$$

The subgroup N of G generated by x^2, y^2 and z^2 is normal in G and it is isomorphic to $\mathbb{Z}^3 \cong \mathbb{Z}x^2 \oplus \mathbb{Z}y^2 \oplus \mathbb{Z}z^2$. Let $q: G \to H = G/N$ denote the quotient map.

$$1 \longrightarrow N \longrightarrow G \xrightarrow{q} H \longrightarrow 1$$

H is isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ with generators are q(x) and q(y).

For later use, we will need the following identities that hold in G.

(2)
$$x^{-1}y = yxy^2z^{-2} = zx^2z^{-2}, \quad x^{-1}z = yz^2,$$

 $y^{-1}x = zx^2, \quad y^{-1}z = x(x^{-2}y^2).$

3.1. Induced representations and the unitary dual of G. Based on Corollary 2.11 we will show that I(G) for the Hantzsche-Wendt group G is not connective. This requires a thorough analysis of the spectrum \hat{G} .

Our basic reference for this section is the book of Kaniuth and Taylor [23]. The unitary dual of G consists of unitary equivalence classes of irreducible unitary representations of G and is denoted by \hat{G} . G acts on $\hat{N} \cong \mathbb{T}^3$ by $g \cdot \chi = \chi(g \cdot g^{-1})$. If we identify the character $\chi \in \hat{N}$ with the point $(\chi(x^2), \chi(y^2), \chi(z^2)) = (u, v, w) \in \mathbb{T}^3$, then the action of G is described as follows:

$$x \cdot (u, v, w) = (u, \bar{v}, \bar{w}), \quad y \cdot (u, v, w) = (\bar{u}, v, \bar{w}), \quad z \cdot (u, v, w) = (\bar{u}, \bar{v}, w).$$

The stabilizer of a character χ is the subgroup G_{χ} of G defined by $G_{\chi} = \{g \in G : \chi(g \cdot g^{-1}) = \chi(\cdot)\}$. It is clear that $N \subset G_{\chi}$ and that there is a bijection from G/G_{χ} onto the orbit of χ . In particular, the orbits of the action of G on \hat{N} can only have length 1, 2 or 4. Mackey has shown that each irreducible representation $\pi \in \hat{G}$ is supported by the orbit of some character $\chi \in \hat{N}$, in the sense that the restriction of π to N is unitarily equivalent to some

multiple m_{π} of the direct sum of the characters in the orbit of χ .

$$\pi_{|_N} \sim m_\pi \bigoplus_{g \in G/G_\chi} \chi(g \cdot g^{-1}).$$

In the sum above g runs through a set of coset representatives.

Mackey's theory has a particularly nice form for virtually abelian discrete groups. Let $\Omega \subset \widehat{N}$ be a subset which intersects each orbit of G exactly once. For each $\chi \in \widehat{N}$, let \widehat{G}_{χ} be the unitary dual of the stabilizer group G_{χ} and denote by $\widehat{G}_{\chi}^{(\chi)}$ the subset of \widehat{G}_{χ} consisting of classes of irreducible representations σ of G_{χ} such the restriction of σ to N is unitarily equivalent to a multiple of χ . Then, according to [23, Thm. 4.28]

Theorem 3.1.
$$\widehat{G} = \left\{ \operatorname{ind}_{G_{\chi}}^{G}(\sigma) \colon \sigma \in \widehat{G_{\chi}}^{(\chi)}, \ \chi \in \Omega \right\}.$$

Let ι be the trivial representation of G. We will prove that $\iota \in \widehat{G}$ is shielded by showing that any sequence $(\pi_n)_n$ of points in $\widehat{G} \setminus {\iota}$ that converges to ι has a subsequence which is convergent to a point $\eta \neq \iota$.

Let $R_{\ell} \subset \widehat{G}$ consist of those classes of irreducible representations which lie over ℓ -orbits, i.e. the orbits of length ℓ . Write \widehat{G} as the disjoint union $\widehat{G} = R_1 \cup R_2 \cup R_4$. It suffices to assume that all the elements π_n belong to the same subset R_{ℓ} . We distinguish the three possible cases for ℓ :

1-orbits. Consider the characters of N of the form $\chi = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$, where $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{\pm 1\}$. These are precisely the points in \hat{N} which are fixed under the action of G. In other words $G_{\chi} = G$. Let $(\pi_n)_n$ be a sequence of elements in $R_1 \subset \hat{G}$ and such that $(\pi_n)_n$ is convergent to ι . Since the restriction of π_n to N is a multiple of a character $\chi_n = (\varepsilon_1(n), \varepsilon_2(n), \varepsilon_3(n))$, it follows that $\varepsilon_1(n) = \varepsilon_2(n) = \varepsilon_3(n) = 1$ for all sufficiently large n and hence since π_n is irreducible, there is m such that $\pi_n = \iota$ for $n \geq m$. Hence there is no sequence in $R_1 \setminus \{\iota\}$ which converges to ι .

2-orbits. The characters $\chi \in \widehat{N}$ with orbits of length two are those $\chi = (u, v, w)$ where precisely only one of the coordinates is not equal to ± 1 . Let us argue first that if $(\pi_n)_n$ is a sequence of elements in $R_2 \subset \widehat{G}$ such that π_n lies over the orbit of $\chi_n = (u_n, v_n, w_n)$ and $(\pi_n)_n$ is convergent to ι , then two of the coordinates u_n, v_n, w_n must be equal to 1 for all sufficiently large n.

Suppose that each π_n lies over the orbit of a character χ_n of the form $\chi_n = (u_n, \varepsilon_2(n), \varepsilon_3(n))$ where $u_n \neq \pm 1$ and $\varepsilon_2(n), \varepsilon_3(n) \in \{\pm 1\}$. Then G_{χ_n} is generated by x, y^2, z^2 and $\{e, y\}$ are coset representatives for G/G_{χ_n} . Since $\pi \models \varepsilon_1 m (\chi_n(x) \oplus \chi_n(u, u^{-1}))$ it follows that

Since $\pi_n|_N \sim m_n(\chi_n(\cdot) \oplus \chi_n(y \cdot y^{-1})))$, it follows that

$$\pi_n(y^2) \sim m_n \begin{pmatrix} \varepsilon_2(n) & 0\\ 0 & \varepsilon_2(n) \end{pmatrix}, \quad \pi_n(z^2) \sim m_n \begin{pmatrix} \varepsilon_3(n) & 0\\ 0 & \varepsilon_3(n) \end{pmatrix}$$

and hence if $(\pi_n)_n$ converges to ι , then we must have $\varepsilon_2(n) = \varepsilon_3(n) = 1$ for all sufficiently large n. The cases $\chi_n = (\varepsilon_1, v_n, \varepsilon_3)$ and $\chi_n = (\varepsilon_1, \varepsilon_2, w_n)$ are treated similarly.

In view of the discussion above, it suffices to focus on characters of N the form $\chi = (u, 1, 1)$. The orbit of χ consists of two points, (u, 1, 1) and $(\bar{u}, 1, 1)$. The corresponding stabilizer G_{χ} is generated by x, y^2 and z^2 . In particular $G_{\chi} = N \cup xN$ and $G = G_{\chi} \cup y G_{\chi}$. The exact sequence

$$1 \longrightarrow N \longrightarrow G_{\chi} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1$$

does not split since G is torsion free. The quotient G/G_{χ} is generated by the coset of y. Let $\sigma \in \widehat{G}_{\chi}$ be an irreducible representation of G_{χ} whose restriction to N is a multiple of χ . Since $\chi(y^2) = \chi(z^2) = 1$, it follows that σ factors through G_{χ}/\mathbb{Z}^2 . Moreover we have a nontrivial central extension

$$1 \longrightarrow \mathbb{Z} \longrightarrow G_{\chi}/\mathbb{Z}^2 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1$$

where the normal subgroup is generated by the image of x^2 under the map $N \to N/\langle y^2, z^2 \rangle$ and the quotient group is generated by the image of q(x) under the map $H \to H/\langle q(y) \rangle$. Since G_{χ}/\mathbb{Z}^2 is an abelian group, σ must be a character such that $\sigma(x)^2 = \sigma(x^2) = \chi(x^2) = u$. Thus $\sigma(x) = a \in \mathbb{T}$ with $a^2 = u$. Let us compute the representation $\pi = \operatorname{ind}_{G_{\chi}}^G(\sigma)$ of G induced by σ . It acts on the Hilbert space

$$H_{\pi} = \{\xi : G \to \mathbb{C} : \xi(gh) = \sigma(h^{-1})\xi(g), \quad g \in G, h \in G_{\chi}\}.$$

Since $G = G_{\chi} \cup y G_{\chi}$, we can identify H_{π} with \mathbb{C}^2 via the isometry $\xi \mapsto (\xi(e), \xi(y))$. Then $\pi(g)\xi = \xi(g^{-1}\cdot)$ can be described using (2) as follows:

$$\begin{aligned} \pi(x)\xi(e) &= \xi(x^{-1}) = \sigma(x)\xi(e) = a\xi(e) \\ \pi(x)\xi(y) &= \xi(x^{-1}y) = \xi(yxy^2z^{-2}) = \sigma(z^2y^{-2}x^{-1})\xi(y) = \bar{a}\xi(y) \\ \pi(y)\xi(e) &= \xi(y^{-1}) = \xi(y \cdot y^{-2}) = \sigma(y^2)\xi(y) = \xi(y) \\ \pi(y)\xi(y) &= \xi(y^{-1} \cdot y) = \xi(e) \end{aligned}$$

which produces the following matrices with respect to the basis given above:

(3)
$$\pi(x) = \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix}, \quad \pi(y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \pi(z) = \begin{pmatrix} 0 & a \\ \bar{a} & 0 \end{pmatrix}$$

Corresponding to the characters (1, v, 1) and (1, 1, w) we obtain the irreducible representations, where we use the isometries $\xi \mapsto (\xi(e), \xi(x))$ and $\xi \mapsto (\xi(e), \xi(y))$ respectively:

(4)
$$\pi(x) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \pi(y) = \begin{pmatrix} b & 0 \\ 0 & \overline{b} \end{pmatrix}, \quad \pi(z) = \begin{pmatrix} 0 & \overline{b} \\ b & 0 \end{pmatrix}, \quad b^2 = v,$$

and

(5)
$$\pi(x) = \begin{pmatrix} 0 & \overline{c} \\ c & 0 \end{pmatrix}, \quad \pi(y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \pi(z) = \begin{pmatrix} c & 0 \\ 0 & \overline{c} \end{pmatrix}, \quad c^2 = w$$

Let $(\pi_n)_n$ be a sequence in R_2 that converges to ι in \widehat{G} . Arguing by symmetry, we may assume that each π_n is given by the formulas (3) corresponding to a sequence of points $u_n \in \mathbb{T}$ with $u_n \notin \{\pm 1\}$. Since $\pi_n \to \iota$ it follows from the equation (3) that $u_n \to 1$. Again from (3) we can compute the limits of the sequences $\pi_n(x)$ and $\pi_n(y)$ in U(2). This gives the representation $\pi: G \to U(2)$:

$$\pi(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \pi(y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

It is clear that π is a representation of G that factors through the left regular representation of $\mathbb{Z}/2$. Decompose π into a direct sum of characters $\pi \sim \iota \oplus \eta$. Then η is not equivalent to ι and $\pi_n \to \eta$ in \widehat{G} .

4-orbits. Let $\chi = (u, v, w) \in \mathbb{T}^3$ be a character of N with $u, v, w \notin \{\pm 1\}$. Its orbit under the action of G consists of four points and $G_{\chi} = N$. Let us compute the representation $\pi = \operatorname{ind}_N^G(\chi)$ of G induced by σ . It acts on the Hilbert space $H_{\pi} = \{\xi : G \to \mathbb{C} : \xi(gh) = \chi(h^{-1})\xi(g), g \in$ $G, h \in N\}$. Thus one can identify H_{π} with \mathbb{C}^4 via the isometry $\xi \mapsto$ $(\xi(e), \xi(x), \xi(y), \xi(z))$. Using the identities (2), we verify that $\pi(g)\xi =$ $\xi(g^{-1} \cdot)$ is described as follows:

$$\begin{aligned} \pi(x)\xi(e) &= \xi(x^{-1}) = \xi(x \cdot x^{-2}) = \chi(x^2)\xi(x) = u\xi(x) \\ \pi(x)\xi(x) &= \xi(e) \\ \pi(x)\xi(y) &= \xi(x^{-1}y) = \xi(zx^2z^{-2}) = \chi(z^2x^{-2})\xi(z) = w\bar{u}\xi(z) \\ \pi(x)\xi(z) &= \xi(x^{-1}z) = \xi(yz^2) = \chi(z^{-2})\xi(y) = \bar{w}\xi(y) \\ \pi(y)\xi(e) &= \xi(y^{-1}) = \xi(y \cdot y^{-2}) = \chi(y^2)\xi(y) = v\xi(y) \\ \pi(y)\xi(x) &= \xi(y^{-1}x) = \xi(zx^2) = \chi(x^{-2})\xi(z) = \bar{u}\xi(z) \\ \pi(y)\xi(y) &= \xi(e) \\ \pi(y)\xi(z) &= \xi(y^{-1}z) = \xi(x \cdot x^{-2}y^2) = \chi(y^{-2}x^2)\xi(x) = \bar{v}u\xi(x) \end{aligned}$$

producing the matrices:

(6)
$$\pi(x) = \begin{pmatrix} 0 & u & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \bar{u}w \\ 0 & 0 & \bar{w} & 0 \end{pmatrix}, \quad \pi(y) = \begin{pmatrix} 0 & 0 & v & 0 \\ 0 & 0 & 0 & \bar{u} \\ 1 & 0 & 0 & 0 \\ 0 & u\bar{v} & 0 & 0 \end{pmatrix}$$

It follows that

(7)
$$\pi(x^2) = \begin{pmatrix} u & 0 & 0 & 0 \\ 0 & u & 0 & 0 \\ 0 & 0 & \bar{u} & 0 \\ 0 & 0 & 0 & \bar{u} \end{pmatrix}, \quad \pi(y^2) = \begin{pmatrix} v & 0 & 0 & 0 \\ 0 & \bar{v} & 0 & 0 \\ 0 & 0 & v & 0 \\ 0 & 0 & 0 & \bar{v} \end{pmatrix}, \quad \pi(z^2) = \begin{pmatrix} w & 0 & 0 & 0 \\ 0 & \bar{w} & 0 & 0 \\ 0 & 0 & \bar{w} & 0 \\ 0 & 0 & 0 & w \end{pmatrix}$$

Let $(\pi_n)_n$ be a sequence in R_4 converging to ι in \widehat{G} . Each π_n is given by the formulas (6) corresponding to a sequence of points $(u_k, v_k, w_k) \in \mathbb{T}^3$ with $u_k, v_k, w_k \notin \{\pm 1\}$. Since $\pi_n \to \iota$ it follows from equation (7) that $u_k, v_k, w_k \to 1$. Again from (6) we can compute the limits of the sequences $\pi_n(x)$ and $\pi_n(y)$ in the space of unitary operators U(4). This gives the representation $\pi: G \to U(4)$:

$$\pi(x) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \pi(y) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

It is clear that π is a representation of G that factors through $H = \mathbb{Z}/2 \times \mathbb{Z}/2$. Decompose π into a direct sum of characters η_i . Since π is not equivalent to a multiple of the trivial representation, it follows that at least one of these characters is not equivalent to ι . On the other hand $\pi_n \to \eta_i$ in \widehat{G} for all i. Combining the above analysis with Corollary 2.11 we obtain immediately

Corollary 3.2. If G is the Hantzsche-Wendt group, then I(G) is not connective.

It was conjectured in [8] that if G is a torsion free discrete amenable group, then $[[I(G), \mathcal{K}]] \cong KK(I(G), \mathbb{C})$. We argue now that this conjecture fails for the Hantzsche-Wendt group. Indeed this follows from the previous corollary in conjunction with the following lemma.

Lemma 3.3. Let G be a residually finite torsion free discrete amenable group which admits a classifying space with finitely generated K-homology group $K_1(BG)$. Then $[[I(G), \mathcal{K}]] \cong KK(I(G), \mathbb{C})$ if and only if I(G) is connective.

Proof. Suppose first that $[[I(G), \mathcal{K}]] \cong KK(I(G), \mathbb{C})$. Since G is amenable and residually finite it follows that $C^*(G)$ is residually finite dimensional. Since G is amenable, G satisfies the Baum-Connes conjecture and $C^*(G)$ satisfies the UCT by results of Higson and Kasparov [22] and Tu [35]. In particular we have a short exact sequence

$$0 \to Ext^{1}(K_{1}(C^{*}(G)), \mathbb{Z}) \to KK(C^{*}(G), \mathbb{C}) \to \operatorname{Hom}(K_{0}((C^{*}(G)), \mathbb{Z}) \to 0$$

Let $\pi_n : C^*(G) \to M_{d(n)}(\mathbb{C})$ be a separating sequence of finite dimensional representations. The restriction of π_n to I(G) will be denoted by σ_n . By [8, Prop. 3.2] $(\pi_n)_* = d(n)\iota_* : K_0(C^*(G)) \to \mathbb{Z}$ and hence $[\sigma_n] \in Ext^1(K_1(I(G)), \mathbb{Z}) \subset KK(I(G), \mathbb{C})$ is a torsion element since $K_1(I(G)) \cong$ $K_1(BG)$ is finitely generated. After replacing π_n by a suitable multiple of itself we have arranged that $[\sigma_n] = 0$ in $KK(I(G), \mathbb{C})$ and hence $[[\sigma_n]] = 0$ in $[[I(G), \mathcal{K}]]$. Since the sequence (σ_n) separates the elements of I(G) it follows that I(G) is connective.

The converse is contained in the main result of [10] which shows that if A is a separable nuclear connective C^* -algebra, then $[[A, \mathcal{K}]] \cong KK(A, \mathbb{C})$. \Box

3.2. Crystallographic groups with cyclic holonomy. In this section we are going to show that torsion free crystallographic groups with cyclic holonomy are connective. Apart from this we isolate a lemma, which proves that I(G) for a group G which is a finite extension of a connective group always contains a "big" connective ideal. In particular, the lemma also holds for the Hantzsche-Wendt group.

The proof of both results uses some tools from the index theory of C^* subalgebras. A reference is [39]. Let Γ and G be discrete groups and let Hbe a finite group. Suppose that they fit into an exact sequence

$$1 \longrightarrow \Gamma \longrightarrow G \xrightarrow{q} H \longrightarrow 1 .$$

Let $E: C^*(G) \to C^*(\Gamma)$ be the faithful conditional expectation [39, Ex. 1.2.3] given on group elements by

$$E(g) = \begin{cases} g & \text{if } g \in \Gamma \\ 0 & \text{else} \end{cases}$$

Choose a lift $g_h \in G$ for each $h \in H$. The pairs (g_h^{-1}, g_h) form a quasi-basis in the sense of [39, Def. 1.2.2]. Let $\mathcal{E} = C^*(G)$ considered as a right Hilbert $C^*(\Gamma)$ -module, where the right action is induced by the inclusion $C^*(\Gamma) \to C^*(G)$ and the inner product is given by $\langle a, b \rangle = E(a^*b)$ [39, Sec. 2.1]. Note that \mathcal{E} is complete [39, Prop. 2.1.5]. The quasi-basis induces an isometric isomorphism of right Hilbert $C^*(\Gamma)$ -modules $u: \mathcal{E} \to \ell^2(H) \otimes C^*(\Gamma)$ with

$$u(a) = \sum_{h} \delta_h \otimes E(g_h a)$$

and inverse $u^* \colon \ell^2(H) \otimes C^*(\Gamma) \to \mathcal{E}$ with $u^*(\delta_h \otimes b) = g_h^{-1} b$. Let $\mathcal{L}_{C^*(\Gamma)}(\mathcal{E})$ be the bounded adjointable operators on \mathcal{E} and denote by $\mathcal{K}_{C^*(\Gamma)}(\mathcal{E})$ the compact ones. Then we have $\mathcal{L}_{C^*(\Gamma)}(\mathcal{E}) \cong \mathcal{K}_{C^*(\Gamma)}(\mathcal{E}) \cong \mathcal{K}(\ell^2(H)) \otimes C^*(\Gamma)$. The left multiplication of $C^*(G)$ on \mathcal{E} induces a *-homomorphism

$$\psi \colon C^*(G) \to \mathcal{K}(\ell^2(H)) \otimes C^*(\Gamma)$$

with matrix entries $\psi_{h',h}(a) = E(g_{h'} a g_h^{-1})$. Suppose we have $a \in C^*(G)$ with $\psi(a) = 0$. Then

$$a = \frac{1}{|H|} \sum_{h,h'} g_{h'}^{-1} E(g_{h'} a g_h^{-1}) g_h = 0$$

Hence, ψ is injective.

Lemma 3.4. Let Γ be a connective group and let H be a finite group. Suppose that the group G fits into a short exact sequence of the form

$$1 \longrightarrow \Gamma \longrightarrow G \xrightarrow{q} H \longrightarrow 1 \ .$$

Then $I(G, H) = \ker(I(q) \colon I(G) \to I(H))$ is connective as well.

Proof. Let $\iota: C^*(\Gamma) \to \mathbb{C}$ be the trivial representation and let ψ be the injective *-homomorphism constructed above. For all $b \in C^*(\Gamma) \subset C^*(G)$ we have $\psi_{h',h}(b) = \delta_{h',h} g_h b g_h^{-1}$. In particular, ψ embeds the ideal J generated by $I(\Gamma)$ into ker(id $\otimes \iota$) = $\mathcal{K}(\ell^2(H)) \otimes I(\Gamma)$, which is connective. Hence J is connective as well. It is clear that $J \subseteq I(G, H)$. Let $x \in I(G, H)$. By the property of the quasi-basis

$$0 = q(x) = \sum_{h \in H} q(E(x g_h^{-1})) h \quad \Rightarrow \quad q(E(x g_h^{-1})) = 0 \quad \forall h \in H \ .$$

Since $I(G, H) \cap C^*(\Gamma) = I(\Gamma)$ we obtain that $E(x g_h^{-1}) \in I(\Gamma)$ for all $h \in H$ and therefore $x = \sum_h E(x g_h^{-1})) g_h \in J$, hence J = I(G, H).

The proof of the second result uses an induction over the rank of the free abelian subgroup based on the following observation.

Lemma 3.5. Let m > 1 and let Γ and G be countable discrete groups that fit into the following short exact sequence

$$1 \longrightarrow \Gamma \longrightarrow G \xrightarrow{\pi} \mathbb{Z}/m\mathbb{Z} \longrightarrow 1 .$$

Suppose that Γ is connective and there are group homomorphisms $\varphi \colon G \to \mathbb{Z}$ and $q \colon \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$, such that $\pi = q \circ \varphi$. Then G is connective as well.

Proof. Let $\psi \colon C^*(G) \to \mathcal{K}(\ell^2(H)) \otimes C^*(\Gamma)$ be the injective *-homomorphism constructed above and let $\iota \colon C^*(\Gamma) \to \mathbb{C}$ be the trivial representation. Observe that $\rho = (\mathrm{id} \otimes \iota) \circ \psi$ satisfies

$$\rho(g\,\gamma)_{h',h} = \iota(E(g_{h'}g\,\gamma\,g_h^{-1})) = \iota(E(g_{h'}g\,g_h^{-1})\,g_h\,\gamma\,g_h^{-1}) = \iota(E(g_{h'}g\,g_h^{-1}))$$

for all $g \in G$ and $\gamma \in \Gamma$. In particular, ρ factors through the *-homomorphism $C^*(G) \to C^*(\mathbb{Z}/m\mathbb{Z})$ induced by π . By assumption this decomposes as

$$C^*(G) \xrightarrow{\varphi} C^*(\mathbb{Z}) \xrightarrow{q} C^*(\mathbb{Z}/m\mathbb{Z})$$
.

Altogether we obtain that ρ decomposes into a direct sum of one-dimensional representations, each of which is homotopic through representations to the trivial one. Hence, to show that G is connective, it suffices to construct a path through discrete asymptotic morphisms connecting a faithful morphism with a direct sum of copies of ρ .

Choose a path witnessing the connectivity of Γ , i.e. a discrete asymptotic morphism

$$H_n: C^*(\Gamma) \to C([0,1]) \otimes M_n(\mathbb{C})$$

such that for $H_n^{(t)} = \operatorname{ev}_t \circ H_n \colon C^*(\Gamma) \to M_n(\mathbb{C})$ we have that $H_n^{(0)}$ is faithful and $H_n^{(1)}$ is a multiple of ι . Then $(\operatorname{id}_{M_m(\mathbb{C})} \otimes H_n) \circ \psi$ has the desired properties. Hence, G is connective. \Box

We need the following elementary fact:

Lemma 3.6. Let a, b > 1 be integers and consider the exact sequence

$$0 \longrightarrow \mathbb{Z}/a\mathbb{Z} \xrightarrow{\cdot b} \mathbb{Z}/ab\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/b\mathbb{Z} \longrightarrow 0$$

with $\pi(x) = x \mod b$. Any generator of $\mathbb{Z}/b\mathbb{Z}$ lifts to a generator of $\mathbb{Z}/ab\mathbb{Z}$.

Proof. Let $\bar{y} \in \mathbb{Z}/b\mathbb{Z}$ be a generator and let $y \in \{0, \ldots, b-1\}$ be a representative. Let p_1, \ldots, p_s be the disctinct prime factors of ab, such that p_1, \ldots, p_r for $r \leq s$ are the ones not dividing y and p_{r+1}, \ldots, p_s divide y. Since gcd(y,b) = 1, the primes p_{r+1}, \ldots, p_s do not divide b. Let $x = y + p_1 \ldots p_r b$. We have for $i \in \{1, \ldots, r\}$ and $j \in \{r+1, \ldots, s\}$

$$x \equiv y \not\equiv 0 \mod p_i ,$$

$$x \equiv p_1 \dots p_r b \not\equiv 0 \mod p_i .$$

Hence gcd(x, ab) = 1 and $x \in \mathbb{Z}/ab\mathbb{Z}$ is a generator with $\pi(x) = \overline{y}$.

To start the induction we need the following lemma.

Lemma 3.7. Let G be a countable torsion free discrete group, which fits into an exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow G \longrightarrow \mathbb{Z}/m\mathbb{Z} \longrightarrow 0$$

Then G is isomorphic to \mathbb{Z} , hence in particular connective.

Proof. This can be proven by calculating $H^2(\mathbb{Z}/m\mathbb{Z},\mathbb{Z})$ for all $\mathbb{Z}/m\mathbb{Z}$ -module structures on \mathbb{Z} , but we give a direct argument here.

Let $x \in G$ be a lift of $1 \in \mathbb{Z}/m\mathbb{Z}$. Then G is generated by x and \mathbb{Z} . Moreover, $x^m \in \mathbb{Z} \cap Z(G)$, where Z(G) denotes the center of G. We have $\operatorname{Aut}(\mathbb{Z}) \cong GL_1(\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. If $t \in \mathbb{Z}$ denotes a generator, we therefore can only have $xtx^{-1} = t^{-1}$ or xt = tx. Suppose the first is true, then

$$x^m = x x^m x^{-1} = x^{-m} \quad \Rightarrow \quad x^{2m} = e$$

contradicting that G is torsion free. Thus, t and x commute and $x^m = t^n$ for some $n \in \mathbb{Z}$. Without loss of generality we can assume gcd(m, n) = 1. Indeed, if $m = m'\ell$ and $n = n'\ell$ with $\ell > 1$, then $(x^{m'}t^{-n'})^{\ell} = e$ and therefore $x^{m'} = t^{n'}$ also holds in G. Consider

$$\alpha \colon G \to \mathbb{Z} \quad ; \quad x^k t^\ell \mapsto k \, n + \ell \, m \; .$$

This is a well-defined group homomorphism, which is easily seen to be bijective as a consequence of gcd(m, n) = 1.

Theorem 3.8. Let G be a countable, torsion free, discrete group, which fits into an exact sequence of the form

$$0 \longrightarrow \mathbb{Z}^n \longrightarrow G \xrightarrow{\pi} \mathbb{Z}/m\mathbb{Z} \longrightarrow 0$$

for some $n, m \in \mathbb{N}$. Then G is connective.

Proof. This will be proven by induction over the rank of the free abelian subgroup. The case n = 1 follows from Lemma 3.7.

Observe that $Z(G) \neq \{e\}$. Indeed, let $x \in G$ be a lift of the generator of $\mathbb{Z}/m\mathbb{Z}$. Then G is generated by \mathbb{Z}^n and x. Moreover, $x^m \neq e$ since G is torsion free and $\pi(x^m)$ is trivial, hence $x^m \in \mathbb{Z}^n$. Thus, x^m commutes with \mathbb{Z}^n and x, hence with all elements of G, i.e. $x^m \in Z(G)$.

This implies that the transfer homomorphism $T: G \to \mathbb{Z}^n$ associated to the finite index subgroup \mathbb{Z}^n is non-trivial. Therefore there exists a surjective group homomorphism $\varphi: G \to \mathbb{Z}$. Let $q: \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ be the canonical quotient homomorphism and let $\overline{\varphi} = q \circ \varphi$. Let $H = \ker(\varphi)$. We have the following commutative diagram with exact rows and columns:



The value of a is chosen in such a way that $\operatorname{Im}(\pi|_H) \cong \mathbb{Z}/a\mathbb{Z}$ and b satisfies m = ab. The homomorphism π' is surjective since π and $\mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/b\mathbb{Z}$ are. The vertical arrow on the left hand side is induced by $\varphi|_{\mathbb{Z}^n}$.

Suppose $H \subset \ker(\pi) = \mathbb{Z}^n$. Then a = 1, b = m and $\pi = \pi' \circ \varphi$. By Lemma 3.5, G is then connective. So we may assume a > 1.

We claim that there is an element $g \in G$ such that $\varphi(g) = 1$ and $\pi(g)$ is a generator of $\mathbb{Z}/m\mathbb{Z}$. This is constructed as follows: If b = 1, we can choose $g \in G$, such that $\varphi(g) = 1$ and modify it by an element in H to achieve that $\pi(g)$ becomes a generator. Otherwise, choose $g' \in G$ such that $\varphi(g') = 1$ and note that $\pi'(\varphi(g'))$ is a generator of $\mathbb{Z}/b\mathbb{Z}$ by surjectivity. We can lift $\pi'(\varphi(g'))$ to a generator $x \in \mathbb{Z}/m\mathbb{Z}$ by Lemma 3.6. Note that $\pi(g') - x \in \mathbb{Z}/a\mathbb{Z}$ and lift this difference to an element $h \in H$. Let $g = g' h^{-1}$. Then $\varphi(g) = \varphi(g') = 1$ and $\pi(g) = \pi(g') - \pi(h) = \pi(g') - \pi(g') + x = x$.

Let $G' = \ker(\bar{\varphi}) = \{g \in G \mid \varphi(g) = \ell \cdot m \text{ for } \ell \in \mathbb{Z}\} \supset H$. Hence, the following diagram has exact rows.

The group G is generated by \mathbb{Z}^n and the element g constructed above. We have $g^m \in \mathbb{Z}^n \cap Z(G)$ and $\varphi(g^m) = m$. In particular, $g^m \in Z(G')$. Let

$$\psi \colon H \times \mathbb{Z} \to G' \quad ; \quad (h,k) \mapsto h \cdot g^{mk}$$

This is a group homomorphism, since g^m is central and it fits into the commutative diagram with exact upper and lower part



proving that ψ is in fact an isomorphism. By the upper row in diagram (8) and Lemma 3.5, the connectivity of G follows if $H \times \mathbb{Z}$, hence H, is connective [10, Thm. 4.1]. But H fits into a short exact sequence of the form

$$0 \to A \to H \to \mathbb{Z}/a\mathbb{Z} \to 0$$

where A is the free abelian kernel of the nonzero homomorphism $\mathbb{Z}^n \to b\mathbb{Z}$ from above, which has rank (n-1). This completes the induction step. \Box

4. Connectivity of Lie group C^* -algebras

In this section we determine which linear connected nilpotent Lie groups and which linear connected reductive Lie groups have connective reduced C^* -algebras. Let us recall that nilpotent connected Lie groups are liminary as shown by Dixmier [13] and Kirillov [27] and semisimple connected Lie groups are liminary as shown by Harish-Chandra [18].

4.1. Solvable and nilpotent Lie groups. A locally compact group N is compactly generated if $N = \bigcup_n V^n$ for some compact subset V of N. Every connected locally compact group is automatically compactly generated. The structure of abelian compactly generated locally compact groups is known. If N is such a group, then $N \cong \mathbb{R}^n \times \mathbb{Z}^m \times K$ for integers $n, m \ge 0$ and K a compact group, [12, Thm. 4.4.2].

Proposition 4.1. If G is a second countable locally compact amenable group (for example a solvable Lie group) whose center contains a noncompact closed connected subgroup, then $C^*(G)$ is connective.

Proof. Let N be a noncompact closed connected subgroup of Z(G). Then, by the structure theorem quoted above, N must have a closed subgroup isomorphic to \mathbb{R} . Consider the central extension:

$$0 \to \mathbb{R} \to G \to H \to 0.$$

Since G is amenable, by [29, Thm. 1.2] (as explained in [15, Lemma 6.3]), $C^*(G)$ has the structure of a continuous field of C^* -algebras over $\widehat{\mathbb{R}} \cong \mathbb{R}$. The desired conclusion follows from Cor. 2.6 since \mathbb{R} has no compact open subsets.

Example 4.2. We give here two examples that complement Proposition 4.1.

(i) Simply connected solvable Lie groups can have discrete noncompact centers. This is the case for $G = \mathbb{C} \rtimes_{\alpha} \mathbb{R}$ where $\alpha : \mathbb{R} \to \operatorname{Aut}(\mathbb{C})$ is defined by $\alpha(t)(z) = e^{it}z$ for $t \in \mathbb{R}$ and $z \in \mathbb{C}$. In this case $Z(G) = \{0\} \times 2\pi\mathbb{Z}$.

Nevertheless in this case $C^*(G)$ is connective. Consider the extension

$$0 \to Z(G) \to G \to G/Z(G) \cong \mathbb{C} \times \mathbb{T} \to 0.$$

Then $C^*(G)$ is a continuous $C(\mathbb{T})$ -algebra whose fiber at 1 is the algebra $C^*(\mathbb{C} \times \mathbb{T})$. Since $C^*(\mathbb{C} \times \mathbb{T}) \cong C_0(\mathbb{R}^2) \otimes c_0(\mathbb{Z})$ is connective, so is $C^*(G)$.

(ii) Both the real and the complex "ax + b" groups

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in F^{\times}, b \in F \right\}$$

where $F = \mathbb{R}$ or $F = \mathbb{C}$ are solvable with trivial center and their C^* -algebras contain a copy of the compacts \mathcal{K} , see [33], and so they are not connective.

Theorem 4.3. Let G be a (real or complex) linear connected nilpotent Lie group. Then $C^*(G)$ is connective if and only if G is not compact.

Proof. We view G as a real Lie group. By [37, Chap. 2, Thm. 7.3], if G is a linear connected nilpotent Lie group, then G decomposes as a direct product $G = T \times N$ of a torus T and a simply connected nilpotent group N. If G is compact, then G = T and $C^*(G)$ is isomorphic to a direct sum of \mathbb{C} so that it is not connective. If G is noncompact, then N is nontrivial and so the center of G is given by $Z(G) = T \times Z(N)$, where the center Z(N) of N is isomorphic to \mathbb{R}^n for some $n \geq 1$. We conclude the proof by applying Proposition 4.1.

Remark 4.4. It is not true that a liminary (CCR) C^* -algebra is connective if and only if does not have nonzero projections. Indeed

$$A = \left\{ f \in C([0,1], M_2(\mathbb{C})) \colon f(0) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \ f(1) = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}, \ \lambda \in \mathbb{C} \right\}$$

does not contain nonzero projections but is not connective since Prim(A) is homeomorphic to a circle S^1 and hence it is compact (and open in itself).

4.2. Reductive Lie groups. A linear connected reductive group G is a closed group of real or complex matrices that is closed under conjugate transpose. In other words G is a closed and selfadjoint subgroup of the general linear group over either \mathbb{R} or \mathbb{C} . A linear connected semisimple group is a linear connected reductive group with finite center [28].

Say $G \subset GL(n, \mathbb{R})$ or $GL(n, \mathbb{C})$. Define $K = G \cap O(n)$, or $K = G \cap U(n)$ in the complex case. If G is linear connected reductive, then K is compact, connected and is a maximal compact subgroup of G [28, Prop.1.2].

Let G = KAN be the Iwasawa decomposition of the linear connected semisimple Lie group G. A is abelian and N is nilpotent and both are closed simply connected subgroups of G, [28, Thm. 5.12].

First we consider the case of complex Lie groups.

Theorem 4.5. If G is a linear connected complex semisimple Lie group, then $C_r^*(G)$ is connective if and only if G is not compact.

Proof. If G is compact, $C_r^*(G)$ is isomorphic to a direct sum of matrix algebras and hence it is not connective as it contains nonzero projections.

Conversely, suppose now that G is a non-compact linear connected semisimple complex Lie group. Note that from the Cartan decomposition G = KAK [28, Thm. 5.20] it follows that since G is non-compact, so is A and therefore $A \cong \mathbb{R}^n$, for some $n \ge 1$.

Let M be the centralizer of A in K. By Lemma 3.3 and Proposition 4.1 of [31], it follows that

$$C_r^*(G) \subset C_0(\widehat{M} \times \widehat{A}, \mathcal{K}).$$

Since $\widehat{M} \times \widehat{A} \cong \widehat{M} \times \mathbb{R}^n$ does not have nonempty compact open subsets, it follows from Proposition 2.7(ii) that $C_0(\widehat{M} \times \widehat{A}, \mathcal{K})$ is connective. This completes the proof since connectivity passes to C^* -subalgebras. \Box

Next we consider the case of linear connected real reductive Lie groups. An element $g \in G$ is semisimple if it can be diagonalized over \mathbb{C} when viewed as a matrix $g \in M_n(\mathbb{C})$.

A closed subgroup H of G is called a Cartan subgroup if it is a maximal abelian subgroup consisting of semisimple elements, [21, p.67]. If G is either compact or a complex Lie group, then all Cartan subgroups of G are connected and they are conjugated inside G. In the general case G has finitely many Cartan subgroups up to conjugacy and Cartan subgroups can have finitely many connected components.

We denote by $\widehat{G}_d \subset \widehat{G}$ the discrete series representations. It consists of unitary equivalence classes of square-integrable representations

$$\sigma: G \to U(H_{\sigma}).$$

Harish-Chandra has shown that the discrete series representations of a semisimple Lie group G are parametrized by compact Cartan subgroups and in particular G has discrete series representations if and only if it has a compact Cartan subgroup, [19, 20].

We recall the following facts from [21, p.72] concerning cuspidal parabolic subgroups. Let H be a Cartan subgroup of G. Then H decomposes as a direct product $H = TA = T \times A$, where T is an abelian compact group and A is a vector group isomorphic to \mathbb{R}^n for $n \ge 0$. The case n = 0 occurs when H is a compact Cartan subgroup. The centralizer of A in G denoted by

$$L = C_G(A) = \{g \in G \colon ga = ag, \forall a \in A\}$$

is a Levi subgroup of G. This means that there is a parabolic subgroup of G of the form P = LN (not unique) with L as Levi subgroup. Since A is central in L, H is a relatively compact Cartan subgroup of L, i.e. H/Z(L) is compact. This implies that L has discrete series representations. Such a parabolic subgroup P = LN is called cuspidal.

One can further decompose L = MA to obtain a Langlands decomposition

$$P = MAN = MA \ltimes N,$$

with N a unipotent group. If H is a compact Cartan subgroup, then L = P = G by [21, p.72].

We will write $P = M_P A_P N_P$ whenever we want emphasize the components of P.

The description of $C_r^*(G)$ relies on the analysis of the unitary principal series representations of G associated to parabolic cuspidal subgroups P(also called the *P*-principal series). They are of the form

$$\operatorname{Ind}_{P}^{G}(\sigma \otimes \omega \otimes 1_{N})$$

where $\sigma \in \widehat{M}_d$ and $\omega \in \widehat{A}$ and 1_N is the trivial representation of N.

Consider two pairs (P_i, σ_i) , i = 1, 2 consisting of cuspidal parabolic subgroups of G and irreducible square-integrable unitary representations of the subgroups M_i , i = 1.2. We say that the pairs are *associated* if there is $g \in G$ such that $gP_1g^{-1} = P_2$ and $\sigma_1(g \cdot g^{-1})$ is unitarily equivalent to σ_2 . This is an equivalence relation [5, Def. 5.2]. We denote by $[P, \sigma]$ the equivalence class of the pair (P, σ) .

The following statement is based on the calculation of $C_r^*(G)$ by A. Wassermann [38] although we don't really use the full strength of his results. An expanded treatment of the structure of $C_r^*(G)$ appears in [5].

Let $J(G) = \bigcap_{\pi \in \widehat{G}_d} \ker(\pi) \subset C_r^*(G)$ be the common kernel of the discrete series representations. The following theorem shows that the K-homology of $C_r^*(G)$ can be described in terms of homotopy classes asymptotic morphisms $C_r^*(G) \to \mathcal{K}$ which factor through J(G) and discrete series representations.

Theorem 4.6. Let G be a linear connected real reductive Lie group. Then $C_r^*(G) \cong J(G) \oplus \bigoplus_{\sigma \in \widehat{G}_d} K(H_{\sigma})$ and J(G) is a connective liminary C^* -algebra. Moreover, the following assertions are equivalent:

- (i) $C_r^*(G)$ is connective,
- (ii) G does not have discrete series representations,
- (iii) G does not have a compact Cartan subgroup,
- (iv) There are no nonzero projections in $C_r^*(G)$.

Proof. As explained in [38, p.560], [5, p.1306] the reduced C^* -algebra of a linear reductive connected Lie group admits an embedding

$$C_r^*(G) \hookrightarrow \bigoplus_{[P,\sigma]} C_0(\widehat{A}_P, \mathcal{K}(H_\sigma)),$$

where the direct sum is over equivalence classes $[P, \sigma]$ as above. It is important to emphasize that if G has a compact Cartan subgroup, then G itself is one of the cuspidal parabolic subgroups and we have:

$$C_r^*(G) \hookrightarrow \bigoplus_{[P,\sigma]} C_0(\widehat{A}_P, \mathcal{K}(H_\sigma)) \oplus \bigoplus_{\sigma \in \widehat{G}_d} \mathcal{K}(H_\sigma),$$

where the first direct sum involves proper cuspidal parabolic subgroups $P = M_P A_P N_P$ and hence dim $(\hat{A}_P) > 0$. Moreover by [38], [5]:

(9)
$$C_r^*(G) \cong J(G) \oplus \bigoplus_{\sigma \in \widehat{G}_d} \mathcal{K}(H_\sigma)$$

where

$$J(G) \hookrightarrow \bigoplus_{[P,\sigma]} C_0(\widehat{A}_P, \mathcal{K}(H_\sigma))$$
.

Hence, J(G) is connective being a subalgebra of a connective C^* -algebra. The first part of the statement follows now from the decomposition (9).

The equivalence $(ii) \Leftrightarrow (iii)$ is Harish-Chandra's result mentioned earlier. In view of the decomposition (9), (ii) implies that $C_r^*(G) = J(G)$ and hence (i) since J(G) is always connective. Connective C^* -algebras do not contain nonzero projections and hence $(i) \Rightarrow (iv)$. Finally by using (9) again, we see that $(iv) \Rightarrow (ii)$ since $\mathcal{K}(H_{\sigma})$ contains nonzero projections if $H_{\sigma} \neq 0$. \Box

4.3. A remark on full C^* -algebras of Lie groups. The full C^* -algebra $C^*(G)$ of a property (T) Lie group G contains nonzero projections and hence it is not connective, see [36]. Nevertheless, inspection of several classes of examples indicates that $C^*(G)$ has interesting connective ideals that arise naturally from the representation theory of G. We postpone a detailed discussion of what is known for another time, but would like to mention two examples.

If G is a connected semisimple Lie group with finite center, then $C^*(G)$ is liminary (or CCR), see [40, p.115].

Proposition 4.7. (a) $C^*(SL_2(\mathbb{C}))$ is connective. (b) $C^*(SL_3(\mathbb{C})) = I(SL_3(\mathbb{C})) \oplus \mathbb{C}$ and $I(SL_3(\mathbb{C}))$ is connective.

Proof. (a) $C^*(SL_2(\mathbb{C}))$ was computed by Fell [16, Thm. 5.4]. We describe now his result. Let Z be the subspace of \mathbb{R}^2 defined by $Z = \bigcup_{n=0}^{\infty} \{n\} \times L_n$ where $L_0 = (-1, \infty)$ and $L_n = (-\infty, \infty)$ for all $n \ge 1$. Endow Z with the induced topology from \mathbb{R}^2 . Let H_0 be a separable infinite dimensional Hilbert space, let $H = H_0 \oplus \mathbb{C}$ and fix a unitary operator $V : H_0 \to H$. Then $C^*(SL_2(\mathbb{C}))$ is isomorphic to

$$\{F \in C_0(Z, \mathcal{K}(H)) \colon F(0, -1) = V^*F(2, 0)V \oplus \lambda, \text{ for some } \lambda \in \mathbb{C}\}\$$

Since Z has no nonempty open compact subsets it follows that $C_0(Z, \mathcal{K}(H))$ is connective and therefore so is its subalgebra $C^*(SL_2(\mathbb{C}))$.

(b) This will be obtained as a consequence of the following result on the structure of $C^*(SL_3(\mathbb{C}))$ obtained by Pierrot [32]. Let $G = SL_3(\mathbb{C})$ and denote by $\lambda_G : C^*(G) \to C_r^*(G)$ the morphism induced by the left regular representation and by $\iota_G : C^*(G) \to \mathbb{C}$ the trivial representation. Pierrot

proved that the kernel J of the morphism $\lambda_G \oplus \iota_G : C^*(G) \to C^*_r(G) \oplus \mathbb{C}$ is a contractible C^* -algebra. The representation ι_G is isolated since G has property (T). Therefore there is an exact sequence

$$0 \to J \to I(G) \to C_r^*(G) \to 0$$

where J is contractible and $C_r^*(G)$ is connective by Theorem 4.5. We conclude that I(G) is connective by applying Theorem 2.4.

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